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REMARKS ON THE HÖLDER-CONTINUITY OF SOLUTIONS TO PARABOLIC EQUATIONS WITH CONIC SINGULARITIES

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This is a note on work of Ladyzhenskaja et al. (AMS 1968) and of Ferretti and Safonov (2013). Using their work line by line, we prove the Hölder-continuity of solutions to linear parabolic equations of mixed type, assuming the coefficient of $\frac{\partial}{\partial t}$ has time-derivative bounded from above. On a Kähler manifold, this Hölder estimate works when the metrics possess conic singularities along a normal crossing divisor.

1. Introduction

Historically, Hölder-continuity of solutions to linear elliptic and parabolic equations (in various cases) has been proved and extensively studied by De Giorgi [1957], Nash [1958], Moser [1964], Krylov and Safonov [1980]. Many other experts have contributed to this topic as well, for example, see [Caffarelli and Cabré 1995; Ferretti and Safonov 2013; Gilbarg and Trudinger 1977; Ladyzhenskaja et al. 1968]. Ferretti and Safonov [2013] give a unified proof of the Hölder-continuity in both the divergence case and nondivergence case. The key is to establish growth properties for the (sub- and super-) level sets of the solutions.

We focus on divergence-form equations. The operator in our main parabolic equation (3) below is exactly the one considered in [Ferretti and Safonov 2013]. The little difference is: the a_0 $\left(\text{coefficient of } \frac{\partial}{\partial t}\right)$ in [Ferretti and Safonov 2013, line 11, page 89] is not allowed to depend on time, but here we allow a_0 in (3) to depend on time.

Our motivation is to study the heat equation associated to a Ricci flow. The Ricci flow is a special time-parametrized family of Riemannian metrics $g(t)$. Given a time-family of Riemannian metrics $g(t)$ over a Euclidean ball B, the associated heat equation reads as

$$\frac{\partial u}{\partial t} - \Delta_g u \triangleq \frac{\partial u}{\partial t} - \frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{\det g_{ij}}\frac{\partial u}{\partial x_j}\right) = f, \tag{1}$$

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where the x_i are the Euclidean coordinates. To estimate the Hölder norm of u, we only care about the L^∞-norm of f, though we can assume that everything involved has higher derivatives. Multiplying (1) by $\sqrt{\det g_{ij}}$, we get

$$\sqrt{\det g_{ij}}\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{\det g_{ij}}\frac{\partial u}{\partial x_j}\right) = F \triangleq f\sqrt{\det g_{ij}}. \tag{2}$$

Let $a_0 \triangleq \sqrt{\det g_{ij}}$ and $a^{ij} \triangleq g^{ij}\sqrt{\det g_{ij}}$; (1) is a special case of (3) and [Ferretti and Safonov 2013, Equation (D), page 89]. Suppose $\det g_{ij}$ is uniformly bounded; the L^∞-norm of f is equivalent to the L^∞-norm of F, thus it makes no difference for the Hölder estimate.

Our main observation (and a one sentence proof of Theorem 1.1) is that when a_0 depends on time and $\frac{\partial \log a_0}{\partial t}$ is bounded from above, the general energy estimates are still true (Lemma 4.5). By the proof in [Ferretti and Safonov 2013], these energy estimates imply the main growth theorem [Ferretti and Safonov 2013, Theorem 5.3]. Moreover, by an idea in [Ladyzhenskaja et al. 1968], [Ferretti and Safonov 2013, Theorem 5.3] directly implies the Hölder continuity of solutions, without involving the Harnack inequality in [Ferretti and Safonov 2013, Theorem 1.5]. We believe these are known by experts. Let

- $Y = (y, s)$ be a space-time point, and $C_r(Y) = B_y(r) \times (s - r^2, s)$ be the parabolic cylinder centered at Y with radius r, where $B_y(r)$ is the usual m-dimensional Euclidean ball.
- Let $[\,\cdot\,]_\alpha$ denote the usual parabolic Hölder seminorm of exponent α (for example, see [Lieberman 1996, (4.1)] for a definition).

The simplest version of our main theorem can be stated as follows.

Theorem 1.1. *Suppose $u \in C^\infty[C_r(Y)]$ solves the following equation* (or the metric heat equation (1) via the correspondence in (2)) *in the classical sense*

$$a_0\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) = f, \tag{3}$$

where a_0, a^{ij} $(1 \le i, j \le m)$ are space-time smooth functions. Suppose

$$\frac{1}{K} \le a_0 \le K, \qquad \frac{\partial \log a_0}{\partial t} \le K, \qquad \frac{I}{K} \le a^{ij} \le KI. \tag{4}$$

Then there exist constants $\alpha(m, K) \in (0, 1)$ and $N(m, K)$ such that

$$r^\alpha [u]_{\alpha, C_{r/2}(Y)} + |u|_{L^\infty[C_{r/2}(Y)]} \le N\left(\frac{|u|_{L^1[C_r(Y)]}}{r^{m+2}} + r^2|f|_{L^\infty[C_r(Y)]}\right).$$

Remark 1.2. When $\frac{\partial \log a_0}{\partial t}$ is not uniformly bounded from above (while the other conditions in Theorem 1.1 hold true), the above uniform Hölder estimate does not hold in general. We refer the interested readers to the beautiful example constructed by Chen and Safonov [2017, Theorems 4.1 and 4.2].

Remark 1.3. In view of the Riemannian geometry setting in (2) and the line right below it, when g_t is a Ricci flow, the upper bound on $\frac{\partial \log a_0}{\partial t}$ means

$$\frac{\partial}{\partial t} d\mathrm{vol}_{g_t} \leq K\, d\mathrm{vol}_{g_t}, \tag{5}$$

where $d\mathrm{vol}_{g_t}$ is the evolving volume form. The K is actually a lower bound for the scalar curvature of g_t. Fortunately, the scalar curvature is usually bounded from below along Ricci flows without any additional condition, see [Cao and Chen 2012, page 5; Hamilton 1995].

Therefore, when the scalar curvature of a Kähler–Ricci flow is not (assumed to be) bounded from above, Theorems 1.1 and 2.2 (as well as the ideas in the proof) might help in obtaining higher-order estimates (convergence) of the Kähler metric as t approaches a singular time or as $t \to \infty$.

Theorem 1.1 can be generalized to heat equations of Kähler-metrics with conic singularities along normal-crossing divisors (Theorem 2.2). We only prove Theorem 2.2, the proof of Theorem 1.1 is the same (by discarding the necessary techniques for the conic singularities, see Claim 4.7 for example).

This note is organized as follows. In Section 2 we define Kähler metrics with conic singularities, and state Theorem 2.2. In Section 3 we prove Theorems 1.1 and 2.2 assuming the main growth theorem (Theorem 5.7) on sublevel sets of subsolutions. In Section 4, we define weak subsolutions and prove energy inequalities for them. This is a preparation for the main growth theorem (Theorem 5.7). In Section 5, we prove the main growth theorem using the energy inequalities and the measure theoretic arguments in [Ferretti and Safonov 2013].

2. The more general version of Theorem 1.1 in Kähler geometry involving conic singularity

In Kähler geometry setting, Theorem 1.1 holds even when the metrics possess conic singularities along analytic hypersurfaces. To state the result, we first give a geometric formulation following [Guenancia and Păun 2016]. Given a closed Kähler manifold M and a divisor $D = \sum_{j=1}^{N} 2\pi(1-\beta_j)D_j$, where each D_j is an irreducible hypersurface and may have self-intersection, suppose D has (no worse than) normal crossing singularities, i.e., there is an open cover of $\operatorname{supp} D$ by neighborhoods $\mathcal{U}_i$ such that in each $\mathcal{U}_i$, $\operatorname{supp} D \cap \mathcal{U}_i = \{z_1 z_2 z_3 \cdots z_k = 0\}$, where $k \leq n$ and $z_1 \cdots z_n$ are holomorphic coordinate functions in $\mathcal{U}_i$.

Definition 2.1. A Kähler metric g (defined away from $\operatorname{supp} D$) is said to be a weak-conic metric with quasi-isometric constant K, if and only if it's Hölder-continuous away from supp D and in each $\mathcal{U}_i$,

$$\frac{g_\beta^k}{K} \leq g \leq K g_\beta^k \quad \text{(quasi-isometric)},$$
$$g_\beta^k = \sum_{j=1}^{k} \frac{\beta_j^2}{|z_j|^{2-2\beta_j}} \, dz_j \otimes d\bar{z}_j + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j. \tag{6}$$

g_β^k is one of the 2 model metrics on $\mathbb{C}^n$ we work with, and in this local setting we abuse notation by denoting supp D as D.

Similarly, a Kähler metric g is called a ϵ-nearly-conic metric with quasi-isometric constant K, if and only if it's Hölder-continuous over the whole M (across supp D) and in each $\mathcal{U}_i$, for $\epsilon > 0$,

$$\frac{g_{\beta,\epsilon}^k}{K} \leq g \leq K g_{\beta,\epsilon}^k,$$
$$g_{\beta,\epsilon}^k = \sum_{j=1}^{k} \frac{\beta_j^2}{(|z_j|^2 + \epsilon^2)^{1-\beta_j}} \, dz_j \otimes d\bar{z}_j + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j, \tag{7}$$

We recall the well known intrinsic polar coordinates of g_β^k. Let

$$\xi_j = r_j e^{\sqrt{-1}\theta_j}, \quad r_j = |z_j|^{\beta_j}, \quad 1 \leq j \leq k.$$

In these polar coordinates the model cone g_β^k is equal to

$$g_\beta^k = \sum_{j=1}^{k} (dr_j^2 + \beta_j^2 r_j^2 \, d\theta_j^2) + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j,$$

and it's quasi-isometric to the Euclidean metric, i.e.,

$$\big(\min_j \beta_j^2\big) g_E \leq g_\beta^k \leq g_E, \quad g_E = \sum_{j=1}^{k} (dr_j^2 + r_j^2 \, d\theta_j^2) + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j. \tag{8}$$

This is important because we want to take advantage of the rescaling and translation invariance of the Euclidean metric.

Similarly, we also have intrinsic polar coordinates for $g_{\beta,\epsilon}^k$. Let s_j be the solution to

$$\frac{ds_j}{d\rho_j} = \frac{\beta_j}{(\rho_j^2 + \epsilon^2)^{(1-\beta_j)/2}}, \quad s_j(0) = 0, \quad \rho_j = |z_j|. \tag{9}$$

Then $\xi_j = s_j e^{\sqrt{-1}\theta_j}$, $1 \le j \le k$ defines the polar coordinates of $g^k_{\beta,\epsilon}$. By [Wang 2016, Lemma 4.3], in these coordinates we have

$$g^k_{\beta,\epsilon} = \sum_{j=1}^{k}(ds_j^2 + a_{j,\epsilon} s_j^2\, d\theta_j^2) + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j, \quad \beta_j^2 < a_{j,\epsilon} \le 1. \tag{10}$$

Hence $g^k_{\beta,\epsilon}$ is also quasi-isometric to the Euclidean metric in its polar coordinate, i.e.,

$$\big(\min_j \beta_j^2\big) g_E \le g^k_{\beta,\epsilon} \le g_E, \quad g_E = \sum_{j=1}^{k}(ds_j^2 + s_j^2\, d\theta_j^2) + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j. \tag{11}$$

Unless specified (via a parentheses or a subsymbol), the constants N and C in this article depend on (at most) n, K, M, D, $\beta'_j s$, and the open cover $\bigcup_i \mathcal{U}_i$. They don't depend on ϵ. Different $N's$ could be different. The real dimension is $m = 2n$ in the Kähler setting.

Theorem 2.2. *Let* $\epsilon \in [0, 1]$.

Part I (local estimate). *Suppose g_t is a time-differentiable family of weak-conic Kähler metrics or of ϵ-nearly-conic metrics, which is defined over a parabolic cylinder $C_r(Y)$ in $\mathbb{C}^n$ under a polar coordinate as below* (7) *or* (9), *respectively. Suppose the quasi-isometric constant of g_t is K,*

$$\frac{\partial}{\partial t}\, d\mathrm{vol}_t \le K\, d\mathrm{vol}_t. \tag{12}$$

and u is a bounded weak solution (in the sense of Definition 4.2) *to*

$$\frac{\partial u}{\partial t} = \Delta_{g_t} u + f \quad \text{over } C_r(Y). \tag{13}$$

Then there exists $\alpha(n, \beta, K) \in (0, 1)$ and $N(n, \beta, K)$ such that

$$r^\alpha [u]_{\alpha, C_{r/2}(Y)} + |u|_{L^\infty, C_{r/2}(Y)} \le N\left(\frac{|u|_{L^1[C_r(Y)]}}{r^{2n+2}} + r^2 |f|_{L^\infty[C_r(Y)]}\right).$$

Part II (global estimate). *In the setting of* (6) *and paragraph above it, suppose all the conditions in* Part I *hold globally on $M \times [0, T]$. Then for all $t_0 \in (0, T)$, there exists an $\alpha(n, \beta, K)$ and $C_{t_0}(n, \beta, K)$ such that*

$$[u]_{\alpha, M\times[t_0,T]} + |u|_{L^\infty(M\times[t_0,T])} \le C_{t_0}(|u|_{L^1(M\times[0,T])} + |f|_{L^\infty(M\times[0,T])}).$$

Remark 2.3. When the divisor is smooth, a weaker version of this Hölder estimate is in section 4 of [Wang 2016]. We hope it's still somewhat valuable to present the proof separately here. The $[u]_\alpha$ is the usual parabolic Hölder seminorm with respect to g^k_β ($g^k_{\beta,\epsilon}$) (see (8)). An important point is that Hölder continuity with respect to the distance of g^k_β ($g^k_{\beta,\epsilon}$) is equivalent to Hölder continuity in the usual sense in holomorphic coordinates (apart from a difference of Hölder exponents). We refer interested readers to [Wang 2016, Lemma 4.4]. Please see Definition 4.2 for the definition of weak solutions (replace SC_r by the underlying domain).

Remark 2.4. Using the Kähler structure, Equation (13) can be written in both divergence and nondivergence form. In this case, we expect that Theorem 2.2 still holds without condition (12).

3. Proof of the main results assuming Theorem 5.7

From now on (and in the subsequent sections), we work in the polar coordinates in (8) and (11). In this coordinate, we do not "see" the conic singularity (except that the coefficients of the equations and solutions are not defined on D).

Definition 3.1. Let C_r^0 denote $C_r(y, s-3r^2)$ (see above Theorem 1.1 on definition of parabolic cylinders). We note that C_r^0 is "earlier" in time than C_r.

Let $|\Omega|$ denote the Euclidean measure of a set $\Omega \in \mathbb{R}^m \times \mathbb{R}$, where $\mathbb{R}^m$ is the spatial direction, and $\mathbb{R}$ is the time direction, i.e., $\mathbb{R}^m \times \mathbb{R}$ is the full parabolic cylinder.

Given a function u on an arbitrary set $\Omega \in \mathbb{R}^m \times \mathbb{R}$, we define

$$\operatorname{osc}_\Omega u \triangleq \sup_\Omega u - \inf_\Omega u.$$

Roughly speaking, to show parabolic Hölder continuity of a function u, it suffices to show that comparing to the oscillation in an arbitrary parabolic cylinder (in the domain of u), the oscillation in the concentric subcylinder of half radius decreases by a fixed amount.

The idea in proving Theorems 1.1 and 2.2 can be explained as follows. For simplicity, we assume $f = 0$ in the main heat equation (3).

For a subsolution to the homogeneous version of (3), the role of the main growth theorem (Theorem 5.7) is to improve a bound on the measure of the superlevel set in an "earlier" parabolic cylinder to a pointwise bound in a "later" cylinder.

The crucial observation is that for any solution u, either u or $-u$ (adjusted by proper constants) admits a bound on the measure of the superlevel set in the "earlier" parabolic cylinder (see Cases 1 and 2 below (15)), hence the condition in the main growth theorem is always satisfied, and it yields the desired decrease of oscillation.

Proof of Theorems 1.1 and 2.2. We only prove (Part I of) Theorem 2.2 as mentioned at the end of the introduction. Notice that y does not have to be in supp D (as long as integration by parts is true, see proof of Lemma 4.5). By the interior L^∞-estimate in Proposition 5.3 which holds for every cylinder and every subsolution, it suffices to show the Hölder norm is bounded by the L^∞-norm, i.e.,

$$r^\alpha [u]_{\alpha, C_{r/2}(Y)} \le N(|u|_{L^\infty[C_r(Y)]} + r^2 |f|_{L^\infty[C_r(Y)]}). \tag{14}$$

By [Lieberman 1996, Lemma 4.6], it suffices to show the oscillation decays for every cylinder C_{2r} and every subsolution u, i.e.,

$$\operatorname{osc}_{C_r} u \leq (1-b)\operatorname{osc}_{C_{2r}} u + 4r^2|f|_{0,C_{2r}}, \quad b = b(n,\beta,K) > 0. \tag{15}$$

By rescaling and translation invariance, it suffices to assume $r = 1,\ s = 0$. By adding a constant, it suffices to assume $0 \leq u \leq h$, where $h \triangleq \operatorname{osc}_{C_2} u$. As in [Ladyzhenskaja et al. 1968], one of the following must hold:

$$\text{Case 1:}\quad \left|\left\{u > \frac{h}{2}\right\} \cap C_1^0\right| \leq \frac{|C_1^0|}{2}; \qquad \text{Case 2:}\quad \left|\left\{u < \frac{h}{2}\right\} \cap C_1^0\right| \leq \frac{|C_1^0|}{2}.$$

We only prove (15) in Case 1 in detail. Case 2 is similar by applying the proof in Case 1 to $h-u$. Consider $\bar{u} = u - t|f|_{0,C_{2r}}$ $(t \in (-4,0))$. Then

$$\begin{aligned} &\frac{\partial \bar{u}}{\partial t} - \Delta_g \bar{u} \leq 0, \quad 0 \leq \bar{u} \leq h + 4|f|_{0,C_2}. \\ &\text{Moreover,}\quad \bar{u} > \frac{h}{2} + 4|f|_{0,C_2} \Rightarrow u > \frac{h}{2}. \end{aligned} \tag{16}$$

Hence the assumption of Case 1 implies

$$\left|\left\{\bar{u} \geq \frac{h}{2} + 4|f|_{0,C_2}\right\} \cap C_1^0\right| \leq \left|\left\{u > \frac{h}{2}\right\} \cap C_1^0\right| \leq \frac{|C_1^0|}{2}.$$

Then Theorem 5.7 $\big($applied to $\bar{u} - \frac{h}{2} - 4|f|_{0,C_2}\big)$, (16), and the above inequality imply that there exists $a(n,\beta,K) > 0$ such that

$$\sup_{C_1}\Big(\bar{u} - \frac{h}{2} - 4|f|_{0,C_2}\Big) \leq (1-a)\sup_{C_2}\Big(\bar{u} - \frac{h}{2} - 4|f|_{0,C_2}\Big) \leq \frac{(1-a)h}{2}.$$

Then $\operatorname{osc}_{C_1} \leq \sup_{C_1} u \leq \sup_{C_1} \bar{u} \leq \big(1 - \frac{a}{2}\big)h + 4|f|_{0,C_2}$. The proof of (15) (under the normalization conditions below it) is complete. □

4. Energy inequalities

We follow closely the definitions and tricks in [Ferretti and Safonov 2013]; the point is that they work equally well in the presence of conic singularity (Definitions 4.1, 4.2, and 4.4). The functions and integrations are all defined away from the divisor D (see the content below (6)). If the notation of a function space does not involve D, we mean the space satisfies the indicated asymptotic property (which should be clear from the context). The sets and (slant) cylinders are standard ones minus D. This does not affect any measure theory, integration, or technique in this article, because the spacewise codimension of D is 2. For the proof of Theorem 1.1, we don't have to chop off any singularity.

Definition 4.1. A slant cylinder $\mathrm{SC}_r(y_0, y_1, T_0, T_1)$, which we abbreviate in most the time as SC_r, is the following set:

$$\mathrm{SC}_r \triangleq \{x \mid |x - y(t)| < r,\ T_0 < t \leq T_1\}, \tag{17}$$

where $y(t) = y_0 + (t - T_0)(y_1 - y_0)/(T_1 - T_0)$. When $y_1 = y_0$, SC_r is just the usual cylinder C_r defined above Theorem 1.1. We define $l \triangleq r(y_1 - y_0)/(T_1 - T_0)$ as the parabolic slope of SC_r. The parabolic slope l is invariant under

- the usual parabolic rescaling (linear multiplication on y_0, y_1, r and quadratic multiplication on T_0, T_1 by the same factor),
- the spacewise translation (on y_0, y_1 by the same displacement),
- and the timewise translation (on T_0 and T_1 by the same displacement).

Definition 4.2. We say u is a weak subsolution to

$$\frac{\partial u}{\partial t} - \Delta_g u \leq 0, \tag{18}$$

in a slant cylinder SC_r if

(1) $u \in C^{2+\alpha,1+\alpha/2}\{\mathrm{SC}_r \setminus D \times [T_0, T_1]\} \cap L^\infty(\mathrm{SC}_r)$;

(2) Inequality (18) holds over $\mathrm{SC}_r \setminus D \times [T_0, T_1]$ in the classical sense.

We call a function η (defined in any bounded space-time domain $\Omega \in \mathbb{C}^n \times (-\infty, \infty)$) tame if $\eta \in C^{1,1}\{\Omega \setminus D \times [T_0, T_1]\} \cap L^\infty(\Omega)$ and the following holds:

$$\frac{\partial \eta}{\partial t} \in L^1(\Omega), \quad \nabla \eta \in L^2(\Omega). \tag{19}$$

Remark 4.3. The $L^\infty(\mathrm{SC}_r)$-requirement in Definition 4.2 is crucial, and is the only global condition. It guarantees (18) holds across the singularity in the sense of integration by parts.

Definition 4.4. Exactly as in [Ferretti and Safonov 2013, Corollary 2.3], we define the cutoff function of u as

$$u_\epsilon = G(u), \tag{20}$$

where G is a function with one variable such that $G(u) = 0$ when $u \leq \epsilon$, $G(u) = u + G(2\epsilon) - 2\epsilon$ when $u \geq 2\epsilon$, and G, G', $G'' \geq 0$. Consequently, we have

$$G(2\epsilon) \leq \epsilon \quad \text{and} \quad \max\{u - 2\epsilon, 0\} \leq u_\epsilon \leq \max\{u - \epsilon, 0\}. \tag{21}$$

The most important feature of u_ϵ is that, if u is a solution to (18), so is u_ϵ, i.e.,

$$\frac{\partial u_\epsilon}{\partial t} - \Delta_g u_\epsilon \leq 0. \tag{22}$$

The cutoff function u_ϵ can be understood as the smoothing of u^+ (nonnegative part of u). We note that in the classical case, u^+ is a subsolution (in proper sense) if u is. The above smoothing is pointwise, thus works in the presence of conic singularities.

Lemma 4.5. *Under the same assumptions in* Part I *of Theorem 2.2* (for any r), *suppose u is a nonnegative weak solution to* (18) *in the sense of Definition 4.2 in a slant cylinder* SC_r, $r \le \frac{1}{100n}$. *Then for any nonnegative tame function η which is compactly supported in* SC_r *spacewisely*, *we have*

$$\int_{\mathbb{C}^n} u\eta^2 \, dV_g|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbb{C}^n} \langle \nabla_g u, \nabla_g \eta^2\rangle \, dV_g \, ds \le \int_{t_1}^{t_2}\int_{\mathbb{C}^n} u \frac{\partial \eta^2}{\partial t} \, dV_g \, ds + K \int_{t_1}^{t_2}\int_{\mathbb{C}^n} u\eta^2 \, dV_g \, ds. \tag{23}$$

Moreover, *we have*

$$\int_{\mathbb{C}^n} u^2\eta^2 \, dV_g|_{t_1}^{t_2} + \tfrac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{C}^n} |\nabla_g u|^2 \eta^2 \, dV_g \, ds \le \int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2 \frac{\partial \eta^2}{\partial t} \, dV_g \, ds + (2K+200)\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2(\eta^2 + |\nabla_g \eta|^2) \, dV_g \, ds, \tag{24}$$

and therefore

$$\int_\Omega |\nabla_g u|^2 \, dV_g \, ds < +\infty, \quad \textit{for any (parabolic) compact subdomain } \Omega \textit{ of } \mathrm{SC}_r. \tag{25}$$

Remark 4.6. By the same proof, the energy estimate of (3) is similar.

Proof of Lemma 4.5. Let r_i be the distance function to the smooth hypersurface D_i. We consider Berndtsson's cutoff function $\psi_{i,\epsilon} = \psi(\epsilon \log(-\log r_i))$, ψ is the standard cutoff function such that $\psi(x) \equiv 1$ when $x \le \frac{1}{2}$, and $\psi(x) \equiv 0$ when $x \ge \frac{4}{5}$. Then

$$\psi_{i,\epsilon} \equiv 0 \quad \text{when } r_i \le e^{-e^{4/(5\epsilon)}}; \qquad \psi_{i,\epsilon} \equiv 1 \quad \text{when } r_i \ge e^{-e^{1/(2\epsilon)}}. \tag{26}$$

Let $\psi_\epsilon = \prod_{i=1,\dots,n} \psi_{i,\epsilon}$.

Claim 4.7. *We have*

$$\lim_{\epsilon \to 0} |\nabla_E \psi_\epsilon|_{L^2(B(1/2))} = 0. \tag{27}$$

The proof of Claim 4.7 is elementary. We only verify it for $\partial\psi_\epsilon/\partial r_1$, the other directional derivatives are similar. We compute

$$\frac{\partial \psi_\epsilon}{\partial r_1} = -\psi' \frac{\epsilon}{r_1 \log r_1} \prod_{i \ne 1} \psi_{i,\epsilon}.$$

Hence in polycylindrical coordinates we find

$$\int_{B(1/2)} |\frac{\partial \psi_\epsilon}{\partial r_1}|^2 \, d\mathrm{vol}_E \le C\epsilon^2 \int_0^{1/2} \frac{1}{r_1(\log r_1)^2} \, dr_1 \le C\epsilon^2.$$

We first prove (24). By definition we have $\lim_{\epsilon\to 0} \psi_\epsilon = 1$ everywhere except on supp D. We multiply both hand sides of (18) by $u\eta^2\psi_\epsilon^2$, then integrate by parts and integrate with respect to time. We obtain

$$\begin{aligned}(28)\quad & \tfrac{1}{2}\int_{\mathbb{C}^n} u^2\eta^2\psi_\epsilon^2 \, dV_g|_{t_1}^{t_2} + \int_{t_1}^{t_2} |\nabla_g u|^2\eta^2\psi_\epsilon^2 \, dV_g \, ds \\ &\le \tfrac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2\eta^2\psi_\epsilon^2 \frac{\partial dV_g}{\partial t} \, ds - 2\int_{t_1}^{t_2}\int_{\mathbb{C}^n} \langle \nabla_g u, \nabla_g \eta\rangle u\eta\psi_\epsilon^2 \, dV_g \, ds \\ &\quad + \tfrac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2 \frac{\partial \eta^2}{\partial t}\psi_\epsilon^2 dV_g \, ds - 2\int_{t_1}^{t_2}\int_{\mathbb{C}^n} \langle \nabla_g u, \nabla_g \psi_\epsilon\rangle u\eta^2\psi_\epsilon \, dV_g \, ds.\end{aligned}$$

Using the Cauchy–Schwartz inequality we deduce that

$$\begin{aligned}(29)\quad & \left|2\int_{t_1}^{t_2}\int_{\mathbb{C}^n} \langle \nabla_g u, \nabla_g \psi_\epsilon\rangle u\eta^2\psi_\epsilon \, dV_g \, ds\right| \\ &\le \tfrac{1}{100}\int_{t_1}^{t_2}\int_{\mathbb{C}^n} |\nabla_g u|^2\psi_\epsilon^2\eta^2 \, dV_g \, ds + 100\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2\eta^2|\nabla_g\psi_\epsilon|^2 \, dV_g \, ds.\end{aligned}$$

Similarly we have

$$\begin{aligned}(30)\quad & \left|2\int_{t_1}^{t_2}\int_{\mathbb{C}^n} \langle \nabla_g u, \nabla_g \eta\rangle u\eta\psi_\epsilon^2 \, dV_g \, ds\right| \\ &\le \tfrac{1}{100}\int_{t_1}^{t_2}\int_{\mathbb{C}^n} |\nabla_g u|^2\psi_\epsilon^2\eta^2 \, dV_g \, ds + 100\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2\psi_\epsilon^2|\nabla_g\eta|^2 \, dV_g \, ds.\end{aligned}$$

Notice that by (5) we have

$$(31)\qquad \int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2\eta^2\psi_\epsilon^2 \frac{\partial dV_g}{\partial t} \, ds \le K\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2\eta^2\psi_\epsilon^2 \, dV_g \, ds.$$

Then

$$\begin{aligned}(32)\quad & \tfrac{1}{2}\int_{\mathbb{C}^n} u^2\eta^2\psi_\epsilon^2 \, dV_g|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbb{C}^n} |\nabla_g u|^2\eta^2\psi_\epsilon^2 \, dV_g \, ds \\ &\le K\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2\eta^2\psi_\epsilon^2 \, dV_g \, ds + \tfrac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2 \frac{\partial \eta^2}{\partial t}\psi_\epsilon^2 \, dV_g \, ds \\ &\quad + \tfrac{1}{100}\int_{t_1}^{t_2}\int_{\mathbb{C}^n} |\nabla_g u|^2\psi_\epsilon^2\eta^2 \, dV_g \, ds + 100\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2\psi_\epsilon^2|\nabla_g\eta|^2 \, dV_g \, ds \\ &\quad + \tfrac{1}{100}\int_{t_1}^{t_2}\int_{\mathbb{C}^n} |\nabla_g u|^2\psi_\epsilon^2\eta^2 \, dV_g \, ds + 100\int_{t_1}^{t_2}\int_{\mathbb{C}^n} u^2\eta^2|\nabla_g\psi_\epsilon|^2 \, dV_g \, ds.\end{aligned}$$

We note that Definition 4.2 requires $u \in L^\infty$, then (27) implies

$$\lim_{\epsilon \to 0} 100 \int_{t_1}^{t_2} \int_{\mathbb{C}^n} u^2 \eta^2 |\nabla_g \psi_\epsilon|^2 \, dV_g \, ds = 0. \tag{33}$$

Let $\epsilon \to 0$ in (32), the proof of (24) and (25) is complete.

Multiplying both hand sides of (18) by $\eta\psi_\epsilon$ and integrating by parts over space-time, (23) is proved similarly. □

By the same proof as for Lemma 4.5 (with Berndtsson's cutoff function as in (27)), the Sobolev embedding theorem is true.

Lemma 4.8 (Sobolev embedding). *Given a function $u \in C^1\{B \setminus D\} \cap L^\infty(B)$, for any cutoff function $\eta \in C_0^1(B)$, the following holds*:

$$\left(\int_B |\eta u|^{\frac{2n}{2n-1}} \, dV_E \right)^{\frac{2n-1}{2n}} \le N(\beta, n) \int_B |\nabla(\eta u)| \, dV_E.$$

Proof. It's true when $\int_B |\nabla(\eta u)| \, dV_E = \infty$. When $\int_B |\nabla(\eta u)| \, dV_E < \infty$, using Berndtsson's cutoff function ψ_ϵ, Claim 4.7, and the same proof as for Lemma 4.5, ηu belongs to $W^{1,1}(B)$ in the usual sense. Then it follows from the usual Sobolev-inequality. □

Remark 4.9. The $N(\beta, n)$ above does not depend on the radius or center of the ball. The only place where we use the Sobolev embedding is (41).

5. Proof of Theorem 5.7 by energy inequalities

Lemma 5.1 (Growth Lemma). *Suppose u is a weak subsolution to* (18) *in a cylinder $C_{2r}(Y)$. Then there exists a $\mu_2(n, \beta, K) > 0$ such that*

$$\frac{|\{u > 0\} \cap C_{2r}(Y)|}{|C_{2r}(Y)|} \le \mu_2 \quad \textit{implies} \quad \sup_{C_r} u \le \tfrac{1}{2} \sup_{C_{2r}} u^+. \tag{34}$$

Proof. The proof is formally the same as for [Ferretti and Safonov 2013, Lemma 4.1]. Since condition (5) is involved, we still give a detailed proof for the reader's convenience. The point is to show that we don't need more on the equation than the energy estimates of subsolutions (Lemma 4.5 and the proof of it). The constants N in this proof only depend on n, β, K.

By rescaling invariance of the subequation (18), it suffices to assume $r = 1$ and $\sup_{C_r} u = 1$. We let μ_2 be small enough. It suffices to prove that for all $Z \notin D$ and $Z \in C_1(Y) = C_1$, under the condition

$$\frac{|\{u > 0\} \cap C_1(Z)|}{|C_1(Z)|} \le \frac{2^{2n+2} |\{u > 0\} \cap C_2(Z)|}{|C_2(Z)|} \le 2^{2n+2} \mu_2 \triangleq \mu_1, \tag{35}$$

the following estimate holds

$$u(Z) \leq \tfrac{1}{2}. \tag{36}$$

We only need to apply Lemma 5.2 (see [Ferretti and Safonov 2013, (3.8), page 33]). Using exactly the induction argument [Ferretti and Safonov 2013, from the last line of page 99 to line 16 of page 100] (only involving Lemma 5.2), we deduce for any integer $j \geq 0$, for some $N(n, \beta, K)$, the following estimate holds when $N\mu_1^{1/(2n+2)} < \frac{1}{2}$.

$$\left|\left\{u > \frac{1-\rho}{2}\right\} \cap C_\rho(Z)\right| \leq \mu_1 \rho^{2n+2} |C_\rho(Z)|, \quad \rho = 2^{-j}. \tag{37}$$

Since $Z \notin D$, (37) directly implies that $u(Z) \leq \frac{1}{2}$. Were this not true, $u(Z) > \frac{1}{2}$ implies that there exists dyadic ρ_0 small enough such that $C_{\rho_0}(Z)$ does not touch the singularity D, and $u > \frac{1}{2}$ over $C_{\rho_0}(Z)$. This contradicts (37). □

Lemma 5.2. *Under the same setting as in Lemma 5.1 and its proof above, for any constant $A \geq 0$, we have*

$$\int_{C_{\rho/2}(Z)} (u - A)_+ \, dV_E \, ds \leq \frac{N}{\rho} |\{u > A\} \cap C_\rho(Z)|^{1+1/(2n+2)}.$$

Proof. By linearity and rescaling invariance of the subequation (18), without loss of generality we can assume $A = 0$ and $\rho = 1$ (note $u \leq 1$). Denote the set $\{(u > 0) \cap C_1(Z)\}$ as E_u, and the spacewise set $\{x | (x, t) \in (u > 0) \cap C_1(Z)\}$ as $Q(t)$. Hence $|E_u| = \int_0^1 |Q(t)| dt$. We need to prove

$$\int_{C_{1/2}(Z)} u_+ \, dV_E \, ds \leq N |E_u|^{1+1/(2n+2)} \tag{38}$$

To show (38) is true, it suffices to show that for any ϵ small enough, u_ϵ satisfies

$$\int_{C_{1/2}(Z)} u_\epsilon \, dV_E \, ds \leq N |E_{u_\epsilon}|^{1+1/(2n+2)}. \tag{39}$$

The advantage of u_ϵ is that it's supported in $Q(t)$, and $0 \leq u_\epsilon \leq 1$. Then integration by parts implies the energy estimates in Lemma 4.5 holds true over $Q(t)$. Let η be the standard cut-off function in $C_1(Z)$ which vanishes near the parabolic boundary; Hölder's inequality and Lemma 4.5 imply

$$\int_B \eta u_\epsilon \, dV_E|_t \leq |Q(t)|^{1/2} \left(\int_B \eta^2 u_\epsilon^2 \, dV_E \right)^{1/2} \Big|_t \leq N E_{u_\epsilon}^{1/2} |Q(t)|^{1/2}. \tag{40}$$

We also have the following bootstrapping estimate on the same term.

$$\begin{aligned}(41)\quad \int_B \eta u_\epsilon \, dV_E &\le \left(\int_B |\eta u_\epsilon|^{2n/(2n-1)} \, dV_E\right)^{(2n-1)/(2n)} |Q(t)|^{1/(2n)} \\ &\le N\left(\int_B |\nabla(\eta u_\epsilon)| \, dV_E\right)|Q(t)|^{1/(2n)} \\ &\le N\left(\int_B |\nabla(\eta u_\epsilon)|^2 dV_E\right)^{1/2} |Q(t)|^{1/(2n)+1/2}, \\ &\qquad\qquad \text{since } \operatorname{supp} \nabla(\eta u_\epsilon) \subset \{u > 0\} \cap B.\end{aligned}$$

By (40), (41), Lemma 4.5, and the Fubini theorem, with the help of (25),

$$\begin{aligned}(42)\quad &\int_{-1}^0 \int_B \eta u_\epsilon \, dV_E \, ds \\ &= \int_{-1}^0 \left(\int_B \eta u_\epsilon \, dV_E\right)^{1/(n+1)} \left(\int_B \eta u_\epsilon \, dV_E\right)^{n/(n+1)} ds \\ &\le N E_{u_\epsilon}^{1/(2n+2)} \int_{-1}^0 |Q(t)|^{(n+2)/(2n+2)} \left(\int_B |\nabla(\eta u_\epsilon)|^2 \, dV_E\right)^{n/(2n+2)} dt \\ &\le N |E_{u_\epsilon}|^{1/(2n+2)} \left(\int_{-1}^0 |Q(t)| \, dt\right)^{(n+2)/(2n+2)} \left(\int_{-1}^0 \int_B |\nabla(\eta u_\epsilon)|^2 \, dV_E \, dt\right)^{n/(2n+2)} \\ &\le N |E_{u_\epsilon}|^{1+1/(2n+2)}\end{aligned}$$

Since $\eta \equiv 1$ over $C_{1/2}(Z)$, the proof is complete. As we've seen, nothing in this proof involves more than Lemma 4.5 on the subsolutions. □

Proposition 5.3. *Suppose u is a weak subsolution to* (18) *in a cylinder $C_r(Y)$, $y \notin D$. Then*

$$(43)\qquad u(Y) \le \frac{N}{|C_r|} \int_{C_r(Y)} u_+ \, dV_E$$

Proof. The proof is exactly as in [Ferretti and Safonov 2013, Theorem 3.4]. The only thing worth mentioning is that we should deal with the singularity D. In [Ferretti and Safonov 2013], they consider the maximal point of $d^\gamma u$, where $\gamma = (2n+2)/p$ and d is the parabolic distance to the parabolic boundary of $C_r(Y)$. However, when a singularity is present, $d^\gamma u$ might not attain maximum away from D. To overcome this, we simply assume $u(Y) > 0$, and use the fact that there exists an almost maximal point away from D. Namely, there exists $X_0 = (x_0, t_0)$ such that $x_0 \notin D$ and

$$(44)\qquad d^\gamma(X_0) u(X_0) \ge \frac{M}{2}, \quad M \triangleq \sup_{C_r} d^\gamma u$$

(we can assume $M > 0$ with out loss of generality). Then the rest of the proof is line by line as [Ferretti and Safonov 2013, line 13 to proof end, page 101], except the μ_1 on line 19 should correspond to $\beta_1 = 2^{-\gamma-2}$, because we have an additional $\frac{1}{2}$ in (44). □

Remark 5.4. As mentioned in [Ferretti and Safonov 2013, Remark 3.5], this proof does *not* involve explicitly the subequation (18). Instead, it only requires the growth lemma (Lemma 5.1). Thus the condition (12) is not involved explicitly in this proof.

Lemma 5.5 (slant cylinder lemma). *Suppose u is a weak subsolution to* (18) *in a slant cylinder* SC_r. *Suppose $u \le 0$ in $B_r \times \{T_0\}$. Then*

$$u(Y) \le (1-\lambda) \sup_{\mathrm{SC}_r} u^+, \tag{45}$$

where $\lambda \in (0,1)$ depends on n, β, an upper bound, $|l|$, on $r|y_1 - y_0|/(T_1 - T_0)$, and upper bounds on $(T_1 - T_0)/r^2$, and K.

Proof. The first paragraph in the proof of Lemma 5.1 also applies here. By translation and rescaling (see Definition 4.1), without changing the parabolic slope, we can transform SC_r to a slant cylinder SC_1 with $r = 1$, $T_0 = 0$, $T_1 = T$, $y_0 = \{0\}$, and $y_1 = y$. We then pull back u and the matrix of the metric g on SC_r to "u" (by abuse of notation) and $\hat{g}$ on SC_1. Thus, u satisfies in SC_1 the following:

$$\frac{\partial u}{\partial t} - \Delta_{\hat{g}} u \le 0 \quad \text{in the sense of Definition 4.2, and} \tag{46}$$

$$\frac{g_E}{K} \le \hat{g} \le K g_E \quad \text{in } \mathrm{SC}_1\,. \tag{47}$$

It suffices to prove (45) for u_ϵ. By rescaling, we can assume $u \le 1$ and $\sup_{\mathrm{SC}_1} u = 1$. Then $0 \le u_\epsilon \le 1-\epsilon$ and $\sup_{\mathrm{SC}_1} u_\epsilon \ge 1 - 3\epsilon$. It suffices to derive an estimate for $v = -\log(1-u_\epsilon)$ which is independent of ϵ. Since u_ϵ satisfies (46), v satisfies

$$\frac{\partial v}{\partial t} - \Delta_{\hat{g}} v \le -|\nabla_{\hat{g}} v|^2 \tag{48}$$

in the sense of Definition 4.2. Let $\underline{\eta}$ be the standard cut-off function in the Euclidean unit ball $B(1)$ which only depends on $|x|^2$. By (the proof of) Lemma 4.5 (replace the 0 on the right hand side of (18) by $-|\nabla_{\hat{g}} v|^2$), using $u_\epsilon \ge 0$, $u_\epsilon|_{t=0} = 0$, by abuse of notation with Lemma 4.5, we consider $\eta = \underline{\eta}[x - y(t)]$ and obtain (similarly to (23))

$$\begin{aligned}\int_{\mathbb{C}^n} v\eta^2\, dV_g\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbb{C}^n} \langle \nabla_g v, \nabla_g \eta^2\rangle\, dV_g\, ds + \int_{t_1}^{t_2}\int_{\mathbb{C}^n} |\nabla_g v|^2 \eta^2\, dV_g\, ds \\ \le \int_{t_1}^{t_2}\int_{\mathbb{C}^n} v \frac{\partial \eta^2}{\partial t}\, dV_g\, ds + K \int_{t_1}^{t_2}\int_{\mathbb{C}^n} v\eta^2\, dV_g\, ds.\end{aligned} \tag{49}$$

We first estimate the term $\int_{t_1}^{t_2}\int_{\mathbb{C}^n} v\frac{\partial\eta^2}{\partial t}\,dV_g\,ds$. It's the same as in [Ferretti and Safonov 2013]. We note that $|\partial\eta^2/\partial t| \le |l||\nabla_E\eta^2|$ (Definition 4.1). Then

$$\left|\int_{t_1}^{t_2}\int_{\mathbb{C}^n} v\frac{\partial\eta^2}{\partial t}\,dV_g\,ds\right| \le |l|K^{2n}\int_{t_1}^{t_2}\int_{\mathbb{C}^n} v|\nabla_E\eta^2|\,dV_E\,ds. \tag{50}$$

Using [Ferretti and Safonov 2013, lines 14–23, page 103], we obtain

$$\int_{\mathbb{C}^n} v|\nabla_E\eta^2|\,dV_E \le N\int_{\mathbb{C}^n}(|v|+|\nabla_E v|)\eta^2\,dV_E. \tag{51}$$

Then the Cauchy–Schwartz inequality and the quasi-isometric condition (47) imply

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_{\mathbb{C}^n} v\frac{\partial\eta^2}{\partial t}\,dV_g\,ds\right| \\ \le \tfrac{1}{100}\int_{t_1}^{t_2}\int_{\mathbb{C}^n}|\nabla_g v|^2\eta^2\,dV_g\,ds+N+N\int_{t_1}^{t_2}\int_{\mathbb{C}^n} v\eta^2\,dV_g\,ds.\end{aligned} \tag{52}$$

For the same reason we have

$$\left|\int_{t_1}^{t_2}\int_{\mathbb{C}^n}\langle\nabla_g v,\nabla_g\eta^2\rangle\,dV_g\,ds\right| \le \tfrac{1}{100}\int_{t_1}^{t_2}\int_{\mathbb{C}^n}|\nabla_g v|^2\eta^2\,dV_g\,ds+N. \tag{53}$$

Then (49), (52), and (53) imply

$$\int_{\mathbb{C}^n} v\eta^2\,dV_g|_{t_1}^{t_2} \le N+N\int_{t_1}^{t_2}\int_{\mathbb{C}^n} v\eta^2\,dV_g\,ds. \tag{54}$$

Let $\int_{\mathbb{C}^n} v\eta^2\,dV_g|_t = I(t)$. Since $I(0)=0$, (54) implies $I(t)$ satisfies the assumption in Lemma 5.6. Hence Lemma 5.6 implies $I(t)\le N$ for all $t\in[0,T]$. Then Proposition 5.3 implies $v(Y)\le N$. Hence for some λ (as in Lemma 5.5) which is independent of ϵ, $u_\epsilon(Y)\le 1-2\lambda\le(1-\lambda)\sup_{\mathrm{SC}_1}u_\epsilon$ when ϵ is small enough. Let $\epsilon\to 0$; the proof of (45) is complete. Again, nothing in this proof involves more than the energy estimates of the subsolutions. □

Lemma 5.6. *Suppose $I(t)$, $t\in[T_0,T_1]$ is an everywhere defined L^∞ function. Suppose $I(t)\ge 0$ for all t, $I(T_0)=0$, and*

$$I(t)\le I(t_1)+N_1\int_{t_1}^{t_2}I(s)\,ds+N_2,\quad \textit{for all } t_1,t_2 \textit{ and } t\in[t_1,t_2]. \tag{55}$$

Then there exists N depending on N_1, N_2, and T_1-T_0, such that $I(t)\le N$.

Proof. Choose a such that $a\le 1/(100N_1)$ and $(T_1-T_0)/a=k_0$ is an integer. Then for $k\le k_0-1$, we deduce

$$\max_{ka\le t\le(k+1)a} I(t)\le \tfrac{1}{2}\max_{ka\le t\le(k+1)a} I(t)+N_2+I(ka),$$

then

$$\max_{ka \le t \le (k+1)a} I(t) \le 2N_2 + 2I(ka). \tag{56}$$

Since $I(T_0) = 0$, the proof is complete by induction. □

Theorem 5.7 (main growth theorem). *Suppose u is a weak subsolution to* (18) *in a cylinder $C_{2r}(Y)$. Suppose*

$$\frac{|\{u > 0\} \cap C_r(y, s - 3r^2)|}{|C_r(y, s - 3r^2)|} \le \frac{1}{2}. \tag{57}$$

Then

$$\sup_{C_r(Y)} u \le (1 - \lambda) \sup_{C_{2r}(Y)} u^+, \quad \text{where } \lambda \in (0, 1) \text{ depends on } n, \beta, K. \tag{58}$$

Assuming the growth lemma and slant cylinder lemma, the proof in [Ferretti and Safonov 2013] goes through without any change. We observe that

> except measure theory which does not involve the subequation (18), the proof of [loc. cit., Theorem 5.3] only depends on the fact that [loc. cit., Theorem 3.3] (Lemma 5.1) and [loc. cit., Lemma 4.1] (Lemma 5.5) hold true for any subsolution (with suitable conditions on initial value or level sets) in any scale.

Therefore, instead of directly quoting [loc. cit.], we sketch the proof of [loc. cit., Theorem 5.3] and Theorem 5.7 for the reader's convenience.

Proof sketch. We screen the center of a parabolic cylinder, i.e., we denote $C_r(Y)$ by C_r. Again, by the rescaling and translation invariance, it suffices to prove it assuming $s = 0$ and $r = 1$, i.e., we shall prove

$$\sup_{C_1} u \le (1 - \lambda) \sup_{C_2} u^+. \tag{59}$$

- In view of Definition 3.1, let C_1^0 denote the cylinder $C_1(y, -3)$ which is "earlier" in t than the target cylinder $C_1 = C_1(y, 0)$ on which we want to prove the estimate.
- Let $\Gamma_u \triangleq \{u \le 0\} \cap C_1^0$, i.e., the subset of C_1^0 on which u is nonpositive.
- Let $\mathcal{A}$ denote the set of all cylinders $C \subset C_2$ such that $|(C \cap \Gamma)/C| \ge 1 - \mu_2$, where μ_2 is the one in Lemma 5.1 (i.e., any such C has a "sufficient large portion" contained in Γ).

The dimension n, cone angle β, K, and the μ_2 in Lemma 5.1 determines 3 numbers $\epsilon_0 > 0$, $R_0 > 0$, and $0 < \beta_2 < 1$ with the following properties (which hold without condition (57)). If there is a cylinder $C_{r_0} \in \mathcal{A}$ with radius $r_0 \ge R_0$, then the growth lemma and slant cylinder lemma routinely yield the desired estimate (59).

If the radius of any cylinder $C \in \mathcal{A}$ is $< R_0$, then measure theory and still the two lemmas imply that

$$\left|\left\{u \le (1-\beta_2)\sup_{C_2} u^+\right\} \cap C_1^0\right| \ge (1+\epsilon_0)|\Gamma_u|. \tag{60}$$

This means that increasing the level by a fixed amount enlarges the sublevel set by a fixed amount of measure.

Then we do an induction argument using the above paragraph successively. We define u_k inductively by

$$u_k = u_{k-1} - (1-\beta_2)\sup_{C_2} u_{k-1}, \quad u_0 = u \text{ (consequently } \Gamma_{u_0} = \Gamma_u).$$

Then $u_k = u - (1-\beta_2^k)\sup_{C_2} u^+$. We note that for any $k \ge 0$, u_k is also a subsolution, and the two alternative possibilities above (and including) (60) apply to any subsolution, particularly to u_k. Therefore for any integer k_0 such that $(1+\epsilon_0)^{k_0}/2 > 1$, either

(1) $\Gamma_{u_k} \ge (1+\epsilon_0)|\Gamma_{u_{k-1}}|$ for all $k \le k_0$, or

(2) there exists a $k \le k_0$ such that u_k satisfies the desired estimate (59).

In case 2 above, u also satisfies the desired estimate (59). In case 1 above, note that condition (57) says $|\Gamma_u| \ge |C_1^0|/2$; we get a contradiction because $\Gamma_{u_{k_0}}$ is a subset of C_1^0, but

$$|\Gamma_{u_{k_0}}| \ge (1+\epsilon_0)|\Gamma_{u_{k_0-1}}| \ge \cdots \ge (1+\epsilon_0)^{k_0}|\Gamma_u| \ge (1+\epsilon_0)^{k_0}\frac{|C_1^0|}{2} > |C_1^0|. \quad \square$$

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References

[Caffarelli and Cabré 1995] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications **43**, American Mathematical Society, Providence, RI, 1995. MR Zbl

[Cao and Chen 2012] H.-D. Cao and Q. Chen, "On locally conformally flat gradient steady Ricci solitons", *Trans. Amer. Math. Soc.* **364**:5 (2012), 2377–2391. MR Zbl

[Chen and Safonov 2017] G. Chen and M. Safonov, "On second order elliptic and parabolic equations of mixed type", *J. Funct. Anal.* **272**:8 (2017), 3216–3237. MR Zbl

[De Giorgi 1957] E. De Giorgi, "Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari", *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* (3) **3** (1957), 25–43. MR Zbl

[Ferretti and Safonov 2001] E. Ferretti and M. V. Safonov, "Growth theorems and Harnack inequality for second order parabolic equations", pp. 87–112 in *Harmonic analysis and boundary value problems* (Fayetteville, AR, 2000), edited by L. Capogna and L. Lanzani, Contemporary Mathematics **277**, American Mathematical Society, Providence, RI, 2001. Zbl

[Ferretti and Safonov 2013] E. Ferretti and M. V. Safonov, "Growth theorems and Harnack inequality for second order parabolic equations", updated version of [Ferretti and Safonov 2001], 2013, Available at http://www-users.math.umn.edu/~safon002/NOTES/FS/GT1.pdf.

[Gilbarg and Trudinger 1977] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften **224**, Springer, 1977. MR Zbl

[Guenancia and Păun 2016] H. Guenancia and M. Păun, "Conic singularities metrics with prescribed Ricci curvature: general cone angles along normal crossing divisors", *J. Differential Geom.* **103**:1 (2016), 15–57. MR Zbl

[Hamilton 1995] R. S. Hamilton, "The formation of singularities in the Ricci flow", pp. 7–136 in *Surveys in differential geometry* (Cambridge, MA, 1993), vol. 2, edited by S.-T. Yau, Int. Press, Cambridge, MA, 1995. MR Zbl

[Krylov and Safonov 1980] N. V. Krylov and M. V. Safonov, "A property of the solutions of parabolic equations with measurable coefficients", *Izv. Akad. Nauk SSSR Ser. Mat.* **44**:1 (1980), 161–175. In Russian; translated in *Math. USSR-Izv.* **16**:1 (1981), 151–164. MR Zbl

[Ladyzhenskaja et al. 1968] O. A. Ladyzhenskaja, V. A. Solonnikov, and N. N. Uralceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs **23**, American Mathematical Society, Providence, RI, 1968. MR Zbl

[Lieberman 1996] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific, River Edge, NJ, 1996. MR Zbl

[Moser 1964] J. Moser, "A Harnack inequality for parabolic differential equations", *Comm. Pure Appl. Math.* **17** (1964), 101–134. MR Zbl

[Nash 1958] J. Nash, "Continuity of solutions of parabolic and elliptic equations", *Amer. J. Math.* **80** (1958), 931–954. MR Zbl

[Wang 2016] Y. Wang, "Smooth approximations of the conical Kähler–Ricci flows", *Math. Ann.* **365**:1-2 (2016), 835–856. MR Zbl

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