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ON SEIFERT FIBERED SPACES BOUNDING DEFINITE MANIFOLDS

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We establish an inequality which gives strong restrictions on when the standard definite plumbing intersection lattice of a Seifert fibered space over S^2 can embed into a standard diagonal lattice, and give two applications. First, we answer a question of Neumann and Zagier on the relationship between Donaldson's theorem and Fintushel–Stern's R-invariant. We also give a short proof of the characterization of Seifert fibered spaces which smoothly bound rational homology $S^1 \times D^3$'s.

1. Introduction

Donaldson's diagonalization theorem [1987] has led to many great successes in understanding several important questions in low dimensional topology, and in knot theory in particular. For example, Donaldson's theorem can often be used to answer questions concerning sliceness, unknotting number, 3-manifolds bounding rational homology balls, and surgery questions. In these cases, one typically uses Donaldson's theorem to obstruct a certain 3-manifold from bounding a certain type of smooth negative definite 4-manifold, with the obstruction taking the form of the existence of a certain map of intersection lattices. However, understanding this obstruction for large families of examples is often highly nontrivial, and can require combinatorial ingenuity.

One appealing application of Donaldson's theorem is to prove the well-known fact that the Poincaré homology sphere $P = S^2\bigl(2; 2, \frac{3}{2}, \frac{5}{4}\bigr)$ does not bound a smooth integral homology 4-ball. This fact can, of course, be proved in many other ways, for example by using Rokhlin's theorem, Fintushel–Stern's R-invariant, or by using the d-invariant coming from Heegaard Floer homology. Assuming that P is oriented to bound the positive E_8 plumbing, the proof by Donaldson's theorem is as follows. If P were the boundary of a smooth integral homology 4-ball W, then we could form a closed positive definite manifold by gluing $-W$ to the positive E_8 plumbing. Donaldson's theorem would then imply that the E_8 intersection form is diagonalizable, which is, of course, untrue. In fact, as the E_8 intersection form

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does not embed into any positive definite diagonal lattice, this argument shows that P does not bound any smooth negative definite 4-manifold. The purpose of this paper is to generalize this argument to other Seifert fibered spaces. We prove the following theorem.

Theorem 1. *Let $Y = S^2\bigl(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\bigr)$ be a Seifert fibered space over S^2 in standard form, that is, with $e \geq 0$, $\frac{p_i}{q_i} > 1$ for all $i \in \{1, 2, \dots, k\}$ and $\varepsilon(Y) \geq 0$. Suppose that Y bounds a smooth 4-manifold W such that $\sigma(W) = -b_2(W)$ and the inclusion induced map $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$ is injective. Then there is a partition of $\{1, 2, \dots, k\}$ into at most e classes such that for each class C,*

$$\sum_{i \in C} \frac{q_i}{p_i} \leq 1.$$

We note that the condition that $\varepsilon(Y) := e - \sum_{i=1}^{k} \frac{q_i}{p_i} \geq 0$ in Theorem 1 guarantees that Y is oriented to bound a positive (semi)definite plumbing 4-manifold. When Y is a rational homology sphere the map $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$ is automatically injective so in this case we are simply obstructing the existence of a negative definite manifold bounding Y. Although we do not discuss the details in this paper, one can easily obtain analogous results for Seifert fibered spaces over any orientable base surface. In our notation, the Poincaré homology sphere oriented to bound the positive E_8 plumbing is $P = S^2\bigl(2; 2, \frac{3}{2}, \frac{5}{4}\bigr)$, see Figure 2. The reader can easily verify that Theorem 1 obstructs P from bounding a negative definite manifold. When $k < 3$, the Seifert fibered spaces are lens spaces which are well known to bound both positive and negative definite smooth 4-manifolds. Finally, we note that the converse to Theorem 1 is not true. The integer homology sphere $S^2\bigl(1; 3, 5, \frac{13}{6}\bigr)$ passes the obstruction, but does not bound a negative definite manifold as it bounds a positive definite plumbing whose intersection form does not embed in a diagonal lattice.

We give two applications of Theorem 1. First, we prove the following theorem.

Theorem 2. *Let $Y = S^2\bigl(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\bigr)$ be a Seifert fibered integral homology sphere in standard form, that is, with $\frac{p_i}{q_i} > 1$ for all $i \in \{1, 2, \dots, k\}$, $e > 0$ and with Y oriented to bound a smooth positive definite plumbing 4-manifold. If Y bounds a smooth negative definite 4-manifold, then $e = 1$.*

In the course of proving Theorem 2, we obtain a positive answer to the following question asked by Neumann and Zagier [1985].

Question. *Let Y be as in Theorem 2. If the intersection form of the plumbing of Y is diagonalizable over $\mathbb{Z}$, must e be equal to 1?*

The motivation for this question comes from the R-invariant. Fintushel and Stern [1985] used gauge theory to define an invariant $R(Y)$ of Seifert fibered integral homology spheres with the property that if $R(Y) > 0$ then Y does not bound a smooth negative definite 4-manifold W with $H_1(W)$ having no 2-torsion. Fintushel and Stern originally gave an expression for $R(Y)$ as a trigonometric sum involving the Seifert invariants of Y. Neumann and Zagier [1985] proved that these sums could be simply evaluated in terms of the central weight e of the standard positive definite plumbing bounding Y, showing that $R(Y) = 2e - 3$. Thus, if $e > 1$ then the R-invariant shows that Y does not bound a smooth negative definite 4-manifold W with $H_1(W)$ having no 2-torsion. In this light, the positive answer to the Neumann–Zagier question implies that this result obtained from the R-invariant is also a consequence of Donaldson's theorem.

We are in fact able to prove a more general version of Theorem 2 which holds for all $|H_1(Y)| \in \{1, 2, 3, 5, 6, 7\}$; see Theorem 8 of Section 5. Some particular cases of Theorem 2 are known. In their original paper, Neumann and Zagier [1985] claimed to have proved the cases when $k = 3$, and when $k = 4$ and $e \neq 3$, but do not provide a proof, remarking that their proof was "clearly not the right proof." The special case when $e = k - 1$ follows from [Lecuona and Lisca 2011, Lemma 3.3].

Finally, we note that a positive answer to the Neumann–Zagier question is a special case of a more general conjecture made by Neumann [1989], stating that if an integral homology sphere Y is given as the boundary of a positive definite plumbing tree Γ and the intersection lattice of Γ is isomorphic to a diagonal lattice, then some vertex of Γ has weight 1. This general form of Neumann's conjecture for graph manifolds remains open.

Lidman and Tweedy [2018, Remark 4.3] asked whether a Seifert fibered integral homology sphere with central weight different from 1 must have nonvanishing Heegaard–Floer d-invariant. As a corollary of Theorem 2, we answer their question positively.

Corollary 3. *Let Y be a Seifert fibered integral homology sphere, and let $e \in \mathbb{Z}$ be the central weight in the standard definite plumbing graph for Y. If $|e| \neq 1$, then $d(Y) \neq 0$.*

As a second application, we give a short proof of the following theorem which, in particular, gives a classification of the Seifert fibered spaces bounding rational homology $S^1 \times D^3$'s.

Theorem 4. *Let Y be a Seifert fibered space over S^2 with*

$$H_*(Y; \mathbb{Q}) \cong H_*(S^1 \times S^2; \mathbb{Q}).$$

The following are equivalent:

(1) *Y is of the form $S^2\big(k; \frac{p_1}{q_1}, \frac{p_1}{p_1-q_1}, \dots, \frac{p_k}{q_k}, \frac{p_k}{p_k-q_k}\big)$, where $k \geq 0$ and $\frac{p_i}{q_i} > 1$ for all $i \in \{1, \dots, k\}$.*

(2) $Y = \partial W$, *where W is a smooth* 4*-manifold with* $H_*(W;\mathbb{Q}) \cong H_*(S^1 \times D^3;\mathbb{Q})$.

(3) *Y is the boundary of smooth* 4*-manifolds* W_+ *and* W_- *such that* $\sigma(W_\pm) = \pm b_2(W_\pm)$ *and each of the inclusion-induced maps* $H_1(Y;\mathbb{Q}) \to H_1(W_\pm;\mathbb{Q})$ *is injective.*

Seifert fibered spaces bounding rational homology $S^1 \times D^3$'s naturally arise in two contexts. First, a Seifert fibered space rational homology $S^1 \times S^2$ which embeds in S^4 necessarily bounds a rational homology $S^1 \times D^3$. Indeed, in this context Donald [2015, Proof of Theorem 1.3] proved the implication (2) implies (1) of Theorem 4. Second, a smoothly slice 2-component Montesinos link has double branched cover a Seifert fibered space over S^2 bounding a rational homology $S^1 \times D^3$. Motivated by trying to determine the slice 2-component Montesinos links, Aceto [2015, Theorem 1.2] also classified Seifert fibered spaces bounding rational homology $S^1 \times D^3$'s.

Much like the proofs by Donald and Aceto, our proof also proceeds by means of Donaldson's theorem. However, their proofs rely on the work of Lisca [2007] which gives a detailed analysis on sums of linear lattices embedding in a full-rank lattice. We give a short proof of Theorem 4 circumventing the reliance on Lisca's work. We obtain the additional equivalent condition (3) in Theorem 4, since our method does not require the lattice embeddings to have full-rank.

Finally, we note that Theorem 1 also plays a key role in the paper [Issa and McCoy 2018], where we analyze which Seifert fibered spaces smoothly embed in S^4, and in particular, completely determine the Seifert fibered spaces

$$Y = S^2(e; r_1, \ldots, r_k) \quad \text{with } r_i \in \mathbb{Q}_{>1} \text{ for all } i, \quad \varepsilon(Y) > 0 \quad \text{and} \quad e > k/2$$

which smoothly embed in S^4.

In Section 2, we recall some standard facts and establish notation and conventions. In Section 3, we prove the key technical theorem used to prove Theorem 1. In Section 4, we analyze when gluing compact 4-manifolds with boundary results in a definite 4-manifold and give a proof of Theorem 1. In Section 5, we prove Theorem 8 answering the Neumann–Zagier question, as well as prove Corollary 3. Finally, in Section 6 we prove Theorem 4 determining the Seifert fibered spaces which bound rational homology $S^1 \times D^3$'s.

2. Preliminaries

In this section we briefly recall some standard facts about Seifert fibered spaces and intersection lattices, and establish notation and conventions. See [Neumann and Raymond 1978] for a more in-depth treatment on Seifert fibered spaces and plumbings.

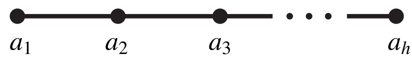

Figure 1. Weighted linear chain representing $r = [a_1, \dots, a_h]^-$.

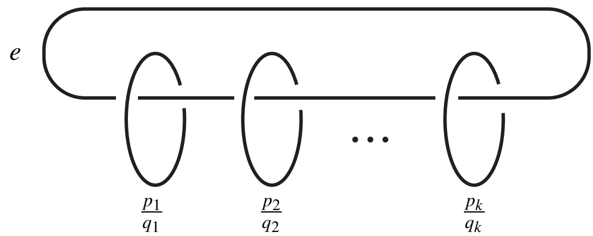

Figure 2. Surgery presentation for the Seifert fibered space $S^2\big(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\big)$.

Given $r \in \mathbb{Q}_{>1}$, there is a unique (negative) continued fraction expansion

$$r = [a_1, \dots, a_h]^- := a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots\ a_{h-1} - \cfrac{1}{a_h}}},$$

where $h \geq 1$ and $a_i \geq 2$ are integers for all $i \in \{1, \dots, h\}$. We associate to r the weighted linear graph (or linear chain) given in Figure 1. We call the vertex with weight labeled by a_i the i-th vertex of the linear chain associated to r, so that the vertex labeled with weight a_1 is the first, or starting vertex of the linear chain.

We denote by $Y = S^2\big(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\big)$ the Seifert fibered space over S^2 given in Figure 2, where $e \in \mathbb{Z}$, and $\frac{p_i}{q_i} \in \mathbb{Q}$ is nonzero for all $i \in \{1, \dots, k\}$. The generalized Euler invariant of Y is given by $\varepsilon(Y) = e - \sum_{i=1}^{k} \frac{q_i}{p_i}$. Every Seifert fibered space Y is (possibly orientation reversing) homeomorphic to one in standard form, i.e., such that $\varepsilon(Y) \geq 0$ and $\frac{p_i}{q_i} > 1$ for all $i \in \{1, \dots, k\}$. We henceforth assume that Y is in standard form. If $\varepsilon(Y) \neq 0$ then Y is a rational homology sphere with $|H_1(Y)| = |p_1 \cdots p_k \varepsilon(Y)|$, and if $\varepsilon(Y) = 0$ then Y is a rational homology $S^1 \times S^2$.

For each $i \in \{1, \dots, k\}$, we have the unique continued fraction expansion $\frac{p_i}{q_i} = [a^i_1, \dots, a^i_{h_i}]^-$ where $h_i \geq 1$ and $a^i_j \geq 2$ are integers for all $j \in \{1, \dots, h_i\}$. We associate to $Y = S^2\big(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\big)$ the weighted star-shaped graph in Figure 3. The i-th leg of the star-shaped graph is the weighted linear subgraph for p_i/q_i generated by the vertices labeled with weights $a^i_1, \dots, a^i_{h_i}$. The degree k vertex labeled with weight e is called the central vertex.

Let Γ be either the weighted star-shaped graph for Y, or a disjoint union of weighted linear graphs. There is an oriented smooth 4-manifold X_Γ given by plumbing D^2-bundles over S^2 according to the weighted graph Γ. We denote by

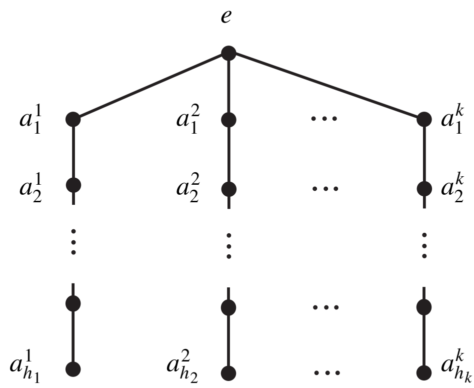

Figure 3. The weighted star-shaped plumbing graph Γ.

$|\Gamma|$ the number of vertices in Γ. Let $N = |\Gamma|$ and denote the vertices of Γ by $v_1, v_2, \ldots, v_N$. The zero-sections of the D^2-bundles over S^2 corresponding to each of $v_1, \ldots, v_N$ in the plumbing together form a natural spherical basis for $H_2(X_\Gamma)$. With respect to this basis, which we call the vertex basis, the intersection form of X_Γ is given by the weighted adjacency matrix Q_Γ with entries Q_{ij}, $1 \le i, j \le N$ given by

$$Q_{ij} = \begin{cases} \mathrm{w}(v_i) & \text{if } i = j, \\ -1 & \text{if } v_i \text{ and } v_j \text{ are connected by an edge,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{w}(v_i)$ is the weight of vertex v_i. Denoting by Q_{X_Γ} the intersection form of X_Γ, we call $(H_2(X_\Gamma), Q_{X_\Gamma}) \cong (\mathbb{Z}^N, Q_\Gamma)$ the intersection lattice of X_Γ (or of Γ). We denote the intersection pairing of two elements $x, y \in \mathbb{Z}^N$ by $x \cdot y = x^T Q_\Gamma y$. Now assume that Γ is the star-shaped plumbing for Y. If $\varepsilon(Y) > 0$ then X_Γ is a positive definite 4-manifold and Γ is the standard positive definite plumbing graph for Y. If $\varepsilon(Y) = 0$, then X_Γ is a positive semidefinite manifold.

Let $\iota : (\mathbb{Z}^N, Q_\Gamma) \to (\mathbb{Z}^m, \mathrm{Id})$, $m > 0$, be a map of lattices, i.e., a $\mathbb{Z}$-linear map preserving pairings, where $(\mathbb{Z}^m, \mathrm{Id})$ is the standard positive diagonal lattice. We call ι a lattice embedding if it is injective. We adopt the following standard abuse of notation. First, for each $i \in \{1, \ldots, N\}$, we identify the vertex v_i with the corresponding i-th basis element of $(\mathbb{Z}^N, Q_\Gamma)$. Moreover, we shall identify an element $v \in (\mathbb{Z}^N, Q_\Gamma)$ with its image $\iota(v) \in (\mathbb{Z}^m, \mathrm{Id})$.

3. The embedding inequality

In this section, we prove Theorem 6 below, which is the key technical result of this paper. In particular, it will be used in the next section to prove Theorem 1. We begin with some continued fraction identities which we will need.

Lemma 5. *Let $\{a_i\}_{i\geq 1}$ be a sequence of integers with $a_i \geq 2$ for all i, and let $p_k/q_k = [a_1, \dots, a_k]^-$ for all $k \geq 1$. Then we have the following identities*:

(a) $q_n p_{n-1} - p_n q_{n-1} = 1$ *for all $n \geq 2$.*

(b) $[a_1, \dots, a_n, x]^- = \dfrac{x p_n - p_{n-1}}{x q_n - q_{n-1}}$, *for all $n \geq 2$ and $x \in \mathbb{R}$ such that both sides are well defined.*

(c) $p_n = \det \begin{pmatrix} a_1 & -1 & 0 & 0 \\ -1 & a_2 & -1 & 0 \\ 0 & -1 & \ddots & -1 \\ 0 & 0 & -1 & a_n \end{pmatrix}$ *and* $q_n = \det \begin{pmatrix} a_2 & -1 & 0 & 0 \\ -1 & a_3 & -1 & 0 \\ 0 & -1 & \ddots & -1 \\ 0 & 0 & -1 & a_n \end{pmatrix}$ *for all $n \geq 2$.*

Proof. For $a \in \mathbb{R}$, let M_a denote the matrix $M_a = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$. If $q/r = [a_2, \dots, a_n]^-$, then $\frac{p_n}{q_n} = a_1 - \frac{r}{q}$. In particular, we have

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = M_{a_1} \begin{pmatrix} q \\ r \end{pmatrix}.$$

Thus, one can inductively show that

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = M_{a_1} \cdots M_{a_n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1}$$

and furthermore that

$$\begin{pmatrix} p_n & -p_{n-1} \\ q_n & -q_{n-1} \end{pmatrix} = M_{a_1} \cdots M_{a_n}. \tag{2}$$

Identity (a) follows by taking determinants of (2) and observing that $\det M_a = 1$ for any a. Identity (b) follows from combining (1) and (2) to get

$$M_{a_1} \cdots M_{a_n} M_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_n & -p_{n-1} \\ q_n & -q_{n-1} \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x p_n - p_{n-1} \\ x q_n - q_{n-1} \end{pmatrix}.$$

The identities in (c) can easily be proven by induction using the observation that

$$\det \begin{pmatrix} a_1 & -1 & 0 \\ -1 & \ddots & -1 \\ 0 & -1 & a_n \end{pmatrix} = a_1 \det \begin{pmatrix} a_2 & -1 & 0 \\ -1 & \ddots & -1 \\ 0 & -1 & a_n \end{pmatrix} - \det \begin{pmatrix} a_3 & -1 & 0 \\ -1 & \ddots & -1 \\ 0 & -1 & a_n \end{pmatrix}. \qquad \square$$

The following theorem is the key technical result of this paper.

Theorem 6. *Let $\iota : (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \to (\mathbb{Z}^m, \mathrm{Id})$ be a map of lattices, where $m > 0$ and Γ is a disjoint union of weighted linear chains representing fractions $\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k} \in \mathbb{Q}_{>1}$. Suppose that there is a unit vector $w \in (\mathbb{Z}^m, \mathrm{Id})$ which pairs nontrivially with* (the image of) *the starting vertex of each linear chain. Then*

$$\sum_{i=1}^{k} \frac{q_i}{p_i} \leq 1.$$

Moreover, if we have equality then w has pairing ± 1 with the starting vertex of each linear chain.

Proof. Let $\{e_1, \dots, e_m\}$ denote the orthonormal basis of coordinates vectors of $(\mathbb{Z}^m, \mathrm{Id})$. Since the unit vectors in $(\mathbb{Z}^m, \mathrm{Id})$ are precisely those vectors of the form $\pm e_i$ where $i \in \{1, \dots, m\}$, by a change of basis if necessary, we may assume that $w = e_1 \in (\mathbb{Z}^m, \mathrm{Id})$. Write $\iota : (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \to (\mathbb{Z}^m, \mathrm{Id})$ as an integer matrix with respect to the vertex basis of $(\mathbb{Z}^{|\Gamma|}, Q_\Gamma)$, and let M be the transpose of this matrix. Since ι preserves intersection pairings we have, $u^T Q_\Gamma v = \iota(u)^T \iota(v) = (M^T u)^T (M^T v) = u^T M M^T v$ for all $u, v \in (\mathbb{Z}^{|\Gamma|}, Q_\Gamma)$. Thus,

$$MM^T = Q_\Gamma = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k \end{pmatrix},$$

where for each $i \in \{1, \dots, k\}$, A_i on the diagonal represents a block matrix of the form

$$A_i = \begin{pmatrix} a_1^i & -1 & 0 & 0 \\ -1 & a_2^i & -1 & 0 \\ 0 & -1 & \ddots & -1 \\ 0 & 0 & -1 & a_{h_i}^i \end{pmatrix},$$

where $[a_1^i, \dots, a_{h_i}^i]^-$ is the continued fraction expansion for p_i/q_i. If a matrix A can be written as a product $M'M'^T$, then[1]

$$\det A \geq 0. \tag{3}$$

We will prove the theorem by applying (3) to a matrix of the form $A = M'M'^T$, where M' is a suitable modification of M.

We may write M in the form

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix},$$

where for all $i \in \{1, \dots, k\}$, M_i is a matrix such that $M_i M_i^T = A_i$. By the assumption that e_1 pairs nontrivially with each of the starting vertices of the linear chains, we may assume that each matrix M_i is nonzero in its top left entry. For each $i \in \{1, \dots, k\}$,

[1] Let $v \neq 0$ be an eigenvector of A with eigenvalue λ. We have $v^T A v = \lambda \|v\|^2 = \|M'^T v\|^2 \geq 0$, thus $\lambda \geq 0$. Since $\det A$ is a product of eigenvalues, this implies that $\det A \geq 0$ as required.

let $p_i/q_i = [a_1^i, \dots, a_{h_i}^i]^-$ be the standard continued fraction expansion and choose M_i' to be the submatrix of M_i obtained by taking the first l_i rows, where l_i is chosen so that the first column w_i of M_i' takes one of the two forms as follows:

(Form 1) If $a_1^i = 2$ and the first column of M_i is of the form $\pm\begin{pmatrix}1 & -1 & \cdots\end{pmatrix}^T$ then we may take $w_i = \pm\begin{pmatrix}1 & -1 & 0 & \cdots & 0 & v\end{pmatrix}^T$, where $v = 0$ only if $M_i' = M_i$.

(Form 2) Otherwise, we may take w_i to be of the form $w_i = \begin{pmatrix}u & 0 & \cdots & 0 & v\end{pmatrix}^T$, where $v = 0$ only if $M_i' = M_i$.

Let M' be the matrix

$$M' = \begin{pmatrix} 1\,0\,\cdots\,0 \\ M_1' \\ \vdots \\ M_k' \end{pmatrix}.$$

Then the product $A = M'M'^T$ takes the form of the block matrix

$$M'M'^T = \begin{pmatrix} 1 & w_1^T & \cdots & w_k^T \\ w_1 & A_1' & & 0 \\ \vdots & & \ddots & \\ w_k & 0 & & A_k' \end{pmatrix}.$$

Claim. *We can write* $\det A$ *in the form*

$$\det A = (P_1 \cdots P_k)\left(1 - \sum_{i=1}^k \frac{Q_i}{P_i}\right),$$

where $P_i = \det A_i' > 0$ *and*

$$Q_i = -\det\begin{pmatrix} 0 & w_i^T \\ w_i & A_i' \end{pmatrix}$$

is a quantity depending only on A_i' *and* w_i.

Proof. Since Γ represents a positive definite lattice, $P_i > 0$ for all i. By the multilinearity of the determinant in both rows and columns we have

$$\det\begin{pmatrix} 1 & w_1^T & \dots & w_k^T \\ w_1 & A_1' & & 0 \\ \vdots & & \ddots & \\ w_k & 0 & & A_k' \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & A_1' & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & A_k' \end{pmatrix} + \sum_{1\le i,j\le k} \det B_{ij},$$

where B_{ij} is the matrix

$$B_{ij} = \begin{pmatrix} 0 & \cdots & w_j^T & \cdots & 0 \\ \vdots & A'_1 & & & \\ w_i & & \ddots & 0 & \\ \vdots & & 0 & \ddots & \\ 0 & & & & A'_k \end{pmatrix}.$$

Since A'_i has full rank, w_i can be expressed as a rational linear combination of the columns of A'_i, and hence $\det B_{ij} = 0$ for all $i \neq j$. For $i \in \{1, \ldots, k\}$, by row and column operations, we can put B_{ii} into the form of a diagonal block matrix with diagonal blocks $\left(\begin{smallmatrix} 0 & w_i^T \\ w_i & A'_i \end{smallmatrix}\right)$, $A'_1, \ldots, A'_{i-1}, A'_{i+1}, \ldots, A'_k$ without changing the determinant. Hence, $\det B_{ii}$ is the product of the determinants of these blocks, that is, $\det B_{ii} = -(P_1 \cdots P_k)\frac{Q_i}{P_i}$. □

Since $P_i > 0$ for all i, the previous claim combined with $\det A \geq 0$ (see (3)) shows that

$$\sum_{i=1}^{k} \frac{Q_i}{P_i} \leq 1. \tag{4}$$

So to prove the inequality in the theorem it suffices to show that $Q_i/P_i \geq q_i/p_i$ for each $i \in \{1, \ldots, k\}$. To do this it suffices to consider some fixed $i \in \{1, \ldots, k\}$. For convenience, let $P/Q = P_i/Q_i$ and $p/q = p_i/q_i = [a_1, a_2, \ldots, a_h]^-$, where $a_j \geq 2$ for all $j \in \{1, \ldots, h\}$, and let $l = l_i$ be the number of rows of A'_i.

Consider the following identity obtained by adding the second row to the first row, and the second column to the first column:

$$\det \begin{pmatrix} 0 & -1 & 1 & \cdots & v \\ -1 & 2 & -1 & & 0 \\ 1 & -1 & a_2 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ v & 0 & & -1 & a_l \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 0 & \cdots & v \\ 1 & 2 & -1 & & 0 \\ 0 & -1 & a_2 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ v & 0 & & -1 & a_l \end{pmatrix}. \tag{5}$$

Recall that w_i takes one of two possible forms. By applying the above identity if w_i takes the form (Form 1)orm 1, we see that regardless of the form that w_i takes, Q is equal to the determinant of a matrix of the following form

$$Q = -\det \begin{pmatrix} 0 & +u & +0 & \cdots & +v \\ u & +a_1 & -1 & & +0 \\ 0 & -1 & +a_2 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ v & +0 & & -1 & +a_l \end{pmatrix}, \tag{6}$$

where if $(u, v) = (\pm 1, \mp 1)$ then either $l > 2$ or $a_1 > 2$. If w_i takes the form (Form 1)orm 1, we define $u \in \{\pm 1\}$ via Equation (6) by applying the identity in (5).

For $j \in \{1, \dots, h\}$, let r_j/s_j denote the continued fraction $[a_1, \dots, a_j]^-$. Note that $P = r_l$.

Claim. $$Q = u^2 s_l + 2uv + v^2 r_{l-1}$$

Proof. Applying cofactor expansion along the first column and first row in (6) gives

$$Q = u^2 C_1 + (-1)^{l+1} uv C_2 + (-1)^{l+1} uv C_3 + v^2 C_4,$$

where

$$C_1 = \det \begin{pmatrix} a_2 & -1 & \cdots & 0 \\ -1 & a_3 & -1 & \\ \vdots & -1 & \ddots & -1 \\ 0 & & -1 & a_l \end{pmatrix}, \quad C_2 = \det \begin{pmatrix} -1 & a_2 & -1 & \cdots & 0 \\ 0 & -1 & a_3 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ & & & -1 & a_{l-1} \\ 0 & & & & -1 \end{pmatrix},$$

$$C_3 = \det \begin{pmatrix} -1 & 0 & & \cdots & 0 \\ a_2 & -1 & & & \\ -1 & a_3 & -1 & & \\ & -1 & \ddots & -1 & \\ 0 & & & a_{l-1} & -1 \end{pmatrix}, \quad C_4 = \det \begin{pmatrix} a_1 & -1 & \cdots & 0 \\ -1 & a_2 & -1 & \\ \vdots & -1 & \ddots & -1 \\ 0 & & -1 & a_{l-1} \end{pmatrix}.$$

Using the continued fraction identities in Lemma 5, we see that $C_1 = s_l$ and $C_4 = r_{l-1}$. Finally, notice that C_2 (resp. C_3) is the determinant of an upper (resp. lower) triangular matrix with $l - 1$ diagonal entries all of which are -1, hence $C_2 = C_3 = (-1)^{l-1}$. □

Claim. *We have $Q/P \geq q/p$ with equality only if $u = \pm 1$ and $v = 0$.*

Proof. Recall that if $v = 0$ then $r_l/s_l = p/q$. Thus if $v = 0$, $Q/P = u^2 q/p$. Since $u \neq 0$, we clearly have $Q/P \geq q/p$ with equality only if $u^2 = 1$, as required.

From now on assume that $v \neq 0$. Notice that if $l = 1$, then we necessarily have $v = 0$. Thus we can further assume that $l \geq 2$. This implies that $s_l \geq 2$ and $r_{l-1} \geq 2$. By completing the square, it follows from the preceding claim that Q can be written in the form

$$Q = (s_l - 1)u^2 + (u + v)^2 + (r_{l-1} - 1)v^2. \tag{7}$$

Since u and v are nonzero integers, it follows from (7) that

$$Q \geq s_l + r_{l-1} - 2.$$

If $r_{l-1} = 2$, then this inequality can be strengthened. It follows from the uniqueness of the negative continued fraction expansions that $r_{l-1} = 2$ only if $l = 2$ and $a_1 = 2$.

By the condition stated immediately after (6) this implies that we do not have $u = -v \in \{\pm 1\}$. Thus we have that either $|u| \geq 2$, $|v| \geq 2$ or $u + v \neq 0$. In any of these eventualities it follows from (7) that

$$Q \geq s_l + r_{l-1} - 1.$$

By combining these two inequalities, we see that there is $\varepsilon \in \{1, 2\}$ such that

$$Q \geq s_l + r_{l-1} - \varepsilon \quad \text{and} \quad r_{l-1} - \varepsilon \geq 1. \tag{8}$$

If $l = h$, or equivalently, $p/q = r_l/s_l$, then $P = p$ and (8) implies that $Q > q = s_l$. It follows that $Q/P > q/p$ in this case. For the rest of the proof, we assume that $p/q \neq r_l/s_l$ and, in particular, that $p/q = [a_1, \dots, a_l, x]^-$, where $x = [a_{l+1}, \dots, a_h]^- > 1$. By Lemma 5 we have

$$\frac{p}{q} = \frac{x r_l - r_{l-1}}{x s_l - s_{l-1}}.$$

Using this identity and (8) we obtain

$$\begin{aligned} \frac{Q}{P} - \frac{q}{p} &\geq \frac{s_l + r_{l-1} - \varepsilon}{r_l} - \frac{x s_l - s_{l-1}}{x r_l - r_{l-1}} \\ &= \frac{(r_{l-1} - \varepsilon)(x r_l - r_{l-1}) + r_l s_{l-1} - s_l r_{l-1}}{r_l (x r_l - r_{l-1})} \\ &= \frac{(r_{l-1} - \varepsilon)(x r_l - r_{l-1}) - 1}{r_l (x r_l - r_{l-1})} \\ &\geq \frac{(x r_l - r_{l-1}) - 1}{r_l (x r_l - r_{l-1})} \\ &> 0, \end{aligned} \tag{9}$$

where we used the identity $r_l s_{l-1} - s_l r_{l-1} = -1$ from Lemma 5 to obtain the third line, $r_{l-1} - \varepsilon \geq 1$ to obtain the fourth line, and finally that $x r_l - r_{l-1} > 1$, which follows from the facts that $r_l \geq r_{l-1} + 1$ and $x > 1$. This gives the desired inequality, proving the claim. □

The claim together with (4) proves that $\sum_{i=1}^{k} \frac{q_i}{p_i} \leq 1$ with equality only if $w = e_1$ has pairing ± 1 with each starting vertex, which completes the proof. □

4. Definite 4-manifolds and the Seifert fibered space inequality

Now we consider when gluing two 4-manifolds can result in a closed definite 4-manifold.

Proposition 7. *Let U_1 and U_2 be connected 4-manifolds with $\partial U_1 = -\partial U_2 = Y$. Then the closed 4-manifold $X = U_1 \cup_Y U_2$ is positive definite if and only if*

(a) *the inclusion-induced map* $(i_1)_* \oplus (i_2)_* : H_1(Y;\mathbb{Q}) \to H_1(U_1;\mathbb{Q}) \oplus H_1(U_2;\mathbb{Q})$ *is injective*, *and*

(b) *for* $i = 1, 2$, U_i *is positive semidefinite*, *or equivalently has signature*

$$\sigma(U_i) = b_2(U_i) + b_1(U_i) - b_3(U_i) - b_2(Y).$$

Proof. In this proof all homology groups will be taken with rational coefficients. First, for $i = 1, 2$, consider the following segment of the long exact sequence in homology of the pair (U_i, Y):

$$0 \to H_3(U_i) \to H_3(U_i, Y) \to H_2(Y) \to H_2(U_i) \xrightarrow{Q} H_2(U_i, Y), \tag{10}$$

where Q can be represented by the intersection form matrix with respect to suitable bases. Hence, the null space of Q is precisely the image of $H_2(Y) \to H_2(U_i)$. By exactness and Lefschetz duality, the rank of the map $H_2(Y) \to H_2(U_i)$ is $b_2(Y) - b_1(U_i) + b_3(U_i)$. This gives an upper bound on the signature of U_i:

$$\sigma(U_i) \le b_2(U_i) + b_1(U_i) - b_3(U_i) - b_2(Y), \tag{11}$$

with equality if and only if U_i is positive semidefinite. Now consider the segment of the Mayer–Vietoris sequence

$$\begin{aligned} 0 \to H_3(U_1) \oplus H_3(U_2) \to H_3(X) \to H_2(Y) \to H_2(U_1) \oplus H_2(U_2) \\ \to H_2(X) \to H_1(Y) \to H_1(U_1) \oplus H_1(U_2) \to H_1(X) \to 0. \end{aligned} \tag{12}$$

The last three terms in this sequence show that

$$b_1(U_1) + b_1(U_2) \le b_1(Y) + b_1(X), \tag{13}$$

with equality if and only if the map induced by the inclusions

$$(i_1)_* \oplus (i_2)_* : H_1(Y) \to H_1(U_1) \oplus H_1(U_2)$$

is injective.

Since the Euler characteristic of an exact sequence is zero, (12) shows that

$$b_2(X) = 2b_1(X) + \sum_{i=1}^{2} \big(b_2(U_i) - b_1(U_i) - b_3(U_i)\big), \tag{14}$$

where we also used that $b_1(Y) = b_2(Y)$ and $b_1(X) = b_3(X)$.

By Novikov additivity, we have that $\sigma(X) = \sigma(U_1) + \sigma(U_2)$. So by summing the inequalities in (11) for $i = 1, 2$ and comparing with (14) we obtain

$$b_2(X) \ge 2(b_1(X) + b_2(Y) - b_1(U_1) - b_1(U_2)) + \sigma(X), \tag{15}$$

with equality if and only if we have equality in (11) for both $i = 1, 2$. Hence, X can be positive definite if and only if

$$b_1(U_1) + b_1(U_2) = b_2(Y) + b_1(X) \tag{16}$$

and we have equality in (11) for $i = 1, 2$. However we have already seen that equality occurs in (13) if and only if $(i_1)_* \oplus (i_2)_*$ is injective. $\square$

This allows us to prove the main theorem.

Theorem 1. *Let $Y = S^2\big(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\big)$ be a Seifert fibered space over S^2 in standard form, that is, with $e \geq 0$, $\frac{p_i}{q_i} > 1$ for all $i \in \{1, 2, \dots, k\}$ and $\varepsilon(Y) \geq 0$. Suppose that Y bounds a smooth 4-manifold W such that $\sigma(W) = -b_2(W)$ and the inclusion induced map $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$ is injective. Then there is a partition of $\{1, 2, \dots, k\}$ into at most e classes such that for each class C,*

$$\sum_{i \in C} \frac{q_i}{p_i} \leq 1.$$

Proof. Let X be the standard positive (semi)definite plumbing 4-manifold with $\partial X = Y$, and let $Z = X \cup_Y -W$. It follows from Proposition 7 that Z is positive definite. Condition (b) of Proposition 7 holds, since $\sigma(W) = -b_2(W)$ implies that $-W$ is positive definite. Thus, Z is a smooth positive definite 4-manifold, so by Donaldson's theorem Z has standard positive diagonal intersection form. The inclusion $X \subset Z$ induces a map $H_2(X) \to H_2(Z)$ which preserves the intersection pairing. Thus, there is a map of lattices $(H_2(X), Q_X) \to (\mathbb{Z}^m, \mathrm{Id})$ for some $m > 0$.

We construct a partition of $\{1, 2, \dots, k\}$ into at most e classes as follows. Denote the orthonormal basis of coordinate vectors of $(\mathbb{Z}^m, \mathrm{Id})$ by $\{e_1, \dots, e_m\}$. For $v \in (H_2(X), Q_X)$, we call $\{e_i : 1 \leq i \leq m, e_i \cdot v \neq 0\}$ the support of v. Without loss of generality, we may assume that the central vertex has support $\{e_1, e_2, \dots, e_n\}$, where $n \leq e$. Let $v_1, v_2, \dots, v_k$ be the vertices of the plumbing adjacent to the central vertex, so that v_i is a vertex belonging to the i-th leg of the plumbing graph (with fraction p_i/q_i). For $i \in \{1, \dots, n\}$, let $B_i = \{1 \leq j \leq k \mid v_j \cdot e_i \neq 0\}$ and define $B_0 = \varnothing$. Let $C_i = B_i \setminus \bigcup_{j<i} B_j$ for $i \in \{1, \dots, n\}$. Then $C_1, \dots, C_n$ are disjoint and $\bigcup_i C_i = \{1, \dots, k\}$. Thus the nonempty classes $\{C_i : C_i \neq \varnothing\}$ form a partition of $\{1, 2, \dots, k\}$ into at most e classes. By definition for each $i \in \{1, 2, \dots, n\}$, the starting vertices of the linear chains indexed by C_i all have support containing the common unit vector e_i. Hence, by Theorem 6, we have that $\sum_{j \in C_i} \frac{q_j}{p_j} \leq 1$. $\square$

5. The Neumann–Zagier question

We prove Theorem 8 below which, when combined with Donaldson's theorem, immediately implies Theorem 2. Note that the following theorem also positively answers the Neumann–Zagier question stated in the introduction.

Theorem 8. *Let $Y = S^2\big(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\big)$, $k \geq 3$, be in standard form, that is, with $\frac{p_i}{q_i} > 1$ for all $i \in \{1, 2, \dots, k\}$, $e > 0$ and with Y bounding a smooth positive definite plumbing X. Suppose that $|H_1(Y)| \in \{1, 2, 3, 5, 6, 7\}$ and the intersection lattice $(H_2(X), Q_X)$ embeds into a positive standard diagonal lattice. Then $e = 1$.*

Proof. For sake of contradiction, assume that $e > 1$. We may apply Theorem 1, noting that the existence of W in the hypothesis of Theorem 1 is only required to ensure that there is a map of lattices of $(H_2(X), Q_X)$ into a positive standard diagonal lattice. Hence, there is a partition $\{C_1, \dots, C_n\}$ of $\{1, \dots, k\}$ into $n \leq e$ classes. Moreover, for each class C, $1 - \sum_{i \in C} \frac{q_i}{p_i} \geq 0$, and we call C *complementary* if equality occurs, and *noncomplementary* otherwise.

We have

$$
\begin{aligned}
(17) \qquad |H_1(Y)| &= p_1 \cdots p_k \cdot \varepsilon(Y) = p_1 \cdots p_k \left(e - \sum_{i=1}^{k} \frac{q_i}{p_i} \right) \\
&= p_1 \cdots p_k \left((e-n) + \sum_{i=1}^{n} \left(1 - \sum_{j \in C_i} \frac{q_j}{p_j} \right) \right) \\
&= p_1 \cdots p_k (e-n) + \sum_{i=1}^{n} a_i \prod_{\substack{1 \leq l \leq k \\ l \notin C_i}} p_l,
\end{aligned}
$$

where $a_i = \big(\prod_{j \in C_i} p_j\big) \cdot \big(1 - \sum_{l \in C_i} \frac{q_l}{p_l}\big)$ is an integer for all $i \in \{1, \dots, n\}$. Notice that all terms in (17) are nonnegative integers. Since we are assuming that $|H_1(Y)| \in \{1, 2, 3, 5, 6, 7\}$, we must have $n = e$, otherwise $|H_1(Y)| \geq p_1 \cdots p_k \geq 2 \cdot 2 \cdot 2 = 8$ since $k \geq 3$.

We claim that $|C_i| \leq k-2$ for some $i \in \{1, \dots, e\}$ with C_i noncomplementary. To see this, we argue as follows. There are $n = e \geq 2$ classes in the partition, and at least one noncomplementary class since $|H_1(Y)| > 0$. If there are two noncomplementary classes then at least one has size at most $k-2$ since $k \geq 3$. If there is only one noncomplementary class, then there is a complementary class which necessarily has size at least 2, and hence the noncomplementary class satisfies the claim.

Combining the above claim with (17), we see that $|H_1(Y)|$ is a sum of integers greater than 1, and at least one of these integers is not prime. For $|H_1(Y)| \in \{1, 2, 3, 5, 6, 7\}$, this is only possible for $|H_1(Y)| = 7$ with decomposition $7 = 3 + 2 \cdot 2$, and for $|H_1(Y)| = 6$ with the two decompositions $|H_1(Y)| = 2 + 2 \cdot 2 = 2 \cdot 3$. We address these cases in turn. For $|H_1(Y)| = 7 = 3 + 2 \cdot 2$, comparing this decomposition with (17), we see that there must exist some noncomplementary C_i with $|C_i| = 2$ and $p_j = 2$ for all $j \in C_i$. However, such a C_i must be complementary since $1 - \frac{1}{2} - \frac{1}{2} = 0$, a contradiction. A similar argument rules out the decomposition $|H_1(Y)| = 2 + 2 \cdot 2$. Finally, in the case $|H_1(Y)| = 2 \cdot 3$, the decomposition implies that there exists a complementary class $C_i = \{a, b\}$ with $p_a = 2$ and $p_b = 3$, which is impossible. □

We obtain the following corollary, answering a question of Lidman and Tweedy [2018, Remark 4.3].

Corollary 3. *Let Y be a Seifert fibered integral homology sphere, and let $e \in \mathbb{Z}$ be the central weight in the standard definite plumbing graph for Y. If $|e| \neq 1$, then $d(Y) \neq 0$.*

Proof. We prove the contrapositive. Assume that $d(Y) = 0$. Note that reversing the orientation of Y simply changes the sign of the weight of the central vertex in the definite plumbing bounding Y. Thus, by reversing the orientation of Y if necessary we assume that Y bounds a smooth negative definite plumbing X^4. Let

$$\mathcal{C} = \left\{\xi \in H_2(X;\mathbb{Z}) \mid \xi \cdot v = v \cdot v \pmod 2 \text{ for all } v \in H_2(X;\mathbb{Z})\right\}$$

be the set of characteristic vectors, and let $n = \mathrm{rk}(H_2(X))$. Elkies [1995] proved that $0 \leq n + \max_{\xi \in \mathcal{C}} \xi \cdot \xi$, with equality if and only if Q_X is diagonalizable over $\mathbb{Z}$. However, it follows from [Ozsváth and Szabó 2003, Theorem 9.6] that $n + \max_{\xi \in \mathcal{C}} \xi \cdot \xi \leq 4d(Y) = 0$. Therefore Q_X is diagonalizable over $\mathbb{Z}$, in particular $(H_2(-X), Q_{-X})$ embeds into a positive standard diagonal lattice. Hence, Theorem 8 implies that $|e| = 1$. □

6. Seifert fibered spaces bounding rational homology $S^1 \times D^3$'s

In this section we prove Theorem 4, which in particular gives a classification of the Seifert fibered spaces which smoothly bound rational homology $S^1 \times D^3$'s. We note that the implication (2) implies (1) was proved by Donald [2015, Proof of Theorem 1.3], and the equivalence of (1) and (2) was shown by Aceto [2015, Theorem 1.2].

Theorem 4. *Let Y be a Seifert fibered space over S^2 with*

$$H_*(Y;\mathbb{Q}) \cong H_*(S^1 \times S^2;\mathbb{Q}).$$

The following are equivalent:

(1) *Y is of the form $S^2\big(k; \frac{p_1}{q_1}, \frac{p_1}{p_1-q_1}, \dots, \frac{p_k}{q_k}, \frac{p_k}{p_k-q_k}\big)$, where $k \geq 0$ and $\frac{p_i}{q_i} > 1$ for all $i \in \{1, \dots, k\}$.*

(2) *$Y = \partial W$, where W is a smooth 4-manifold with $H_*(W;\mathbb{Q}) \cong H_*(S^1 \times D^3;\mathbb{Q})$.*

(3) *Y is the boundary of smooth 4-manifolds W_+ and W_- such that $\sigma(W_\pm) = \pm b_2(W_\pm)$ and each of the inclusion-induced maps $H_1(Y;\mathbb{Q}) \to H_1(W_\pm;\mathbb{Q})$ is injective.*

Proof. First suppose that (1) holds, that is, $Y = S^2\big(k; \frac{p_1}{q_1}, \frac{p_1}{p_1-q_1}, \dots, \frac{p_k}{q_k}, \frac{p_k}{p_k-q_k}\big)$, where $k \geq 0$ and $\frac{p_i}{q_i} \in \mathbb{Q}_{>1}$ for all $i \in \{1, \dots, k\}$. By Rolfsen twisting, Y can be put into the form $S^2\big(0; \frac{p_1}{q_1}, -\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}, -\frac{p_k}{q_k}\big)$. Let $M = S^2\big(0; \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_k}{q_k}\big)$, let M° be the 3-manifold with torus boundary given by removing a tubular neighborhood of a regular fiber of M and let $W = M \times [0, 1]$. Then $\partial W = M^\circ \cup_\partial -M^\circ$ is the double of M°, which is precisely Y. Finally, notice that

$$H_*(W; \mathbb{Q}) = H_*(M^\circ; \mathbb{Q}) = H_*(S^1 \times D^3; \mathbb{Q}),$$

where the last equality follows from the fact that M is a rational homology S^3 and M° is obtained by removing a neighborhood of a simple closed curve from M. This proves (2).

The implication (2) implies (3) holds by taking $W_\pm = W$ and noting that $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$ is injective by the long exact sequence of the pair (Y, W).

Finally assume that (3) holds. Hence, Y is the boundary of smooth 4-manifolds W_+ and W_- satisfying $\sigma(W_\pm) = \pm b_2(W_\pm)$ and such that the inclusion induced maps $H_1(Y) \to H_1(W_\pm)$ are injective. Write Y as $S^2\big(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\big)$ with $k \geq 3$ and $\frac{p_i}{q_i} \in \mathbb{Q}_{>1}$ for all $i \in \{1, \dots, k\}$. Notice that $-Y = S^2\big(k - e; \frac{p_1}{p_1-q_1}, \dots, \frac{p_k}{p_k-q_k}\big)$, and Y is of the form given in (1) if and only if $-Y$ is of this form. Thus, by reversing the orientations of both Y and $W_\pm$ if necessary, we may assume that $e \geq \frac{k}{2}$.

By Theorem 1, there is a partition $\{C_1, \dots, C_n\}$ of $\{1, \dots, k\}$ into $n \leq e$ classes such that for each class C, $1 - \sum_{i \in C} \frac{q_i}{p_i} \geq 0$. Since Y is a rational homology $S^1 \times S^2$, we thus have

$$\begin{aligned} 0 = p_1 \cdots p_k \cdot \varepsilon(Y) &= p_1 \cdots p_k \left(e - \sum_{i=1}^{k} \frac{q_i}{p_i} \right) \\ &= p_1 \cdots p_k \left((e - n) + \sum_{i=1}^{n} \left(1 - \sum_{j \in C_i} \frac{q_j}{p_j} \right) \right), \end{aligned}$$

where all terms in the sum are nonnegative. Hence, we must have $n = e$ and $1 - \sum_{i \in C} \frac{q_i}{p_i} = 0$, for all $i \in \{1, \dots, n\}$. This implies that $|C_i| \geq 2$ for all $i \in \{1, \dots, n\}$. Thus, there are at least $2n = 2e$ legs, so $e \leq \frac{k}{2}$. However, by assumption $e \geq \frac{k}{2}$ so $e = \frac{k}{2}$ and $|C_i| = 2$ for all $i \in \{1, \dots, n\}$. Thus, $C_1, \dots, C_n$ partition $\{1, \dots, k\}$ into pairs of indices indexing pairs of fractions of the form $\frac{p}{q}, \frac{p}{p-q} \in \mathbb{Q}_{>1}$, and thus (1) holds. □

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References

[Aceto 2015] P. Aceto, "Rational homology cobordisms of plumbed 3-manifolds", preprint, 2015. arXiv

[Donald 2015] A. Donald, "Embedding Seifert manifolds in S^4", *Trans. Amer. Math. Soc.* **367**:1 (2015), 559–595. MR Zbl

[Donaldson 1987] S. K. Donaldson, "The orientation of Yang–Mills moduli spaces and 4-manifold topology", *J. Differential Geom.* **26**:3 (1987), 397–428. MR Zbl

[Elkies 1995] N. D. Elkies, "A characterization of the $\mathbb{Z}^n$ lattice", *Math. Res. Lett.* **2**:3 (1995), 321–326. MR Zbl

[Fintushel and Stern 1985] R. Fintushel and R. J. Stern, "Pseudofree orbifolds", *Ann. of Math.* (2) **122**:2 (1985), 335–364. MR Zbl

[Issa and McCoy 2018] A. Issa and D. McCoy, "Smoothly embedding Seifert fibered spaces in S^4", preprint, 2018. arXiv

[Lecuona and Lisca 2011] A. G. Lecuona and P. Lisca, "Stein fillable Seifert fibered 3-manifolds", *Algebr. Geom. Topol.* **11**:2 (2011), 625–642. MR Zbl

[Lidman and Tweedy 2018] T. Lidman and E. Tweedy, "A note on concordance properties of fibers in Seifert homology spheres", *Canad. Math. Bull.* **61**:4 (2018), 754–767. MR Zbl

[Lisca 2007] P. Lisca, "Sums of lens spaces bounding rational balls", *Algebr. Geom. Topol.* **7** (2007), 2141–2164. MR Zbl

[Neumann 1989] W. D. Neumann, "On bilinear forms represented by trees", *Bull. Austral. Math. Soc.* **40**:2 (1989), 303–321. MR Zbl

[Neumann and Raymond 1978] W. D. Neumann and F. Raymond, "Seifert manifolds, plumbing, μ-invariant and orientation reversing maps", pp. 163–196 in *Algebraic and geometric topology* (Santa Barbara, CA, 1977), edited by K. C. Millett, Lecture Notes in Math. **664**, Springer, Berlin, 1978. MR Zbl

[Neumann and Zagier 1985] W. D. Neumann and D. Zagier, "A note on an invariant of Fintushel and Stern", pp. 241–244 in *Geometry and topology* (College Park, MD, 1983/84), edited by J. Alexander and J. Harer, Lecture Notes in Math. **1167**, Springer, Berlin, 1985. MR Zbl

[Ozsváth and Szabó 2003] P. Ozsváth and Z. Szabó, "Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary", *Adv. Math.* **173**:2 (2003), 179–261. MR Zbl

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