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Effective simultaneous rational approximation to pairs of real quadratic numbers

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To the memory of Naum Ilich Feldman (1918–1994)

Let ξ, ζ be quadratic real numbers in distinct quadratic fields. We establish the existence of effectively computable, positive real numbers τ and c such that, for every integer q with $q > c$, we have

$$\max\{\|q\xi\|, \|q\zeta\|\} > q^{-1+\tau},$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

1. Introduction and results

Let ξ be an irrational real number. The real number μ is an irrationality measure for ξ if there exists a positive real number $c(\xi)$ such that every rational number p/q with $q \geq 1$ satisfies

$$\left|\xi - \frac{p}{q}\right| > \frac{c(\xi)}{q^{\mu}}.$$

If, moreover, the constant $c(\xi)$ is effectively computable, then μ is an effective irrationality measure for ξ. We denote by $\mu(\xi)$ the infimum of the irrationality measures for ξ and call it the irrationality exponent of ξ and we denote by $\mu_{\text{eff}}(\xi)$ the infimum of the effective irrationality measures for ξ and call it the effective irrationality exponent of ξ. It follows from the theory of continued fractions that $\mu(\xi) \geq 2$ and an easy covering argument shows that equality holds for almost all ξ, with respect to the Lebesgue measure. Furthermore, if ξ is real algebraic of degree $d \geq 2$, then Liouville's inequality implies that $\mu_{\text{eff}}(\xi) \leq d$, while Roth's theorem asserts that $\mu(\xi) = 2$. To get better upper bounds for the effective irrationality exponents of algebraic numbers is a notoriously challenging problem.

The first result of this type was obtained by Alan Baker [1964], who established that $\mu_{\text{eff}}(\sqrt[3]{2}) \leq 2.955$, but his method applies only to a very restricted class of algebraic numbers. A few years later, Feldman [1971], by means of a refinement of the lower bounds for linear forms in logarithms of algebraic numbers established by Baker, proved that the effective irrationality exponent of an arbitrary real algebraic number of degree greater than 2 is strictly less than its degree; see also [Bilu and Bugeaud 2000] for a proof depending on lower bounds for linear forms in only two logarithms. Subsequently, Bombieri [1993; 2002] gave an alternative proof of Feldman's result, completely independent of the theory of linear forms in logarithms and based on the Thue–Siegel principle. Further results can be found in [Bugeaud 2018a]; see in particular Section 4.10.

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In this note, we are concerned with the simultaneous approximation to pairs of real numbers by rational numbers having the same denominator. We extend the above definition of (effective) irrationality exponent as follows. Let ξ, ζ be real numbers such that $1, \xi, \zeta$ are linearly independent over the rational numbers. The real number μ is a simultaneous irrationality measure for the pair (ξ, ζ) if there exists a positive real number $c(\xi, \zeta)$ such that, for every integer triple (p, q, r) with $q \geq 1$, we have

$$\max\left\{\left|\xi - \frac{p}{q}\right|, \left|\zeta - \frac{r}{q}\right|\right\} > \frac{c(\xi, \zeta)}{q^{\mu}}.$$

If, moreover, the constant $c(\xi, \zeta)$ is effectively computable, then μ is an effective irrationality measure for the pair (ξ, ζ). We denote by $\mu(\xi, \zeta)$ the infimum of the irrationality measures for the pair (ξ, ζ) and call it the irrationality exponent of the pair (ξ, ζ), and we denote by $\mu_{\mathrm{eff}}(\xi, \zeta)$ the infimum of the effective irrationality measures for the pair (ξ, ζ) and call it the effective irrationality exponent of the pair (ξ, ζ).

Let ξ, ζ be real numbers such that $1, \xi, \zeta$ are linearly independent over the rational numbers. An easy application of Minkowski's theorem implies that $\mu(\xi, \zeta) \geq \frac{3}{2}$ and a covering lemma shows that equality holds for almost all pairs (ξ, ζ), with respect to the planar Lebesgue measure. Schmidt [1967] established that $\mu(\xi, \zeta) = \frac{3}{2}$ if ξ and ζ are both real and algebraic. His result is ineffective and gives no better information on $\mu_{\mathrm{eff}}(\xi, \zeta)$ than the obvious inequality

$$\mu_{\mathrm{eff}}(\xi, \zeta) \leq \max\{\mu_{\mathrm{eff}}(\xi), \mu_{\mathrm{eff}}(\zeta)\}.$$

The particular case where ξ and ζ are quadratic numbers in distinct number fields is of special interest. The obvious upper bound $\mu_{\mathrm{eff}}(\xi, \zeta) \leq 2$ has been improved in some cases, in particular by Rickert [1993], who established among other results that

$$\max\left\{\left|\sqrt{2} - \frac{p}{q}\right|, \left|\sqrt{3} - \frac{r}{q}\right|\right\} > \frac{10^{-7}}{q^{1.913}} \quad \text{for integers } p, q, r \geq 1,$$

and subsequently by Bennett [1995; 1996]. The method used in these papers applies only to a very restricted class of pairs (ξ, ζ) of quadratic numbers.

The purpose of the present note is to show how the theory of linear forms in logarithms (or, alternatively, Bombieri's method) allows us to improve the trivial upper bound $\mu_{\mathrm{eff}}(\xi, \zeta) \leq 2$ for all quadratic real numbers ξ and ζ in distinct quadratic fields.

Theorem 1.1. *Let ξ, ζ be real quadratic numbers in distinct quadratic fields. Let R_ξ and R_ζ denote the regulators of the fields $\mathbb{Q}(\xi)$ and $\mathbb{Q}(\zeta)$, respectively. There exists an absolute, positive, effectively computable real number c_1 such that*

$$\mu_{\mathrm{eff}}(\xi, \zeta) \leq 2 - (c_1 R_\xi R_\zeta)^{-1}. \tag{1-1}$$

In particular, if a, b are positive integers such that none of a, b, and ab is a perfect square, then there exists an absolute, positive, effectively computable real number c_2 such that

$$\mu_{\mathrm{eff}}(\sqrt{a}, \sqrt{b}) \leq 2 - (c_2\sqrt{ab}(\log a)(\log b))^{-1}.$$

The last assertion of Theorem 1.1 is an immediate consequence of the first one, since for any squarefree integer $D \geq 2$ the regulator R_D of the quadratic field generated by $\sqrt{D}$ satisfies

$$R_D < \sqrt{D}(1 + \log \sqrt{D}); \tag{1-2}$$

see, e.g., Theorem 13.4 in [Hua 1982].

Theorem 1.1 is by no means surprising. It is ultimately a consequence of the quantity B', which has its origin in [Feldman 1968; 1971] and is the key tool for his effective improvement of Liouville's bound; see Theorem 2.1 and the discussion below it. Other consequences of the quantity B' can be found in [Bugeaud 2018a] and in the recent papers [Bugeaud 2018b; Bugeaud and Evertse 2017; Bugeaud et al. 2018].

We present a proof of Theorem 1.1 together with a proof of a slightly weaker version of it, with $R_\xi R_\zeta$ replaced by $R_\xi R_\zeta \log(R_\xi R_\zeta)$ in (1-1). For the latter result, we apply an estimate for linear forms in three logarithms, while the former is derived from a result of Bombieri [1993] (and can also be derived from an estimate for linear forms in only two logarithms). This is in accordance with the improvements on Liouville's bound obtained by these two methods. Namely, for an algebraic number ξ of degree d at least equal to 3, denoting by R_ξ the regulator of the number field generated by ξ, it follows from the theory of linear forms in logarithms and from Bombieri's method, respectively, that there exist absolute, effectively computable, positive real numbers c_3 and c_4 such that

$$\mu_{\text{eff}}(\xi) \leq d - (c_3 R_\xi \log R_\xi)^{-1},$$
$$\mu_{\text{eff}}(\xi) \leq d - (c_4 R_\xi)^{-1};$$

see, e.g., [Bugeaud 1998].

The last assertion of Theorem 1.1 is equivalent to the following statement on systems of Pellian equations.

Theorem 1.2. *Let a, b be positive integers such that none of a, b, and ab is a perfect square. Let u, v be nonzero integers. There exists an effectively computable, absolute real number c_5 such that all the solutions in positive integers x, y, z of the system of Pellian equations*

$$x^2 - ay^2 = u, \quad z^2 - by^2 = v$$

satisfy

$$\max\{x, y, z\} \leq (\max\{|u|, |v|, 2\})^{c_5 \sqrt{ab} (\log a)(\log b)}.$$

2. Auxiliary results

As usual, $h(\alpha)$ denotes the (logarithmic) Weil height of the algebraic number α. Our auxiliary result for the proof of (a slightly weaker version of) Theorems 1.1 and 1.2 is a particular case of Theorem 2.1 of [Bugeaud 2018a], which essentially reproduces a theorem of Waldschmidt [1993; 2000].

Theorem 2.1. *Let $n \geq 1$ be an integer. Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers. Let $b_1, \dots, b_n$ be integers with $b_n \neq 0$. Let D be the degree over $\mathbb{Q}$ of the number field $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$. Let $A_1, \dots, A_n$ be real numbers with*

$$\log A_j \geq \max\left\{h(\alpha_j), \frac{\mathrm{e}}{D}|\log \alpha_j|, \frac{1}{D}\right\}, \quad 1 \leq j \leq n.$$

Let B' be a real number satisfying

$$B' \geq 3D, \quad B' \geq \max_{1 \leq j \leq n-1} \left\{ \frac{|b_n|}{\log A_j} + \frac{|b_j|}{\log A_n} \right\}.$$

If $b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n$ is nonzero, then we have

$$\log |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| \geq -2^{n+26} n^{3n+9} D^{n+2} \log(3D) \log A_1 \cdots \log A_n \log B'.$$

The quantity B' in Theorem 2.1, which replaces the quantity

$$B = \max\{3D, |b_1|, \ldots, |b_n|\}$$

occurring in earlier estimates of Baker, originates in [Feldman 1968; 1971]. It is a consequence of the use of the functions $x \mapsto \binom{x}{k}$ instead of $x \mapsto x^k$ in the construction of the auxiliary function. The key point is the presence of the factor $\log A_n$ in the denominator in the definition of B'. It is of great interest when $b_n = 1$ and $\log A_n$ is large, since it then allows us, roughly speaking, to replace B by $B/(\log A_n)$.

The auxiliary result for the proof of Theorems 1.1 and 1.2 is a particular case of Theorem 2 of [Bombieri 1993]. Actually, since the dependence in the parameters d and κ occurring in this theorem has been improved in [Bugeaud 1998], we choose to quote below a particular case of Théorème 1 of that paper.

Theorem 2.2. *Let $\mathbb{K}$ be a real number field of degree d. Let Γ be a finitely generated subgroup of $\mathbb{K}^*$ and consider a system $\xi_1, \ldots, \xi_t$ of generators of Γ/tors. Let ξ in Γ, A in $\mathbb{K}^*$ and $\kappa > 0$ be such that $\kappa \leq 1$ and*

$$0 < |1 - A\xi| < \mathrm{e}^{-\kappa h(A\xi)} < 1.$$

Setting

$$C = 4.10^{19} d^4 \frac{(\log 3d)^7}{\kappa} \log^* \frac{d}{\kappa}, \quad Q = (2tC)^t \prod_{i=1}^{t} h(\xi_i),$$

we have the upper bound

$$h(\xi) \leq 10Q \max\{h(A), Q\}.$$

Bombieri's original proof of Theorem 2.2 (up to the dependence on d and κ) is independent of the theory of linear forms in logarithms. An alternative proof, given in [Bugeaud 1998], depends on lower estimates for linear forms in two logarithms (a careful reader can observe that, while the proof of Théorème 1 of [Bugeaud 1998] rests on estimates for linear forms in three logarithms, estimates for linear forms in two logarithms are enough to establish Theorem 2.2 above, and even with a better numerical constant, since we have assumed that $\mathbb{K}$ is a real number field) combined with a lemma of geometry of numbers from [Bombieri 1993]. To deduce Theorem 2.2 from estimates for linear forms in two logarithms, the crucial ingredient is ultimately the presence of the factor B' in these estimates.

3. Proofs

We start with the proof of (a slightly weaker version of) Theorem 1.2. Let a, b be positive integers such that $1, \sqrt{a}, \sqrt{b}$ are linearly independent over the rationals. Let u, v be nonzero integers and consider the system of Pellian equations

$$x^2 - ay^2 = u, \quad z^2 - by^2 = v \quad \text{in positive integers } x, y, z. \tag{3-1}$$

Set

$$U = \max\{|u|, |v|, 2\} \quad \text{and} \quad X = \max\{x, y, z\}.$$

It is well known [Baker and Davenport 1969; Pinch 1988] that the theory of linear forms in logarithms allows us to bound effectively X in terms of U. Our goal is to show that we can get a bound which is polynomial in U.

Let ε and η be the fundamental totally positive units of the rings of integers of the fields $\mathbb{Q}(\sqrt{a})$ and $\mathbb{Q}(\sqrt{b})$, respectively, normalized to be greater than 1. We note that ξ and η are at least equal to $(1+\sqrt{5})/2$.

Let x, y, and z be positive integers satisfying (3-1). Since the norm over $\mathbb{Q}$ of $x + y\sqrt{a}$ (resp. $z + y\sqrt{b}$) is u (resp. v), there exist nonnegative integers m, n and algebraic numbers α in $\mathbb{Q}(\sqrt{a})$ and β in $\mathbb{Q}(\sqrt{b})$ such that

$$\begin{gathered} \alpha \ge |\alpha^\sigma|, \quad \beta \ge |\beta^\sigma|, \quad \alpha\varepsilon^{-1} \le |\alpha^\sigma|\varepsilon, \quad \beta\eta^{-1} \le |\beta^\sigma|\eta, \\ x + y\sqrt{a} = \alpha\varepsilon^m, \quad \text{and} \quad z + y\sqrt{b} = \beta\eta^n, \end{gathered} \tag{3-2}$$

where the superscript σ denotes the Galois conjugacy.

Since $\varepsilon^\sigma = \varepsilon^{-1}$ and $\eta^\sigma = \eta^{-1}$, we have

$$\begin{aligned} 2y\sqrt{a} &= \alpha\varepsilon^m - \alpha^\sigma\varepsilon^{-m}, \\ 2y\sqrt{b} &= \beta\eta^n - \beta^\sigma\eta^{-n}. \end{aligned}$$

Set

$$\Lambda = \left|\alpha\beta^{-1}\sqrt{\frac{b}{a}}\,\varepsilon^m\eta^{-n} - 1\right| = \left|\alpha^\sigma\beta^{-1}\sqrt{\frac{b}{a}}\,\varepsilon^{-m}\eta^{-n} - \beta^\sigma\beta^{-1}\eta^{-2n}\right|. \tag{3-3}$$

Clearly, Λ is nonzero.

Set

$$U_0 = \max\{U, ab, \varepsilon^2, \eta^2\} \tag{3-4}$$

Observe that $\alpha = |u|/|\alpha^\sigma|$, $\beta = |v|/|\beta^\sigma|$, (3-2), and (3-4) imply

$$\alpha^2 \le |u|\varepsilon^2 \le U_0^2, \quad \beta^2 \le |v|\eta^2 \le U_0^2, \tag{3-5}$$

and

$$\begin{aligned} h\left(\alpha\beta^{-1}\sqrt{\frac{b}{a}}\right) &\le h(\alpha) + h(\beta) + h(\sqrt{a}) + h(\sqrt{b}) \\ &\le \log\alpha + \log\beta + \frac{\log a}{2} + \frac{\log b}{2} \le 3\log U_0. \end{aligned}$$

Assume first that

$$\max\{m\log\varepsilon, n\log\eta\} \ge 12\log U_0. \tag{3-6}$$

Observe that (3-3), (3-4), and (3-5) imply

$$\log\Lambda \le -n\log\eta + 2\log U_0, \tag{3-7}$$

and

$$|m \log \varepsilon - n \log \eta| \le 4 \log U_0; \tag{3-8}$$

thus, by (3-6), we get

$$\log \Lambda \le -\max\{m \log \varepsilon, n \log \eta\} + 6 \log U_0 \le -\frac{\max\{m \log \varepsilon, n \log \eta\}}{2}. \tag{3-9}$$

It then follows from Theorem 2.1 applied with $\alpha_1 = \varepsilon,\ \alpha_2 = \eta,\ \alpha_3 = \alpha\beta^{-1}\sqrt{b/a}$ that

$$\log \Lambda \gg -(\log U_0)(\log \varepsilon)(\log \eta) \log^* \frac{\max\{m, n\}}{\log U_0}, \tag{3-10}$$

where we write $\log^*$ for the function $\max\{1, \log\}$. Here and below, the numerical constant implied by $\ll$ is positive, absolute, and effectively computable.

The combination of (3-9) with (3-10) gives

$$\max\{m \log \varepsilon, n \log \eta\} \ll (\log U_0)(\log \varepsilon)(\log \eta) \log^* \frac{\max\{m, n\}}{\log U_0}.$$

We deduce that

$$X \ll \max\{m \log \varepsilon, n \log \eta\} \ll (\log \varepsilon)(\log \eta) \log^*(\max\{\log \varepsilon, \log \eta\}) \log U_0,$$

while $X \ll \log U_0$ if (3-6) is not satisfied.

Consequently, no matter if (3-6) holds or not, there exist an effectively computable positive real number C_1, depending only on a and b, and an effectively computable positive, absolute real number c_6 such that

$$X \le C_1 U^{c_6 (\log \varepsilon)(\log \eta) \log^*(\max\{\log \varepsilon, \log \eta\})}. \tag{3-11}$$

Combined with the upper bound (1-2), this gives Theorem 1.2 up to an extra logarithmic factor.

For the proof of (a slightly weaker version of) Theorem 1.1, without any loss of generality, we may assume that ξ, η are square roots of positive integers a, b as above. Then, keeping our notation, it follows from (3-11) that there exist effectively computable positive real numbers C_2 and C_3, depending only on a and b, such that

$$\begin{aligned}
\max\left\{\left|\sqrt{a} - \frac{x}{y}\right|, \left|\sqrt{b} - \frac{z}{y}\right|\right\} &\ge \frac{C_2}{y^2} \max\{|x^2 - ay^2|, |z^2 - by^2|\} \\
&\ge \frac{C_2}{2y^2} \left(\frac{X}{C_1}\right)^{1/(c_6 (\log \varepsilon)(\log \eta) \log^*(\max\{\log \varepsilon, \log \eta\}))} \\
&\ge \frac{C_3}{y^{2 - 1/(c_6 (\log \varepsilon)(\log \eta) \log^*(\max\{\log \varepsilon, \log \eta\}))}}.
\end{aligned}$$

Combined with (1-2), this completes the proof of Theorem 1.1 up to an extra logarithmic factor.

It remains for us to explain how to deduce Theorems 1.1 and 1.2 from Theorem 2.2, applied with Γ being the subgroup generated by ε and η,

$$A = \alpha\beta^{-1}\sqrt{b/a}, \quad \xi_1 = \varepsilon, \quad \xi_2 = \eta, \quad \text{and} \quad \xi = \varepsilon^m \eta^{-n}.$$

Note that

$$h(A\xi) \le h(A) + m \log \varepsilon + n \log \eta \le 3 \log U_0 + m \log \varepsilon + n \log \eta. \tag{3-12}$$

Assume that (3-6) holds. By combining (3-6), (3-7), (3-8), and (3-12) we get

$$\log \Lambda \ll -\log U_0 - m \log \varepsilon - n \log \eta \ll -h(A\xi).$$

It then follows from Theorem 2.2 that

$$h(\xi) \ll ((\log \varepsilon)(\log \eta) h(A) + (\log \varepsilon)^2 (\log \eta)^2).$$

Since $h(A) \leq 3 \log U_0$ and

$$X \ll \max\{m \log \varepsilon, n \log \eta\} \leq 4h(\xi),$$

there exist an effectively computable positive real number C_4, depending only on a and b, and an effectively computable positive, absolute real number c_7 such that

$$X \leq C_4 U^{c_7 (\log \varepsilon)(\log \eta)}. \tag{3-13}$$

By increasing c_7 and C_4 if necessary, we see that (3-13) also holds if (3-6) is not satisfied. Then, proceeding as below (3-11), we establish Theorems 1.1 and 1.2.

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YANN BUGEAUD:

bugeaud@math.unistra.fr
Institut de Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg et CNRS, Strasbourg, France

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In memoriam
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