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We prove that the probability measure induced by the BBM flow satisfies a logarithmic Sobolev type inequality. Precisely, we suppose the initial data u_0 induces a Gaussian measure on H^s with $s \in \left[1-\frac{\gamma}{2}, \frac{\gamma}{2}\right]$ for $\gamma \in \left(\frac{3}{2}, 2\right]$. Then the induced measure ν under BBM flow satisfies, for any ε small enough,

$$\mathbb{E}_\nu\left[f^2 \log \frac{f^2}{\mathbb{E}_\nu[f^2]}\right] \leq C_\varepsilon \left(\mathbb{E}_\nu\left[|\nabla f|^{2+\varepsilon}_{H^{s+\gamma/2}}\right]\right)^{\frac{2}{2+\varepsilon}},$$

where C_ε is an ε-dependent constant.

1. Introduction

(Gaussian) Logarithmic Sobolev inequalities were first introduced by Gross [1975] for Gaussian measures on finite-dimensional spaces. They turned out to be effective tools for analysis on manifolds. For infinite-dimensional manifolds, thanks to its dimensionless character, logarithmic Sobolev inequalities seem to be similar to these classical ones. Indeed, logarithmic Sobolev inequalities were proved for infinite-dimensional spaces equipped with Gaussian measures [Da Prato 2006], for some infinite-dimensional spaces equipped with certain weighted Gaussian measures [Ledoux 2001], and even for some measures induced by certain transformations [Üstünel 2010] (these inequalities were also established on path spaces [Hsu 1997] and loop groups [Gross 1991]). Here we establish logarithmic Sobolev-type inequalities for measures induced by some flows associated to BBM equation.

Consider the generalized BBM model equation

$$\text{(1-1)} \quad \begin{cases} \partial_t u + \partial_t |\partial_x|^\gamma u + \partial_x(u + u^2) = 0, \\ u(0) = u_0, \end{cases} \quad \text{where } u : (t, x) \in \mathbb{R} \times \mathbb{T} \mapsto u(t, x) \in \mathbb{R}.$$

One can find in Section 4A that (1-1) is quasiglobally well-posed. Precisely, for fixed $\gamma \in \left(\frac{3}{2}, 2\right]$ and $s \in \left[1-\frac{\gamma}{2}, \frac{\gamma}{2}\right]$, if $u_0 \in H^s$ and $\widetilde{T} \in (0, \infty)$, then (1-1) has a

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solution in $C([0,\widetilde{T}];H^s)$. We denote by $\Phi(t)$ the flow associated to (1-1). Now suppose that the initial data u_0 is given by

$$u_0=\phi_s(\omega,x)=\sum_{n\neq 0}\frac{g_n(\omega)}{|n|^{s+\gamma/2}}e^{inx},$$

where $g_n=\overline{g_{-n}}$ and $(g_n)_{n>0}$ is a sequence of independent standard complex Gaussian random variables on some proper probability space $(\Omega,\mathcal{A},\mathbb{P})$. Then the map $\omega\mapsto u_0$ induces a Gaussian measure on H^s, which we denote by μ_s. The classical theory asserts that μ_s satisfies a logarithmic Sobolev inequality (see [Üstünel 2010] for a proof in the case of Wiener space)

$$\mathbb{E}_{\mu_s}\left[f^2\log\frac{f^2}{\mathbb{E}_{\mu_s}[f^2]}\right]\leq C\mathbb{E}_{\mu_s}[|\nabla f|^2_{H^{s+\gamma/2}}]$$

for any $f\in W^{1,2}(H^s,\mathbb{R})$, where C is a universal constant. In this manuscript, we consider whether or not the measure $\nu:=(\Phi(t))_*\mu_s$ satisfies inequalities of this type. We here give some partial answer to this question and our main result is:

Theorem 1.1. *Let $\gamma\in\left(\frac{3}{2},2\right]$ and $s\in\left[1-\frac{\gamma}{2},\frac{\gamma}{2}\right]$. Assume also $t\in[0,\widetilde{T}]$ is fixed and denote $T:=\Phi(t)$. Then there exists some constant C, such that the induced measure $\nu=T_*\mu_s$ satisfies for functions defined on H^s*

$$\mathbb{E}_\nu\left[f^2\log\frac{f^2}{\mathbb{E}_\nu[f^2]}\right]\leq C\mathbb{E}_{\mu_s}\left[|\nabla f\circ T|^2_{H^{s+\gamma/2}}(1+\|\cdot\|^2_{H^s})\right].$$

Furthermore, by invoking Fernique's theorem, for $\varepsilon>0$, there exists $C_\varepsilon>0$ such that ν satisfies a log-Sobolev type inequality with a loss of integrability

$$\mathbb{E}_\nu\left[f^2\log\frac{f^2}{\mathbb{E}_\nu[f^2]}\right]\leq C_\varepsilon\left(\mathbb{E}_\nu\left[|\nabla f|^{2+\varepsilon}_{H^{s+\gamma/2}}\right]\right)^{\frac{2}{2+\varepsilon}},$$

where $f\in W^{1,2+\varepsilon}(H^s,\mathbb{R})$.

The article is organized as follows. In Section 2 and Section 3, we prove that the logarithmic Sobolev inequality is preserved under the flows generated by certain ODEs in finite and infinite-dimensional spaces respectively. That is, the induced measures still satisfy logarithmic Sobolev inequalities both in finite and infinite dimensional cases. Then in Section 4A we prove the existence of the dynamics of BBM equation, and in Section 4B we prove Theorem 1.1.

2. The flow generated by vector fields in the finite-dimensional case

Let $\left(\mathbb{R}^d, d\mu(x)=\frac{1}{(2\pi)^{d/2}}e^{-x^2/2}\,dx\right)$ be the Gaussian space, then there holds the classical logarithmic Sobolev inequality

$$\int f^2\log\frac{f^2}{\int f^2\,d\mu}d\mu\leq 2\int|\nabla f|^2\,d\mu. \tag{2-1}$$

Suppose $T : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible diffeomorphism; then it induces a new probability measure $\nu = T_*\mu$ on $\mathbb{R}^d$. By denoting $\mathbb{E}_\mu[\,\cdot\,] = \int \cdot \, d\mu$, and applying (2-1) with $f \circ T$, we have

$$\begin{aligned}\mathbb{E}_\nu\left[f^2 \log \frac{f^2}{\mathbb{E}_\nu[f^2]}\right] &= \mathbb{E}_\mu\left[(f\circ T)^2 \log \frac{(f\circ T)^2}{\mathbb{E}_\mu[(f\circ T)^2]}\right] \\ &\le 2\mathbb{E}_\mu[|\nabla(f\circ T)|^2] \\ &\le 2\mathbb{E}_\mu[|\nabla f \circ T|^2 \cdot |\nabla T|^2] \\ &\le 2c\mathbb{E}_\nu[|\nabla f|^2],\end{aligned}$$

provided that, for some constant $c > 0$,

$$|\nabla T| \le c, \quad \mu - a.e. \tag{2-2}$$

We are now in a position to claim:

Proposition 2.1. *Suppose $T : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible diffeomorphism and it satisfies the assumption* (2-2). *Then for the induced probability measure $\nu = T_*\mu$ there holds, for some other constant C,*

$$\mathbb{E}_\nu\left[f^2 \log \frac{f^2}{\mathbb{E}_\nu[f^2]}\right] \le C\mathbb{E}_\nu[|\nabla f|^2].$$

Next we suppose that the transformation $T_t : \mathbb{R}^d \ni x \mapsto U_t(x) \in \mathbb{R}^d$ is the flow map associated to the ODE

$$\begin{cases} \dfrac{d}{dt} U_t(x) = B(U_t(x)), \\ U_0(x) = x \end{cases} \tag{2-3}$$

under the condition that B is C^1 and also globally Lipschitzian. In the following, we are going to seek some condition on the vector field B such that $|\nabla T|$ is $(T_t)_*\mu$-almost surely bounded by some constant C. By differentiating the flow equation (2-3) in the space variable, we arrive at

$$\begin{cases} \dfrac{d}{dt} \nabla U_t(x) = \nabla B(U_t(x)) \cdot \nabla U_t(x), \\ \nabla U_0(x) = \mathrm{Id}. \end{cases} \tag{2-4}$$

By the assumption that B is globally Lipschitzian, the ODE system (2-4) is globally well-posed and its solution can be written as

$$\nabla U_t(x) = \mathrm{Id} + \int_0^t \nabla B(U_\tau(x)) \cdot \nabla U_\tau(x)\, d\tau. \tag{2-5}$$

Hence we have

$$\|\nabla U_t\| \le 1 + \left| \int_0^t \|\nabla B\| \times \|\nabla U_\tau(x)\| \, d\tau \right|.$$

With L being the Lipschitz constant of B, we have

$$\|\nabla U_t\| \le 1 + \left| \int_0^t L \|\nabla U_\tau(x)\| \, d\tau \right|.$$

By Gronwall's inequality, we get

$$\|\nabla U_t\| \le e^{L|t|}.$$

This last estimate verifies the assumption (2-2), with an upper bound depending on the time. Thus we can state:

Proposition 2.2. *Let $T_t : x \mapsto T_t(x) = U_t(x)$, where $U_t(x)$ is the flow map defined by* (2-3). *Assume that B is a C^1 Lipschitz vector field with Lipschitz constant L. Then for any $t \in (-\infty, +\infty)$, the induced measure $\nu_t = (T_t)_*\mu$ satisfies a logarithmic Sobolev inequality with some constant depending on c and t. In particular, for some given time T, and for any $t \in [-|T|, |T|]$, the measure ν_t satisfies a logarithmic Sobolev inequality with a uniform constant $C = C(T)$.*

3. The flow generated by vector fields in the infinite dimensional case

Suppose that $W = C_0([0, 1]; \mathbb{R})$ is the space of continuous functions vanishing at 0. We equip W with supremum norm $\|\cdot\|_{C_0}$, then $(W, \|\cdot\|_{C_0})$ is a Banach space. Consider its Cameron-Martin space H defined by

$$H = \left\{ u \in W : u' \text{ exists and } \int_0^1 |u'(\tau)|^2 \, d\tau < \infty \right\}.$$

We supply H an inner product $(u, v) = \int_0^1 u'(\tau) v'(\tau) \, d\tau$ for $u, v \in H$. We select an orthonormal basis $\{e_1, \dots, e_k, \dots\}$ in H, such that all these e_k's are from a subspace H_0 of H

$$H_0 = \{h \in H : h'' \text{ is a signed measure}\}.$$

For example we can take this orthonormal basis to be the Faber–Schauder system. Then the linear continuous functional $x \mapsto (x, e_k)$ for any k defined on H can be extended as a continuous functional defined on W. We also supply a Gaussian or Wiener measure μ on W via the formula

$$\int_W e^{i(x,h)} \, d\mu(x) = e^{-\frac{1}{2}|h|_H^2} \qquad \text{for all } h \in H.$$

For any $n \geq 1$, denote by V_n the linear envelope of $\{e_1, e_2, \dots, e_n\}$. Suppose that we are given a function $f(x) = F(x_1, x_2, \dots, x_k, \dots)$,[1] where $x_i = (e_i, x)$ for all $i \geq 1$, we define its restriction to V_n by

$$f_n(x) = F(x_1, x_2, \dots, x_n).$$

An equivalent way to define this restriction is via the Hermite polynomials. For any $k \geq 0$, the Hermite polynomial $H_k(y)$ on $\mathbb{R}$ is defined by

$$H_k(y) = \frac{(-1)^k}{\sqrt{k!}} e^{y^2/2} \frac{d^k}{dy^k} e^{-y^2/2}, \quad y \in \mathbb{R}.$$

We define the Hermite polynomials on W by

$$H_k(x) = \Pi_i H_{k_i}((e_i, x)),$$

where $k = (k_1, \dots, k_n, \dots), k_i \geq 0, |k| = \sum k_i < \infty$. Then $\{H_k(x)\}_{k \in \mathbb{N}^{\mathbb{N}}}$ is an orthonormal basis of $L^2(W, \mathbb{R})$. We denote

$$C_n = \{k = (k_1, \dots, k_n, \dots) | k_q = 0, \text{ for all } q > n\}.$$

Then the restriction of f to V_n can be expressed as $f_n(x) = \sum_{k \in C_n} c_k H_k(x)$.

For any $\phi \in L^2(W, \mu)$, we define its H-derivative $\nabla_h \phi$ for any $h \in H$ provided that the following limit exists

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}\, \phi(x + \varepsilon h) =: \nabla_h \phi(x) = \langle \nabla \phi, h \rangle.$$

We can see that $\nabla \phi(x)$ is actually in $H^* = H$, that is, it is an H-valued random variable. More generally, if X is another Banach space, then $\nabla \phi(x)$ is indeed an element in $\mathcal{L}(H, X)$, the space of linear operators from H to X. We next define the Sobolev spaces $W^{1,p}$ on W as the collection of these functions f on W such that $f \in L^p(W)$ and their derivatives satisfy $\|\nabla f\|_H \in L^p(W, d\mu)$.

With these notions and notations, we are ready to study the following infinite-dimensional ODE

$$\begin{cases} \dfrac{d}{dt} U_t(x) = B(U_t(x)), \\ U_0(x) = x, \end{cases} \tag{3-1}$$

where B is a vector field over W. We can write the equation in the integral form

$$U_t(x) = x + \int_0^t B(U_\tau(x))\, d\tau.$$

Hence, by the Cameron–Martin theorem, it is expected that the induced measure $\nu_t = (U_t)_* \mu$ is absolutely continuous with respect to μ if B is Cameron–Martin space-valued. Indeed this was studied by Cruzeiro [1983], and he obtained this

[1] This expression is unique due to the fact that $\{e_1, e_2, \dots\}$ is a Schauder basis.

absolute continuity under some exponential integrability condition, which gives a sense of the Radon–Nikodym derivative. The successive question is whether or not there exist some other properties of the measure, which remain (almost) invariant under the flow associated to (3-1)? Notice that initially the measure is a Gaussian one, which satisfies a logarithmic Sobolev inequality. We are thus lead to consider the preservation of inequalities of this type.

To address this question, we begin with the statement of well-posedness of the flow associated to (3-1), whose proof is standard.

Proposition 3.1. *Let $B : x \in W \mapsto B(x) \in W$ be C^1, in the general sense rather than H-derivative, and globally Lipschitzian. Then* (3-1) *defines an invertible global flow on W. Moreover, for fixed t, the flow map is actually a C^1-diffeomorphism, and so is its inverse.*

Under the conditions of the above proposition, the induced measure $\nu_t = (U_t)_*\mu$ is also a probability measure on W, but it may not be absolutely continuous with respect to μ. For example, we take $B(x) = x$ on W, and the flow U_t is

$$U_t(x) = e^t x,$$

and hence the induced measure is just a scaling of μ. In the infinite-dimensional case, it is well-known that ν_t is singular to μ for any $t \neq 0$! But ν_t is still a Gaussian measure, and hence it still satisfies a log-Sobolev type inequality.

In the rest of this section, B is assumed to be Lipschitzian and H-valued. In this case, we can perform the trick used in the finite-dimensional ODE case. Substituting the composed function $f \circ T$ in the log-Sobolev inequality for μ leads us to

$$\begin{aligned}\mathbb{E}_\nu\left[f^2 \log \frac{f^2}{\mathbb{E}_\nu[f^2]}\right] &= \mathbb{E}_\mu\left[(f \circ T)^2 \log \frac{(f \circ T)^2}{\mathbb{E}_\mu[(f \circ T)^2]}\right] \\ &\le C\mathbb{E}_\mu[|\nabla f|_H^2 \cdot |\nabla T|_H^2],\end{aligned}$$

where $T := U_t$ for fixed t. Then we are going to estimate the upper bound of the operator norm of ∇T. For any $h \in H$, differentiating the ODE (3-1) in the direction h, we get

$$\begin{cases}\dfrac{d}{dt}\nabla_h U_t(x) = (\nabla B(U_t(x)), \nabla_h U_t(x)), \\ \nabla_h U_0(x) = h,\end{cases}$$

or, the equivalent form in the space $\mathcal{L}(H, H)$,

$$\begin{cases}\dfrac{d}{dt}\nabla .U_t(x) = (\nabla B(U_t(x)), \nabla .U_t(x)), \\ \nabla .U_0(x) = \mathrm{Id}.\end{cases}$$

The succeeding estimates are the same as that in the finite-dimensional ODE case. Finally we get the desired result.

4. The flow generated by the BBM model equation

4A. ***Existence of the global transformation.*** Consider the generalized BBM model equation

$$\begin{cases} \partial_t u + \partial_t |\partial_x|^\gamma u + \partial_x(u+u^2) = 0, \\ u(0) = u_0, \end{cases} \tag{4-1}$$

where u is a real-valued function defined on $\mathbb{R}_t \times \mathbb{T}^1$. Observing that $\int_{\mathbb{T}} u\,dx$ is preserved, we thus assume $\int_{\mathbb{T}} u\,dx = 0$ and work on the space H^s of functions of Sobolev regularity s with mean zero. By denoting $\varphi(\partial_x) = \partial_x/(1+|\partial_x|^\gamma)$, we rewrite (4-1) as

$$\partial_t u = -\varphi(\partial_x)(u+u^2). \tag{4-2}$$

By integrating on the time interval $[0, t]$, we can also write the equation in the integral form

$$u(t) = u_0 - \int_0^t \varphi(\partial_x)(u(\tau)+u^2(\tau))d\tau. \tag{4-3}$$

We notice that, if we are using fixed point argument to solve (4-3), the main obstacle is the one brought by the nonlinear term u^2. To deal with this nonlinearity, we invoke the following lemma.

Lemma 4.1. *Fix $\gamma > \frac{3}{2}$. Let $0 \le \alpha, \beta \le s$ such that $2s - \alpha - \beta < \gamma - \frac{3}{2}$. Then for any $u \in H^\alpha$, $v \in H^\beta$, we have*

$$\|\varphi(\partial_x)(uv)\|_{H^s} \le C\|u\|_{H^\alpha}\,\|v\|_{H^\beta},$$

where C is a finite positive constant, depending on s, α and β.

Proof. We follow the ideas in [Bona and Tzvetkov 2009; Roumégoux 2010] to prove this lemma. Indeed, it suffices to prove, for any $w \in L^2$, there exists some universal constant C such that

$$\left|\sum_k \frac{\langle k\rangle^s k}{1+|k|^\gamma} \sum_l \hat{u}(k-l)\hat{v}(l)\overline{\hat{w}(k)}\right| \le C\|u\|_{H^\alpha}\,\|v\|_{H^\beta}\,\|w\|_{L^2}. \tag{4-4}$$

Denoting the left-hand side of (4-4) by I, we then do the following calculations:

$$\begin{aligned} I &= \left|\sum_l \langle l\rangle^\beta \hat{v}(l) \sum_k \frac{1}{\langle k\rangle^{-s+\alpha+\beta}} \frac{\langle k\rangle^s k}{(1+|k|^\gamma)} \frac{\langle k\rangle^{-s+\alpha+\beta}}{\langle k-l\rangle^\alpha \langle l\rangle^\beta} \big(\langle k-l\rangle^\alpha \hat{u}(k-l)\big)\overline{\hat{w}(k)}\right| \\ &\le \|v\|_{H^\beta} \left\|\sum_k \frac{1}{\langle k\rangle^{-s+\alpha+\beta}} \frac{\langle k\rangle^s k}{(1+|k|^\gamma)} \frac{\langle k\rangle^{-s+\alpha+\beta}}{\langle k-l\rangle^\alpha \langle l\rangle^\beta} \big(\langle k-l\rangle^\alpha \hat{u}(k-l)\big)\overline{\hat{w}(k)}\right\|_{\ell^2_l} \end{aligned}$$

$$\leq C\|u\|_{H^\alpha}\|v\|_{H^\beta}\left\|\frac{1}{\langle k\rangle^{\gamma-1-2s+\alpha+\beta}}\overline{\hat{w}(k)}\right\|_{\ell_k^1},$$

where in the first inequality we used Hölder's inequality, and in the second one we used Young's inequality and the fact that the quantity $(\langle k\rangle^{-s+\alpha+\beta})/(\langle k-l\rangle^\alpha\langle l\rangle^\beta)$ is bounded by some constant C on $(k,l)\in\mathbb{Z}^2$ provided $\alpha,\beta\in[0,s]$. To finish the proof, we use Hölder's inequality again

$$\left\|\frac{1}{\langle k\rangle^{\gamma-1-2s+\alpha+\beta}}\overline{\hat{w}(k)}\right\|_{\ell_k^1}\leq\|w\|_{L^2}\left\|\frac{1}{\langle k\rangle^{\gamma-1-2s+\alpha+\beta}}\right\|_{\ell_k^2}\leq\widetilde{C}\|w\|_{L^2},$$

where $\widetilde{C}=\|1/\langle k\rangle^{\gamma-1-2s+\alpha+\beta}\|_{\ell_k^2}$, which is finite provided that $\gamma-1-2s+\alpha+\beta>\frac{1}{2}$, i.e., $\gamma-\frac{3}{2}>2s-\alpha-\beta$. This last condition is just the assumption in the statement of the lemma. Thus this completes the proof of Lemma 4.1. □

Remark 4.2. In particular, if we take $\alpha=\beta=s$, the inequality in the lemma reads

$$\|\varphi(\partial_x)(uv)\|_{H^s}\leq C\|u\|_{H^s}\|v\|_{H^s}.$$

Here we need γ to be strictly bigger than $\frac{3}{2}$. Furthermore, this inequality cannot hold for $\gamma\leq\frac{3}{2}$ and $s=0$, as shown by Chenmin Sun [2015]. In a later paper, we will talk more about this inequality in more general cases.

With the help of these bilinear estimates, we prove the following local well-posedness result.

Theorem 4.3 (local well-posedness). *Fix $\gamma>\frac{3}{2}$ and $s\geq 0$. Then for any $u_0\in H^s$,* (4-1) *is locally well-posed in $X_T^s:=C(0,T;H^s)$ provided T is sufficiently small.*

Proof. Taking X_T^s-norm on both sides of (4-3), we get the following estimates by using Lemma 4.1 with $\alpha=\beta=s$:

$$\|u\|_{X_T^s}\leq\|u_0\|_{H^s}+T(\|u\|_{X_T^s}+\|u\|_{X_T^s}^2). \tag{4-5}$$

By taking $R=2\|u_0\|_{H^s}$, the map defined by the right-hand side of (4-3) is onto $B_R\subset X_T^s$ for T sufficiently small. Suppose v is another solution with the same initial data, we estimate

$$\|u-v\|_{X_T^s}\leq T(1+\|u+v\|_{X_T^s})\|u-v\|_{X_T^s}\leq T(1+2R)\|u-v\|_{X_T^s}. \tag{4-6}$$

Choosing $T=C(1+\|u_0\|_{H^s})^{-1}$ with C a small constant, we see that the solution map defined on the right-hand side of (4-3) is a contraction map. Therefore, by the contraction mapping principle, there is a solution u to (4-3) in X_T^s for T sufficiently small. □

Remark 4.4. We see that the length of the time interval is just of size $(1+\|u_0\|_{H^s})^{-1}$. This contradicts the general expectation that we can get the solution on any long time interval if we let the size of the initial data be sufficiently small. To remedy this expectation, we need to rewrite (4-2) in the Duhamel form as

$$u(t)=e^{-t\varphi(\partial_x)}u_0-\int_0^t e^{-(t-\tau)\varphi(\partial_x)}\varphi(\partial_x)(u^2(\tau))\,d\tau.$$

Doing the same estimates as above, we have the following estimates analogous to estimates (4-5) and (4-6):

$$\begin{aligned}\|u\|_{X_T^s}&\le\|u_0\|_{H^s}+T\|u\|_{X_T^s}^2,\\ \|u-v\|_{X_T^s}&\le T\big(\|u\|_{X_T^s}+\|v\|_{X_T^s}\big)\|u-v\|_{X_T^s}.\end{aligned}$$

Taking $R=2\|u_0\|_{H^s}$ and then playing the fixed-point argument, we get the desired size of the existence time interval. In particular, for any $T>0$, there exists $\delta>0$, such that for any data $u_0\in H^s$, as long as $\|u_0\|_{H^s}\le\delta$, there exists solution $u(t)$ to (4-1) up to time T.

Next we are going to study the large time existence. For any $u_0\in H^s$ and any $T>0$, we take N a sufficiently large positive integer such that

$$\sum_{|k|\ge N}\langle k\rangle^{2s}|\widehat{u_0}(k)|^2\le T^{-2}.$$

Denote $v_0=\sum_{|k|\ge N}\widehat{u_0}(k)e^{ikx}$. Then by Remark 4.4, there exists a unique solution v in X_T^s to (4-1), issued from v_0. Furthermore, the solution v is of size $\frac{1}{T}$ in X_T^s. Decompose $u_0=v_0+w_0$. Then if we want to solve (4-1), we only need to solve

$$\begin{cases}\partial_t w=-\varphi(\partial_x)(w+2vw+w^2),\\ w(0)=w_0.\end{cases}\tag{4-7}$$

Suppose w solves (4-7) up to time T. Then $u:=v+w$ solves (4-1) on the time interval $[0,T]$. As w_0 consists of only the first N-frequencies, it belongs to H^r for any $r>0$. But we only treat this in $H^{\frac{\gamma}{2}}$, which is just what we need. In this case (4-7) is locally well-posed. Indeed, by writing (4-7) in its Duhamel form

$$w(t)=e^{-t\varphi(\partial_x)}w_0-\int_0^t e^{-(t-\tau)\varphi(\partial_x)}\phi(\partial_x)(2v(\tau)w(\tau)+w^2(\tau))\,d\tau=:L(w),\tag{4-8}$$

we play the fixed point argument as follows. Before doing this, we need the following lemma, which allows us to deal with the nonlinear term $\varphi(\partial_x)(vw)$.

Lemma 4.5. *Fix $\alpha>\frac{1}{2}$. Let $s\in[0,\alpha]$. Then for any $u\in H^\alpha$ and $v\in H^s$, we have for some positive constant $C(\alpha,s)$*

$$\|\varphi(\partial_x)(uv)\|_{H^{s+\gamma-1}}\le C(\alpha,s)\|u\|_{H^\alpha}\,\|v\|_{H^s}.$$

Proof. We prove this lemma essentially along the same lines as in [Bona and Tzvetkov 2009]. By the smoothing effect of $\varphi(\partial_x)$ in the L^2-based Sobolev spaces, we only need to show

$$\|uv\|_{H^s} \le C(\alpha, s)\|u\|_{H^\alpha}\|v\|_{H^s},$$

for $u \in H^\alpha$ and $v \in H^s$. This last inequality follows exactly from the fact that elements of H^α are multipliers in H^s for $\alpha > \frac{1}{2}$ and $0 \le s \le \alpha$. □

Suppose that S is a positive time to be selected. We estimate for $t \in [0, S]$

$$\begin{aligned}\|L(w)\|_{X_S^{\gamma/2}} &\le \|w_0\|_{H^{\gamma/2}} + \int_0^t \|\varphi(\partial_x)(2vw + w^2)\|_{H^{\gamma/2}}\,d\tau \\ &\le \|w_0\|_{H^{\gamma/2}} + \int_0^t (\|\varphi(\partial_x)(wv)\|_{H^{\gamma/2}} + \|w\|^2_{X_S^{\gamma/2}})\,d\tau \\ &\le \|w_0\|_{H^{\gamma/2}} + S(\|w\|^2_{X_S^{\gamma/2}} + \|w\|_{X_S^{\gamma/2}}\|v\|_{X_S^{1-\gamma/2}}),\end{aligned}$$

where in the last inequality, we have used Lemma 4.5. Under the assumption that $s \ge 1 - \gamma/2$, we have

$$\|L(w)\|_{X_S^{\gamma/2}} \le \|w_0\|_{H^{\gamma/2}} + S(\|w\|^2_{X_S^{\gamma/2}} + \|w\|_{X_S^{\gamma/2}}\|v\|_{X_S^s}). \tag{4-9}$$

A similar argument gives us the estimate

$$\|L(w_1) - L(w_2)\|_{X_S^{\gamma/2}} \le CS(\|v\|_{X_S^s} + \|w_1\|_{X_S^{\gamma/2}} + \|w_2\|_{X_S^{\gamma/2}})\|w_1 - w_2\|_{X_S^{\gamma/2}}. \tag{4-10}$$

Thus by selecting

$$R = 2\|w_0\|_{H^{\gamma/2}} \sim N^{\gamma/2-s}\|u_0\|_{H^s},$$

and choosing $S \sim c/(\|w_0\|_{H^{\gamma/2}} + 1/T)$ with c sufficiently small, the map L is a contraction map onto $B_R \subset X_S^{\gamma/2}$. Hence it has a solution w to (4-7) up to time S.

In order to establish the large time existence of the solution of (4-7), we need to establish the following *a priori* estimate, which can be viewed as an almost conservation law.

Multiplying the first equation in (4-7) by $(1 + |\partial_x|^\gamma)w$ and integrating on the circle, we get

$$\frac{d}{dt}\frac{1}{2}\|w\|^2_{H^{\gamma/2}} = -\int_{\mathbb{T}} \varphi(\partial_x)(w + 2wv + w^2)(1 + |\partial_x|^\gamma)w\,dx.$$

On one hand, by the definition of $\varphi(\partial_x)$ and integration by parts, we have that

$$\int_{\mathbb{T}} \varphi(\partial_x)(w^j)(1 + |\partial_x|^\gamma)w\,dx = 0 \quad \text{for } j = 1, 2.$$

On the other hand, by the self-adjointness of $(1+|\partial_x|^\gamma)^{1/2}$ and the Cauchy–Schwartz inequality, we have

$$\left|\int_{\mathbb{T}} \varphi(\partial_x)(wv)(1+|\partial_x|^\gamma)w\,dx\right| \le \|\varphi(\partial_x)(vw)\|_{H^{\gamma/2}}\,\|w\|_{H^{\gamma/2}}.$$

By using Lemma 4.5 with $s = 1-\frac{\gamma}{2}$ and $\alpha = \frac{\gamma}{2}$, we have

$$\left|\int_{\mathbb{T}} \varphi(\partial_x)(wv)(1+|\partial_x|^\gamma)w\,dx\right| \le \|v\|_{H^{1-\gamma/2}}(\|w\|_{H^{\gamma/2}})^2.$$

Thus, by combining these two points, and using the assumption that $s \ge 1-\frac{\gamma}{2}$, we get

$$\frac{d}{dt}\frac{1}{2}\|w\|^2_{H^{\gamma/2}} \le \|w\|_{H^{\gamma/2}} + \|w\|^2_{H^{\gamma/2}}\,\|v\|_{H^s}.$$

A usage of Gronwall's inequality gives us, for any $t \in [0, T]$,

$$\|w\|_{H^{\gamma/2}} \le \|w_0\|_{H^{\gamma/2}}\,e^{\int_0^T (1+\|v(\tau)\|_{H^s})\,d\tau} \le e^{1+T}\,\|w_0\|_{H^{\gamma/2}}. \tag{4-11}$$

Therefore, by the *a priori* estimate (4-11), we can solve (4-7) on the interval $[0, S]$, with S of size

$$\frac{c}{\|w_0\|_{H^{\gamma/2}}e^{1+T} + 1/T}.$$

Thanks to this *a priori* bound, we can solve (4-7) on the succeeding interval $[S, 2S]$ with initial data $w(S)$ obtained in the previous step. Because S does not depend at which step we solve the equation, we can repeat the above procedure until we arrive at some interval $[kS, (k+1)S]$ such that $(k+1)S \ge T$. That is to say, we can solve (4-1) up to time T and also validate the estimate by using the assumption $s \le \frac{\gamma}{2}$

$$\|u\|_{X^s_T} \le \frac{1}{T} + N^{\gamma/2-s}e^{1+T} \tag{4-12}$$

up to some constants. Therefore, we are in a position to state:

Theorem 4.6. *Fix* $\gamma \in \left(\frac{3}{2}, 2\right]$. *Let* $1-\frac{\gamma}{2} \le s \le \frac{\gamma}{2}$. *Then for any* $u_0 \in H^s$ *and any* $T > 0$, *there exists a unique solution* u *to* (4-1) *in* $C([0, T]; H^s)$. *Furthermore, there exists some* $N_0 \in \mathbb{N}$ *such that for all* $t \in [0, T]$, *we have*

$$\|u(t)\|_{H^s} \le \frac{1}{T} + e^{1+T}N_0^{\gamma/2-s}.$$

Remark 4.7. To get long time existence, we used the local well-posedness and Lemma 4.5. To establish the local well-posed result we need to use the assumption $\gamma > \frac{3}{2}$, while we do not need the assumption in Lemma 4.5. This assumption just arises when we use Lemma 4.1 to deal with the nonlinearity. So this motivates us to seek some other conditions that are sufficient to achieve local well-posedness.

There are two such conditions:

(1) $\gamma \geq 1$ and $s > \frac{1}{2}$, which works due to the fact that H^s is an algebra when $s > \frac{1}{2}$ and $\varphi(\partial_x)$ is a smoothing operator;

(2) $\gamma > \frac{5}{4}$ and $s > \frac{1}{4}$, which works thanks to the fact that, in this case, $u \in H^s$ implies $u^2 \in H^{s-1/4}$ and hence $\varphi(\partial_x)(u^2) \in H^s$.

But both of the above conditions cannot guarantee the large time well-posedness for the regularity $s < (\gamma - 1)/2$, so we omit the detailed discussion here.

4B. ***Some kind of LSI.*** In the following, we consider the special random initial data

$$u_0 = \phi_s(\omega, x) = \sum_{n \neq 0} \frac{g_n(\omega)}{|n|^{s+\gamma/2}} e^{inx},$$

where $g_n = \overline{g_{-n}}$ and $(g_n)_{n>0}$ is a sequence of independent standard complex Gauss random variables on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then one can show that the map $\omega \mapsto \phi_s(\omega, x)$ induces a Gaussian measure μ_s on H^s, and the space $H^{s+\gamma/2}$ with $\gamma > 1$ is the Cameron–Martin space of the Gaussian probability space (H^s, μ_s). In this case, the triple $(H^{s+\gamma/2}, H^s, \mu_s)$ is called an *abstract Wiener space*. Furthermore, one can also define the H-derivative as in the case of classical Wiener space. Under these notions, we actually have an infinite-dimensional log-Sobolev type inequality with some constant c

$$\mathbb{E}_{\mu_s}\left[f^2 \log \frac{f^2}{\mathbb{E}_{\mu_s}[f^2]} \right] \leq c\mathbb{E}_{\mu_s}[|\nabla f|^2_{H^{s+\gamma/2}}], \tag{4-13}$$

for any $f \in W^{1,2}(H^s, \mathbb{R})$, the space of real functions on H^s, which is $L^2(\mu_s)$-integrable together with its $H^{s+\gamma/2}$-derivatives.

As $u_0 = \phi_s(\omega, x)$ lies in H^s almost surely, the flow $\Phi(t)$ is defined almost surely everywhere. In order to study the preservation of log-Sobolev type inequalities, we are going to study the linearization, in the direction v_0, of the solution $\Phi(t)(u_0)$ as follows.[2]

For brevity, we denote the solution $\Phi(t)(u_0) := S(t)(u_0) + K(t)_{u_0}$. Then

$$(D\Phi(t))_{u_0}(v_0) = S(t)(v_0) + (DK(t))_{u_0}(v_0),$$

where

$$(DK(t))_{u_0}(v_0) = -2 \int_0^t S(t-\tau)\big((1 + |\partial_x|^\gamma)^{-1} \partial_x(\Phi(\tau)(u_0)v(\tau))\big)\, d\tau,$$

where $v(t)$ solves the linearized equation

$$\begin{cases} \partial_t v + \partial_t |\partial_x|^\gamma v + \partial_x v + 2\partial_x(\Phi(t)(u_0)v) = 0, \\ v|_{t=0} = v_0. \end{cases} \tag{4-14}$$

[2]Here we follow [Tzvetkov 2015].

Proposition 4.8. *Let γ, s and u_0 be as in Theorem 4.6, then for any $v_0 \in H^{s+\gamma/2}$, we have that $v(t)$ is also in $H^{s+\gamma/2}$ for any t in the life span of $u(t)$. Furthermore, we have the bound*

$$\|v(t)\|_{H^{s+\gamma/2}} \le C(t)\|v_0\|_{H^{s+\gamma/2}}.$$

Next we are going to show that, as an operator parametrized by u_0, $(DK(t))$ is bounded from $H^{s+\gamma/2}$ into itself.

$$\begin{aligned}\|(DK(t))_{u_0}v_0\|_{H^{s+\gamma/2}} &\le \Big\|\int_0^t S(t-\tau)\varphi(\partial_x)\big(\Phi(\tau)(u_0)v(\tau)\big)\,d\tau\Big\|_{H^{s+\gamma/2}}\\ &\le \int_0^t \|\varphi(\partial_x)(\Phi(\tau)(u_0)v(\tau))\|_{H^{s+1-\gamma/2+\gamma-1}}\,d\tau\\ &\le \int_0^t \|\Phi(\tau)(u_0)\|_{H^s}\,\|v(\tau)\|_{H^{s+1-\gamma/2}}\,d\tau.\end{aligned}$$

Under the condition that $\gamma > \frac{3}{2}$ and hence $s-\frac{\gamma}{2}+1 \le \frac{\gamma}{2}+s$, we have

$$\|DK(t)_{u_0}\|_{H^{s+\gamma/2}\to H^{s+\gamma/2}} \le C\|u_0\|_{H^s}. \tag{4-15}$$

By denoting $T := \Phi(t) : u \to \Phi(t)(u)$ for fixed t, we do the following calculations

$$\begin{aligned}\mathbb{E}_\nu\Big[f^2\log\frac{f^2}{\mathbb{E}_\nu[f^2]}\Big] &= \mathbb{E}_\mu\Big[(f\circ T)^2\log\frac{(f\circ T)^2}{\mathbb{E}_\mu[(f\circ T)^2]}\Big]\\ &\le c\mathbb{E}_\mu[|\nabla(f\circ T)|^2_{H^{s+\gamma/2}}] \le c\mathbb{E}_\mu\big[|\nabla f\circ T|^2_{H^{s+\gamma/2}}(1+\|\cdot\|^2_{H^s})\big].\end{aligned}$$

Then thanks to Fernique's theorem, we arrive at the following log-Sobolev type inequality, with a loss of integrability:

Proposition 4.9. *Let $\gamma \in \big(\frac{3}{2}, 2\big]$ and $s \ge 1-\frac{\gamma}{2}$. Then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that the induced measure $\nu = (\Phi(t))_*\mu_s$ satisfies a log-Sobolev type inequality*

$$\mathbb{E}_\nu\Big[f^2\log\frac{f^2}{\mathbb{E}_\nu[f^2]}\Big] \le C_\varepsilon(\mathbb{E}_\nu[|\nabla f|^{2+\varepsilon}_{H^{s+\gamma/2}}])^{\frac{2}{2+\varepsilon}}. \tag{4-16}$$

Proof. Given $\varepsilon > 0$, we first estimate

$$\begin{aligned}\mathbb{E}_\nu\Big[f^2\log\frac{f^2}{\mathbb{E}_\nu[f^2]}\Big] &\le c\mathbb{E}_\mu\big[|\nabla f\circ T|^2_{H^{s+\gamma/2}}(1+\|\cdot\|^2_{H^s})\big]\\ &\le c\big(\mathbb{E}_\mu[|\nabla f\circ T|^{2+\varepsilon}_{H^{s+\gamma/2}}]\big)^{\frac{2}{2+\varepsilon}}\big(\mathbb{E}_\nu[(1+\|\cdot\|^2_{H^s})^{\frac{2+\varepsilon}{\varepsilon}}]\big)^{\frac{\varepsilon}{2+\varepsilon}},\end{aligned}$$

where in the last inequality, we have used Hölder's inequality. Noticing that

$$\big(\mathbb{E}_\mu[|\nabla f\circ T|^{2+\varepsilon}_{H^{s+\gamma/2}}]\big)^{\frac{2}{2+\varepsilon}} = \big(\mathbb{E}_\nu[|\nabla f|^{2+\varepsilon}_{H^{s+\gamma/2}}]\big)^{\frac{2}{2+\varepsilon}},$$

we only need to control

$$\big(\mathbb{E}_\nu[(1+\|\cdot\|^2_{H^s})^{\frac{2+\varepsilon}{\varepsilon}}]\big)^{\frac{\varepsilon}{2+\varepsilon}}.$$

Recall that Fernique's theorem [Üstünel 2010, Chapter 9] states that for some small positive $c > 0$,

$$\mathbb{E}_\mu[e^{c\|\cdot\|_{H^s}^2}] < +\infty.$$

Consequently, for any positive integer k, we have

$$\mathbb{E}_\mu[\|\cdot\|_{H^s}^k] < +\infty. \tag{4-17}$$

Letting k be the smallest integer that is bigger than $(2+\varepsilon)/\varepsilon$, we have

$$\mathbb{E}_\mu\big[(1+\|\cdot\|_{H^s}^2)^{\frac{2+\varepsilon}{\varepsilon}}\big] \le \mathbb{E}_\mu\big[(1+\|\cdot\|_{H^s}^2)^k\big] = \sum_{j=0}^{k}\binom{2k}{2j}\mathbb{E}_\mu[\|\cdot\|_{H^s}^{2j}],$$

which is finite thanks to (4-17). This completes the proof of Proposition 4.9. □

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