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Generalized colored circular palindromic compositions

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We derive the generating function (g.f.) of the number of colored circular palindromic compositions of N with K parts in terms of the g.f. of an input sequence a that determines how many different colors each part of the composition can have. As a result, we get the g.f. of the number of all colored circular palindromic compositions of N. Using the latter formula and the g.f. of the number of colored circular compositions, we may easily derive the g.f. of the number of all colored dihedral compositions of N.

1. Introduction

A *linear composition* of a positive integer N with length K is a K-tuple $(\lambda_1, \dots, \lambda_K) \in \mathbb{Z}_{>0}^K$ such that

$$N = \lambda_1 + \cdots + \lambda_K.$$

Here the numbers $\lambda_1, \dots, \lambda_K$ are called *parts* of the composition. For example, (1, 2, 3, 3) and (3, 3, 2, 1) are two different linear compositions of $N = 9$ with $K = 4$ parts each.

Cyclic or circular compositions of N with length K are equivalence classes on the set of all linear compositions of N with length K such that two compositions are equivalent if and only if one can be obtained from the other by a cyclic shift. For example, {(1, 2, 3, 3), (3, 1, 2, 3), (3, 3, 1, 2), (2, 3, 3, 1)} and {(3, 3, 2, 1), (1, 3, 3, 2), (2, 1, 3, 3), (3, 2, 1, 3)} are two different cyclic or circular compositions of $N = 9$ with $K = 4$ parts each.

Cyclic or circular compositions were studied, for example, in [Ferrari and Zagaglia Salvi 2018; Gibson et al. 2018; Hadjicostas 2016; 2017; Knopfmacher and Robbins 2010; Sommerville 1909; Zagaglia Salvi 1999].

A (*bilaterally*) *symmetric cyclic composition* is a *circular palindrome*, that is, a circular or cyclic composition with at least one axis of (reflective) symmetry. Such circular or cyclic palindromic compositions were studied in [Bower 2010; Hadjicostas and Zhang 2017; Sommerville 1909; Williamson 1972].

For example, {(1, 3, 2, 3), (3, 1, 3, 2), (2, 3, 1, 3), (3, 2, 3, 1)} is a circular palindromic composition of $N = 9$ with $K = 4$ parts with one axis of symmetry (through 1 and 2). On the other hand, {(1, 2, 1, 2), (2, 1, 2, 1)} is a circular palindromic composition of $N = 6$ with $K = 4$ parts and two axes of symmetry (through the two 1's and through the two 2's). Finally, {(1, 1, 2, 2), (2, 1, 1, 2), (2, 2, 1, 1), (1, 2, 2, 1)} is a circular palindromic composition of $N = 6$ with $K = 4$ parts and one axis of symmetry (that passes through no part).

Dihedral compositions of N with length K are equivalence classes on the set of all linear compositions of N with length K such that two compositions are equivalent if and only if one can be obtained from the

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other by a cyclic shift or a reversal of order. Such compositions were studied, for example, in [Hadjicostas 2017; Knopfmacher and Robbins 2013; Zagaglia Salvi 1999]. For example,

$$\{(1,2,3,3),(3,1,2,3),(3,3,1,2),(2,3,3,1),(3,3,2,1),(1,3,3,2),(2,1,3,3),(3,2,1,3)\}$$

is a dihedral composition of $N=9$ with $K=4$ parts.

A *colored linear composition* of N with K parts according to an *input sequence*

$$a=(a(m):m\in\mathbb{Z}_{>0}),\quad \text{where } a(m)\in\mathbb{Z}_{\geq 0} \text{ for each } m\in\mathbb{Z}_{>0},$$

is a K-tuple $((\lambda_1,m_1),\dots,(\lambda_K,m_K))$ such that $\lambda_i,m_i\in\mathbb{Z}_{>0}$ for $i=1,\dots,K$ with

$$N=\lambda_1+\cdots+\lambda_K \quad\text{and}\quad 1\leq m_i\leq a(\lambda_i) \quad \text{for } i=1,\dots,K. \tag{1-1}$$

Note that, when $a(m)=0$ for some $m\in\mathbb{Z}_{>0}$, no part λ_i of the colored linear compositions of N we are considering (according to the input sequence a) can equal m.

For example, $(1_2,2_3,1_3,2_1)$ is a colored linear composition of $N=6$ with $K=4$ parts, where the parts have colors 2, 3, 3 and 1, respectively. This is a colored composition with respect to any input sequence $a=(a(m):m\in\mathbb{Z}_{>0})$ that satisfies $a(m)\in\mathbb{Z}_{\geq 0}$ for each $m\in\mathbb{Z}_{>0}$ and the inequalities $a(1)\geq 2$ and $a(2)\geq 2$.

Colored cyclic compositions, *colored* (*bilaterally*) *symmetric cyclic compositions* (i.e., *colored palindromic cyclic compositions*), and *colored dihedral compositions* of N with length K can be defined much as above. Colored compositions (of any kind) were studied in [Agarwal 2000; 2003; Bower 2010; Heubach and Mansour 2010, Section 3.5; Gibson et al. 2018].

For example, $\{(1_2,2_3,1_3,2_1),(2_3,1_3,2_1,1_2),(1_3,2_1,1_2,2_3),(2_1,1_2,2_3,1_3)\}$ is a circular nonpalindromic (not bilaterally symmetric) colored composition of $N=6$ with $K=4$ parts. On the other hand, $\{(1_2,2_3,1_3,2_3),(2_3,1_3,2_3,1_2),(1_3,2_3,1_2,2_3),(2_3,1_2,2_3,1_3)\}$ is a circular palindromic colored composition of $N=6$ with $K=4$ parts and one axis of symmetry (through 1_2 and 1_3).

Consider the collection $\mathcal{C}(N,K;a)$ of all colored compositions of N with K parts according to an *input sequence* $a=(a(m):m\in\mathbb{Z}_{>0})$. Consider a partition $\mathcal{P}=\mathcal{P}(N,K;a)$ of $\mathcal{C}(N,K;a)$ into (nonempty) *equivalence classes*. Denote by $b^{\mathcal{P}}(N,K)=b^{\mathcal{P}}(N,K;a)$ the total number of such equivalence classes in $\mathcal{P}$. Throughout the paper, instead of using $\mathcal{P}$, we use different superscripts (e.g., L, PL, CP, etc.) to denote the partition $\mathcal{P}$ of $\mathcal{C}(N,K;a)$. In particular, when each equivalence class in $\mathcal{P}$ has only one element, we use the superscript L (which stands for *linear* compositions).

Given $K\in\mathbb{Z}_{>0}$ and an input sequence $a=(a(m):m\in\mathbb{Z}_{>0})$, consider a sequence of partitions $\mathcal{P}_{K,a}=(\mathcal{P}(N,K;a):N\in\mathbb{Z}_{>0})$ and the set

$$\bigcup_{N\in\mathbb{Z}_{>0}}\mathcal{C}(N,K;a)$$

of all colored compositions of positive integers with K parts such that $\mathcal{P}(N,K;a)$ is a partition of $\mathcal{C}(N,K;a)$ for each $N\in\mathbb{Z}_{>0}$. We call the sequence

$$b_K=(b(N,K):N\in\mathbb{Z}_{>0})$$

the corresponding *output sequence*, where for simplicity we have dropped the superscript $\mathcal{P}_{K,a}$ from b_K and the superscript $\mathcal{P}(N,K;a)$ from $b(N,K)$.

Given $a = (a(m) : m \in \mathbb{Z}_{>0})$, we may also consider a family of partitions

$$\mathcal{P}_a = (\mathcal{P}(N, K; a) : N, K \in \mathbb{Z}_{>0} \text{ with } 1 \le K \le N)$$

and the set

$$\bigcup_{\substack{N,K \in \mathbb{Z}_{>0} \\ 1 \le K \le N}} \mathcal{C}(N, K; a)$$

of all colored compositions with input sequence a such that $\mathcal{P}(N, K; a)$ is a partition of $\mathcal{C}(N, K; a)$ for each pair (N, K). The N-th term of the sequence

$$b = (b(N) : N \in \mathbb{Z}_{>0}) = \left(\sum_{K=1}^{N} b(N, K) : N \in \mathbb{Z}_{>0} \right)$$

gives the *total* number of equivalence classes of colored compositions of N according to the family of partitions $(\mathcal{P}(N, K; a) : 1 \le K \le N)$. Again, for simplicity, we have dropped the superscript $\mathcal{P}_a$ from b and $b(N)$ and the superscript $\mathcal{P}(N, K; a)$ from $b(N, K)$.

Throughout the paper, for integers N and K, we set (for convenience)

$$b(N, K) = 0 \quad \text{when } K < 1 \text{ or } K > N. \tag{1-2}$$

Following [Bower 2010], we denote the (formal) *generating functions* (g.f.'s) of the three sequences a, b_K, and b by

$$A(x) = \sum_{m=1}^{\infty} a(m)\, x^m, \quad B_K(x) = \sum_{N=1}^{\infty} b(N, K)\, x^N,$$
$$B(x) = \sum_{N=1}^{\infty} b(N)\, x^N = \sum_{N=1}^{\infty} \sum_{K=1}^{N} b(N, K)\, x^N,$$

respectively. The following trivial result connects $B_K(x)$ with $B(x)$ under mild assumptions.

Proposition 1.1. *If $b(N, K) = 0$ for all $K \ge 2$ and $N \in \{1, \dots, K-1\}$, then*

$$B(x) = \sum_{K=1}^{\infty} B_K(x).$$

As mentioned above, we use the superscript L to denote linear colored compositions of N according to some input sequence $a = (a_m : m \in \mathbb{Z}_{>0})$ with g.f. $A(x)$. It is well known that

$$\begin{aligned} B_K^{\mathrm{L}}(x) &= \sum_{N=1}^{\infty} b^{\mathrm{L}}(N, K)\, x^N = A(x)^K \quad \text{for } K \in \mathbb{Z}_{>0}, \\ B^{\mathrm{L}}(x) &= \sum_{N=1}^{\infty} b^{\mathrm{L}}(N)\, x^N = \frac{A(x)}{1 - A(x)}. \end{aligned} \tag{1-3}$$

This is the INVERT transform in [Bernstein and Sloane 1995, p. 61] and the AIK transform in [Bower 2010].

Example 1.2. When the input sequence is $a = (m : m \in \mathbb{Z})$, then we are dealing with the so-called *m-colored* (*linear*) *compositions* of N studied in [Abrate et al. 2014; Agarwal 2000; Heubach and Mansour 2010, Section 3.5]. In such a case, $A(x) = x/(1-x)^2$ with

$$B_K^{\mathrm{L}}(x) = \frac{x^K}{(1-x)^{2K}} \quad \text{for } K \in \mathbb{Z}_{>0} \qquad \text{and} \qquad B^{\mathrm{L}}(x) = \frac{x}{x^2 - 3x + 1}. \tag{1-4}$$

From this, we may easily deduce that

$$b^{\mathrm{L}}(N, K) = \binom{N+K-1}{2K-1} \quad \text{for } K \in \mathbb{Z}_{>0} \qquad \text{and} \qquad b^{\mathrm{L}}(N) = F_{2N}, \tag{1-5}$$

where F_n is the n-th *Fibonacci number*. Other properties of "n-color compositions" can be found in [Agarwal 2003; Guo 2012; Sachdeva and Agarwal 2017].

The organization of the paper is as follows. In Section 2, we study colored linear palindromic compositions and derive their generating functions (see Theorems 2.1 and 2.2). The material in that section is needed for the material in Section 3, which is the main section of the paper. In Section 3, we study colored circular palindromic compositions and derive their generating functions (see Theorems 3.1 and 3.2, which generalize results in [Hadjicostas and Zhang 2017]). In particular, we prove that, if $b^{\mathrm{CP}}(N)$ is the total number of colored circular palindromic compositions of N with input sequence $a = (a(m) : m \in \mathbb{Z}_{>0})$, then the g.f. of the sequence $(b^{\mathrm{CP}}(N) : N \in \mathbb{Z}_{>0})$ is given by

$$\sum_{N=1}^{\infty} b^{\mathrm{CP}}(N)\, x^N = \frac{(1 + A(x))^2}{2(1 - A(x^2))} - \frac{1}{2}, \tag{1-6}$$

where $A(x) = \sum_{m=1}^{\infty} a(m)\, x^m$.

Equation (1-6) is important because it allows the calculation of the g.f. of the number of colored dihedral compositions of N, say $b^D(N)$. When $A(x)$ is the g.f. of the input sequence $a = (a(m) : m \in \mathbb{Z}_{>0})$, then it can be proved (using Möbius inversion) that the g.f. of the number of colored circular compositions of N is $-\sum_{d=1}^{\infty} (\phi(d)/d) \log(1 - A(x^d))$, where $\phi(d)$ is the Euler totient function. We omit the details of the proof of this result, but see, for example, [Flajolet and Sedgewick 2009; Flajolet and Soria 1991]. The g.f. of the number of colored dihedral compositions is then given by

$$\sum_{N=1}^{\infty} b^D(N)\, x^N = -\frac{1}{2} \sum_{d=1}^{\infty} \frac{\phi(d)}{d} \log(1 - A(x^d)) + \frac{(1 + A(x))^2}{4(1 - A(x^2))} - \frac{1}{4}. \tag{1-7}$$

2. Colored linear palindromic compositions

The study of *linear palindromic compositions* goes back all the way to the 19th century work [MacMahon 1893]. These compositions have been studied by several mathematicians since then. They are compositions $(\lambda_1, \dots, \lambda_K)$ of N with K parts such that $\lambda_{K+1-i} = \lambda_i$ for $i = 1, \dots, K$. Hadjicostas and Zhang [2017] called these compositions *type-I palindromic compositions* and denoted their number by $P^{L_1}(N, K)$. MacMahon [1893] proved that, for $n, k \in \mathbb{Z}_{>0}$ with $1 \le k \le n$,

$$P^{L_1}(2n, 2k) = P^{L_1}(2n, 2k-1) = P^{L_1}(2n-1, 2k-1) = \binom{n-1}{k-1},$$

while $P^{L_1}(2n-1, 2k) = 0$. We shall not use the notation P^{L_1} in this paper.

Using the superscript PL for colored linear palindromic compositions and the superscript L for (general) linear compositions of N with K parts according to the input sequence $a = (a(m) : m \in \mathbb{Z}_{>0})$ in both cases, it is easy to prove (e.g., see the LPAL transform in [Bower 2010]) that:

- If N and K are even, then $b^{\mathrm{PL}}(N, K) = b^{\mathrm{L}}(N/2, K/2)$.
- If N is odd and K is even, then $b^{\mathrm{PL}}(N, K) = 0$.
- If $K = 1$, then $b^{\mathrm{PL}}(N, K = 1) = a(N)$.
- If N is even and K is odd ≥ 3, then

$$b^{\mathrm{PL}}(N, K) = \sum_{0<i<N/2} a(2i)\, b^{\mathrm{L}}\Big(\frac{N}{2} - i, \frac{K-1}{2}\Big).$$

- If N and K are both odd with $K \geq 3$, then

$$b^{\mathrm{PL}}(N, K) = \sum_{0<i<N/2} a(2i-1)\, b^{\mathrm{L}}\Big(\frac{N+1}{2} - i, \frac{K-1}{2}\Big).$$

Theorem 2.1. *For fixed $K \in \mathbb{Z}_{>0}$, the g.f. of the sequence $(b^{\mathrm{PL}}(N, K) : N \in \mathbb{Z}_{>0})$ is given by*

$$B_K^{\mathrm{PL}}(x) = \begin{cases} A(x^2)^{K/2} & \text{if } K \text{ is even}, \\ A(x)A(x^2)^{(K-1)/2} & \text{if } K \text{ is odd}. \end{cases}$$

Proof. Assume first K is even. Since $b^{\mathrm{PL}}(N, K) = 0$ when N is odd, we get

$$B_K^{\mathrm{PL}}(x) = \sum_{s=1}^{\infty} b^{\mathrm{L}}\Big(s, \frac{K}{2}\Big)\,(x^2)^s = A(x^2)^{K/2}.$$

Assume K is odd. If $K = 1$, the result is trivial because $B_{K=1}^{\mathrm{PL}}(x) = A(x)$. If $K \geq 3$, then

$$\begin{aligned} B_K^{\mathrm{LP}}(x) &= \sum_{m=1}^{\infty}\sum_{i=1}^{m-1} a(2i)\, b^{\mathrm{L}}\Big(m-i, \frac{K-1}{2}\Big)\, x^{2m} + \sum_{m=0}^{\infty}\sum_{i=1}^{m} a(2i-1)\, b^{\mathrm{L}}\Big(m+1-i, \frac{K-1}{2}\Big)\, x^{2m+1} \\ &= \sum_{i=1}^{\infty} a(2i)x^{2i} \sum_{m=i+1}^{\infty} b^{\mathrm{L}}\Big(m-i, \frac{K-1}{2}\Big)\, x^{2(m-i)} \\ &\qquad + \sum_{i=1}^{\infty} a(2i-1)x^{2i-1} \sum_{m=i}^{\infty} b^{\mathrm{L}}\Big(m-i+1, \frac{K-1}{2}\Big)\, x^{2(m-i)+2}. \end{aligned}$$

If we let $A_{\mathrm{E}}(x) = \sum_{i=1}^{\infty} a(2i)x^{2i}$ and $A_{\mathrm{O}}(x) = \sum_{i=1}^{\infty} a(2i-1)x^{2i-1}$, we then get

$$\begin{aligned} B_K^{\mathrm{LP}}(x) &= A_{\mathrm{E}}(x) \sum_{\ell=1}^{\infty} b^{\mathrm{L}}\Big(\ell, \frac{K-1}{2}\Big)\, x^{2\ell} + A_{\mathrm{O}}(x) \sum_{\ell=0}^{\infty} b^{\mathrm{L}}\Big(\ell+1, \frac{K-1}{2}\Big)\, x^{2(\ell+1)} \\ &= A_{\mathrm{E}}(x)A(x^2)^{(K-1)/2} + A_{\mathrm{O}}(x)A(x^2)^{(K-1)/2} = A(x)A(x^2)^{(K-1)/2}. \end{aligned}$$ □

Theorem 2.2. *The g.f. of the sequence $(b^{\mathrm{PL}}(N) : N \in \mathbb{Z}_{>0})$ is given by*

$$B^{\mathrm{LP}}(x) = \sum_{N=1}^{\infty} b^{\mathrm{PL}}(N)\, x^N = \frac{A(x) + A(x^2)}{1 - A(x^2)}.$$

Proof. By Proposition 1.1,

$$\sum_{N=1}^{\infty} b^{\mathrm{PL}}(N)\, x^N = \sum_{K=1}^{\infty}\Biggl(\sum_{N=1}^{\infty} b^{\mathrm{PL}}(N, K)\, x^N\Biggr)$$
$$= \sum_{s=1}^{\infty} A(x^2)^{2s/2} + \sum_{s=0}^{\infty} A(x)A(x^2)^{((2s+1)-1)/2} = \frac{A(x)+A(x^2)}{1-A(x^2)}. \qquad \square$$

Example 2.3. Consider again the m-colored compositions from Example 1.2 with input sequence $a = (m : m \in \mathbb{Z}_{>0})$ and input g.f. $A(x) = x/(1-x)^2$. Using Theorems 2.1 and 2.2, we can easily prove that

$$B_K^{\mathrm{PL}}(x) = \frac{x^K}{(1-x^2)^K} \qquad \text{for } K \text{ even},$$
$$B_K^{\mathrm{PL}}(x) = \frac{x^K(1+x)^2}{(1-x^2)^{K+1}} \qquad \text{for } K \text{ odd},$$
$$B^{\mathrm{PL}}(x) = \frac{x(x^2+3x+1)}{x^4-3x^2+1} = \frac{x(x^2+3x+1)}{(x^2+x-1)(x^2-x-1)}.$$

It follows that the number of m-colored (linear) palindromic compositions of N with K parts is given by

$$b^{\mathrm{PL}}(N, K) = \begin{cases} \dfrac{1+(-1)^N}{2}\dbinom{\lfloor \frac{N}{2}\rfloor + \frac{K}{2} - 1}{K-1} & \text{if } K \text{ is even},\\[2ex] \dbinom{\lfloor \frac{N}{2}\rfloor + \frac{K-1}{2}}{K} + \dbinom{\lceil \frac{N}{2}\rceil + \frac{K-1}{2}}{K} & \text{if } K \text{ is odd}. \end{cases}$$

Here, $\lfloor a\rfloor$ and $\lceil a\rceil$ denote the floor and ceiling of $a \in \mathbb{R}$, respectively. In addition, the total number of m-colored (linear) palindromic compositions of N is given by

$$b^{\mathrm{PL}}(N) = \begin{cases} 3F_N & \text{if } N \text{ is even},\\ F_{N-1} + F_{N+1} & \text{if } N \text{ is odd}. \end{cases}$$

Example 2.4. Using a combinatorial argument and using g.f.'s, Mansour and Shattuck [2014] proved that, for $N \in \mathbb{Z}_{>0}$, the number of F_m-compositions of N equals the *Pell number* p_N, which is defined by the recurrence

$$p_0 = 0, \qquad p_1 = 1, \qquad \text{and} \qquad p_N = 2p_{N-1} + p_{N-2} \quad \text{for } N \geq 2. \tag{2-1}$$

Of course, using g.f.'s, the proof of this claim is very easy: we have $a(m) = F_m$ for each $m \in \mathbb{Z}_{>0}$, and as a result, $A(x) = x/(1-x-x^2)$. It follows that the g.f. of the number $b^{\mathrm{L}}(N)$ of colored linear compositions of N with respect to the input sequence $a = (F_m : m \in \mathbb{Z}_{>0})$ is

$$B^{\mathrm{L}}(x) = \frac{A(x)}{1-A(x)} = \frac{x}{1-2x-x^2},$$

which is the g.f. of the Pell numbers defined by (2-1).

By Theorem 2.2, the g.f. of the number $b^{\mathrm{LP}}(N)$ of colored linear palindromic compositions of N with respect to the input sequence $a = (F_m : m \in \mathbb{Z}_{>0})$ is

$$B^{\mathrm{LP}}(x) = \frac{A(x) + A(x^2)}{1 - A(x^2)} = \frac{x(1 + x - 2x^2 - x^3 - x^4)}{(1 - x - x^2)(1 - 2x^2 - x^4)} = \frac{2(x+1)}{1 - x - x^2} - \frac{2 + 3x + x^3}{1 - 2x^2 - x^4}.$$

It follows immediately that

$$b^{\mathrm{LP}}(N) = \begin{cases} 2F_{N+2} - 2p_{N/2+1} & \text{if } N \text{ is even,} \\ 2F_{N+2} - 3p_{(N+1)/2} - p_{(N-1)/2} & \text{if } N \text{ is odd} \end{cases} = 2F_{N+2} - p_{\lfloor N/2+1 \rfloor} - p_{\lceil N/2+1 \rceil}.$$

For example, the $b^{\mathrm{LP}}(5) = 9$ linear palindromic F_m-compositions of $N = 5$ are

$$(5_1),\ (5_2),\ (5_3),\ (5_4),\ (5_5),\ (1_1, 3_1, 1_1),\ (1_1, 3_2, 1_1),\ (2_1, 1_1, 2_1),\ (1_1, 1_1, 1_1, 1_1, 1_1).$$

Remark 2.5. Hadjicostas and Zhang [2017] considered also *type-II palindromic compositions* of N with K, denoted by $P^{\mathrm{L}_2}(N, K)$, which are compositions $(\lambda_1, \dots, \lambda_K)$ of N of length K that satisfy $\lambda_i = \lambda_{K+2-i}$ for $i = 2, \dots, K$; that is, $(\lambda_1, \lambda_2, \dots, \lambda_K) = (\lambda_1, \lambda_K, \dots, \lambda_2)$. (For $K = 1$, it is assumed that $(\lambda_1) = (N)$ is a linear palindromic composition of both types.) Again, we shall not use the notation $P^{\mathrm{L}_2}(N, K)$ in this paper (since the superscript L in this paper denotes a general linear composition, not necessarily palindromic).

3. Colored circular palindromic compositions

Circular palindromic compositions or *circular symmetrical compositions* were originally studied in [Sommerville 1909]. Hadjicostas and Zhang [2017] defined them as equivalence classes (with respect to cyclic shifts) on the set of linear compositions of N with K parts that contain at least one palindromic composition of type I or type II (see Remark 2.5 in this paper). Williamson [1972] called them *bilaterally symmetric cyclic compositions*, while Bower [2010] called them *circular palindromes*.

In this paper, we use the superscript CP for colored circular palindromes (as opposed to PL used for colored linear palindromes and L for general colored linear compositions). In order to find the g.f.s of the sequences $(b^{\mathrm{CP}}(N, K) : N \in \mathbb{Z}_{>0})$ and $(b^{\mathrm{CP}}(N) : N \in \mathbb{Z}_{>0})$, we need first to express these quantities in terms of $b^{\mathrm{L}}(N, K)$ or $b^{\mathrm{LP}}(N, K)$ (for which we know the g.f.'s from previous sections).

Using input sequence $a = (a(m) : m \in \mathbb{Z})$, for $N, K \in \mathbb{Z}_{>0}$, it is clear that

$$b^{\mathrm{CP}}(N, K) = b^{\mathrm{PL}}(N, K) \quad \text{when } K \text{ is odd.} \tag{3-1}$$

When K is even and N is odd, we have (e.g., see [Bower 2010])

$$b^{\mathrm{CP}}(N, K) = \begin{cases} \sum_{\substack{i \text{ odd}, j \text{ even} \\ i+j=N}} a(i)\, a(j) & \text{if } K = 2, \\ \sum_{\substack{i \text{ odd}, j \text{ even} \\ i+j<N}} a(i)\, a(j)\, b^{\mathrm{L}}\Big(\dfrac{N-i-j}{2}, \dfrac{K}{2} - 1\Big) & \text{if } K \geq 4. \end{cases} \tag{3-2}$$

When N and K are both even, the situation is more complicated. Following [Bower 2010], we divide the possible output configurations (with parts of the composition on a circle) into two kinds:

<u>Case 1</u>: those configurations for which the axis of symmetry passes through no parts of the composition.

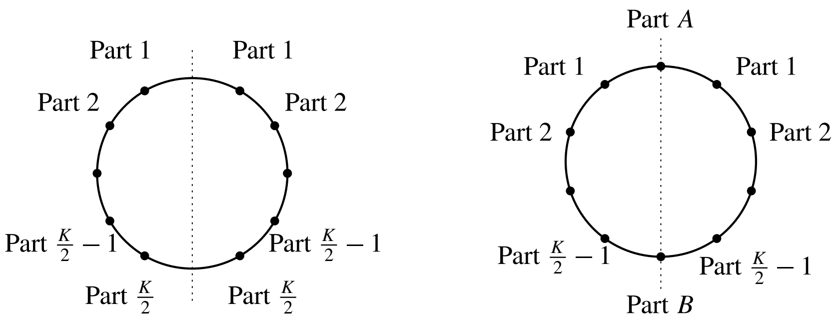

Figure 1. Case 1, left: no parts joined. Case 2, right: two parts joined.

Case 2: those configurations for which the axis of symmetry passes through two parts, which we label Part A and Part B.

See Figure 1. Because two circular configurations of parts are equivalent if one can be obtained from the other through cyclic rotation, it is possible for a configuration to belong to both categories.

We have the following formula for $b^{\mathrm{CP}}(N, K)$ when both N and K are even:

$$b^{\mathrm{CP}}(N, K) = \frac{I_N + J_N}{2} + P_N + S_N + M_N.$$

The quantities I_N, J_N, P_N, S_N, and M_N are defined by [Bower 2010] (see his CPAL transform):

- $I_N = b^{\mathrm{L}}(N/2, K/2)$ (no parts are joined).
- $J_N = a(N/2)$ when $K = 2$ and

$$J_N = \sum_{0<i<N/2} a(i)\, b^{\mathrm{L}}\Big(\frac{N-2i}{2}, \frac{K}{2} - 1\Big)$$

  when $K \geq 4$ (the two parts joined, A and B, are identical).

- $P_N = \begin{cases} \sum_{\substack{i,j \text{ even} \\ j>i,\, i+j=N}} a(i)\, a(j) & \text{when } K = 2, \\ \sum_{\substack{i,j \text{ even} \\ j>i,\, i+j<N}} a(i)\, a(j)\, b^{\mathrm{L}}\Big(\frac{N-i-j}{2}, \frac{K}{2} - 1\Big) & \text{when } K \geq 4 \end{cases}$

  (the two parts joined, A and B, are even and have different values).

- $S_N = \begin{cases} \sum_{\substack{i,j \text{ odd} \\ j>i,\, i+j=N}} a(i)\, a(j) & \text{when } K = 2, \\ \sum_{\substack{i,j \text{ odd} \\ j>i,\, i+j<N}} a(i)\, a(j)\, b^{\mathrm{L}}\Big(\frac{N-i-j}{2}, \frac{K}{2} - 1\Big) & \text{when } K \geq 4 \end{cases}$

  (the two parts joined, A and B, are odd and have different values).

- $M_N = \begin{cases} \frac{1}{2}\Big(a\Big(\frac{N}{2}\Big)^2 - a\Big(\frac{N}{2}\Big)\Big) & \text{when } K = 2, \\ \frac{1}{2} \sum_{0<i<N/2} (a(i)^2 - a(i))\, b^{\mathrm{L}}\Big(\frac{N-2i}{2}, \frac{K}{2} - 1\Big) & \text{when } K \geq 4 \end{cases}$

  (the two parts joined, A and B, have the same value but different colors).

Theorem 3.1. *For fixed $K \in \mathbb{Z}_{>0}$, the g.f. of the sequence $(b^{\mathrm{CP}}(N, K) : N \in \mathbb{Z}_{>0})$ is given by*

$$B_K^{\mathrm{CP}}(x) = \sum_{N=1}^{\infty} b^{\mathrm{CP}}(N, K)\, x^N = \begin{cases} \frac{1}{2} A(x^2)^{(K/2)-1}(A(x)^2 + A(x^2)) & \text{if } K \text{ is even,} \\ A(x)A(x^2)^{(K-1)/2} & \text{if } K \text{ is odd.} \end{cases}$$

Proof. In view of (3-1) and Theorem 2.1, we only need to prove the theorem when K is even. Let $B_K^{\mathrm{O}}(x)$ and $B_K^{\mathrm{E}}(x)$ be the contribution to the g.f. $B_K^{\mathrm{CP}}(x) = \sum_{N=1}^{\infty} b^{\mathrm{CP}}(N, K)\, x^N$ from the terms $b^{\mathrm{CP}}(N, K)$ with odd and even indexes N, respectively. We claim that

$$B_K^{\mathrm{O}}(x) = A_{\mathrm{O}}(x)A_{\mathrm{E}}(x)A(x^2)^{K/2-1}, \tag{3-3}$$

$$B_K^{\mathrm{E}}(x) = \tfrac{1}{2} A(x^2)^{K/2-1}[A(x^2) + A_{\mathrm{E}}(x)^2 + A_{\mathrm{O}}(x)^2], \tag{3-4}$$

where $A_{\mathrm{O}}(x) = \sum_{m=0}^{\infty} a(2m+1)\, x^{2m+1}$ and $A_{\mathrm{E}}(x) = \sum_{m=1}^{\infty} a(2m)\, x^{2m}$. From (3-3) and (3-4), we get

$$B_K^{\mathrm{CP}}(x) = B_K^{\mathrm{O}}(x) + B_K^{\mathrm{E}}(x) = \tfrac{1}{2} A(x^2)^{K/2-1}[A(x^2) + A(x)^2],$$

and this would complete the proof of theorem.

Proof of equation (3-3): We use (3-2). For $K = 2$, we have

$$B_{K=2}^{\mathrm{O}}(x) = \sum_{t=1}^{\infty} \sum_{\substack{s \ge 0,\, r \ge 1 \\ s+r=t}} a(2s+1)\, a(2r)\, x^{2t+1} = \sum_{s=0}^{\infty} \sum_{r=1}^{\infty} a(2s+1)\, a(2r)\, x^{2r} x^{2s+1} = A_{\mathrm{O}}(x)A_{\mathrm{E}}(x)A(x^2)^{2/2-1}.$$

For K even ≥ 4, we have

$$\begin{aligned} B_K^{\mathrm{O}}(x) &= \sum_{t=2}^{\infty} \sum_{s=0}^{t-2} \sum_{r=1}^{t-1-s} a(2s+1)\, a(2r)\, b^{\mathrm{L}}\Big(t-r-s, \frac{K}{2}-1\Big)\, x^{2t+1} \\ &= \sum_{s=0}^{\infty} \sum_{r=1}^{\infty} a(2s+1)\, a(2r)\, x^{2r} x^{2s+1} \sum_{t=r+s+1}^{\infty} b^{\mathrm{L}}\Big(t-r-s, \frac{K}{2}-1\Big)\, x^{2(t-r-s)} \\ &= A_{\mathrm{O}}(x)A_{\mathrm{E}}(x)A(x^2)^{K/2-1}, \end{aligned}$$

which proves (3-3).

Proof of equation (3-4): We calculate the contributions of the terms I_N, J_N, P_N, S_N, and M_N to the generating function $B_K^{\mathrm{E}}(x)$. For $T \in \{I, J, P, S, M\}$, denote the corresponding contribution to the g.f. $B_K^{\mathrm{E}}(x)$ by $B_K^T(x)$. We claim that

$$B_K^I(x) = \sum_{m=1}^{\infty} I_{2m} x^{2m} = A(x^2)^{K/2},$$

$$B_K^J(x) = \sum_{m=1}^{\infty} J_{2m} x^{2m} = \Big(\sum_{i=1}^{\infty} a(i) x^{2i}\Big) A(x^2)^{K/2-1},$$

$$B_K^P(x) = \sum_{m=1}^{\infty} P_{2m} x^{2m} = \Big(\sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} a(2i)\, a(2j)\, x^{2(i+j)}\Big) A(x^2)^{K/2-1},$$

$$B_K^S(x) = \sum_{m=1}^{\infty} S_{2m} x^{2m} = \Big(\sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} a(2i+1)\, a(2j+1)\, x^{2(i+j+1)}\Big) A(x^2)^{K/2-1},$$

$$B_K^M(x) = \sum_{m=1}^{\infty} M_{2m} x^{2m} = \frac{1}{2}\Big(\sum_{i=1}^{\infty} (a(i)^2 - a(i))\, x^{2i}\Big) A(x^2)^{K/2-1}.$$

Below we prove the formulae for $B^J_K(x)$ and $B^S_K(x)$. The proofs for the rest are similar, and hence we omit them.

For $B^J_K(x)$, when $K=2$, we have

$$B^J_{K=2}(x)=\sum_{m=0}^{\infty} J_{2m}x^{2m}=\sum_{m=0}^{\infty} a\Big(\frac{2m}{2}\Big)x^{2m}=\Big(\sum_{i=1}^{\infty} a(i)x^{2i}\Big)A(x^2)^{2/2-1}.$$

For K even ≥ 4, we have

$$\begin{aligned}B^J_K(x)&=\sum_{m=1}^{\infty} J_{2m}x^{2m}=\sum_{m=1}^{\infty}\Big(\sum_{0<i<m} a(i)\, b^{\mathrm{L}}\Big(\frac{2m-2i}{2},\,\frac{K}{2}-1\Big)\Big)x^{2m}\\&=\sum_{i=1}^{\infty} a(i)x^{2i}\sum_{m=i+1}^{\infty} b^{\mathrm{L}}\Big(m-i,\,\frac{K}{2}-1\Big)\,x^{2(m-i)}\\&=\sum_{i=1}^{\infty} a(i)x^{2i}\sum_{\ell=1}^{\infty} b^{\mathrm{L}}\Big(\ell,\,\frac{K}{2}-1\Big)x^{2\ell}=\Big(\sum_{i=1}^{\infty} a(i)x^{2i}\Big)A(x^2)^{K/2-1}.\end{aligned}$$

For $B^S_K(x)$, when $K=2$, we have

$$\begin{aligned}B^S_{K=2}(x)&=\sum_{m=1}^{\infty} S_{2m}x^{2m}=\sum_{m=2}^{\infty}\sum_{\substack{s>t\geq 0\\ s+t=m-1}} a(2t+1)a(2s+1)x^{2m}\\&=\Big(\sum_{t=0}^{\infty}\sum_{s=t+1}^{\infty} a(2s+1)\,a(2t+1)\,x^{2(s+t+1)}\Big)A(x^2)^{2/2-1}.\end{aligned}$$

For K even ≥ 4, we have

$$\begin{aligned}B^S_K(x)&=\sum_{m=1}^{\infty} S_{2m}x^{2m}=\sum_{m=1}^{\infty}\Big(\sum_{\substack{i,j\text{ odd}\\ j>i,\, i+j<2m}} a(i)\,a(j)\,b^{\mathrm{L}}\Big(\frac{2m-i-j}{2},\,\frac{K}{2}-1\Big)\Big)x^{2m}\\&=\sum_{t=0}^{\infty}\sum_{\ell=t+1}^{\infty}\sum_{m=t+\ell+2}^{\infty} a(2t+1)\,a(2\ell+1)\,b^{\mathrm{L}}\Big(m-t-\ell-1,\,\frac{K}{2}-1\Big)x^{2m}.\end{aligned}$$

Therefore,

$$\begin{aligned}B^S_K(x)&=\sum_{t=0}^{\infty}\sum_{\ell=t+1}^{\infty} a(2t+1)\,a(2\ell+1)\,x^{2(t+\ell+1)}\\&\qquad\times\sum_{m=t+\ell+2}^{\infty} a(2t+1)\,a(2\ell+1)\,b^{\mathrm{L}}\Big(m-t-\ell-1,\,\frac{K}{2}-1\Big)\,x^{2(m-t-\ell-1)}\\&=\sum_{t=0}^{\infty}\sum_{\ell=t+1}^{\infty} a(2t+1)\,a(2\ell+1)\,x^{2(t+\ell+1)}\sum_{s=1}^{\infty} b^{\mathrm{L}}\Big(s,\,\frac{K}{2}-1\Big)\,x^{2s}\\&=\Big(\sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} a(2i+1)\,a(2j+1)\,x^{2(i+j+1)}\Big)A(x^2)^{K/2-1}.\end{aligned}$$

In a similar way, we can prove the formulae for $B_K^I(x)$, $B_K^P(x)$, and $B_K^M(x)$. We then have

$$\begin{aligned} B_K^{\mathrm{E}}(x) &= \tfrac{1}{2}(B_K^I(x) + B_K^J(x)) + B_K^P(x) + B_K^S(x) + B_K^M(x) \\ &= A(x^2)^{K/2-1} T_K(x), \end{aligned}$$

where

$$\begin{aligned} T_K(x) &= \frac{1}{2}\Big(A(x^2) + \sum_{i=1}^{\infty} a(i) x^{2i}\Big) + \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} a(2i)\,a(2j)\,x^{2(i+j)} \\ &\qquad + \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} a(2i+1)\,a(2j+1)\,x^{2(i+j+1)} + \frac{1}{2}\sum_{i=1}^{\infty}(a(i)^2 - a(i))\,x^{2i} \\ &= \frac{1}{2}\Big(A(x^2) + \sum_{i=1}^{\infty} a(i) x^{2i} + A_{\mathrm{E}}(x)^2 - \sum_{m=1}^{\infty} a(2m)^2 x^{4m} + A_{\mathrm{O}}(x)^2 \\ &\qquad - \sum_{m=0}^{\infty} a(2m+1)^2 x^{2(2m+1)} + \sum_{i=1}^{\infty}(a(i)^2 - a(i))\,x^{2i}\Big) \\ &= \frac{1}{2}(A(x^2) + A_{\mathrm{E}}(x)^2 + A_{\mathrm{O}}(x)^2). \end{aligned}$$

This finishes the proof of (3-4) and the proof of the theorem. □

Theorem 3.2. *The g.f. of the sequence* $(b^{\mathrm{CP}}(N) : N \in \mathbb{Z}_{>0})$ *is given by*

$$B^{\mathrm{CP}}(x) = \sum_{N=1}^{\infty} b^{\mathrm{CP}}(N)\,x^N = \frac{(1 + A(x))^2}{2(1 - A(x^2))} - \frac{1}{2}.$$

Proof. By Proposition 1.1,

$$\begin{aligned} \sum_{N=1}^{\infty} b^{\mathrm{CP}}(N)\,x^N &= \sum_{K=1}^{\infty}\Big(\sum_{N=1}^{\infty} b^{\mathrm{CP}}(N, K)\,x^N\Big) \\ &= \frac{1}{2}\sum_{s=1}^{\infty} A(x^2)^{(2s/2)-1}(A(x)^2 + A(x^2)) + \sum_{s=0}^{\infty} A(x) A(x^2)^{(2s+1-1)/2} \\ &= \frac{A(x)^2 + A(x^2) + 2A(x)}{2(1 - A(x^2))} = \frac{(1 + A(x))^2}{2(1 - A(x^2))} - \frac{1}{2}. \end{aligned}$$

□

A special case of Theorems 3.1 and 3.2 has to do with circular palindromic compositions of a positive integer with parts belonging to a subset E of $\mathbb{Z}_{>0}$. (Several similar results for various kinds of linear compositions were surveyed in [Heubach and Mansour 2004].) Hadjicostas and Zhang [2017, Theorem 2.6 and Corollary 2.9] proved the following result using different methods. We give a new proof of this result using Theorems 3.1 and 3.2 above.

Theorem 3.3. *Let* $E \subseteq \mathbb{Z}_{>0}$. *For each pair of positive integers* N *and* K, *let* $P_E^R(N, K)$ *be the number of circular palindromic compositions of* N *with length* K *whose parts belong to* E. *Let also* $P_E^R(N)$ *be the total number of circular palindromic compositions of* N *with parts in* E. *Then*:

(a) *For* $K \in \mathbb{Z}_{>0}$,

$$\sum_{N\ge 1} P_E^R(N,K)\,x^N = \begin{cases} \Big(\sum_{m\in E} x^m\Big)\Big(\sum_{m\in E} x^{2m}\Big)^{(K-1)/2} & \text{if } K \text{ is odd},\\ \frac{1}{2}\Big(\Big(\sum_{m\in E} x^m\Big)^2 + \Big(\sum_{m\in E} x^{2m}\Big)\Big)\Big(\sum_{m\in E} x^{2m}\Big)^{(K)/2-1} & \text{if } K \text{ is even}. \end{cases}$$

(b) *We have*

$$\sum_{N\ge 1} P_A^R(N)\,x^N = \frac{\big(1+\sum_{m\in E} x^m\big)^2}{2\big(1-\sum_{m\in E} x^{2m}\big)} - \frac{1}{2}.$$

Proof. Define the input sequence $a = (a(m) : m \ge 1)$ by

$$a(m) = \begin{cases} 1 & \text{if } m \in E,\\ 0 & \text{if } m \notin E. \end{cases}$$

Then the g.f. of sequence a is $A(x) = \sum_{x\in E} x^m$. We have

$$P_E^R(N;K) = b^{\mathrm{CP}}(N,K) \quad \text{and} \quad P_E^R(N) = b^{\mathrm{CP}}(N).$$

The theorem then follows from Theorems 3.1 and 3.2 above. □

Example 3.4. Consider again the m-colored compositions from Example 1.2 with input sequence $a = (m : m \in \mathbb{Z}_{>0})$ and input g.f. $A(x) = x/(1-x)^2$. Using Theorems 3.1 and 3.2, we can easily prove that

$$B_K^{\mathrm{CP}}(x) = \frac{x^K(1+x^2)(1+x)^2}{(1-x^2)^{K+2}} \qquad \text{for } K \text{ even},$$

$$B_K^{\mathrm{CP}}(x) = \frac{x^K(1+x)^2}{(1-x^2)^{K+1}} \qquad \text{for } K \text{ odd},$$

$$B^{\mathrm{CP}}(x) = \frac{x(x^4+x^3-2x^2+x+1)}{(x^4-3x^2+1)(1-x)^2} = \frac{x(x^4+x^3-2x^2+x+1)}{(x^2+x-1)(x^2-x-1)(1-x)^2}.$$

Thus, the number of m-colored circular palindromic compositions of N with K parts is given by

$$b^{\mathrm{CP}}(N,K) = \begin{cases} \binom{\lfloor \frac{N-1}{2}\rfloor + \frac{K}{2}}{K+1} + \binom{\lceil \frac{N-1}{2}\rceil + \frac{K}{2}}{K+1} + \binom{\lfloor \frac{N+1}{2}\rfloor + \frac{K}{2}}{K+1} + \binom{\lceil \frac{N+1}{2}\rceil + \frac{K}{2}}{K+1} & \text{if } K \text{ is even},\\ \binom{\lfloor \frac{N}{2}\rfloor + \frac{K-1}{2}}{K} + \binom{\lceil \frac{N}{2}\rceil + \frac{K-1}{2}}{K} & \text{if } K \text{ is odd}. \end{cases}$$

In addition, the total number of m-colored circular palindromic compositions of N is given by

$$b^{\mathrm{CP}}(N) = F_{N+4} + (-1)^N F_{N-4} - 2N \quad \text{for } N \ge 4$$

with $b^{\mathrm{CP}}(1) = 1$, $b^{\mathrm{CP}}(2) = 3$ and $b^{\mathrm{CP}}(3) = 6$.

Example 3.5. Consider again Example 2.4 with input sequence $a = (F_m : m \in \mathbb{Z}_{>0})$ and input g.f. $A(x) = x/(1-x-x^2)$, which extends an example from [Mansour and Shattuck 2014]. Using Theorem 3.2, we can prove that the g.f. of the number $b^{\mathrm{CP}}(N)$ of colored circular palindromic F_m-compositions of N is given by

$$B^{\mathrm{CP}}(x) = \frac{x(1-3x^2+x^4+x^5+x^6)}{(1-x-x^2)^2(1-2x^2-x^4)} = -\frac{5(x+2)}{1-x-x^2} + \frac{1}{(1-x-x^2)^2} + \frac{9+14x+4x^2+6x^3}{1-2x^2-x^4}.$$

It follows that

$$b^{\mathrm{CP}}(N) = -5(F_N + 2F_{N+1}) + \tfrac{1}{5}((N+3)F_{N+1} + (N+1)F_{N+3}) + g(N),$$

where

$$g(N) = \begin{cases} 4p_{N/2} + 9p_{N/2+1} & \text{if } N \text{ is even,} \\ 6p_{(N-1)/2} + 14p_{(N+1)/2} & \text{if } N \text{ is odd.} \end{cases}$$

Here p_n denotes the n-th Pell number defined by (2-1). For example, the $b^{\mathrm{CP}}(5) = 15$ circular palindromic F_m-compositions of $N = 5$ are

$$(5_1),\ (5_2),\ (5_3),\ (5_4),\ (5_5),\ (1_1, 4_1),\ (1_1, 4_2),\ (1_1, 4_3),\ (2_1, 3_1),\ (2_1, 3_2),$$
$$(1_1, 3_1, 1_1),\ (1_1, 3_2, 1_1),\ (2_1, 1_1, 2_1),\ (1_1, 1_1, 1_1, 2_1),\ (1_1, 1_1, 1_1, 1_1, 1_1),$$

where we have listed only one representative from each equivalence class.

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