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We consider compact constant mean curvature surfaces with boundary immersed in a mean convex region of the Euclidean space or in the unit sphere. We prove that the weak Morse index is bounded from below by a linear function of the genus and number of boundary components.

1. Introduction

Let W be a Riemannian manifold with nonempty boundary such that its boundary ∂W is a union of smooth hypersurfaces. Let $M \subset W$ be a compact constant mean curvature hypersurface such that M intersects the regular part of ∂W along its boundary in a right angle. It is well known that such hypersurfaces are critical points of the area functional for variations of M that preserve the enclosed volume and keep the boundary freely on ∂W. We recall that the variations allowed of M are variations $\phi : (-\varepsilon, \varepsilon) \times M \to W$ whose immersions $\phi_t : M \to W$ satisfy $\phi_t(\text{int } M) \subset \text{int } W$ and $\phi_t(\partial M) \subset \text{int } \partial W$ for all $t \in (-\varepsilon, \varepsilon)$; see [Ros and Vergasta 1995, Section 1]. These hypersurfaces arise in many geometrical and physical problems and are referred as *free boundary CMC hypersurfaces* (FBCMC hypersurfaces, for short). They have been studied since the 19th century and still form a very active topic in differential geometry. We refer the reader to the books of Finn [1986] and López [2013] for a nice introduction to this subject.

An important problem about FBCMC hypersurfaces is to classify those ones that are *stable*, that is, whose second variation of the area is nonnegative for volume preserving variations. For instance, in the case that W is a geodesic ball in a space form, a well-known conjecture asserts that the totally geodesic ball and the spherical caps are the only solutions. It was confirmed by Ros and Vergasta [1995] and Nunes [2017] in dimension two, and more recently by Wang and Xia [2019] in any dimension and also for capillary hypersurfaces. Other results on stable FBCMC hypersurfaces can be found for instance in [Ainouz and Souam 2016; Athanassenas

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1987; Barbosa 2018; Choe and Koiso 2016; Li and Xiong 2017; 2018; López 2014; Ros 2008; Ros and Souam 1997].

When M is a stable FBCMC surface immersed in a mean convex region $W \subset \mathbb{R}^3$, Ros [2008, Theorem 9] showed that there are just a few possibilities for the genus and the number of boundary components of M (see Corollary 1.2 below).

When M is not stable, there is a Schrödinger operator L associated to the second variation of the area which has nonzero *weak Morse index* (see Section 2 for precise definitions). Geometrically, the weak Morse index of M is the number of directions whose volume preserving variations decrease area. It will be denoted by $\mathrm{Ind}_{\mathrm{w}}(M)$.

The case of free boundary minimal surfaces is of special interest and the works of Fraser and Schoen [2011; 2013; 2016] have motivated much research in this case. For free boundary minimal hypersurfaces all variations keeping the boundary freely in the boundary are allowed, not only volume preserving variations. In this setting Sargent [2017] proved that if M is a free boundary minimal surface immersed in a convex region $W \subset \mathbb{R}^3$ then the index is bounded from below by $(2g+k-1)/3$, where g is the genus of M and k is the number of boundary components. In [Ambrozio et al. 2018], Ambrozio, Carlotto and Sharp proved that it is valid in weakly mean convex domains in $\mathbb{R}^3$. In fact, their results apply more generally to domains in Riemannian manifolds with boundary satisfying suitable curvature conditions. In higher dimensions, they also obtained lower bounds for the index in terms of the dimension of the first relative homology group with real coefficients. The technique presented in these results uses the coordinates of harmonic forms as test functions and is inspired by previous works on eigenvalue estimates and index estimates for minimal hypersurfaces without boundary; see [Ros 2006; Savo 2010; Ambrozio et al. 2018].

Following these lines, in this paper, we obtain lower bounds for the weak Morse index of FBCMC surfaces in weakly mean convex regions of the Euclidean space or the unit sphere. More precisely, our results are the following.

Theorem 1.1. *Let W be a region of $\mathbb{R}^3$ such that its boundary is a union of smooth weakly mean convex surfaces, and let M^2 be a compact, orientable, FBCMC surface immersed in the mean convex side of W and whose boundary intersects the regular part of ∂W. If M has genus g and k boundary components, then*

$$\mathrm{Ind}_{\mathrm{w}}(M) \geq \frac{2g+k-4}{6}.$$

As an immediate consequence we obtain the result of Ros cited above:

Corollary 1.2 [Ros 2008, Theorem 9]. *Under the conditions of Theorem 1.1, if M is stable, then the only possibilities for g and k are*

(1) $g=0$ *and* $k \leq 4$;

(2) $g=1$ *and* $k=1$ *or* 2.

In the case of FBCMC surfaces immersed in weakly mean convex domains of $\mathbb{S}^3$, the result reads as follows:

Theorem 1.3. *Let W be a region in the unit sphere $\mathbb{S}^3$ such that its boundary is a union of smooth weakly mean convex surfaces, and let M^2 be a compact, orientable, FBCMC surface immersed in the mean convex side of W and whose boundary intersects the regular part of ∂W. If M has genus g and k boundary components,*

$$\mathrm{Ind}_{\mathrm{w}}(M) \geq \frac{2g+k-5}{8}.$$

Corollary 1.4. *Under the conditions of Theorem 1.3, if M is stable, then the only possibilities for g and k are*

(1) $g=0$ *and* $k \leq 5$;

(2) $g=1$ *and* $k \leq 3$;

(3) $g=2$ *and* $k=1$.

This paper is organized as follows. In Section 2, we present some definitions and basic results to be used in the proofs. Section 3 is devoted to computing the Jacobi operator of the test functions given by the coordinates of harmonic forms. In Section 4, we present the proof of Theorem 1.1. The proof of Theorem 1.3 is analogous and it is sketched in Section 5.

2. Preliminaries

Let us denote by W a connected domain of the Euclidean space $\mathbb{R}^3$ which is not necessarily compact. For simplicity, let us assume that W has smooth boundary and fix a unit normal vector field ν along each component of ∂W. We recall that the second fundamental form and the mean curvature of ∂W with respect to ν are defined respectively by

$$II^{\partial W}(X,Y) = \langle -D_X \nu, Y\rangle, \qquad \text{for } X, Y \in T\partial W,$$

and

$$H^{\partial W} = \tfrac{1}{2}\,\mathrm{tr}\, II^{\partial W},$$

where D is the Levi-Civita connection in the Euclidean space. Since the boundary ∂W is orientable, at a point on ∂W we have two choices for ν, one pointing inward from W and the other one pointing outward from W. From now on, we fix the vector field ν pointing outward from W. In this case, we say that W is *weakly convex* if $II^{\partial W}$ is nonpositive defined. If $H^{\partial W} \leq 0$, W is said to be *weakly mean convex*.

Let $x : M \to W$ be a compact oriented surfaced with boundary which is properly immersed, that is, $x(M) \cap W = x(\partial M)$. Fixing a unit normal vector field N along x, we denote by A the shape operator associated to the second fundamental form II^M of M with respect to N, namely

$$\begin{aligned} AX &= -D_X N, && \text{for } X \in TM, \\ II^M(X,Y) &= \langle AX, Y\rangle, && \text{for } X, Y \in TM. \end{aligned}$$

We say that M is free boundary if $x(\partial M)$ meets ∂W orthogonally.

From now on, let us assume that W is a mean convex domain of $\mathbb{R}^3$ and M is a *free boundary constant mean curvature surface* properly immersed in W. Such surfaces are critical points of the area functional for normal variations whose variational vector field is given by $X = uN$, where $u \in \mathcal{F}$ and

$$\mathcal{F} = \left\{ u : M \to \mathbb{R} : u \text{ is smooth up to the boundary and } \int_M u\, dM = 0 \right\}.$$

The second variation of area functional is given by the quadratic form $Q : C^\infty(M) \times C^\infty(M) \to \mathbb{R}$ (see [Ros and Vergasta 1995; Ros 2008]),

$$Q(u,u) = \int_M (u\Delta u - \|A\|^2 u^2)\, dM + \int_{\partial M} (u\eta(u) + II^{\partial W}(N,N)u^2)\, ds.$$

Here η is the outward unit conormal vector field on ∂M, that is, the unique unit vector field on ∂M that is tangent to M, normal to ∂M and pointing outward from M. Note that, under our notations, the free boundary condition means that $\eta = \nu$ along ∂M. We point out that in this paper we are using the geometric definition of the Laplacian operator, that is, $\Delta u = \operatorname{div} \nabla u$, where $\operatorname{div} X = -\operatorname{tr} \nabla X$.

The *weak Morse index* of M, denoted by $\mathrm{Ind_w}(M)$, is defined as the maximal dimension of a subspace of $\mathcal{F}$ on which $Q|_{\mathcal{F}}$ is negative definite. Geometrically, the index indicates the number of directions whose variations decrease area. In particular, we say that M is *stable* if the weak Morse index is zero.

We say that $u \in \mathcal{F}$ is an eigenfunction of $Q|_{\mathcal{F}}$ associated to the eigenvalue $\lambda \in \mathbb{R}$ if and only if $Q|_{\mathcal{F}}(u,v) = \lambda \int_M uv\, dM$ for all $v \in \mathcal{F}$. This is equivalent to saying that u solves the following eigenvalue problem:

$$\begin{cases} Lu = \lambda u & \text{in } M, \\ \dfrac{\partial u}{\partial \eta} = -II^{\partial W}(N,N)u & \text{on } \partial M, \end{cases} \tag{2-1}$$

where $L : \mathcal{F} \to \mathcal{F}$ is given by

$$Lu = Ju - \frac{1}{\mathrm{vol}(M)} \int_M Ju\, dM \tag{2-2}$$

and $J = \Delta - \|A\|^2$ is the Jacobi operator. We conclude that the weak Morse index coincides with the number of negative eigenvalues of the boundary problem (2-1).

Also, it is well known that such eigenvalues are given in a nondecreasing sequence $\lambda_1^L \le \lambda_2^L \le \cdots \le \lambda_k^L \le \cdots \nearrow \infty$ associated to a $L^2(M)$-orthonormal basis,

$$\{\phi_1, \phi_2, \ldots, \phi_k, \ldots\} \quad \text{of} \quad L^2(M) \cap \mathcal{F}$$

of solutions of the eigenvalue problem (2-1), satisfying the min-max characterization

$$\lambda_k^L = \min_{u \in \mathcal{J}^{k-1} \setminus \{0\}} \frac{Q|_{\mathcal{F}}(u, u)}{\int_M u^2 dM},$$

where $\mathcal{J}^{k-1} = \langle \phi_1, \ldots, \phi_{k-1} \rangle^{\perp}$.

In order to give lower bounds for the weak Morse index of M in terms of its topological invariants we will construct admissible eigenfunctions in $\mathcal{F}$ using harmonic vector fields, or equivalently harmonic 1-forms. Let us denote by $i : \partial M \to M$ the inclusion map. We also set $\mathcal{H}_T^1(M)$ the space of closed and coclosed 1-forms that are tangential at ∂M, that is,

$$\mathcal{H}_T^1(M) := \{w \in \Omega^1(M), dw = 0, \delta w = 0 \text{ and } i_\eta w = 0 \text{ along } \partial M\}.$$

Here d is the exterior derivative operator and δ is the interior derivative operator defined by $\delta = -\star d \star$, where $\star : \Omega^1(M) \to \Omega^1(M)$ is the Hodge star operator.

It is important to note that $\mathcal{H}_T^1(M)$ coincides with the space of harmonic 1-forms satisfying the *absolute boundary conditions*, that is

$$\mathcal{H}_T^1(M) = \{w \in \Omega^1(M), \Delta w = 0, i_\eta w = 0 \text{ and } i_\eta dw = 0 \text{ along } \partial M\}.$$

This space is closed related to the topology of the underline manifold. In fact we have the following result; see [Ambrozio et al. 2018] or [Sargent 2017].

Lemma 2.1. *Let M^2 be a compact, orientable surface with nonempty boundary ∂M. If M has genus g and $k \ge 1$ boundary components, then*

$$\dim \mathcal{H}_T^1(M) = 2g + k - 1.$$

3. Test functions and harmonic vector fields

Denoting by $\mathcal{E} = \{\bar{E}_1, \bar{E}_2, \bar{E}_3\}$ the canonical basis in $\mathbb{R}^3$ we will consider $E_i := \bar{E}_i - \langle \bar{E}_i, N \rangle N$, the orthogonal projection of $\bar{E}_i$ on TM. We also consider the smooth support functions $g_i : M \to \mathbb{R}$, $g_i := \langle \bar{E}_i, N \rangle$, for $1 \le i \le 3$.

Given a 1-form ω on M we denote by ξ its dual vector field, that is, $\xi^\flat = \omega$. Abusing notation slightly, we denote by $\star\xi$ the vector field dual of $\star\omega$. In the following, we will use the coordinates of ξ and $\star\xi$ as test functions. Namely, for each $1 \le i \le 3$, we define $w_i, \ \bar{w}_i : M \to \mathbb{R}$ as

$$w_i := \omega(E_i) = \langle E_i, \xi \rangle \quad \text{and} \quad \bar{w}_i := \star\omega(E_i) = \langle E_i, \star\xi \rangle.$$

To compute the Jacobi operator of w_i and $\bar{w}_i$ we need the following lemma of local nature proved in [Cavalcante and de Oliveira 2020]; see also [Ros 2007].

Lemma 3.1. *Let M^2 be an orientable CMC surface in $\mathbb{R}^3$. Then, using the above notation we have*

$$\Delta w_i = (\|A\|^2 - 4H^2)w_i + 2H\langle AE_i, \xi\rangle - 2g_i\langle A, \nabla\xi\rangle + \langle E_i, \Delta\xi\rangle,$$
$$\Delta \overline{w}_i = (\|A\|^2 - 4H^2)\overline{w}_i + 2H\langle AE_i, \star\xi\rangle - 2g_i\langle A, \nabla\star\xi\rangle + \langle E_i, \Delta\star\xi\rangle,$$

for $1 \le i \le 3$.

Now we note that when the vector field ξ is harmonic and tangential along ∂M its coordinates are admissible functions to compute the weak Morse index of CMC surfaces. More precisely we have:

Lemma 3.2. *If $\xi \in TM$ is a harmonic vector field which is tangential in ∂M, then $w_i \in \mathcal{F}$, that is,*

$$\int_M w_i \, dM = 0,$$

for $1 \le i \le 3$.

Proof. Note that $E_i = \nabla x_i$, $1 \le i \le 3$, where $x = (x_1, x_2, x_3) : M \to W$ is the immersion map. Then we have

$$\int_M w_i \, dM = \int_M \langle \nabla x_i, \xi\rangle \, dM$$
$$= \int_M x_i \operatorname{div}\xi \, dM + \int_{\partial M} x_i \langle \xi, \eta\rangle \, ds = 0.$$

In fact, $\operatorname{div}\xi = 0$ since ξ is harmonic, and $\langle \xi, \eta\rangle = 0$ since ξ tangential to ∂M. □

Remark 3.3. In general the functions $\overline{w}_i$, $1 \le i \le 3$, do not have mean value zero. However, we will see in Section 4 that if $\dim \mathcal{H}^1_T(M)$ is large enough then we can choose ξ such that

$$\int_M \overline{w}_i \, dM = 0, \qquad \text{for } 1 \le i \le 3.$$

We conclude this section by computing the boundary term of the quadratic form Q on w_i and $\overline{w}_i$,.

Lemma 3.4. *If $\xi \in TM$ is a vector field such that its dual 1-form satisfies the absolute boundary condition, then*

$$\sum_i \int_{\partial M} (w_i \eta(w_i) + II^{\partial W}(N, N) w_i^2) \, ds = 2 \int_{\partial M} H^{\partial W} \|\xi\|^2 \, ds, \tag{3-1}$$

$$\sum_i \int_{\partial M} (\overline{w}_i \eta(\overline{w}_i) + II^{\partial W}(N, N) \overline{w}_i^2) \, ds = 2 \int_{\partial M} H^{\partial W} \|\xi\|^2 \, ds. \tag{3-2}$$

Proof. We first note that for any vector field $X \in TM$ we have

$$\begin{aligned}\langle \nabla_\eta E_i, X\rangle &= \eta\langle \bar{E}_i, X\rangle - \langle \bar{E}_i, \nabla_\eta X\rangle\\ &= \langle \bar{E}_i, D_\eta X - \nabla_\eta X\rangle\\ &= \langle \bar{E}_i, N\rangle\langle X, A\eta\rangle.\end{aligned}$$

Let ω be the dual 1-form of the vector field ξ. Since $i_\eta d\omega = 0$ we have

$$0 = d\omega(\eta, \xi) = \langle \nabla_\eta \xi, \xi\rangle - \langle \nabla_\xi \xi, \eta\rangle.$$

Thus,

$$\begin{aligned}\sum_i \int_{\partial M} w_i \eta(w_i)\, ds &= \sum_i \int_{\partial M} w_i(\langle \nabla_\eta \xi, E_i\rangle + \langle \bar{E}_i, N\rangle\langle \xi, A\eta\rangle)\, ds\\ &= \int_{\partial M} \langle \nabla_\eta \xi, \xi\rangle\, ds = \int_{\partial M} \langle \nabla_\xi \xi, \eta\rangle\, ds\\ &= -\int_{\partial M} \langle \nabla_\xi \eta, \xi\rangle\, ds = \int_{\partial M} II^{\partial W}(\xi, \xi)\, ds.\end{aligned}$$

Since ξ and N form an orthogonal basis of the tangent space of ∂W along ∂M we conclude the proof by noting that

$$II^{\partial W}(\xi, \xi) + II^{\partial W}(N, N)\|\xi\|^2 = 2H^{\partial W}\|\xi\|^2.$$

The proof of assertion (3-2) follows the same steps as above, noting additionally that the Levi-Civita connection ∇ commutes with the Hodge star operator $\star$. □

4. Proof of Theorem 1.1

Proof. The proofs follow the same spirit as our proofs in [Cavalcante and de Oliveira 2020] but take into account the boundary term. Let $\xi_1, \xi_2, \dots, \xi_m$, be the first m eigenfunctions of the Hodge Laplacian Δ on M, which satisfy the absolute boundary condition [Gilkey et al. 1999, Theorem 1.5.4]. Set $\mathcal{L}_m^\Delta = \operatorname{span}\{\xi_1, \dots, \xi_m\}$ the vector space generated by these functions. By Lemma 2.1, we know that $\dim \mathcal{H}_T^1(M) = 2g + k - 1$. Let us assume that $m \geq 2g - k - 1$, and so $\mathcal{H}_T^1(M)$ is a subspace of $\mathcal{L}_m^\Delta$.

Next, we choose an orthonormal basis of $L^2(M)$ given by eigenfunctions of the operator L defined in (2-2), say $\{\phi_1, \phi_2, \dots, \phi_k, \dots\}$. We denote by $\mathcal{J}^n := \langle \phi_1, \dots, \phi_n\rangle^\perp$ the linear subspace of $\mathcal{F}$ orthogonal to the first n eigenfunctions of L.

Initially, we look for harmonic forms $\xi \in \mathcal{L}_m^\Delta$ such that the functions $w_i, \bar{w}_i \in \mathcal{J}^{\alpha-1}$, for some $\alpha \in \mathbb{N}$ and $i \in \{1, 2, 3\}$. It is equivalent to find a solution to the following system with $6(\alpha - 1)$ homogenous linear equations in the variable ξ:

$$\int_M w_i \phi_k\, dM = \int_M \bar{w}_i \phi_k\, dM = 0, \tag{4-1}$$

where $1 \leq i \leq 3$ and $1 \leq k \leq \alpha - 1$. In particular, if $m(\alpha) := \dim\ \mathcal{L}_m^\Delta > 6(\alpha - 1)$, then the system (4-1) has at least one nontrivial solution $\xi \in \mathcal{L}_m^\Delta$ such that $w_i, \overline{w}_i \in \mathcal{J}^{\alpha-1}$ for all $1 \le i \le 3$. By min-max characterization we have

$$\lambda_\alpha^J \int_M w_i^2\, dM \le Q(w_i, w_i) \quad \text{and} \quad \lambda_\alpha^J \int_M \overline{w}_i^2\, dM \le Q(\overline{w}_i, \overline{w}_i).$$

Now, using Lemma 3.1 we get

$$\begin{aligned}\lambda_\alpha^J \int_M w_i^2\, dM \le -4H^2 \int_M w_i^2\, dM + 2H \int_M \langle E_i, A\xi\rangle w_i\, dM \\ + \int_M \langle E_i, \Delta\xi\rangle w_i\, dM - 2\int_M g_i\langle A, \nabla\xi\rangle w_i\, dM \\ + \int_{\partial M} (w_i \eta(w_i) + II^{\partial W(N,N)} w_i^2)\, ds.\end{aligned}$$

Summing up $i = 1, 2, 3$ and using Lemma 3.4 we obtain

$$\begin{aligned}\lambda_\alpha^J \int_M \|\xi\|^2\, dM \le -4H^2 \int_M \|\xi\|^2\, dM + 2H \int_M \langle A\xi, \xi\rangle\, dM \\ + \int_M \langle \Delta\xi, \xi\rangle\, dM + 2\int_{\partial M} H^{\partial W} \|\xi\|^2\, dM.\end{aligned}$$

Applying the same arguments to the test functions $\overline{w}_i$ we get

$$\begin{aligned}\lambda_\alpha^J \int_M \|\xi\|^2\, dM \le -4H^2 \int_M \|\xi\|^2\, dM + 2H \int_M \langle A\star\xi, \star\xi\rangle\, dM \\ + \int_M \langle \Delta\star\xi, \star\xi\rangle\, dM + 2\int_{\partial M} H^{\partial W} \|\xi\|^2\, ds.\end{aligned}$$

Then, summing these last two inequalities and noting that $\langle A\xi, \xi\rangle + \langle A\star\xi, \star\xi\rangle = 2H\|\xi\|^2$, we have

$$\begin{aligned}\text{(4-2)} \quad \lambda_\alpha^J \int_M \|\xi\|^2 dM \le +2\int_{\partial M} H^{\partial W}\|\xi\|^2 dM - 2H^2 \int_M \|\xi\|^2 dM \\ + \tfrac{1}{2} \int_M (\langle \Delta\xi, \xi\rangle + \langle \Delta\star\xi, \star\xi\rangle) dM.\end{aligned}$$

Finally, if $\xi \in \mathcal{L}_m^\Delta$ we get $\xi = \sum_i \alpha_i \xi_i$ and therefore

$$\text{(4-3)} \qquad \int_M \langle \Delta\star\xi, \star\xi\rangle dM = \int_M \langle \Delta\xi, \xi\rangle dM = \lambda_{m(\alpha)} \int_M \|\xi\|^2 dM.$$

Substituting (4-3) into (4-2) and using the fact that $H^{\partial W} \le 0$ we obtain

$$\lambda_\alpha^J \le -2H^2 + \lambda_{m(\alpha)}^\Delta,$$

where $m(\alpha) > 6(\alpha - 1)$. This concludes the first part of Theorem 1.1.

In order to get the lower bound for the weak Morse index of M we take $\mathcal{J}^{\alpha-1} := \langle \phi_1, \dots, \phi_{\alpha-1} \rangle$, where $\phi_1, \dots, \phi_{\alpha-1}$ are the first eigenfunctions of the eigenvalue equation (2-1). From Lemma 3.2 we know that if $\xi \in \mathcal{H}_T^1(M)$, then the test functions w_1, w_2 and w_3, belong to $\mathcal{F}$.

We look for vector fields $\xi \in \mathcal{H}_T^1(M)$ such that for $1 \le i \le 3$, the test functions $w_i, \bar{w}_i$ are in $\mathcal{J}^{\alpha-1}$, for some $\alpha \in \mathbb{N}$, and $\bar{w}_i$ is in $\mathcal{F}$. In this case, we have the following system with $6\alpha - 3$ homogeneous linear equations in the variable ξ:

$$\int_M \bar{w}_i = \int_M w_i \phi_k = \int_M \bar{w}_i \phi_k = 0, \tag{4-4}$$

where $1 \le i \le 3$ and $1 \le k \le \alpha - 1$.

If $\dim \mathcal{H}_T^1(M) = 2g + k - 1 > 6\alpha - 3$, then the system (4-4) has at least one nontrivial solution $\xi \in \mathcal{H}_T^1(M)$. Following the same steps as above we get

$$\lambda_\alpha^L \int_M \|\xi\|^2 \le -2H^2 \int_M \|\xi\|^2.$$

This implies that $\lambda_\alpha^L < 0$ and then $\mathrm{Ind}_{\mathrm{w}}(M) \ge \alpha$. Since α can be chosen as the largest integer such that $2g + k - 1 > 6\alpha - 3$ we get

$$\mathrm{Ind}_{\mathrm{w}}(M) \ge \frac{2g + k - 4}{6}. \qquad \square$$

5. Proof of Theorem 1.3

Proof. Composing the immersion $x : M \to W \subset \mathbb{S}^3$ with the canonical immersion of the unit sphere into the Euclidean space, we may consider $x : M \to \mathbb{R}^4$. Let $\mathcal{E} = \{\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4\}$ be the canonical basis in $\mathbb{R}^4$ and $E_i := \bar{E}_i - \langle \bar{E}_i, N\rangle N - \langle \bar{E}_i, x\rangle x$, the orthogonal projections of $\bar{E}_i$ on TM. Choosing $\nu = -x$ as an orientation of $\mathbb{S}^3$ we have

$$D_Y X - \nabla_Y X = \langle AX, Y\rangle N + \langle X, Y\rangle \nu, \qquad X, Y \in TM,$$

and also

$$\begin{aligned}\langle \nabla_X E_i, Y\rangle &= X\langle \bar{E}_i, Y\rangle - \langle \bar{E}_i, \nabla_X Y\rangle = \langle \bar{E}_i, D_Y X - \nabla_Y X\rangle \\ &= \langle AX, Y\rangle \langle \bar{E}_i, N\rangle + \langle X, Y\rangle \langle \bar{E}_i, \nu\rangle.\end{aligned}$$

Using that $\langle \xi, N\rangle = \langle \star\xi, N\rangle = \langle \xi, \nu\rangle = \langle \star\xi, \nu\rangle = 0$ we get

$$\begin{aligned}\sum_i \int_{\partial M} w_i \eta(w_i)\, ds &= \sum_i \int_{\partial M} w_i (\langle \nabla_\eta \xi, E_i\rangle) ds \\ &\quad + \sum_i \int_{\partial M} w_i (\langle \bar{E}_i, N\rangle \langle \xi, A\eta\rangle + \langle \eta, \xi\rangle \langle \bar{E}_i, \nu\rangle)\, ds \\ &= \int_{\partial M} \langle \nabla_\eta \xi, \xi\rangle\, ds + \int_{\partial M} \langle \xi, N\rangle \langle \xi, A\eta\rangle + \int_{\partial M} \langle \eta, \xi\rangle \langle \xi, \nu\rangle \\ &= \int_{\partial M} \langle \nabla_\eta \xi, \xi\rangle\, ds,\end{aligned}$$

and analogously

$$\sum_i \int_{\partial M} \bar{w}_i \eta(\bar{w}_i)\, ds = \int_{\partial M} \langle \nabla_\eta \star\xi, \star\xi \rangle\, ds = \int_{\partial M} \langle \nabla_\eta \xi, \xi \rangle\, ds.$$

So, Lemma 3.4 holds in the spherical case. The Laplacian of the test functions w_i and $\bar{w}_i$ are given by (see [Cavalcante and de Oliveira 2020])

$$\Delta w_i = (\|A\|^2 - 4H^2) w_i + 2H\langle AE_i, \xi\rangle - 2g_i\langle A, \nabla\xi\rangle + \langle E_i, \Delta\xi\rangle - 2\langle x, \bar{E}_i\rangle \operatorname{div}\xi,$$
$$\Delta \bar{w}_i = (\|A\|^2 - 4H^2) \bar{w}_i + 2H\langle AE_i, \bar{\xi}\rangle - 2g_i\langle A, \nabla\bar{\xi}\rangle + \langle E_i, \Delta\bar{\xi}\rangle - 2\langle x, \bar{E}_i\rangle \operatorname{div}\bar{\xi}.$$

Under these considerations, and taking into account that the Jacobi operator for immersions into $\mathbb{S}^3$ is given by $J = \Delta - (\|A\|^2 + 2)$ the proof follows as in the proof of Theorem 1.1. □

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