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On approximations of solutions of  
the equation  $P(z, \ln z) = 0$  by algebraic numbers

Alexander Galochkin and Anastasia Godunova



# On approximations of solutions of the equation $P(z, \ln z) = 0$ by algebraic numbers

Alexander Galochkin and Anastasia Godunova

The paper is devoted to studying how well solutions of an equation  $P(z, \ln z) = 0$ , where  $P(x, y) \in \mathbb{Z}[x, y]$ , can be approximated with algebraic numbers. We prove a new bound with the help of a construction due to K. Mahler.

The length of a polynomial is the sum of the absolute values of its coefficients. The length of an algebraic number is the length of its canonical polynomial. Let  $\ln z$  be an arbitrary branch of the logarithm. The main result of this paper is the following theorem.

**Theorem 1.** *Suppose*

$$P(x, y) \in \mathbb{Z}[x, y], \quad \deg_x P = d_1, \quad \deg_y P = d_2, \quad d_1 d_2 \neq 0, \\ \zeta \in \mathbb{C}, \quad P(\zeta, y) \not\equiv 0, \quad P(\zeta, \ln \zeta) = 0.$$

*Then, for every  $\varepsilon > 0$ , the inequality*

$$|\zeta - \theta| < \exp\left(- (4 + \varepsilon) d_1 d_2 \frac{\ln^2 L(\theta)}{\ln \ln L(\theta)}\right) \quad (1)$$

*admits only finitely many solutions in algebraic  $\theta$  such that*

$$\kappa = \deg \theta = o(\ln \ln L(\theta)), \quad \text{i.e.,} \quad \kappa < \alpha(L) \cdot \ln \ln L, \quad \lim_{L \rightarrow \infty} \alpha(L) = 0. \quad (2)$$

The length of  $\theta$  can be replaced in (1) by its height  $H(\theta)$ , as

$$L(\theta) \leq (\kappa + 1)H(\theta).$$

N. I. Feldman [1964] proved a theorem on approximations of the solutions of the equation  $P(z, e^z) = 0$  by algebraic numbers. His result was improved in [Galochkin 1972]. A result similar to (1) can be obtained from [Nesterenko and Waldschmidt 1996, Theorem 5] but with a constant greater than 4 in the exponent. Our proof is based on Mahler's construction [1932a; 1932b; 1967] with a special choice of parameters.

**Lemma 2.** *Suppose  $P(x) \in \mathbb{Z}[x]$ ,  $\theta$  is an algebraic number, and  $P(\theta) \neq 0$ . Then*

$$|P(\theta)| \geq L(P)^{1 - \deg \theta} L(\theta)^{- \deg P},$$

*where  $L(P)$  and  $L(\theta)$  are the lengths of  $P$  and  $\theta$  respectively.*

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The proof can be found, for instance, in [Feldman 1981].

**Lemma 3.** *Let  $m, n$  be positive integers. For each  $k = \overline{0, n}$  set*

$$\Phi_k(t) = (t - m)^{k+1} \prod_{j=0}^{m-1} (t - j)^{n+1}, \quad R_k(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{e^{tz}}{\Phi_k(t)} dt, \quad (3)$$

where  $\Gamma$  is the circle  $|t| = m(n+1) + m$ . Then

$$\text{ord}_{z=0} R_k(z) = m(n+1) + k, \quad (4)$$

$$|R_k(z)| < (m(n+1) + m)e^{(m(n+1)+m)|z|} (m(n+1))^{-m(n+1)}, \quad (5)$$

$$R_k(z) = P_{k0}(z) + P_{k1}(z)e^z + \dots + P_{km}(z)e^{mz}, \quad P_{kj}(z) \in \mathbb{Q}[z], \quad (6)$$

$$n_{kj} = \deg P_{kj} = (\text{ord}_{t=j} \Phi_k(t)) - 1 = \begin{cases} n & \text{for } j = \overline{0, m-1}, \\ k & \text{for } j = m, k = \overline{0, n}. \end{cases} \quad (7)$$

Set  $p_{kj}(z) = b^n n! (m!)^{n+1} P_{kj}(z)$ , where  $b = \text{lcm}(1, 2, \dots, m)$ . Then

$$p_{kj}(z) \in \mathbb{Z}[z], \quad L(p_{kj}) < e^{\gamma_1 mn} n!, \quad \text{where } \gamma_1 \text{ is an absolute constant.} \quad (8)$$

*Proof.* We have

$$R_k(z) = \sum_{s=0}^{\infty} \frac{a_{ks}}{s!} z^s, \quad a_{ks} = \frac{1}{2\pi i} \oint_{\Gamma} \frac{t^s}{\Phi_k(t)} dt,$$

$a_{ks} = 0$  for  $s < n_k$ ,  $a_{k, n_k} \neq 0$ ,  $n_k = m(n+1) + k$ , which proves (4).

Inequality (5) follows from the estimate

$$|\Phi_k(t)| \geq (m(n+1))^{m(n+1)}$$

and an obvious estimate on the integral in (3).

We have

$$R_k(z) = \sum_{j=0}^m I_{kj}, \quad I_{kj} = \frac{1}{2\pi i} e^{jz} \oint_{|t-j|=1/2} \frac{e^{(t-j)z}}{\Phi_k(t)} dt = e^{jz} \sum_{s=0}^{n_{kj}} \frac{a_{kjs}}{s!} z^s,$$

where

$$a_{kjs} = \frac{1}{2\pi i} \oint_{|t-j|=1/2} \frac{(t-j)^s}{\Phi_k(t)} dt, \quad a_{k, j, n_{kj}} \neq 0, \quad P_{kj}(z) = \sum_{s=0}^{n_{kj}} \frac{a_{kjs}}{s!} z^s. \quad (9)$$

This proves both (6) and (7).

Since  $|t-j| = \frac{1}{2}$ , we have

$$|\Phi_k(t)| > (m!)^{n+1} e^{-\gamma_2 mn}.$$

Thus, by (9)

$$|a_{kjs}| < (m!)^{-n-1} e^{\gamma_3 mn}. \quad (10)$$

Let us use the substitution  $t - j = bu$ , where  $b = \text{lcm}(1, 2, \dots, m) = e^{O(m)}$ , in order to transform the integral in (9). Then for  $l \neq j$

$$t - l = bu + j - l = (j - l) \left( 1 - \frac{bu}{l - j} \right).$$

This substitution gives  $a_{kjs} = A_{kjs} B_{kjs}$ , where

$$A_{kjs} = b^{s-n_{kj}} \prod_{l=0, l \neq j}^m (j - l)^{-n_{kl}-1},$$

$$B_{kjs} = \oint_{|u|=(2b)^{-1}} \prod_{l=0, l \neq j}^m \left( 1 - \frac{bu}{l - j} \right)^{-n_{kl}-1} u^{s-n_{kj}-1} du,$$

with  $n_{lj}$  defined by (7).

The coefficients of the series

$$\left( 1 - \frac{bu}{l - j} \right)^{-1} = \sum_{r=0}^{\infty} \left( \frac{b}{l - j} \right)^r u^r$$

are integers; hence  $B_{kjs} \in \mathbb{Z}$  and  $b^n(m!)^{n+1} A_{kjs} \in \mathbb{Z}$ . Taking into account (9) and (10), we get (8).  $\square$

**Lemma 4.** *There exist polynomials  $B_{ks}(u)$  which, together with the corresponding form*

$$V_k(u) = \sum_{s=0}^n B_{ks}(u) (\ln u)^s,$$

*enjoy the properties*

$$B_{ks}(u) \in \mathbb{Z}[u], \quad k = \overline{0, n}, \quad (11)$$

$$\deg B_{ks} = \begin{cases} m & \text{for } k \leq s, \\ m - 1 & \text{for } k > s, \end{cases} \quad (12)$$

$$\Delta(u) = \det |B_{ks}|_{k,s=\overline{0,n}} = \lambda(u-1)^{m(n+1)}, \quad \lambda \neq 0, \quad (13)$$

$$L(B_{ks}(u)) < e^{\gamma_4 mn} n!, \quad (14)$$

$$|V_k(u)| < e^{\gamma_5 |u| mn} n^{-mn}. \quad (15)$$

*Proof.* Let us set

$$V_k(u) = b^n(n!)(m!)^{n+1} R_k(\ln u) = \sum_{j=0}^m p_{kj}(\ln u) u^j = \sum_{s=0}^n B_{ks}(u) (\ln u)^s. \quad (16)$$

Statements (11), (12), (14), and (15) follow from Lemma 3. Thus, it remains to prove (13).

First, let us assume that  $|u - 1| < 1$  and that  $\ln 1 = 0$ . In this case we have by (4)

$$R_k(z) = z^{m(n+1)+k} T_k(z), \quad T_k(0) \neq 0,$$

whence, taking into account that  $\ln 1 = 0$ , we get

$$V_k(u) = (\ln u)^{m(n+1)+k} F_k(u) = (u - 1)^{m(n+1)+k} G_k(u), \quad G_k(1) \neq 0.$$

It follows from (12) that  $\Delta(u) \not\equiv 0$  and that

$$\deg \Delta(u) = m(n+1).$$

Replacing the first column with the one consisting of  $V_0(u), V_1(u), \dots, V_n(u)$  preserves the determinant. Hence

$$\text{ord}_{u=1} \Delta(u) \geq m(n+1),$$

which implies (13).

Moving along a path around the origin changes  $\ln u$ , but it does not change  $B_{ks}(u)$ . Therefore, it does not change  $\Delta(u)$ . Thus, (13) holds for every branch of the logarithm.  $\square$

**Theorem 5.** *Suppose*

$$P(x, y) \in \mathbb{Z}[x, y], \quad \deg_x P = d_1, \quad \deg_y P = d_2, \quad d_1 d_2 \neq 0.$$

*Then, for every  $\varepsilon > 0$  and every  $r > 0$ , the inequality*

$$|P(\theta, \ln \theta)| < \exp\left(- (4 + \varepsilon) d_1 d_2 \frac{\ln^2 L(\theta)}{\ln \ln L(\theta)}\right) \quad (17)$$

*admits only finitely many solutions in algebraic  $\theta$  such that*

$$|\theta| < r \quad \text{and} \quad \kappa = \deg \theta = o(\ln \ln L(\theta)) \quad \text{as } L(\theta) \rightarrow \infty. \quad (18)$$

Note that Theorem 1 follows from Theorem 5. Indeed, for all but finitely many  $\theta$  Theorem 5 provides

$$|P(\theta, \ln \theta)| \geq \exp\left(- (4 + \varepsilon) d_1 d_2 \frac{\ln^2 L(\theta)}{\ln \ln L(\theta)}\right).$$

Hence

$$|P(\theta, \ln \theta)| = |P(\zeta, \ln \zeta) - P(\theta, \ln \theta)| = \left| \int_{\theta}^{\zeta} P'(t, \ln t) dt \right| < \gamma_6 |\zeta - \theta|,$$

and we can assume that  $|\zeta - \theta| < 1$ ,  $r = |\zeta| + 1$ . Thus, it remains to prove Theorem 5.

*Proof of Theorem 5.* Let us take

$$m = \left\lceil \frac{d_1}{d_2} n \right\rceil, \quad n > d_2. \quad (19)$$

Then by (14) and (15) we have

$$L(B_{ks}(u)) < e^{\gamma_4 m n} n! < e^{\gamma_7 n^2}, \quad |V_k(u)| < e^{\gamma_7 n^2} n^{-d_1 d_2^{-1} n^2}. \quad (20)$$

Let  $\theta$  be an algebraic number,  $\kappa = \deg \theta$ ,  $L = L(\theta)$ . We may assume that

$$\theta \neq 0, \quad \theta \neq 1, \quad P(\theta, y) \not\equiv 0.$$

This excludes finitely many values of  $\theta$ . The values

$$W_k(\theta) = (\ln \theta)^k P(\theta, \ln \theta) = \sum_{s=0}^n A_{ks}(\theta) (\ln \theta)^s, \quad k = \overline{0, v}, \quad v = n - d_2, \quad (21)$$

of the corresponding forms at  $1, \ln \theta, \dots, (\ln \theta)^n$  are linearly independent. Moreover, we have  $|A_{ks}(\theta)| < e^{\gamma_8 n}$ . Hence we can choose  $d_2$  values among  $V_0(\theta), \dots, V_n(\theta)$  (say,  $V_1(\theta), \dots, V_{d_2}(\theta)$ ) which are linearly

independent with the values from (21) and such that

$$D(\theta) = \begin{vmatrix} A_{00}(\theta) & A_{01}(\theta) & \cdots & A_{0n}(\theta) \\ \vdots & \vdots & & \vdots \\ A_{v0}(\theta) & A_{v1}(\theta) & \cdots & A_{vn}(\theta) \\ B_{10}(\theta) & B_{11}(\theta) & \cdots & B_{1n}(\theta) \\ \vdots & \vdots & & \vdots \\ B_{d_2,0}(\theta) & B_{d_2,1}(\theta) & \cdots & B_{d_2,n}(\theta) \end{vmatrix} \neq 0. \quad (22)$$

Consider the determinant

$$D(u) = \begin{vmatrix} A_{00}(u) & A_{01}(u) & \cdots & A_{0n}(u) \\ \vdots & \vdots & & \vdots \\ A_{v0}(u) & A_{v1}(u) & \cdots & A_{vn}(u) \\ B_{10}(u) & B_{11}(u) & \cdots & B_{1n}(u) \\ \vdots & \vdots & & \vdots \\ B_{d_2,0}(u) & B_{d_2,1}(u) & \cdots & B_{d_2,n}(u) \end{vmatrix}$$

as a polynomial of  $u$ . By (11), (12), (19), (20), and (22),

$$D(u) \in \mathbb{Z}[u], \quad D(\theta) \neq 0, \quad \deg D(u) \leq nd_1 + md_2 \leq 2nd_1, \quad L(D(u)) < e^{\gamma_9 n^2}. \quad (23)$$

By Lemma 2,

$$|D(\theta)| \geq e^{(1-\varkappa)\gamma_9 n^2} L^{-2d_1 n} > e^{-\gamma_9 \varkappa n^2} L^{-2d_1 n}, \quad L = L(\theta). \quad (24)$$

On the other hand,

$$D(\theta) = \begin{vmatrix} W_0(\theta) & A_{01}(\theta) & \cdots & A_{0n}(\theta) \\ \vdots & \vdots & & \vdots \\ W_v(\theta) & A_{v1}(\theta) & \cdots & A_{vn}(\theta) \\ V_1(\theta) & B_{11}(\theta) & \cdots & B_{1n}(\theta) \\ \vdots & \vdots & & \vdots \\ V_{d_2}(\theta) & B_{d_2,1}(\theta) & \cdots & B_{d_2,n}(\theta) \end{vmatrix};$$

i.e.,

$$D(\theta) = \sum_{k=0}^v W_k(\theta) M_k(\theta) + \sum_{l=1}^{d_2} V_l(\theta) N_l(\theta),$$

where  $M_k(\theta)$  and  $N_l(\theta)$  are the cofactors of the first column of  $D(\theta)$ . It follows from (20), (21), and (23) that

$$\begin{aligned} |W_k(\theta)| &< e^{\gamma_8 n} |P(\theta, \ln \theta)|, & |V_l(\theta)| &< e^{\gamma_7 n^2} n^{-d_1 d_2^{-1} n^2}, \\ |M_k(\theta)| &< e^{\gamma_{10} n^2}, & |N_l(\theta)| &< e^{\gamma_{10} n^2}. \end{aligned}$$

Hence

$$|D(\theta)| < e^{\gamma_{11} n^2} |P(\theta, \ln \theta)| + e^{\gamma_{11} n^2} n^{-d_1 d_2^{-1} n^2}.$$

Taking into account (24), we get

$$e^{-2d_1 n \ln L} < e^{\gamma_{12} \varkappa n^2} |P(\theta, \ln \theta)| + e^{\gamma_{12} \varkappa n^2 - d_1 d_2^{-1} n^2 \ln n}. \quad (25)$$



Given an arbitrary  $\varepsilon > 0$ , let us set

$$n = \left\lceil \left(2 + \frac{\varepsilon}{4}\right) d_2 \frac{\ln L}{\ln \ln L} \right\rceil.$$

Then

$$\begin{aligned} 2d_1 n \ln L &\sim \left(4 + \frac{\varepsilon}{2}\right) d_1 d_2 \frac{\ln^2 L}{\ln \ln L} \quad \text{as } L \rightarrow \infty, \\ d_1 d_2^{-1} n^2 \ln n &\sim \left(4 + \varepsilon + \frac{\varepsilon^2}{16}\right) d_1 d_2 \frac{\ln^2 L}{\ln \ln L} \quad \text{as } L \rightarrow \infty. \end{aligned}$$

Hence due to restrictions (18),

$$\gamma_{12} \kappa n^2 = o\left(\frac{\ln^2 L}{\ln \ln L}\right).$$

Thus, for  $L$  large enough we have  $e^{\gamma_{12} \kappa n^2 - d_1 d_2^{-1} n^2 \ln n} < \frac{1}{2} e^{-2d_1 n \ln L}$ . Combining this with (25), we get

$$|P(\theta, \ln \theta)| > \frac{1}{2} e^{-\gamma_{12} \kappa n^2 - 2d_1 n \ln L} > \exp\left(-\left(4 + \varepsilon\right) d_1 d_2 \frac{\ln^2 L}{\ln \ln L}\right),$$

which implies that inequality (17) has finitely many solutions.

Theorems 5 and 1 are proved. □

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