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It is known that there exists a Mori dream space such that the Mori chamber decomposition of its effective cone is strictly finer than the stable base locus decomposition. In other words, stable base loci of line bundles do not contain enough information to separate different Mori chambers in general. Here we show, however, that different Mori chambers can be separated if the scheme structures of the base loci are taken into account. More precisely, we show that for any two distinct Mori chambers Γ and Γ', the asymptotic order of vanishing along some divisorial valuation is linear on Γ and on Γ' respectively, but not simultaneously on their union $\Gamma \cup \Gamma'$. Two toric examples are given to illustrate our result: the first one exhibits two adjacent Mori chambers where the base schemes have the same underlying set but different embedded components; the second one shows that it is not always possible to separate two adjacent Mori chambers by the asymptotic order of vanishing along an associated component of the base schemes.

1. Introduction

Let X be a complex projective variety, and denote by $N^1(X)$ the group of numerical equivalence classes of Cartier divisors on X. Hu and Keel [2000] introduced the following notion of *Mori dream spaces*:

Definition 1. Let $D_1, \dots, D_r$ be Cartier divisors on X whose numerical equivalence classes form a $\mathbb{Z}$-basis of $N^1(X)$. The ring

$$\operatorname{Cox}(X) = \bigoplus_{(m_1,\dots,m_r)\in\mathbb{Z}^r} H^0(X, m_1D_1 + \cdots + m_rD_r)$$

is called a *Cox ring* of X.

Definition 2. A *Mori dream space* is a normal $\mathbb{Q}$-factorial projective variety X such that $\operatorname{Pic}(X)$ is a finitely generated abelian group and $\operatorname{Cox}(X)$ is a finitely generated $\mathbb{C}$-algebra (this does not depend on the choice of basis $D_1, \dots, D_r$).

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Examples of Mori dream spaces include $\mathbb{Q}$-factorial projective toric varieties [Hu and Keel 2000] and Fano varieties [Birkar et al. 2010]. The name "Mori dream space" comes from the fact that the birational geometry (or Mori theory) of such spaces can be described in terms of combinatorial data. Specifically, Hu and Keel showed the following:

Theorem 3 [Hu and Keel 2000]. *Let X be a Mori dream space. Then the effective cone of X has a finite rational polyhedral chamber decomposition, called the Mori chamber decomposition, such that the chambers correspond bijectively to all the birational contractions of X with $\mathbb{Q}$-factorial images. The birational contraction $X \dashrightarrow X_\Gamma$ that corresponds to a Mori chamber Γ is equal to the Iitaka fibration*

$$\phi_D : X \dashrightarrow \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, mD)$$

for any Cartier divisor D whose numerical equivalence class lies in the interior of Γ.

The effective cone of a Mori dream space can also be decomposed into chambers according to the stable base loci of the line bundles involved. This decomposition, called the *stable base locus decomposition*, is strictly coarser than the Mori chamber decomposition in general [Laface et al. 2018]. In other words, even if Γ and Γ' are two distinct Mori chambers, line bundles in their respective interiors can still have the same stable base locus. Our main result shows, however, that they cannot have the same "stable base scheme". More precisely, we show that the asymptotic order of vanishing along some divisorial valuation is linear on Γ and on Γ' respectively, but not simultaneously on their union $\Gamma \cup \Gamma'$: see Theorem 5 for the complete statement, and Definition 4 for the definition of asymptotic orders of vanishing. We give two toric examples to illustrate our result: Example 7 exhibits two adjacent Mori chambers where the base schemes have the same underlying set but different embedded components. In this case, the asymptotic order of vanishing along the embedded component separates the two chambers. Even more subtle behavior is possible, as shown in Example 8, where two adjacent Mori chambers have base schemes with exactly the same associated components, and the asymptotic order of vanishing along any associated component is linear across both chambers. In this case, the divisor whose valuation separates the two chambers cannot be obtained by any (ordinary) blowup along an associated component.

2. Main result

To state our result, let us first recall the notion of *asymptotic orders of vanishing* due to Ein, Lazarsfeld, Mustață, Nakamaye, and Popa [Ein et al. 2006].

Definition 4. Let X be a normal complex projective variety, and let $K(X)$ denote the function field of X. Let v be a discrete valuation of $K(X)$ over $\mathbb{C}$, and let Z be the center of v on X. For any effective big Cartier divisor D on X, we define:

(1) $v(D) = v(f)$, where f is a local equation of D at the generic point of Z.

(2) $v(|D|) = \min\{v(D') \mid D' \in |D|\} = v(D')$ for general $D' \in |D|$.

(3) $v(\|D\|) = \lim_{m\to\infty} v(|mD|)/m$. This is called the *asymptotic order of vanishing* of D along v. One can also define $v(\|D\|)$ for any big $\mathbb{Q}$-divisor D by taking m to be sufficiently divisible.

It is proved in [Ein et al. 2006, Theorem A] that $v(\|D\|)$ depends only on the numerical equivalence class of D, so it induces a function on the set of numerical equivalence classes of big $\mathbb{Q}$-divisors. Moreover, this function extends uniquely to a continuous function on the cone of numerical equivalence classes of big $\mathbb{R}$-divisors.

A discrete valuation v of $K(X)/\mathbb{C}$ is said to be *divisorial* if there exist a birational morphism $\widetilde{X} \to X$ from a normal projective variety $\widetilde{X}$ and a prime divisor E on $\widetilde{X}$ such that $v = \mathrm{ord}_E$ is the order of vanishing along E. For example, if $Y \subset X$ is an irreducible subvariety of X, then the order of vanishing along Y is a divisorial valuation of $K(X)/\mathbb{C}$, because $\mathrm{ord}_Y = \mathrm{ord}_E$, where E is the exceptional divisor of the normalized blowup of X along Y.

Our main result states that different Mori chambers of a Mori dream space can be separated by the asymptotic orders of vanishing along divisorial valuations.

Theorem 5. *Let X be a Mori dream space.*

(1) *For any discrete valuation v of $K(X)/\mathbb{C}$, the asymptotic order-of-vanishing function $v(\|\cdot\|)$ is linear on each Mori chamber of X.*

(2) *If Γ and Γ' are two distinct* (not necessarily adjacent) *Mori chambers of X, there exists a divisorial valuation v of $K(X)/\mathbb{C}$ such that the function $v(\|\cdot\|)$ is not linear on $\Gamma \cup \Gamma'$.*

Remark 6. A previous result similar to Theorem 5(i) is [Ein et al. 2006, Theorem D], which assumes only that $\mathrm{Cox}(X)$ is a finitely generated $\mathbb{C}$-algebra and concludes that the effective cone of X has a chamber decomposition such that $v(\|\cdot\|)$ is linear on each chamber. Note that, however, even when X is a Mori dream space, the chamber decomposition constructed in the proof of [Ein et al. 2006, Theorem D] is not the Mori chamber decomposition in general, but rather a refinement thereof. One interpretation of our Theorem 5 is that for a Mori dream space X, the Mori chamber decomposition is the "coarsest" chamber decomposition of the effective cone of X such that $v(\|\cdot\|)$ is linear on each chamber for every discrete valuation v of $K(X)/\mathbb{C}$.

Proof of Theorem 5. (1) Let Γ be a Mori chamber of X, and let $\rho_1, \dots, \rho_\ell$ be its extremal rays. By [Hu and Keel 2000, Lemma 2.8 and Proposition 1.11(2)], there exist Cartier divisors $D_1, \dots, D_\ell$ on X such that the class of D_i lies in ρ_i and that the canonical map

$$H^0(X, D_1)^{\otimes m_1} \otimes \cdots \otimes H^0(X, D_\ell)^{\otimes m_\ell} \to H^0(X, m_1 D_1 + \cdots + m_\ell D_\ell)$$

is surjective for all $m_i \geq 0$. This implies that

$$v(|m_1 D_1 + \cdots + m_\ell D_\ell|) = m_1 v(|D_1|) + \cdots + m_\ell v(|D_\ell|)$$

for all $m_i \geq 0$, so

$$v(\|m_1 D_1 + \cdots + m_\ell D_\ell\|) = m_1 v(|D_1|) + \cdots + m_\ell v(|D_\ell|)$$

for all $m_i \geq 0$. Hence $v(\|\cdot\|)$ is linear on Γ.

(2) Let $\rho_1, \dots, \rho_\ell$ be the extremal rays of Γ, and let $\rho'_1, \dots, \rho'_{\ell'}$ be the extremal rays of Γ'. By [Hu and Keel 2000, Lemma 2.8 and Proposition 1.11(2)], there exist Cartier divisors $D_1, \dots, D_\ell$ and $D'_1, \dots, D'_{\ell'}$ on X such that the class of D_i lies in ρ_i, the class of D'_j lies in ρ'_j, and that the canonical maps

$$H^0(X, D_1)^{\otimes m_1} \otimes \cdots \otimes H^0(X, D_\ell)^{\otimes m_\ell} \to H^0(X, m_1 D_1 + \cdots + m_\ell D_\ell),$$
$$H^0(X, D'_1)^{\otimes n_1} \otimes \cdots \otimes H^0(X, D'_{\ell'})^{\otimes n_{\ell'}} \to H^0(X, n_1 D'_1 + \cdots + n_{\ell'} D'_{\ell'}),$$

are surjective for all $m_i, n_j \geq 0$. Let $\widetilde{X}$ be a smooth projective variety with a birational morphism $\pi : \widetilde{X} \to X$ that resolves the base schemes of $|D_i|$ and $|D'_j|$ for all i, j. Then π also resolves the base scheme of every divisor $D \in \Lambda \cup \Lambda'$, where

$$\Lambda = \{m_1 D_1 + \cdots + m_\ell D_\ell \mid m_1, \dots, m_\ell \in \mathbb{N}\},$$
$$\Lambda' = \{n_1 D'_1 + \cdots + n_{\ell'} D'_{\ell'} \mid n_1, \dots, n_{\ell'} \in \mathbb{N}\}.$$

For $D \in \Lambda \cup \Lambda'$, let $F(D)$ and $M(D)$ be the divisors on $\widetilde{X}$ defined by

$$F(D) = \text{the fixed part of } |\pi^* D|,$$
$$M(D) = \pi^* D - F(D) = \text{the free part of } \pi^* D.$$

If $E \subset \widetilde{X}$ is a prime divisor, let v_E denote the divisorial valuation of $K(X)/\mathbb{C}$ given by the order of vanishing along E. Then

$$v_E(|D|) = \text{the coefficient of } E \text{ in } F(D).$$

It thus follows from

$$F(m_1 D_1 + \cdots + m_\ell D_\ell) = m_1 F(D_1) + \cdots + m_\ell F(D_\ell)$$

that $v_E(|D|) = v_E(\|D\|)$ for all $D \in \Lambda$, and similarly for all $D \in \Lambda'$. Hence

$$F(D) = \sum_{E \subset \widetilde{X} \text{ prime div.}} v_E(|D|)E = \sum_{E \subset \widetilde{X} \text{ prime div.}} v_E(\|D\|)E$$

for all $D \in \Lambda \cup \Lambda'$.

Suppose, for a contradiction, that $v_E(\|\cdot\|)$ is linear on $\Gamma \cup \Gamma'$ for all prime divisors $E \subset \widetilde{X}$. Then the map

$$\Lambda \cup \Lambda' \to N^1(\widetilde{X}), \quad D \longmapsto F(D) = \sum_{E \subset \widetilde{X} \text{ prime div.}} v_E(\|D\|)E$$

extends to a linear map

$$F : N^1(X)_{\mathbb{R}} \to N^1(\widetilde{X})_{\mathbb{R}},$$

where $N^1(X)_{\mathbb{R}} = N^1(X) \otimes \mathbb{R}$ and $N^1(\widetilde{X})_{\mathbb{R}} = N^1(\widetilde{X}) \otimes \mathbb{R}$. Since the pullback map $\pi^* : N^1(X) \to N^1(\widetilde{X})$ is obviously linear, the map

$$\Lambda \cup \Lambda' \to N^1(\widetilde{X}), \quad D \longmapsto M(D) = \pi^* D - F(D)$$

also extends to a linear map

$$M : N^1(X)_{\mathbb{R}} \to N^1(\widetilde{X})_{\mathbb{R}}.$$

Let D and D' be Cartier divisors on X whose classes lie in the interiors of Γ and Γ', respectively, and let $\phi_D : X \dashrightarrow X_\Gamma$ and $\phi_{D'} : X \dashrightarrow X_{\Gamma'}$ be the corresponding Iitaka fibrations. Then ϕ_D and $\phi_{D'}$ lift to semiample fibrations $\phi_{M(D)} : \widetilde{X} \to X_\Gamma$ and $\phi_{M(D')} : \widetilde{X} \to X_{\Gamma'}$. If $C \subset \widetilde{X}$ is a curve, then

$$\begin{aligned} C \text{ is contracted by } \phi_{M(D)} &\iff C \cdot M(D) = 0 \\ &\iff C \cdot M(D_1) = \cdots = C \cdot M(D_\ell) = 0 \\ &\iff C \cdot \widetilde{D} = 0 \quad \text{for all } \widetilde{D} \in M(N^1(X)_{\mathbb{R}}). \end{aligned}$$

Similarly, C is contracted by $\phi_{M(D')}$ if and only if $C \cdot \widetilde{D} = 0$ for all $\widetilde{D} \in M(N^1(X)_{\mathbb{R}})$. This means that $\phi_{M(D)}$ and $\phi_{M(D')}$ contract the same curves, so $\phi_{M(D)} = \phi_{M(D')}$ and $X_\Gamma = X_{\Gamma'}$ [Debarre 2001, Proposition 1.14]. But by Theorem 3, $X_\Gamma = X_{\Gamma'}$ only when $\Gamma = \Gamma'$, so this is a contradiction. □

3. Examples

Example 7. Let Y be the $\mathbb{Q}$-factorial toric variety given in [Laface et al. 2018, Example 3.4]: its fan has 6 rays which are spanned by the column vectors $v_1, \dots, v_6$ of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{pmatrix},$$

and the maximal cones of Y have the following indices: $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{2, 4, 5\}$, $\{1, 2, 6\}$, $\{2, 5, 6\}$, $\{1, 3, 6\}$, $\{3, 4, 6\}$, $\{4, 5, 6\}$. Let $D_1, \ldots, D_6$ be the torus-invariant divisors of Y. One can check that the cone $\mathcal{C}_1 \subseteq N^1(Y)_{\mathbb{R}}$ given in [Laface et al. 2018, Example 3.4] is spanned by the numerical equivalence classes of the divisors D_2, $D_2 + 2D_5$, and $D_1 + D_2 + D_5$, and the cone $\mathcal{C}_2 \subseteq N^1(Y)_{\mathbb{R}}$ is spanned by the numerical equivalence classes of the divisors D_2, $D_2 + 2D_1$, and $D_1 + D_2 + D_5$.

It was mentioned in [Laface et al. 2018, Example 3.4] that $\mathcal{C}_1$ and $\mathcal{C}_2$ are Mori chambers of Y whose interior divisors have the same stable base locus D_2. We will show, however, that the base schemes of the divisors in $\mathcal{C}_1$ have an embedded component $D_2 \cap D_3$, whereas the base schemes of the divisors in $\mathcal{C}_2$ have an embedded component $D_2 \cap D_4$. In particular, the asymptotic order of vanishing along $D_2 \cap D_3$ is not linear on $\mathcal{C}_1 \cup \mathcal{C}_2$.

We choose the numerical equivalence classes of $D_2 + 2D_5$, $D_2 + 2D_1$, and D_2 as basis of $N^1(Y)_{\mathbb{R}}$, so that the cones $\mathcal{C}_1$ and $\mathcal{C}_2$ are described by simple inequalities

$$\mathcal{C}_1 = \big\{r_1(D_2 + 2D_5) + r_2(D_2 + 2D_1) + r_3 D_2 \mid r_1, r_2, r_3 \geq 0, r_1 \geq r_2\big\};$$
$$\mathcal{C}_2 = \big\{r_1(D_2 + 2D_5) + r_2(D_2 + 2D_1) + r_3 D_2 \mid r_1, r_2, r_3 \geq 0, r_1 \leq r_2\big\}.$$

Let x_1, x_2, x_3 be the characters of the torus that are dual to v_1, v_2, v_3. In other words, x_1, x_2, x_3 are the coordinate functions of the open affine space $U \subseteq Y$ corresponding to the maximal cone spanned by v_1, v_2, v_3. The linear system $|D|$ of a divisor

$$D = r_1(D_2 + 2D_5) + r_2(D_2 + 2D_1) + r_3 D_2$$

is generated by all effective divisors of the form

$$\begin{aligned}
&D + \operatorname{div}(x_1^{m_1} x_2^{m_2} x_3^{m_3}) \\
&= (m_1 + 2r_2)D_1 + (m_2 + r_1 + r_2 + r_3)D_2 + m_3 D_3 + (-m_1 + m_2 + m_3)D_4 \\
&\qquad\qquad + (-m_1 + 2m_2 + 2r_1)D_5 + (m_1 - 4m_2 - 2m_3)D_6 \\
&= n_1 D_1 + n_2 D_2 + n_3 D_3 + (-n_1 + n_2 + n_3 - r_1 + r_2 - r_3)D_4 \\
&\qquad\qquad + (-n_1 + 2n_2 - 2r_3)D_5 + (n_1 - 4n_2 - 2n_3 + 4r_1 + 2r_2 + 4r_3)D_6.
\end{aligned}$$

The base scheme of $|D|$ is the scheme-theoretic intersection of these effective divisors. On the affine space U, the divisor $D_i|_U$ is defined by x_i for $i = 1, 2, 3$, while $D_i|_U = \varnothing$ for $i = 4, 5, 6$. Hence the base ideal $\mathfrak{b}(|D|)_U$ of $|D|$ on U is generated by all monomials $x_1^{n_1} x_2^{n_2} x_3^{n_3}$ such that

$$-n_1 + n_2 + n_3 \geq r_1 - r_2 + r_3, \tag{1}$$
$$-n_1 + 2n_2 \geq 2r_3, \tag{2}$$
$$n_1 - 4n_2 - 2n_3 \geq -4r_1 - 2r_2 - 4r_3. \tag{3}$$

We claim that $\mathfrak{b}(|D|)_U$ has the following primary decomposition

$$\mathfrak{b}(|D|)_U = \begin{cases} \langle x_2\rangle^{r_3} \cap \langle x_2, x_3\rangle^{r_1-r_2+r_3} & \text{if } D \in \mathcal{C}_1; \\ \langle x_2\rangle^{r_3} & \text{if } D \in \mathcal{C}_2. \end{cases}$$

Recall that $D \in \mathcal{C}_1$ if and only if $r_1, r_2, r_3 \geq 0$ and $r_1 \geq r_2$. In this case, the ideal $\langle x_2\rangle^{r_3} \cap \langle x_2, x_3\rangle^{r_1-r_2+r_3}$ is generated by all monomials $x_1^{n_1}x_2^{n_2}x_3^{n_3}$ such that $n_1 = 0$, $n_2 \geq r_3$, and $n_2+n_3 = r_1-r_2+r_3$, and it is easy to see that all such triples (n_1, n_2, n_3) satisfy the inequalities (1), (2), and (3). Hence $\langle x_2\rangle^{r_3} \cap \langle x_2, x_3\rangle^{r_1-r_2+r_3} \subseteq \mathfrak{b}(|D|)_U$. On the other hand, $\mathfrak{b}(|D|)_U$ is generated by monomials $x_1^{n_1}x_2^{n_2}x_3^{n_3}$ with (n_1, n_2, n_3) satisfying (1), (2), and (3), and such monomials lie in $\langle x_2\rangle^{r_3} \cap \langle x_2, x_3\rangle^{r_1-r_2+r_3}$ since $n_2 \geq r_3$ by (2) and $n_2 + n_3 \geq r_1 - r_2 + r_3$ by (1). The case $D \in \mathcal{C}_2$ can be checked similarly.

The linear map that sends v_1 to v_5, v_3 to v_4, and fixes v_2 induces an involution on the fan of Y, and hence on Y. This involution swaps $\mathcal{C}_1$ and $\mathcal{C}_2$, and maps U to the open affine space V corresponding to the maximal cone spanned by v_5, v_2, v_4. It follows that if y_5, y_2, y_4 are the characters of the torus that are dual to v_5, v_2, v_4, then the base ideal $\mathfrak{b}(|D|)_V$ of $|D|$ on V has the following primary decomposition:

$$\mathfrak{b}(|D|)_V = \begin{cases} \langle y_2\rangle^{r_3} & \text{if } D \in \mathcal{C}_1; \\ \langle y_2\rangle^{r_3} \cap \langle y_2, y_4\rangle^{r_2-r_1+r_3} & \text{if } D \in \mathcal{C}_2. \end{cases}$$

Finally, similar computation on the open affine space corresponding to the maximal cone $\{2, 3, 4\}$ confirms that $D_2 \cap D_3 \cap D_4$ is not an embedded point of the base scheme of $|D|$. Hence the primary decomposition of the base ideal $\mathfrak{b}(|D|)$ is

$$\mathfrak{b}(|D|) = \begin{cases} \mathscr{I}(D_2)^{r_3} \cap \mathscr{I}(D_2 \cap D_3)^{r_1-r_2+r_3} & \text{if } D \in \mathcal{C}_1; \\ \mathscr{I}(D_2)^{r_3} \cap \mathscr{I}(D_2 \cap D_4)^{r_2-r_1+r_3} & \text{if } D \in \mathcal{C}_2. \end{cases}$$

One sees that the base scheme of $|D|$ varies "linearly" as D varies in each chamber, but a "flip of embedded component" occurs when D crosses the wall.

Example 8. Even more subtle behavior is possible by slightly modifying Example 7. Let $v_1 = (1, 0, 0)$, $v_2 = (1, 2, -2)$, $v_3 = (0, 0, 1)$, $v_4 = (-1, 1, 1)$, $v_5 = (-1, 2, 0)$, and $v_6 = \frac{1}{5}(-v_1 - v_2 - v_3 - v_4 - v_5) = (0, -1, 0)$. Consider the complete fans given by

$$\Delta(3) = \{\sigma_{123}, \sigma_{234}, \sigma_{245}, \sigma_{136}, \sigma_{346}, \sigma_{456}, \sigma_{256}, \sigma_{126}\},$$

$$\Delta_1(3) = \{\sigma_{134}, \sigma_{145}, \sigma_{125}, \sigma_{136}, \sigma_{346}, \sigma_{456}, \sigma_{256}, \sigma_{126}\},$$

$$\Delta_2(3) = \{\sigma_{135}, \sigma_{345}, \sigma_{125}, \sigma_{136}, \sigma_{346}, \sigma_{456}, \sigma_{256}, \sigma_{126}\};$$

see Figure 1.

Let X, X_1, and X_2 be the toric varieties associated to the fans Δ, Δ_1, and Δ_2, respectively. Let $\mathcal{C}_i \subseteq N^1(X)_{\mathbb{R}}$ be the nef cone of X_i for $i = 1, 2$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are adjacent Mori chambers of X. We will show that the base schemes of divisors

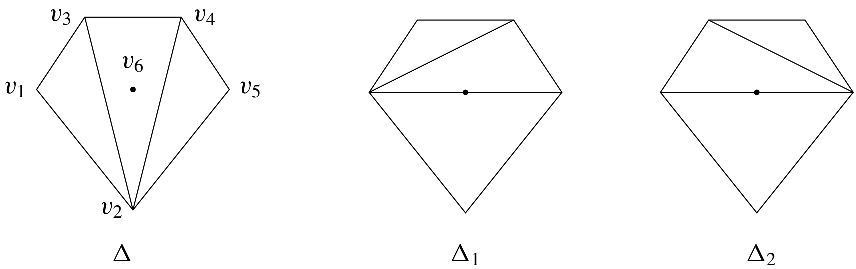

Figure 1. Figure for Example 8.

in the interior of $\mathcal{C}_1$ have exactly the same associated components as those in $\mathcal{C}_2$. Moreover, the asymptotic order of vanishing along any associated component is linear on $\mathcal{C}_1 \cup \mathcal{C}_2$.

Since the base scheme varies linearly in each Mori chamber, to compute it for any given chamber, it is enough to compute it for a full-dimensional subcone of that chamber. It is straightforward to check that all divisors $D = m_1 D_1 + m_2 D_2 + m_5 D_5$ on X, where

$$(m_1, m_2, m_5) \in 12\mathbb{N}^3, \quad m_5 \geq m_1, \quad m_2 \geq m_1 + 2m_5$$

are Cartier and span a full-dimensional subcone of $\mathcal{C}_1$. Moreover, since every associated component of $\mathfrak{b}(|D|)$ intersects the open set U_{234}, we can do our computation on this open patch.

The dual cone of σ_{234} is

$$\sigma_{234}^{\vee} = \operatorname{Cone}\big((1, 1, 0), (4, 1, 3), (-2, 1, 0)\big).$$

Using similar argument as in Example 7, we get

$$\mathfrak{b}(|D|) = I_{23} \cap I_{24} \subseteq \mathbb{C}[\sigma_{234}^{\vee} \cap \mathbb{Z}^3],$$

where

$$I_{23} = \Big\{\chi^u \;\Big|\; u = u_2(1, 1, 0) + u_3(4, 1, 3) + u_4(-2, 1, 0),\ \tfrac{3u_2}{m_2 - 3m_1} + \tfrac{12u_3}{m_2 - 3m_1} \geq 1, \\ \tfrac{3u_2}{m_2 - 2m_1 - m_5} + \tfrac{6u_3}{m_2 - 2m_1 - m_5} \geq 1\Big\},$$

and

$$I_{24} = \Big\{\chi^u \;\Big|\; u = u_2(1, 1, 0) + u_3(4, 1, 3) + u_4(-2, 1, 0), \\ \tfrac{3u_2}{m_2 - m_1 - 2m_5} + \tfrac{6u_4}{m_2 - m_1 - 2m_5} \geq 1\Big\}.$$

Here I_{23} is $I_{\sigma_{23}}$-primary, and I_{24} is $I_{\sigma_{24}}$-primary, so $\mathfrak{b}(|D|) = I_{23} \cap I_{24}$ is a primary decomposition. From this, one can compute the order of vanishing

$$\operatorname{ord}_{V(\sigma_{ij})}(|D|) = \min\{\langle u, v_i\rangle + \langle u, v_j\rangle \mid \chi^u \in I_{ij}\}, \quad (i, j) = (2, 3) \text{ or } (2, 4).$$

The result is

$$\operatorname{ord}_{V(\sigma_{23})}(|D|) = \frac{m_2 - m_5 - 2m_1}{2}, \quad \text{and} \quad \operatorname{ord}_{V(\sigma_{24})}(|D|) = \frac{m_2 - 2m_5 - m_1}{2}.$$

By symmetry, all divisors $D' = m_1 D_1 + m_2 D_2 + m_5 D_5$ on X, where

$$(m_1, m_2, m_5) \in 12\mathbb{N}^3, \quad m_1 \geq m_5, \quad m_2 \geq m_5 + 2m_1$$

are Cartier and span a full-dimensional subcone of $\mathcal{C}_2$, and there is a primary decomposition

$$\mathfrak{b}(|D'|) = I'_{23} \cap I'_{24} \subseteq \mathbb{C}[\sigma_{234}^{\vee} \cap \mathbb{Z}^3],$$

where

$$I'_{24} = \Big\{ \chi^u \;\Big|\; u = u_2(1,1,0) + u_3(4,1,3) + u_4(-2,1,0),\ \tfrac{3u_2}{m_2 - 3m_5} + \tfrac{12u_4}{m_2 - 3m_5} \geq 1, \\ \tfrac{3u_2}{m_2 - 2m_5 - m_1} + \tfrac{6u_4}{m_2 - 2m_5 - m_1} \geq 1 \Big\},$$

and

$$I'_{23} = \Big\{ \chi^u \;\Big|\; u = u_2(1,1,0) + u_3(4,1,3) + u_4(-2,1,0), \\ \tfrac{3u_2}{m_2 - m_5 - 2m_1} + \tfrac{6u_3}{m_2 - m_5 - 2m_1} \geq 1 \Big\}.$$

Furthermore,

$$\operatorname{ord}_{V(\sigma_{24})}(|D'|) = \frac{m_2 - m_1 - 2m_5}{2}, \quad \text{and} \quad \operatorname{ord}_{V(\sigma_{23})}(|D'|) = \frac{m_2 - 2m_1 - m_5}{2}.$$

So the asymptotic orders of vanishing $\operatorname{ord}_{V(\sigma_{23})}(\|\cdot\|)$ and $\operatorname{ord}_{V(\sigma_{24})}(\|\cdot\|)$ are both linear on $\mathcal{C}_1 \cup \mathcal{C}_2$.

Let Δ' be the star subdivision of the fan Δ with respect to the ray spanned by $4v_3 + v_2$. Let $X' \to X$ be the corresponding toric morphism, and let $E \subseteq X'$ be the exceptional divisor. One can check that the Mori chambers $\mathcal{C}_1$ and $\mathcal{C}_2$ are separated by the divisorial valuation v_E of $K(X)/\mathbb{C}$ given by the order of vanishing along E. In fact, for a divisor $D = m_1 D_1 + m_2 D_2 + m_5 D_5$ on X with $(m_1, m_2, m_5) \in 12\mathbb{N}^3$,

$$v_E(|D|) = \begin{cases} \dfrac{m_2 - 3m_1}{4} & \text{if } D \in \mathcal{C}_1; \\ \dfrac{m_2 - 2m_1 - m_5}{4} & \text{if } D \in \mathcal{C}_2. \end{cases}$$

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