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**ON COMMUTING BILLIARDS IN HIGHER-DIMENSIONAL  
SPACES OF CONSTANT CURVATURE**

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# ON COMMUTING BILLIARDS IN HIGHER-DIMENSIONAL SPACES OF CONSTANT CURVATURE

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We consider two nested billiards in  $\mathbb{R}^d$ ,  $d \geq 3$ , with  $C^2$ -smooth strictly convex boundaries. We prove that if the corresponding actions by reflections on the space of oriented lines commute, then the billiards are confocal ellipsoids. This together with the previous analogous result of the author in two dimensions solves completely the commuting billiard conjecture due to Sergei Tabachnikov. The main result is deduced from the classical theorem due to Marcel Berger which says that in higher dimensions only quadrics may have caustics. We also prove versions of Berger's theorem and the main result for billiards in spaces of constant curvature (space forms).

## 1. Introduction

**1A. Main result.** Let  $\Omega_a \Subset \Omega_b \subset \mathbb{R}^d$  be two nested bounded domains with smooth strictly convex boundaries  $a = \partial\Omega_a$  and  $b = \partial\Omega_b$ . Consider the corresponding billiard transformations  $\sigma_a, \sigma_b$  acting on the space of oriented lines in space by reflection as follows. Each  $\sigma_f$ ,  $f = a, b$ , acts as identity on the lines disjoint from  $f$ . For each oriented line  $l$  intersecting  $f$  we take its last intersection point  $x$  with  $f$  in the sense of orientation: the orienting arrow of the line  $l$  at  $x$  is directed outside  $\Omega_f$ . The image  $\sigma_f(l)$  is the line obtained by reflection of the line  $l$  from the hyperplane  $T_x f$ : the angle of incidence equals the angle of reflection. The line  $\sigma_f(l)$  is oriented by a tangent vector at  $x$  directed inside  $\Omega_f$ . This is a continuous mapping that is smooth on the space of lines intersecting  $f$  transversely.

**Remark 1.1.** The above action can be defined for a convex billiard in any Riemannian manifold; the billiard reflection acts on the space of oriented geodesics.

Recall, see, e.g., [Berger 1995; Tabachnikov 2005], that a pencil of *confocal quadrics* in a Euclidean space  $\mathbb{R}^d$  is a one-dimensional family of quadrics defined

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in some orthogonal coordinates  $(x_1, \dots, x_d)$  by equations

$$\sum_{j=1}^d \frac{x_j^2}{a_j^2 + \lambda} = 1; \quad a_j \in \mathbb{R} \text{ are fixed; } \lambda \in \mathbb{R} \text{ is the parameter.}$$

It is known that *any two confocal elliptic or ellipsoidal billiards* commute [Tabachnikov 2005, p. 59, Corollary 4.6; 1994, p. 58]. Sergei Tabachnikov [1994, p. 58] stated the conjecture affirming the converse: any two commuting nested convex billiards are confocal ellipses (ellipsoids). In two dimensions this conjecture was proved in [Glutsyuk 2017a, Theorem 5.21, p. 231] for piecewise  $C^4$ -smooth boundaries. Here we prove it in higher dimensions in  $\mathbb{R}^d$  and in spaces of constant curvature (space forms).

**Theorem 1.2.** *Let two nested strictly convex  $C^2$ -smooth closed hypersurfaces in  $\mathbb{R}^d$ ,  $d \geq 3$ , be such that the corresponding billiard transformations commute. Then they are confocal ellipsoids.*

To extend Theorem 1.2 to spaces of constant curvature, let us recall the notions of space forms and (confocal) quadrics in them.

**Definition 1.3.** A *space form* is a complete connected Riemannian manifold of constant curvature.

**Remark 1.4.** We will deal only with simply connected space forms. It is well-known that they are the Euclidean space  $\mathbb{R}^d$ , the unit sphere  $S^d \subset \mathbb{R}^{d+1}$  in the Euclidean space and the hyperbolic space  $\mathbb{H}^d$  (up to normalization of the metric by constant scalar factor, which changes neither geodesics, nor reflections). It is known that the hyperbolic space  $\mathbb{H}^d$  admits a standard model in the Minkowski space  $\mathbb{R}^{d+1}$ . Finally, each space form  $\Sigma$  we will be dealing with is realized as an appropriate hypersurface in the space  $\mathbb{R}^{d+1}$  with coordinates  $x = (x_0, \dots, x_d)$  equipped with a suitable quadratic form

$$\langle Gx, x \rangle, \quad G \text{ is a symmetric } (d+1) \times (d+1)\text{-matrix.}$$

Here  $\langle x, x \rangle := \sum_j x_j^2$ .

Euclidean case:  $G = \text{diag}(0, 1, \dots, 1)$ ,  $\Sigma = \mathbb{R}^d = \{x_0 = 1\}$ .

Spherical case:  $G = \text{Id}$ ,  $\Sigma = S^d = \{\langle Gx, x \rangle = 1\}$ ,  $\langle Gx, x \rangle = \sum_j x_j^2$ .

Hyperbolic case:  $G = \text{diag}(-1, 1, \dots, 1)$ ,  $\Sigma = \mathbb{H}^d = \{\langle Gx, x \rangle = -1\} \cap \{x_0 > 0\}$ .

The metric on each hypersurface  $\Sigma$  is the restriction to  $T\Sigma$  of the quadratic form  $\langle Gx, x \rangle$  on the ambient space. It is well-known that the geodesics on  $\Sigma$  are its intersections with two-dimensional vector subspaces in  $\mathbb{R}^{d+1}$ . Completely geodesic  $k$ -dimensional submanifolds in  $\Sigma$  are its intersections with  $(k+1)$ -dimensional vector subspaces in  $\mathbb{R}^{d+1}$ .

**Definition 1.5** [Veselov 1990, p. 84]. A *quadric* in  $\Sigma$  is a hypersurface

$$S = \Sigma \cap \{\langle Qx, x \rangle = 0\}, \quad Q \text{ is a symmetric matrix.}$$

The *pencil of confocal quadrics* associated to a symmetric matrix  $Q$  is the family of quadrics

$$S_\lambda = \Sigma \cap \{\langle Q_\lambda x, x \rangle = 0\}, \quad Q_\lambda = (Q - \lambda G)^{-1}, \quad \lambda \in \mathbb{R}.$$

**Definition 1.6.** A germ of  $C^2$ -smooth hypersurface  $S$  in a space form  $\Sigma$  at a point  $p$  is *strictly convex*, if it has quadratic tangency with its tangent completely geodesic hypersurface  $\Gamma_p$ , that is, there exists a constant  $C > 0$  such that for every  $q \in S$  close to  $p$  one has

$$\text{dist}(q, \Gamma_p) > C \|q - p\|^2; \text{ here } \|q - p\| = \text{dist}(q, p).$$

**Theorem 1.7.** Let  $d \geq 3$ , and let  $\Sigma$  be a simply connected  $d$ -dimensional space form: either  $\mathbb{R}^d$ , or the unit sphere, or the hyperbolic space. Let two nested strictly convex  $C^2$ -smooth closed hypersurfaces in  $\Sigma$  be such that the corresponding billiard transformations commute. Then they are confocal quadrics.

Theorem 1.2 follows from Theorem 1.7.

Theorem 1.2 can be deduced from a classical theorem due to Marcel Berger [1995] concerning billiards in  $\mathbb{R}^d$ ,  $d \geq 3$ , which states that only billiards bounded by quadrics may have caustics (see Definition 1.8 for the notion of caustic), and the caustics are their confocal quadrics. To prove Theorem 1.7 in full generality, we extend Berger's theorem to the case of billiards in space forms (Theorem 1.10 stated in Section 1C and proved in Section 2) and then deduce Theorem 1.7 in Section 3. A local version of Theorem 1.7 is proved in Section 4. In Section 5 we present some open problems.

**1B. Historical remarks.** Commuting billiards are closely related to problems of classification of integrable billiards, see [Tabachnikov 1994]. It is known that elliptic and ellipsoidal billiards are integrable, see [Veselov 1988, Proposition 4; Tabachnikov 2005, Chapter 4], and this also holds for non-Euclidean ellipsoids in spheres and in the Lobachevsky (hyperbolic) space of any dimension, see [Veselov 1990, the corollary on p. 95]. The famous Birkhoff conjecture states that in two dimensions the converse is true. Namely, it deals with the so-called *Birkhoff caustic-integrable* convex planar billiards with smooth boundary, that is, billiards for which there exists a foliation by closed caustics (a one-parameter family of nested closed caustics  $\Gamma_p$ ,  $p > 0$ ) in an interior neighborhood of the boundary, and the boundary itself is the leaf  $\Gamma_0$  of this foliation. The Birkhoff conjecture states that the only Birkhoff caustic-integrable billiards are ellipses. The Birkhoff conjecture was first stated in print in [Poritsky 1950], where it was proved under the additional assumption that for any two nested caustics in the above family  $\Gamma_p$  the smaller

one is a caustic for the billiard in the bigger one. Poritsky's assumption implies that *the initial billiard map in  $\Gamma_0$*  (being restricted to the set of those lines that are disjoint from some given caustic  $\Gamma_p$  with  $p > 0$ ) *commutes with the billiard in every caustic  $\Gamma_q$* . This follows by the arguments presented in [Tabachnikov 2005, Section 4, pp. 58–59].

The set of lines intersecting the given convex billiard is a topological cylinder called the *phase cylinder*. One of the most famous results on the Birkhoff conjecture is a theorem of M. Bialy [1993], who proved that if the phase cylinder of the billiard map is foliated (almost everywhere) by noncontractible closed curves which are invariant under the billiard map, then the boundary is a circle. In [Bialy 2013] he proved the same result for billiards on surfaces of nonzero constant curvature. A local version of the Birkhoff conjecture for integrable deformations of ellipses was recently solved in [Avila et al. 2016; Kaloshin and Sorrentino 2018b]. The recent solution of its polynomial version (stated and partially studied in [Bolotin 1992]) is a result of [Bialy and Mironov 2017a; 2017b; Glutsyuk 2017b; 2018]. For a historical survey of the Birkhoff conjecture see [Tabachnikov 2005, Section 5, p. 95], the recent surveys [Bialy and Mironov 2018; Kaloshin and Sorrentino 2018a] and the papers [Kaloshin and Sorrentino 2018b; Glutsyuk 2017b]. Dynamics in billiards in two and higher dimensions with piecewise smooth boundaries consisting of confocal quadrics was studied in [Dragovich and Radnovich 2010].

### 1C. Berger's theorem and its extension to billiards in space forms.

**Definition 1.8.** Let  $a, b$  be two nested strictly convex closed hypersurfaces in a Riemannian manifold  $E$ : the hypersurface  $b$  bounds a relatively compact domain in  $E$  whose interior contains  $a$ . We say that  $a$  is a *caustic* for the hypersurface  $b$ , if the image of each oriented geodesic tangent to  $a$  by the reflection  $\sigma_b$  from  $b$  is again a geodesic tangent to  $a$ .

**Remark 1.9.** It is well-known that if  $a, b$  are two confocal ellipses (ellipsoids) in Euclidean space, then the smaller one is a caustic for the bigger one. In the plane this is the classical Proclus–Poncelet theorem. In higher dimensions this theorem is due to Jacobi; see [Staude 1914, p. 80]. Similar statement holds in any space form; see, e.g., [Veselov 1990, Theorem 3].

We will deduce Theorem 1.7 from the following theorem, which implies that *in every space form only quadrics have caustics, and the caustics of each quadric  $S$  are exactly the quadrics confocal to  $S$* .

**Theorem 1.10.** Let  $d \geq 3$ , and let  $\Sigma$  be a  $d$ -dimensional simply connected space form. Let  $S, U \subset \Sigma$  be germs of  $C^2$ -smooth hypersurfaces at points  $B$  and  $A \neq B$  respectively with nondegenerate second fundamental forms. Let the geodesic  $AB$  be tangent to  $U$  at  $A$  and transversal to  $S$  at  $B$ . Let  $C \in \Sigma \setminus \{B\}$ , and let a vector

tangent to the geodesic  $AB$  at  $B$  be reflected from the hyperplane  $T_B S$  to a tangent vector to the geodesic  $BC$ . Let there exist a germ of  $C^2$ -smooth hypersurface  $V$  at  $C$  tangent to  $BC$  at  $C$  such that each geodesic close to  $AB$  and tangent to  $U$  be reflected from the hypersurface  $S$  to a geodesic tangent to  $V$ . Then  $S$  is a piece of a quadric  $b$ , and  $U$ ,  $V$  are pieces of one and the same quadric confocal to  $b$ .

**Remark 1.11.** In the case when  $\Sigma = \mathbb{R}^d$ , [Theorem 1.10](#) was proved by Berger [\[1995\]](#).

## 2. Caustics of hypersurfaces in space forms: Proof of [Theorem 1.10](#)

The proof of [Theorem 1.10](#) for space forms essentially follows Berger's proof [\[1995\]](#) for the Euclidean case. In [Section 2A](#) we first prove that the hypersurfaces  $U$  and  $V$  are pieces of the same quadric denoted by  $U$ . Then in [Section 2B](#) we show that  $S$  is a quadric confocal to  $U$ , using the fact that it is an integral hypersurface of a finite-valued hyperplane distribution: the field of symmetry hyperplanes in  $T_x \Sigma$ ,  $x \in \Sigma$ , for the geodesic cones  $K_x$  circumscribed about the quadric  $U$  with vertex at  $x$ .

### 2A. The hypersurfaces $U$ and $V$ and circumscribed cones.

**Theorem 2.1.** *In the conditions of [Theorem 1.10](#) the hypersurfaces  $U$  and  $V$  are pieces of one and the same quadric.*

[Theorem 2.1](#) is proved below following [\[Berger 1995\]](#). As in [\[loc. cit.\]](#), we first prove that for every  $y \in S$  the geodesic cone with vertex  $y$  tangent to  $U$  is a quadratic cone tangent to both  $U$  and  $V$  ([Lemma 2.4](#)). Afterwards we apply a result from [\[Berger 1995\]](#) (proved via arguments using projective duality, and stated below as [Lemma 2.12](#)), showing that if the latter statement holds, then  $U$  and  $V$  lie in the same quadric.

Let  $\pi : \Sigma \rightarrow \mathbb{RP}^d$  denote the restriction to  $\Sigma$  of the tautological projection  $\mathbb{R}^{d+1} \setminus \{0\} \rightarrow \mathbb{RP}^d$ . It is a diffeomorphism onto the image  $\pi(\Sigma)$  in nonspherical cases and a degree two covering over  $\mathbb{RP}^d$  in the spherical case. Let  $g$  denote the metric on  $\pi(\Sigma)$  that is the (well-defined) pushforward of the space form metric. Note that the geodesics (completely geodesic subspaces) for the metric  $g$  are the intersections of projective lines (respectively, projective subspaces) with  $\pi(\Sigma)$ . In order to reduce the proof to the Euclidean case treated in [\[Berger 1995\]](#), we use the following property of the metric  $g$ .

**Proposition 2.2.** *For every point  $y \in \pi(\Sigma)$  there exist an affine chart  $\mathbb{R}^d \subset \mathbb{RP}^d$  centered at  $y$  and a Euclidean metric on  $\mathbb{R}^d$  (compatible with the affine structure) that has the same 1-jet at  $y$ , as the metric  $g$ .*

*Proof.* Without loss of generality we assume that  $y = (1 : 0 : \cdots : 0)$ , that is, the isometry group of the space form  $\Sigma$  acts transitively, and the projection  $\pi$  conjugates its action on  $\Sigma$  with its action on  $\mathbb{RP}^d$  by projective transformations

(since the isometry group is a subgroup in  $\mathrm{GL}_{d+1}(\mathbb{R})$ ). Thus, in the standard affine chart  $\mathbb{R}^d = \{x_0 = 1\}$  the point  $y$  is the origin. The metric  $g$  is invariant under the orthogonal transformations of the chart  $\mathbb{R}^d$ , since the metric of the space form is invariant under the rotations around the  $x_0$ -axis. The metric  $g$  on  $T_y\mathbb{R}^d$  coincides with the standard Euclidean metric of the chart  $\mathbb{R}^d$ , by definition. The last two statements together imply that the 1-jets of both metrics at  $y = 0$  coincide. This proves the proposition.  $\square$

**Corollary 2.3.** *Let  $U \subset \Sigma$  be a germ of hypersurface with nondegenerate second fundamental form. Then its projection  $\pi(U)$  has nondegenerate second fundamental form in any affine chart  $\mathbb{R}^d$  with respect to the standard Euclidean metric.*

*Proof.* The corollary follows from [Proposition 2.2](#) and invariance of the property of having nondegenerate second fundamental form under projective transformations. Indeed, each germ of projective hypersurface is tangent to some quadric with order 3 (which is not unique). The 2-jet of a quadric determines completely whether it is regular or not. Nondegeneracy of the second fundamental form is equivalent to regularity of the tangent quadric. The space of regular quadrics is invariant under projective transformations. This proves the corollary.  $\square$

In what follows in the present subsection we identify the hypersurfaces  $S, U, V$  and their points with their projection images: for simplicity the projection images  $\pi(S), \pi(U), \pi(B)$  etc. will be denoted by the symbols  $S, U, B, \dots$

**Lemma 2.4.** *Let  $S, U, V \subset \mathbb{RP}^d$  be the tautological projection images of the same hypersurfaces in  $\Sigma$ , as in [Theorem 1.10](#) (see the above paragraph). For every  $y \in S$  there exists a quadratic cone  $K_y \subset \mathbb{RP}^d$  (i.e., given by the zero locus of a homogeneous quadratic polynomial) with vertex at  $y$  that is tangent to both hypersurfaces  $U$  and  $V$ .*

The proof of [Lemma 2.4](#) given below follows [[Berger 1995](#), Section 2].

Let  $\sigma_g : (T\mathbb{RP}^d)|_S \rightarrow (T\mathbb{RP}^d)|_S$  denote the involution acting as the symmetry of each space  $T_y\mathbb{RP}^d$ ,  $y \in S$ , with respect to the hyperplane  $T_yS$  in the metric  $g$ . Its action on the projectivized tangent spaces  $\mathbb{RP}_y^{d-1} = \mathbb{P}(T_y\mathbb{RP}^d)$  induces its action on the space of projective lines in  $\mathbb{RP}^d \supset S$  intersecting  $S$  transversely and so that the intersection point is unique: if  $\ell$  intersects  $S$  at a point  $y$ , then

$$\hat{\ell} := \sigma_g(\ell)$$

is the line through  $y$  that is symmetric to  $\ell$  in the above sense.

For every  $y \in \Sigma$  set

$$M_y := \text{the space of projective lines through } y \text{ that are tangent to } U.$$

It suffices to prove the statement of [Lemma 2.4](#) for an arbitrary point  $y \in S$  satisfying the following statements:

**Proposition 2.5** (stated in [Berger 1995, pp. 110–111]). *There exists an open and dense subset of points  $y \in S \subset \mathbb{RP}^d$  for which there exists an open and dense subset  $M_y^0 \subset M_y$  of lines  $\ell$  satisfying the following statements:*

- (i) *The line  $\ell$  is quadratically tangent to  $U$ , (i.e.,  $\ell$  is not an asymptotic direction of the hypersurface  $U$  at the tangency point). The projective hyperplane containing  $\ell$  and tangent to  $U$  at the latter point is not orthogonal to  $T_y S$ .*
- (ii) *The line  $\hat{\ell} = \sigma_g(\ell)$  is quadratically tangent to  $V$  at a point, where the second fundamental form of the hypersurface  $V$  is nondegenerate.*
- (iii) *The lines  $\ell$  and  $\hat{\ell}$  are transversal to  $T_y S$  and their above tangency points with  $U$  and  $V$  are distinct from the point  $y$ .*

*Proof.* Statement (i) holds for an open and dense subset of lines  $\ell \in M_y$ , since the second fundamental form of the hypersurface  $U$  is nondegenerate (by assumptions and Corollary 2.3). Statement (iii) also holds for a generic  $\ell \in M_y$ , whenever  $y \notin U \cup V$ . Let us show that statement (ii) also holds generically.

Let  $y \in S$ ,  $y \notin U \cup V$ , and let  $\ell$  be a line through  $y$  satisfying assumption (i). Then the cone  $K_y$  with vertex  $y$  containing  $\ell$  and circumscribed about  $U$  is tangent to  $U$  along a  $(n-2)$ -dimensional submanifold  $\mathcal{T}_U \subset U$ .

The correspondence sending a point  $p \in \mathcal{T}_U$  to the projective hyperplane tangent to  $U$  at  $p$  (i.e., to the projective hyperplane tangent to the cone along the line  $yp$ ) is a local immersion to the space of hyperplanes through  $y$ . Or equivalently, the correspondence sending a line  $L \subset K_y$  through  $y$  to the projective hyperplane tangent to  $K_y$  along  $L$  is a local immersion. This follows from nondegeneracy of the second fundamental form of the hypersurface  $U$ . This implies a similar statement for the symmetric cone  $\widehat{K}_y = \sigma_g(K_y)$  circumscribed about  $V$ : the correspondence sending each line  $\widehat{L} \subset \widehat{K}_y$  through  $y$  to the hyperplane tangent to  $\widehat{K}_y$  along the line  $\widehat{L}$  is a local immersion to the space of hyperplanes through  $y$ .

Suppose now that a line  $\widehat{L} \subset \widehat{K}_y$  through  $y$  is quadratically tangent to  $V$  at a point  $q$ . Then the above immersivity statement for the symmetric cone together with quadraticity of tangency imply nondegeneracy of the second fundamental form of the hypersurface  $V$  at the point  $q$ . It is clear that for a generic choice of the point  $y \in S$  and a line  $L \subset K_y$  through  $y$  the corresponding symmetric line  $\widehat{L} = \sigma_g(L)$  is quadratically tangent to  $V$ . This proves the proposition.  $\square$

**Convention 2.6.** In the proof of Lemma 2.4 without loss of generality we assume that  $y = B$ , and there exists a line  $\ell$  through  $B$  that is transversal to  $S$  and satisfies statements (i)–(iii) of Proposition 2.5. Without loss of generality we assume that  $A$  is the tangency point of the line  $\ell$  with  $U$ , and  $C$  is the tangency point of the symmetric line  $\hat{\ell} = \sigma_g(\ell)$  with  $V$ , where  $A, C \neq B$ . Fix an affine chart  $\mathbb{R}^d \subset \mathbb{RP}^d$  centered at  $B$  and equipped with an Euclidean metric whose 1-jet at  $B$  coincides with the 1-jet of the metric  $g$  (Proposition 2.2).



Consider a smooth deformation  $x(t) \in S$  of the point  $B$ ,  $x(0) = B$ , and a smooth deformation  $p(t) \in U$  of the point  $A$ ,  $p(0) = A$ , such that the line  $\ell(t) = x(t)p(t)$  is tangent to  $U$  at  $p(t)$ ,  $t \in [0, 1)$ . Then the line  $\hat{\ell}(t) = \sigma_g(\ell(t))$  symmetric to  $\ell(t)$  in the metric  $g$  is tangent to the hypersurface  $V$  at some point  $q(t)$ ,  $q(0) = C$ , that depends smoothly on the parameter  $t$  (assumptions (i)–(iii)). We will show that the property that every deformation  $x(t)$  extends to a pair of deformations  $p(t)$  and  $q(t)$  as above implies that the cone  $K_y$  tangent to both  $U$  and  $V$  is quadratic. To do this, consider the projective hyperplanes  $\mathcal{U}$  and  $\mathcal{V}$  through  $B$  containing the lines  $\ell(0) = BA$  and  $\hat{\ell}(0) = BC$  respectively:  $\mathcal{U}$  is tangent to  $U$  at  $A$ , and  $\mathcal{V}$  is tangent to  $V$  at  $C$ .

**Remark 2.7.** Let  $\mathcal{U}$  and  $\mathcal{V}$  be as above. The tangent subspaces  $T_B\mathcal{U}$ ,  $T_B\mathcal{V} \subset T_B\mathbb{RP}^d$  are  $\sigma_g$ -symmetric. Indeed, consider the germs of the cones circumscribed about the hypersurfaces  $U$  and  $V$  with vertex  $B$  and containing the lines  $\ell(0)$  and  $\hat{\ell}(0)$  respectively: we take the germs of the above cones at the latter lines. The  $\sigma_g$ -symmetry permutes the cones, by statement (ii) of [Proposition 2.5](#), which holds for an open and dense set of lines through  $B$ . The hyperplanes  $\mathcal{U}$  and  $\mathcal{V}$  are tangent to the cones along the lines  $\ell(0)$  and  $\hat{\ell}(0)$  respectively, by construction. Hence they are also  $\sigma_g$ -symmetric, as are the cones, and so are their tangent spaces  $T_B\mathcal{U}$  and  $T_B\mathcal{V}$ .

The latter tangent spaces intersect on a codimension 2 subspace  $H \subset T_B\mathbb{R}^d$  lying in  $T_BS$ , by symmetry and statement (i):

$$(2-1) \quad H = T_B\mathcal{U} \cap T_BS = T_B\mathcal{V} \cap T_BS.$$

For every deformation  $x(t)$ ,  $p(t)$ ,  $q(t)$  as above one has

$$(2-2) \quad u = x'(0) \in T_BS, \quad v = p'(0) \in T_A\mathcal{U}, \quad w = q'(0) \in T_C\mathcal{V}.$$

This motivates the following definition:

**Definition 2.8.** Let  $S$  be a germ of hypersurface at a point  $B \in \mathbb{R}^d \subset \mathbb{RP}^d$ . Let  $g$  be a positive definite scalar product on the bundle  $T\mathbb{R}^d|_S$ . Let  $\ell$  be a projective line through  $B$  that is transversal to  $T_BS$ , and let  $H \subset T_BS$  be a vector subspace of codimension one (codimension two in  $T_B\mathbb{R}^d$ ). Let  $A \in \ell$ ,  $C \in \hat{\ell} = \sigma_g(\ell)$ ,  $A, C \neq B$ . Let  $\mathcal{U}$  and  $\mathcal{V}$  denote the projective hyperplanes through  $B$  that are tangent to  $H$  and such that  $\ell \subset \mathcal{U}$ ,  $\hat{\ell} \subset \mathcal{V}$ . Let

$$u \in T_BS, \quad u \neq 0, \quad v \in T_A\mathcal{U}, \quad w \in T_C\mathcal{V}.$$

We say that  $(B, \ell, H, u, A, v, C, w)$  is a *Berger tuple* with base point  $B$ , if there exist germs of  $C^1$ -smooth curves of points  $x(t) \in S$ ,  $p(t), q(t) \in \mathbb{RP}^d$ ,  $x(0) = B$ ,  $p(0) = A$ ,  $q(0) = C$ , such that statements (2-2) hold and for every small  $t$  the lines  $x(t)p(t)$  and  $x(t)q(t)$  are  $\sigma_g$ -symmetric.

**Proposition 2.9.** *The property of being a Berger tuple depends only on the 1-jet of the metric  $g$ . Namely, let  $S$  be a germ of hypersurface at a point  $B \in \mathbb{R}^d \subset \mathbb{RP}^d$ . Let  $g_1$  and  $g_2$  be two positive definite scalar products on the bundle  $(T\mathbb{R}^d)|_S$  that have the same 1-jet at  $B$ . Then any Berger tuple for the metric  $g_1$  with base point  $B$  is a Berger tuple for the metric  $g_2$  and vice versa.*

*Proof.* The proposition follows from definition and smoothness of the dependence of the reflection  $\sigma_g$  on the parameters of the metric  $g$ : if two metrics have the same 1-jets at  $B$ , then the corresponding reflections acting in  $T_y\mathbb{R}^d$  differ by a quantity  $o(y - B)$ .  $\square$

**Theorem 2.10** [Berger 1995, Section 2]. *Let  $S$  be a germ of hypersurface at a point  $B \in \mathbb{R}^d \subset \mathbb{RP}^d$ . Consider the standard Euclidean metric on the affine chart  $\mathbb{R}^d$ , and let  $S$  have nondegenerate second fundamental form. Let  $\ell$  be a line through  $B$  transversal to  $T_BS$ . Then there exist only a finite number  $k \leq d - 1$  of codimension one vector subspaces  $H = H_1(\ell), \dots, H_k(\ell) \subset T_BS$  such that for every  $u \in T_BS$ ,  $u \neq 0$  the triple  $(\ell, H, u)$  extends to a Berger tuple  $(B, \ell, H, u, A, v, C, w)$  for the Euclidean metric. The number  $k$  depends only on the second fundamental form of the hypersurface  $S$  at  $B$ . The subspaces  $H_j(\ell)$  are uniquely determined by the line  $\ell$  and the second fundamental form.*

**Proposition 2.11** [Berger 1995, p. 114]. *In the conditions of Theorem 2.10 consider the tautological projection  $\pi_B : \mathbb{R}^d \setminus \{B\} \rightarrow \mathbb{RP}^{d-1}$  to the space of lines through  $B$ . For every line  $\ell$  through  $B$  the corresponding projection  $\pi_B(\ell \setminus \{B\}) \in \mathbb{RP}^{d-1}$  will be denoted by  $[\ell]$ . For every  $\ell$  transversal to  $S$  let  $\Delta_j(\ell) \subset T_B\mathbb{RP}^d = \mathbb{R}^d$  denote the codimension 1 vector subspace spanned by  $H_j(\ell)$  and  $\ell$ . Let  $\tilde{\Delta}_j([\ell]) = \pi_B(\Delta_j(\ell) \setminus \{0\}) \subset \mathbb{RP}^{d-1}$  denote its tautological projection, which is a projective hyperplane through  $[\ell]$ . Set*

$$\mathcal{D}_j([\ell]) := T_{[\ell]}\tilde{\Delta}_j([\ell]) \subset T_{[\ell]}\mathbb{RP}^{d-1}.$$

*The subspaces  $\mathcal{D}_1([\ell]), \dots, \mathcal{D}_k([\ell]) \subset T_{[\ell]}\mathbb{RP}^{d-1}$  form a  $k$ -valued hyperplane distribution  $\mathcal{D}$  on  $\mathbb{RP}^{d-1}$ , whose all integral surfaces are quadrics. Moreover, let  $\tilde{S}$  be a quadric tangent to  $S$  at  $B$  with order 3: having the same second fundamental form at  $B$ . The  $\pi_B$ -preimages of the above quadrics in  $\mathbb{RP}^{d-1}$  (i.e., the preimages of the integral hypersurfaces) are cones with vertex at  $B$  that are tangent to the quadrics confocal to  $\tilde{S}$ .*

*Proof of Lemma 2.4.* Let  $K$  be the cone with vertex at  $y = B$  circumscribed about the hypersurface  $U$ . Let  $A$  be a point of tangency of the cone  $K$  with  $U$ . Set  $\ell = BA$ ,  $\hat{\ell} = \sigma_g(\ell)$ . Let  $C$  denote the point of tangency of the line  $\hat{\ell}$  with  $V$ . (We suppose that the assumptions of Convention 2.6 hold.) Then for every germ of smooth curve  $x(t) \subset S$ ,  $x(0) = B$ , there exist curves  $p(t) \subset U$  and  $q(t) \subset V$ ,  $p(0) = A$ ,  $q(0) = C$ ,

such that the lines  $x(t)p(t)$  and  $x(t)q(t)$  are tangent to  $U$  and  $V$  at  $p(t)$  and  $q(t)$  respectively and  $\sigma_g$ -symmetric.

Let  $\mathcal{U}, \mathcal{V} \subset \mathbb{RP}^d$  be the previously defined projective hyperplanes through  $B$  tangent to  $U$  and  $V$  at  $A$  and  $C$  respectively, and  $H = T_B\mathcal{U} \cap T_B\mathcal{V} \subset T_BS$ ; see (2-1). The tuple  $(B, \ell, H, x'(0), A, p'(0), C, q'(0))$  is a Berger tuple for the metric  $g$ , by definition. Therefore, it is also a Berger tuple for the Euclidean metric as well (Proposition 2.9). This together with Theorem 2.10 implies that  $H = H_j(\ell)$  for some  $j$ .

The cone  $K$  is tangent along the line  $\ell$  to the hyperplane generated by  $\ell$  and  $H = H_j$ , by definition. Therefore, at  $[\ell] = \pi_B(\ell \setminus \{B\})$ , the tautological projection  $\tilde{K} = \pi_B(K \setminus \{B\}) \subset \mathbb{RP}^{d-1}$  is tangent to the corresponding hyperplane  $\mathcal{D}_j([\ell])$  from Proposition 2.11. Finally,  $\tilde{K}$  is an integral hypersurface of the multivalued hyperplane distribution  $\mathcal{D}$  from Proposition 2.11, and hence, lies in a quadric  $\Gamma(U)$ . The preimage  $\pi_B^{-1}(\Gamma(U))$  is a quadratic cone  $K_B$  with vertex  $B$  that contains  $K$  and is  $\sigma_g$ -symmetric, being a cone tangent to a quadric confocal to  $\tilde{S}$ ; see Proposition 2.11. (Recall that for any given quadric  $\tilde{S}$  and  $B \in \tilde{S}$  a cone with vertex  $B$  circumscribed about a quadric confocal to  $\tilde{S}$  is symmetric with respect to the hyperplane tangent to  $\tilde{S}$  at  $B$ .) Similarly, the punctured cone  $\sigma_g(K) \setminus \{B\}$  tangent to  $V$  is projected to a quadric  $\Gamma(V)$ , and  $\sigma_g(K)$  lies in a quadratic cone. The latter quadratic cone coincides with  $K_B$ , by symmetry. This proves Lemma 2.4.  $\square$

**Lemma 2.12** [Berger 1995, Section 3]. *Let  $U, V, S$  be  $C^2$ -smooth germs of hypersurfaces in  $\mathbb{RP}^d$  with nondegenerate second fundamental forms. Let for every  $x \in S$  there exist a quadratic cone  $K_x$  with vertex at  $x$  that is tangent to both  $U$  and  $V$ . Then  $U$  and  $V$  are pieces of one and the same quadric.*

*Proof of Theorem 2.1.* For every  $y \in S$  close enough to  $B$  there exists a quadratic cone  $K_y$  with vertex at  $y$  circumscribed about both  $U$  and  $V$  (Lemma 2.4). Applying this statement to an open and dense subset of points  $y \in S$  satisfying genericity assumptions from Convention 2.6 together with Lemma 2.12 yield that  $U$  and  $V$  are pieces of one and the same quadric. Theorem 2.1 is proved.  $\square$

**2B. Symmetry hyperplanes of circumscribed cones and confocal quadrics.** Here we prove the following lemma and then deduce Theorem 1.10 from it.

**Lemma 2.13** (A generalization of an analogous statement in [Berger 1995, p. 109]). *Let  $\Sigma$  be a simply connected space form of dimension at least three. Let  $U \subset \Sigma$  be a quadric with nondegenerate second fundamental form. For every  $y \in \Sigma \setminus U$  let  $K_y$  denote the geodesic cone circumscribed about the quadric  $U$  with the vertex at  $y$  (i.e., the union of geodesics through  $y$  that are tangent to  $U$ ). We identify the cone  $K_y$  with the cone  $\tilde{K}_y \subset T_y\Sigma$  of vectors tangent to the above geodesics via the exponential mapping  $\exp : T_y\Sigma \rightarrow \Sigma$ . Let  $S \subset \Sigma$  be a germ of hypersurface at a*

point  $B \notin U$  with nondegenerate second fundamental form such that for every  $y \in S$  the cone  $\tilde{K}_y$  is symmetric with respect to the hyperplane  $T_y S$ . Then  $S$  is a quadric confocal to  $U$ .

In the proof of [Lemma 2.13](#) we use the following lemma. To state it, let us recall that the orthogonal polarity in  $\mathbb{R}^{d+1}$  is the correspondence sending each vector subspace to its orthogonal complement with respect to the standard Euclidean scalar product. The orthogonal polarity in codimension one, which sends codimension one vector subspaces to their orthogonal lines, induces a projective duality  $\mathbb{RP}^{d*} \rightarrow \mathbb{RP}^d$  sending hyperplanes to points. It sends each hypersurface  $S \subset \mathbb{RP}^d$  to its dual  $S^*$ : the family of points dual to the hyperplanes tangent to  $S$ .

**Definition 2.14.** Consider a scalar product  $\langle Gx, x \rangle$  on  $\mathbb{R}^{d+1}$  defining a space form. Orthogonality with respect to the latter scalar product will be called *G-orthogonality*. Let  $V \subset \mathbb{R}^{d+1}$  be a subspace that is *not isotropic*: this means that the restriction to  $V$  of the scalar product  $\langle Gx, x \rangle$  is a nondegenerate quadratic form (or equivalently, that  $V$  is not tangent to the light cone  $\{\langle Gx, x \rangle = 0\}$ ). The *pseudosymmetry* with respect to  $V$  is the linear involution  $I_V : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$  that preserves the above scalar product on  $\mathbb{R}^{d+1}$  and whose fixed point set coincides with  $V$ : it acts trivially on  $V$  and as a central symmetry in the  $G$ -orthogonal subspace.

**Lemma 2.15.** Let  $V \subset \mathbb{R}^{d+1}$  be a nonisotropic vector subspace. Let  $k < d + 1$ . Consider the action  $I_{V,k} : G(k, d + 1) \rightarrow G(k, d + 1)$  of the pseudosymmetry with respect to  $V$  on the Grassmannian of  $k$ -subspaces. The orthogonal polarity  $L \mapsto L^\perp$  conjugates the actions  $I_{V,k}$  and  $I_{V^\perp, d+1-k}$ .<sup>1</sup>

*Proof.* This lemma seems to be well-known to specialists. In three dimensions it follows from [[Bolotin 1992](#), formula (15), p. 23; [Kozlov and Treshchëv 1991](#), formula (3.12), p. 140]. Let us present its proof for completeness of presentation. As it is shown below, [Lemma 2.15](#) is implied by the two following propositions.

**Proposition 2.16.** Let  $G$  be a real symmetric  $(d + 1) \times (d + 1)$ -matrix such that  $G^3 = G$ . Let two nonisotropic subspaces  $V, W \subset \mathbb{R}^{d+1}$  of complementary dimensions be  $G$ -orthogonal. Then their **Euclidean** orthogonal complements  $V^\perp$  and  $W^\perp$  are also nonisotropic and  $G$ -orthogonal.

*Proof.* The condition of the proposition implies that the restrictions of the linear operator  $G$  to  $V$  and  $W$  have zero kernels and

$$(2-3) \quad GV = W^\perp, \quad GW = V^\perp.$$

Thus, to prove  $G$ -orthogonality of the latter subspaces, it suffices to show that

$$\langle G^2 v, Gw \rangle = \langle G^3 v, w \rangle = 0 \quad \text{for every } v \in V \text{ and } w \in W.$$

<sup>1</sup>Everywhere below the orthogonality sign  $\perp$  means orthogonality with respect to the standard Euclidean scalar product.

The first equality follows from symmetry of the matrix  $G$ . The second one follows from  $G$ -orthogonality of the subspaces  $V$  and  $W$  and the equality  $G^3 = G$ . The subspaces (2-3) are nonisotropic, since the restrictions to them of the scalar product  $\langle Gx, x \rangle$  are isomorphic to its restrictions to  $V$  and  $W$  via the operator  $G$ : for every  $v_1, v_2 \in V$  one has  $\langle G(Gv_1), Gv_2 \rangle = \langle Gv_1, v_2 \rangle$ , since  $G^3 = G$ . **Proposition 2.16** is proved.  $\square$

**Proposition 2.17.** *Let  $\langle Gx, x \rangle$  be a scalar product on  $\mathbb{R}^{d+1}$  defining a space form. Let  $k \in \{1, \dots, d\}$ ,  $V \subset \mathbb{R}^{d+1}$  be a nonisotropic subspace, and let  $W \subset \mathbb{R}^{d+1}$  be its  $G$ -orthogonal complement.<sup>2</sup> Let  $N_k(V) \subset G(k, d+1)$  denote the subset of those vector  $k$ -subspaces in  $\mathbb{R}^{d+1}$  that are direct sums of some subspaces  $\ell_1 \subset V$  and  $\ell_2 \subset W$ . The pseudosymmetry  $I_V$  induces a nontrivial projective involution  $\mathbb{RP}^d \rightarrow \mathbb{RP}^d$  and acts trivially on  $N_k(V)$ . Vice versa, every nontrivial projective involution acting trivially on  $N_k(V)$  is the projectivization of the pseudosymmetry  $I_V$ .*

*Proof.* The first statement of the proposition is obvious. Let us prove the second one. Let  $F : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$  be a linear transformation whose projectivization is a nontrivial involution acting trivially on  $N_k(V)$ . Without loss of generality we assume that  $F^2 = \pm \text{Id}$ . For every vector subspace  $L \subset V$  of dimension between 1 and  $k$  the transformation  $F$  preserves the subset in  $N_k(V)$  consisting of the  $k$ -subspaces containing  $L$ . Their intersection being equal to  $L$ ,  $F$  preserves  $L$ . The same statement holds for  $L \subset W$ . Therefore, the restriction of the transformation  $F$  to any of the subspaces  $V$  and  $W$  is a homothety. The coefficients of the homotheties on  $V$  and  $W$  are equal to  $\pm 1$ , since  $F^2 = \text{Id}$  up to sign. The signs of the latter coefficients are opposite, since the projectivization of the transformation  $F$  is nontrivial. Hence,  $F = \pm I_V$ . This proves the proposition.  $\square$

Let us now return to the proof of **Lemma 2.15**. The action of a linear automorphism  $F : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$  on all the vector subspaces of all the dimensions is conjugated via the orthogonal polarity to the similar action of the inverse  $(F^*)^{-1}$  to the conjugate operator  $F^*$  (with respect to the Euclidean scalar product). In the case, when  $F$  is an involution, so is  $F^* = (F^*)^{-1}$ . Let  $W$  be the  $G$ -orthogonal complement of the subspace  $V$ .

**Claim.** *The conjugate operator  $F = I_V^*$  acts trivially on  $N_{d+1-k}(V^\perp)$ .*

*Proof.* The orthogonal polarity sends each  $k$ -subspace  $\Pi = \ell_1 \oplus \ell_2 \in N_k(V)$ ,  $\ell_1 \subset V$ ,  $\ell_2 \subset W$ , to the intersection of two subspaces  $L_j = L_j(\Pi) = \ell_j^\perp$ :

$$(2-4) \quad \begin{aligned} L_1 \supset V^\perp, \quad L_2 \supset W^\perp, \quad \Pi^\perp = L_1 \cap L_2, \\ \dim(\Pi^\perp) = \dim L_1 + \dim L_2 - (d+1) = d+1-k. \end{aligned}$$

<sup>2</sup> The  $G$ -orthogonal complement  $W$  to a nonisotropic subspace  $V$  is always a vector subspace complementary to  $V$ . In the non-Euclidean cases  $W$  is automatically nonisotropic. In the Euclidean case, when the matrix  $G$  is degenerate,  $W$  contains the kernel of the matrix  $G$ : the  $x_0$ -axis.

The transformation  $F$  fixes  $\Pi^\perp$ , by construction and since the pseudosymmetry  $I_V$  fixes  $\Pi$  (Proposition 2.17). The intersection  $\Pi^\perp$  is the direct sum of the subspaces  $L_1 \cap W^\perp$  and  $L_2 \cap V^\perp$ , which follows from the inclusions (2-4) and the fact that  $W^\perp$  and  $V^\perp$  are complementary subspaces, as are  $V$  and  $W$ . Hence,  $\Pi^\perp$  lies in  $N_{d+1-k}(V^\perp)$ . Vice versa, each point in  $N_{d+1-k}(V^\perp)$  can be represented as the intersection  $\Pi^\perp$  of some subspaces  $L_1$  and  $L_2$  containing  $V^\perp$  and  $W^\perp$  respectively. Therefore,  $F$  acts trivially on all of  $N_{d+1-k}(V^\perp)$ . The claim is proved.  $\square$

The operator  $F = I_V^*$  is a projectively nontrivial involution, as is  $I_V$ . It coincides with  $I_{V^\perp}$  up to sign, by the claim and Proposition 2.17. This together with the discussion preceding the claim implies the statement of Lemma 2.15.  $\square$

*Proof of Lemma 2.13.* Consider the tautological projection  $\pi : \mathbb{R}^{d+1} \setminus \{0\} \rightarrow \mathbb{RP}^d$ , the images  $\pi(S), \pi(U) \subset \mathbb{RP}^d$  and the hypersurfaces in  $\mathbb{RP}^d$  projective-dual to them with respect to the orthogonal polarity. For simplicity the latter projective-dual hypersurfaces will be denoted by  $S^*$  and  $U^*$  respectively. Let  $\tilde{S}, \tilde{U}, \tilde{S}^*, \tilde{U}^* \subset \mathbb{R}^{d+1}$  denote the complete  $\pi$ -preimages in  $\mathbb{R}^{d+1}$  of the hypersurfaces  $\pi(S), \pi(U), S^*$  and  $U^*$  respectively: the cones in  $\mathbb{R}^{d+1} \setminus \{0\}$  defined by the latter hypersurfaces. Recall that  $\pi(U)$  and  $U^*$  are dual quadrics; thus one can write

$$U^* = \{\langle Qx, x \rangle = 0\}, \quad Q \text{ is a real symmetric } (d+1) \times (d+1)\text{-matrix.}$$

For every  $y \in S$  let  $\mathcal{T}_y S \subset \mathbb{RP}^d$  denote the projective hyperplane tangent to  $\pi(S)$  at  $\pi(y)$ . Define the following vector subspaces in  $\mathbb{R}^{d+1}$ :

$$\Pi_y := \pi^{-1}(\mathcal{T}_y S) \cup \{0\} \subset \mathbb{R}^{d+1}, \quad L_y := \Pi_y^\perp,$$

$$V_y := \text{the one-dimensional subspace } \pi^{-1}(\pi(y)) \cup \{0\} \subset \Pi_y, \quad W_y := V_y^\perp.$$

The subspaces  $L_y$  and  $W_y$  are nonisotropic. Indeed, in the case when  $\Sigma$  is non-Euclidean, this follows from obvious nonisotropy of their orthogonal subspaces  $\Pi_y$  and  $V_y$  and the fact that in the non-Euclidean case the orthogonal complement to a nonisotropic subspace is also nonisotropic. The latter statement follows from footnote 2 and Proposition 2.16. In the case when  $\Sigma$  is Euclidean, if, to the contrary, either  $L_y$ , or  $W_y$  contained the  $x_0$ -axis, this would imply that either  $\Pi_y$  or  $V_y$  lies in the coordinate  $(x_1, \dots, x_d)$ -subspace, and hence, is disjoint from  $\Sigma$ . This is obviously impossible.

**Claim 1.** *The quadric  $U^*$  is regular, i.e., the matrix  $Q$  is nondegenerate. The hyperplane section  $\tilde{U}^* \cap W_y$  is invariant under the pseudosymmetry with respect to the one-dimensional vector subspace  $L_y \subset W_y$ .*

*Proof.* The first statement (nondegeneracy) follows from nondegeneracy of the second fundamental form of the quadric  $U$ . The inclusion  $L_y \subset W_y$  follows from definition. Recall that the cone  $\tilde{K}_y$  is symmetric with respect to the hyperplane  $\mathcal{T}_y S$ , i.e., the preimage  $\pi^{-1}(K_y)$  is pseudosymmetric with respect to  $\Pi_y$ , by assumption.

The latter statement is equivalent to the second statement of the claim, by duality and [Lemma 2.15](#).  $\square$

The restriction to the  $d$ -dimensional vector subspace  $W_y$  of the scalar product  $\langle Gx, x \rangle$  is nondegenerate (nonisotropy), and there exist  $d$  values

$$\lambda = \lambda_1(y), \dots, \lambda_d(y)$$

(taken with multiplicity, some of them may coincide) such that the restriction to  $W_y$  of the scalar product  $\langle (Q - \lambda G)x, x \rangle$  is degenerate. Thus, the  $d$ -dimensional vector subspace  $W_y$  is the  $G$ -orthogonal direct sum of kernels of the scalar products  $\langle (Q - \lambda_j(y)G)x, x \rangle|_{W_y}$ .

**Claim 2.** *For every  $y \in S$  the pseudosymmetry line  $L_y$  lies in the kernel of some of the scalar products  $\langle (Q - \lambda_j(y)G)x, x \rangle|_{W_y}$ .*

*Proof.* The scalar product  $\langle Qx, x \rangle|_{W_y}$  is invariant under the pseudosymmetry with respect to the line  $L_y$ . Indeed, the latter pseudosymmetry is an involution preserving the zero locus (light cone)  $\tilde{U}^* \cap W_y = \{\langle Qx, x \rangle = 0\} \cap W_y$  (Claim 1), and hence, it preserves the above scalar product up to sign. Let us show that the sign is also preserved. For an open and dense subset of points  $y \in S$  one has  $\langle Qx, x \rangle \neq 0$  on  $L_y \setminus \{0\}$ : equivalently (via duality), the tangent hyperplane  $T_y S$  is not tangent to  $U$ . Indeed, the latter statement holds for an open and dense subset of points  $y \in S$ , since  $S \cap U = \emptyset$  and a (germ of) hypersurface is uniquely defined by the family of its tangent hyperplanes (well-definedness of the dual hypersurface). Thus, for the above  $y$  the pseudosymmetry fixes the nonzero quadratic form  $\langle Qx, x \rangle|_{L_y}$ , since the points of the line  $L_y$  are fixed. This together with the above discussion implies that the above-mentioned sign, and hence the scalar product  $\langle Qx, x \rangle|_{W_y}$  are preserved for all  $y \in S$ .

For every  $\lambda_j(y)$  the kernel of the form  $\langle (Q - \lambda_j(y)G)x, x \rangle|_{W_y}$  is invariant under the above pseudosymmetry, by invariance of the scalar products  $\langle Qx, x \rangle$  and  $\langle Gx, x \rangle$ . This is possible only in the case, when the pseudosymmetry line  $L_y$  lies in some of the kernels, which form an orthogonal direct sum decomposition of the subspace  $W_y$ . This proves Claim 2.  $\square$

**Remark 2.18.** The subspace  $W_y$  and hence, the corresponding kernels from Claim 2 depend only on  $y$  and are well-defined for all  $y \in \Sigma$ .

Due to Claim 2, the following two cases are possible.

**Case 1:** For an open and dense subset  $S_0$  of points  $y \in S$  the line  $L_y$  coincides with a one-dimensional kernel corresponding to a simple eigenvalue  $\lambda_j(y)$ . Let us show that in this case  $S$  lies in a quadric confocal to  $U$ . Indeed then there exist a neighborhood  $Y = Y(B) \subset \Sigma$  of the base point  $B$  of the hypersurface  $S$  and an open and dense subset  $Y_0 \subset Y$  containing  $S_0$  such that the correspondence  $y \mapsto L_y$  extends to a family of lines depending analytically on  $y \in Y_0$ : these lines are some of the kernels



mentioned in the above remark. This implies that the corresponding hyperplanes  $\Pi_y := L_y^\perp$  also depend analytically on  $y$  and thus, induce a field of hyperplanes  $T = T(y) = \Pi_y \cap T_y \Sigma$  on  $Y_0$ . The hypersurface  $S_0$  is its integral hypersurface.

**Subcase 1.1:**  $U$  is a generic quadric. Then for a generic point  $y \in \Sigma$  (here “generic” means “outside an algebraic subset”)

- there are exactly  $d$  quadrics through  $y$  confocal to  $U$ , and any two of them are orthogonal at  $y$ ;
- the corresponding eigenvalues  $\lambda_j(y)$  are simple and the corresponding  $d$  kernels in  $W_y$  are one-dimensional.

Recall that the tangent hyperplanes at  $y$  of the above confocal quadrics are symmetry hyperplanes for the cone  $K_y$ , since  $U$  is a caustic for its confocal quadrics. Therefore, the orthogonal polarity  $\Pi_y \mapsto L_y$  induces a one-to-one correspondence between the above tangent hyperplanes and kernels. This implies that for a generic  $y \in Y_0$  the integral hypersurface of the hyperplane field  $T$  through  $y$  is a confocal quadric to  $U$ . Passing to limit, as  $y$  tends to a point of the integral hypersurface  $S$ , we get that  $S$  is a confocal quadric as well.

**Subcase 1.2:**  $U$  is a general regular quadric. Then it is a limit of generic quadrics  $U_n$  in the above sense. For each  $U_n$  the integral hypersurfaces of the corresponding above hyperplane field  $T_n$  are quadrics confocal to  $U_n$ . Passing to limit, as  $n \rightarrow \infty$ , we get the same statement for the hyperplane field  $T$  associated to  $U$ . Hence,  $S$  is a quadric confocal to  $U$ .

**Case 2:** There exists an open subset of points  $y \in S$  for which  $L_y$  lies in at least two-dimensional kernel of the form  $\langle (Q - \lambda_j(y))x, x \rangle|_{W_y}$ , corresponding to a multiple eigenvalue  $\lambda_j(y)$ . In this case the latter kernel contains at least two linearly independent vectors  $w_1, w_2 \in W_y$ , and by definition, both of them are orthogonal to the hyperplane  $W_y$  with respect to the scalar product  $\langle (Q - \lambda G)x, x \rangle$ ,  $\lambda = \lambda_j(y)$ . Hence, their appropriate nonzero linear combination  $w = a_1 w_1 + a_2 w_2$  is orthogonal to the whole ambient space  $\mathbb{R}^{d+1}$  with respect to the same scalar product. Therefore,  $w$  lies in the kernel of the same scalar product taken on all of  $\mathbb{R}^{d+1}$ , and thus,  $\lambda$  is such that the matrix  $Q - \lambda G$  is degenerate: then we'll call such a  $\lambda$  a *global eigenvalue*. The number of global eigenvalues  $\lambda$  is at most  $d+1$ , and all of them are independent on  $y$ .

Finally, there exist a global eigenvalue  $\lambda$  and an open subset  $S_0 \subset S$  such that for every  $y \in S_0$  one has  $\langle (Q - \lambda)x, x \rangle \equiv 0$  on  $L_y$ , since  $L_y$  lies in the kernel of the restriction to  $W_y$  of the scalar product  $\langle (Q - \lambda)x, x \rangle$ .

Thus, for  $y \in S_0$  the projections  $p(y) = \pi(L_y \setminus \{0\}) \in \mathbb{RP}^d$  lie in a degenerate quadric  $\Gamma \subset \mathbb{RP}^d$  defined by the equation  $\langle (Q - \lambda)x, x \rangle = 0$ . The points  $p(y)$  form the dual hypersurface  $S_0^*$ , by definition. Hence,  $S_0^*$  lies in a degenerate quadric  $\Gamma$ . This contradicts nondegeneracy of the second fundamental form of the hypersurface  $S$ . Hence, the case under consideration is impossible. [Lemma 2.13](#) is proved.  $\square$



*Proof of Theorem 1.10.* The hypersurfaces  $U$  and  $V$  lie in the same quadric in  $\Sigma$ , which will be now denoted by  $U$  (Theorem 2.1). The quadric  $U$  is a caustic for the hypersurface  $S$ : for every  $y \in S$  the cone of geodesics through  $y$  that are tangent to  $U$  is symmetric with respect to the hyperplane tangent to  $T_y S$ . Therefore,  $S$  is a quadric confocal to  $U$ , by Lemma 2.13. This proves Theorem 1.10.  $\square$

### 3. Commuting billiards and caustics: Proof of Theorem 1.7

**Proposition 3.1.** *Let  $\Sigma$  be a space form of constant curvature of dimension  $d \geq 2$ . Let two nested strictly convex  $C^2$ -smooth closed hypersurfaces  $a, b \subset \Sigma$ ,  $a \Subset \Omega_b$  (see the notations at the beginning of the paper) be such that the corresponding billiard transformations  $\sigma_a$  and  $\sigma_b$  commute. Then  $a$  is a caustic for the hypersurface  $b$ .*

*Proof.* Let  $\Pi_a$  denote the open subset of geodesics in  $\Sigma$  that are disjoint from the hypersurface  $a$ . Its boundary  $\partial\Pi_a$  consists of those geodesics that are tangent to  $a$ . A geodesic  $L$  is fixed by  $\sigma_a$ , if and only if  $L \in \bar{\Pi}_a$ , i.e.,  $L$  is either disjoint from  $a$ , or tangent to  $a$ . In this case  $\sigma_b\sigma_a(L) = \sigma_b(L) = \sigma_a\sigma_b(L)$ , and thus,  $\sigma_b(L)$  is a fixed point of the transformation  $\sigma_a$ . This implies that  $\sigma_b(\bar{\Pi}_a) \subset \bar{\Pi}_a$ . The subset  $\bar{\Pi}_a$  is invariant under two transformations acting on oriented geodesics: the reflection  $\sigma_b$  and the transformation  $J$  of the orientation change. The transformations  $J$  and  $J \circ \sigma_b$  are involutions. Hence, they are homeomorphisms of the whole space of oriented geodesics in  $\Sigma$ . Their restrictions to the common invariant subset  $\bar{\Pi}_a$  should be also a homeomorphism: an involution acting on a set is obviously always bijective. Therefore, each of them sends the boundary  $\partial\Pi_a$  onto itself homeomorphically, and the same is true for their composition  $\sigma_b = J \circ (J \circ \sigma_b)$ :

$$\sigma_b(\partial\Pi_a) = \partial\Pi_a.$$

The latter equality means exactly that  $a$  is a caustic for the hypersurface  $b$ . The proposition is proved.  $\square$

*Proof of Theorems 1.7 and 1.2.* Let  $a, b \subset \Sigma$  be two nested strictly convex  $C^2$ -smooth closed hypersurfaces in a space form  $\Sigma$  with commuting billiard transformations,  $a \Subset \Omega_b$ ,  $\dim \Sigma \geq 3$ . Then  $a$  is a caustic for the hypersurface  $b$ , by Proposition 3.1. This means that, for every point  $B \in b$  and  $A \in a$  such that the line  $AB$  is tangent to  $a$  at  $A$ , the image  $\sigma_b(AB)$  of the line  $AB$  (oriented from  $A$  to  $B$ ) is a line through  $B$  tangent to  $a$ . Recall that  $a$  and  $b$  are strictly convex, which implies that their second fundamental forms are sign-definite and thus, nondegenerate. Therefore, for every  $A$  and  $B$  as above the germs at  $A$  and  $B$  of the hypersurfaces  $U = a$  and  $S = b$  respectively satisfy the conditions of Theorem 1.10, with  $V$  being the germ of the hypersurface  $a$  at its point  $D$  of tangency with the line  $\sigma_b(AB)$ . Hence, for every  $A$  and  $B$  as above the germ  $(S, B)$  lies in a quadric, and the germs  $(U, A)$ ,  $(V, D)$  lie in one and the same quadric confocal to  $S$ . This implies that  $b$  is a quadric, and  $a$  is a quadric confocal to  $b$ . Theorems 1.7 and 1.2 are proved.  $\square$

#### 4. A tangential local version of Theorem 1.7

**Theorem 4.1.** *Let  $d \geq 3$ . Let  $(U, A)$ ,  $(S, B)$ ,  $(V, D)$  be germs of  $C^2$ -smooth hypersurfaces in a  $d$ -dimensional space form  $\Sigma$  at points  $A, B$  and  $D$ . Let  $B \neq A, D$ , and let  $U$  and  $S$  have nondegenerate second fundamental forms. For every  $Z = U, S, V$  consider the action of the reflection  $\sigma_Z$  on the oriented geodesics that intersect  $Z$ , defined as at the beginning of the paper: we reflect the geodesic at its last intersection point with the hypersurface  $Z$ . Let  $L_0$  be a geodesic through  $B$  transversal to  $S$  and quadratically tangent to  $U$  at  $A$  (we orient it from  $A$  to  $B$ ), and let its image  $\sigma_S(L_0)$  be quadratically tangent to  $V$  at  $D$ . Let  $W$  be a small neighborhood of the geodesic  $L_0$  in the space of oriented geodesics; in particular, each point in  $W$  represents a geodesic intersecting  $S$  transversally. Let  $\Pi_W \subset W$  denote the subset of those geodesics that intersect  $U$ . For every  $L \in \Pi_W$  let the image  $\sigma_S(L)$  intersect  $V$ ; more precisely, we suppose that the compositions  $\sigma_S \circ \sigma_U$  and  $\sigma_V \circ \sigma_S$  are well-defined on  $\Pi_W$ . Let the latter compositions be identically equal on  $\Pi_W$ . Then  $S$  lies in a quadric  $b$ , and  $U, V$  lie in one and the same quadric confocal to  $b$ .*

*Proof.* Every geodesic  $L$  tangent to  $U$  and close enough to  $L_0$  lies in  $\Pi_W$ . Its image  $\sigma_U(L)$  coincides with  $L$  (by definition), and hence,  $\sigma_S \circ \sigma_U(L) = \sigma_S(L) = \sigma_V \circ \sigma_S(L)$ . Thus, the geodesic  $\sigma_S(L)$ , which should intersect  $V$  by assumption, is fixed by  $\sigma_V$ . Hence it is tangent to  $V$  (at the last point of its intersection with  $V$ ). Finally, the germs of hypersurfaces  $U, S$  and  $V$  satisfy the conditions of Theorem 1.10. Therefore,  $S$  lies in a quadric  $b$ , and  $U, V$  lie in one and the same quadric confocal to  $b$ , by Theorem 1.10. This proves Theorem 4.1.  $\square$

#### 5. Open problems

The billiards in space forms are particular cases of the projective billiards introduced in [Tabachnikov 1997]. The main results of the present paper (Theorem 1.10 extending Berger's result on caustics [1995], Theorem 1.7 on commuting billiards) are proved for billiards in space forms. It would be interesting to extend them to projective billiards.

**Problem 1** (appeared as a result of our discussion with Sergei Tabachnikov). Let  $S \subset \mathbb{R}^d$ ,  $d \geq 3$  be a germ of hypersurface at a point  $B$  equipped with a field  $\Lambda$  of one-dimensional subspaces  $\Lambda_y \subset T_y \mathbb{R}^d$ ,  $y \in S$ , transversal to  $S$ . Consider the family of linear involutions  $\sigma_y : T_y \mathbb{R}^d \rightarrow T_y \mathbb{R}^d$ ,  $y \in S$ , that fix each point of the hyperplane  $T_y S$  and have  $\Lambda_y$  as an eigenline with eigenvalue  $-1$ . Let there exist two germs of hypersurfaces  $U$  and  $V$  at points  $A, C \neq B$  respectively such that the lines  $AC, BC$  are tangent to  $U$  and  $V$  at points  $A$  and  $C$  respectively and for every  $y \in S$  each line through  $y$  that is tangent to  $U$  is reflected by  $\sigma_y$  to a line tangent to  $V$ . (The defined action of the reflections  $\sigma_y$  on oriented lines transversal

to  $S$  is called the *projective billiard transformation*, and the pair  $(S, \Lambda)$  is called a *projective billiard*; see [Tabachnikov 1997].) Is it true that then  $U$  and  $V$  lie in one and the same quadric?

**Problem 2** (Tabachnikov). Classify commuting nested pairs of projective billiards in  $\mathbb{R}^d$ ,  $d \geq 2$ .

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ALEXEY GLUTSYUK

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Los Angeles, CA 90095-1555  
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Los Angeles, CA 90095-1555  
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Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[popa@math.ucla.edu](mailto:popa@math.ucla.edu)

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Department of Mathematics  
Princeton University  
Princeton NJ 08544-1000  
[yang@math.princeton.edu](mailto:yang@math.princeton.edu)

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University of California  
Riverside, CA 92521-0135  
[chari@math.ucr.edu](mailto:chari@math.ucr.edu)

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Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[liu@math.ucla.edu](mailto:liu@math.ucla.edu)

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