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We introduce a differential graded Hopf algebroid associated to a proper Lie group action on an oriented manifold with boundary and prove that the cyclic cohomology of this Hopf algebroid is equal to the relative de Rham cohomology of invariant differential forms. When the action is cocompact, by investigating the boundaryless double of an oriented manifold with boundary, we prove that the cyclic cohomology of the above Hopf algebroid is of finite dimension.

1. Introduction

Hopf algebroids were introduced by Lu [1996] to generalize the notion of Hopf algebras. Connes and Moscovici [2001] developed a beautiful theory of cyclic cohomology for a Hopf algebroid, and used it to study the transverse index theory.

When Γ is a discrete group acting on a smooth manifold M, Kaminker and Tang [2009] showed that the graded commutative algebra of differential forms on the action groupoid $M \rtimes \Gamma$ is a differential graded Hopf algebroid with the coalgebra and antipode structures defined by taking the dual of the groupoid structure.

In the case of a Lie group G action, Tang, Yao and Zhang [Tang et al. 2013] considered the algebra $\mathscr{H}(G, M)$ of differential forms valued functions on G and defined a differential graded Hopf algebroid structure on this algebra. When the G-action is proper, they proved that the cyclic cohomology groups of this Hopf algebroid are equal to the de Rham cohomology groups of G-invariant differential forms. While when G is a Lie group acting properly and holomorphically on a complex manifold M, Zhang [2018] introduced two Hopf algebroids $\mathscr{H}(G, M; \bar{\partial})$ and $\mathscr{H}(G, \Omega^{0,\bullet}(M); \bar{\partial})$, and proved that the cyclic cohomology of each Hopf algebroid is equal to the Dolbeault cohomology of G-invariant differential forms.

In this paper, we let G be a Lie group acting on an oriented manifold M with boundary ∂M. We construct a differential graded Hopf algebroid $\mathscr{H}(G, \widetilde{\Omega}^*_D(M); d)$, where $\widetilde{\Omega}^k_D(M)$ is the subalgebra of $\Omega^k(M)$ generated by the Dirichlet k-forms and the multiplicative identity 1. When the G-action is proper, we are able to compute

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the cyclic cohomology of this Hopf algebroid, which is equal to the relative de Rham cohomology of G-invariant differential forms on M. When the G-action is cocompact, we investigate the boundaryless double $\widetilde{M}$ of M and prove that the relative de Rham cohomology groups of G-invariant differential forms on M are isomorphic to the de Rham cohomology groups of some special differential forms on $\widetilde{M}$. Then using a result of Tang, Yao and Zhang [Tang et al. 2013], we can prove that the cyclic cohomology of $\mathscr{H}(G, \Omega^*_D(M); d)$ is finite dimensional.

Theorem 1.1. *Let G be a Lie group acting on an oriented manifold M with boundary ∂M, and assume that the G-action is proper and cocompact. Then the cyclic cohomology groups of $\mathscr{H}(G, \widetilde{\Omega}^*_D(M); d)$ are of finite dimension.*

This paper is organized as follows. In Section 2, we introduce the differential graded Hopf algebroid $\mathscr{H}(G, \widetilde{\Omega}^*_D(M); d)$ and compute its cyclic cohomology groups. In Section 3, we investigate the boundaryless double $\widetilde{M}$ of M and give a proof of Theorem 1.1.

2. Hopf algebroid $\mathscr{H}(G, \widetilde{\Omega}^*_D(M); d)$ and its cyclic cohomology

Hopf algebroids and cyclic cohomology. First, we recall the definition of Hopf algebroids introduced by Lu [1996]. We will only work in the category of topological algebras here, and by tensor product $\otimes$ we always mean topological tensor product.

Following [Kaminker and Tang 2009], let A and B be unital topological algebras. A topological bialgebroid structure on A, over B, consists of the following data:

(1) A continuous algebra homomorphism $\alpha : B \to A$ called the *source map* and a continuous algebra anti-homomorphism $\beta : B \to A$ called the *target map*, satisfying
$$\alpha(a)\beta(b) = \beta(b)\alpha(a) \quad \text{for all } a, b \in B.$$
Let $A \otimes_B A$ be the quotient of $A \otimes A$ by the right $A \otimes A$ ideal generated by $\beta(a) \otimes 1 - 1 \otimes \alpha(a)$ for all $a \in B$.

(2) A continuous B-B bimodule map $\Delta : A \to A \otimes_B A$, called the *coproduct*, satisfying
   (a) $\Delta(1) = 1 \otimes_B 1$;
   (b) $(\Delta \otimes_B \mathrm{Id})\Delta = (\mathrm{Id} \otimes_B \Delta)\Delta : A \to A \otimes_B A \otimes_B A$;
   (c) $\Delta(a)(\beta(b) \otimes 1 - 1 \otimes \alpha(b)) = 0$ for $a \in A$, $b \in B$;
   (d) $\Delta(a_1 a_2) = \Delta(a_1)\Delta(a_2)$ for $a_1, a_2 \in A$.

(3) A continuous B-B bimodule map $\epsilon : A \to B$, called the *counit*, satisfying
   (a) $\epsilon(1) = 1$;
   (b) $\ker \epsilon$ is a left A ideal;
   (c) $(\epsilon \otimes_B \mathrm{Id})\Delta = (\mathrm{Id} \otimes_B \epsilon)\Delta = \mathrm{Id} : A \to A$.

A topological *Hopf algebroid* is a topological bialgebroid A, over B, which admits a continuous algebra anti-isomorphism $S : A \to A$ satisfying

(1) $S \circ \beta = \alpha$;

(2) $m_A(S \otimes \mathrm{Id})\Delta = \beta\epsilon S : A \to A$, with $m_A : A \otimes A \to A$ the multiplication on A;

(3) there is a linear map $\gamma : A \otimes_B A \to A \otimes A$ such that

 (a) if $\pi : A\otimes A \to A\otimes_B A$ is the natural projection, $\pi\gamma = \mathrm{Id} : A\otimes_B A \to A\otimes_B A$;

 (b) $m_A(\mathrm{Id} \otimes S)\gamma\Delta = \alpha\epsilon : A \to A$.

A topological *para-Hopf* algebroid is a topological bialgebroid A, over B, which admits a continuous algebra anti-isomorphism $S : A \to A$ such that

(1) $S^2 = \mathrm{Id}$ and $S\beta = \alpha$;

(2) $m_A(S \otimes_B \mathrm{Id})\Delta = \beta\epsilon S : A \to A$;

(3) $S(a^{(1)})^{(1)} a^{(2)} \otimes_B S(a^{(1)})^{(2)} = 1 \otimes_B S(a)$.

In the above formula, we have used Sweedler's notation for the coproduct

$$\Delta(a) = a^{(1)} \otimes_B a^{(2)}.$$

In the above definition, if A and B are differential graded algebras and all of the above maps are compatible with the differentials and of degree 0, then one would have a differential graded (para) Hopf algebroid; see [Gorokhovsky 2002].

In this paper, we will only deal with para-Hopf algebroids. As pointed out in [Kowalzig 2009, Section 2.6.13], any para-Hopf algebroid defined above is a Hopf algebroid. Therefore, for simplicity, we will abbreviate "para-Hopf algebroid" to "Hopf algebroid" in the sequel.

We now recall the cyclic module $A^\natural$ for $(A, B, \alpha, \beta, \Delta, \epsilon, S)$ introduced by Connes and Moscovici [2000].

Define

$$C^0 = B, \quad C^n = \underbrace{A \otimes_B A \otimes_B \cdots \otimes_B A}_{n},\ n \geq 1.$$

Then faces and degeneracy operators can be defined as follows:

$$\delta_0(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = 1 \otimes_B a^1 \otimes_B \cdots \otimes_B a^{n-1};$$

$$\delta_i(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B \Delta a^i \otimes_B \cdots \otimes_B a^{n-1}, \quad 1 \leq i \leq n-1;$$

$$\delta_n(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B a^{n-1} \otimes_B 1;$$

$$\sigma_i(a^1 \otimes_B \cdots \otimes_B a^{n+1}) = a^1 \otimes_B \cdots \otimes_B a^i \otimes_B \epsilon(a^{i+1}) \otimes_B a^{i+2} \otimes_B \cdots \otimes_B a^{n+1}.$$

The cyclic operators are defined by

$$\tau_n(a^1 \otimes_B \cdots \otimes_B a^n) = (\Delta^{n-1} S(a^1))(a^2 \otimes \cdots \otimes a^n \otimes 1).$$

Let

$$\begin{aligned}
&\beta'_n : C^n \to C^{n+1}, && \beta'_n := \sum_{i=0}^{n} (-1)^i \delta_i, \\
&\beta_n : C^n \to C^{n+1}, && \beta_n := \beta'_n + (-1)^{n+1} \delta_{n+1}, \\
&\lambda_n : C^n \to C^n, && \lambda_n := (-1)^n \tau_n, \\
&N_n : C^n \to C^n, && N_n := \sum_{i=0}^{n} \lambda_n^i.
\end{aligned}$$

According to [Kowalzig 2009, Chapter 1],

$$\begin{tikzcd}
 & {} & {} & {} & {} & \\
0 \arrow[r] & C^2 \arrow[r, "1-\lambda"] \arrow[u, "\beta"] & C^2 \arrow[r, "N"] \arrow[u, "\beta'"] & C^2 \arrow[r, "1-\lambda"] \arrow[u, "\beta"] & C^2 \arrow[r, "N"] \arrow[u, "\beta'"] & \cdots \\
0 \arrow[r] & C^1 \arrow[r, "1-\lambda"] \arrow[u, "\beta"] & C^1 \arrow[r, "N"] \arrow[u, "\beta'"] & C^1 \arrow[r, "1-\lambda"] \arrow[u, "\beta"] & C^1 \arrow[r, "N"] \arrow[u, "\beta'"] & \cdots \\
0 \arrow[r] & C^0 \arrow[r, "1-\lambda"] \arrow[u, "\beta"] & C^0 \arrow[r, "N"] \arrow[u, "\beta'"] & C^0 \arrow[r, "1-\lambda"] \arrow[u, "\beta"] & C^0 \arrow[r, "N"] \arrow[u, "\beta'"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

is a bicomplex. In this complex, the columns are periodic of order 2; for p even, the p-th column is the Hochschild complex $(C^\bullet, \beta)$; in case p is odd, the respective column is the acyclic complex $(C^\bullet, \beta')$. The Hochschild cohomology of $A^\natural$ is defined to be the cohomology of $(C^\bullet, \beta)$ and its cyclic cohomology is defined to be the cohomology of the total complex of the bicomplex.

The cyclic (resp. Hochschild) cohomology of $(A, B, \alpha, \beta, \Delta, \epsilon, S)$ is defined to be the cyclic (resp. Hochschild) cohomology of $A^\natural$.

Remark 2.1. If A, B are differential graded algebras, and $(A, B, \alpha, \beta, \Delta, \epsilon, S, d)$ is a differential graded Hopf algebroid with the differential d, then the cyclic cohomology of $(A, B, \alpha, \beta, \Delta, \epsilon, S, d)$ is defined to be the cohomology of the total complex of a tricomplex as in [Gorokhovsky 2002].

Hopf algebroid $\mathscr{H}(G, \widetilde{\Omega}^*_D(M); d)$. Assume that G is a Lie group acting on an oriented manifold M with boundary ∂M. Let $\Omega^k_D(M)$ be the space of Dirichlet k-forms — those satisfying $i^*\omega = 0$ where $i : \partial M \hookrightarrow M$ is the inclusion of the boundary. We denote by $\widetilde{\Omega}^*_D(M)$ the subalgebra of $\Omega^*(M)$ generated by $\Omega^*_D(M)$ and the multiplicative identity 1.

Now, we define B to be the algebra $\widetilde{\Omega}^*_D(M)$, and A to be the algebra of B-valued functions on G. Then both A and B are differential graded algebras with the differential d.

For any smooth function a on M and any group element g in G, let

$$(g^*(a))(x) := a(x \cdot g).$$

Similar to the construction of Hopf algebroid $\mathscr{H}(G, M)$ in [Tang et al. 2013], we can define the source and target maps $\alpha, \beta : B \to A$ as follows:

$$\alpha(b)(g) = b \quad \text{and} \quad \beta(b)(g) = g^*(b).$$

The space $A \otimes_B A$ is isomorphic to the space of $\widetilde{\Omega}^*_D(M)$-valued functions on $G \times G$ when we consider the projective tensor product, i.e.,

$$(\phi \otimes_B \psi)(g_1, g_2) = \phi(g_1)(g_1^*(\psi(g_2))),$$

for any $\phi, \psi \in A$. Then we can define the coproduct $\Delta : A \to A \otimes_B A$ satisfying

$$\Delta(\phi)(g_1, g_2) = \phi(g_1 g_2),$$

and the counit map $\epsilon : A \to B$ satisfying $\epsilon(\phi) = \phi(1)$ for any $\phi \in A$.

It is easy to check that $(A, B, \alpha, \beta, \Delta, \epsilon, d)$ is a differential graded topological bialgebroid.

Define the antipode S by

$$S(\phi)(g) = g^*(\phi(g^{-1})).$$

We can compute that

$$\begin{aligned}
S^2(\phi)(g) &= g^*(S(\phi)(g^{-1})) = g^*((g^{-1})^*\phi(g)) = \phi(g);\\
(S\beta(b))(g) &= g^*(\beta(b)(g^{-1})) = g^*((g^{-1})^*(b)) = b;\\
(m_A(S\otimes \mathrm{Id})\Delta)(\phi)(g) &= g^*\phi(g^{-1}g) = g^*\phi(1) = g^*(S(\phi)(1)) = (\beta\epsilon S)(\phi)(g);\\
\big(S(a^{(1)})^{(1)}\otimes_B S(a^{(1)})^{(2)}\big)(a^{(2)}\otimes 1)(g_1, g_2) &= (g_1g_2)^*(a((g_1g_2)^{(-1)}g_1))\\
&= g_1^*(g_2^*(a(g_2^{-1}))) = 1\otimes_B S(a)(g_1, g_2).
\end{aligned}$$

Hence, $(A, B, \alpha, \beta, \Delta, \epsilon, S, d)$ is a differential graded Hopf algebroid; we denote it by $\mathscr{H}(G, \widetilde{\Omega}^*_D(M); d)$.

Hopf cyclic cohomology of $\mathscr{H}(G, \widetilde{\Omega}^*_D(M); d)$. Now we compute the cyclic cohomology of the differential graded Hopf algebroid $\mathscr{H}(G, \widetilde{\Omega}^*_D(M); d)$.

Let ζ be any $\widetilde{\Omega}^*_D(M)$-valued function on $G^{\times n}$. Then

$$\delta_i(\zeta)(g_1,\dots,g_{n+1}) = \begin{cases} g_1^*(\zeta(g_2,\dots,g_{n+1})), & i=0,\\ \zeta(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}), & 1\le i\le n,\\ \zeta(g_1,\dots,g_n), & i=n+1,\end{cases}$$

$$\sigma_i(\zeta)(g_1,\dots,g_{n-1}) = \zeta(g_1,\dots,g_i,1,g_{i+1},\dots,g_{n-1}),\quad 0\le i\le n-1,$$

$$\tau_n(\zeta)(g_1,\dots,g_n) = (g_1\dots g_n)^*\zeta\big((g_1g_2\dots g_n)^{-1},g_1,\dots,g_{n-1}\big),$$

$$\beta_n(\zeta)(g_1,\dots,g_{n+1}) = g_1^*(\zeta(g_2,\dots,g_{n+1})) + \sum_{i=1}^{n}(-1)^i\zeta(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}) + (-1)^{n+1}\zeta(g_1,\dots,g_n).$$

We denote by $\Omega^k_D(M)^G$ the space of G-invariant Dirichlet k-forms, and let $H^q_r(M,G)$ be the q-th cohomology group of the complex $(\Omega^\bullet_D(M)^G, d)$. Then the cyclic cohomology $HC^\bullet(\mathscr{H}(G,\widetilde{\Omega}^*_D(M);d))$ can be computed as follows:

Proposition 2.2. *Let G be a Lie group acting on an oriented manifold M with boundary ∂M. If the G-action is proper, then*

$$HC^k(\mathscr{H}(G,\widetilde{\Omega}^*_D(M);d)) = \bigoplus_{p\ge 0} H^{k-2p}_r(M,G) \qquad \textit{for } k \textit{ odd};$$

$$HC^k(\mathscr{H}(G,\widetilde{\Omega}^*_D(M);d)) = \bigoplus_{p\ge 0} H^{k-2p}_r(M,G)\oplus\mathbb{R} \quad \textit{for } k \textit{ even}.$$

Proof. Let $dg := dm(g)$ be a fixed left invariant Haar measure on G. Since the G-action is proper, according to [Bourbaki 2004], there exists a smooth, nonnegative function $c(x)$ on M whose support intersects every orbit of the action in a compact subset, satisfying

$$\int_G c^2(x\cdot g)\,dg = 1.$$

According to [Crainic 2003, Section 2.1, Proposition 1], let ζ be any $\widetilde{\Omega}^*_D(M)$-valued function on $G^{\times k}$ with $k\ge 1$, which induces an $\widetilde{\Omega}^*_D(M)$-valued function $h(\zeta)$ on $G^{\times(k-1)}$ satisfying

$$h(\zeta)(g_1,\dots,g_{k-1})(x) = \int_G g^*(\zeta(g^{-1},g_1,\dots,g_{k-1}))(x)\cdot c^2(x\cdot g)\,dg.$$

Then one can easily get

$$h\circ\beta_k + \beta_{k-1}\circ h = \mathrm{Id}.$$

Now, according to the theory of homological algebra (see [Rotman 1979]), the Hopf cyclic cohomology of $\mathscr{H}(G,\widetilde{\Omega}^*_D(M);d)$ is isomorphic to the cohomology of the total complex of the following bicomplex:

$$
\begin{tikzcd}
 & & & & & \\
0 \arrow[r] & \Omega_D^2(M)^G \arrow[u, "d"] \arrow[r] & 0 \arrow[u] \arrow[r] & \Omega_D^2(M)^G \arrow[u, "d"] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & \Omega_D^1(M)^G \arrow[u, "d"] \arrow[r] & 0 \arrow[u] \arrow[r] & \Omega_D^1(M)^G \arrow[u, "d"] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & \Omega_D^0(M)^G \oplus \mathbb{R} \arrow[u, "d"] \arrow[r] & 0 \arrow[u] \arrow[r] & \Omega_D^0(M)^G \oplus \mathbb{R} \arrow[u, "d"] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Then we can easily get

$$HC^k(\mathscr{H}(G, \widetilde{\Omega}_D^*(M); d)) = \bigoplus_{p\geq 0} H_r^{k-2p}(M, G) \qquad \text{for } k \text{ odd;}$$

$$HC^k(\mathscr{H}(G, \widetilde{\Omega}_D^*(M); d)) = \bigoplus_{p\geq 0} H_r^{k-2p}(M, G) \oplus \mathbb{R} \quad \text{for } k \text{ even.} \qquad \square$$

3. The boundaryless double $\widetilde{M}$ and a proof of Theorem 1.1

Throughout this section, let G be a locally compact Lie group acting on the left of an n-dimensional oriented manifold M with boundary ∂M. Assume that the G-action is proper and cocompact.

The boundaryless double $\widetilde{M}$. First, we need to comment a little on the geometric structure near the boundary ∂M.

If N is a compact oriented manifold with boundary ∂N, a basic result called the *collar neighborhood theorem* states that the boundary ∂N has an open neighborhood which is diffeomorphic to the product $\partial N \times [0, 1)$; see [Wloka et al. 1995, Section 5.12].

In our noncompact settings, similar to [Wloka et al. 1995], there exists a smooth vector field V on M such that $V(p)$ is inward for any $p \in \partial M$. Since the G-action is proper, we can assume that V is G-invariant; see [Mathai and Zhang 2010, Section 2]. This vector field has a local 1-parameter group of integral curves, and the integral curves with initial conditions on the boundary define a smooth map from a G-invariant neighborhood of $\partial M \times \{0\}$ in $\partial M \times \mathbb{R}_+$ to M. Since the derivative map over each point of $\partial M \times \{0\}$ is an isomorphism, then the integral flow is locally a diffeomorphism at boundary points. Finally, since the G-action is cocompact, we can choose a small ε such that the restriction of the integral flow to $\partial M \times [0, \varepsilon)$ is a diffeomorphism.

Then we have the following property:

Proposition 3.1. *There exists a smooth diffeomorphism*

$$\Upsilon : \partial M \times [0, 1) \to W,$$

where W is a G-invariant neighborhood of ∂M in M. Moreover, the diffeomorphism Υ should be G-equivariant.

Let the two copies of M be denoted $M \times \{1\}$ and $M \times \{-1\}$. The boundaryless double $\widetilde{M}$ of M can be constructed from the set $(M\times\{1\})\cup(M\times\{-1\})$ by identifying

$$(x, 1) \cong (x, -1) \quad \text{for all } x \in \partial M.$$

Then, $\widetilde{M}$ is naturally an n-dimensional C^0-manifold without boundary; see [Schwarz 1995, Chapter 1]. We use the notation $[(x, \pm 1)]$ for the points of $\widetilde{M}$.

Let $\pi : (M\times\{1\})\cup(M\times\{-1\}) \to \widetilde{M}$ be the canonical map $(x, \pm 1) \mapsto [(x, \pm 1)]$, and $\jmath_\pm : M \to \widetilde{M}$ be the inclusions $x \mapsto [(x, \pm 1)]$. Then the G-equivariant diffeomorphism Υ in Proposition 3.1 induces a map $\widetilde{\Upsilon} : \partial M \times (-1, 1) \to \widetilde{W}$ defined by

$$\widetilde{\Upsilon}(x, t) = \begin{cases} \jmath_+ \circ \Upsilon(x, t) & \text{if } (x, t) \in \partial M \times [0, 1), \\ \jmath_- \circ \Upsilon(x, -t) & \text{if } (x, t) \in \partial M \times (-1, 0], \end{cases}$$

where $\widetilde{W} = \pi\big((W \times \{1\}) \cup (W \times \{-1\})\big)$ is an open set in $\widetilde{M}$.

Similar to the construction in [Wloka et al. 1995, Theorem 5.77], we can easily define a C^∞ structure on $\widetilde{M}$.

Proposition 3.2. *There is a unique C^∞ structure on $\widetilde{M}$ such that both inclusions $\jmath_\pm : M \to \widetilde{M}$ are C^∞, and such that $\widetilde{\Upsilon} : \partial M \times (-1, 1) \to \widetilde{W}$ is a diffeomorphism.*

Equip $M \times \{1\}$ with the same orientation as M, and $M \times \{-1\}$ with the opposite orientation. Then $\widetilde{M}$ is oriented. We define the reflection

$$\gamma : \widetilde{M} \to \widetilde{M},$$
$$[(x, \pm 1)] \mapsto [(x, \mp 1)].$$

Then γ is a smooth map which exchange the orientation of $\widetilde{M}$.

The G-action on M naturally induces a G-action on $\widetilde{M}$ defined by

$$g \cdot [(x, \pm 1)] := [(g \cdot x, \pm 1)] \quad \text{for all } g \in G,\ [(x, \pm 1)] \in \widetilde{M}.$$

One can easily prove that this G-action is smooth, proper and cocompact and that the diffeomorphism $\widetilde{\Upsilon}$ is G-equivariant.

Let $\Omega^k(\widetilde{M})^G$ denote the space of G-invariant k-forms on $\widetilde{M}$, and define

$$\Omega^k(\widetilde{M})^{\widetilde{G}} := \{\omega \in \Omega^k(\widetilde{M})^G | \gamma^*\omega = -\omega\}.$$

We denote by $H^q(\widetilde{M}, \widetilde{G})$ the q-th cohomology group of the complex $(\Omega^\bullet(\widetilde{M})^{\widetilde{G}}, d)$.

An isomorphism between $H_r^q(M, G)$ and $H^q(\widetilde{M}, \widetilde{G})$. The G-equivariant inclusion $\jmath_+ : M \to \widetilde{M}$ induces an injection $\jmath_+^* : (\Omega^\bullet(\widetilde{M})^{\widetilde{G}}, d) \to (\Omega^\bullet(M)^G, d)$. We define

$$\Omega^\bullet(M)^{\widetilde{G}} := \{\jmath_+^*\eta \mid \eta \in \Omega^\bullet(\widetilde{M})^{\widetilde{G}}\} \subset \Omega^\bullet(M)^G.$$

Now let $\omega \in \Omega^k(M)^G$, $\tilde{\eta} \in \Omega^k(\widetilde{M})^{\widetilde{G}}$, and let $\boldsymbol{x} = (x_1, \dots, x_{n-1})$ be any local coordinates on ∂M.

According to Proposition 3.1, $(\boldsymbol{x}, t)$ are local coordinates on M with $t \in [0, 1)$. In terms of local coordinates $(\boldsymbol{x}, t)$, we can write

$$\omega(\boldsymbol{x}, t) = \sum_{\sharp I = k} \omega_I(\boldsymbol{x}, t)\, dx_I + \sum_{\sharp J = k-1} \omega_J(\boldsymbol{x}, t)\, dt \wedge dx_J \quad \text{for } t \in [0, 1), \tag{3-1}$$

where for each multiindex $I = \{i_1, \dots, i_k\}$,

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Here, if $\sharp I = n$, we mean that $\omega_I(\boldsymbol{x}, t) \equiv 0$, while if $\sharp J = -1$, we mean $\omega_J(\boldsymbol{x}, t) \equiv 0$. Then $\omega \in \Omega_D^k(M)^G$ if and only if $\omega_I(\boldsymbol{x}, 0) = 0$ for all $\sharp I = k$ and for any local coordinates $\boldsymbol{x}$ on ∂M.

According to Proposition 3.2, $(\boldsymbol{x}, t)$ are local coordinates on $\widetilde{M}$ with $t \in (-1, 1)$. In terms of local coordinates, we can write

$$\tilde{\eta}(\boldsymbol{x}, t) = \sum_{\sharp I = k} \tilde{\eta}_I(\boldsymbol{x}, t)\, dx_I + \sum_{\sharp J = k-1} \tilde{\eta}_J(\boldsymbol{x}, t)\, dt \wedge dx_J \quad \text{for } t \in (-1, 1).$$

Then

$$(\jmath_+^*\tilde{\eta})(\boldsymbol{x}, t) = \sum_{\sharp I = k} \tilde{\eta}_I(\boldsymbol{x}, t)\, dx_I + \sum_{\sharp J = k-1} \tilde{\eta}_J(\boldsymbol{x}, t)\, dt \wedge dx_J \quad \text{for } t \in [0, 1). \tag{3-2}$$

Since $\gamma^*\tilde{\eta} = -\tilde{\eta}$, it follows that

$$\tilde{\eta}_I(\boldsymbol{x}, -t) = -\tilde{\eta}_I(\boldsymbol{x}, t) \quad \text{and} \quad \tilde{\eta}_J(\boldsymbol{x}, -t) = \tilde{\eta}_J(\boldsymbol{x}, t) \quad \text{for } t \in (-1, 1). \tag{3-3}$$

Then

$$\tilde{\eta}_I(\boldsymbol{x}, 0) \equiv 0 \Rightarrow \jmath_+^*\tilde{\eta} \in \Omega_D^k(M)^G \Rightarrow \Omega^k(M)^{\widetilde{G}} \subset \Omega_D^k(M)^G.$$

Using (3-1)–(3-3), one can easily prove that $\omega \in \Omega^k(M)^{\widetilde{G}}$ if and only if

i) the Taylor series of $\omega_I(\boldsymbol{x}, t)$ about t at $t = 0$ includes only odd powers for all $\sharp I = k$ and for any local coordinates $\boldsymbol{x}$ on ∂M;

ii) the Taylor series of $\omega_J(\boldsymbol{x}, t)$ about t at $t = 0$ includes only even powers for all $\sharp J = k - 1$ and for any local coordinates $\boldsymbol{x}$ on ∂M.

Since $\Omega^\bullet(M)^{\widetilde{G}} \subset \Omega^\bullet_D(M)^G$, the injection $\jmath_+^* : (\Omega^\bullet(\widetilde{M})^{\widetilde{G}}, d) \to (\Omega^\bullet_D(M)^G, d)$ induces an homomorphism $\jmath_+^* : H^q(\widetilde{M}, \widetilde{G}) \to H^q_r(M, G)$. More precisely, we have the following result:

Proposition 3.3. *The homomorphism $\jmath_+^* : H^q(\widetilde{M}, \widetilde{G}) \to H^q_r(M, G)$ is actually an isomorphism for any integer $q \geq 0$.*

Proof. Let

$$\Omega^k_D(M, G, \widetilde{G}) := \Omega^k_D(M)^G / \Omega^k(M)^{\widetilde{G}}.$$

We denote the q-th cohomology group of the complex $(\Omega^\bullet_D(M, G, \widetilde{G}), d)$ by $H^q_r(M, G, \widetilde{G})$. Since the short exact sequence

$$0 \to (\Omega^\bullet(\widetilde{M})^{\widetilde{G}}, d) \to (\Omega^\bullet_D(M)^G, d) \to (\Omega^\bullet_D(M, G, \widetilde{G}), d) \to 0$$

induces a long exact sequence

$$\cdots \to H^q(\widetilde{M}, \widetilde{G}) \to H^q_r(M, G) \to H^q_r(M, G, \widetilde{G}) \to H^{q+1}(\widetilde{M}, \widetilde{G}) \to \cdots,$$

we only need to prove that $H^q_r(M, G, \widetilde{G}) = 0$ for all $q \geq 0$, i.e., for any $\alpha \in \Omega^q_D(M)^G$ with $d\alpha \in \Omega^{q+1}(M)^{\widetilde{G}}$, we need to construct a smooth section $\beta \in \Omega^{q-1}_D(M)^G$ such that $d\beta - \alpha \in \Omega^q(M)^{\widetilde{G}}$.

Let $\boldsymbol{x} = (x_1, \dots, x_{n-1})$ be any local coordinates on ∂M. In terms of local coordinates $(\boldsymbol{x}, t)$ on M, we can write

$$\alpha(\boldsymbol{x}, t) = \sum_{\sharp I = q} \alpha_I(\boldsymbol{x}, t)\, dx_I + \sum_{\sharp J = q-1} \alpha_J(\boldsymbol{x}, t)\, dt \wedge dx_J \quad \text{for } t \in [0, 1), \tag{3-4}$$

where $\alpha_I(\boldsymbol{x}, 0) = 0$ for all $\sharp I = q$. Let

$$\widehat{\beta}(\boldsymbol{x}, t) = \sum_{\sharp J = q-1} \Big(\int_0^t \alpha_J(\boldsymbol{x}, s)\, ds \Big)\, dx_J \quad \text{for } t \in [0, 1). \tag{3-5}$$

One can easily see that the above construction is independent of the choice of local coordinates $\boldsymbol{x}$ on ∂M, so $\widehat{\beta}$ is a G-invariant $(q-1)$-form on W.

Let ρ be a smooth function on $[0, 1)$ such that $\rho\,|_{[0,1/3]} = 1$ while $\rho\,|_{[2/3,1)} = 0$. Then ρ induces a G-invariant smooth function $\hat{\rho}$ on $\partial M \times [0, 1)$ satisfying

$$\hat{\rho}(x, t) = \rho(t) \quad \text{for } (x, t) \in \partial M \times [0, 1).$$

Define

$$\beta(p) = \begin{cases} \hat{\rho}(\Upsilon^{-1}(p)) \cdot \hat{\beta}(p) & \text{if } p \in W, \\ 0 & \text{if } p \in M \setminus W. \end{cases} \tag{3-6}$$

Then β is a G-invariant $(q-1)$-form on M.

According to (3-5) and (3-6), in terms of local coordinates $(\boldsymbol{x}, t)$, we have

$$\beta(\boldsymbol{x}, t) = \sum_{\sharp J = q-1} \Big(\int_0^t \alpha_J(\boldsymbol{x}, s)\, ds \Big) dx_J \quad \text{for } t \in \big[0, \tfrac{1}{3}\big). \tag{3-7}$$

Then

$$\beta(\boldsymbol{x}, 0) \equiv 0 \Rightarrow \beta \in \Omega_D^{q-1}(M)^G.$$

Combining (3-4) and (3-7), we get

$$\begin{aligned}(d\beta - \alpha)&(\boldsymbol{x}, t)\\ &= \sum_{\sharp J = q-1, i} \Big(\int_0^t \frac{\partial \alpha_J(\boldsymbol{x}, s)}{\partial x_i} ds \Big) dx_i \wedge dx_J - \sum_{\sharp I = q} \alpha_I(\boldsymbol{x}, t)\, dx_I \quad \text{for } t \in \big[0, \tfrac{1}{3}\big).\end{aligned}$$

Observe that the expression on the right-hand side contains no terms with a factor dt, so letting $\theta = d\beta - \alpha$, we can write

$$\theta(\boldsymbol{x}, t) = \sum_{\sharp I = q} \theta_I(\boldsymbol{x}, t)\, dx_I \quad \text{for } t \in \big[0, \tfrac{1}{3}\big).$$

Then

$$d\theta(\boldsymbol{x}, t) = \sum_{\sharp I = q, i} \frac{\partial \theta_I(\boldsymbol{x}, t)}{\partial x_i} dx_i \wedge dx_I + \sum_{\sharp I = q} \frac{\partial \theta_I(\boldsymbol{x}, t)}{\partial t} dt \wedge dx_I \quad \text{for } t \in \big[0, \tfrac{1}{3}\big).$$

Since $d\alpha \in \Omega^{q+1}(M)^{\widetilde{G}}$, we have $d\theta = d(d\beta - \alpha) = -d\alpha \in \Omega^{q+1}(M)^{\widetilde{G}}$.

According to the discussion preceding Proposition 3.3, we get that the Taylor series of $\partial\theta_I(\boldsymbol{x}, t)/\partial t$ about t at $t = 0$ includes only even powers for all $\sharp I = q$. Since $\alpha \in \Omega_D^q(M)^G$ and $\beta \in \Omega_D^{q-1}(M)^G$, we have $\theta \in \Omega_D^q(M)^G$, then

$$\theta_I(\boldsymbol{x}, 0) = 0 \quad \text{for all } \sharp I = q.$$

Hence, the Taylor series of $\theta_I(\boldsymbol{x}, t)$ about t at $t = 0$ include only odd powers for all $\sharp I = q$, which implies that $\theta \in \Omega^q(M)^{\widetilde{G}}$. □

A proof of Theorem 1.1. Now we give a proof of Theorem 1.1. According to Propositions 2.2 and 3.3, we only need to prove that the cohomology groups $H^\bullet(\widetilde{M}, \widetilde{G})$ are of finite dimension.

Following [Tang et al. 2013], since G acts on $\widetilde{M}$ properly, we can assume that $\widetilde{M}$ is endowed with a G-invariant metric $g^{T\widetilde{M}}$ which is also invariant under the reflection γ. Then $\Omega^\bullet(\widetilde{M})$ carry the natural inner product such that for any $\alpha, \beta \in \Omega^\bullet(\widetilde{M})$ with compact supports,

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta, \tag{3-8}$$

where $*$ is the usual de Rham Hodge operator.

Since the G-action is cocompact, there exists a compact subset Y of $\widetilde{M}$ such that $G(Y) = \bigcup_{g\in G} g \cdot Y = \widetilde{M}$; see [Phillips 1989, Lemma 2.3]. Let U, U' be two open subsets of $\widetilde{M}$ such that they are both invariant under the reflection γ and that the closures $\overline{U}$ and $\overline{U'}$ are both compact in $\widetilde{M}$, and that

$$Y \subset U \subset \overline{U} \subset U' \subset \overline{U'}.$$

Let $f : \widetilde{M} \to [0, 1]$ be a smooth function such that $f|_U = 1$, supp(f)$\subset U'$ and that f is invariant under γ. Let

$$f\Omega^\bullet(\widetilde{M})^G := \{fs \mid s \in \Omega^\bullet(\widetilde{M})^G\}.$$

Then one can define the operators

$$d_f : f\Omega^k(\widetilde{M})^G \to f\Omega^{k+1}(\widetilde{M})^G,$$
$$f\alpha \mapsto f d\alpha,$$

and $d_f^* : f\Omega^{k+1}(\widetilde{M})^G \to f\Omega^k(\widetilde{M})^G$ satisfying

$$(d_f(f\alpha), f\beta) = (f\alpha, d_f^*(f\beta)),$$

for any $f\alpha \in f\Omega^k(\widetilde{M})^G$ and $f\beta \in f\Omega^{k+1}(\widetilde{M})^G$; see [Tang et al. 2013, Section 3].

Let

$$\Delta_f = d_f d_f^* + d_f^* d_f,$$

and let $\mathcal{H}_f^\bullet(\widetilde{M})^G$ denote the kernel of the operator Δ_f. Then Tang, Yao and Zhang [Tang et al. 2013] proved that Δ_f has essentially the same properties as the standard Laplace–Beltrami operator on a compact manifold:

Proposition 3.4. *For any integer $k \geq 0$, $\dim \mathcal{H}_f^k(\widetilde{M})^G < \infty$, and thus the orthogonal projection*

$$H_f : f\Omega^k(\widetilde{M})^G \to \mathcal{H}_f^k(\widetilde{M})^G$$

is well-defined, and there exists a unique operator, the Green operator $\mathfrak{G}$,

$$\mathfrak{G} : f\Omega^k(\widetilde{M})^G \to f\Omega^k(\widetilde{M})^G,$$

with $\mathfrak{G}(\mathcal{H}_f^k(\widetilde{M})^G) = 0$, $d_f \circ \mathfrak{G} = \mathfrak{G} \circ d_f$, $d_f^* \circ \mathfrak{G} = \mathfrak{G} \circ d_f^*$ and

$$\mathrm{Id} = H_f + \Delta_f \circ \mathfrak{G}$$

on $f\Omega^k(\widetilde{M})^G$. Furthermore, $\mathfrak{G}$ commutes with any linear operator that commutes with Δ_f.

Remark 3.5. Since the metric $g^{T\widetilde{M}}$ and the cut-off function f are both invariant under the reflection γ, according to the definitions of d_f and d_f^*, one can easily prove that

$$\gamma^* \circ d_f = d_f \circ \gamma^* \quad \text{and} \quad \gamma^* \circ d_f^* = d_f^* \circ \gamma^*.$$

on $f\Omega^\bullet(\widetilde{M})^G$. Then according to Proposition 3.4, both $\mathfrak{G}$ and H_f commute with γ^*.

Now define

$$\mathcal{H}_f^\bullet(\widetilde{M})^{\widetilde{G}} := \{f\omega \in \mathcal{H}_f^\bullet(\widetilde{M})^G \mid \gamma^*(f\omega) = -f\omega\}.$$

Then $\dim \mathcal{H}_f^\bullet(\widetilde{M})^{\widetilde{G}} < \infty$. Moreover, we have the following result:

Proposition 3.6. *For any integer $k \geq 0$, the orthogonal projection H_f induces an isomorphism*

$$H : H^k(\widetilde{M}, \widetilde{G}) \to \mathcal{H}_f^k(\widetilde{M})^{\widetilde{G}}.$$

Proof. Define

$$f\Omega^\bullet(\widetilde{M})^{\widetilde{G}} := \{f\omega \mid \omega \in \Omega^\bullet(\widetilde{M})^{\widetilde{G}}\} \subset f\Omega^\bullet(\widetilde{M})^G.$$

Suppose $\alpha \in \Omega^k(\widetilde{M})^{\widetilde{G}}$ with $d\alpha = 0$. Then $f\alpha \in f\Omega^k(\widetilde{M})^{\widetilde{G}}$ satisfies

$$d_f(f\alpha) = f d\alpha = 0.$$

According to Proposition 3.4 and Remark 3.5, we have

$$f\alpha = d_f d_f^* \mathfrak{G}(f\alpha) + d_f^* d_f \mathfrak{G}(f\alpha) + H_f(f\alpha),$$

where $d_f d_f^* \mathfrak{G}(f\alpha)$, $d_f^* d_f \mathfrak{G}(f\alpha)$ and $H_f(f\alpha)$ are elements in $f\Omega^k(\widetilde{M})^{\widetilde{G}}$ and mutually orthogonal to each other with respect to the inner product (3-8). Since

$$\big(f\alpha, d_f^* d_f \mathfrak{G}(f\alpha)\big) = \big(d_f(f\alpha), d_f \mathfrak{G}(f\alpha)\big) = 0,$$

we have

$$f\alpha = d_f d_f^* \mathfrak{G}(f\alpha) + H_f(f\alpha). \tag{3-9}$$

Letting $[\alpha]$ denote the cohomology class in $H^k(\widetilde{M}, \widetilde{G})$ determined by α, we define $H([\alpha])$ to be $H_f(f\alpha)$.

If $\alpha = d\beta$ with $\beta \in \Omega^{k-1}(\widetilde{M})^{\widetilde{G}}$, we have

$$d_f(f\beta) = f d\beta = f\alpha.$$

Then

$$\big(f\alpha, H_f(f\alpha)\big) = \big(d_f(f\beta), H_f(f\alpha)\big) = \big(f\beta, d_f^*(H_f(f\alpha))\big) = 0.$$

According to (3-9), we get

$$H([\alpha]) = H_f(f\alpha) = 0.$$

This means that H is a well-defined map from $H^k(\widetilde{M}, \widetilde{G})$ to $\mathcal{H}_f^k(\widetilde{M})^{\widetilde{G}}$.

When $H([\alpha]) = H_f(f\alpha) = 0$, according to (3-9), we have $f\alpha = d_f d_f^* \mathfrak{G}(f\alpha)$. Write $d_f^* \mathfrak{G}(f\alpha) = f\beta$. Then

$$f\beta \in f\Omega^{k-1}(\widetilde{M})^{\widetilde{G}} \quad \text{and} \quad f\alpha = d_f(f\beta).$$

Hence, $\alpha = d\beta$. This implies that H is injective.

For any $f\beta \in \mathcal{H}_f^k(\widetilde{M})^{\widetilde{G}}$, since

$$d_f(f\beta) = f d\beta = 0,$$

we have $d\beta = 0$, which implies that β is a smooth closed form in $\Omega^k(\widetilde{M})^{\widetilde{G}}$ with $H([\beta]) = H_f(f\beta) = f\beta$. Then H is onto. □

Theorem 1.1 is a corollary of Propositions 2.2, 3.3, 3.4, and 3.6.

References

[Bourbaki 2004] N. Bourbaki, *Integration*, vol. II: Chapters 7–9, Elements of Mathematics, Springer, Berlin, 2004. MR Zbl

[Connes and Moscovici 2000] A. Connes and H. Moscovici, "Cyclic cohomology and Hopf algebra symmetry", *Lett. Math. Phys.* **52**:1 (2000), 1–28. MR Zbl

[Connes and Moscovici 2001] A. Connes and H. Moscovici, "Differentiable cyclic cohomology and Hopf algebraic structures in transverse geometry", pp. 217–255 in *Essays on geometry and related topics*, vol. 1, edited by E. Ghys et al., Monogr. Enseign. Math. **38**, Enseignement Math., Geneva, 2001. MR Zbl

[Crainic 2003] M. Crainic, "Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes", *Comment. Math. Helv.* **78**:4 (2003), 681–721. MR Zbl

[Gorokhovsky 2002] A. Gorokhovsky, "Secondary characteristic classes and cyclic cohomology of Hopf algebras", *Topology* **41**:5 (2002), 993–1016. MR Zbl

[Kaminker and Tang 2009] J. Kaminker and X. Tang, "Hopf algebroids and secondary characteristic classes", *J. Noncommut. Geom.* **3**:1 (2009), 1–25. MR Zbl

[Kowalzig 2009] N. Kowalzig, *Hopf algebroids and their cyclic theory*, Ph.D. thesis, Utrecht University, 2009, available at https://www.math.ru.nl/~landsman/Niels.pdf.

[Lu 1996] J.-H. Lu, "Hopf algebroids and quantum groupoids", *Internat. J. Math.* **7**:1 (1996), 47–70. MR Zbl

[Mathai and Zhang 2010] V. Mathai and W. Zhang, "Geometric quantization for proper actions", *Adv. Math.* **225**:3 (2010), 1224–1247. MR Zbl

[Phillips 1989] N. C. Phillips, *Equivariant K-theory for proper actions*, Pitman Research Notes in Mathematics Series **178**, Longman Scientific & Technical, Harlow, England, 1989. MR Zbl

[Rotman 1979] J. J. Rotman, *An introduction to homological algebra*, Pure and Applied Mathematics **85**, Academic Press, New York, 1979. MR Zbl

[Schwarz 1995] G. Schwarz, *Hodge decomposition—a method for solving boundary value problems*, Lecture Notes in Mathematics **1607**, Springer, 1995. MR Zbl

[Tang et al. 2013] X. Tang, Y.-J. Yao, and W. Zhang, "Hopf cyclic cohomology and Hodge theory for proper actions", *J. Noncommut. Geom.* **7**:3 (2013), 885–905. MR Zbl

[Wloka et al. 1995] J. T. Wloka, B. Rowley, and B. Lawruk, *Boundary value problems for elliptic systems*, Cambridge University Press, 1995. MR Zbl

[Zhang 2018] X. Zhang, "Hopf cyclic cohomology and Hodge theory for proper actions on complex manifolds", *Front. Math. China* **13**:5 (2018), 1189–1214. MR Zbl

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