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We introduce the notion of a twisted differential operator of given radius relative to an endomorphism σ of an affinoid algebra A. We show that this notion is essentially independent of the choice of the endomorphism σ. As a particular case, we obtain an explicit equivalence between modules endowed with a usual integrable connection (i.e., differential systems) and modules endowed with a σ-connection of the same radius (this concept generalizes both finite difference and q-difference systems). Moreover, this equivalence preserves cohomology and in particular solutions.

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Introduction

Confluence is the process that consists in replacing a differential equation with a finite difference or q-difference equation in order to get a good approximation of the solutions. In ultrametric analysis, it happens that one can do a lot better: with some mild extra hypothesis, there exists a finite difference or q-difference equation (or more generally, as we will see, a *difference equation*) that has exactly the *same* solutions. Actually, there exists a one to one correspondence between differential

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equations on one side and difference equations on the other (that induces an equality on the solutions).

The last years have seen a lot of progress in this direction. For example, working over the Robba ring (which can be thought of as an analog with respect to an algebraic curve of the usual formal completion of the local ring at a point), a celebrated result of Yves André [2002] states that a linear differential equation admitting a Frobenius action can be trivialized by an étale extension of the Robba ring to which one adds a formal logarithm. Among other applications, this permits the computation of the tannakian group of the category of those differential equations, thus providing a complete classification of these objects.

Shortly after, André and Lucia Di Vizio [2004] computed the tannakian group of q-difference equations (with Frobenius) over the Robba ring, and proved the surprising and unexpected fact that it is the same as the tannakian group of differential equations over that ring. As a corollary, they obtained an equivalence of both categories, which André and Di Vizio were in fact able to compute explicitly. More precisely, they found a finite étale extension of the Robba ring trivializing simultaneously the given differential equation and the associated q-difference equation, proving as a consequence that the differential equation and the q-difference equation have the same solutions in that finite étale extension. One of the tools they used was the systematic theory of p-adic q-difference equations developed by Di Vizio [2004], including the introduction of a twisted Taylor formula.

Not much later, Andrea Pulita [2008] could prove a general equivalence between differential equations and q-difference equations on an affinoid open subset of the line, doing also away with the need for a Frobenius action. His main idea was to use formal solutions as a bridge between the two worlds, remarking that the equality of solutions remains true for convergent Taylor solutions, and that this fact alone is sufficient to recover the equivalence. More recently Pulita [2017] has improved this result in various directions. In particular, replacing the multiplication by q by an automorphism sufficiently close to the identity, he has extended the equivalence to a general curve. One basic tool used by Pulita is to consider the ring of functions that converge on a tube around the diagonal of the product of the curve with itself.

Our approach consists in generalizing the notion of differential operator, as we did in the algebraic situation in [Le Stum and Quirós 2018a], so that it includes, besides the usual differential operators, both finite difference and q-difference operators.

We start with an arbitrary affinoid algebra R over a fixed nontrivially valued complete ultrametric field K and an affinoid R-algebra A endowed with an (automatically continuous) endomorphism σ. This is what we call a *twisted affinoid R-algebra* and, as Pulita, we think of the twist σ as a deformation (that for us can be almost arbitrary) of the identity. We are in the realm of rigid geometry and it

is possible to define twisted differential operators in the spirit of Grothendieck's EGA-IV. Namely, twisted differential operators will be the dual of appropriate spaces of functions defined on tubes around the diagonal. Let us remark that our constructions may be glued and applied to (suitable) *twisted rigid analytic varieties*.

Our main results on the solutions of twisted differential equations will hold on a twisted affinoid R-algebra A of relative dimension one that admits what we call a *σ-coordinate* x, a concept that generalizes the notion of étale coordinate x when A is smooth over R and $\sigma = \mathrm{Id}_A$. In particular, the cases $\sigma(x) = x + h$ and $\sigma(x) = qx$ will provide finite difference and q-difference operators respectively (they can be unified, and called simply *difference equations*, by using $\sigma(x) = qx + h$), while $\sigma = \mathrm{Id}_A$ gives the usual differential operators *à la Grothendieck*.

With some conditions on A and σ, one can define the ring $D^{(\eta)}_{A/R,\sigma}$ of σ-differential operators of radius η on A. We show in Theorem 6.3 that if τ is any other such R-endomorphism of A, then there exists a canonical isomorphism

$$D^{(\eta)}_{A/R,\sigma} \simeq D^{(\eta)}_{A/R,\tau}.$$

Again, this is not isolated from previous work, since we also use twisted Taylor morphisms. In fact our key tool to prove the equality is to show, in Proposition 6.1 (which is quite similar to Lemma 4.3.3 of [Pulita 2017]), that the Taylor structures for η and τ coincide. On the other hand, our isomorphism is totally explicit, in the sense that we have, for example

$$\partial_\sigma = \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} (\sigma(x) - \tau^i(x)) \right) \partial_\tau^{[k]}.$$

From the particular cases $\tau = \mathrm{Id}_A$ and $\sigma(x) = qx + h$, we deduce our confluence Theorem 7.6 (see also theorem 6.3(ii) of [Pulita 2008]). It would be necessary to introduce some more vocabulary in order to state now this theorem in full generality. Nevertheless, as an illustration, we can indicate the following corollary. It makes the link with the original result of André and Di Vizio, as well as with the work of Pulita, since it is very similar to Proposition 15.1 of [André and Di Vizio 2004] both in the content and in the spirit (they show that the Tannakian groups are the same, while we prove that the "classification rings" are the same) and also to Theorem 4.3.1 of [Pulita 2017].

Theorem. *Let K be a nontrivially valued complete ultrametric field of characteristic zero and X the closed annulus $r_1 \le |x| \le r$. Let $q, h \in K$ and $\eta \ge 0$ be such that*

$$\|h\| \le \eta < r_1, \quad \|1 - q\| \le \frac{\eta}{r},$$

and q is not *a root of unity. Then, X is globally stable under the endomorphism $\sigma(x) = qx + h$, and there exists a fully faithful functor*

$$\nabla\text{-Mod}^{(\eta^\dagger)}(X) \hookrightarrow \sigma\text{-Mod}(X)$$

from the category of modules with connection whose (naive) *radius of convergence is at least η, to the category of σ-modules.*

We want to emphasize again the fact that, even if this functor already exists in Pulita's work, our methods provide explicit formulas: calling ∇ the connection, we will have

$$\sigma(s) = \sum_{k=0}^{\infty} \frac{((q-1)x+h)^k}{k!} \nabla^k(s).$$

Moreover, not only do we see that the solutions are the same, but we obtain an isomorphism on the full cohomology $H^*_\nabla(M) = H^*_\sigma(M)$ (recall that the solutions are given by the cohomology in degree 0). With the tools we have developed the idea is not hard:

$$H^*_\nabla(M) \simeq \mathrm{Ext}^*_{D^{(\eta^\dagger)}}(A, M) \simeq \mathrm{Ext}^*_{D^{(\eta^\dagger)}_\sigma}(A, M) \simeq H^*_\sigma(M).$$

Note that we need here a dagger version of our rings.

Notice also that we exclude the case where q is a root of unity. Actually, there exists also a formal confluence theorem (Theorem 9.13 of [Le Stum and Quirós 2018a]) in this situation but it is a lot more technical. This is however very interesting because, as the first author showed with Michel Gros in [Gros and Le Stum 2014] (see also the more recent [Gros et al. 2019]), there exists a quantum Simpson correspondence when q is a root of unity. One can hope that an ultrametric version of these theorems could provide an ultrametric Simpson correspondence. One could also use the analytic density lemma (Lemma 5.2 below) and the Azumaya nature of the ring of quantum differential operators in order to attack Dixmier's conjecture (Problem 11.1 of [Dixmier 1968]). Let us also mention that some constructions that appear in recent work of Bhatt, Morrow and Scholze [Bhatt et al. 2018] on integral p-adic Hodge Theory are examples of the concepts we discuss, and our explicit formulas might be useful to provide explicit objects in their theory and can also serve as computational tests of their results. We intend to come back to these questions in the future.

To wrap up this introduction, we summarize what we consider to be the main improvements with respect to previous work, in particular [Di Vizio 2004; André and Di Vizio 2004; Pulita 2008; 2017].

- The theory developed in this paper works over any nontrivially valued complete ultrametric field K, while only characteristic zero was considered by André, Di Vizio and Pulita.

- Our deformation endomorphisms σ need not be bijective.
- We do not work only on curves over K, but on affinoid R-algebras of relative dimension one over an arbitrary affinoid K-algebras R, allowing the treatment step by step of higher dimensional varieties.
- We take into account general D-modules, while André, Di Vizio and Pulita only considered vector bundles with connection.
- We prove that the entire de Rham cohomology is preserved by deformation in any degree (solutions are indeed the de Rham cohomology in degree zero).
- We have managed to provide explicit formulas of a combinatorial nature.

Notation and conventions

We fix a nontrivially valued complete ultrametric field K and refer the reader to [Bosch et al. 1984] for the theory of affinoid algebras. When we consider a real number, we always assume that some power of it lies in $|K|$. We recall that an *affinoid algebra* is a quotient A (by an ideal) of a *Tate algebra*

$$K\{T_1, \dots, T_N\} := \left\{ \sum_{i \geq 0} a_{\underline{i}} T^{\underline{i}}, \quad |a_i| \to 0 \right\}.$$

The quotient norm (of the Gauss norm) turns A into a Banach algebra and any morphism of affinoid algebras is automatically continuous (and any ideal is closed) for any Banach structures. We recall that a *finite A-module* is a quotient M (by a submodule) of a free module A^r. The quotient norm turns it into a Banach A-module and any morphism between finite A-modules is continuous (and any submodule is closed) for any Banach structure. Note however that tensor product of affinoid algebras has to be replaced by *completed tensor product* in order to stay inside the category of affinoid algebras.

Throughout the article, R denotes an affinoid K-algebra (implicitly endowed with a fixed Banach norm), that will play the role of a base.

We also want to recall from Definition 2.5 of [Le Stum and Quirós 2015], that the *quantum binomial coefficients* are defined by the *quantum Pascal identities* (in a ring R for some $q \in R$)

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$$

for $n, k \geq 1$ with initial conditions

$$\binom{n}{0}_q = 1 \quad \text{and} \quad \binom{0}{k}_q = 0 \quad \text{for } k \neq 0.$$

We will also consider the *quantum integers* $(n)_q := \binom{n}{1}_q$ and the *quantum factorials* $(n)_q! := (n)_q(n-1)_q \cdots (2)_q(1)_q$. We will use the prefix "q-" instead of the attribute *quantum* when we want to specify the parameter.

1. Twisted calculus

We explain here how some material introduced in [Le Stum and Quirós 2018b; 2018a] has to be modified (or not) in order to work in the setting of affinoid algebras. We recall that R denotes an affinoid algebra over a fixed nontrivially valued complete ultrametric field K.

1A. *Twisted affinoid algebras.* We define a *twisted affinoid R-algebra* as an affinoid R-algebra A endowed with an (automatically continuous) endomorphism σ. They form a category in the obvious way. A Banach norm on a twisted affinoid algebra will be called *contractive* if σ is contractive for this norm. For example, when A is reduced, then the spectral norm is contractive no matter what σ is.

Lemma 1.1. *If A is a twisted affinoid R-algebra and $\pi : R\{T_1, \dots, T_N\} \twoheadrightarrow A$ is a surjective homomorphism of R-algebras, then the following are equivalent*:

(1) *the quotient norm is contractive*,

(2) *σ lifts to $R\{T_1, \dots, T_N\}$*.

Proof. The second condition means that there exists for all $i = 1, \dots, N$, $F_i \in R\{T_1, \dots, T_N\}$ such that $\|F_i\| \le 1$ and $(\sigma \circ \pi)(T_i) = \pi(F_i)$. This is equivalent to saying that, for the quotient norm, we have for all $i = 1, \dots, N$, $\|(\sigma \circ \pi)(T_i)\| \le 1$. This means that σ is contractive. □

Examples. (i) Let

$$A := R\{x/r, r_1/x\} := \left\{ \sum_{-\infty}^{\infty} a_n x^n, \quad a_n \in R \text{ and } \begin{cases} \|a_n\| r^n \to 0 & \text{when } n \to \quad \infty \\ \|a_n\| r_1^n \to 0 & \text{when } n \to -\infty \end{cases} \right\}$$

with $0 < r_1 \le r$. This is the ring of functions on the closed relative annulus and it comes with the usual norm

$$\left\| \sum_{-\infty}^{\infty} a_n x^n \right\| := \max\bigl\{ \|a_n\| r^n,\ n \ge 0;\ \|a_n\| r_1^n,\ n \le 0 \bigr\}.$$

Giving an R-algebra endomorphism σ of A amounts to giving an element $\sigma(x) = \sum_{-\infty}^{\infty} a_n x^n \in A$ with the condition $r_1 \le \|\sigma(x)\| \le r$, which can be rewritten

$$r_1 \le \max\bigl\{ \|a_n\| r^n,\ n \ge 0;\ \|a_n\| r_1^n,\ n \le 0 \bigr\} \le r.$$

(ii) As a particular case, we may look for an endomorphism of the form $\sigma(x) = qx + h$ with $q, h \in R$. Then, the conditions become

$$\|q\| \leq 1 \quad \text{and} \quad \|h\| \leq r, \tag{1}$$

and

$$\text{either} \quad \|q\| \geq \frac{r_1}{r} \quad \text{or} \quad \|h\| \geq r_1. \tag{2}$$

(iii) Note that such an endomorphism need not be bijective. Actually, this happens exactly when $q \in R^\times$ and

$$\|q\| = \|q^{-1}\| = 1 \quad \text{and} \quad \|h\| \leq r.$$

1B. ***Twisted modules.*** As we did in [Le Stum and Quirós 2018b; 2018a] in the algebraic situation, one can extend many constructions on affinoid R-algebras (case $\sigma = \mathrm{Id}_A$) to analog constructions on *twisted* affinoid R-algebras (with nontrivial endomorphism σ). We will then use the attribute *twisted* (or the prefix "σ-") and replace it with the word *usual* (or remove the prefix) in the case $\sigma = \mathrm{Id}_A$. Also, we will try to make the notation as light as possible by not mentioning A and R but we will add the subscript "A/R" when necessary.

As a first example, we may consider *twisted A-modules*, which are *finite A-modules* M endowed with a σ-linear endomorphism σ_M. Note that σ_M is automatically continuous since it splits as $M \to \sigma^* M \to M$. We will denote by $\sigma\text{-Mod}(A/R)$ the category of twisted A-modules. In order to better understand these objects, it is convenient to introduce the *twisted polynomial ring* $A[T]_\sigma$: this is the noncommutative polynomial ring with the commutation rule $Tz = \sigma(z)T$ when $z \in A$. Then, there exists an equivalence of categories (σ_M acts as multiplication by T):

$$\sigma\text{-Mod}(A/R) \simeq A\text{-finite } A[T]_\sigma\text{-Mod}$$

Moreover, it induces an isomorphism:

$$\mathrm{H}^*_\sigma(M) := \mathrm{H}^*[M \xrightarrow{1-\sigma_M} M] \simeq \mathrm{Ext}^*_{A[T]_\sigma}(A, M).$$

1C. ***Twisted derivations.*** A *twisted derivation* on A with values in a finite A-module M is an R-linear map $D : A \to M$ satisfying the *twisted Leibniz rule*:

$$\text{for all } z_1, z_2 \in A, \quad D(z_1 z_2) = z_2 D(z_1) + \sigma(z_1) D(z_2).$$

Proposition 1.2. *A twisted derivation D on a twisted affinoid R-algebra A is automatically continuous.*

Proof. Let us fix a surjection $\pi : R\{\underline{T}\} \twoheadrightarrow A$ (we use the standard multiindex notation) and endow A with the quotient norm. We set $\widetilde{D} = D \circ \pi$ and $\tilde{\sigma} = \sigma \circ \pi$.

If we denote by D_0 the restriction of $\widetilde{D}$ to the polynomial ring $R[\underline{T}]$, then we have $\|D_0\| \leq \|\tilde{\sigma}\| \max_{i=1}^{N} \|D_0(T_i)\| < +\infty$. This is proved by induction on the total degree:

$$\|D_0(\underline{T}^{\underline{n}} T_i)\| = \|\pi(T_i) D_0(\underline{T}^{\underline{n}}) + \tilde{\sigma}(\underline{T}^{\underline{n}}) D_0(T_i)\| \leq \max\{\|D_0(\underline{T}^{\underline{n}})\|, \|\tilde{\sigma}\| \|D_0(T_i)\|\}.$$

It follows that D_0 is continuous and extends therefore uniquely to a continuous map $\widehat{D}_0 : R\{\underline{T}\} \to D$. We only have to show now that $\widetilde{D} = \widehat{D}_0$, or equivalently, if we set $E := \widetilde{D} - \widehat{D}_0$, that $E = 0$. It follows from Proposition 3 in Section 7.1.1 of [Bosch et al. 1984] that if $\mathfrak{m}$ is a maximal ideal of $R\{\underline{T}\}$ and $\mathfrak{m}_0 := \mathfrak{m} \cap R[\underline{T}]$, then we have $\mathfrak{m} = \mathfrak{m}_0 R\{\underline{T}\}$ and for all $n \in \mathbb{N}$, $R\{\underline{T}\} = R[\underline{T}] + \mathfrak{m}^n$. Thus, if $F \in R\{\underline{T}\}$, we can write $F = P + QG$ with $P, Q \in R[\underline{T}]$ and $Q \in \mathfrak{m}^n$. Therefore, since the restriction of E to $R[\underline{T}]$ is zero, we see that

$$E(F) = E(P) + \pi(Q)E(G) + \tilde{\sigma}(G)E(Q) = \pi(Q)E(G).$$

Thus, if we consider M as an $R\{\underline{T}\}$-module via π, we have $E(F) \in \mathfrak{m}^n M$. This being true for all $\mathfrak{m}$ and all n, and M being finite over $R\{\underline{T}\}$, it implies that $E(F) = 0$. $\square$

We will denote by $\mathrm{Der}_\sigma(A, M)$ the submodule of all twisted derivations on A with values in the finite A-module M and we will also write $\mathrm{T}_{A,\sigma} := \mathrm{Der}_\sigma(A, A)$.

One can build on this notion and define for a finite A-module M the notion *twisted derivation* D of M: this is an R-linear map $D : M \to M$ satisfying the *twisted Leibniz rule*: there exists a twisted derivation[1] (that we also denote by D) of A such that

$$\text{for all } z \in A,\ s \in M, \quad D(zs) = D(z)s + \sigma(z)D(s).$$

One can check as in Proposition 1.2 that such a twisted derivation is automatically continuous. We will denote by $\mathrm{Der}_\sigma(M)$ the submodule of all twisted derivations of M.

We may also consider the ring of *small twisted differential operators* $\overline{\mathrm{D}}_\sigma$ which is the smallest subring of $\mathrm{End}_{R\text{-cont}}(A) := \mathrm{Hom}_{R\text{-cont}}(A, A)$ that contains both A and $\mathrm{T}_{A,\sigma}$. As an example, the endomorphism σ itself is a small twisted differential operator (because $1 - \sigma$ is a twisted derivation). If M is a $\overline{\mathrm{D}}_\sigma$-module, then it has a natural action by twisted derivations, but such an action does *not* extend in general to a $\overline{\mathrm{D}}_\sigma$-module structure (curvature *and* p-curvature phenomena).

1D. *Linearization.* We will denote by $P := A \widehat{\otimes}_R A$ the completed tensor product of A by itself and by I the ideal of the multiplication map $P \to A$. We will see

[1] A twisted derivation D of A is not uniquely determined by the twisted derivation of M: this only holds up to the annihilator of M.

P as an A-module through the action on the *left* and simply write $z := z \otimes 1$. By contrast, we will write $\tilde{z} = 1 \otimes z$ and call *Taylor map* the morphism

$$\theta : A \to P, \quad z \mapsto \tilde{z}$$

that defines the action on the right. Recall that I is generated as an ideal by the image of the *differentiation map*

$$\mathrm{d} : A \to P, \quad z \mapsto \tilde{z} - z$$

(which is also continuous). We will also need below the *comultiplication map*

$$\delta : A \widehat{\otimes}_R A \to A \widehat{\otimes}_R A \widehat{\otimes}_R A, \quad z_1 \otimes z_2 \mapsto z_1 \otimes 1 \otimes z_2$$

that will be seen as a map $\delta : P \to P \widehat{\otimes}'_A P$ (the $\otimes'$ means that we use the *right* action on the left factor — we will keep this convention throughout).

If M, N are two finite A-modules, then $\mathrm{Hom}_{R\text{-cont}}(M, N)$ is naturally a P-module for $z_1 \tilde{z}_2 \cdot \varphi = z_1 \circ \varphi \circ z_2$ (where z_i stands here for multiplication by z_i). One can remark that $\mathrm{d}z \cdot \varphi = [\varphi, z]$ is then the commutator. Moreover, there exists a *linearization* isomorphism

$$\mathrm{Hom}_{R\text{-cont}}(M, N) \simeq \mathrm{Hom}_{A\text{-cont}}(P \otimes'_A M, N), \quad \varphi \mapsto \widetilde{\varphi}$$

with the relations

$$\varphi(s) = \widetilde{\varphi}(1 \otimes' s) \quad \text{and} \quad \widetilde{\varphi}(f \otimes' s) = (f \cdot \varphi)(s).$$

If $\varphi : M \to N$ and $\psi : L \to M$ are two continuous R-linear maps between finite A-modules, then we have

$$\widetilde{\varphi \circ \psi} : \qquad P \otimes'_A L \xrightarrow{\delta} P \widehat{\otimes}'_A P \otimes'_A L \xrightarrow{\mathrm{Id} \otimes \widetilde{\psi}} P \otimes'_A M \xrightarrow{\widetilde{\varphi}} N.$$

1E. ***Twisted principal parts.*** We extend σ to P by setting $\sigma_P := \sigma_A \widehat{\otimes}_R \mathrm{Id}_A$. Then, we consider for $n \in \mathbb{N}$, the *twisted product*

$$I^{(n)} := I \sigma(I) \cdots \sigma^{n-1}(I)$$

(as ideals) and the *ring of twisted principal parts of order n*: $P_{(n)} := P / I^{(n+1)}$ (we will write $(n)_\sigma$ in place of (n) if we want to make clear the dependence on σ and remove the parenthesis when $\sigma = \mathrm{Id}_A$). We also introduce the *module of twisted differential forms* $\Omega^1_\sigma = I / I^{(2)}$ and the *ring of (all) twisted principal parts* $\widehat{P}_\sigma := \varprojlim P_{(n)}$.

Proposition 1.3. *If A is a twisted affinoid R-algebra, then $P_{A/R,(n)_\sigma}$ is a finite A-module* (for the left structure). *If we assume that σ is finite, then $P_{A/R,(n)_\sigma}$ is also a finite A-module for the right structure.*

Proof. One first checks that there exists, for each $n \in \mathbb{N}$, a short exact sequence

$$0 \longrightarrow \sigma^n(I) \longrightarrow P \longrightarrow A \longrightarrow 0, \qquad z_1\tilde{z}_2 \longmapsto z_1\sigma^n(z_2). \tag{3}$$

Since $I^{(n)}$ is finitely generated as a P-module, it follows that $I^{(n)}/I^{(n+1)} := I^{(n)}/I^{(n)}\sigma^n(I)$ is finitely generated as an A-module. We may then proceed by induction on n. If we consider now the action of A on the right on P, then we need to endow A with its A-module structure via σ^n in order to keep A-linearity in the exact sequence (3). We may then proceed in the same way in order to show that $P_{(n)}$ is also finite for its right structure as long as σ itself is finite. □

Since we need it below, we also want to mention that we always have

$$\sigma^n(\mathrm{d}z) \cdot \varphi = [\varphi, z]_{\sigma^n} := \varphi \circ z - \sigma^n(z) \circ \varphi$$

so that $\|\sigma^n(\mathrm{d}z) \cdot \varphi\| \leq \|\varphi\| \, \|z\|$ when σ is contractive.

Finally, note that if $A \to B$ is any morphism of twisted affinoid R-algebras, then there exists, by functorially, natural morphisms

$$\text{for all } n \in \mathbb{N}, \quad B \otimes_A P_{A/R,(n)_\sigma} \to P_{B/R,(n)_\sigma}$$

and

$$B \otimes_A \Omega^1_{A/R,\sigma} \to \Omega^1_{B/R,\sigma}.$$

1F. ***Twisted connections.*** It follows from Proposition 1.3 that Ω^1_σ is a finite A-module. The (continuous) differentiation map $\mathrm{d} : A \to P$ takes values inside I and we will still denote by $\mathrm{d} : A \to \Omega^1_\sigma$ the composition with the projection. This is the universal continuous twisted derivation in the sense that it induces an isomorphism

$$\mathrm{Hom}_A(\Omega^1_\sigma, M) \simeq \mathrm{Der}_\sigma(A, M)$$

for any finite A-module M.

A *twisted connection* on a finite A-module M is an R-linear map

$$\nabla : M \to M \otimes_A \Omega^1_\sigma$$

that satisfies the twisted Leibniz rule

$$\text{for all } z \in A, \ s \in M, \quad \nabla(zs) = s \otimes d(z) + \sigma(z)\nabla(s).$$

A twisted connection is automatically continuous because it is induced by a $P_{(1)}$-linear map $\epsilon : P_{(1)} \otimes'_A M \to M \otimes_A P_{(1)}$ in the sense that $\epsilon(1 \otimes' s) = s \otimes 1 + \nabla(s)$. Finite A-modules endowed with a twisted connection form a category ∇_σ-Mod(A/R) (with *horizontal maps* as morphisms). Any twisted connection on an A-module M gives rise to a linear action by twisted derivations. Moreover, if Ω^1_σ is a flat A-module, then we obtain an equivalence that preserves cohomology (this is shown

as in Proposition 3.9 of [Le Stum and Quirós 2018a]). This is the case in practice, and we will use the vocabulary of ∇_σ-modules, which is more pleasant.

1G. ***Twisted differential operators.*** Let M, N be two finite A-modules. A *twisted differential operator (of infinite radius)*[2] *of degree at most* n is an R-linear map $\varphi : M \to N$ whose A-linearization $\widetilde{\varphi} : P \otimes'_A M \to N$ factors through $P_{(n)} \otimes'_A M$. Since $P_{(n)}$ is a finite A-module, a twisted differential operator is automatically continuous. We will then denote the induced map by $\widetilde{\varphi}_n : P_{(n)} \otimes'_A M \to N$. We obtain for each $n \in \mathbb{N}$, a finite A-module

$$\mathrm{Diff}^{(\infty)}_{\sigma,n}(M, N) \simeq \mathrm{Hom}_A(P_{(n)} \otimes'_A M, N)$$

and we will write

$$\mathrm{Diff}^{(\infty)}_{\sigma}(M, N) = \bigcup_{n\in\mathbb{N}} \mathrm{Diff}^{(\infty)}_{\sigma,n}(M, N).$$

Note that, by definition, the action of P on $\mathrm{Hom}_{R\text{-cont}}(M, N)$ induces an action of $\widehat{P}_\sigma$ on $\mathrm{Diff}^{(\infty)}_{\sigma}(M, N)$. Actually, a continuous R-linear map φ falls into $\mathrm{Diff}^{(\infty)}_{\sigma,n}(M, N)$ if and only if $\sigma^n(I) \cdot \varphi \subset \mathrm{Diff}^{(\infty)}_{\sigma,n-1}(M, N)$. Twisted differential operators are stable under composition (this is shown as in Proposition 4.6 of [Le Stum and Quirós 2018a]) and this turns $\mathrm{D}^{(\infty)}_{\sigma} := \mathrm{Diff}^{(\infty)}_{\sigma}(A, A)$ into a subring of $\mathrm{End}_{R\text{-cont}}(A)$ containing $\overline{\mathrm{D}}_\sigma$. In particular, any $\mathrm{D}^{(\infty)}_{\sigma}$-module gives rise to a $\overline{\mathrm{D}}_\sigma$-module. But, again, this is not an equivalence in general, even for finite A-modules in characteristic 0 (p-curvature problem when the q-characteristic is positive [Le Stum and Quirós 2015]). Note also that the action of $\mathrm{D}^{(\infty)}_{\sigma}$ on an A-finite $\mathrm{D}^{(\infty)}_{\sigma}$-module M induces a continuous $\widehat{P}_\sigma$-linear ring homomorphism

$$\mathrm{D}^{(\infty)}_{\sigma} \to \mathrm{Diff}^{(\infty)}_{\sigma}(M, M).$$

1H. ***Twisted Taylor structures.*** One can show that the comultiplication map $\delta : P \to P \widehat{\otimes}'_A P$ induces for all $m, n \in \mathbb{N}$ a comultiplication map

$$\delta_{(m),(n)} : P_{(m+n)} \to P_{(m)} \otimes'_A P_{(n)}.$$

A *twisted Taylor structure* on a finite A-module M is a family of compatible *twisted Taylor maps* $\theta_n : M \to M \otimes_A P_{(n)}$. More precisely, we require for θ_n to be A-linear when the target is endowed with the A-module structure coming from the *right* structure of $P_{(n)}$, and compatibility means that

$$\text{for all } m, n \in \mathbb{N}, \quad (\theta_n \otimes \mathrm{Id}_{P_{(m)}}) \circ \theta_m = (\mathrm{Id}_M \otimes \delta_{(n),(m)}) \circ \theta_{m+n}.$$

[2] We prefer to say "infinite radius" (later on we will also talk about "finite radius") rather than "infinite level", which is the term we used in [Le Stum and Quirós 2018a]. The notion of level is simply a logarithmic version of the notion of radius.

We shall then write

$$\hat{\theta} := \varprojlim \theta_n : M \mapsto M \otimes_A \widehat{P}_\sigma$$

and call $\hat{\theta}(s)$ the *twisted Taylor series* of $s \in M$. A twisted Taylor structure on M induces a $D_\sigma^{(\infty)}$-module structure via

$$\varphi s = \sum_{k=0}^{n} \widetilde{\varphi}(f_k) s_k \quad \text{if} \quad \theta_n(s) = \sum_{k=0}^{n} s_k \otimes f_k,$$

when the twisted differential operator φ has order n. Moreover, we obtain an equivalence of categories when all the $P_{(n)}$ are flat A-modules (see Section 5 of [Le Stum and Quirós 2018a]).

1I. ***Twisted coordinate.*** We call $x \in A$ a *twisted coordinate* if all the maps

$$A[\xi]_{\leq n} \to P_{(n)}, \quad \xi \mapsto \overline{\tilde{x} - x} \tag{4}$$

(where the left hand side denotes the set of polynomials of degree at most n) are bijective. We call x a *quantum coordinate* if, moreover, there exists $q, h \in R$ such that $\sigma(x) = qx + h$. We will say *q-coordinate* when we want to make explicit the value of q.

Examples. (i) If A is formally smooth over R and $\sigma = \mathrm{Id}_A$, then a quantum coordinate is the same thing as an étale coordinate.

(ii) It follows from Proposition 1.5 below that if A is the ring of functions on an affinoid domain X of the R-line, x is a (usual) coordinate on the R-line, and $\sigma(x)$ is a convergent power series on X, then x is a twisted coordinate on A (but we still have to check that X is stable under σ). In the particular case $\sigma(x) = qx + h$, this is a quantum coordinate.

(iii) This applies to a *standard affinoid domain* as in [Pulita 2008]: a union of subsets of the form

$$X := \mathbb{D}_R^+(c, r) \Big\backslash \left(\bigcup_{i=1}^{h} \mathbb{D}_R^-(c_i, r_i) \right). \tag{5}$$

In other words, A will be a *standard affinoid algebra*: a product of rings of the form

$$R\left\{ \frac{x-c}{r}, \frac{r_1}{x-c_1}, \dots, \frac{r_h}{x-c_h} \right\}.$$

When $R = K$ is an algebraically closed complete ultrametric field, any affinoid domain of the line is standard.

(iv) The simplest nontrivial case is an annulus centered at the origin

$$X := \mathbb{D}^+(0, r) \setminus \mathbb{D}^-(0, r_1)$$

and we fall back onto our above example $A := R\{x/r, r_1/x\}$.

Alternatively, one can show that x is a twisted coordinate on A if and only if there exists canonical isomorphisms of A-algebras

$$A[\xi]/\xi^{(n)} \simeq P_{(n)} \quad \text{with} \quad \xi^{(n)} := \prod_{i=0}^{n-1}(\xi + x - \sigma^n(x)). \tag{6}$$

When x is a twisted coordinate on A, then Ω^1_σ is free of dimension one generated by the image of ξ and there exists therefore a unique twisted derivation $\partial_{A,\sigma}$ on A such that $\partial_{A,\sigma}(x) = 1$. In particular, an action by twisted derivations on a finite A-module M is simply given by an R-linear endomorphism $\partial_{M,\sigma}$ satisfying

$$\text{for all } z \in A,\ s \in M, \quad \partial_{M,\sigma}(zs) = \partial_{A,\sigma}(z)s + \sigma(z)\partial_{M,\sigma}(s).$$

1J. ***Twisted Weyl algebra.*** When x is a twisted coordinate, the *Ore extension* D_σ of A by σ and $\partial_{A,\sigma}$ is called the *twisted Weyl algebra* of A: this is the free A-module (of infinite rank) with formal basis ∂_σ^k and commutation rule

$$\partial_\sigma \circ z = \partial_{A,\sigma}(z) + \sigma(z)\partial_\sigma.$$

Note that it depends on x because $\partial_{A,\sigma}$ does. One easily sees that here exists an equivalence of categories

$$\nabla_\sigma\text{-Mod}(A/R) \simeq A\text{-finite } \mathrm{D}_{A/R,\sigma}\text{-Mod}.$$

Moreover, it induces an isomorphism:

$$\mathrm{H}^*_{\partial_\sigma}(M) := \mathrm{H}^*[M \xrightarrow{\partial_\sigma} M] \simeq \mathrm{Ext}^*_{D_\sigma}(A, M).$$

On the other hand, there exists a canonical map

$$A[T]_\sigma \to \mathrm{D}_\sigma, \quad T \mapsto 1 - (x - \sigma(x))\partial_\sigma. \tag{7}$$

We call x *strong* when $x - \sigma(x) \in A^\times$ (note that, this cannot happen when $\sigma = \mathrm{Id}_A$). If this is the case, then the morphism (7) is an isomorphism. It follows that, in general, there exists a functor

$$\nabla_\sigma\text{-Mod}(A/R) \to \sigma\text{-Mod}(A/R)$$

and that, when x is strong, this is an equivalence providing an isomorphism

$$H^*_\sigma(M) = H^*_{\partial_\sigma}(M).$$

To summarize, when $\sigma \neq \mathrm{Id}_A$, there is essentially no difference between an $A[T]_\sigma$-module, a twisted module, a module with an action by twisted derivations, a module with a twisted connection or a D_σ-module (and these identifications are compatible

with cohomology). We will see later that, with some extra conditions, we can add $D_\sigma^{(\infty)}$-modules, and even what we will call $D_\sigma^{(\eta)}$-modules, to the list.

1K. ***Standard twisted differential operators.*** When x is a twisted coordinate on A, we have

$$\widehat{P}_\sigma \simeq A[[\xi]]_\sigma := \varprojlim A[\xi]/\xi^{(n)} \quad \text{and} \quad \mathrm{D}_\sigma^{(\infty)} \simeq \varinjlim_n \mathrm{Hom}_A(A[\xi]/\xi^{(n)}, A). \tag{8}$$

We will denote by $\{\partial_\sigma^{[k]}\}_{k\in\mathbb{N}}$ the dual basis to $\{\xi^{(k)}\}_{k\in\mathbb{N}}$ and call $\partial_\sigma^{[k]}$ the *standard twisted differential operator* of order k. Note that the action of $\widehat{P}_\sigma$ on $\mathrm{D}_\sigma^{(\infty)}$ composed with evaluation at 1 gives back the paring of A-modules

$$\begin{aligned} \widehat{P}_\sigma \times \mathrm{D}_\sigma^{(\infty)} &\longrightarrow \mathrm{D}_\sigma^{(\infty)} \longrightarrow A, \\ (\xi^{(n)}, \partial_\sigma^{[k]}) &\longmapsto 1, \quad \text{if } n = k \text{ and } 0 \text{ otherwise.} \end{aligned}$$

One can also prove explicit identities such as

$$\text{for all } k \in \mathbb{N} \setminus \{0\}, \quad \partial_\sigma^{[k]} \circ x = \sigma(x)\partial_\sigma^{[k]} + \partial_\sigma^{[k-1]}, \tag{9}$$

from which we can derive

$$\text{for all } k \in \mathbb{N} \setminus \{0\}, \quad \xi \cdot \partial_\sigma^{[k]} = \partial_\sigma^{[k-1]} - (x - \sigma(x))\partial_\sigma^{[k]}. \tag{10}$$

Also, if M is an A-finite $\mathrm{D}_\sigma^{(\infty)}$-module and $s \in M$, then its twisted Taylor series will be given by

$$\hat{\theta}(s) = \sum_{k=0}^{\infty} \partial_\sigma^{[k]}(s) \otimes \xi^{(k)} \in M \otimes_A \widehat{P}_\sigma,$$

providing an a posteriori justification for the terminology.

When x is actually a q-coordinate, there exists an epi-mono factorization

$$\mathrm{D}_\sigma \twoheadrightarrow \overline{\mathrm{D}}_\sigma \hookrightarrow \mathrm{D}_\sigma^{(\infty)}$$

sending ∂_σ^k to $(k)_q!\partial_\sigma^{[k]}$. When all positive q-integers are invertible (for example if $q \in K$ is not a root of unity), then we get equalities $D_\sigma = \overline{D}_\sigma = D_\sigma^{(\infty)}$. In particular, in this situation, we obtain a sequence of equivalences

$$\nabla_\sigma\text{-Mod}(A/R) \simeq A\text{-finite } \mathrm{D}_{A/R,\sigma}\text{-Mod} \simeq A\text{-finite } \mathrm{D}_{A/R,\sigma}^{(\infty)}\text{-Mod}$$

inducing isomorphisms

$$H_{\partial_\sigma}^*(M) = \mathrm{Ext}_{\mathrm{D}_\sigma}^*(A, M) = \mathrm{Ext}_{\mathrm{D}_\sigma^{(\infty)}}^*(A, M).$$

When q is a root of unity, this is not true anymore because then some power ∂_σ^p will act trivially on A.

1L. ***Formal density.*** When x is a twisted coordinate, we denote by $K^{[k]}$ the submodule (freely) generated by all $\partial_\sigma^{[i]}$ for $i \geq k$, and we consider the completion

$$\widehat{\mathrm{D}}_\sigma^{(\infty)} =: \varprojlim \mathrm{D}_\sigma^{(\infty)} / K^{[k]}$$

which is *not* a ring in general. Note however that formula (10) turns $\widehat{\mathrm{D}}_\sigma^{(\infty)}$ into an $A[\xi]$-module. Just as in Lemma 7.3 of [Le Stum and Quirós 2018a], we can show that the second isomorphism in (8) extends to

$$\widehat{\mathrm{D}}_\sigma^{(\infty)} \xrightarrow{\simeq} \mathrm{Hom}_A(A[\xi], A) \tag{11}$$

where the right hand side does *not* depend on σ. We call this the *formal density lemma*. One can derive from it the *formal deformation map* and the *formal confluence theorems* as in [Le Stum and Quirós 2018a] but this is not what we are interested in here.

Recall that we have

$$\mathrm{D}_\sigma^{(\infty)} := \left\{ \sum_{0 \leq k \ll \infty} z_k \partial_\sigma^{[k]}, \quad z_k \in A \right\} \subset \widehat{\mathrm{D}}_\sigma^{(\infty)} := \left\{ \sum_{k=0}^{\infty} z_k \partial_\sigma^{[k]}, \quad z_k \in A \right\}.$$

Our goal is to use the topology of A in order to describe some A-modules that lie in between, that have a ring structure (such as $\mathrm{D}_\sigma^{(\infty)}$) and, at the same time, are essentially independent of σ (such as $\widehat{\mathrm{D}}_\sigma^{(\infty)}$).

1M. ***Quantum analogs.*** When x is a q-coordinate, we have

$$\text{for all } k, l \in \mathbb{N}, \quad \partial_\sigma^{[k]} \circ \partial_\sigma^{[l]} = \binom{k+l}{l}_q \partial_\sigma^{[k+l]}.$$

More generally, q-analogs will appear systematically in formulas.

Example. When $\sigma(x) = qx$, we have

$$\partial_\sigma^{[k]}(x^n) = \begin{cases} \binom{n}{k}_q x^{n-k} & \text{for } k \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Proposition 1.4. *If $q \in R$, we have*

$$\left\| \binom{n}{k}_q \right\| \leq \max\{1, \|q\|^{k(n-1)}\}.$$

The case $k = 1$ reads $\|(n)_q\| \leq \max\{1, \|q\|^{n-1}\}$.

Proof. By induction, we will have for $n, k \geq 1$,

$$\begin{aligned} \left\| \binom{n}{k}_q \right\| &\leq \max\left\{ \left\| \binom{n-1}{k-1}_q \right\|, \ \|q\|^k \left\| \binom{n-1}{k}_q \right\| \right\} \\ &\leq \max\left\{1, \ \|q\|^{(k-1)(n-2)}, \ \|q\|^k, \ \|q\|^{k+k(n-2)}\right\} \\ &= \max\{1, \|q\|^{k(n-1)}\}. \end{aligned}$$

□

1N. ***Globalization and localization.*** A *twisted (rigid) analytic* variety over R is a rigid analytic variety X which is defined over R and endowed with an endomorphism s (also defined over R). Any admissible open subset U of X which is *stable* under s inherits a twisted structure. We call X *suitable* if there exists an admissible open covering by stable affinoid subsets.

Example. (i) The functor $A \mapsto \mathrm{Spm}(A)$ induces an equivalence between twisted affinoid algebras and (automatically suitable) twisted affinoid varieties.

(ii) The rigid analytic projective line $\mathbb{P}^{1,\mathrm{an}}_R$ is a twisted analytic variety for

$$(x : y) \mapsto (qx + hy : y) \quad \text{(the quantum line).}$$

It is suitable as long as $\|q\| = \|q^{-1}\| = 1$ (but not otherwise).

(iii) Another example of suitable twisted analytic variety is given by an open annulus

$$\mathbb{A}_R(r_1^-, r^-) = \left\{x \in \mathbb{A}^{1,\mathrm{an}}_R, \quad r_1 < |x| < r\right\}$$

with $x \mapsto qx + h$ as long as $\|q\| \leq 1$, $\|h\| < r$ and either $\|q\| \geq r_1/r$ or $\|h\| < r_1$.

The whole above theory extends in a straightforward way from twisted affinoid algebras to twisted analytic varieties. For example, a *twisted module* on X is a coherent module $\mathcal{F}$ endowed with a morphism $s^*\mathcal{F} \to \mathcal{F}$. A *twisted derivation* is an R-linear map that satisfies the Leibniz rule on any *stable* admissible open subset. Cohomology is meant as Čech cohomology with respect to *stable* (affinoid) coverings. And so on.

One can give a global definition for the notion of *twisted principal parts*. We consider X as a subvariety of $X \times_R X$ via the diagonal embedding $i_X : X \hookrightarrow X \times_R X$ and we set $s_P := s \times \mathrm{Id}_X$. We define[3]

$$X^{(n+1)} = X \cup s_P^{-1}(X) \cup \cdots (s_P^n)^{-1}(X),$$

we let $\mathcal{I}_X^{(n+1)}$ be the ideal of $X^{(n+1)}$ in $X \times_R X$, and we set

$$\mathcal{P}_{X,(n)} = \iota_X^{-1}(\mathcal{O}_{X \times_R X}/\mathcal{I}_X^{(n+1)}).$$

This is a coherent module on X and it corresponds to $P_{A,(n)}$ when $X = \mathrm{Spm}(A)$. We will also write $\widehat{\mathcal{P}}_{X,s} := \varprojlim \mathcal{P}_{X,(n)}$ and consider the sheaf of *twisted differential forms* $\Omega^1_{X/R,s}$ as well as the sheaf $D^{(\infty)}_{X/R,s}$ of *twisted differential operators (of infinite radius)*.

Alternatively, when X is suitable, we could use the next proposition in order to glue from the affinoid case but we'd rather prove it as a consequence of the global definition.

[3] We mean "geometric" inverse image and union: nilpotents may play a role.

Proposition 1.5. *Let A be a twisted affinoid R-algebra and $f_0, \dots, f_r \in A$ with no common zeroes. If $B := A\{f_1/f_0, \dots, f_r/f_0\}$ is a twisted A-algebra, then for each $n \in \mathbb{N}$, we have*

$$B \otimes_A P_{A/R,(n)_\sigma} \simeq P_{B/R,(n)_\sigma}.$$

Proof. Recall that B is the affinoid algebra associated to the admissible open subset U defined by

$$|f_1(x)|, \dots, |f_r(x)| \leq |f_0(x)|.$$

Saying that B is a twisted A-algebra means that U is a stable admissible open subset of X. Now, if we let $j : U \hookrightarrow X$ denote the inclusion map, then we will have $j^{-1}\mathcal{P}_{X,(n)} = \mathcal{P}_{U,(n)}$ and taking global sections provides the expected isomorphism. □

In the situation of the proposition, there exists also natural isomorphisms

$$B \otimes_A \Omega^1_{A/R,\sigma} \simeq \Omega^1_{B/R,\sigma} \quad \text{and} \quad B \otimes_A D^{(\infty)}_{A/R,\sigma} \simeq D^{(\infty)}_{B/R,\sigma}.$$

We insist on the fact that the whole theory is local as long as we stick to *stable* subsets (although s must exist globally). For example a global section x of $\mathcal{O}_X$ is a twisted coordinate if and only if there exits an admissible covering by stable open subsets $X = \bigcup_{i \in I} X_i$ such that for each $i \in I$, $x_{|X_i}$ is a twisted coordinate on X_i. When this is the case, if U is a stable admissible open subset, then $x_{|U}$ is automatically a twisted coordinate on U.

We turn now to the *localization* process: we give ourselves a decreasing sequence of stable admissible open subsets $U_0 \supset \cdots \supset U_k \supset \cdots$ and we let

$$A := \varinjlim \Gamma(U_k, \mathcal{O}_X) \quad \text{and} \quad P_{A,(n)} := \varinjlim \Gamma(U_k, \mathcal{P}_{X,(n)}).$$

We may then develop a theory over A which happens to be exactly the limit of the theory at each step (a "continuous" theory).

Example. (i) If $c \in X$ satisfies $s(c) = c$ and has a cofinal system of stable admissible open neighborhoods U_k, then $A = \mathcal{O}_{X,c}$.

(ii) If $\|q\| \leq 1$ and $\|h\| < 1$, then we may consider $U_k := \mathbb{A}_R(r_k^-, 1^-)$ with $s(x) = qx + h$ when $r_k \overset{<}{\to} 1$. The limit is the Robba ring $\mathcal{R}_R$ which is thus turned into a (continuous) twisted algebra. Note that it is necessary for σ to be bijective on $\mathcal{R}_R$ to also assume that $\|q^{-1}\| \leq 1$ (in which case $\|q\| = \|q^{-1}\| = 1$).

2. Twisted principal parts of finite radius

We recall that R denotes an affinoid algebra over a fixed nontrivially valued complete ultrametric field K. We let (A, σ) be a twisted affinoid R-algebra with twisted coordinate x and fixed contractive norm.

Definition 2.1. The *x-radius* of σ is

$$\rho_{A,x}(\sigma) := \|x - \sigma(x)\|.$$

We will usually drop the subscripts A and x and might even just denote by ρ the x-radius of σ on A. We will be mostly interested in conditions of the type $\eta \geq \rho$ in which η is some nonnegative real number.

Lemma 2.2. *For all $n \in \mathbb{N}$, we have $\rho(\sigma^n) \leq \rho(\sigma)$.*

Proof. Since σ is a contraction, we will have by induction on n,

$$\begin{aligned}\|x - \sigma^{n+1}(x)\| &\leq \max\big\{\|x - \sigma^n(x)\|,\ \|\sigma^n(x - \sigma(x))\|\big\}\\ &\leq \max\{\rho(\sigma^n), \rho(\sigma)\}\\ &= \rho(\sigma).\end{aligned}$$

□

Examples. (i) We have $\rho(\mathrm{Id}_A) = 0$ and it follows that the condition $\eta \geq \rho$ is always satisfied when $\sigma = \mathrm{Id}_A$.

(ii) When $A := R\{x/r, r_1/x\}$ and $\sigma(x) = qx + h$ (so that conditions (1) and (2) in the example (ii) on page 529 are satisfied), we have

$$x - \sigma(x) = (1 - q)x - h$$

and it follows that

$$\rho(\sigma) = \max\{\|1 - q\|r, \|h\|\}.$$

Thus, we see that $\eta \geq \rho$ if and only if

$$\|1 - q\| \leq \frac{\eta}{r} \quad \text{and} \quad \|h\| \leq \eta. \tag{13}$$

Note that the first condition will always be satisfied in the finite difference equation case ($q = 1$), while the second will always be satisfied if we are in the q-difference equation case ($h = 0$).

We consider now the affinoid algebra

$$A\{\xi/\eta\} := \left\{\sum_{n=0}^{\infty} z_n \xi^n, \quad z_n \in A \text{ and } \|z_n\|\eta^n \to 0 \text{ when } n \to \infty\right\}$$

of functions that converge on the relative closed disk of radius η. This is a Banach algebra for the sup norm

$$\left\|\sum_{n=0}^{\infty} z_n \xi^n\right\|_\eta = \max \|z_n\|\eta^n. \tag{14}$$

Remark. If $\eta \geq \rho$, then the σ-linear morphism of rings

$$\xi \mapsto \xi + \sigma(x) - x$$

turns $A\{\xi/\eta\}$ into a twisted affinoid R-algebra because

$$\|\xi + \sigma(x) - x\|_\eta = \max\{\eta, \rho\} = \eta.$$

We will still denote this map by σ so that $\sigma(\xi) = \xi + \sigma(x) - x$.

Lemma 2.3. *If $\eta \geq \rho$, then the A-linear map*

$$\xi^n \mapsto \xi^{(n)} := \xi\sigma(\xi)\cdots\sigma^{n-1}(\xi)$$

defines an isometric automorphism of the A-module $A\{\xi/\eta\}$.

Proof. Since we assume that $\eta \geq \rho$, it follows from Lemma 2.2 that for all $i \in \mathbb{N}$, we have $\|x - \sigma^i(x)\| \leq \eta$. Thus we see that

$$\xi^{(n)} := \prod_0^{n-1}\big(\xi + (x - \sigma^i(x))\big) = \xi^n + f_n, \tag{15}$$

where $f_n \in A[\xi]_{<n}$ and $\|f_n\|_\eta \leq \eta^n$. Hence, the unique A-linear endomorphism of $A[\xi]$ that send ξ^n to $\xi^{(n)}$ is an isometry that preserves the degree. It extends therefore uniquely to an isometry from $A\{\xi/\eta\}$ onto itself. □

An *orthogonal Schauder basis* for a normed A-module M is a family $\{s_n\}_{n\in\mathbb{N}}$ of elements of M such that any $s \in M$ can be uniquely written as a convergent sum

$$s = \sum_{n=0}^{\infty} z_n s_n$$

with $z_n \in A$ and $\|s\| = \sup\{\|z_n\|\|s_n\|\}$.

For example, one immediately sees that $\{\xi^n\}_{n\in\mathbb{N}}$ is an orthogonal Schauder basis for the A-module $A\{\xi/\eta\}$ with the sup norm (14).

Proposition 2.4. *When $\eta \geq \rho$, $\{\xi^{(n)}\}_{n\in\mathbb{N}}$ is an orthogonal Schauder basis for the A-module $A\{\xi/\eta\}$.*

Proof. This follows from Lemma 2.3 and the fact that $\{\xi^n\}_{n\in\mathbb{N}}$ is an orthogonal Schauder basis for $A\{\xi/\eta\}$. □

There exists an analytic version of the isomorphism (6):

Proposition 2.5. *Assume that $\eta \geq \rho$. Then, for all $n \in \mathbb{N}$, there exists an isomorphism of A-algebras*

$$A\{\xi/\eta\}/\xi^{(n)} \xrightarrow{\simeq} P_{(n)}, \quad \xi \mapsto \overline{\tilde{x} - x}.$$

Proof. Since we know that (6) holds, we only have to prove that the map

$$A[\xi]_{\leq n} \xrightarrow{\simeq} A[\xi]/\xi^{(n)} \to A\{\xi/\eta\}/\xi^{(n)}$$

is bijective. But this follows from Proposition 2.4 (or Lemma 2.3 if you want). □

Corollary 2.6. *Assume that $\eta \geq \rho$. Then, if M is any finite A-module, there exists a canonical embedding*

$$M \otimes_A A\{\xi/\eta\} \hookrightarrow M \otimes_A \widehat{P}_\sigma.$$

Proof. Using Proposition 2.5, one easily sees that there exists such a map. Injectivity means that if $\sum s_i \otimes f_i \equiv 0 \mod \xi^{(n+1)}$ for all $n \in \mathbb{N}$, then $\sum s_i \otimes f_i = 0$ in $M \otimes_A A\{\xi/\eta\}$. But this follows from the fact that the $\xi^{(n)}$ form an orthogonal Schauder basis (and the properties of completed direct sums and tensor products). □

Definition 2.7. If $\eta \geq \rho$, then the image of $A\{\xi/\eta\}$ in $\widehat{P}_\sigma$ is called the ring of *twisted principal parts of radius (at least) η*.

In practice we will identify $A\{\xi/\eta\}$ with its image and call this ring itself the ring of twisted principal parts of radius η, exactly as we identify $A[\![\xi]\!]_\sigma$ with $\widehat{P}_\sigma$.

Remarks. (i) When X is a twisted analytic variety and x is a twisted coordinate on X, one can define the *local x-radius* $\tilde{\rho}_{X,x}$ of X as follows. If $\mathcal{X}$ is an admissible open covering of X by stable affinoid subsets $X_i = \mathrm{Spm}(A_i)$ and each A_i is endowed with a contractive norm, then we let $\rho_{\mathcal{X},x} := \sup_i \rho_{A_i,x}$. We may then set $\tilde{\rho}_{X,x} := \min_{\mathcal{X}} \rho_{\mathcal{X},x}$. Note that in the case $X = \mathrm{Spm}(A)$ we have $\tilde{\rho}_{X,x} \leq \rho_{A,x}$. Actually, when A is reduced, we have

$$\tilde{\rho}_{X,x} = \|x - \sigma(x)\|_{\mathrm{sp}}.$$

(ii) The last corollary globalizes as follows. If $\eta \geq \tilde{\rho}$ and $\mathcal{F}$ is any coherent $\mathcal{O}_X$-module, then there exists a canonical embedding

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X\{\xi/\eta\} \hookrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \widehat{\mathcal{P}}_{X,s}.$$

Examples. (i) In the case of the projective line, there exists no (global) twisted coordinate.

(ii) For an open (or closed) annulus with the usual twist, there exists an obvious twisted coordinate and we have $\tilde{\rho} = \max(\|q-1\|r, \|h\|)$. The conditions for $\eta \geq \tilde{\rho}$ read as above:

$$\|1-q\| \leq \eta/r \quad \text{and} \quad \|h\| \leq \eta.$$

3. Radius of convergence

We recall that R denotes an affinoid algebra over a fixed nontrivially valued complete ultrametric field K. We still let (A, σ) be a twisted affinoid R-algebra with fixed contractive norm and x a be twisted coordinate on A. We denote by ρ the x-radius of σ on A and by $\partial_\sigma^{[k]}$ the standard twisted differential operator of order k associated to x.

Definition 3.1. Let M be an A-finite $\mathrm{D}_\sigma^{(\infty)}$-module and $\eta \in \mathbb{R}$.

(1) Let $s \in M$. Then,

(a) s is *η-convergent* if

$$\|\partial_\sigma^{[k]}(s)\|\eta^k \to 0 \quad \text{when } k \to +\infty,$$

(b) the *radius of convergence of s* is

$$\mathrm{Rad}(s) = \sup\{\eta,\, s \text{ is } \eta\text{-convergent}\},$$

(c) s is *$\eta\dagger$-convergent* if $\mathrm{Rad}(s) \geq \eta$.

(2) (a) M is *η-convergent* if all $s \in M$ are η-convergent.

(b) The *radius of convergence of M* is

$$\mathrm{Rad}(M) = \inf_{s \in M} \mathrm{Rad}(s).$$

(c) M is *$\eta\dagger$-convergent* if all $s \in M$ are $\eta\dagger$-convergent.

Remarks. (i) Alternatively, M is η-convergent if and only if

$$\text{for all } s \in M, \quad \|\partial_\sigma^{[k]}(s)\|\eta^k \to 0 \quad \text{when } k \to +\infty,$$

we have

$$\mathrm{Rad}(M) = \sup\{\eta,\, M \text{ is } \eta\text{-convergent}\},$$

and M is *$\eta\dagger$-convergent* if and only if $\mathrm{Rad}(M) \geq \eta$.

(ii) We have explicit formulas

$$\mathrm{Rad}(s) = \underline{\lim}_k \|\partial_\sigma^{[k]}(s)\|^{-1/k} \quad \text{and} \quad \mathrm{Rad}(M) = \inf_{s \in M} \underline{\lim}_k \|\partial_\sigma^{[k]}(s)\|^{-1/k}.$$

(iii) If s (resp. M) is η-convergent then s (resp. M) is $\eta\dagger$-convergent. There exists a partial converse: s (resp. M) is $\eta\dagger$-convergent if and only if s (resp. M) is η'-convergent for all $\eta' < \eta$.

(iv) When $\sigma = \mathrm{Id}_A$, we recover standard notions (see Proposition 4.4.11 of [Le Stum 2007], for example). More precisely, if $\mathcal{X}$ is a smooth affine formal $\mathcal{V}$-scheme with an étale coordinate x, $\mathcal{M}$ is a coherent $\mathcal{O}_{\mathcal{X}_K}$-module and $M := \Gamma(\mathcal{X}, \mathcal{M})$, then a connection on $\mathcal{M}$ is convergent (in the sense of rigid cohomology) if and only if it is $1\dagger$-convergent in our sense.

(v) This notion of radius of convergence should not be confused with the notion of *radius of convergence at a generic point* which is more subtle. For example, if $A = R\{x, r_1/x\}$ with $r_1 < 1$ and $\sigma = \mathrm{Id}_A$, we have $\mathrm{Rad}(A) = r_1 \neq 1$ (see an example below). But the trivial connection on A must clearly be "convergent". This phenomenon appears because, even if A is smooth, it has *bad* reduction.

(vi) It is important to notice that our notion of radius of convergence is *dependent* on the choice of the twisted coordinate x. This is however a local notion, as we will see later.

Examples. (i) Assume that $A = R\{x\}$ and $\sigma(x) = qx$ with $\|q\| \leq 1$. Then, it follows from Proposition 1.4 that we always have $\left\|\binom{n}{k}_q\right\| \leq 1$. Thus, thanks to formula (12), we see that

$$\text{for all } k \leq n \in \mathbb{N}, \quad \|\partial_\sigma^{[k]}(x^n)\| = \left\|\binom{n}{k}_q x^{n-k}\right\| \leq 1$$

(and $\partial_\sigma^{[k]}(x^n) = 0$ for $k > n$). It follows that if $z = \sum_{n=0}^{\infty} a_n x^n \in R\{x\}$, then we have

$$\|\partial_\sigma^{[k]}(z)\| \leq \max \|a_n \partial_\sigma^{[k]}(x^n)\| \leq \max_{n \geq k} \|a_n\| \to 0,$$

and we see that A is η-convergent for any $\eta \leq 1$.

(ii) We consider now the situation

$$R = K, \quad A = K\{x/r, r_1/x\} \quad \text{and} \quad \sigma(x) = qx + h$$

with $0 < r_1 \leq r$. We first notice that formula (9) above implies that for all $n \in \mathbb{Z}$ and $k > 0$, we have

$$\partial_\sigma^{[k]}(x^{n+1}) = \sigma^k(x)\partial_\sigma^{[k]}(x^n) + \partial_\sigma^{[k-1]}(x^n). \tag{16}$$

Since we must have $|\sigma^k(x)| \leq r$, we obtain by induction that, for $n \geq 0$, we have $|\partial_\sigma^{[k]}(x^n)| \leq r^{n-k}$. Using equality (16) again, we can also prove that for $n \leq 0$, we have $|\partial_\sigma^{[k]}(x^n)| \leq r_1^{n-k}$. More precisely, since we always have $|\sigma^k(x)| \geq r_1$, then by induction on $k - n$, we obtain

$$|\partial_\sigma^{[k]}(x^n)| \leq \frac{1}{|\sigma^k(x)|} r_1^{n-k+1} \leq r_1^{n-k}.$$

It follows that, if $z = \sum_{-\infty}^{\infty} a_n x^n \in A$, we have

$$|\partial_\sigma^{[k]}(z)| \leq \max_n |a_n \partial_\sigma^{[k]}(x^n)| \leq \frac{1}{r_1^k} \max_n \{|a_n| r^n, |a_n| r_1^n\} = \frac{1}{r_1^k}|z|.$$

Thus we see that A is η-convergent as long as $\eta < r_1$ (one can show that we do have $\mathrm{Rad}(A) = r_1$). For further use, note that A is η-convergent for some $\eta \geq \rho$ if and only if

$$|1 - q| \leq \frac{\eta}{r} \quad \text{and} \quad |h| \leq \eta < r_1.$$

When this is the case, then σ is always an (isometric) automorphism.

Lemma 3.2. *Let M be an A-finite $\mathrm{D}_\sigma^{(\infty)}$-module and $\eta \geq \rho$. Then, $s \in M$ is η-convergent if and only if its twisted Taylor series*

$$\hat{\theta}(s) := \sum_{k=0}^{\infty} \partial_\sigma^{[k]}(s) \otimes \xi^{(k)}$$

falls inside $M \otimes_A A\{\xi/\eta\} \subset M \otimes \widehat{P}_\sigma$.

Proof. This is immediate from the definitions since the $\xi^{(k)}$ form an orthogonal Schauder basis of $A\{\xi/\eta\}$ by Proposition 2.4. □

Proposition 3.3. *Let M be an A-finite $\mathrm{D}_\sigma^{(\infty)}$-module and $\eta \geq \rho$. Then, M is η-convergent if and only if the twisted Taylor map factors as*

$$\begin{tikzcd} M \arrow[r, "\hat{\theta}"] \arrow[dr, "\theta_\eta"'] & M \otimes_A \widehat{P}_\sigma \\ & M \otimes_A A\{\xi/\eta\} \arrow[u, hook] \end{tikzcd}$$

□

Definition 3.4. This morphism θ_η is called the *twisted Taylor map of radius* η of M.

Remarks. (i) If M is an A-finite $\mathrm{D}_\sigma^{(\infty)}$-module with Taylor map θ, one may set

$$\|\theta\|_\eta := \sup_{k \in \mathbb{N},\, s \in M \setminus \{0\}} \left\{ \frac{\|\partial_\sigma^{[k]}(s)\| \eta^k}{\|s\|} \right\} \in [1, +\infty].$$

Then, we see that M is η-convergent if and only if $\|\theta\|_\eta < \infty$, and in that case $\|\theta_\eta\| = \|\theta\|_\eta$. In particular, the twisted Taylor map of radius η is automatically continuous.

(ii) It is quite simple to globalize the notion of η-convergence and define it on any twisted analytic variety X (with no reference to a specific norm) endowed with a fixed twisted coordinate x. More precisely, assume that $\mathcal{F}$ is an $\mathcal{O}_X$-coherent $\mathcal{D}_{X,s}^{(\infty)}$-module and that $\eta \geq \tilde{\rho}_{X,x}$. Then, a global section of $\mathcal{F}$ will be called *η-convergent* if and only if its twisted Taylor series falls inside $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X\{\xi/\eta\})$.

(iii) This notion of η-convergence is local on X. More precisely, a global section m of $\mathcal{F}$ is η-convergent if and only if there exists an admissible open covering by stable open subsets $X = \bigcup_{i \in I} X_i$ such that for each $i \in I$, $m_{|X_i}$ is η-convergent (this is simply the sheaf property). When this is the case, if U is a stable admissible open subset of X, then $m_{|U}$ is automatically η-convergent.

(iv) The module $\mathcal{F}$ itself will be called *η-convergent* if all its *global* sections are η-convergent. Be careful that this notion is *not* local anymore as the problem already appears when $\sigma = \mathrm{Id}_X$ and $\mathcal{F} = \mathcal{O}_X$ (see example (ii)enumi below).

Example. (i) If $X := \mathbb{A}_K(r_1^-, r^-)$ is an open annulus and $x \mapsto qx + h$, then the conditions for the structural sheaf $\mathcal{O}_X$ to be η-convergent (which by definition now requires $\eta \geq \tilde{\rho}$) read

$$|1-q| \leq \frac{\eta}{r} \quad \text{and} \quad |h| \leq \eta \leq r_1.$$

(ii) As a particular case, we may consider the pointed open disk

$$\mathbb{D}_K^\times(0, 1^-) = \mathbb{A}_K(0^-, 1^-).$$

In this situation, we have $r_1 = 0$ and the structural sheaf *cannot* be η-convergent (unless $\eta = 0$ and $\sigma = \mathrm{Id}$). On the contrary, on the full disk, the structural sheaf is always η-convergent (as long as $|h| \leq \eta \leq 1$ and $|1-q| \leq \eta/r$).

Proposition 3.5. *Being η-convergent for an A-finite $\mathrm{D}_\sigma^{(\infty)}$-module is stable under subobject, quotient and extension.*

Proof. Since both $A\{\xi/\eta\}$ and $\widehat{P}_\sigma$ are flat A-modules, if we are given an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of A-finite $\mathrm{D}_\sigma^{(\infty)}$-modules, we will have a commutative diagram with exact lines

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\hat{\theta}'"] & M \arrow[r] \arrow[d, "\hat{\theta}"] & M'' \arrow[r] \arrow[d, "\hat{\theta}''"] & 0 \\
0 \arrow[r] & M' \otimes \widehat{P}_\sigma \arrow[r] & M \otimes \widehat{P}_\sigma \arrow[r] & M'' \otimes \widehat{P}_\sigma \arrow[r] & 0 \\
0 \arrow[r] & M' \otimes A\{\xi/\eta\} \arrow[r] \arrow[u, hook] & M \otimes A\{\xi/\eta\} \arrow[r] \arrow[u, hook] & M'' \otimes A\{\xi/\eta\} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

Using Proposition 3.3, the assertion results from an easy diagram chasing. □

Remarks. (i) As a particular case, we see that A is η-convergent if and only if the twisted Taylor map of radius η

$$\theta_\eta : A \to A\{\xi/\eta\}, \quad z \mapsto \tilde{z} := \textstyle\sum_{k=0}^\infty \partial_\sigma^{[k]}(z)\xi^{(k)} \tag{17}$$

is well defined. Note that if A is the ring of functions on an affinoid domain of the R-line, and x is a coordinate on the R-line, then there may exist at most one morphism of R-algebras $A \to A\{\xi/\eta\}$ sending x to $\tilde{x}$.

(ii) If A is η-convergent and we linearize the twisted Taylor map of radius η, then we obtain an A-linear map

$$\tilde{\theta}_\eta : P \to A\{\xi/\eta\}, \quad z_1 \otimes z_2 \mapsto z_1\theta_\eta(z_2)$$

and we have a commutative diagram

$$\begin{tikzcd} & & A \arrow[dl, "\theta"'] \arrow[d, "\theta_\eta"] \arrow[dr, "\hat{\theta}"] & \\ A[\xi] \arrow[r, hook] & P \arrow[r] & A\{\xi/\eta\} \arrow[r, hook] & \widehat{P}_\sigma \end{tikzcd} \tag{18}$$

(iii) The twisted Taylor map of radius η of A is a morphism of affinoid R-algebras: actually, all the maps in the diagram (18) are morphisms of rings (and even morphisms of A-algebras on the bottom line).

4. Twisted differential operators of finite radius

We recall that R denotes an affinoid algebra over a fixed nontrivially valued complete ultrametric field K. We let (A, σ) be an η-convergent twisted affinoid R-algebra with respect to a twisted coordinate x and a given contractive norm (this implies $\eta \geq \rho := \rho_x(\sigma)$).

Definition 4.1. The A-module structure induced on $A\{\xi/\eta\}$ by the twisted Taylor map of radius η will be called the *right structure*.

Remember that, when we write $A\{\xi/\eta\} \otimes' -$, it means that we use the right structure on the left hand side:

$$f \otimes' zs = \theta_\eta(z) f \otimes' s.$$

Lemma 4.2. *If M, N are two finite A-modules, then the obvious map*

$$M \to A\{\xi/\eta\} \otimes'_A M, \quad s \mapsto 1 \otimes' s$$

induces an injective P-linear map

$$\mathrm{Hom}_{A\text{-cont}}(A\{\xi/\eta\} \otimes'_A M, N) \hookrightarrow \mathrm{Hom}_{R\text{-cont}}(M, N).$$

Proof. Recall from the commutative diagram (18) that the inclusion of $A[\xi]$ into $A\{\xi/\eta\}$ splits as

$$A[\xi] \to P \xrightarrow{\tilde{\theta}_\eta} A\{\xi/\eta\}$$

It follows that the image of $\tilde{\theta}_\eta$ is dense and therefore we have an injective map

$$\mathrm{Hom}_{A\text{-cont}}(A\{\xi/\eta\} \otimes'_A M, N) \hookrightarrow \mathrm{Hom}_{A\text{-cont}}(P \otimes'_A M, N).$$

Notice that the right hand side is nothing but $\mathrm{Hom}_{R\text{-cont}}(M, N)$. □

Definition 4.3. Assume that M, N are two finite A-modules. An R-linear map $\varphi : M \to N$ is called a *twisted differential operator of radius η* if it extends to a (necessarily unique) continuous A-linear map $\widetilde{\varphi}_\eta : A\{\xi/\eta\} \otimes'_A M \to N$, called its *$\eta$-linearization*.

Note that uniqueness follows from Lemma 4.2. We will denote by $\mathrm{Diff}_\sigma^{(\eta)}(M, N)$ the set of all twisted differential operators of radius η.

Proposition 4.4. *Assume that M, N are two finite A-modules. Then,* $\mathrm{Diff}_\sigma^{(\eta)}(M, N)$ *is a P-submodule of* $\mathrm{Hom}_{R\text{-cont}}(M, N)$ *containing* $\mathrm{Diff}_\sigma^{(\infty)}(M, N)$ *and*

$$\mathrm{Hom}_{A\text{-cont}}(A\{\xi/\eta\} \otimes'_A M, N) \simeq \mathrm{Diff}_\sigma^{(\eta)}(M, N). \tag{19}$$

Proof. The last assertion is an immediate consequence of the definition (one uses Lemma 4.2 though), the first one results from the fact that restriction is P-linear and the middle one is obtained from the canonical surjections $A\{\xi/\eta\} \to P_{(n)}$. $\square$

Remarks. (i) The isomorphism (19) turns $\mathrm{Diff}_\sigma^{(\eta)}(M, N)$ into a normed $A\{\xi/\eta\}$-module so that $\|\varphi\|_\eta = \|\widetilde{\varphi}_\eta\|$.

(ii) Since $A\{\xi/\eta\}$ has an orthogonal Schauder basis and M and N are finite A-modules, one easily sees that $\mathrm{Diff}_\sigma^{(\eta)}(M, N)$ is a Banach A-module.

(iii) If $\varphi : M \to N$ is a twisted differential operator of radius η, then we have the commutative diagrams

$$\begin{tikzcd}
M \arrow[rr, "\varphi"] \arrow[d, hook] & & N \\
A\{\xi/\eta\} \otimes'_A M \arrow[urr, "\widetilde{\varphi}_\eta"] \arrow[d, hook] & & \\
P \otimes'_A M \arrow[uurr, "\widetilde{\varphi}"] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A\{\xi/\eta\} \times \mathrm{Diff}_\sigma^{(\eta)}(M, N) \arrow[rr] \arrow[d, hook] & & \mathrm{Diff}_\sigma^{(\eta)}(M, N) \arrow[d, hook] \\
P \times \mathrm{Hom}_{R\text{-cont}}(M, N) \arrow[rr] & & \mathrm{Hom}_{R\text{-cont}}(M, N)
\end{tikzcd}$$

$$(f, \varphi) \longmapsto f \cdot \varphi$$

and we have

$$\widetilde{\varphi}_\eta(f \otimes' s) = \widetilde{\varphi}(f \otimes' s) = (f \cdot \varphi)(s).$$

Proposition 4.5. *There exists a unique map δ_η making commutative the following diagram*:

$$\begin{tikzcd}
A\widehat{\otimes}_R A \arrow[r, equal] \arrow[d, "\delta"] & \mathrm{P} \arrow[r] \arrow[d, "\delta"] & A\{\xi/\eta\} \arrow[r, two heads] \arrow[d, "\delta_\eta"] & P_{(m+n)} \arrow[d, "\delta_{(m),(n)}"] \\
A\widehat{\otimes}_R A\widehat{\otimes}_R A \arrow[r, equal] & \mathrm{P}\widehat{\otimes}'_A\mathrm{P} \arrow[r] & A\{\xi/\eta\}\widehat{\otimes}'_A A\{\xi/\eta\} \arrow[r, two heads] & P_{(m)} \otimes'_A P_{(n)}
\end{tikzcd}$$

Moreover, δ_η *is a morphism of affinoid* R*-algebras of norm* 1.

Proof. Uniqueness as well as being a morphism of rings follow immediately from the fact that the canonical map $A\{\xi/\eta\} \to \widehat{P}_\sigma$ is an injective morphism of rings. Also, necessarily, one must have $\delta_\eta(\xi) = 1 \otimes' \xi + \xi \otimes' 1$. Hence, existence and the assertion on the norm follow from the inequality $\|\delta(\xi)\|_\eta \le \|\xi\|_\eta = \eta$ if we write $\|-\|_\eta$ also for the tensor product norm. □

Definition 4.6. The morphism δ_η is called the *comultiplication map* on $A\{\xi/\eta\}$.

Proposition 4.7. *Twisted differential operators of radius* η *are stable under composition and we always have* $\|\varphi \circ \psi\|_\eta \le \|\varphi\|_\eta \|\psi\|_\eta$. *In particular*, $\mathrm{Diff}_\sigma^{(\eta)}(M, M)$ *is a sub-*R*-algebra of* $\mathrm{End}_{R\text{-cont}}(M)$ *which is a Banach* R*-algebra.*

Proof. If $\varphi : M \to N$ and $\psi : L \to M$ are two twisted differential operators of radius η, then there exists a commutative diagram

$$\begin{tikzcd}
P \otimes'_A L \arrow[r, "\delta"] \arrow[d] & P \widehat{\otimes}'_A P \otimes'_A L \arrow[r, "\mathrm{Id} \otimes' \widetilde{\psi}"] \arrow[d] & P \otimes'_A M \arrow[r, "\widetilde{\varphi}"] \arrow[d] & N \arrow[d] \\
A\{\xi/\eta\} \otimes'_A L \arrow[r, "\delta_\eta"] & A\{\xi/\eta\} \widehat{\otimes}'_A A\{\xi/\eta\} \otimes'_A L \arrow[r, "\mathrm{Id} \otimes \widetilde{\psi}_\eta"] & A\{\xi/\eta\} \otimes'_A M \arrow[r, "\widetilde{\varphi}_\eta"] & N
\end{tikzcd} \tag{20}$$

and we know that the upper line is exactly $\widetilde{\varphi \circ \psi}$. Moreover, we have

$$\|\varphi \circ \psi\|_\eta = \|\widetilde{(\varphi \circ \psi)}_\eta\| \le \|\widetilde{\varphi}_\eta\| \, \|\mathrm{Id} \otimes \widetilde{\psi}_\eta\| \, \|\delta_\eta\| = \|\widetilde{\varphi}_\eta\| \, \|\widetilde{\psi}_\eta\| = \|\varphi\|_\eta \|\psi\|_\eta. \qquad \square$$

Proposition 4.8. *Let* M *be a finite* A*-module and* $\theta_\eta : M \to M \otimes_A A\{\xi/\eta\}$ *an* A*-linear map with respect to the right structure of* $A\{\xi/\eta\}$. *Then* θ_η *is a twisted Taylor map of radius* η *if and only if the diagram*

$$\begin{tikzcd}
M \arrow[r, "\theta_\eta"] \arrow[d, "\theta_\eta"] & M \otimes_A A\{\xi/\eta\} \arrow[d, "\theta_\eta \otimes' \mathrm{Id}"] \\
M \otimes_A A\{\xi/\eta\} \arrow[r, "\mathrm{Id} \otimes \delta_\eta"] & M \otimes_A A\{\xi/\eta\} \widehat{\otimes}'_A A\{\xi/\eta\}
\end{tikzcd} \tag{21}$$

is commutative.

Proof. We already know that the category of (A-finite) $\mathrm{D}_\sigma^{(\infty)}$-modules M is equivalent to the category of finite A-modules endowed with a twisted Taylor structure. And we saw in Proposition 3.3 that the twisted Taylor map will factor (uniquely) through $M \otimes_A A\{\xi/\eta\}$ if and only if M is η-convergent. Finally, Proposition 4.5 implies the commutativity of the diagram. The converse also follows from the same proposition. □

Remarks. (i) One could define abstractly a twisted Taylor map of radius η as a map $M \to M \otimes_A A\{\xi/\eta\}$ making commutative the diagram (21). Then, the proposition

says that we would obtain a category which is equivalent to the category of A-finite $\mathrm{D}_\sigma^{(\infty)}$-modules that are η-convergent.

(ii) In the particular case $M = A$, the proposition implies that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\theta_\eta"] \arrow[d, "\theta_\eta"] & A\{\xi/\eta\} \arrow[d, "\theta_\eta \otimes' \mathrm{Id}"] \\ A\{\xi/\eta\} \arrow[r, "\delta_\eta"] & A\{\xi/\eta\} \widehat{\otimes}'_A A\{\xi/\eta\} \end{tikzcd}$$

5. Rings of twisted differential operators of finite radius

We recall that R denotes an affinoid algebra over a fixed nontrivially valued complete ultrametric field K. We let (A, σ) be an η-convergent twisted affinoid R-algebra with respect to a twisted coordinate x and a contractive norm. We denote by $\partial_\sigma^{[k]}$ the standard twisted differential operator of order k associated to x.

Definition 5.1. The *ring of twisted differential operators of radius* η is

$$\mathrm{D}_\sigma^{(\eta)} := \mathrm{Diff}_\sigma^{(\eta)}(A, A).$$

Since, by Proposition 4.7, $\mathrm{D}_\sigma^{(\eta)}$ is the subring of $\mathrm{End}_{R\text{-cont}}(A)$ made of all continuous R-linear endomorphisms φ such that there exists a (necessarily unique) continuous A-linear form $\widetilde{\varphi}_\eta : A\{\xi/\eta\} \to A$ with $\varphi = \widetilde{\varphi}_\eta \circ \theta_\eta$, we get for free the analytic version of the density lemma, [Le Stum and Quirós 2018a, Lemma 7.2]:

Lemma 5.2 (analytic density). *There exists a canonical isomorphism of Banach A-modules*

$$\mathrm{D}_\sigma^{(\eta)} \simeq \mathrm{Hom}_{A\text{-cont}}(A\{\xi/\eta\}, A). \qquad \square$$

Recall also from the same Proposition 4.7 that $\mathrm{D}_\sigma^{(\eta)}$ is actually a Banach algebra for the norm induced by the density lemma:

$$\|\varphi\|_\eta := \|\widetilde{\varphi}_\eta\| := \sup_{f \neq 0} \|\widetilde{\varphi}_\eta(f)\| / \|f\|.$$

Example. If we apply the analytic density lemma to $A = R\{x\}$ and $\eta = 1$, we see that

$$\mathrm{D}_\sigma^{(1)} \simeq \mathrm{End}_{R\text{-cont}}(A),$$

which means that any continuous endomorphism of $R\{x\}$ is a bounded differential operator.

Corollary 5.3. *The A-module* $\mathrm{Hom}_{A\text{-cont}}(A\{\xi/\eta\}, A)$ *is a* (noncommutative) *Banach R-algebra for the multiplication defined by*

$$\Phi\Psi: \ A\{\xi/\eta\} \xrightarrow{\delta_\eta} A\{\xi/\eta\}\widehat{\otimes}'_A A\{\xi/\eta\} \xrightarrow{\mathrm{Id}\otimes'\Psi} A\{\xi/\eta\} \xrightarrow{\Phi} A. \qquad \square$$

By duality, from the bottom line of diagram (18), one obtains a series of inclusion maps

$$\begin{tikzcd} \mathrm{D}_\sigma^{(\infty)} \arrow[r, hook] & \mathrm{D}_\sigma^{(\eta)} \arrow[r, hook] \arrow[dr, hook] & \widehat{\mathrm{D}}_\sigma^{(\infty)} \\ & & \mathrm{End}_{R\text{-cont}}(A) \arrow[u] \end{tikzcd} \tag{22}$$

and the right upper map is explicitly given by

$$\varphi \mapsto \sum_{k=0}^{\infty} \widetilde{\varphi}_\eta(\xi^{(k)})\partial_\sigma^{[k]}.$$

Proposition 5.4. *The inclusion map* $\mathrm{D}_\sigma^{(\eta)} \hookrightarrow \widehat{\mathrm{D}}_\sigma^{(\infty)}$ *induces an isometric isomorphism of Banach A-modules*

$$\mathrm{D}_\sigma^{(\eta)} \xrightarrow{\simeq} \left\{\textstyle\sum_{k=0}^{\infty} z_k\partial_\sigma^{[k]}, \quad \exists C>0,\ \forall k\in\mathbb{N},\ \|z_k\|\le C\eta^k\right\}$$

for the sup norm

$$\left\|\sum_{k=0}^{\infty} z_k\partial_\sigma^{[k]}\right\|_\eta = \sup\{\|z_k\|/\eta^k\}.$$

Proof. Let $\varphi \in \mathrm{D}_\sigma^{(\eta)}$. Since $\widetilde{\varphi}_\eta$ is continuous, we have for all $k\in\mathbb{N}$,

$$\|\widetilde{\varphi}_\eta(\xi^{(k)})\| \le \|\widetilde{\varphi}_\eta\|\,\|\xi^{(k)}\| = \|\widetilde{\varphi}_\eta\|\eta^k$$

and it follows that the canonical map $\mathrm{D}_\sigma^{(\eta)} \hookrightarrow \widehat{\mathrm{D}}_\sigma^{(\infty)}$ takes its values inside the right hand side as asserted. It also follows that the corresponding map has norm at most 1. Conversely, assume that we are given some $\sum_{k=0}^{\infty} w_k\partial_\sigma^{[k]}$ of norm (at most) C in the right hand side. Then, there exists a unique $\varphi\in\mathrm{D}_\sigma^{(\eta)}$ such that

$$\widetilde{\varphi}_\eta\left(\sum_{k=0}^{\infty} z_k\xi^{(k)}\right) = \sum_{k=0}^{\infty} z_k w_k \in A$$

because $\|z_k w_k\| \le C\|z_n\|\eta^k \to 0$. This defines an inverse map for our inclusion and the same inequality also implies that $\|\varphi\|\le C$. This shows that our map is an isometry. $\square$

Remarks. (i) We will usually identify $\mathrm{D}_\sigma^{(\eta)}$ with its image inside $\widehat{\mathrm{D}}_\sigma^{(\infty)}$. Note that we could have chosen to define it this way, but we do want an A-algebra and not merely an A-module.

(ii) There exists a (left perfect) topological pairing of Banach A-modules

$$A\{\xi/\eta\} \times \mathrm{D}_\sigma^{(\eta)} \to A, \quad (\xi^{(n)}, \partial_\sigma^{[k]}) \mapsto \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

(iii) The action of $A\{\xi/\eta\}$ on $\mathrm{D}_\sigma^{(\eta)}$ is given by

$$\xi \cdot \sum_{k=0}^{\infty} z_k \partial_\sigma^{[k]} = \sum_{k=0}^{\infty} \big(z_{k+1} - z_k(x - \sigma^k(x))\big)\partial_\sigma^{[k]}, \tag{24}$$

and one recovers the paring (23) by composition with evaluation at 1.

Proposition 5.5. *An η-convergent A-finite $\mathrm{D}_\sigma^{(\infty)}$-module M extends canonically to a $\mathrm{D}_\sigma^{(\eta)}$-module.*

Proof. The process is standard but requires some care. We first consider the natural $A\{\xi/\eta\}$-linear map

$$\begin{aligned} M \otimes_A A\{\xi/\eta\} &\longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(A\{\xi/\eta\}, A), M), \\ s \otimes f &\longmapsto (\Phi \mapsto \Phi(f)s). \end{aligned} \tag{25}$$

Next, we extend linearly the twisted Taylor map of radius η of M in order to get an $A\{\xi/\eta\}$-linear map $A\{\xi/\eta\} \otimes' M \to M \otimes A\{\xi/\eta\}$ and we compose on the left hand side. Finally, we use restriction to continuous maps and the isomorphism $D_\sigma^{(\eta)} \simeq \mathrm{Hom}_{A\text{-cont}}(A\{\xi/\eta\}, A)$ on the right hand side, and we obtain an $A\{\xi/\eta\}$-linear map

$$A\{\xi/\eta\} \otimes'_A M \to \mathrm{Hom}_A(\mathrm{D}_\sigma^{(\eta)}, M),$$

or equivalently, an $A\{\xi/\eta\}$-linear map

$$\mathrm{D}_\sigma^{(\eta)} \to \mathrm{Hom}_A(A\{\xi/\eta\} \otimes' M, M) = \mathrm{Diff}_\sigma^{(\eta)}(M, M).$$

Using the commutativity of (21), one can check that this is a morphism of rings. By construction, it is compatible with the natural map

$$\mathrm{D}_\sigma^{(\infty)} \to \mathrm{Diff}_\sigma^{(\infty)}(M, M)$$

that gives the action of $\mathrm{D}_\sigma^{(\infty)}$ on M. □

There is no exact converse to this proposition because the canonical map

$$M \otimes_A A\{\xi/\eta\} \to \mathrm{Hom}_A(\mathrm{D}_\sigma^{(\eta)}, M)$$

is far from being bijective in general (the right hand side is too big as the case $M = A$ shows). However, there exists a partial converse that will be used later:

Proposition 5.6. *If M is an A-finite $\mathrm{D}_\sigma^{(\eta)}$-module then M is $\eta\dagger$-convergent.*

Proof. We have to show that M is η'-convergent whenever $\eta' < \eta$.

It follows from Banach open mapping theorem that M is actually a topological $\mathrm{D}_\sigma^{(\eta)}$-module (see lemme 4.1.2 of [Berthelot 1996], for example). As a consequence, for all $s \in M$, the map

$$\mathrm{D}_\sigma^{(\eta)} \to M, \quad \varphi \mapsto \varphi(s)$$

will be a continuous A-linear map. This means that there exists a constant C such that for all $\varphi \in \mathrm{D}_\sigma^{(\eta)}$, we have $\|\varphi(s)\| \leq C\|\varphi\|\,\|s\|$. Therefore, for all $k \in \mathbb{N}$, we have $\|\partial_\sigma^{[k]}(s)\| \leq C\|s\|/\eta^k$, and $\|\partial_\sigma^{[k]}(s)\|\eta'^k \leq C\|s\|(\eta'/\eta)^k \to 0$. □

6. Deformation

We recall that R denotes an affinoid algebra over a fixed nontrivially valued complete ultrametric field K. As before, we let (A, σ) be a twisted affinoid R-algebra with twisted coordinate x. We denote by ρ the x-radius of σ on A with respect to x and some contractive norm. We also fix some $\eta \geq \rho$ such that A is η-convergent.

The next result is fundamental in the sense that it shows that the whole theory is in some sense completely independent of the choice of σ. In the case $\tau = \mathrm{id}_A$, this is quite similar to Lemma 4.3.3 of [Pulita 2017].

Proposition 6.1. *Assume that x is also a twisted coordinate for some other endomorphism τ of A and that $\eta \geq \rho(\tau)$. Then, A is also η-convergent with respect to τ with the* same *twisted Taylor morphism as σ.*

Proof. We may consider the following commutative diagram

$$\begin{tikzcd}
 & & A \arrow[dl, "\theta"'] \arrow[d, "\theta_\eta"] \\
A[\xi] \arrow[r, hook] \arrow[d, two heads] & P \arrow[r] \arrow[d, two heads] & A\{\xi/\eta\} \arrow[d, two heads] \\
A[\xi]/\xi^{(n)_\tau} \arrow[r, equal] \arrow[rr, bend right] & P_{(n)_\tau} & A\{\xi/\eta\}/\xi^{(n)_\tau}
\end{tikzcd}$$

(where θ_η denotes the twisted Taylor map of radius η with respect to σ). A quick look shows that the composition of the right vertical maps does not depend on σ. Moreover, since we assume that $\rho(\tau) \leq \eta$, then the curved arrow is an isomorphism

by Proposition 2.5, and going to the limit on n provides us with a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\theta"'] \arrow[d, "\theta_\eta"] \\ P \arrow[d] & A\{\xi/\eta\} \arrow[dl, hook] \\ \widehat{P}_\tau & \end{tikzcd}$$

It then follows from Proposition 3.3 that A is η-convergent with respect to τ and that θ_η is the twisted Taylor map of radius η with respect to τ. □

Corollary 6.2. *The ring structure of* $\mathrm{Hom}_{A\text{-cont}}(A\{\xi/\eta\}, A)$ *does* ***not*** *depend on* σ.

Proof. This follows from the definition in Corollary 5.3 of the multiplication rule on the A-module $\mathrm{Hom}_{A\text{-cont}}(A\{\xi/\eta\}, A)$ and Proposition 6.1. □

Theorem 6.3 (deformation). *Let τ be another R-endomorphism of A such that x is also a τ-coordinate. If $\eta \geq \rho(\tau)$, then there exists an isometric A-linear isomorphism of R-algebras*

$$\mathrm{D}^{(\eta)}_{A/R,\sigma} \xrightarrow{\simeq} \mathrm{D}^{(\eta)}_{A/R,\tau}.$$

that only depends on x.

Proof. This follows from Corollary 6.2 since the analytic density isomorphisms provided by Lemma 5.2 are A-linear isomorphisms of R-algebras. □

As a consequence of the theorem, we can (and will) *identify* two twisted differential operators of radius η with respect to two different endomorphisms when they have the same linearization.

Remark. This isomorphism is explicit in the sense that, as in Proposition 7.6 of [Le Stum and Quirós 2018a], we have, for example,

$$\partial_\sigma = \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} \big(\sigma(x) - \tau^i(x)\big) \right) \partial_\tau^{[k]}.$$

The particular case $\tau = \mathrm{Id}_A$ is worth stating separately:

Corollary 6.4. *If A is formally smooth* (of relative dimension one) *over R and x is also an étale coordinate on A, then there exists an isometric A-linear isomorphism of R-algebras*

$$\mathrm{D}^{(\eta)}_{\sigma} \xrightarrow{\simeq} \mathrm{D}^{(\eta)}$$

that only depends on x. □

This is the *analytic deformation map*.

Remark. Again, this isomorphism is explicit. For example, as shown in Corollary 7.7 of [Le Stum and Quirós 2018a], we will have

$$\partial_\sigma = \sum_{k=1}^{\infty} (\sigma(x) - x)^{k-1} \partial^{[k]} \quad \text{and} \quad \sigma = \sum_{k=0}^{\infty} (\sigma(x) - x)^k \partial^{[k]}.$$

7. Confluence

We recall that R denotes an affinoid algebra over a fixed nontrivially valued complete ultrametric field K. We still let (A, σ) be a twisted affinoid R-algebra with twisted coordinate x. We denote by ρ the x-radius of σ on A with respect to x and some contractive norm. We also fix some $\eta > \rho$ such that A is $\eta\dagger$-convergent

Definition 7.1. The ring of *twisted differential operators of radius* $\eta\dagger$ is

$$\mathrm{D}_\sigma^{(\eta\dagger)} := \varinjlim_{\eta' < \eta} \mathrm{D}_\sigma^{(\eta')}.$$

Remarks. (i) We have the following description:

$$\begin{aligned} \mathrm{D}_\sigma^{(\eta\dagger)} &= \left\{ \sum_{k=0}^{\infty} z_k \partial_\sigma^{[k]}, \quad \exists \eta' < \eta,\ \exists C > 0, \forall k \in \mathbb{N},\ \|z_k\| \le C\eta'^k \right\} \\ &= \left\{ \sum_{k=0}^{\infty} z_k \partial_\sigma^{[k]}, \quad \exists \eta' < \eta,\ \frac{\|z_k\|}{\eta'^k} \to 0 \right\}. \end{aligned}$$

(ii) When $\sigma = \mathrm{Id}_A$ (so that we drop σ from our notation), this definition is closely related to the theory of arithmetic differential operators of Pierre Berthelot (see proposition 2.4.4 of [Berthelot 1996]). More precisely, if we let $\mathcal{X}$ be a (one dimensional) smooth affine formal $\mathcal{V}$-scheme with étale coordinate x and if we set $A := \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}\mathbb{Q}})$, then we have

$$\Gamma(\mathcal{X}, \mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}) = \mathrm{D}_{A/K}^{(1\dagger)}.$$

(iii) More generally, still assuming $\sigma = \mathrm{Id}_A$, let $\mathcal{X}_0$ be a smooth formal $\mathcal{V}$-scheme with étale coordinate x, $\mathcal{X}$ a blowing up of $\mathcal{X}_0$ centered in an ideal containing π^k (π a uniformizer) and $\mathcal{Y} = \mathrm{Spf}(A)$ an affine open subset of $\mathcal{X}$. Then, with the notation of [Huyghe et al. 2019], we have

$$\Gamma(\mathcal{Y}, \mathcal{D}^\dagger_{\mathcal{X},k}) = \mathrm{D}_{A/K}^{(\eta\dagger)}$$

with $\eta = |\pi|^k$.

Proposition 7.2. *The category of A-finite $\mathrm{D}_\sigma^{(\eta\dagger)}$-modules is equivalent* (isomorphic) *to the subcategory of all $\eta\dagger$-convergent A-finite $\mathrm{D}_\sigma^{(\infty)}$-modules.*

Proof. This follows from Proposition 5.6 and 5.5. □

Theorem 7.3 (Confluence). *Let τ be another R-endomorphism of A such that x is also a τ-coordinate. If $\eta > \rho(\tau)$, then the categories of $\eta\dagger$-convergent A-finite $\mathrm{D}_\sigma^{(\infty)}$-modules with respect to σ and τ are canonically equivalent.*

Proof. This follows from the deformation Theorem 6.3 and Proposition 7.2. □

Definition 7.4. Assume that x is a q-coordinate on A and that all positive q-integers are invertible in R. Then, a ∇_σ-module M on A is said to be *η'-convergent* if

$$\text{for all } s \in M, \qquad \left\| \frac{\nabla_\sigma^k(s)}{(k)_q!} \right\| \eta'^k \to 0 \quad \text{when } k \to \infty,$$

and $\eta\dagger$-convergent if it is η'-convergent whenever $\eta' < \eta$.

We will denote by $\nabla_\sigma\text{-Mod}^{(\eta\dagger)}(A/R)$ the full subcategory of all finite ∇_σ-modules M that are $\eta\dagger$-convergent.

The condition that positive q-integers are invertible is necessary for the definition to make sense. If q belongs to a subfield of R, then this happens exactly when q is not a root of unity or else if $q = 1$ and $\mathrm{Char}(K) = 0$.

Proposition 7.5. *Assume that x is a q-coordinate and that all positive q-integers are invertible. If A is $\eta\dagger$-convergent, then there exists an equivalence of categories*

$$\nabla_\sigma\text{-Mod}^{(\eta\dagger)}(A/R) \simeq A\text{-finite } \mathrm{D}_{A/R,\sigma}^{(\eta\dagger)}\text{-Mod}$$

which is compatible with cohomology in the sense that

$$\mathrm{H}^*_{\partial_\sigma}(M) = \mathrm{Ext}^*_{D_\sigma^{(\eta\dagger)}}(A, M).$$

Proof. We know that, if D_σ denotes the twisted Weyl algebra of (A, σ), then the category of finite ∇_σ-modules is equivalent to the category of finite D_σ-modules. Moreover in our situation, we have $\mathrm{D}_\sigma = \mathrm{D}_\sigma^{(\infty)}$. Therefore, since for all $k \in \mathbb{N}$, we have $\partial_\sigma^k = (k)_q!\partial_\sigma^{[k]}$, we can identify η-convergent ∇_σ-modules and η-convergent $\mathrm{D}_\sigma^{(\infty)}$-modules. Hence, the first result follows from Proposition 7.2. In order to prove the second one, we may use the Spencer resolution first introduced in [Spencer 1969]:

$$[D_\sigma^{(\eta\dagger)} \xrightarrow{\partial_\sigma} D_\sigma^{(\eta\dagger)}] \simeq A$$

(the map is multiplication on the *right* by ∂_σ). More precisely, there exists an obvious short exact sequence

$$0 \to D_\sigma^{(\eta\dagger)} \xrightarrow{\partial_\sigma} D_\sigma^{(\eta\dagger)} \to A \to 0$$

which provides the exact sequence

$$0 \to \mathrm{Ext}^1_{D_\sigma^{(\eta\dagger)}}(A, M) \to M \xrightarrow{\partial_\sigma} M \to \mathrm{Hom}_{D_\sigma^{(\eta\dagger)}}(A, M) \to 0$$

upon taking cohomology. □

As a consequence of the confluence theorem, we obtain the following theorem (which is essentially Theorem 6.3(ii) of [Pulita 2008]) that we state with all the necessary hypothesis for the convenience of the reader:

Theorem 7.6 (confluence-bis). *Let K be a nontrivially valued complete ultrametric field of characteristic zero. Let R be an affinoid K-algebra and $q \in R$ such that all positive q-integers are invertible. Let A be a formally smooth affinoid R-algebra* (of relative dimension one) *such that A is $\eta\dagger$-convergent with respect to some étale coordinate x for some $\eta \geq 0$. Assume that x is a q-coordinate for an R-algebra endomorphism σ of A with $\eta \geq \rho_x(\sigma)$.*

Then there exists an equivalence of categories

$$\nabla_\sigma\text{-Mod}^{(\eta\dagger)}(A/R) \simeq \nabla\text{-Mod}^{(\eta\dagger)}(A/R)$$

which is compatible with cohomology on both sides.

Proof. We are in a situation of applying Proposition 7.5 to both σ itself and to Id_A. In order to prove the theorem, it is therefore sufficient to show that there exists an isomorphism

$$\mathrm{D}^{(\eta\dagger)}_{A/R,\sigma} \simeq \mathrm{D}^{(\eta\dagger)}_{A/R}.$$

But this follows from Corollary 6.4. □

The next corollary is very similar to Proposition 15.1 of [André and Di Vizio 2004] both in the content and in the spirit (they show that the Tannakian groups are the same, while we prove that the "classification rings" are the same). This is also very similar to Theorem 4.3.1 of [Pulita 2017] (as we have already remarked, his Lemma 4.3.3 corresponds to our Proposition 6.1).

Corollary 7.7. *In the situation of the theorem, there exists a functor*

$$\nabla\text{-Mod}^{(\eta\dagger)}(A/R) \to \sigma_A\text{-Mod}(A/R). \tag{26}$$

When x is a strong quantum coordinate for σ, this functor is fully faithful and compatible with cohomology on both sides. □

Proof. Compose the equivalence in Theorem 7.6 with the functor built from morphism (7) in Section 1J. □

In the latter situation, we may identify an $\eta\dagger$-convergent ∇-module with the corresponding quantum module.

Remarks. (i) The hypothesis on K are satisfied when $K = \mathbb{Q}_p$ or $K = \mathbb{C}_p$ or $K = \mathbb{Q}((T))$ for example.

(ii) The hypothesis on q is satisfied for example if $q \in K$ and q is not a root of 1.

(iii) The extra condition in the corollary of x being a strong quantum coordinate is satisfied if $x \in A^\times$ and $1 - q \in R^\times$.

(iv) When $R = K$ and $A = K\{x/r, r_1/x\}$, the conditions on A (and σ) can be summarized (when q is not a root of 1) as

$$|h| \leq \eta < r_1 \quad \text{and} \quad |1-q| \leq \frac{\eta}{r}.$$

(v) The functor (26) is completely explicit: if we are given a finite A-module M, endowed an $\eta\dagger$-convergent connection, then the corresponding twisted module will be given by

$$\sigma_M(s) = \sum_{k=0}^{\infty} \frac{((q-1)x+h)^k}{k!} \partial_M^k(s). \tag{27}$$

In other words, it is obtained by replacing ξ by $\sigma(x) - x$ in the twisted Taylor series:

$$\sigma_M(s) = \theta_\eta(s)(\sigma(x) - x).$$

Formally, we may also write $\sigma_M = \exp((\sigma(x) - x)\partial_M)$.

(vi) In the case $h = 0$, one can also use the logarithmic derivative: if we set

$$q^{x\partial} := \sum_{k=0}^{\infty} \log(q)^k (x\partial)^k,$$

then we have for all $n \in \mathbb{N}$,

$$q^{x\partial}(x^n) = \sum_{k=0}^{\infty} \log(q)^k (x\partial)^k (x^n) = \sum_{k=0}^{\infty} \log(q)^k n^k x^n = q^n x^n = \sigma(x^n)$$

and it formally follows that

$$\sigma_M(s) = (q^{x\partial_M})(s) \quad \left(= \sum_{k=0}^{\infty} \log(q)^k (x\partial_M)^k(s) \right).$$

(vii) Compatibility with cohomology means that $H^*_\nabla(M) = H^*_\sigma(M)$. In particular, the solutions of the differential system associated to the ∇-module M are exactly the same as the solutions of the functional system associated to the σ-module M.

Examples. (i) If we consider the differential equation $\partial(s) = cs$, then we will have $\partial^k(s) = c^k s$ for all $k \in \mathbb{N}$ and σ will be multiplication by

$$\sum_{k=0}^{\infty} \frac{c^k((q-1)x+h)^k}{k!} = \exp\big(c((q-1)x+h)\big).$$

Of course, the differential equation $\partial(s) = cs$ has the same solution $\exp(cx)$ as the functional equation

$$s(qx+h) = \exp\big(c((q-1)x+h)\big)s(x).$$

(ii) If we consider now the differential equation $\partial(s) = \frac{a}{x}s$, then we will have

$$\partial^k(s) = \frac{a(a-1)\cdots(a-k+1)}{x^k}s$$

for all $k \in \mathbb{N}$ and σ will be the multiplication by

$$\sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!}\Big(q + \frac{h}{x} - 1\Big)^k = \Big(q + \frac{h}{x}\Big)^a.$$

Again, one easily checks that the differential equation $\partial(s) = \frac{a}{x}s$ has the same solution x^a as the functional equation

$$s(qx + h) = \Big(q + \frac{h}{x}\Big)^a s(x).$$

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BERNARD LE STUM
INSTITUT DE RECHERCHE MATHEMATIQUE (IRMAR)
UNIVERSITE DE RENNES I
CAMPUS DE BEAULIEU
RENNES
FRANCE
bernard.le-stum@univ-rennes1.fr

ADOLFO QUIRÓS
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD AUTÓNOMA DE MADRID
CAMPUS DE CANTOBLANCO
MADRID
SPAIN
adolfo.quiros@uam.es

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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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