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Weakly distinguishing graph polynomials on addable properties

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A graph polynomial P is weakly distinguishing if for almost all finite graphs G there is a finite graph H that is not isomorphic to G with $P(G) = P(H)$. It is weakly distinguishing on a graph property $\mathcal{C}$ if for almost all finite graphs $G \in \mathcal{C}$ there is $H \in \mathcal{C}$ that is not isomorphic to G with $P(G) = P(H)$. We give sufficient conditions on a graph property $\mathcal{C}$ for the characteristic, clique, independence, matching, and domination and ξ polynomials, as well as the Tutte polynomial and its specializations, to be weakly distinguishing on $\mathcal{C}$. One such condition is to be addable and small in the sense of C. McDiarmid, A. Steger and D. Welsh (2005). Another one is to be of genus at most k.

1. Introduction and preliminaries

Unless otherwise stated, we only consider simple (finite, loopless, undirected graphs with no parallel edges) graphs with vertices labeled $1, \dots, n$. For a graph $G = (V(G), E(G))$ and $A \subseteq E(G)$, we denote by $G\langle A\rangle$ the graph $(V(G), A)$, and by $k(A)$ the number of connected components of $G\langle A\rangle$. A graph property is a family of graphs that is closed under isomorphisms. For a graph property $\mathcal{C}$, denote by $\mathcal{C}(n)$ the graphs of order n in $\mathcal{C}$. We only consider properties such that $\mathcal{C}(n)$ is nonempty for all sufficiently large n.

Let P be a graph polynomial. We say that two non isomorphic graphs G and H are P-mates if $P(G) = P(H)$, and that G is P unique if it has no P mates. P is *trivial* if all graphs G, H are P-mates. P is *complete* if all graphs G are P-unique.

In this paper we investigate conditions which imply that almost all graphs in a graph property $\mathcal{C}$ have a P-mate. More formally, we give the following definitions:

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Let P be a graph polynomial, and denote by $\mathcal{G}(n)$ the family of graphs of order n with vertices labeled $1, \dots, n$, and by $U_P(n)$ the set of P unique graphs of order n.

Definition 1.1. P is *weakly distinguishing* if

$$\lim_{n\to\infty} \frac{|U_P(n)|}{|\mathcal{G}(n)|} = 0$$

and P is *almost complete* if

$$\lim_{n\to\infty} \frac{|U_P(n)|}{|\mathcal{G}(n)|} = 1$$

In [Bollobás et al. 2000], Bollobás, Pebody and Riordan conjectured:

Conjecture 1 (BPR conjecture). The chromatic and Tutte polynomials are almost complete.

In [Makowsky and Zhang 2019], the analogous question for r-regular hypergraphs was considered, and for $r \geq 3$ the conjecture was refuted. In [Noy 2003] it was observed, as a remark in the conclusions, that the independence polynomial $\mathrm{In}(G; x)$, discussed in Section 7, is weakly distinguishing on all finite graphs. In [Makowsky and Rakita 2019], it was proven that an infinite number of graph polynomials, among them the independence, clique and harmonious polynomials, are weakly distinguishing.

A natural way to approach the question whether a graph polynomial P is weakly distinguishing, almost complete or otherwise, is to ask, given a graph property $\mathcal{C}$, whether almost all graphs in $\mathcal{C}$ are P unique in $\mathcal{C}$. Let $\mathcal{C}$ be a graph property and denote by $U_{P,\mathcal{C}}(n) = \{G \in \mathcal{C}(n) : G \text{ has no } P\text{-mate in } \mathcal{C}\}$.

Definition 1.2. P is *weakly distinguishing on* $\mathcal{C}$ if

$$\lim_{n\to\infty} \frac{|U_{P,\mathcal{C}}(n)|}{|\mathcal{C}(n)|} = 0,$$

and P is *almost complete on* $\mathcal{C}$ if

$$\lim_{n\to\infty} \frac{|U_{P,\mathcal{C}}(n)|}{|\mathcal{C}(n)|} = 1.$$

In this paper we prove that many well studied graph polynomials are weakly distinguishing on infinitely many graph properties $\mathcal{C}$, listed in Example 3.2. However, all these properties $\mathcal{C}$ are small (in the sense that they are sets of measure 0 in the collection of all graphs), so these results do not imply that any of the above polynomials are weakly distinguishing or almost complete for arbitrary properties $\mathcal{C}$.

The graph polynomials which we show are weakly distinguishing for $\mathcal{C}$ include the characteristic polynomial, the domination polynomial, and the ξ-polynomial, which is a generalization of the both the matching and the Tutte polynomial. These three polynomials are mutually incomparable in distinctive power by Proposition 2.3. Once we know that ξ is weakly distinguishing for a graph property $\mathcal{C}$, this holds also for all the graph polynomials which are substitution instances of ξ in $\mathcal{C}$. This includes the matching polynomials, the independence polynomial, the chromatic polynomial, the Tutte polynomial, the Euler polynomial, and many others; see Section 7.

This paper is organized as follows: In Section 2 we introduce a method to compare distinctive power of graph polynomials and formulate how to use it (Lemma A). In Section 3, we give a graph theoretic background on addable properties, and graphs of genus at most k, and formulate our main tools, Lemma B and Lemma C. In Sections 4 and 5 we prove our results: Theorems 4.4 and 4.7 for the characteristic polynomial, Theorems 5.5 and 5.8 for the domination polynomial. In Section 6 we prove the same for the ξ-polynomial and in Section we apply Lemma A to derive the corresponding results for the matching polynomials, the independence polynomial, the chromatic polynomial, the Tutte polynomial, the Euler polynomial, and many others. In Section 8 we draw conclusions and present open problems.

2. Comparing graph polynomials

In this section we provide a tool (Lemma A) which allows us to show that many graph polynomials are weakly distinguishing.

Definition 2.1. Let $\mathcal{C}$ be a graph property, P and Q be two graph polynomials and G and H two finite graphs.

(i) G and H are *similar* if they have the same number of vertices, edges and connected components.

(ii) $P <^{\mathcal{C}}_{d.p} Q$ or Q *is at least as distinctive as* P *in* $\mathcal{C}$ if for all graphs $G, H \in \mathcal{C}$, $Q(G; \bar{x}) = Q(H; \bar{x})$ implies $P(G; \bar{y}) = P(H; \bar{y})$.

(iii) $P \sim^{\mathcal{C}}_{d.p} Q$ or P *and* Q *are of the same distinctive power in* $\mathcal{C}$ if $P <^{\mathcal{C}}_{d.p} Q$ and $Q <^{\mathcal{C}}_{d.p} P$.

(iv) $P <^{\mathcal{C}}_{s.d.p} Q$ or P *and* Q *are of the same distinctive power in* $\mathcal{C}$ *on similar graphs* if if for all similar graphs $G, H \in \mathcal{C}$, $Q(G; \bar{x}) = Q(H; \bar{x})$ implies $Q(G; \bar{y}) = Q(H; \bar{y})$.

(v) $P \sim^{\mathcal{C}}_{s.d.p} Q$ if $P <^{\mathcal{C}}_{s.d.p} Q$ and $Q <^{\mathcal{C}}_{s.d.p} P$.

(vi) For all graph properties $\mathcal{C}$ $P <^{\mathcal{C}}_{d.p} Q$ implies $P <^{\mathcal{C}}_{s.d.p} Q$.

If $\mathcal{C}$ consists of all finite graphs, we omit it.

The partial preorders $<_{d.p.}$ and $<_{s.d.p.}$ between graph polynomials (or general graph invariants) are studied extensively in [Makowsky et al. 2019]. A complete (trivial) graph polynomial is a maximal (minimal) element with respect to $<_{d.p.}$

Example 2.2. (i) The chromatic polynomial $\chi(G; x)$ and the Tutte polynomial satisfy $\chi(G; x) <_{s.d.p.} T(G; x, y)$ but not $\chi(G; x) <_{d.p.} T(G; x, y)$; see Section 7, because $T(G; x, y)$ does not determine the order of G in the presence of isolated vertices.

(ii) The characteristic polynomial $P_A(G; x)$ and the matching polynomial (defect aka acyclic) $\mu(G; x)$ from Section 7 are $d.p.$-equivalent on forests; see [Godsil and Gutman 1981].

(iii) Let $\bar{G}$ be the (loopless) complement graph of a simple graph G, and $P(G; \bar{x})$ be a (possibly multivariate) graph polynomial. Put $\bar{P}(G; \bar{x}) = P(\bar{G}, \bar{x})$. Then $\bar{P}(G; \bar{x}) \sim_{d.p.} P(G, \bar{x})$.
If we relativize this to a graph property $\mathcal{C}$, $\bar{P}(G; \bar{x}) \sim^{\mathcal{C}}_{d.p.} P(G, \bar{x})$ holds provided $\mathcal{C}$ is closed under complement graphs.

Proposition 2.3. *The following graph polynomials are pairwise incomparable by* $<_{s.d.p}$:

(i) *The chromatic polynomial* $\chi(G; x)$ *and the independence polynomial* $\mathrm{In}(G; x)$ *from Section 7.*

(ii) *The characteristic polynomial* $P_A(G, x)$ *from Sections 4 and the chromatic polynomial* $\chi(G; x)$.

(iii) *The characteristic polynomial* $P_A(G; x)$ *and the independence polynomial* $\mathrm{In}(G; x)$.

(iv) *The characteristic polynomial* $P_A(G, x)$ *and the domination polynomial* $\mathrm{Dom}(G; x)$. *from Sections 4 and 5.*

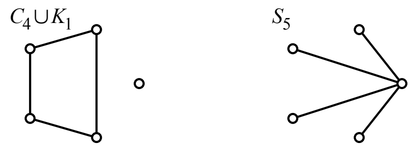

Figure 1. P_A-mates.

Figure 2. A path of length 5.

(v) *The characteristic polynomial and the ξ-polynomial from Section 6.*

(vi) *The domination polynomial and the ξ-polynomial.*

Proof. The first three examples are from [Makowsky et al. 2019].

For (iv)-(vi) we first note that all graphs of order less than 8 and all trees of order less than 10 are ξ-unique; see [Trinks 2012].

(A) Consider the graphs $C_4 \sqcup K_1$ and S_5 in Figure 1. We have

$$P_A(S_5, x) = P_A(C_4 \sqcup K_1, x) = x^2(x+2)(x-2), \mathrm{Dom}(S_5, 1) = 1, \quad \mathrm{Dom}(C_4 \sqcup K_1, 1) = 0,$$

and $\xi(S_5, x) \neq \xi(C_4 \sqcup K_1, x)$, since they are of order 5.

(B) Let P_5 and $\hat{P}_5$ be the graphs of order 5 shown in Figure 2. Then $\mathrm{Dom}(P_5, x) = \mathrm{Dom}(\hat{P}_5, x)$, since every dominating set of $\hat{P}_5$ is also a dominating set of P_5. Further,

$$P_A(P_5, x) = -x(x-1)(x+1)(x^2-3) \neq P_A(\hat{P}_5, x) = -x(x^2-x-3)(x^2+x-1),$$

hence $P_A(P_5, x) \neq P_A(\hat{P}_5, x)$. Thus $\xi(P_5, x) \neq \xi(\hat{P}_5, x)$, since both P_5 and $\hat{P}_5$ are ξ-unique.

(C) The graphs G_1 and G_2 of Figure 3 are of order 10. Therefore $\xi(G_1, x) = \xi(G_2, x)$. We also have

$$\begin{aligned}
\mathrm{Dom}(G_1; x) &= x^{10} + 10x^9 + 40x^8 + 82x^7 + 92x^6 + 56x^5 + 16x^4, \\
\mathrm{Dom}(G_2; x) &= x^{10} + 10x^9 + 41x^8 + 86x^7 + 94x^6 + 48x^5 + 9x^4, \\
P_A(G_1; x) &= x^2(x^4 - x^3 - 4x^2 + 2x + 3)(x^4 + x^3 - 4x^2 - 2x + 3), \\
P_A(G_2; x) &= x^2(x-1)(x+1)(x^2-2)(x^4 - 5x^2 + 3),
\end{aligned}$$

Hence $\mathrm{Dom}(G_1, x) \neq \mathrm{Dom}(G_2, x)$ and $P_A(G_1, x) \neq P_A(G_2, x)$.

Now (A) and (B) proves (iv), (A) and (C) proves (v), and (B) and (C) proves (vi). $\square$

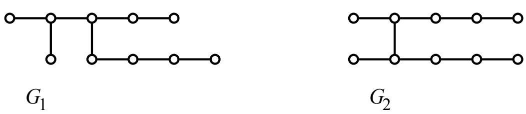

Figure 3. ξ-mates.

In Section 7 we use the following observation:

Lemma A. *Let $P(G;\bar{x})$ and $Q(G;\bar{y})$ two graph polynomials and $\mathcal{C}$ and $\mathcal{D}$ be graph properties with $\mathcal{D} \subseteq \mathcal{C}$. Assume that $P <^{\mathcal{C}}_{s.d.p} Q$ and Q is weakly distinguishing in $\mathcal{C}$. Then also P is weakly distinguishing in $\mathcal{C}$ but not necessarily in $\mathcal{D}$.*

Proof. Clearly, $P <^{\mathcal{C}}_{s.d.p} Q$ implies that $U_P(n) \subseteq U_Q(n)$. Hence

$$\lim_{n\to\infty} \frac{|U_P(n)|}{|\mathcal{G}(n)|} \leq \lim_{n\to\infty} \frac{|U_Q(n)|}{|\mathcal{G}(n)|}. \qquad \square$$

3. Addable properties and graphs of genus k

We discuss properties on which the Tutte, domination, matching, ξ, clique and characteristic polynomials are weakly distinguishing. These properties were studied in [McDiarmid, Steger, Welsh 2005; McDiarmid 2008].

Small addable classes.

Definition 3.1. A graph property $\mathcal{A}$ is *decomposable* if it is closed under disjoint union, and for all $G \in \mathcal{A}$ every component of G is in $\mathcal{A}$.

A graph property $\mathcal{A}$ is *bridge addable* if for each graph $G \in \mathcal{A}$ and every two vertices u, v in different components of G the graph obtained by adding an edge between u and v is also in $\mathcal{A}$.

A graph property $\mathcal{A}$ is *addable* if it is decomposable and bridge addable.

Example 3.2. The following properties are easily seen to be addable:

- planar graphs,
- outerplanar graphs,
- series-parallel graphs,
- graphs with tree width at most k for $k \geq 2$,
- k-colorable graphs for $k \geq 2$,
- graphs with no cycles of length greater than k,
- graphs with no K_k minor for $k \geq 2$.

Let $\mathcal{A}$ be a graph property. If $\mathcal{A}$ is minor closed (that is, closed under deletion of vertices and edges, and under contraction of edges), the Graph Minor theorem says that $\mathcal{A}$ is characterized by a finite set of forbidden minors (see [Lovász 2006] for more on graph minors and the graph minor theorem). We can characterize addable minor closed graph properties in terms of their forbidden minors:

Proposition 3.3 [McDiarmid 2009]. *Let $\mathcal{A}$ be a minor closed graph property. Then $\mathcal{A}$ is addable if and only if each excluded minor of $\mathcal{A}$ is 2-connected.*

Note that any nonempty, minor closed addable graph property $\mathcal{A}$ contains all forests, as it contains the graph with a single vertex, and is closed under taking disjoint unions of graphs in $\mathcal{A}$ and under adding edges between different components.

In addition to being addable, we will require the properties we consider to be small:

Definition 3.4. Let $\mathcal{A}$ be a graph property, and denote by $\mathcal{A}_n$ the graphs of order n in $\mathcal{A}$. We say that a graph property $\mathcal{A}$ is small if there exists a constant $a > 0$ such that $|\mathcal{A}_n| \leq a^n n!$ for all sufficiently large n.

The following result is convenient:

Theorem 3.5 [Norine et al. 2006]. *Let $\mathcal{C}$ be a proper minor closed graph property. Then $\mathcal{C}$ is small.*

Let H be a graph on vertex set $\{1, \ldots, h\}$ and let G be a graph on the vertex set $\{1, \ldots, n\}$ where $n > h$. Let $W \subseteq V(G)$ with $|W| = h$, and let the root r_W denote the least element in W. We say that W is a pendant appearance of H in G if (a) the increasing bijection from $\{1, \ldots, h\}$ to W gives an isomorphism between H and and the induced subgraph $G[W]$ of G, and (b) there exists exactly one edge between W and $V(G) - W$, and this edge is incident with the root r_W. Our method for proving graph polynomials are weakly distinguishing will relay on the following theorem:

Theorem 3.6 [McDiarmid, Steger, Welsh 2005]. *Let $\mathcal{C}$ be a non-empty, addable, and small graph property, let $H \in \mathcal{C}$ be a connected graph, and let R_n be a random graph selected uniformly at random from the graphs of order n in $\mathcal{C}$. Denote by $f_H(R_n)$ the number of pendant appearances of H in R_n. Then there are constants $\alpha > 0, n_0 \in \mathbb{N}$ such that for all $n > n_0$,*

$$\mathbb{P}[f_H(R_n) \leq \alpha n] < e^{-\alpha n}$$

For our purposes, we shall only need a weaker corollary of Theorem 3.6:

Corollary 3.7. *Let $\mathcal{A}$ be an addable proper minor closed graph property, H a fixed connected graph in $\mathcal{A}$. Denote by $\mathcal{A}'_n$ the set of all n vertex graphs $G \in \mathcal{A}$ that have at least one pendent appearance of H. Then* $\lim_{n\to\infty} |\mathcal{A}'_n|/|\mathcal{A}_n| = 1$.

All the properties listed in Example 3.2 are addable and minor closed, hence the corollary applies to them. Using this corollary, we get:

Lemma B. *Let P be a graph polynomial, and $\mathcal{A}$ a small addable graph property. If there is a fixed $H \in \mathcal{A}$ such that every graph $G \in \mathcal{A}$ with a pendant appearance of H has a P mate $G' \in \mathcal{A}$, then P is weakly distinguishing on $\mathcal{A}$.*

Proof. By Corollary 3.7, almost all graphs $G \in \mathcal{A}$ have a pendant appearance of H, and hence almost all graphs $G \in \mathcal{A}$ have a P-mate, so P is weakly distinguishing on $\mathcal{A}$. □

Graphs with genus at most k. Let G be a graph. The genus of G, denoted $g(G)$, is the minimal genus of a surface in which G can be embedded. (For more on graphs embedded in surfaces see [Lando and Zvonkin 2004; Mohar and Thomassen 2001], for example.) It is easy to see that $g(G)$ is well defined, as for all G, $g(G) \leq |E(G)|$. For $k \in \mathbb{N}$, denote by $\mathcal{C}_k$ the property of graphs with genus at most k. Note that in general $\mathcal{C}_k$ is not addable - for instance, the genus of K_5, the complete graph with 5 vertices, is 1, but the genus of a disjoint union of two copies of K_5 has genus 2- so we can not apply Theorem 3.6 to $\mathcal{C}_k$. However, the same result does hold for $\mathcal{C}_k$ with a slight modification:

Theorem 3.8 [McDiarmid 2008]. *Let H be a connected graph, $k \in \mathbb{N}$, and let R_n be a random graph selected uniformly at random from the graphs of order n in $\mathcal{C}_k$. Denote by $f_H(R_n)$ the number of pendant appearances of H in R_n. Then there are constants $\alpha > 0, n_0 \in \mathbb{N}$ such that for all $n > n_0$,*

$$\mathbb{P}[f_H(R_n) \leq \alpha n] < e^{-\alpha n}$$

Again, we only need a weaker corollary of this theorem:

Corollary 3.9. *Let H be a fixed connected planar graph, $k \in \mathbb{N}$ and let $\mathcal{A} = \mathcal{C}_k$ and let $\mathcal{A}'_n$ be the set of all n vertex graphs $G \in \mathcal{A}$ that have at least one pendent appearance of H. Then $\lim_{n\to\infty} |\mathcal{A}'_n|/|\mathcal{A}_n| = 1$.*

Lemma C. *Let P be a graph polynomial, and $\mathcal{C}_k$ the property of graphs with genus at most k. If there is a fixed planar graph H such that every graph G with a pendant appearance of H has a P mate $G' \in \mathcal{C}_k$, then P is weakly distinguishing on $\mathcal{C}_k$.*

Proof. By Corollary 3.9, almost all graphs $G \in \mathcal{C}_k$ have a pendant appearance of H, and hence almost all graphs $G \in \mathcal{C}_k$ have a P-mate, so P is weakly distinguishing on $\mathcal{C}_k$. □

4. The characteristic polynomial

Definition 4.1. Let G be a graph, and denote by A_G its adjacency matrix. The characteristic polynomial of G, denoted $P_A(G)$ is defined as the characteristic polynomial of A_G. The set of roots of $P_A(G)$ is referred to as the spectrum of G.

A P_A-unique graph is usually referred to as a graph determined by its spectrum, and if G and H are two non isomorphic graphs such that $P_A(G) = P_A(H)$, G and H are said to be cospectral.

The characteristic polynomial and particularly its roots are widely studied; see [Brouwer and Haemers 2012], for example, for an introduction. A classic result of Schwenk states that almost all trees are not determined by their spectrum (see [Schwenk 1973] for details). Using Corollary 3.7, we can extend this result to all small addable graph properties.

We use the following recurrence relation for the characteristic polynomial:

Lemma 4.2 (see [Clarke 1970]). *Let G_1 and G_2 be two graphs, and let $v_1 \in V(G_1)$,$v_2 \in V(G_2)$. Denote by H the graph obtained from the disjoint union of G_1 and G_2 by adding an edge from v_1 to v_2. Then*

$$P_A(H, x) = P_A(G_1, x) P_A(G_2, x) - P_A(G_1 - v_1, x) P_A(G_2 - v_2, x) \tag{4-1}$$

We now take two graphs considered in [Schwenk 1973], labeled R and S in Figure 4. They are isomorphic, and we can check that

$$P_A(S - v, x) = P_A(R - u, x) = x^8 - 6x^6 + 10x^4 - 4x^2.$$

Lemma 4.3. *Let G_S be a graph with a pendent appearance of S rooted at v. Denote by G_R the graph obtained from G by replacing the pendant appearance of S in G with a pendant appearance of R, rooted at u. Then G_S and G_R are cospectral mates.*

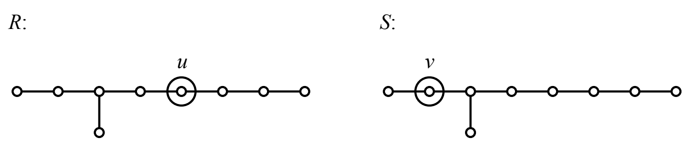

Figure 4. Cospectral graphs.

Proof. Denote by $w \in V(G_S)$ the vertex adjacent to v that is not in the pendant appearance of S, and by H the graph obtained from G_S by deleting all the vertices in the pendant appearance of S. By applying relation (4-1) to G_S with the edge wv, we get

$$P_A(G_S, x) = P_A(S, x)P_A(H, x) - P_A(S - v, x)P_A(H - w, x)$$

By applying relation (4-1) to G_R with the edge wu we get

$$P_A(G_R, x) = P_A(R, x)P_A(H, x) - P_A(R - u, x)P_A(H - w, x)$$

But since $P_A(R, x) = P_A(S, x)$ and $P_A(S - v, x) = P_A(R - u, x)$, we get that $P_A(G_S, x) = P_A(G_R, x)$, so G_S and G_R are cospectral mates. □

As unrooted graphs, S and R are isomorphic. Thus, we have:

Theorem 4.4. *Let $\mathcal{A}$ be a small addable graph property such that $S \in \mathcal{A}$. Then P_A is weakly distinguishing on $\mathcal{A}$.*

Proof. From lemmas B and 4.3 we get that P_A is weakly distinguishing on $\mathcal{A}$. □

Corollary 4.5. *Let $\mathcal{A}$ be a proper minor closed addable graph property. Then P_A is weakly distinguishing on $\mathcal{A}$.*

Proof. Since all forests are in $\mathcal{A}$, and S is a tree, by Theorems 4.4 and 3.5, P_A is weakly distinguishing on $\mathcal{A}$. □

Since all the properties in list 3.2 are addable and minor closed, we have:

Corollary 4.6. *P_A is weakly distinguishing on all the properties listed in Example 3.2.*

Similarly, we get:

Theorem 4.7. *Denote by $\mathcal{C}_k$ the class of graphs of genus at most k. Then P_A is weakly distinguishing on $\mathcal{C}_k$ for all $k \in \mathbb{N}$.*

Proof. Since S is a tree, from lemmas C and 4.3 we get that P_A is weakly distinguishing on $\mathcal{C}_k$. □

Instead of the characteristic polynomial P_A, which is based on the adjacency matrix, one can also look at the analogue graph polynomial P_L which is based on the Laplacian matrix $L_G = D_G - A_G$, where D_G is the diagonal matrix with diagonal elements $d_{v,v} = deg(v)$ for each $v \in V(G)$. Now $P_L(G; x)$ is the characteristic polynomial of L_G. An early survey about the Laplacian polynomial may be found in [Mohar 1991]. The graph polynomials are $d.p.$-equivalent on regular graphs, but $d.p.$-incomparable on all finite graphs; see [Brouwer and Haemers 2012] and [Makowsky et al. 2019]. Our proofs of Theorems 4.4 and 4.7 do not work for $P_L(G; x)$.

Problem 4.1. Find two graphs R and S that can be used two prove analogues of Theorems 4.4 and 4.7.

Figure 5. A path of length 5.

5. The domination polynomial

The *domination polynomial* is the generating function of dominating sets in a given graph G. More formally:

Definition 5.1. Let G be a graph. A set $S \subseteq V(G)$ is called a dominating set if for every $v \in V(G)$, either $v \in S$ or v has a neighbor $u \in S$. Define $\mathrm{Dom}(G; x) = \sum_{S \subseteq V(G)} x^{|S|}$, where the sum is over all dominating sets of G.

The domination polynomial was extensively studied in recent years; see [Alikhani and Peng 2014] for a survey.

Lemma 5.2 [Kotek et al. 2012]. *Let G be a graph. We say a vertex $v \in V(G)$ is a stem if it has a neighbor $u \in V(G)$ with $deg(u) = 1$. Assume G has two stems $u, u' \in G$, $(u, u') \in E(G)$, and denote by G' the graph resulting from G by deleting the edge (u, u'). Then for every set $S \subseteq V(G)$, S is a dominating set of G if and only if S is a dominating set of G'.*

Proof. If S is a dominating set of G', it is clearly also a dominating set of G. On the other hand, let S be a dominating set of G. After deleting (u, u'), the only vertices that are perhaps not dominated by S are u, u'. But u has a vertex of degree 1 as a neighbor, so either it or u are in S, and in either case u is dominated. The same applies to u'. So S is a dominating set of G'. □

Theorem 5.3. *Every graph G that has two distinct stems has a* Dom*-mate.*

Proof. This is a direct consequence of the lemma. □

Corollary 5.4. *Let G be a graph that has a pendant appearance of P_5 rooted at r* (see Figure 5), *and let $\hat{G}$ be the graph obtained from G by replacing the pendant appearance of P_5 with a pendant appearance of $\hat{P}_5$ rooted at r. Then G and $\hat{G}$ are* Dom*-mates.*

Theorem 5.5. *Let $\mathcal{A}$ be a small addable graph property such that $P_5, \hat{P}_5 \in \mathcal{A}$. Then* Dom *is weakly distinguishing on $\mathcal{A}$.*

Proof. From Lemma B and Corollary 5.4 we get that Dom is weakly distinguishing on $\mathcal{A}$. □

Corollary 5.6. *Let $\mathcal{A}$ be a proper minor closed addable graph property, such that $\hat{P}_5 \in \mathcal{A}$. Then* Dom *is weakly distinguishing on $\mathcal{A}$*

Proof. Since all forests are in $\mathcal{A}$, $P_5 \in \mathcal{A}$, and hence from Theorems 5.5 and 3.5 Dom is weakly distinguishing on $\mathcal{A}$. □

As in the previous sections, we have:

Corollary 5.7. Dom *is weakly distinguishing on all the properties listed in Example 3.2.*

Similarly, we get:

Theorem 5.8. *Denote by $\mathcal{C}_k$ the property of graphs of genus less than k. Then* Dom *is weakly distinguishing on $\mathcal{C}_k$ for all $k \in \mathbb{N}$.*

Proof. Since P_5 and $\hat{P}_5$ are planar, from Lemma C and Corollary 5.4 we get that Dom is weakly distinguishing on $\mathcal{C}_k$. □

6. The ξ polynomial

The ξ polynomial, introduced in [Averbouch et al. 2008; 2010], generalizes the matching and the Tutte polynomial. It is defined via a recurrence relation:

Let G be a graph and e an edge of G. We denote by $G - x$ the graph resulting from G by deleting the edge e, by G/e the graph resulting from G by contracting the edge e, and by $G \dagger e$ the graph resulting from G by extracting the edge e, i.e., deleting e together with its adjacent vertices.

Definition 6.1. Let G be a graph. Define the trivariate polynomial $\xi(G; x, y, z)$ using the recursive relation

$$\xi(G; x, y, z) = \xi(G - e; x, y, z) + y\xi(G/e; x, y, z) + z\xi(G \dagger e; x, y, z), \tag{6-1}$$
$$\xi(G \sqcup H; x, y, z) = \xi(G; x, y, z)\xi(H; x, y, z)$$

with the base conditions $\xi(K_1; x, y, z) = x$ and $\xi(\varnothing; x, y, z) = 1$.

An alternative representation of the ξ polynomial was introduced in [Trinks 2012]:

Definition 6.2. Let G be a graph. The covered components polynomial $C(G; x, y, z)$ of G is defined as

$$C(G; x, y, z) = \sum_{A \subseteq E(G)} x^{k(G\langle A\rangle)} y^{|A|} z^{c(G\langle A\rangle)},$$

where $G\langle A\rangle$ is the graph $(V(G), A)$, $k(G\langle A\rangle)$ is the number of connected components of $G\langle A\rangle$ and $c(G\langle A\rangle)$ is the number of covered connected components of $G\langle A\rangle$, that is connected components in $G\langle A\rangle$ with at least one edge.

We sometimes write $C(G)$ for $C(G; x, y, z)$ to simplify long expressions. The two definitions are connected by the following result:

Proposition 6.3 [Trinks 2012]. *For all graphs G,*

$$C(G; x, y, z) = \xi(G; x, y, xyz - xy). \tag{6-2}$$

Corollary 6.4. *Let G and H be graphs. Then $C(G; x, y, z) = C(H; x, y, z)$ if and only if $\xi(G; x, y, z) = \xi(H; x, y, z)$*

We will use the following recurrence relation:

Theorem 6.5 [Trinks 2012]. *Let G_1, G_2 be graphs, and $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. Let H be the graph obtained from G_1 and G_2 by identifying v_1 with v_2. Then*

$$C(G) = \Big(\frac{1}{xz} + \frac{2}{x}\Big)C(G_1)C(G_2) + \Big(-\frac{1}{z} - 1\Big)\big(C(G_1)C(G_2 - v_2) + C(G_1 - v_1)C(G_2)\big) + \Big(\frac{x}{z} + x\Big)C(G_1 - v_1)C(G_2 - v_2). \tag{6-3}$$

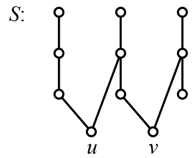

Figure 6. A tree with pseudosimilar vertices.

We next apply Theorem 6.5 to exhibit a pair of C mates.

Lemma 6.6. *Consider the graph S in Figure 6. Let G_v be a graph with a pendent appearance of S rooted at v, and let G_u be the graph obtained from G_v by replacing the pendant appearance of S rooted at v by a pendant appearance of S rooted at u. Then G_v and G_u are C mates.*

Proof. We apply relation (6-3) to G_v, with $G_1 = S$ and G_2 being the graph H_1 obtained from G_v by replacing the pendant appearance of S with a single vertex w. We get

$$C(G_v) = \Big(\frac{1}{xz} + \frac{2}{x}\Big)C(S)C(H_1) + \Big(-\frac{1}{z} - 1\Big)\big(C(S)C(H_1 - w) + C(S - v)C(H_1)\big) + \Big(\frac{x}{z} + x\Big)C(S - v)C(H_1 - w). \quad (6\text{-}4)$$

Similarly, we apply relation (6-3) to G_u, with $G_1 = S$ and G_2 being the graph H_2 obtained from G_u by replacing the pendant appearance of S with a single vertex w. We get

$$C(G_u) = \Big(\frac{1}{xz} + \frac{2}{x}\Big)C(S)C(H_2) + \Big(-\frac{1}{z} - 1\Big)\big(C(S)C(H_2 - w) + C(S - u)C(H)\big) + \Big(\frac{x}{z} + x\Big)C(S - u)C(H_2 - w).$$

Note that $H_1 \cong H_2$, $H_1 - w \cong H_2 - w$, and $S - v \cong S - u$, and hence $C(G_v) = C(G_u)$. □

As a direct consequence of Lemma 6.6 and Corollary 6.4, we have:

Corollary 6.7. *Let G_v be a graph with a pendent appearance of S (depicted in Figure 6) rooted at v, and let G_u be the graph obtained from G_v by replacing the pendant appearance of S rooted at v by a pendant appearance of S rooted at u. Then G_v and G_u are ξ mates.*

From Lemma B and Corollary 6.7 we get:

Theorem 6.8. *Let $\mathcal{A}$ be a small addable graph property such that $S \in \mathcal{A}$. Then ξ is weakly distinguishing on $\mathcal{A}$.*

Corollary 6.9. *Let $\mathcal{A}$ be a proper minor closed addable graph property. Then ξ is weakly distinguishing on $\mathcal{A}$.*

Proof. Since all forests are in $\mathcal{A}$, $S \in \mathcal{A}$, and hence from Theorems 6.8 and 3.5, M is weakly distinguishing on $\mathcal{A}$. □

As in the previous section, we get:

Corollary 6.10. *ξ is weakly distinguishing on all the properties listed in Example 3.2.*

Similarly, we get:

Theorem 6.11. *Denote by $\mathcal{C}_k$ the class of graphs of genus less than k. Then ξ is weakly distinguishing on $\mathcal{C}_k$ for all $k \in \mathbb{N}$.*

Proof. Since S is a tree, from lemmas C and 6.7 we get that ξ is weakly distinguishing on $\mathcal{C}_k$. □

7. ξ-invariants

We investigate consequences of Theorems 6.8 and 6.11 using Lemma A. From Theorem 6.8 we get:

Corollary 7.1. *Let $\mathcal{A}$ be a small addable graph property such that $S \in \mathcal{A}$, and let P be a graph polynomial such that $P <^{\mathcal{C}}_{s.d.p} \xi$. Then P is weakly distinguishing on $\mathcal{A}$.*

Corollary 7.2. *Let $\mathcal{A}$ be a proper minor closed addable graph property, and let P be a graph polynomial such that $P <^{\mathcal{C}}_{s.d.p} \xi$. Then P is weakly distinguishing on $\mathcal{A}$.*

From Theorem 6.11 we get:

Corollary 7.3. *Let $\mathcal{C}_k$ the class of graphs of genus less than k, and let P be a graph polynomial such that $P <^{\mathcal{C}}_{s.d.p} \xi$. Then P is weakly distinguishing on $\mathcal{C}_k$ for all $k \in \mathbb{N}$.*

We next consider results from the literature showing that there are many graph polynomials P where we can apply these results. Among them we find

(i) the generalized chromatic polynomial from [Dohmen et al. 2003], including the chromatic polynomial;

(ii) the matching polynomials;

(iii) the independence polynomial, including the vertex cover polynomial;

(iv) the Tutte polynomial, including the flow polynomial, the reliability polynomial and the Euler polynomial.

The generalized chromatic polynomial. The generalized chromatic polynomial $\mathrm{GC}(G; x, y)$ was introduced in [Dohmen et al. 2003]. and Z. Let Y and Z be two disjoint sets of colors. A generalized coloring of a graph $G = (V(G), E(G))$ is a map $c : V(G) \to (Y \sqcup Z)$ such that for all $(u, v) \in E$, if $c(u) \in Y$ and $c(v) \in Y$, then $c(u) = c(v)$. Y is called the set of proper colors. For two positive integers $x \geq y$, the value of the polynomial $\mathrm{GC}(G; x, y)$ is the number of generalized colorings by x colors, where y of them are proper. The chromatic polynomial $\chi(G; x)$ is obtained for the case $x = y$.

Theorem 7.4 [Averbouch et al. 2010, Proposition 22]. *For all graphs G,*

$$\mathrm{GC}(G; x, y) = \xi(G; x, -1, x - y).$$

Therefore $\chi <^{\mathcal{C}}_{d.p} CG <^{\mathcal{C}}_{d.p} \xi$.

The matching polynomial. Several variants of the matching polynomial are considered in the literature:

Definition 7.5. Let G be a graph with n vertices. A *matching* in G is a spanning subgraph of G in which every connected component is either an isolated vertex or two vertices connected by a single edge. We say a matching is of size k if it has exactly k edges. Denote by $m_k(G)$ the number of k matchings in G.

The *matching acyclic polynomial* (also known as the matching defect polynomial) is defined as

$$\mu(G;x)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k m_k(G)x^{n-2k}$$

The *matching generating polynomial* is defined as

$$g(G;x)=\sum_{k=0}^{\lfloor n/2\rfloor} m_k(G)x^k$$

The *bivariate matching polynomial* of G is defined as

$$M(G;w_1,w_2)=\sum_{k=0}^{\lfloor n/2\rfloor} m_k(G)w_1^{n-2k}w_2^k$$

For an introduction to the bivariate matching polynomial, see [Farrell 1979]. For a recent survey on the acyclic matching polynomial; see [Gutman 2016].

For our purposes, we note the following fact:

Fact 7.1. All of the polynomials in Definition 7.5 are of the same distinctive power on similar graphs:.

$$\mu \sim_{s.d.p} g \sim_{s.d.p} M.$$

In fact, we also have $\mu \sim_{d.p} M$, but for the edgeless graphs E_n of order n we have $g(E_n,x)=g(E_m,x)$ for all $m,n\geq 1$, but $\mu(E_n,x)\neq\mu(E_m,x)$ for $n\neq m$.

Thus we will only consider the bivariate matching polynomial.

Theorem 7.6 [Averbouch et al. 2010, Proposition 20]. *For all graphs G*

$$M(G;x,y)=\xi(G;x,0,y);$$

therefore $M <_{s.d.p} \xi$.

Independence and clique polynomial. The independence polynomial is defined as

$$\mathrm{In}(G;x)=\sum_{A\subseteq V(G)} x^{|A|}$$

where the graph induced graph $G[A]$ is edgless.

The vertex cover polynomial $V(G;x)$ is defined as

$$\mathrm{VC}(G;x)=\sum_{A\subseteq V(G)} x^{|A|}$$

where A is a vertex cover of G.

The following is taken from (but possibly not originally due to) [Trinks 2012].

Proposition 7.7. *For all graphs G we have*:

(i) $\mathrm{In}(G;x)=\mathrm{GC}(G;x+1,1)$.

(ii) $\mathrm{VC}(G;x)=x^n\,\mathrm{In}\big(G;\frac{1}{x}\big)$.

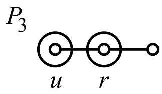

Figure 7. A path of length 3.

Hence, VC $\sim_{s.d.p}$ In $<_{s.d.p}$ GC.

The clique polynomial is defined as

$$\mathrm{Cl}(G; x) = \sum_{A \subseteq V(G)} x^{|A|},$$

where the graph induced graph $G[A]$ is a complete graph.

We note that $\mathrm{In}(G; x) = \mathrm{Cl}(\bar{G})$ for simple graphs. Therefore In $\sim_{d.p}$ Cl by Example 2.2(iii).

Both $\mathrm{In}(G; x)$ and $\mathrm{Cl}(G; x)$ were shown in [Makowsky and Rakita 2019] to be weakly distinguishing on all finite graphs. In the light of the above, $\mathrm{In}(G; x)$ is also weakly distinguishing on small addable graph properties, and on graphs of genus at most k.

This also holds if $\mathcal{C}$ is addable, small, and closed under complements. If $\mathcal{C}$ is addable and small, but not closed under complements we can use the following lemma:

Lemma 7.8. *Let G_r be a graph such that G has a pendant appearance of P_3* (see Figure 7) *rooted at r, and G_u the graph obtained from G_r by replacing the pendant appearance of P_3 by a pendant appearance of P_3 rooted at u. Then G_r and G_u are clique mates.*

Proof. For all $k > 2$, G_r and G_u have the same number of k cliques (since no k clique can include vertices from the pendant appearance), and G and G' have the same number of vertices and edges, and hence the same number of 1 and 2 cliques. So $\mathrm{Cl}(G_r; x) = \mathrm{Cl}(G_u; x)$. □

From Lemmas B and 7.8 we get:

Theorem 7.9. *Let $\mathcal{A}$ be a small addable graph property such that $P_3 \in \mathcal{A}$. Then* Cl *is weakly distinguishing on $\mathcal{A}$.*

The Tutte polynomial. Let $G = (V(G), E(G))$ be a graph. The Tutte polynomial $T(G; x, y)$ is defined by

$$T(G; x, y) = \sum_{A \subseteq E(G)} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V(G)|}.$$

The partition function $Z(G; q, w)$ is defined by

$$Z(G; q, w) = \sum_{A \subseteq E(G)} q^{k(F)} w^{|F|}.$$

They are related by the equation

$$T(G; x, y) = (x-1)^{-k(E)}(y-1)^{-|V(G)|} Z(G; (x-1)(y-1), (y-1)).$$

The chromatic polynomial $\chi(G; x)$ can be obtained from the Tutte polynomial by

$$\chi(G; x) = (-1)^{|V(G)|-k(G)} x^{k(G)} T(G; (1-x), 0)$$

From this we get:

Proposition 7.10. $\chi <_{s.d.p} T \sim_{s.d.p} Z <_{d.p} C \sim_{d.p.} \xi$.

A graph is *Eulerian* if all its vertices have even degree. It does not have to be connected. The Euler polynomial $\mathcal{E}(G; x)$ of a graph is defined by

$$\mathcal{E}(G; x) = \sum_{\substack{A \subseteq E(G) \\ (V,A) \text{ is Eulerian}}} x^{|A|};$$

it is related to the Tutte polynomial as follows [Aigner 2007, Chapter 10, p. 468]:

$$\mathcal{E}(G; x) = (1-x)^{|E(G)-|V(G)|+k(G)} x^{|V(G)|-k(G)} T\Big(G; \frac{1}{x}, \frac{1+x}{1-x}\Big).$$

Hence we get:

Proposition 7.11. $\mathcal{E} <_{s.d.p} T$.

The flow polynomial $\mathrm{Fl}(G, x)$ and the reliability polynomial $R(G; p)$ are related to the Tutte polynomial by

$$\mathrm{Fl}(G; x) = (-1)^{|E(G)-|V(G)|+k(G)} T(G; 0, 1-x)$$
$$R(G; p) = (p)^{|E(G)-|V(G)|+k(G)} (1-p)^{|V(G)|-k(G)} T\Big(G; 1, \frac{1}{p}\Big).$$

Hence we get:

Proposition 7.12. $\mathrm{Fl} <_{s.d.p} T$ *and* $R <_{s.d.p} T$.

8. Conclusion

We have shown that many graph polynomials are weakly distinguishing on all proper minor closed addable graph properties, including many interesting properties such as planar graphs, graphs with tree width at most k, and K_k free graphs (for $4 < k \in \mathbb{N}$). In addition, we proved that the domination, characteristic, and the edge elimination polynomial $\xi(G; x, y, z)$ are weakly distinguishing on the properties of graphs with genus less then k for all $k \in \mathbb{N}$. This also applies to graph polynomials derivable from ξ, such as the generalized chromatic polynomial, Tutte polynomial and its variants, the matching, the independence, and the clique polynomials.

Our results relied on the fact that proper minor closed addable properties are, in a sense, small, and so they do not settle the question whether the domination, characteristic or Tutte polynomials are weakly distinguishing, almost complete, or otherwise on all graphs.

We have shown that for the above graph polynomials P, the sequence $\alpha_P^{\mathcal{C}}(n) = \dfrac{|U_P^{\mathcal{C}}(n)|}{|\mathcal{C}(n)|}$ tends to 0 as n tends to infinity.

The following questions are natural extensions of the work in this paper:

Problem 8.1. What can be said about $\alpha_P^{\mathcal{C}}(n)$ when $\mathcal{C}$ is assumed to be a hereditary or monotone graph property?

Problem 8.2. Can we find a graph polynomial P and a graph property $\mathcal{C}$ such that we can prove that $\alpha_P^{\mathcal{C}}(n) \geq \beta$ for all sufficiently large n for some fixed $\beta \in (0, 1]$?

Problem 8.3. For P one of the above polynomials and $\mathcal{A}$ a proper minor closed addable property, select a random graph G_n uniformly at random in $\mathcal{A}_n$, and denote $[G_n] = \{H \in \mathcal{A}_n : P(G) = P(H)\}$. What can be said about the limit distribution of the random variable $|[G_n]|$?

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