Moscow Journal of Combinatorics and Number Theory

2020
vol. 9 no. 4

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msp

Moscow Journal of Combinatorics and Number Theory
Vol. 9, No. 4, 2020
https://doi.org/10.2140/moscow.2020.9.435

On approximations of solutions of the equation $P(z, \ln z) = 0$ by algebraic numbers

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The paper is devoted to studying how well solutions of an equation $P(z, \ln z) = 0$, where $P(x, y) \in \mathbb{Z}[x, y]$, can be approximated with algebraic numbers. We prove a new bound with the help of a construction due to K. Mahler.

The length of a polynomial is the sum of the absolute values of its coefficients. The length of an algebraic number is the length of its canonical polynomial. Let $\ln z$ be an arbitrary branch of the logarithm. The main result of this paper is the following theorem.

Theorem 1. *Suppose*

$$P(x, y) \in \mathbb{Z}[x, y], \quad \deg_x P = d_1, \quad \deg_y P = d_2, \quad d_1 d_2 \neq 0,$$
$$\zeta \in \mathbb{C}, \quad P(\zeta, y) \not\equiv 0, \quad P(\zeta, \ln \zeta) = 0.$$

Then, for every $\varepsilon > 0$, the inequality

$$|\zeta - \theta| < \exp\left(-(4+\varepsilon) d_1 d_2 \frac{\ln^2 L(\theta)}{\ln \ln L(\theta)}\right) \tag{1}$$

admits only finitely many solutions in algebraic θ such that

$$\varkappa = \deg \theta = o(\ln \ln L(\theta)), \quad \text{i.e.,} \quad \varkappa < \alpha(L) \cdot \ln \ln L, \quad \lim_{L \to \infty} \alpha(L) = 0. \tag{2}$$

The length of θ can be replaced in (1) by its height $H(\theta)$, as

$$L(\theta) \leq (\varkappa + 1) H(\theta).$$

N. I. Feldman [1964] proved a theorem on approximations of the solutions of the equation $P(z, e^z) = 0$ by algebraic numbers. His result was improved in [Galochkin 1972]. A result similar to (1) can be obtained from [Nesterenko and Waldschmidt 1996, Theorem 5] but with a constant greater than 4 in the exponent. Our proof is based on Mahler's construction [1932a; 1932b; 1967] with a special choice of parameters.

Lemma 2. *Suppose $P(x) \in \mathbb{Z}[x]$, θ is an algebraic number, and $P(\theta) \neq 0$. Then*

$$|P(\theta)| \geq L(P)^{1-\deg \theta} L(\theta)^{-\deg P},$$

where $L(P)$ and $L(\theta)$ are the lengths of P and θ respectively.

MSC2010: 11J82.
Keywords: Diophantine approximation, algebraic numbers, logarithms.

The proof can be found, for instance, in [Feldman 1981].

Lemma 3. *Let m, n be positive integers. For each $k=\overline{0,n}$ set*

$$\Phi_k(t)=(t-m)^{k+1}\prod_{j=0}^{m-1}(t-j)^{n+1},\qquad R_k(z)=\frac{1}{2\pi i}\oint_\Gamma \frac{e^{tz}}{\Phi_k(t)}\,dt, \tag{3}$$

where Γ is the circle $|t|=m(n+1)+m$. Then

$$\operatorname{ord}_{z=0}R_k(z)=m(n+1)+k, \tag{4}$$

$$|R_k(z)|<(m(n+1)+m)e^{(m(n+1)+m)|z|}(m(n+1))^{-m(n+1)}, \tag{5}$$

$$R_k(z)=P_{k0}(z)+P_{k1}(z)e^z+\dots+P_{km}(z)e^{mz},\qquad P_{kj}(z)\in\mathbb{Q}[z], \tag{6}$$

$$n_{kj}=\deg P_{kj}=(\operatorname{ord}_{t=j}\Phi_k(t))-1=\begin{cases} n & \textit{for } j=\overline{0,m-1},\\ k & \textit{for } j=m,\ k=\overline{0,n}.\end{cases} \tag{7}$$

Set $p_{kj}(z)=b^n n!\,(m!)^{n+1}P_{kj}(z)$, where $b=\operatorname{lcm}(1,2,\dots,m)$. Then

$$p_{kj}(z)\in\mathbb{Z}[z],\qquad L(p_{kj})<e^{\gamma_1 mn}n!,\qquad \textit{where } \gamma_1 \textit{ is an absolute constant.} \tag{8}$$

Proof. We have

$$R_k(z)=\sum_{s=0}^{\infty}\frac{a_{ks}}{s!}z^s,\qquad a_{ks}=\frac{1}{2\pi i}\oint_\Gamma\frac{t^s}{\Phi_k(t)}\,dt,$$

$a_{ks}=0$ for $s<n_k$, $a_{k,n_k}\neq 0$, $n_k=m(n+1)+k$, which proves (4).

Inequality (5) follows from the estimate

$$|\Phi_k(t)|\geq (m(n+1))^{m(n+1)}$$

and an obvious estimate on the integral in (3).

We have

$$R_k(z)=\sum_{j=0}^{m}I_{kj},\qquad I_{kj}=\frac{1}{2\pi i}e^{jz}\oint_{|t-j|=1/2}\frac{e^{(t-j)z}}{\Phi_k(t)}\,dt=e^{jz}\sum_{s=0}^{n_{kj}}\frac{a_{kjs}}{s!}z^s,$$

where

$$a_{kjs}=\frac{1}{2\pi i}\oint_{|t-j|=1/2}\frac{(t-j)^s}{\Phi_k(t)}\,dt,\qquad a_{k,j,n_{kj}}\neq 0,\qquad P_{kj}(z)=\sum_{s=0}^{n_{kj}}\frac{a_{kjs}}{s!}z^s. \tag{9}$$

This proves both (6) and (7).

Since $|t-j|=\frac{1}{2}$, we have

$$|\Phi_k(t)|>(m!)^{n+1}e^{-\gamma_2 mn}.$$

Thus, by (9)

$$|a_{kjs}|<(m!)^{-n-1}e^{\gamma_3 mn}. \tag{10}$$

Let us use the substitution $t - j = bu$, where $b = \mathrm{lcm}(1, 2, \dots, m) = e^{O(m)}$, in order to transform the integral in (9). Then for $l \neq j$

$$t - l = bu + j - l = (j - l)\left(1 - \frac{bu}{l - j}\right).$$

This substitution gives $a_{kjs} = A_{kjs}B_{kjs}$, where

$$A_{kjs} = b^{s-n_{kj}} \prod_{l=0,\, l\neq j}^{m} (j - l)^{-n_{kl}-1},$$

$$B_{kjs} = \oint_{|u|=(2b)^{-1}} \prod_{l=0,\, l\neq j}^{m} \left(1 - \frac{bu}{l - j}\right)^{-n_{kl}-1} u^{s-n_{kj}-1}\, du,$$

with n_{lj} defined by (7).

The coefficients of the series

$$\left(1 - \frac{bu}{l - j}\right)^{-1} = \sum_{r=0}^{\infty} \left(\frac{b}{l - j}\right)^r u^r$$

are integers; hence $B_{kjs} \in \mathbb{Z}$ and $b^n(m!)^{n+1}A_{kjs} \in \mathbb{Z}$. Taking into account (9) and (10), we get (8). □

Lemma 4. *There exist polynomials $B_{ks}(u)$ which, together with the corresponding form*

$$V_k(u) = \sum_{s=0}^{n} B_{ks}(u)(\ln u)^s,$$

enjoy the properties

$$B_{ks}(u) \in \mathbb{Z}[u], \quad k = \overline{0, n}, \tag{11}$$

$$\deg B_{ks} = \begin{cases} m & \text{for } k \leq s, \\ m - 1 & \text{for } k > s, \end{cases} \tag{12}$$

$$\Delta(u) = \det |B_{ks}|_{k,s=\overline{0,n}} = \lambda(u - 1)^{m(n+1)}, \quad \lambda \neq 0, \tag{13}$$

$$L(B_{ks}(u)) < e^{\gamma_4 mn} n!\,, \tag{14}$$

$$|V_k(u)| < e^{\gamma_5 |u| mn} n^{-mn}. \tag{15}$$

Proof. Let us set

$$V_k(u) = b^n(n!)(m!)^{n+1}R_k(\ln u) = \sum_{j=0}^{m} p_{kj}(\ln u)u^j = \sum_{s=0}^{n} B_{ks}(u)(\ln u)^s. \tag{16}$$

Statements (11), (12), (14), and (15) follow from Lemma 3. Thus, it remains to prove (13).

First, let us assume that $|u - 1| < 1$ and that $\ln 1 = 0$. In this case we have by (4)

$$R_k(z) = z^{m(n+1)+k}T_k(z), \quad T_k(0) \neq 0,$$

whence, taking into account that $\ln 1 = 0$, we get

$$V_k(u) = (\ln u)^{m(n+1)+k}F_k(u) = (u - 1)^{m(n+1)+k}G_k(u), \quad G_k(1) \neq 0.$$

It follows from (12) that $\Delta(u) \not\equiv 0$ and that

$$\deg \Delta(u) = m(n+1).$$

Replacing the first column with the one consisting of $V_0(u), V_1(u), \ldots, V_n(u)$ preserves the determinant. Hence

$$\operatorname{ord}_{u=1} \Delta(u) \geq m(n+1),$$

which implies (13).

Moving along a path around the origin changes $\ln u$, but it does not change $B_{ks}(u)$. Therefore, it does not change $\Delta(u)$. Thus, (13) holds for every branch of the logarithm. □

Theorem 5. *Suppose*

$$P(x, y) \in \mathbb{Z}[x, y], \quad \deg_x P = d_1, \quad \deg_y P = d_2, \quad d_1 d_2 \neq 0.$$

Then, for every $\varepsilon > 0$ and every $r > 0$, the inequality

$$|P(\theta, \ln\theta)| < \exp\left(-(4+\varepsilon)d_1 d_2 \frac{\ln^2 L(\theta)}{\ln\ln L(\theta)}\right) \tag{17}$$

admits only finitely many solutions in algebraic θ such that

$$|\theta| < r \qquad \text{and} \qquad \varkappa = \deg\theta = o(\ln\ln L(\theta)) \quad \text{as } L(\theta) \to \infty. \tag{18}$$

Note that Theorem 1 follows from Theorem 5. Indeed, for all but finitely many θ Theorem 5 provides

$$|P(\theta, \ln\theta)| \geq \exp\left(-(4+\varepsilon)d_1 d_2 \frac{\ln^2 L(\theta)}{\ln\ln L(\theta)}\right).$$

Hence

$$|P(\theta, \ln\theta)| = |P(\zeta, \ln\zeta) - P(\theta, \ln\theta)| = \left|\int_\theta^\zeta P'(t, \ln t)\,dt\right| < \gamma_6|\zeta - \theta|,$$

and we can assume that $|\zeta - \theta| < 1$, $r = |\zeta| + 1$. Thus, it remains to prove Theorem 5.

Proof of Theorem 5. Let us take

$$m = \left[\frac{d_1}{d_2} n\right], \quad n > d_2. \tag{19}$$

Then by (14) and (15) we have

$$L(B_{ks}(u)) < e^{\gamma_4 mn} n! < e^{\gamma_7 n^2}, \quad |V_k(u)| < e^{\gamma_7 n^2} n^{-d_1 d_2^{-1} n^2}. \tag{20}$$

Let θ be an algebraic number, $\varkappa = \deg\theta$, $L = L(\theta)$. We may assume that

$$\theta \neq 0, \quad \theta \neq 1, \quad P(\theta, y) \not\equiv 0.$$

This excludes finitely many values of θ. The values

$$W_k(\theta) = (\ln\theta)^k P(\theta, \ln\theta) = \sum_{s=0}^{n} A_{ks}(\theta)(\ln\theta)^s, \quad k = \overline{0, v}, \ v = n - d_2, \tag{21}$$

of the corresponding forms at $1, \ln\theta, \ldots, (\ln\theta)^n$ are linearly independent. Moreover, we have $|A_{ks}(\theta)| < e^{\gamma_8 n}$. Hence we can choose d_2 values among $V_0(\theta), \ldots, V_n(\theta)$ (say, $V_1(\theta), \ldots, V_{d_2}(\theta)$) which are linearly independent with the values from (21) and such that

$$D(\theta) = \begin{vmatrix} A_{00}(\theta) & A_{01}(\theta) & \cdots & A_{0n}(\theta) \\ \vdots & \vdots & & \vdots \\ A_{v0}(\theta) & A_{v1}(\theta) & \cdots & A_{vn}(\theta) \\ B_{10}(\theta) & B_{11}(\theta) & \cdots & B_{1n}(\theta) \\ \vdots & \vdots & & \vdots \\ B_{d_2,0}(\theta) & B_{d_2,1}(\theta) & \cdots & B_{d_2,n}(\theta) \end{vmatrix} \neq 0. \tag{22}$$

Consider the determinant

$$D(u) = \begin{vmatrix} A_{00}(u) & A_{01}(u) & \cdots & A_{0n}(u) \\ \vdots & \vdots & & \vdots \\ A_{v0}(u) & A_{v1}(u) & \cdots & A_{vn}(u) \\ B_{10}(u) & B_{11}(u) & \cdots & B_{1n}(u) \\ \vdots & \vdots & & \vdots \\ B_{d_2,0}(u) & B_{d_2,1}(u) & \cdots & B_{d_2,n}(u) \end{vmatrix}$$

as a polynomial of u. By (11), (12), (19), (20), and (22),

$$D(u) \in \mathbb{Z}[u], \quad D(\theta) \neq 0, \quad \deg D(u) \leq nd_1 + md_2 \leq 2nd_1, \quad L(D(u)) < e^{\gamma_9 n^2}. \tag{23}$$

By Lemma 2,

$$|D(\theta)| \geq e^{(1-\varkappa)\gamma_9 n^2} L^{-2d_1 n} > e^{-\gamma_9 \varkappa n^2} L^{-2d_1 n}, \quad L = L(\theta). \tag{24}$$

On the other hand,

$$D(\theta) = \begin{vmatrix} W_0(\theta) & A_{01}(\theta) & \cdots & A_{0n}(\theta) \\ \vdots & \vdots & & \vdots \\ W_v(\theta) & A_{v1}(\theta) & \cdots & A_{vn}(\theta) \\ V_1(\theta) & B_{11}(\theta) & \cdots & B_{1n}(\theta) \\ \vdots & \vdots & & \vdots \\ V_{d_2}(\theta) & B_{d_2,1}(\theta) & \cdots & B_{d_2,n}(\theta) \end{vmatrix};$$

i.e.,

$$D(\theta) = \sum_{k=0}^{v} W_k(\theta) M_k(\theta) + \sum_{l=1}^{d_2} V_l(\theta) N_l(\theta),$$

where $M_k(\theta)$ and $N_l(\theta)$ are the cofactors of the first column of $D(\theta)$. It follows from (20), (21), and (23) that

$$|W_k(\theta)| < e^{\gamma_8 n} |P(\theta, \ln\theta)|, \quad |V_l(\theta)| < e^{\gamma_7 n^2} n^{-d_1 d_2^{-1} n^2},$$
$$|M_k(\theta)| < e^{\gamma_{10} n^2}, \quad |N_l(\theta)| < e^{\gamma_{10} n^2}.$$

Hence

$$|D(\theta)| < e^{\gamma_{11} n^2} |P(\theta, \ln\theta)| + e^{\gamma_{11} n^2} n^{-d_1 d_2^{-1} n^2}.$$

Taking into account (24), we get

$$e^{-2d_1 n \ln L} < e^{\gamma_{12} \varkappa n^2} |P(\theta, \ln\theta)| + e^{\gamma_{12} \varkappa n^2 - d_1 d_2^{-1} n^2 \ln n}. \tag{25}$$

Given an arbitrary $\varepsilon > 0$, let us set

$$n = \left[\left(2+\frac{\varepsilon}{4}\right)d_2\frac{\ln L}{\ln\ln L}\right].$$

Then

$$2d_1 n\ln L \sim \left(4+\frac{\varepsilon}{2}\right)d_1d_2\frac{\ln^2 L}{\ln\ln L} \qquad \text{as } L\to\infty,$$

$$d_1d_2^{-1}n^2\ln n \sim \left(4+\varepsilon+\frac{\varepsilon^2}{16}\right)d_1d_2\frac{\ln^2 L}{\ln\ln L} \quad \text{as } L\to\infty.$$

Hence due to restrictions (18),

$$\gamma_{12}\varkappa n^2 = o\left(\frac{\ln^2 L}{\ln\ln L}\right).$$

Thus, for L large enough we have $e^{\gamma_{12}\varkappa n^2 - d_1d_2^{-1}n^2\ln n} < \frac{1}{2}e^{-2d_1 n\ln L}$. Combining this with (25), we get

$$|P(\theta,\ln\theta)| > \tfrac{1}{2}e^{-\gamma_{12}\varkappa n^2 - 2d_1 n\ln L} > \exp\left(-(4+\varepsilon)d_1d_2\frac{\ln^2 L}{\ln\ln L}\right),$$

which implies that inequality (17) has finitely many solutions.

Theorems 5 and 1 are proved. □

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Received 30 Dec 2019. Revised 11 Feb 2020.

ALEXANDER GALOCHKIN:
aigalochkin@yandex.ru
Department of Number Theory, Moscow Lomonosov State University, Moscow, Russia

ANASTASIA GODUNOVA:
icq13@mail.ru
Department of Number Theory, Moscow Lomonosov State University, Moscow, Russia

MJCNT — published in partnership with the
Moscow Institute of Physics and Technology

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Moscow Journal of Combinatorics and Number Theory

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Moscow Journal of Combinatorics and Number Theory (ISSN 2640-7361 electronic, 2220-5438 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

MJCNT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
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Moscow Journal of Combinatorics and Number Theory

vol. 9 no. 4 2020

In memoriam
N. I. Feldman