

Pacific Journal of Mathematics

**THE MODULI SPACE OF REAL VECTOR BUNDLES OF
RANK TWO OVER A REAL HYPERELLIPTIC CURVE**

THOMAS JOHN BAIRD AND SHENGDA HU

Volume 304 No. 1

January 2020

THE MODULI SPACE OF REAL VECTOR BUNDLES OF RANK TWO OVER A REAL HYPERELLIPTIC CURVE

THOMAS JOHN BAIRD AND SHENGDA HU

The Desale–Ramanan theorem is an isomorphism between the moduli space of rank two vector bundles over complex hyperelliptic curve and the variety of linear subspaces in an intersection of two quadrics. We prove a real version of this theorem for the moduli space of real vector bundles over a real hyperelliptic curve. We then apply this result to study the topology of the moduli space, proving that it is relatively spin and identifying the diffeomorphism type for genus two curves. Our results lay the groundwork for future study of the quantum homology of these moduli spaces.

1. Introduction

Given a Riemann surface Σ of genus $g \geq 2$ and a line bundle $\xi \rightarrow \Sigma$ of odd degree, the moduli space U_ξ of stable rank 2 vector bundles $E \rightarrow \Sigma$ with fixed determinant $\wedge^2 E \cong \xi$ is a nonsingular projective variety of dimension $3g - 3$ which is naturally endowed with a Kähler metric.

Given an antiholomorphic involution $\tau : \Sigma \rightarrow \Sigma$, there is an induced antiholomorphic and antisymplectic involution on U_ξ . The fixed point set $U_\xi^\tau \subseteq U_\xi$ is a totally geodesic real Lagrangian submanifold of U_ξ . The points of U_ξ^τ correspond to holomorphic bundles over Σ which admit a real structure [Biswas et al. 2010; Schaffhauser 2012]. The topology of U_ξ^τ has been investigated using Atiyah–Bott–Kirwan type methods: the mod 2 Betti numbers were calculated in [Baird 2014; 2017; Liu and Schaffhauser 2013] and when $g \geq 3$ the rational cohomology ring was calculated in [Baird 2017].

In the current paper, we study U_ξ^τ using an entirely different method in the special case when Σ is a hyperelliptic curve. Recall that a hyperelliptic curve is a 2-fold ramified cover $\pi : \Sigma \rightarrow \mathbb{C}P^1$. If Σ has genus $g \geq 2$, then π has $2g + 2$ ramification points $W \subseteq \mathbb{C}P^1$ also known as the Weierstrass points. Choose affine coordinates $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ and let $\lambda_1, \dots, \lambda_{2g+2} \in \mathbb{C}$ denote the coordinates of W . The following is due to Desale and Ramanan [1976/77]; see also [Narasimhan and Ramanan 1969; Newstead 1968] for the genus 2 case.

MSC2010: 14D20, 14P05.

Keywords: real vector bundle, hyperelliptic curve, moduli space.

Theorem 1.1 (Desale–Ramanan). *If Σ is hyperelliptic, then U_ξ is isomorphic to the subvariety of the Grassmannian $X \subseteq \text{Gr}_{g-1}(\mathbb{C}^{2g+2})$ consisting of $(g-2)$ -linear subspaces contained in the intersection of the two quadrics $Z(q_0) \cap Z(q_1) \subset \mathbb{C}P^{2g+1}$, where*

$$\begin{aligned} q_0 &= x_1^2 + \cdots + x_{2g+2}^2, \\ q_1 &= \lambda_1 x_1^2 + \cdots + \lambda_{2g+2} x_{2g+2}^2. \end{aligned}$$

Our main result (Theorem 4.1) is a real version of the Desale–Ramanan theorem. Namely, we prove that if (Σ, τ) is a real hyperelliptic curve, then U_ξ^τ is isomorphic to the subvariety of $\text{Gr}_{g-1}(\mathbb{R}^{2g+2})$ consisting of planes contained in the intersection of two explicit real quadrics. For genus $g = 2$ curves, this result was proven by Shuguang Wang [1995].

The rest of the paper is devoted to applications of this result. In Section 5 we produce formulas for the Stiefel–Whitney classes of U_ξ^τ and use these to show that U_ξ^τ is relatively spin as a Lagrangian submanifold of U_ξ . This implies that it has a well-defined quantum homology ring with integer coefficients [Biran and Cornea 2009].

In Section 6 we identify the diffeomorphism type of U_ξ^τ for all genus two examples. Previously, only the \mathbb{Z}_2 -Betti numbers were known.

2. Real hyper-elliptic curves

A hyperelliptic curve, $\pi : \Sigma \rightarrow \mathbb{C}P^1$, is a 2-fold ramified cover over the complex projective line. We assume always that Σ has genus $g \geq 2$. Such a curve admits an involution $\iota : \Sigma \rightarrow \Sigma$, which interchanges the two sheets of the cover and thus identifies $\Sigma/\iota \cong \mathbb{C}P^1$. The ramification points $W \subset \mathbb{C}P^1$ are called the Weierstrass points. Choosing affine coordinates on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ we can represent W as a set of $2g+2$ points in the complex plane \mathbb{C} (we assume always $\infty \notin W$).

A real structure on a Riemann surface Σ is an antiholomorphic involution τ , that is a smooth map $\tau : \Sigma \rightarrow \Sigma$ which reverses the complex structure and satisfies $\tau^2 = Id_\Sigma$. The fixed point set of the involution, $\Sigma_\mathbb{R} := \Sigma^\tau$, is diffeomorphic to disjoint union of circles. In this paper, we consider real structures that are compatible with the hyperelliptic projection, i.e.,

$$\pi \circ \tau = \tau_{P^1} \circ \pi$$

with respect to the standard real involution τ_{P^1} on $\mathbb{C}P^1$, defined by $\tau_{P^1}([z : w]) := [\bar{z} : \bar{w}]$. A hyperelliptic curve admits a compatible real structure if and only if $\tau_{P^1}(W) = W$. If this happens, then Σ admits a pair of compatible real structures τ and $\tau \circ \iota = \iota \circ \tau$, where ι is the hyperelliptic involution. For consistency, we will always choose τ so that $\infty \in \pi(\Sigma^\tau)$ and $\infty \notin \pi(\Sigma^{\tau \circ \iota})$. A typical situation for genus $g = 2$ is illustrated in Figures 1 and 2 below. The ramification points $W \subset \mathbb{C} \subset \mathbb{C}P^1$,

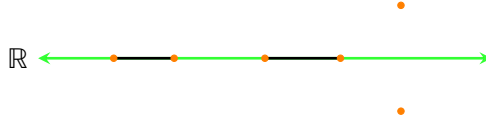


Figure 1. A real structure τ .

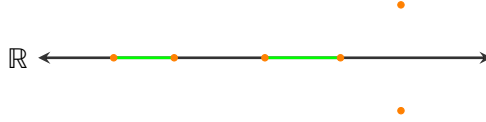


Figure 2. A real structure $\iota \circ \tau$, compatible with τ , where ι is the hyperelliptic involution.

drawn in orange, are arranged symmetrically about the real axis. The respective images $\pi(\Sigma^\tau)$ and $\pi(\Sigma^{\tau \circ \iota})$ are drawn in green.

Given a real curve (Σ, τ) the fixed point set Σ^τ is a union of disjoint circles called the *real circles*. The topological type of a real curve (Σ, τ) is determined by the genus of g , the number of path components of Σ^τ and whether or not $\Sigma \setminus \Sigma^\tau$ is connected; see [Griffiths and Harris 1978]. The following proposition classifies real hyperelliptic curves topologically.

Proposition 2.1. *Decompose*

$$W = W_0 \cup W_+ \cup W_-,$$

where W_0, W_+, W_- are those Weierstrass points in $w \in \mathbb{C}$ whose imaginary part is respectively zero, positive, negative. Clearly W_+ and W_- have equal cardinality so W_0 has even cardinality. Let $2n = \#W_0$.

- (i) If $n > 0$, then $\pi_0(\Sigma^\tau) = \pi_0(\Sigma^{\tau \circ \iota}) = n$.
- (ii) If $n = 0$, then $\pi_0(\Sigma^\tau) = 1$ and $\pi_0(\Sigma^{\tau \circ \iota}) = 0$.
- (iii) If $1 \leq n \leq g$, then both $\Sigma \setminus \Sigma^\tau$ and $\Sigma \setminus \Sigma^{\tau \circ \iota}$ are connected. If $n = g + 1$, then both $\Sigma \setminus \Sigma^\tau$ and $\Sigma \setminus \Sigma^{\tau \circ \iota}$ are disconnected. If $n = 0$, then $\Sigma \setminus \Sigma^\tau$ is disconnected and $\Sigma \setminus \Sigma^{\tau \circ \iota}$ is connected.

Proof. Statements (i) and (ii) are pretty clear.

For statement (iii), observe that $\Sigma \setminus \Sigma^\tau$ is connected if and only if it is possible to draw a closed loop in $\mathbb{C} \setminus W$ which encloses an odd number of points in W and crosses the real line. Then it is a simple matter of looking at pictures. \square

Remark 2.2. Proposition 2.1 implies that for genus $g \geq 4$, there exist topological types of real curves that cannot be realized as hyperelliptic curves. Namely, those curves for which $1 < \pi_0(\Sigma^\tau) < g + 1$ and $\Sigma \setminus \Sigma^\tau$ is disconnected.

Real line bundles over real hyperelliptic curves. Let (Σ, τ) be a real curve. A real line bundle ξ over (Σ, τ) is a line bundle that admits an antiholomorphic lift

$$\begin{array}{ccc} \xi & \xrightarrow{\tilde{\tau}} & \xi \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\tau} & \Sigma \end{array}$$

such that $\tilde{\tau}^2 = Id_\xi$. If $\tilde{\tau}$ exists then by Schur's lemma it is uniquely determined up to multiplication by a unit scalar. The fixed point set $\xi^{\tilde{\tau}} \rightarrow \Sigma^\tau$ is an \mathbb{R}^1 -bundle over Σ^τ . The Stiefel–Whitney class $w_1(\xi^{\tilde{\tau}}) \in H^1(\Sigma^\tau; \mathbb{Z}_2)$ depends only on (Σ, τ, ξ) , not on the choice of lift $\tilde{\tau}$. If $C \subset \Sigma^\tau$ is a real circle, then the following are equivalent:

- (i) $\xi^{\tilde{\tau}}|_C$ is nonorientable (i.e., a Möbius band).
- (ii) $w_1(\xi^{\tilde{\tau}})(C) = 1$.

We call C odd with respect to ξ if it satisfies these equivalent conditions. If k is the number of odd circles with respect to ξ then by [Biswas et al. 2010, Proposition 4.1]

$$(2-1) \quad \deg(\xi) \equiv w_1(\xi^{\tilde{\tau}})(\Sigma^\tau) \equiv k \pmod{2}.$$

The Desale–Ramanan theorem requires the degree of ξ to be odd, which implies that our real line bundle ξ will always have an odd number of odd circles. In particular, we need only consider (Σ, τ) for which Σ^τ is nonempty. By [Biswas et al. 2010, Proposition 4.1] if $\Sigma^\tau \neq \emptyset$, then real line bundles exist over (Σ, τ) in all degrees. Tensoring by a real line bundle L over (Σ, τ) determines an isomorphism

$$U_\xi^\tau \cong U_{\xi \otimes L^{\otimes 2}}^\tau$$

and $\deg(\xi \otimes L^{\otimes 2}) = \deg(\xi) + 2\deg(L)$. Therefore in studying U_ξ^τ we may assume without loss of generality that ξ has degree $2g + 1$.

Given a real line bundle ξ with real structure $\tilde{\tau}$, there always exists a $\tilde{\tau}$ -invariant meromorphic section which determines a real divisor $D = \sum_i m_i p_i$, where $m_i \in \mathbb{Z}$ is the multiplicity of the zero of s at $p_i \in \Sigma$. Observe that $\pi(D) = D$. A real circle in $C \subseteq \Sigma^\tau$ is odd with respect to ξ if and only if $\sum_{p_i \in C} m_i$ is odd.

3. The Desale–Ramanan theorem

We review what we need from [Desale and Ramanan 1976/77].

The moduli space of bundles as a subvariety of a Grassmannian. Let $\pi : \Sigma \rightarrow \mathbb{C}P^1$ be a hyperelliptic curve of genus g with hyperelliptic involution ι . Let W be the set of $2g + 2$ Weierstrass points for Σ . We abuse notation and identify $W = \pi^{-1}(W) \subset \Sigma$. Fix a line bundle $\xi \rightarrow \Sigma$ of degree $2g + 1$.

Let $E \in U_\xi$ be a stable bundle with determinant isomorphic to ξ . There is a natural map

$$H^0(\Sigma, E \otimes \iota^* E) \rightarrow \bigoplus_{w \in W} E_w \otimes \iota^* E_w = \bigoplus_{w \in W} E_w \otimes E_w$$

defined by restriction and identifying $\iota^* E_w = E_w$. This map is ι -equivariant. Restricting to the -1 eigenspaces yields a map

$$(3-1) \quad H^0(\Sigma, E \otimes \iota^* E)^{-\iota} \rightarrow \bigoplus_{w \in W} (E_w \otimes E_w)^{-\iota} = \bigoplus_{w \in W} \wedge^2 E_w \cong \bigoplus_{w \in W} \xi_w,$$

where the last isomorphism uses $\wedge^2 E \cong \xi$, so is only natural up to a nonzero scalar in \mathbb{C}^* . Desale and Ramanan proved that (3-1) is injective, and $H^0(\Sigma, E \otimes \iota^* E)^{-\iota}$ has fixed dimension $g + 3$ for all E . We therefore obtain a morphism

$$U_\xi \rightarrow \text{Gr}_{g+3}(V^*),$$

where $V^* = \bigoplus_{w \in W} \xi_w^*$. Applying duality, we can think of this as a morphism

$$\phi : U_\xi \rightarrow \text{Gr}_{g-1}(V),$$

where $V := \bigoplus_{w \in W} \xi_w^*$. Desale and Ramanan proved that ϕ is an embedding, so U_ξ is isomorphic with the image $\text{im}(\phi)$. Moreover, they proved that $\text{im}(\phi)$ is equal to the set of $(g - 2)$ -linear subspaces in $P(V)$ that lie in the intersection of two quadrics Q_0 and Q_1 . We construct these quadrics in the next section.

Now suppose we endow Σ with an antiholomorphic involution τ which commutes with ι . This means in particular that τ permutes the Weierstrass points. If ξ is a real line bundle over (Σ, τ) , then $V = \bigoplus_{w \in W} \xi_w^*$ inherits an antilinear involution inducing an involution on $\text{Gr}_{g-1}(V)$. It is clear from the construction that ϕ is τ -equivariant.

Line bundles and quadrics. A pencil of quadratic forms on a vector space V is determined by a surjective linear map $Q : S^2(V) \rightarrow U$, where U is a vector space of dimension two. In terms of a basis for U , this is defined by a linearly independent pair of quadratic forms $Q_0, Q_1 : S^2(V) \rightarrow \mathbb{C}$.

Given a line bundle ξ of degree $2g + 1$ over the hyperelliptic curve Σ , we want to construct a pencil of quadratic forms on V , where

$$(3-2) \quad V := \bigoplus_{w \in W} \xi_w^*.$$

We do this by constructing a linear map

$$Q : S^2(V) \rightarrow H^0(h^{-2g-1}(W)),$$

where h is the hyperplane bundle over $\mathbb{C}P^1$ and W is the Weierstrass divisor. Our

pencil will be diagonal with respect to the decomposition (3-2) so it is determined by morphisms

$$q_w : (\xi_w^*)^{\otimes 2} \rightarrow H^0(h^{-2g-1}(W))$$

for each $w \in W$. We define q_w to be the composition of morphisms (3-4)–(3-6) defined below. Fix an isomorphism

$$(3-3) \quad \phi : \xi^* \otimes \iota^* \xi^* \cong \pi^*(h)^{-2g-1}.$$

At a Weierstrass point $w \in W$, ϕ restricts to an isomorphism

$$(3-4) \quad (\xi_w^*)^{\otimes 2} = \xi_w^* \otimes \iota^* \xi_w^* \cong \pi^*(h)_w^{-2g-1} = h_w^{-2g-1}$$

(recall we abuse notation identifying w and $\pi(w)$). Evaluation at w determines an isomorphism

$$(3-5) \quad h_w^{-2g-1} \cong H^0(h^{-2g-1}(W - w)),$$

and we have the natural inclusion

$$(3-6) \quad H^0(h^{-2g-1}(W - w)) \hookrightarrow H^0(h^{-2g-1}(W)).$$

Choose affine coordinates on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ with variable t and Weierstrass points $\{t_w \in \mathbb{C}\}$. In these coordinates we have

$$H^0(h^{-2g-1}(W)) = \{(at + b)R(t) \mid a, b \in \mathbb{C}\},$$

where $R(t) = t^{2g+1} \prod_{w \in W} (t - t_w)^{-1}$. The quadratic forms Q_0, Q_1 are defined by the identity $Q = (Q_0 t - Q_1)R(t)$.

Let u be a meromorphic section of ξ which may be chosen to have no poles or zeros on W so that $\{u_w \mid w \in W\}$ is a basis for V . Let D be the divisor of u and express

$$\pi(D) = \sum m_i \alpha_i$$

in affine coordinates $\alpha_i \in \mathbb{C} \subset \mathbb{C}P^1$, where $\sum m_i = -2g - 1$. Then up to a scalar (which we may fix to equal one), $u \otimes \iota^*(u)$ is sent by ϕ to the pull-back of $\prod_i (t - \alpha_i)^{m_i}$. Tracing through the definition of q_w we get

$$q_w(u_w^2) = (t - t_w) \prod_i (t_w - \alpha_i)^{m_i} \prod_{w' \neq w} (t_w - t_{w'}) R(t).$$

Therefore

$$(3-7) \quad Q_0(u_w^2) := \prod_i (t_w - \alpha_i)^{m_i} \prod_{w' \neq w} (t_w - t_{w'}),$$

$$(3-8) \quad Q_1(u_w^2) := t_w Q_0(u_w^2).$$

We recover the expression in Theorem 1.1 simply by replacing the basis u_w with $(1/\sqrt{Q_0(u_w^2)})u_w$. To get the real version, we must choose a different basis.

4. Main result

Suppose now that $\pi : \Sigma \rightarrow \mathbb{C}P^1$ is a real hyperelliptic curve with antiholomorphic involution τ compatible with the standard involution τ_{P^1} on $\mathbb{C}P^1$. Suppose this lifts to a real structure $\tilde{\tau}$ on ξ and choose a $\tilde{\tau}$ -invariant meromorphic section u . We want to replace the basis $\{u_w \mid w \in W\}$ of V by a basis whose elements are fixed by τ and with respect to which the expressions for Q_0 and Q_1 simplify.

Given affine coordinates $\mathbb{C} \cup \{\infty\}$, recall that if $\#W_0 = 2n > 0$, then $\pi(\Sigma^\tau) \cap \mathbb{C}$ is a union of n disjoint closed intervals in \mathbb{R} , two of which are half infinite. Declare such an interval $I \subseteq \mathbb{R}$ to be odd with respect to $\pi(D) = \sum_i m_i \alpha_i$ if $\sum_{\alpha_i \in I} m_i$ is odd. Note that a bounded interval is odd if and only if it is the image of an odd circle for ξ . To simplify what follows we require when $n > 0$ that the negative half infinite interval be even. This can always be arranged by changing the affine coordinate chart using a Möbius transformation.

Theorem 4.1. *Suppose that W_0 consists of real numbers $r_1 < r_2 < \dots < r_{2n}$ while W_+ consists of complex numbers $a_1 + ib_1, \dots, a_{g+1-n} + ib_{g+1-n}$, where $b_i > 0$. Assume that $\infty \in \pi(\Sigma^\tau)$. There is a choice of coordinates $x_1, \dots, x_{2r}, z_1, \dots, z_s, w_1, \dots, w_s$ on V , in which the quadratic forms Q_0 and Q_1 are expressed as polynomials*

$$(4-1) \quad q_0 := \sum_{i=1}^{2n} \epsilon_i (x_{2i-1}^2 - x_{2i}^2) + \sum_{j=1}^s (z_j^2 - w_j^2),$$

$$(4-2) \quad q_1 := \sum_{i=1}^{2n} \epsilon_i (r_{2i-1} x_{2i-1}^2 - r_{2i} x_{2i}^2) + \sum_{j=1}^s (a_j z_j^2 - a_j w_j^2 + 2b_j z_j w_j),$$

where $\epsilon_i \in \{\pm 1\}$ equals $(-1)^{(N_i+1)}$, where N_i is the number of odd intervals to the right of r_{2i-1} . This convention implies $\epsilon_1 = 1$.

Proof. Clearly τ_{P^1} preserves both $\pi(D)$ and W . Let ν be the permutation of W such that $\tau_{P^1}(w) = \nu(w)$. The induced antilinear involution on $V = \bigoplus_{w \in W} \xi_w^*$ sends a linear combination $\sum_{w \in W} \lambda_w u_w$ to $\sum_{w \in W} \bar{\lambda}_w u_{\nu(w)}$.

Introduce the notation $R_w e^{i\theta_w} := Q_0(u_w^2)$, where $R_w > 0$ and $e^{i\theta_w} \in [0, 2\pi)$. This is well-defined because from (3-7) we see $Q_0(u_w^2) \neq 0$. Define the basis

$$B := \{v_w \mid w \in W_0\} \cup \{v'_w, v''_w \mid w \in W_+\},$$

where

$$\begin{aligned} v_w &:= \frac{1}{\sqrt{R_w}} u_w, & v'_w &:= \frac{e^{-i\theta_w/2}}{\sqrt{2R_w}} u_w + \frac{e^{i\theta_w/2}}{\sqrt{2R_w}} u_{\nu(w)}, \\ v''_w &:= -i \left(\frac{e^{-i\theta_w/2}}{\sqrt{2R_w}} u_w - \frac{e^{i\theta_w/2}}{\sqrt{2R_w}} u_{\nu(w)} \right). \end{aligned}$$

Clearly each basis vector in B is fixed by conjugation. It is straightforward to check that Q_0 is diagonal in this basis, with

$$Q_0(v_w^2) = e^{i\theta_w}, \quad Q_0((v'_w)^2) = 1, \quad Q_0((v''_w)^2) = -1,$$

while Q_1 satisfies

$$\begin{aligned} Q_1(v_w^2) &= t_w e^{i\theta_w}, \\ Q_1((v'_w)^2) &= (t_w + \bar{t}_w)/2 = \operatorname{Re}(t_w), \\ Q_1((v''_w)^2) &= -(t_w + \bar{t}_w)/2 = -\operatorname{Re}(t_w), \\ Q_1(v'_w v''_w) &= -i(t_w - \bar{t}_w)/2 = \operatorname{Im}(t_w) \end{aligned}$$

with all other entries zero. Finally, if $w \in W_0$, then $e^{i\theta_w} = (-1)^N$, where

$$N = \#\{w' \in W_0 \mid t_{w'} > t_w\} + \sum_{\alpha_i \in \mathbb{R}, \alpha_i > t_w} m_i.$$

Therefore $e^{i\theta_w} = \epsilon_i$ if $t_w = r_{2i-1}$ and $e^{i\theta_w} = -\epsilon_i$ if $t_w = r_{2i}$ as desired. \square

5. Stiefel–Whitney classes

Let $G_{\mathbb{C}} := \operatorname{Gr}_{g-1}(\mathbb{C}^{2g+2})$ with tautological bundle $V_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$. Let $G := \operatorname{Gr}_{g-1}(\mathbb{R}^{2g+2})$ and V be their real counterparts. We saw in Section 3 that there is a τ -equivariant embedding $\phi : U_{\xi} \hookrightarrow G_{\mathbb{C}}$, which restricts to an embedding of U_{ξ}^{τ} into G . The image of ϕ coincides with the set of $(g-1)$ -planes on which the quadratic forms Q_0, Q_1 both vanish.

Proposition 5.1. *The image of ϕ equals the zero locus $Z(s)$ of a τ -invariant section $s \in H^0(G; S^2(V_{\mathbb{C}}^*)^{\oplus 2})$ which intersects the zero section transversely. Therefore $U_{\xi}^{\tau} \cong Z(s)^{\tau} = Z(s^{\tau})$, where s^{τ} is restricted section of $S^2(V^*)$.*

Proof. The identification of $Z(s)$ with linear subspaces of Q_0, Q_1 is explained in [Borcea 1990]. Transversality follows from [Borcea 1990, Corollary 2.2] because the pencil of quadrics $xQ_0 + yQ_1$ is generic. Invariance under τ is clear. \square

Corollary 5.2. *Let (Σ_0, τ_0) and (Σ_1, τ_1) be real hyperelliptic curves of the same genus $g \geq 2$ equipped with real line bundles ξ_1 and ξ_2 respectively of degree $2g+1$. Then $U_{\xi_0}^{\tau_0}$ and $U_{\xi_1}^{\tau_1}$ are cobordant.*

Proof. There exist real sections $s_0, s_1 \in \Gamma(S^2(V^*))$ such that $U_{\xi_i}^{\tau_i} = Z(s_i) := s_i^{-1}(0)$. Choose a homotopy $s : \operatorname{Gr}_{g-1}(\mathbb{R}^{2g+2}) \times I \rightarrow S^2(V^*)$ from s_0 to s_1 which intersects the zero section transversally. Then $Z(s)$ provides a cobordism between $Z(s_0) = M_0$ and $Z(s_1) = M_1$. \square

Let $N := \text{im}(\phi)^\tau \cong U_\xi^\tau$. We have the isomorphism of vector bundles

$$(5-1) \quad TG|_N \cong TN \oplus S^2(V^*|_N) \oplus S^2(V^*|_N),$$

which can be used to compute Stiefel–Whitney classes.

Proposition 5.3. *Let $N \cong U_\xi^\tau$ be as above. The total Stiefel–Whitney class of the tangent bundle of N is equal to*

$$(5-2) \quad w(TN) = w(V^*|_N)^{2g+2} w(V|_N \otimes V^*|_N)^{-1} w(S^2(V^*)|_N)^{-2}.$$

In particular, we have

$$\begin{aligned} w_1(U_\xi^\tau) &= 0, \\ w_2(U_\xi^\tau) &= (g+1)\phi^*(w_1)^2, \end{aligned}$$

where w_i is the tautological Stiefel–Whitney class in $H^i(G; \mathbb{Z}_2)$.

Proof. By (5-1) and the Whitney sum formula, we get

$$w(TN) = w(TG|_N) w(S^2(V^*)|_N)^{-2}.$$

We have the well-known isomorphism $TG \cong \text{Hom}(V, W)$, where V and W are the tautological bundle and its orthogonal complement respectively. In particular, $V \oplus W \cong G \times \mathbb{R}^{2g+2}$. Therefore

$$TG \oplus \text{Hom}(V, V) = \text{Hom}(V, \mathbb{R}^{2g+2}) = V^* \oplus \cdots \oplus V^*,$$

so by the Whitney sum formula

$$w(TG) = w(V^*)^{2g+2} w(V \otimes V^*)^{-1}$$

proving (5-2). By definition

$$w(V) = w(V^*) = 1 + w_1 + w_2 + \cdots + w_{g-1}.$$

A simple calculation using the splitting principle gives

$$\begin{aligned} w(V \otimes V^*) &= 1 + (rk(V) - 1)w_1^2 + O(3) \\ &= 1 + gw_1^2 + O(3), \\ w(S^2(V^*))^2 &= 1 + w_1(S^2(V^*))^2 + O(3) \\ &= 1 + gw_1^2 + O(3), \end{aligned}$$

where $O(3)$ is a sum of terms in degree three or higher. The values of $w_i(U_\xi^\tau)$ follow by direct calculation and functoriality. \square

Before stating the next corollary we to introduce some terminology.

Definition 5.4. Let L be a Lagrangian submanifold in a symplectic manifold (M, ω) . We call L

- relatively pin if $w_2(L)$ lies in the image of $H^2(M; \mathbb{Z}_2) \rightarrow H^2(L; \mathbb{Z}_2)$.
- relatively spin if it is relatively pin and orientable.
- spin if it is orientable and $w_2(L) = 0$.

Corollary 5.5. *The Lagrangian submanifold $U_\xi^\tau \subset U_\xi$ is relatively spin for all $g \geq 2$ and is spin if g is odd.*

Proof. That U_ξ^τ is orientable follows from $w_1(U_\xi^\tau) = 0$ which is proven in Proposition 5.3. Consider the commutative diagram of inclusions

$$\begin{array}{ccc} U_\xi & \xrightarrow{\phi} & G_{\mathbb{C}} \\ \uparrow & & \uparrow i \\ U_\xi^\tau & \xrightarrow{\phi} & G \end{array}$$

We have $w_2(U_\xi^\tau) = (g+1)\phi^*(w_1)^2$ by Proposition 5.3, and we have $w_1^2 = i^*(u)$, where u is the pull-back of the generator of $H^2(G_{\mathbb{C}}; \mathbb{Z}_2)$ [Milnor and Stasheff 1974, Problem 15-A]. Commutativity completes the argument. \square

Remark 5.6. We note that the moduli space U_ξ is monotone with minimal Chern number 2, which implies that U_ξ^τ is monotone with minimal Maslov number greater than or equal to two; see [Baird 2017, Theorem 1.6]. It follows from [Biran and Cornea 2009] that the quantum homology of the Lagrangian submanifold U_ξ^τ is well defined over the Novikov ring with integer coefficients. It is well known that Dehn twist on Σ induces fibered Dehn twist of U_ξ ; see, e.g., [Wehrheim and Woodward 2016]. For the real curve (Σ, τ) , we can consider a real Dehn twist. In this case, the corresponding moduli spaces of real bundles can be related by a Lagrangian cobordism in Lefschetz fibration; cf. [Biran and Cornea 2017]. We hope to pursue this in future work.

6. The genus 2 case

Given a generic intersection of real quadrics $X = Z(q_0) \cap Z(q_1) \subseteq \mathbb{R}P^N$ we can form the double cover $\tilde{X} \rightarrow X$ by pulling back the double cover $S^N \rightarrow \mathbb{R}P^N$. The diffeomorphism types of such \tilde{X} were classified by Gómez Gutiérrez and López de Medrano [2014], which we review below.

Suppose q_0 and q_1 are determined by symmetric real matrices A and B . Up to a small perturbation that doesn't affect the diffeomorphism type, we may assume that $A^{-1}B$ is diagonalizable with distinct eigenvalues. Introduce real variables $x_1, \dots, x_r, u_1, v_1, \dots, u_s, v_s$, where each x_i corresponds to the eigenspace for a real eigenvalue of $A^{-1}B$ and each pair u_i, v_i corresponds to the real part of the sum

of eigenspaces for a complex conjugate pair of eigenvalues. Then the coefficients of q_0, q_1 can be continuously varied without changing the diffeomorphism type to a pair of quadratic forms

$$(6-1) \quad p_0 = \sum_{i=1}^r a_i x_i^2 + \sum_{j=1}^s (u_j^2 - v_j^2),$$

$$(6-2) \quad p_1 = \sum_{i=1}^r b_i x_i^2 + \sum_{j=1}^s 2u_j v_j,$$

where the b_i/a_i are the real eigenvalues of $A^{-1}B$.

Consider the set of points $\lambda_i = (a_i, b_i) \in \mathbb{R}^2$. The fact that the intersection is generic implies that $(0, 0)$ does not lie on the line segment joining λ_i and λ_j for any pair $i, j \in \{1, \dots, r\}$. If the coefficients λ_i are continuously varied without violating this property, then the diffeomorphism type of the intersection doesn't change. The λ_i can in this way be put into a standard form, such that all of the λ_i lie on $2l + 1$ roots of unity (in $\mathbb{R}^2 = \mathbb{C}$) for a minimal value l . This determines a cyclically ordered partition $r = n_1 + \dots + n_{2l+1}$, where $n_i > 0$ counts the multiplicity of λ_i at the i -th root of unity. The diffeomorphism type of \tilde{X} , hence also X , is determined by s and the partition $r = n_1 + \dots + n_{2l+1}$.

Proposition 6.1. *Suppose that (Σ, τ) is a real hyperelliptic curve of genus $g = 2$ with $2n$ real Weierstrass points and let ξ be a real line bundle with k many odd circles. Then the quadric intersection type of $Z(q_0) \cap Z(q_1)$ and the diffeomorphism type of the double cover \tilde{U}_ξ^τ are as follows:*

(n, k)	quadric intersection type	diffeomorphism type of \tilde{U}_ξ^τ
$(0, 1)$	$s = 3, r = 0$	$\mathbb{R}P^3$
$(1, 1)$	$s = 2, r = 2$	$S^1 \times S^2$
$(2, 1)$	$s = 1, r = 1 + 1 + 2$	$\#_3(S^1 \times S^2)$
$(3, 1)$	$s = 0, r = 1 + 1 + 1 + 1 + 2$	$\#_5(S^1 \times S^2)$
$(3, 3)$	$s = 0, r = 2 + 2 + 2$	T^3

Proof. Comparing formulas (4-1), (4-2) with (6-1), (6-2), we see that $r = 2n$ is equal to the number of real Weierstrass points and $2s$ is the number of nonreal Weierstrass points. The partitions can be worked out by hand case-by-case. The diffeomorphism types follows from the main theorem of [Gómez Gutiérrez and López de Medrano 2014]. \square

Gómez Gutiérrez and López de Medrano do not provide a general formula for the diffeomorphism type of the intersection of projective quadrics X itself, but it is not hard to determine it for our examples.

Theorem 6.2. *Suppose that (Σ, τ) is a real hyperelliptic curve of genus 2 with $2n$ real Weierstrass points and let ξ be a real line bundle with k many odd circles (as for Proposition 6.1). The diffeomorphism type of U_ξ^τ is*

(n, k)	diffeomorphism type of U_ξ^τ
$(0, 1)$	$L(4, 1)$
$(1, 1)$	$S^1 \times S^2$
$(2, 1)$	$\#_2(S^1 \times S^2)$
$(3, 1)$	$\#_3(S^1 \times S^2)$
$(3, 3)$	T^3

Proof. For the case $(0, 1)$ we know that U_ξ^τ is diffeomorphic to the projective quadric defined by

$$\sum_{i=1}^3 (u_i^2 - v_i^2) = \sum_{i=1}^3 2u_i v_i = 0.$$

If we regard these as affine equations and impose the extra affine condition

$$\sum_{i=1}^3 (u_i^2 + v_i^2) = 1,$$

then these define the unit tangent bundle of S^2 . Taking the projective quotient gives the unit tangent bundle of $\mathbb{R}P^2$ which is diffeomorphic to the lens space $L(4, 1)$; see [Konno 2002].

The cases $(1, 1)$, $(2, 1)$, and $(3, 1)$ are each diffeomorphic to an intersection of quadrics of the form

$$\begin{aligned} x_1^2 + x_2^2 + F(x_3, x_4, x_5, x_6) &= 0 \\ 0 + F(x_3, x_4, x_5, x_6) &= 0 \end{aligned}$$

which admits an $SO(2)$ -action defined by rotating the coordinates x_1, x_2 . Note that points in the intersection must either be fixed by this rotation (if $x_1 = x_2 = 0$) or have trivial stabilizer (because the first equation implies there are no nonzero solutions of the form $(a_1, a_2, 0, 0, 0, 0)$). In all three cases it is easy to verify that the fixed point set is nonempty. Since by Corollary 5.5 we also know that U_ξ^τ is orientable, a result of Raymond [1968, Theorem 1] implies that U_ξ^τ is diffeomorphic to a connected sum of copies of $S^1 \times S^2$. The mod 2 Betti numbers of U_ξ^τ were calculated in [Baird 2017], from which we can determine the number of copies of $S^1 \times S^2$ occurring in each case.

The case $(3, 3)$ actually admits an effective action by the 3-torus $SO(2)^3$ (see [Gómez Gutiérrez and López de Medrano 2014, §4.1]) and therefore must be diffeomorphic to a 3-torus. \square

Remark 6.3. The case $(3, 1)$ was considered by Saveliev and Wang [2011]. They correctly proved U_ξ^τ has rational Poincaré polynomial equal to $1 + 3t + 3t^2 + t^3$, but is not homeomorphic to T^3 .

Remark 6.4. In the proof of Theorem 6.2, we made use of the existence of torus actions on (a manifold diffeomorphic to) U_ξ^τ . These circle actions may be related to the Jeffrey–Weitsman torus action [1992] obtained from Goldman’s integral system [1986]. The Narasimhan–Seshadri theorem determines a diffeomorphism between U_ξ and the twisted representation variety $M = \text{Hom}_{-1}(\pi_1(\Sigma), \text{SU}(2))/\text{SU}(2)$. Jeffrey and Weitsman produced Hamiltonian $\text{SO}(2)$ actions defined on dense open subsets of M corresponding to embedded circles in the Riemann surface Σ . These $\text{SO}(2)$ actions commute whenever the embedded circles are disjoint. The action is not defined everywhere on M because the Hamiltonian functions are not everywhere differentiable. However, in unpublished work by the first author it is shown that for an embedded circle coinciding with an odd circle for a real curve (Σ, τ) , the corresponding $\text{SO}(2)$ -action restricts to a globally defined circle action on the submanifold M^τ identified with U_ξ^τ . Therefore M^τ admits a torus action of rank equal to the number of odd circles. This may correspond to the torus actions employed above.

Acknowledgements

The authors would like to thank Joel Kamnitzer, Luis Haug and Francois Charette for insightful discussions. Thanks also to Gómez Gutiérrez and Lopez de Medrano for answering our questions about their work. This research was supported in part by NSERC Discovery grants.

References

- [Baird 2014] T. Baird, “Moduli spaces of vector bundles over a real curve: $\mathbb{Z}/2$ -Betti numbers”, *Canad. J. Math.* **66**:5 (2014), 961–992. MR Zbl
- [Baird 2017] T. J. Baird, “Moduli spaces of vector bundles with fixed determinant over a real curve”, preprint, 2017. arXiv
- [Biran and Cornea 2009] P. Biran and O. Cornea, “A Lagrangian quantum homology”, pp. 1–44 in *New perspectives and challenges in symplectic field theory*, edited by M. Abreu et al., CRM Proc. Lecture Notes **49**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl
- [Biran and Cornea 2017] P. Biran and O. Cornea, “Cone-decompositions of Lagrangian cobordisms in Lefschetz fibrations”, *Selecta Math. (N.S.)* **23**:4 (2017), 2635–2704. An earlier version is available on arxiv:1504.00922v2. MR Zbl
- [Biswas et al. 2010] I. Biswas, J. Huisman, and J. Hurtubise, “The moduli space of stable vector bundles over a real algebraic curve”, *Math. Ann.* **347**:1 (2010), 201–233. MR Zbl
- [Borcea 1990] C. Borcea, “Deforming varieties of k -planes of projective complete intersections”, *Pacific J. Math.* **143**:1 (1990), 25–36. MR Zbl
- [Desale and Ramanan 1976/77] U. V. Desale and S. Ramanan, “Classification of vector bundles of rank 2 on hyperelliptic curves”, *Invent. Math.* **38**:2 (1976/77), 161–185. MR Zbl

- [Goldman 1986] W. M. Goldman, “Invariant functions on Lie groups and Hamiltonian flows of surface group representations”, *Invent. Math.* **85**:2 (1986), 263–302. MR Zbl
- [Gómez Gutiérrez and López de Medrano 2014] V. Gómez Gutiérrez and S. López de Medrano, “Topology of the intersections of quadrics II”, *Bol. Soc. Mat. Mex.* (3) **20**:2 (2014), 237–255. MR Zbl
- [Griffiths and Harris 1978] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience, New York, 1978. MR Zbl
- [Jeffrey and Weitsman 1992] L. C. Jeffrey and J. Weitsman, “Bohr–Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula”, *Comm. Math. Phys.* **150**:3 (1992), 593–630. MR Zbl
- [Konno 2002] T. Konno, “Unit tangent bundle over two-dimensional real projective space”, *Nihonkai Math. J.* **13**:1 (2002), 57–66. MR Zbl
- [Liu and Schaffhauser 2013] C.-C. M. Liu and F. Schaffhauser, “The Yang–Mills equations over Klein surfaces”, *J. Topol.* **6**:3 (2013), 569–643. MR Zbl
- [Milnor and Stasheff 1974] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies **76**, Princeton University Press, 1974. MR Zbl
- [Narasimhan and Ramanan 1969] M. S. Narasimhan and S. Ramanan, “Moduli of vector bundles on a compact Riemann surface”, *Ann. of Math.* (2) **89** (1969), 14–51. MR Zbl
- [Newstead 1968] P. E. Newstead, “Stable bundles of rank 2 and odd degree over a curve of genus 2”, *Topology* **7** (1968), 205–215. MR Zbl
- [Raymond 1968] F. Raymond, “Classification of the actions of the circle on 3-manifolds”, *Trans. Amer. Math. Soc.* **131** (1968), 51–78. MR Zbl
- [Saveliev and Wang 2011] N. Saveliev and S. Wang, “On real moduli spaces of holomorphic bundles over M -curves”, *Topology Appl.* **158**:3 (2011), 344–351. MR Zbl
- [Schaffhauser 2012] F. Schaffhauser, “Real points of coarse moduli schemes of vector bundles on a real algebraic curve”, *J. Symplectic Geom.* **10**:4 (2012), 503–534. MR Zbl
- [Wang 1995] S. Wang, “Classification of real moduli spaces over genus 2 curves”, *Geom. Dedicata* **57**:2 (1995), 207–215. MR Zbl
- [Wehrheim and Woodward 2016] K. Wehrheim and C. T. Woodward, “Exact triangle for fibered Dehn twists”, *Res. Math. Sci.* **3** (2016), art. id. 17, 75 pp. MR Zbl

Received October 29, 2018. Revised February 22, 2019.

THOMAS JOHN BAIRD
 DEPARTMENT OF MATHEMATICS & STATISTICS
 MEMORIAL UNIVERSITY OF NEWFOUNDLAND
 ST. JOHN’S NL
 CANADA
 tbaird@mun.ca

SHENGDA HU
 DEPARTMENT OF MATHEMATICS
 WILFRID LAURIER UNIVERSITY
 WATERLOO ON
 CANADA
 shu@wlu.ca

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Matthias Aschenbrenner
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
matthias@math.ucla.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2020 is US \$520/year for the electronic version, and \$705/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2020 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 304 No. 1 January 2020

The moduli space of real vector bundles of rank two over a real hyperelliptic curve	1
THOMAS JOHN BAIRD and SHENGDA HU	
Bounds of double zeta-functions and their applications	15
DEBIKA BANERJEE, T. MAKOTO MINAMIDE and YOSHIO TANIGAWA	
Commensurability growth of branch groups	43
KHALID BOU-RABEE, RACHEL SKIPPER and DANIEL STUDENMUND	
Asymptotic orders of vanishing along base loci separate Mori chambers	55
CHIH-WEI CHANG and SHIN-YAO JOW	
Local Langlands correspondence in rigid families	65
CHRISTIAN JOHANSSON, JAMES NEWTON and CLAUS SORENSEN	
PseudoindeX theory and Nehari method for a fractional Choquard equation	103
MIN LIU and ZHONGWEI TANG	
Symmetry and nonexistence of positive solutions for fractional Choquard equations	143
PEI MA, XUDONG SHANG and JIHUI ZHANG	
Decomposability of orthogonal involutions in degree 12	169
ANNE QUÉGUINER-MATHIEU and JEAN-PIERRE TIGNOL	
Zelevinsky operations for multisegments and a partial order on partitions	181
PETER SCHNEIDER and ERNST-WILHELM ZINK	
Langlands parameters, functoriality and Hecke algebras	209
MAARTEN SOLLEVELD	
On the archimedean local gamma factors for adjoint representation of GL_3 , part I	303
FANGYANG TIAN	
An explicit CM type norm formula and effective nonvanishing of class group L-functions for CM fields	347
LIYANG YANG	



0030-8730(2020)304:1;1-S