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We construct a tensor product on Freyd's universal abelian category $\mathbf{Ab}(C)$ attached to an additive tensor category or a ⊗-quiver and establish a universal property. This is used to give an alternative construction for the tensor product on Nori motives.

Introduction

In the late 1990s, Nori made a spectacular proposal for an unconditional definition of an abelian category of motives and a motivic Galois group over a field of characteristic zero. It has two main inputs:

(1) The existence of a universal abelian category attached to a fixed representation of a quiver.

(2) His basic lemma (known earlier to Beilinson and Vilonen) which shows the existence of an algebraically defined "skeletal filtration" on an affine algebraic variety.

The first part is enough to give the definition of the category. The second is needed

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in order to establish the tensor structure. In a third step, we pass from effective motives to all motives and check rigidity.

The motivic Galois group is its Tannaka dual. However, all steps are intrinsically linked together. The proof of the existence of the abelian category is done by constructing a suitable coalgebra. The tensor product is defined by turning this coalgebra into a bialgebra. After localisation, it is shown to be even a Hopf algebra — the Hopf algebra of the motivic Galois group. Indeed, the proof given in full detail in [Huber and Müller-Stach 2017] gives as a byproduct a full proof of Tannaka duality.

Meanwhile there have been a couple of alternative approaches to the first step of the above program; see [Barbieri-Viale et al. 2018; Barbieri-Viale 2017; Barbieri-Viale and Prest 2018; Ivorra 2017]. They are more general and arguably simpler. However, these references did not address tensor products.

In this paper we explain how the approach of [Barbieri-Viale and Prest 2018] can be used to handle tensor categories and tensor functors. We show that if $(C, \otimes)$ is an additive tensor category then Freyd's universal abelian category $\mathrm{Ab}(C)$ carries an induced right-exact tensor structure which is also universal in a certain sense (the exact statement is Proposition 1.10).

Given a module M on C (i.e., an additive functor into an abelian tensor category), this induces, under additional technical assumptions, a tensor structure on the universal abelian category $\mathcal{A}(M)$ for the module M. This is again universal; see Proposition 1.13. The results can also be reformulated in terms of representations of quivers (see Section 2, in particular Theorem 2.10) bringing it even closer to the shape of Nori's original results. Our results are a lot more general in allowing modules with values in quite general abelian categories. We get back Nori's case as the case of the representation of a quiver in the category of modules over a Dedekind domain or even a noetherian ring of homological dimension at most 2.

We also show how to apply our results to Nori motives. This can be done by using his original quiver of good pairs. Alternatively, we start with the more canonical tensor category of geometric motives in the sense of Voevodsky. However, the functor H^0_B used in the definition of Nori motives is *not* a tensor functor, in contrast with the graded functor H^*_B. It remains to check that the Künneth components are motivic. This step of the construction relies on Nori's basic lemma. We give an abstract criterion in Section 3. It is applied to Nori motives in Section 4: we obtain a Tannakian category and define the motivic Galois group as its Tannaka dual. We find this more natural than defining the category as representations of the motivic Galois group.

We feel that the nature of the argument and the role of the basic lemma become a lot clearer in this new description. However, its main advantage is the great generality in choosing the target category $\mathcal{A}$. For example, we can easily define Nori motives over a base by using the Betti-realisation of triangulated motives into constructible sheaves. In the follow-up [Barbieri-Viale and Prest 2020], we take a more axiomatic approach, using many-sorted languages such that (co)homology theories are models of certain regular theories in that language. All this applies to several different geometric situations.

Notation. By a tensor category $(C, \otimes)$ we mean a category C provided with a functor $\otimes : C \times C \to C$ satisfying an associativity constraint and with $\mathbf{1}$ a unit object; in addition, also a commutativity constraint can be required, e.g., see [Deligne and Milne 1982, §1]. This is often called a nonstrict tensor category. By an additive (resp. abelian) tensor category we mean a tensor category $(C, \otimes)$ such that C is additive (resp. abelian) and $\otimes$ is a biadditive functor; see [Deligne and Milne 1982, Definition 1.15]. Tensor functors are not assumed strict. Tensor functors between additive tensor categories are assumed to be additive. We denote by $\mathbb{Q}$-vsp. the tensor category of $\mathbb{Q}$-vector spaces.

If $\mathcal{A}$ is an abelian category, we denote by $\operatorname{gr} \mathcal{A}$ the associated category of $\mathbb{Z}$-graded objects. If, in addition, $(\mathcal{A}, \otimes)$ carries a tensor structure, we equip $(\operatorname{gr} \mathcal{A}, \otimes)$ with the induced tensor structure. If the tensor product is commutative, we choose the commutativity constraint on $\operatorname{gr} \mathcal{A}$ such that the product becomes graded anticommutative.

For an additive category C we shall consider the additive functors from C to the category Ab of abelian groups as (left) C-modules. We shall denote by C-mod the category of finitely presented C-modules; see, e.g., [Prest 2011, Chapters 2 and 3].

1. Universal abelian tensor categories

Let C be an additive category. We denote by $\operatorname{Ab}(C)$ the universal abelian category on C; see [Freyd 1966; Prest 2011, Chapter 4]. We may refer to it as Freyd's abelian category. It comes with a canonical fully faithful functor $C \hookrightarrow \operatorname{Ab}(C)$. Recall that this functor is universal with respect to additive functors into abelian categories, e.g., see [Barbieri-Viale and Prest 2018, Theorem 1.1].

Thus, for $M : C \to \mathcal{A}$ an additive functor into some abelian category $\mathcal{A}$, we obtain an induced exact functor $\widetilde{M} : \operatorname{Ab}(C) \to \mathcal{A}$, unique to natural equivalence.

We denote by $\mathcal{A}(M)$ the quotient of $\operatorname{Ab}(C)$ by the Serre subcategory which is the kernel of $\widetilde{M}$; we also denote by $\widetilde{M} : \mathcal{A}(M) \to \mathcal{A}$ the induced faithful exact functor:

$$\begin{array}{ccc} C & \longrightarrow & \operatorname{Ab}(C) \\ & & \downarrow \\ {\scriptstyle M}\downarrow & \swarrow{\scriptstyle \widetilde{M}} & \mathcal{A}(M) \\ & \swarrow{\scriptstyle \widetilde{M}} & \\ \mathcal{A} & & \end{array}$$

We shall refer to $\mathcal{A}(M)$ as the *universal abelian category defined by M*, according with [Barbieri-Viale and Prest 2018, §1.1]. In fact, this abelian category $\mathcal{A}(M)$ is universal for (i.e., initial among) all abelian categories together with a faithful exact functor into $\mathcal{A}$ which extends M. Note that, in the case where $\mathcal{A}$ is the category of finitely generated modules over a commutative noetherian ring R, this recovers Nori's abelian category (see [Huber and Müller-Stach 2017, Chapter 7] and compare with [Barbieri-Viale and Prest 2018, §1.2]). For later use, we introduce:

Definition 1.1. Let C be additive and let $C \to \mathrm{Ab}(C)$ be Freyd's abelian category. We denote by $\mathrm{Ab}(C)^\flat$ the smallest full subcategory containing the objects in the image of C and closed under kernels.

Remark 1.2. The universal abelian category $\mathrm{Ab}(C)$ can be constructed explicitly as the category (C-mod)-mod (see, e.g., [Prest 2011, 4.3]). In this construction, $\mathrm{Ab}(C)^\flat$ is, because C-mod has cokernels and every object of C-mod is the cokernel of a morphism between representables, precisely the image of C-mod under the (contravariant) Yoneda embedding into $\mathrm{Ab}(C)$. All objects of $\mathrm{Ab}(C)^\flat$ are representable functors, and hence, by the Yoneda lemma, projective. The above definition is independent of this description.

Let $(C, \otimes)$ be an additive tensor category; see [Deligne and Milne 1982, §1]. Consider an (additive) tensor functor $M : (C, \otimes) \to (\mathcal{A}, \otimes)$ where $(\mathcal{A}, \otimes)$ is an abelian tensor category. We want to equip the above universal abelian category $\mathcal{A}(M)$ with a natural tensor structure $(\mathcal{A}(M), \otimes)$ such that $\widetilde{M} : (\mathcal{A}(M), \otimes) \to (\mathcal{A}, \otimes)$ is turned into a tensor functor. We proceed in several steps.

Multilinear functors. By definition, $\mathrm{Ab}(C)$ has a universal property with respect to additive functors. In fact, this extends to biadditive and even multiadditive functors, even though we lose some properties.

We first recall a well-known property of injective resolutions.

Lemma 1.3. *Let $\mathcal{A}$ be an abelian category, $f : X \to Y$ a morphism in $\mathcal{A}$. Assume*

$$0 \to X \to I_0 \to I_1 \quad \text{and} \quad 0 \to Y \to J_0 \to J_1$$

are exact and that all I_k, J_k are injective. Then there are lifts $f_0 : I_0 \to J_0$, $f_1 : I_1 \to J_1$ making the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & I_0 & \xrightarrow{d} & I_1 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ 0 & \longrightarrow & Y & \longrightarrow & J_0 & \longrightarrow & J_1 \end{array}$$

commute. Moreover, if (g_0, g_1) is a second lift, then there is $h : I_1 \to J_0$ such that

$$f_0 - g_0 = h \circ d.$$

Proof. Our complexes are the starting bits of injective resolutions and h is the beginning of a chain homotopy. The assertion is usually proved as the first step of the proof of existence of a lift of f to an injective resolution and that the lift is unique up to chain homotopy; see for example [Mac Lane 1963, Theorem 6.1]. □

As usual, we also have the dual statement for left resolutions by projectives.

Proposition 1.4. *Let $C_1, \dots, C_n$ be additive categories and let $\mathcal{A}$ be an abelian category.*

(1) *Let $F : C_1 \times \cdots \times C_n \to \mathcal{A}$ be a multilinear functor, i.e., additive in each argument. Then F extends to a multilinear functor*

$$\widetilde{F} : \mathrm{Ab}(C_1) \times \mathrm{Ab}(C_2) \times \cdots \times \mathrm{Ab}(C_n) \to \mathcal{A}$$

which is right-exact in each argument. Fix j and for $i \neq j$ choose $X_i \in \mathrm{Ab}(C_i)^\flat$ (see Definition 1.1). Then $\widetilde{F}(X_1, \dots, -, \dots, X_n)$ is exact as a functor on $\mathrm{Ab}(C_j)$.

(2) *The functor F is uniquely determined up to unique isomorphism of functors by these properties.*

(3) *Let $\alpha : F_1 \to F_2$ be a transformation of multilinear functors $C_1 \times \cdots \times C_n \to \mathcal{A}$ and $\widetilde{F}_1$ and $\widetilde{F}_2$ their extensions to $\mathrm{Ab}(C_1) \times \cdots \times \mathrm{Ab}(C_n)$. Then there is a transformation of functors $\tilde{\alpha} : \widetilde{F}_1 \to \widetilde{F}_2$ extending α. It is unique.*

Proof. Recall that $\mathrm{Ab}(C_i) = (C_i\text{-mod})\text{-mod}$ and that the universal functor factors

$$C_i \to (C_i\text{-mod})^{\mathrm{op}} \to (C_i\text{-mod})\text{-mod}$$

where both steps are given by the Yoneda embedding. As pointed out in Remark 1.2 the subcategory $\mathrm{Ab}(C_i)^\flat$ agrees with the image of $(C_i\text{-mod})^{\mathrm{op}}$.

All statements are shown in two steps. In the first we extend to a functor

$$F' : (C_1\text{-mod})^{\mathrm{op}} \times (C_2\text{-mod})^{\mathrm{op}} \times \cdots \times (C_n\text{-mod})^{\mathrm{op}} \to \mathcal{A}$$

which will be multilinear and left-exact in each argument. In the second step, which is actually dual to the first, we extend F' to $\widetilde{F}$.

We first show uniqueness. This will make clear why the formula that we use in the construction is correct. Let E be any extension of F to $\mathrm{Ab}(C_i)$ with the exactness property of (1). Let $X_i \in (C_i\text{-mod})^{\mathrm{op}}$. We argue by descending induction on the number of X_i which are in the image of C_i, i.e., of the form $(A_i, -)^{\mathrm{op}}$. If they all are, then $E(X_1, \dots, X_n) = F(A_i, \dots, A_n)$ by assumption. Assume that E is uniquely determined if at least m of the X_i are corepresentable. After reordering we have to consider the tuple $(X_1, \dots, X_n)$ with $X_i = (A_i, -)^{\mathrm{op}}$ for $i < m$. By definition, there is an injective corepresentation

$$0 \to X_m \to (A_m, -)^{\mathrm{op}} \to (B_m, -)^{\mathrm{op}}.$$

By (1), the functor $E(X_1,\dots,X_{m-1},-,X_{m+1},\dots,X_n)$ is exact. Hence we have an exact sequence

$$0\to E(X_1,\dots,X_n)\to E(X_1,\dots,X_{m-1},(A_m,-)^{\mathrm{op}},X_{m+1},\dots,X_n)$$
$$\to E(X_1,\dots,X_{m-1},(B_m,-)^{\mathrm{op}},X_{m+1},\dots,X_n).$$

By induction the two terms on the right are uniquely determined up to unique isomorphism. As a kernel, $E(X_1,\dots,X_n)$ is again uniquely determined up to unique isomorphism. By induction, this shows uniqueness if all arguments are in C_i-mod.

The dual argument for a right-exact E and representations gives uniqueness for arguments in $\mathrm{Ab}(C_i)$.

We turn to the construction of F'. Let $X_i\in(C_i\text{-mod})^{\mathrm{op}}$. By definition, these objects have an injective copresentation

$$0\to X_i\to(A_i,-)^{\mathrm{op}}\to(B_i,-)^{\mathrm{op}}.$$

We choose such a presentation for each object $X_i\in(C_i\text{-mod})^{\mathrm{op}}$. The uniqueness proof suggests $F'(X_1,\dots,X_n)\subset F(A_1,\dots,A_n)$ as an iteration of kernels. The same object is given by the formula

$$F'(X_1,\dots,X_n):=\mathrm{Ker}\Big(F(A_1,\dots,A_n)\to\bigoplus_{m=1}^{n}F(A_1,\dots,A_{m-1},B_m,A_{m+1},\dots,A_n)\Big).$$

In other words, applying F to the n-tuple of complexes $(A_i,-)^{\mathrm{op}}\to(B_i,-)^{\mathrm{op}}$ we obtain an n-fold complex. The above is H^0 of its total complex.

Let $1\le i\le n$. For a morphism $X_i\to Y_i$ in $(C_i\text{-mod})^{\mathrm{op}}$ we choose a lift to the copresentations as in Lemma 1.3. This induces a morphism $F'(X_1,\dots,X_n)\to F'(X_1,\dots,Y_i,\dots,X_n)$. It is independent of the lift because any two such morphisms differ by h as in Lemma 1.3. This makes F' a functor in each variable.

The dual argument via projective presentations gives a right-exact extension to $\mathrm{Ab}(C_i)$.

A diagram chase shows that the functor $\widetilde{F}$ has the exactness property claimed in (1) because F' is left-exact, $\widetilde{F}$ right-exact and every object Y of $\mathrm{Ab}(C_i)$ has a projective resolution of the form

$$0\leftarrow Y\leftarrow P^0\leftarrow P^1\leftarrow P^2\leftarrow 0$$

with $P^i\in(C_i\text{-mod})^{\mathrm{op}}$.

Now let $\alpha:F_1\to F_2$ be a transformation of functors. Going through the above construction, we get induced $\alpha':F_1'\to F_2'$ and then $\tilde{\alpha}:\widetilde{F}_1\to\widetilde{F}_2$. The uniqueness argument for the functors also gives the uniqueness of the transformation. □

Remark 1.5. Unexpectedly the extension $\widetilde{F}$ *fails* to be exact in each argument. For a counterexample, see Example 1.12 below.

Remark 1.6. In a general enriched-category setting, lifting monoidal structure to functor categories can be found in [Bunge 1969] and [Day 1970].

This applies in particular to additive tensor categories.

Definition 1.7. Let $(C, \otimes)$ be an additive tensor category. We extend the functor $\otimes : C \times C \to C$ defining

$$\otimes : \mathrm{Ab}(C) \times \mathrm{Ab}(C) \to \mathrm{Ab}(C)$$

as the extension of $C \times C \to C \hookrightarrow \mathrm{Ab}(C)$ of Proposition 1.4.

Proposition 1.8. *Let* $(\mathrm{Ab}(C), \otimes)$ *be Freyd's category together with the functor in Definition 1.7. Then*:

(1) $(\mathrm{Ab}(C), \otimes)$ *is an abelian tensor category.*

(2) *The tensor product is right-exact by construction. The objects in* $\mathrm{Ab}(C)^\flat$ *are flat*, *i.e.*, *acyclic with respect to* $\otimes$.

(3) *If the tensor structure on C is commutative then so is the tensor structure on* $\mathrm{Ab}(C)$.

Proof. Right-exactness and acyclicity are special cases of Proposition 1.4. Let $\mathbf{1}$ be the unit object of C. By definition it comes with a transformation of functors $u : \mathbf{1} \otimes - \to \mathrm{id}$ on C. Let $[\mathbf{1}]$ be its image in $\mathrm{Ab}(C)$. In explicit formulas this means $[\mathbf{1}] = ((\mathbf{1}, -), -)$. Then $[\mathbf{1}]$ with the induced transformation is the unit of $\mathrm{Ab}(C)$. The equivalences used to express the associativity constraint on C^3 (see [Deligne and Milne 1982, §1]) induce equivalences on $\mathrm{Ab}(C)^3$. In detail: Let

$$F_1 : C^3 \xrightarrow{\otimes \circ (\mathrm{id}, \otimes)} C \quad \text{and} \quad F_2 : C^3 \xrightarrow{\otimes \circ (\otimes, \mathrm{id})} C.$$

The associativity constraint is a functorial isomorphism $\alpha : F_1 \to F_2$. By abuse of notation we use the same notation for their composition with the inclusion $C \to \mathrm{Ab}(C)$. Note that α is still a functorial isomorphism. The functor $\mathrm{Ab}(C)^3 \to \mathrm{Ab}(C)$ given by $(X, Y, Z) \mapsto X \otimes (Y \otimes Z)$ is right-exact in each argument and exact as a functor in one variable if the other entries are flat. By the uniqueness property of Proposition 1.4 it agrees with $\widetilde{F}_1$. The same argument also applies to F_2. Again by Proposition 1.4, the transformation extends to a transformation $\tilde{\alpha}$. This is our associativity constraint. We need to check that a certain diagram of functors on $\mathrm{Ab}(C)^4$ involving $\otimes$ and $\tilde{\alpha}$ commutes. This holds by the uniqueness part of Proposition 1.4 applied to functors $C^4 \to C$.

We argue similarly for the commutativity constraint if there is one on C. □

Definition 1.9. For an abelian tensor category, with a right-exact tensor product, a $\flat$-*subcategory* is a full additive subcategory of flat objects (i.e., acyclic with respect to the tensor product) which is closed under kernels. If $(\mathcal{A}, \otimes)$ is such an abelian tensor category we shall denote by $\mathcal{A}^\flat \subseteq \mathcal{A}$ some $\flat$-subcategory.

As a consequence of Proposition 1.8 we have that $\mathrm{Ab}(C)^\flat \subset \mathrm{Ab}(C)$ as in Definition 1.1 is a $\flat$-subcategory.

Proposition 1.10 (universal property). *Let C be an additive tensor category. Let $\mathcal{A}$ be an abelian tensor category with a right-exact tensor product. Let $M : (C, \otimes) \to (\mathcal{A}, \otimes)$ be a tensor functor. In addition, assume that M factors via $\mathcal{A}^\flat \subseteq \mathcal{A}$ a $\flat$-subcategory (see Definition 1.9). Then $\widetilde{M} : (\mathrm{Ab}(C), \otimes) \to (\mathcal{A}, \otimes)$ is a tensor functor. The triple $(\mathrm{Ab}(C), \mathrm{Ab}(C)^\flat, \otimes)$ is universal with this property, and in particular unique.*

Proof. Let $M : (C, \otimes) \to (\mathcal{A}, \otimes)$ be a tensor functor. We have to compare

$$\mathrm{Ab}(C) \times \mathrm{Ab}(C) \to \mathrm{Ab}(C) \to \mathcal{A}$$

and

$$\mathrm{Ab}(C) \times \mathrm{Ab}(C) \to \mathcal{A} \times \mathcal{A} \to \mathcal{A}.$$

Both are right-exact in each argument (this is where right-exactness of the tensor product on $\mathcal{A}$ is used) and agree on $C \times C$.

As in the proof of Proposition 1.4, we extend M in two steps: first to $(C\text{-mod})^{\mathrm{op}}$, then to $\mathrm{Ab}(C) = (C\text{-mod})\text{-mod}$. The second step is unproblematic as it only uses the right-exactness. In the first step, we need to check the action on (certain) kernels. Let $X_1, X_2 \in (C\text{-mod})^{\mathrm{op}}$ with resolutions

$$0 \to X_i \to (A_i, -)^{\mathrm{op}} \to (B_i, -)^{\mathrm{op}}.$$

By definition

$$0 \to M'(X_i) \to M(A_i) \to M(B_i)$$

is exact. By assumption $M(A_i), M(B_i)$ and hence also $M'(X_i)$ are in $\mathcal{A}^\flat$. In particular, consider the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M'(X_1) \otimes M'(X_2) & \longrightarrow & M'(X_1) \otimes M(A_2) & \longrightarrow & M'(X_1) \otimes M(B_2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M(A_1) \otimes M'(X_2) & \longrightarrow & M(A_1) \otimes M(A_2) & \longrightarrow & M(A_1) \otimes M(B_2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M(B_1) \otimes M'(X_2) & \longrightarrow & M(B_1) \otimes M(A_2) & \longrightarrow & M(B_1) \otimes M(B_2)
\end{array}$$

All rows and columns are exact because they arise by tensoring an exact sequence with a flat object. This implies

$$\begin{aligned}M'(X_1)\otimes M'(X_2)\\ &= \mathrm{Ker}\big(M(A_1)\otimes M(A_2)\to (M(A_1)\otimes M(B_2))\oplus(M(B_1)\otimes M(A_2))\big)\\ &= M'(X_1\otimes X_2).\end{aligned}$$

The triple $(\mathrm{Ab}(C), \mathrm{Ab}(C)^\flat, \otimes)$ itself satisfies the assumptions of the universal property; hence it is universal and as such unique. □

Remark 1.11. There are a number of interesting cases where the assumptions of Proposition 1.10 and Definition 1.9 are satisfied. However, they are not as general as one could hope for.

(1) If $\otimes$ is exact on $\mathcal{A}$, then $\mathcal{A}^\flat = \mathcal{A}$ clearly satisfies the assumptions.

(2) If $C_1 \to C_2$ is a $\otimes$-functor between additive tensor categories, by composition, we may consider $M : C_1 \to \mathcal{A}^\flat = \mathrm{Ab}(C_2)^\flat \subset \mathcal{A} = \mathrm{Ab}(C_2)$ which satisfies the assumptions; then, by the universal property, we get an exact tensor functor $\widetilde{M} : \mathrm{Ab}(C_1) \to \mathrm{Ab}(C_2)$.

(3) The assumptions are satisfied if $\mathcal{A} = R$-mod for a Dedekind domain R where $\mathcal{A}^\flat$ is the $\flat$-subcategory of projective finitely generated R-modules, i.e., torsion free finitely generated modules, and $M : C \to \mathcal{A}^\flat$ is any tensor functor. In particular this is true for $R = \mathbb{Z}$.

(4) They are not satisfied for $\mathcal{A} = R$-mod for a general noetherian commutative ring R and the subcategory of projective finitely generated R-modules, which is not a $\flat$-subcategory if the global dimension of R is > 2. See Example 1.12.

Example 1.12. Let C be the category with objects $(\mathbb{Z}/4)^n$ for $n \geq 0$ and morphisms given by homomorphisms of abelian groups.

Let $\mathcal{A} = \mathbb{Z}/4$-mod. In this case is possible to compute all objects explicitly. The functor $M \mapsto M^\vee = \mathrm{Hom}(M, \mathbb{Z}/4)$ is an antiequivalence of C with itself. We have C-mod $\cong \mathbb{Z}/4$-mod with $C \to \mathbb{Z}/4$-mod given by $M \mapsto M^\vee$. Hence $\mathrm{Ab}(C)$ is the category of finitely presented presheaves on $\mathbb{Z}/4$-mod. Objects are uniquely determined by the values of these presheaves on the groups $\mathbb{Z}/4$ and $\mathbb{Z}/2$. Direct computation will show:

(1) $\otimes$ is not biexact on $\mathrm{Ab}(C)$.

(2) The tensor functor $\mathrm{Ab}(C) \to \mathcal{A}$ induced by the inclusion functor $C \to \mathcal{A}$ is not a tensor functor.

By Auslander–Reiten theory (see, e.g., [Assem et al. 2006, §IV.6, p. 149]) the simple objects of the category $\mathrm{Ab}(C)$ have the form $(X, -)/\mathrm{rad}(X, -)$ for X an indecomposable $\mathbb{Z}/4$-module. So there are two simple objects, S and T say, and these are such that $S(\mathbb{Z}/4) = \mathbb{Z}/2$, $S(\mathbb{Z}/2) = 0$ and $T(\mathbb{Z}/4) = 0$, $T(\mathbb{Z}/2) = \mathbb{Z}/2$. Noting the exact sequence $0 \to \mathbb{Z}/2 \xrightarrow{j} \mathbb{Z}/4 \xrightarrow{p} \mathbb{Z}/2 \to 0$ and considering the maps $(p, -)$ and $(j, -)$ in $\mathrm{Ab}(C)$, it can be easily checked that $\mathrm{rad}(\mathbb{Z}/4, -) = (\mathbb{Z}/2, -)$ and that $(\mathbb{Z}/2, -)$ has length 2, with socle S. The remaining indecomposable objects of $\mathrm{Ab}(C)$ may then be computed (see, for example, [Prest 2012, 4.3]): there are five of them, all of them subquotients of the two representable functors. They are $(\mathbb{Z}/4, -)$, $(\mathbb{Z}/2, -)$, the two simples S, T and $(\mathbb{Z}/4, -)/S$.

Now consider the exact functor $\widetilde{\mathbb{Z}/4} : \mathrm{Ab}(C) \to \mathbb{Z}/4\text{-mod}$. This is evaluation of an object of $\mathrm{Ab}(C)$, considered as a functor on $\mathbb{Z}/4$-mod, at $\mathbb{Z}/4$, hence is 0 only on T among those five indecomposables. Therefore its kernel is the Serre subcategory which consists of direct sums of copies of T. In order to compute $T \otimes T$, we apply the definition of the tensor product on $\mathrm{Ab}(C)$ using the projective presentation

$$(\mathbb{Z}/4, -) \xrightarrow{(j,-)} (\mathbb{Z}/2, -) \xrightarrow{\pi_T} T \to 0$$

of T and, checking

$$(\mathrm{id}_{\mathbb{Z}/2}, -) \otimes (j, -) = 0,$$

we obtain $T \otimes T = (\mathbb{Z}/2, -)$, which is not in the kernel of $\widetilde{\mathbb{Z}/4}$, so this is not a tensor functor. As part of the computation of $T \otimes T$ one sees that

$$T \otimes (\mathbb{Z}/2, -) = (\mathbb{Z}/2, -).$$

So applying $T \otimes -$ to the monomorphism

$$(\mathbb{Z}/2, -) \xrightarrow{(p,-)} (\mathbb{Z}/4, -)$$

gives $(\mathbb{Z}/2, -) \to T$ which is not monic, showing that $\otimes$ is not exact on $\mathrm{Ab}(C)$.

This implies that we cannot expect a different, exact, tensor product on $\mathrm{Ab}(C)$ extending the tensor product on C — by the universal property the identity would have to be a tensor functor.

Tensor structures on $\mathcal{A}(M)$. Consider $(\mathcal{A}, \otimes)$ an abelian tensor category with a right-exact tensor product.

For the sake of exposition we now drop explicit reference to $\otimes$ if unnecessary.

Proposition 1.13. *Let C be an additive tensor category, $\mathcal{A}$ an abelian tensor category with a right-exact tensor product, and $M : C \to \mathcal{A}$ an additive tensor functor. Further assume that M factors through a $\flat$-subcategory $\mathcal{A}^\flat \subset \mathcal{A}$ (see Definition 1.9).*

(1) *Then $\mathcal{A}(M)$ carries a canonical tensor structure such that the faithful exact functor $\widetilde{M} : \mathcal{A}(M) \to \mathcal{A}$ is a tensor functor.*

(2) *If in addition, the tensor structures on C and $\mathcal{A}$ are commutative and the tensor functor is symmetric, then the tensor product on $\mathcal{A}(M)$ is symmetric.*

(3) *If in addition, the tensor structure on C is rigid and the tensor product and the* Hom*-functor on $\mathcal{A}$ are exact in both arguments, the same is true for $\mathcal{A}(M)$.*

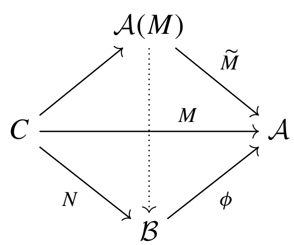
Proof. We need to check that the tensor functor on $\mathrm{Ab}(C)$ (see Proposition 1.8) factors via an induced tensor structure on $\mathcal{A}(M)$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ab}(C) \times \mathrm{Ab}(C) & \xrightarrow{\otimes} & \mathrm{Ab}(C) \\ {\scriptstyle \widetilde{M}\times\widetilde{M}}\downarrow & & \downarrow{\scriptstyle \widetilde{M}} \\ \mathcal{A} \times \mathcal{A} & \xrightarrow{\otimes} & \mathcal{A} \end{array}$$

by Proposition 1.10. This implies that the kernel of $\mathrm{Ab}(C) \to \mathcal{A}$ is a $\otimes$-ideal. Hence the tensor product induces one on $\mathcal{A}(M)$. Associativity, unit and symmetry are immediate from the properties of the tensor structure on $\mathrm{Ab}(C)$.

We turn to rigidity. By assumption, every object X of C has a strong dual. By the criterion formulated in [Levine 1998, Part I, IV, Proposition 1.1.9] the existence of a dual for X can be characterized by the existence of unit and counit maps satisfying some compatibilities. In particular, this property is functorial, hence the image of X in $\mathcal{A}(M)$ also has a strong dual. Consider the full subcategory of $\mathcal{A}(M)$ consisting of objects with a strong dual. It contains all objects in the image of C. Under our assumptions on $\mathcal{A}$, the tensor product on $\mathcal{A}(M)$ is exact in both arguments and hence the subcategory is closed under kernels and cokernels. Hence it is an abelian subcategory of $\mathcal{A}(M)$ containing the image of C, hence it agrees with $\mathcal{A}(M)$. □

Proposition 1.14 (universal property). *Let C, $\mathcal{A}^\flat \subset \mathcal{A}$, and M be as defined in Proposition 1.13. In addition, let $\mathcal{B}$ be another abelian tensor category with $\flat$-subcategory $\mathcal{B}^\flat$, let $N : C \to \mathcal{B}$ be an additive tensor functor which factors through $\mathcal{B}^\flat$ and let $\phi : \mathcal{B} \to \mathcal{A}$ be a faithful exact functor mapping $\mathcal{B}^\flat$ to $\mathcal{A}^\flat$ such that $\phi \circ N = M$:*



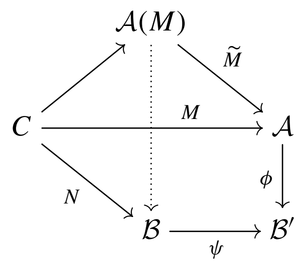
Then there exists a unique faithful exact tensor functor $\Phi : \mathcal{A}(M) \to \mathcal{B}$ making the diagram commute.

In particular, the universal property characterises $(\mathcal{A}(M), \widetilde{M})$ uniquely up to unique equivalence of categories.

Proof. The universal property of $\mathrm{Ab}(C)$ (see Proposition 1.10) gives us a similar statement, but with $\mathrm{Ab}(C)$ instead of $\mathcal{A}(M)$. The kernels of $\mathrm{Ab}(C) \to \mathcal{A}$ and $\mathrm{Ab}(C) \to \mathcal{B}$ agree because ϕ is faithful and exact. Hence $\mathcal{A}(M) = \mathcal{A}(N)$ and $\Phi = \widetilde{N}$. □

By a simple trick that we learned from Arapura [2013], this can be upgraded to a more general one.

Corollary 1.15 (generalised universal property). *Let C, $\mathcal{A}^\flat \subset \mathcal{A}$, and M be as in Proposition 1.13. In addition, let $\mathcal{B}$ be another abelian tensor category with $\flat$-subcategory $\mathcal{B}^\flat$, $N : C \to \mathcal{B}$ be an additive tensor functor which factors through $\mathcal{B}^\flat$. Let $(\mathcal{B}', \mathcal{B}'^\flat)$ be a third abelian tensor category, $\phi : \mathcal{A} \to \mathcal{B}'$ and $\psi : \mathcal{B} \to \mathcal{B}'$ faithful exact tensor functors respecting the $\flat$-subcategories. Finally, let $F : \psi \circ N \to \phi \circ M$ be an isomorphism of functors*:



Then there exists a faithful exact tensor functor $\Phi : \mathcal{A}(M) \to \mathcal{B}$ making the diagram commute up to isomorphism of functors.

Proof. Let $\mathcal{C}$ be a tensor category with objects of the form (A, B, f) where $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $f : \psi(B) \to \phi'(A)$. This category is abelian with kernels and cokernels taken componentwise. We equip it with a right-exact tensor product $(A, B, f) \otimes (A', B', f') = (A \otimes A', B \otimes B', f \otimes f')$. Let $\mathcal{C}^\flat$ be the subcategory of (A, B, f) with $A \in \mathcal{A}^\flat$, $B \in \mathcal{B}^\flat$.

Let $N' : C \to \mathcal{C}$ be the additive tensor functor $X \mapsto (M(X), N(X), F_X)$. We apply the universal property of Proposition 1.14 to N' and the forgetful functor $\phi : \mathcal{C} \to \mathcal{A}$. We define Φ as $\widetilde{N}'$ composed with the forgetful functor to $\mathcal{B}$. In other words, $\widetilde{N}'(X) = (\widetilde{M}(X), \widetilde{N}(X), F_X)$. The isomorphism of functors is given by F_X. □

Example 1.16. A possible application is with $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}'$ the categories of k-vector spaces, L-vector spaces and L'-vector spaces, respectively, for field extensions L'/k and L'/L.

Remark 1.17. This is a version of Nori's result on the tensor structure on his abelian category; see [Huber and Müller-Stach 2017, Proposition 8.1.5]. It is much stronger in allowing general abelian categories $\mathcal{A}$ as targets. In loc. cit. it was claimed that the original construction works for functors $C \to R$-proj (where the latter is the category of finitely generated projective modules over a noetherian ring R). However, as Paranjape pointed out, the proof is only correct if kernels of maps between projective modules are projective, i.e., if the global dimension of R is at most 2.

2. Universal ⊗-representation

We want to extend our results to representations of quivers. Given the results of the previous section, this means to extend tensor structures from a quiver to the additive category generated by it.

Recall from [Borceux 1994, Definition 5.1.5] or [Gabriel and Riedtmann 1979] the concept of a quiver "with relations", i.e., a quiver (a collection of vertices and directed edges) with a set of commutativity conditions or linear relations between paths (= compositions of directed edges). In this sense:

Definition 2.1. A ⊗-*quiver* is a quiver D with relations, plus the following data $(\mathrm{id}, \otimes, \alpha, \beta, \beta', \mathbf{1}, u)$ with,

(1) for every vertex v, a distinguished self-edge $\mathrm{id} : v \to v$;

(2) for every pair of vertices (v, w), a vertex denoted $v \otimes w$ in D;

(3) for every edge $e : v \to v'$ and vertex w, an edge $e \otimes \mathrm{id} : v \otimes w \to v' \otimes w$ and an edge $\mathrm{id} \otimes e : w \otimes v \to w \otimes v'$;

(4) for every pair of vertices u, v, a distinguished edge $\alpha_{u,v} : u \otimes v \to v \otimes u$;

(5) for every triple of vertices u, v, w, a distinguished edge $\beta_{u,vw} : u \otimes (v \otimes w) \to (u \otimes v) \otimes w$ and also $\beta'_{u,vw} : (u \otimes v) \otimes w \to u \otimes (v \otimes w)$;

(6) a distinguished vertex $\mathbf{1}$;

(7) for every vertex, distinguished edges $u_v : v \to \mathbf{1} \otimes v$ and $u'_v : \mathbf{1} \otimes v \to v$;

and the relations

(1) $\mathrm{id}_v \otimes \mathrm{id}_v = \mathrm{id}_{v \otimes v}$;

(2) $\mathrm{id}_v = e_v$ where e_v is the empty path for every vertex v;

(3) $(e \otimes \mathrm{id}) \circ (\mathrm{id} \otimes e') = (\mathrm{id} \otimes e') \circ (e \otimes \mathrm{id})$ for all pairs of edges e, e';

(4) $\alpha_{v,w} \circ \alpha_{w,v} = \mathrm{id}$ for all vertices v, w;

(5) $(\mathrm{id} \otimes \gamma) \circ \alpha = \alpha \circ (\gamma \otimes \mathrm{id})$ and $(\gamma \otimes \mathrm{id}) \circ \alpha = \alpha \circ (\mathrm{id} \otimes \gamma)$ for all edges γ;

(6) $\beta_{u,vw} \circ \beta'_{uv,w} = \mathrm{id}$ and $\beta'_{uv,w} \circ \beta_{u,vw} = \mathrm{id}$;

(7) $\beta \circ (\gamma \otimes (\mathrm{id} \otimes \mathrm{id})) = ((\gamma \otimes \mathrm{id}) \otimes \mathrm{id}) \circ \beta$ for all edges γ and analogously in the second and third argument;

(8) (pentagon axiom) for all vertices x, y, z, t the relation

$$\begin{array}{ccccc}
x \otimes (y \otimes (z \otimes t)) & \xrightarrow{\beta} & (x \otimes y) \otimes (z \otimes t) & \xrightarrow{\beta} & ((x \otimes y) \otimes z) \otimes t \\
\Big\downarrow{\scriptstyle \mathrm{id} \otimes \beta} & & & & \Big\uparrow{\scriptstyle \beta \otimes \mathrm{id}} \\
x \otimes ((y \otimes z) \otimes t) & & \xrightarrow{\qquad\quad \beta \qquad\quad} & & (x \otimes (y \otimes z)) \otimes t
\end{array}$$

(9) for all vertices x, y, z the relation

$$\begin{array}{ccccc}
x\otimes(y\otimes z) & \xrightarrow{\beta} & (x\otimes y)\otimes z & \xrightarrow{\alpha} & z\otimes(x\otimes y) \\
{\scriptstyle \mathrm{id}\otimes\alpha}\downarrow & & & & \downarrow{\scriptstyle \beta} \\
x\otimes(z\otimes y) & \xrightarrow{\beta} & (x\otimes z)\otimes y & \xrightarrow{\alpha\otimes\mathrm{id}} & (z\otimes x)\otimes y
\end{array}$$

(10) $u_v \circ u'_v = \mathrm{id}$ and $u'_v \circ u_v = \mathrm{id}$ for all vertices v;

(11) for all edges $e : v \to v'$ the relation

$$\begin{array}{ccc}
v' & \xrightarrow{u} & \mathbf{1}\otimes v' \\
{\scriptstyle e}\uparrow & & \uparrow{\scriptstyle \mathrm{id}\otimes e} \\
v & \xrightarrow{u} & \mathbf{1}\otimes v
\end{array}$$

Remark 2.2. This data is modelled after the notion of a commutative product structure on a diagram with identities; see [Huber and Müller-Stach 2017, Definition 8.1.3] and the variant in [loc. cit., Remark 8.1.6]. The axioms for the associativity and commutativity constraint and unitality are the usual ones for a commutative tensor category; see [Deligne and Milne 1982, §1].

It is more general than the notion of a monoidal quiver introduced by Bruguières [2004, Section 5.2].

Recall [Barbieri-Viale and Prest 2018] where a universal representation

$$\Delta : D \to \mathrm{Ab}(D)$$

is constructed for any quiver D. It is given by the composition

$$\Delta : D \to \mathcal{P}(D) \to \mathbb{Z}D \to \mathbb{Z}D^+ \to \mathrm{Ab}(\mathbb{Z}D^+) = \mathrm{Ab}(D),$$

where (in the notation of [Barbieri-Viale and Prest 2018, §1]) $\mathcal{P}(D)$ is the path category, $\mathbb{Z}D$ the preadditive enrichment of $\mathcal{P}(D)$ and $\mathbb{Z}D^+$ its additive completion.

We now repeat the same chain with tensor categories. Let $(D, \otimes)$ be a $\otimes$-quiver. We define the $\otimes$-path category $\mathcal{P}(D)^{\otimes}$ as the quotient of the path category by the relations of $(D, \otimes)$. We define

$$\otimes : \mathcal{P}(D) \times \mathcal{P}(D) \to \mathcal{P}(D)$$

on objects as prescribed by the tensor structure. Let

$$\Gamma = \gamma_1 \circ \cdots \circ \gamma_n$$

and

$$\Delta = \delta_1 \circ \cdots \circ \gamma_m$$

be paths. We define

$$\Gamma \otimes \Delta = (\gamma_1 \otimes \mathrm{id}) \circ \cdots \circ (\gamma_n \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_1) \circ \cdots \circ (\mathrm{id} \otimes \delta_m).$$

For example, for $n = m = 1$ and $\gamma : v \to v',\ \delta : w \to w'$, we have

$$\begin{array}{ccc} v \otimes w & \xrightarrow{\mathrm{id}\otimes\delta} & v \otimes w' \\ \Big\downarrow{\scriptstyle \gamma\otimes\mathrm{id}} & \searrow^{\gamma\otimes\delta} & \Big\downarrow{\scriptstyle \gamma\otimes\mathrm{id}} \\ v' \otimes w & \xrightarrow[\delta\otimes\mathrm{id}]{} & v' \otimes w' \end{array}$$

where we have by definition set the diagonal to be the path via the top right corner. In $\mathcal{P}(D)^{\otimes}$ this agrees with the path via the bottom left corner because of the relation (3).

Lemma 2.3. *Let $(D, \otimes)$ be a $\otimes$-quiver. Then $\mathcal{P}(D)^{\otimes}$ is a tensor category.*

Proof. Property (3) of a tensor structure ensures that $\otimes$ is a functor on $\mathcal{P}(D)^{\otimes}$. The other axioms make sure that the commutativity constraint α and the associativity constraint β are isomorphisms and satisfy the properties of a commutative tensor category. The relations on u_v ensure that $v \to \mathbf{1} \otimes v$ is an isomorphism and the functor $\mathbf{1} \otimes -$ is an equivalence of categories. □

Definition 2.4. Let $(D, \otimes)$ be a $\otimes$-quiver. We put $\mathbb{Z}D^{\otimes}$ and $\mathbb{Z}D^{\otimes,+}$ the preadditive and additive hull of $\mathcal{P}(D)^{\otimes}$. Denote by $\mathrm{Ab}(D)^{\otimes}$ Freyd's abelian category of $\mathbb{Z}D^{\otimes,+}$.

Proposition 2.5. *$\mathbb{Z}D^{\otimes}$, $\mathbb{Z}D^{\otimes,+}$ and $\mathrm{Ab}(D)^{\otimes}$ with the bilinear extension of $\otimes$ are commutative tensor categories. The canonical functor $\mathbb{Z}D^{+} \to \mathbb{Z}D^{\otimes,+}$ induces a Serre quotient $\pi : \mathrm{Ab}(D) \to \mathrm{Ab}(D)^{\otimes}$.*

Proof. The statements on the additive and preadditive category are obvious. The statement on the abelian category is Proposition 1.8. The claim on the Serre quotient is granted by the following general fact. □

Lemma 2.6. *Let D_1 be a quiver with relations and D the underlying quiver. Then $\pi : \mathrm{Ab}(D) \to \mathrm{Ab}(D_1)$ is a Serre quotient.*

This is well known but for the convenience of the reader we give the simple argument directly.

Proof. Consider $\mathrm{Ab}(D)/\operatorname{Ker}(\pi)$. By construction this is an exact subcategory of $\mathrm{Ab}(D_1)$, hence it remains to check that the inclusion is full and essentially surjective.

The quiver D has a canonical representation in $\mathrm{Ab}(D)/\operatorname{Ker}\pi$. All relations in D_1 are satisfied, hence it is even a representation of D_1. By the universal property this yields an exact functor $\mathrm{Ab}(D_1) \to \mathrm{Ab}(D)/\operatorname{Ker}\pi$. By the uniqueness part of the universal property, its composition with the inclusion into $\mathrm{Ab}(D_1)$ is isomorphic to the identity. In particular, the inclusion is full and essentially surjective, and hence an equivalence of categories. □

We now turn to the universal property. The obvious approach is to consider representations $T : D \to \mathcal{A}$ where all relations in D are mapped to identities in $\mathcal{A}$. However, this is too rigid for most applications. We follow the approach of [Huber and Müller-Stach 2017, Definition 8.1.3].

Definition 2.7. Let D be a $\otimes$-quiver, and $\mathcal{A}$ a commutative tensor category. A *tensor representation* or $\otimes$*-representation* for short, is a representation $T : D \to \mathcal{A}$ of the underlying quiver together with the choice of an isomorphism $\kappa_0 : \mathbf{1} \to T(\mathbf{1})$ and of natural isomorphisms

$$\kappa : T(u) \otimes T(v) \xrightarrow{\simeq} T(u \otimes v)$$

for all vertices $u, v \in D$, functorial in each variable and compatible with the associativity and commutativity constraints and the unit in the obvious way.

Proposition 2.8. *Let* $(D, \otimes)$ *be a* $\otimes$*-quiver.*

(1) $D \to \mathcal{P}(D)^{\otimes}$ *is the universal* $\otimes$*-representation into a commutative tensor category.*

(2) $D \to \mathbb{Z}D^{\otimes,+}$ *is the universal* $\otimes$*-representation into an additive commutative tensor category.*

Proof. The universal properties for $\mathcal{P}(D)^{\otimes}$ and $\mathbb{Z}D^{\otimes,+}$ are obvious. □

Theorem 2.9. *Let* $(D, \otimes)$ *be a* $\otimes$*-quiver.*

(1) *The natural assignment* $\Delta^{\otimes} : D \to \mathrm{Ab}(D)^{\otimes}$ *is a* $\otimes$*-representation into an abelian tensor category with right-exact tensor product.*

(2) *It takes values in the subcategory* $(\mathrm{Ab}(D)^{\otimes})^{\flat}$ *of Definition 1.1. Moreover, this is a* $\flat$*-subcategory* (*see Definition 1.9*).

(3) *The category* $\mathrm{Ab}(D)^{\otimes}$ *is universal with this property.*

In detail: *Let* $T : D \to \mathcal{A}$ *be a* $\otimes$*-representation via* κ *in an abelian tensor category with a right-exact tensor*, *which factors through a* $\flat$*-subcategory* $\mathcal{A}^{\flat} \subseteq \mathcal{A}$. *Then there is an induced exact tensor functor* $\widetilde{M}^{\otimes} : \mathrm{Ab}(D)^{\otimes} \to \mathcal{A}$.

Proof. $\mathcal{A} = \mathrm{Ab}(D)^{\otimes}$ is an abelian tensor category by Proposition 2.5 and $\Delta^{\otimes}$ is a $\otimes$-representation by construction. It factors via the additive category $\mathbb{Z}D^{\otimes,+}$. Hence property (2) follows from Proposition 1.8. To see the second statement, note that if

$$(B, -) \xrightarrow{(f,-)} (A, -)$$

is a morphism in $(\mathrm{Ab}(D)^{\otimes})^{\flat}$ then its kernel is

$$(C, -) \xrightarrow{(g,-)} (B, -)$$

where $B \xrightarrow{g} C$ is the cokernel of $A \xrightarrow{f} B$.

The induced functor

$$M^{\otimes} : \mathbb{Z}D^{\otimes,+} \to \mathcal{A}$$

satisfies the assumptions in Proposition 1.10. Thus it induces the tensor functor $\widetilde{M}^{\otimes} : \mathrm{Ab}(D)^{\otimes} \to \mathcal{A}$ such that $T = \widetilde{M}^{\otimes}\Delta^{\otimes}$. □

Recall the universal representation theorem stated in [Barbieri-Viale and Prest 2018]. For $T : D \to \mathcal{A}$ any representation of a quiver in an abelian category $\mathcal{A}$ there is an induced additive functor

$$M : \mathbb{Z}D^{+} \to \mathcal{A}$$

and a corresponding $\widetilde{M} : \mathrm{Ab}(D) \to \mathcal{A}$ such that

$$\widetilde{T} : D \to \mathcal{A}(T) := \mathcal{A}(M) = \mathrm{Ab}(D)/\operatorname{Ker}\widetilde{M}$$

is the induced universal representation (see [Barbieri-Viale and Prest 2018, §1.3]). For a $\otimes$-quiver D, together with a $\otimes$-representation T in an abelian tensor category $\mathcal{A}$, as in Theorem 2.9, we have now constructed a factorisation via an exact tensor functor $\widetilde{M}^{\otimes}$ on $\mathrm{Ab}(D)^{\otimes}$. Hence we get a tensorial refinement of the universal representation theorem. This also implies the existence of a tensor structure on the universal abelian category $\mathcal{A}(T)$ attached to the representation. Note that this is really $\mathcal{A}(T)$; in contrast to $\mathcal{P}(D)^{\otimes}$ etc., no $\otimes$-adornment is needed.

Theorem 2.10. *Let $T : D \to \mathcal{A}$ be a representation in an abelian tensor category with a right-exact tensor, which factors through a $\flat$-subcategory $\mathcal{A}^{\flat} \subseteq \mathcal{A}$, with the following additional properties:*

(i) $(D, \otimes)$ *is a $\otimes$-quiver.*

(ii) T *is a $\otimes$-representation in $\mathcal{A}^{\flat} \subseteq \mathcal{A}$ via κ.*

Then Nori's universal abelian category $\mathcal{A}(T)$ carries a right-exact tensor product and $\widetilde{M} : \mathcal{A}(T) \to \mathcal{A}$ is a tensor functor (here M is the additive functor induced by T and $\widetilde{M}$ is the faithful exact functor induced by M; see also Proposition 1.13). It is universal among such representations into abelian tensor categories $\mathcal{B}$ compatible with T (cf. the statement of Proposition 1.14) via a faithful exact tensor functor $\mathcal{B} \to \mathcal{A}$.

Proof. By the universal property in Theorem 2.9, there is a canonical exact tensor functor $\widetilde{M}^{\otimes} : \mathrm{Ab}(D)^{\otimes} \to \mathcal{A}$. Hence $\operatorname{Ker}\widetilde{M}^{\otimes}$ is a Serre subcategory and a tensor ideal. Denoting by $\mathcal{A}(T)^{\otimes}$ the Serre quotient $\mathrm{Ab}(D)^{\otimes}/\operatorname{Ker}\widetilde{M}^{\otimes}$ we have obtained a tensor category with the universal property as claimed. Furthermore, by the universal property of $\mathcal{A}(T)$, there is also an exact faithful functor

$$\mathcal{A}(T) \to \mathcal{A}(T)^{\otimes}.$$

We claim that it is an equivalence of abelian categories. The canonical additive functor $\mathbb{Z}D^+ \to \mathbb{Z}D^{\otimes,+}$ induces an exact functor $\pi : \mathrm{Ab}(D) \to \mathrm{Ab}(D)^{\otimes}$ such that $\widetilde{M}^{\otimes} \circ \pi = \widetilde{M}$ by the uniqueness in the universal property of Freyd's construction (see [Barbieri-Viale and Prest 2018, Theorem 1.1]). The faithful exact functor $\bar{\pi} : \mathrm{Ab}(D)/\operatorname{Ker}\pi \xrightarrow{\simeq} \mathrm{Ab}(D)^{\otimes}$ is an equivalence by Proposition 2.5.

Thus, the composition $\mathrm{Ab}(D) \xrightarrow{\pi} \mathrm{Ab}(D)^{\otimes} \to \mathrm{Ab}(D)^{\otimes}/\operatorname{Ker}(\widetilde{M}^{\otimes})$ is essentially surjective and is equivalent to the composition $\mathrm{Ab}(D) \to \mathcal{A}(T) \to \mathcal{A}(T)^{\otimes}$ since they have equivalent compositions with the faithful functor $\mathcal{A}(T)^{\otimes} \to \mathcal{A}$. So $\mathcal{A}(T) \to \mathcal{A}(T)^{\otimes}$ also is essentially surjective and hence an equivalence. □

Remark 2.11. The universal property can be upgraded analogously to Corollary 1.15.

Remark 2.12. In the special case where $\mathcal{A}$ is the category of finitely generated modules over a Dedekind domain, this gives back Nori's original result as formulated for example in [Huber and Müller-Stach 2017]. The same case (actually in the more restrictive setting of monoidal quivers) is also handled by Bruguières [2004, Theorem 3]. His conditions P1 and P2 are analogous to our factorisation via $\mathcal{A}^{\flat}$. Both [Huber and Müller-Stach 2017] and [Bruguières 2004] are based on the explicit description of the universal abelian category as comodules or modules.

Signs. In many cases, notably in Nori's original application, we do not start with a tensor representation but with a tensor representation with signs. We explain the necessary modifications, following again the approach of [Huber and Müller-Stach 2017, Definition 8.1.3].

Definition 2.13. A *graded quiver* is a quiver together with a function $|\cdot|$ assigning to each vertex a degree in $\mathbb{Z}/2\mathbb{Z}$. For an edge $e : v \to w$ we put $|e| = |w| - |v|$. A *graded $\otimes$-quiver* is a graded quiver together with the data of a $\otimes$-quiver such that $|v \otimes w| = |v| + |w|$ and $|\mathbf{1}| = 0$. The relations are the same as for a $\otimes$-quiver, except for relation (3), which is replaced by

(3′) $(e \otimes \mathrm{id}) \circ (\mathrm{id} \otimes e') = (-1)^{|e|\,|e'|}(\mathrm{id} \otimes e') \circ (e \otimes \mathrm{id})$ for all pairs of edges e, e'.

The grading on D induces gradings on $\mathcal{P}(D)$, $\mathbb{Z}D$, and $\mathbb{Z}D^+$. In the case of the additive hull this means that every object is equipped with a decomposition into an even and an odd part. Note that morphisms are *not* required to preserve the degree. Recall that part of the data of a $\otimes$-quiver is the choice of edges $\alpha_{v,w} : v \otimes w \to w \otimes v$.

Definition 2.14. Let $(D, \otimes)$ be a graded $\otimes$-quiver.

(1) We define $\mathbb{Z}D^{\otimes,\mathrm{sgn}}$ as the quotient of the category $\mathbb{Z}D$ modulo the relations of a graded $\otimes$-quiver. It is equipped with the tensor product $\otimes^{\mathrm{sgn}}$ which agrees with $\otimes$ on objects and for morphisms $\gamma : v \to v'$, $\delta : w \to w'$,

$$\gamma \otimes^{\mathrm{sgn}} \delta = (-1)^{|\gamma|\,|w|} \gamma \otimes \delta,$$

with associativity constraint $\beta^{\mathrm{sgn}}_{u,vw} = \beta_{u,vw}$ and commutativity constraint given by

$$\alpha^{\mathrm{sgn}}_{v,w} = (-1)^{|v|\,|w|}\alpha_{v,w} : v \otimes w \to w \otimes v$$

for all objects v, w.

(2) Let $\mathbb{Z}D^{\otimes,\mathrm{sgn},+}$ be the category $\mathbb{Z}D^{\otimes,+}$ with tensor structure given by the additive extension from $\mathbb{Z}D^{\otimes,\mathrm{sgn}}$.

(3) Set $\mathrm{Ab}(D)^{\otimes,\mathrm{sgn}} = \mathrm{Ab}(\mathbb{Z}D^{\otimes,\mathrm{sgn},+})$ for the universal abelian category attached to $\mathbb{Z}D^{\otimes,\mathrm{sgn},+}$.

Remark 2.15. Note that $\mathbb{Z}D^{\otimes}$ is different from $\mathbb{Z}D^{\otimes,\mathrm{sgn}}$ even as an additive category.

Lemma 2.16. *$\mathbb{Z}D^{\otimes,\mathrm{sgn}}$ and $\mathbb{Z}D^{\otimes,\mathrm{sgn},+}$ are well-defined tensor categories.*

Proof. It suffices to consider $\mathbb{Z}D^{\otimes,\mathrm{sgn}}$. We have to check that $\otimes^{\mathrm{sgn}}$ satisfies the axioms of a commutative tensor category. Condition (3′) ensures functoriality of $\otimes^{\mathrm{sgn}}$. It is tedious but straightforward to see that β and α are functorial. For example, for $\gamma : x \to x'$, $\delta : y \to y'$ the diagram reads

$$\begin{array}{ccc} x \otimes y & \xrightarrow{(-1)^{|x|\,|y|}\alpha} & y \otimes x \\ {\scriptstyle (-1)^{|\gamma||y|}\gamma\otimes\delta}\downarrow & & \downarrow{\scriptstyle (-1)^{|\delta||x|}\delta\otimes\gamma} \\ x' \otimes y' & \xrightarrow{(-1)^{|x'|\,|y'|}\alpha} & y' \otimes x' \end{array}$$

It does not commute on the level of $\mathcal{P}(D)$. In order to check that it commutes in $\mathbb{Z}D^{\otimes,\mathrm{sgn}}$, it is enough to treat the two special cases $\gamma = \mathrm{id}$ or $\delta = \mathrm{id}$ separately because $(\gamma, \delta) = (\gamma, \mathrm{id}) \circ (\mathrm{id}, \delta)$. In each of these cases the diagram commutes in $\mathcal{P}(D)$.

The pentagon axiom (concerning associativity) holds because it is a relation on D and no signs are involved. Unitality is preserved because $\mathbf{1}$ is of degree 0. The hexagon axiom reads

$$\begin{array}{ccccc} x \otimes (y \otimes z) & \xrightarrow{\beta} & (x \otimes y) \otimes z & \xrightarrow{(-1)^{(|x|+|y|)|z|}\alpha} & z \otimes (x \otimes y) \\ {\scriptstyle \mathrm{id}\otimes(-1)^{|y|\,|z|}\alpha}\downarrow & & & & \downarrow{\scriptstyle \beta} \\ x \otimes (z \otimes y) & \xrightarrow{\beta} & (x \otimes z) \otimes y & \xrightarrow{(-1)^{|x|\,|z|}\alpha\otimes\mathrm{id}} & (z \otimes x) \otimes y \end{array}$$

It commutes because the hexagon axiom holds for $\otimes$. ☐

Again, we turn to representations. Following [Huber and Müller-Stach 2017, Definition 8.1.3]:

Definition 2.17. Let $(D, \otimes)$ be a graded $\otimes$-quiver. Let $\mathcal{A}$ be an additive commutative tensor category. A *graded tensor representation* of $(D, \otimes)$ is a representation

$T : D \to \mathcal{A}$ of the underlying quiver together with the choice of an isomorphism $\kappa_0 : \mathbf{1} \to T(\mathbf{1})$ and of natural isomorphisms

$$\kappa : T(u) \otimes T(v) \xrightarrow{\simeq} T(u \otimes v)$$

for all vertices $u, v \in D$, functorial in each variable and compatible with the associativity constraint and the unit in the obvious way and such that:

(1) For all vertices v, w,

$$\begin{array}{ccc} T(v \otimes w) & \xrightarrow{T(\alpha)} & T(w \otimes v) \\ {\scriptstyle\kappa}\uparrow & & \uparrow{\scriptstyle\kappa} \\ T(v) \otimes T(w) & \longrightarrow & T(w) \otimes T(v) \end{array}$$

commutes where the bottom arrow is $(-1)^{|v||w|}$ times the commutativity constraint in $\mathcal{A}$.

(2) For all edges $\gamma : v \to v'$ and vertices w,

$$\begin{array}{ccc} T(v \otimes w) & \xrightarrow{T(\gamma \otimes \mathrm{id})} & T(v' \otimes w) \\ {\scriptstyle\kappa}\uparrow & & \uparrow{\scriptstyle\kappa} \\ T(v) \otimes T(w) & \xrightarrow{T(\gamma) \otimes \mathrm{id}} & T(v') \otimes T(w) \end{array}$$

commutes up to the factor $(-1)^{|\gamma||w|}$.

(3) For all edges $\gamma : v \to v'$ and vertices w,

$$\begin{array}{ccc} T(w \otimes v) & \xrightarrow{T(\mathrm{id} \otimes \gamma)} & T(w \otimes v') \\ {\scriptstyle\kappa}\uparrow & & \uparrow{\scriptstyle\kappa} \\ T(w) \otimes T(v) & \xrightarrow{\mathrm{id} \otimes T(\gamma)} & T(w) \otimes T(v') \end{array}$$

commutes (without signs).

The following is a graded analogue of Proposition 2.8(2).

Proposition 2.18. *Let $(D, \otimes)$ be a graded $\otimes$-quiver. The natural map $D \to \mathbb{Z}D^{\otimes,\mathrm{sgn},+}$ is the universal graded $\otimes$-representation of $(D, \otimes)$. In detail: it is a graded $\otimes$-representation and if $T : D \to \mathcal{A}$ is a graded tensor representation in an additive commutative tensor category $\mathcal{A}$ then T factors uniquely through an induced additive tensor functor as shown*

$$\begin{array}{ccc} D & \longrightarrow & \mathbb{Z}D^{\otimes,\mathrm{sgn},+} \\ & {\scriptstyle T}\searrow & \downarrow{\scriptstyle M^{\otimes,\mathrm{sgn}}} \\ & & \mathcal{A} \end{array}$$

Proof. The argument is the same as in the ungraded case. Relation (3′) is forced by the signs in the graded tensor representation. □

Now consider the category $\mathrm{Ab}(D)^{\otimes,\mathrm{sgn}}$ as in Definition 2.14(3).

Theorem 2.19. *The graded analogue of Theorem 2.9 is satisfied by the category* $\mathrm{Ab}(D)^{\otimes,\mathrm{sgn}}$.

Proof. The proof is as in the ungraded case. □

Finally:

Theorem 2.20. *Let $T : D \to \mathcal{A}$ be a representation in an abelian tensor category with a right-exact tensor, which factors through a $\flat$-subcategory $\mathcal{A}^\flat \subset \mathcal{A}$ with the following additional properties:*

(1) $(D, \otimes)$ *is a graded $\otimes$-quiver.*

(2) T *is a graded $\otimes$-representation in $\mathcal{A}^\flat \subset \mathcal{A}$ via κ.*

Then Nori's universal abelian category $\mathcal{A}(T)$ carries a right-exact tensor product and $\widetilde{M} : \mathcal{A}(T) \to \mathcal{A}$ is a tensor functor (here M is the additive functor induced by T and $\widetilde{M}$ is the faithful exact functor induced by M; see also Proposition 1.13). It is universal among such representations into abelian tensor categories $\mathcal{B}$ compatible with T (cf. Proposition 1.14) via a faithful exact tensor functor $\mathcal{B} \to \mathcal{A}$.

Proof. Compare with the proof of Theorem 2.10. If T is such a graded tensor representation, we get $M^{\otimes,\mathrm{sgn}} : \mathbb{Z}D^{\otimes,\mathrm{sgn},+} \to \mathcal{A}$ and also an induced exact tensor functor $\widetilde{M}^{\otimes,\mathrm{sgn}} : \mathrm{Ab}(D)^{\otimes,\mathrm{sgn}} \to \mathcal{A}$. Denote by $\mathcal{A}(T)^{\otimes,\mathrm{sgn}}$ the quotient of $\mathrm{Ab}(D)^{\otimes,\mathrm{sgn}}$ by the kernel of $\widetilde{M}^{\otimes,\mathrm{sgn}}$. We have that $\mathcal{A}(T) \to \mathcal{A}(T)^{\otimes,\mathrm{sgn}}$ is an equivalence. □

Remark 2.21. Again, the universal property can be upgraded analogously to Corollary 1.15.

3. Homological functors

We return to the case of additive categories, but specialise further by considering triangulated categories and homological functors.

Proposition 3.1 [Neeman 2001, Theorem 5.1.18]. *Let $\mathcal{T}$ be a triangulated category. Then there is an abelian category $\mathrm{Ab}^\Delta(\mathcal{T})$ and a homological functor $[-] : \mathcal{T} \to \mathrm{Ab}^\Delta(\mathcal{T})$ such that every homological functor $\mathcal{T} \to \mathcal{A}$ into an abelian category factors uniquely via an exact functor $\mathrm{Ab}^\Delta(\mathcal{T}) \to \mathcal{A}$.*

Neeman uses the notation $\mathcal{A}(\mathcal{T})$, which we have reserved for Nori's abelian category. By construction, $\mathrm{Ab}^\Delta(\mathcal{T})$ is the subcategory of finitely presented objects in the category of presheaves of abelian groups on $\mathcal{T}$. It is obtained from the image of the Yoneda functor by adding all cokernels. Hence every object of $\mathrm{Ab}^\Delta(\mathcal{T})$ is the cokernel of a morphism of objects in the image of the Yoneda functor and this is a projective resolution. Note, however, see [Neeman 2001, §5.2, Appendix C], that the category $\mathrm{Ab}^{\Delta}(\mathcal{T})$ is typically not well-powered. Our constructions and results in Section 1 do not require the initial categories C to be well-powered, so those results apply here.

Proposition 3.2. *Let $\mathcal{T}$ be a tensor triangulated category. Then $\mathrm{Ab}^{\Delta}(\mathcal{T})$ carries a right-exact tensor product. If $\mathcal{A}$ is an abelian tensor category with right-exact tensor and $\mathcal{T} \to \mathcal{A}$ is a homological tensor functor, then the natural functor $\mathrm{Ab}^{\Delta}(\mathcal{T}) \to \mathcal{A}$ is a tensor functor.*

Proof. We extend from representable objects to cokernels by using projective resolutions $[B] \to [A] \to X \to 0$, where $[A]$ denotes the image of A in $\mathrm{Ab}^{\Delta}(\mathcal{T})$. The arguments are dual to the ones used in the first part of the proof of Proposition 1.4. The associativity constraint etc. are constructed in the same way as in the proof of Proposition 1.8. The compatibility of $\mathrm{Ab}^{\Delta}(\mathcal{T}) \to \mathcal{A}$ with the tensor structure holds for representable arguments and extends to cokernels by right-exactness. □

Remark 3.3. (1) This was already proved by Balmer, Krause and Stevenson in [Balmer et al. 2020, Proposition A.14] for compactly generated tensor triangulated categories for the smaller category of all presheaves on $\mathcal{T}^c$ which is universal for all homological functors commuting with colimits, see [Krause 2000, Section 2].

(2) Applying the universal property of $\mathrm{Ab}(\mathcal{T})$ to $\mathcal{T} \to \mathrm{Ab}^{\Delta}(\mathcal{T})$, we obtain an exact functor $\mathrm{Ab}(\mathcal{T}) \to \mathrm{Ab}^{\Delta}(\mathcal{T})$ but it is not clear whether this is a tensor functor. This may well be false since the kernel of a map between representable functors in $\mathrm{Ab}^{\Delta}(\mathcal{T})$ might not be tensor-flat.

Proposition 3.4. *Let $\mathcal{T}$ be a tensor triangulated category, $\mathcal{A}$ an abelian tensor category with a right-exact tensor product, and $M : \mathcal{T} \to \mathcal{A}$ a homological functor and tensor functor.*

(1) *Then $\mathcal{A}(M)$ carries a canonical right-exact tensor structure such that the faithful exact functor $\widetilde{M} : \mathcal{A}(M) \to \mathcal{A}$ is a tensor functor.*

(2) *If in addition, the tensor structures on $\mathcal{T}$ and $\mathcal{A}$ are commutative and the tensor functor is symmetric, then the tensor product on $\mathcal{A}(M)$ is symmetric.*

(3) *If in addition, the tensor structure on $\mathcal{T}$ is rigid and the tensor product and the* Hom-*functor on $\mathcal{A}$ are exact in both arguments, the same is true for $\mathcal{A}(M)$.*

Proof. The proof is the same as for Proposition 1.13, but with $\mathrm{Ab}(\mathcal{T})$ and the tensor product of Proposition 1.8 replaced with $\mathrm{Ab}^{\Delta}(\mathcal{T})$ and the tensor product of Proposition 3.2. □

Remark 3.5. If both Propositions 1.13 and 3.4 apply, then by the universal property of Proposition 3.4 the tensor structures agree because they agree on objects in the image of $\mathcal{T}$.

Künneth components. We now consider the following situation modelled for the application to Nori motives. Let $\mathcal{T}$ be a triangulated category, $\mathcal{A}$ an abelian category and $R:\mathcal{T}\to D^b(\mathcal{A})$ an exact functor. We abbreviate $H^i_R := H^i\circ R$ and $H^*_R := \bigoplus H^i_R$. The latter is understood with values in $\operatorname{gr}\mathcal{A}$. Let $\mathcal{A}(H^*_R)$ be the universal abelian category defined by H^*_R, and $\mathcal{A}(H^0_R)$ that defined by H^0_R. The commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{H^*_R} & \operatorname{gr}\mathcal{A} \\ & \searrow^{H^0_R} & \Big\downarrow (-)^0 \\ & & \mathcal{A} \end{array}$$

induces a functor $\mathcal{A}(H^*_R)\to\mathcal{A}(H^0_R)$. We also have $\widetilde{H^*_R}:\mathcal{A}(H^*_R)\to\operatorname{gr}\mathcal{A}$.

Definition 3.6. In the above situation let $\mathcal{A}_0(H^*_R)\subset\mathcal{A}(H^*_R)$ be the full subcategory of objects $X\in\mathcal{A}(H^*_R)$ with $\widetilde{H^*_R}(X)\in\operatorname{gr}\mathcal{A}$ concentrated in degree 0.

The subcategory is abelian and closed under subquotients and extensions.

Remark 3.7. We are interested in the case where $\mathcal{T}$ is a triangulated tensor category, $\mathcal{A}$ is an abelian tensor category with an exact tensor product and R is a tensor functor. Then H^*_R is a tensor functor, but H^0_R is not. Hence while $\mathcal{A}(H^*_R)$ is a tensor category by the results of Section 1, this does not follow for $\mathcal{A}(H^0_R)$. It is, however, true for $\mathcal{A}_0(H^*_R)$. In good cases, it will be equivalent to $\mathcal{A}(H^0_R)$, giving the latter the tensor structure that we want.

Proposition 3.8. *Let $\mathcal{T}$, $\mathcal{A}$ and R be as above. Assume in addition that R can be lifted to an exact functor*

$$R:\mathcal{T}\to D^b(\mathcal{A}_0(H^*_R)).$$

Then the natural functor

$$\mathcal{A}_0(H^*_R)\to\mathcal{A}(H^0_R)$$

is an equivalence of categories.

Proof. We abbreviate $\mathcal{A}' := \mathcal{A}_0(H^*_R)$. By assumption, there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{T} & \longrightarrow & D^b(\mathcal{A}') & \longrightarrow & D^b(\mathcal{A}) \\ & \searrow & \Big\downarrow H^0 & & \Big\downarrow H^0 \\ & & \mathcal{A}' & \longrightarrow & \mathcal{A} \end{array}$$

The functor $\widetilde{H^0}_R:\mathcal{A}(H^0_R)\to\mathcal{A}$ is faithful and exact by construction. The same is true for $\widetilde{H_R^*} : \mathcal{A}' \to \operatorname{gr}\mathcal{A}$. By definition, this functor takes values in degree 0; hence $\mathcal{A}' \to \mathcal{A}$ is also faithful and exact. This implies that the universal categories defined by $H^0 : \mathcal{T} \to \mathcal{A}'$ and $H_R^0 : \mathcal{T} \to \mathcal{A}$ agree. This gives $\mathcal{A}(H_R^0) \to \mathcal{A}'$ inverse to the inclusion. □

Corollary 3.9. *Let $\mathcal{T}$ be a tensor triangulated category. Let $\mathcal{A}$ be an abelian tensor category with an exact tensor product. Let $R : \mathcal{T} \to D^b(\mathcal{A})$ be a tensor triangulated functor. Assume in addition, that R factors via $D^b(\mathcal{A}_0(H_R^*))$. Then $\mathcal{A}(H_R^0)$ carries a natural tensor structure such that $\mathcal{A}(H_R^0) \to \mathcal{A}$ is a tensor functor. If the tensor product on $\mathcal{T}$ is rigid and* $\operatorname{Hom}_{\mathcal{A}}$ *exact in both variables, then the tensor product on $\mathcal{A}(H_R^0)$ is rigid as well.*

Proof. Combining Proposition 3.8 with the strategy of Remark 3.7 gives the tensor structure. If the tensor product on $\mathcal{T}$ is rigid and $\operatorname{Hom}_{\mathcal{A}}$ exact, then by Proposition 1.13 the tensor product on $\mathcal{A}(H_R^*)$ is rigid as well. Hence every object X of $\mathcal{A}_0(H_R^*)$ has a dual $X^\vee$ in $\mathcal{A}(H_R^*)$. The object $X^\vee$ is actually in $\mathcal{A}_0(H_R^*)$, as we can test by applying the forgetful functor to $\operatorname{gr}\mathcal{A}$. □

Remark 3.10. (1) The use of the *bounded* derived category in the above argument is not very important. We can drop the assumption, if arbitrary direct sums exist in $\mathcal{A}$. This is needed in order to write down the Künneth formula or, equivalently, the tensor structure on $D(\mathcal{A})$.

(2) We may also replace $D^b(\mathcal{A})$ by a tensor triangulated category equipped with a t-structure (compatible with the tensor structure) with heart $\mathcal{A}$ without any change in the arguments.

Integral coefficients. What we have done so far does not apply to $\mathcal{A} = \mathbb{Z}$-mod because its tensor product is not exact. However, there is a version of the above criterion for integral coefficients.

Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{A}$ be an abelian tensor category with a right-exact tensor product such that its derivation on $D^b(\mathcal{A})$ exists. Let $\mathcal{A}^\flat \subset \mathcal{A}$ be a $\flat$-subcategory as in Definition 1.9. Let $R : \mathcal{T} \to D^b(\mathcal{A})$ be a tensor functor. Note that $H_R^* : \mathcal{T} \to \operatorname{gr}\mathcal{A}$ is no longer a tensor functor because $H^* : D^b(\mathcal{A}) \to \operatorname{gr}\mathcal{A}$ is not. However, we have the following lemma:

Lemma 3.11. *In this situation, let $\mathcal{T}^\flat \subset \mathcal{T}$ be the full subcategory of objects with H_R^* in* $\operatorname{gr}\mathcal{A}^\flat$. *Then $\mathcal{T}^\flat$ is a tensor category and*

$$H_R^*|_{\mathcal{T}^\flat} : \mathcal{T}^\flat \to \operatorname{gr}\mathcal{A}^\flat$$

is a tensor functor which satisfies the assumptions of the universal property in Proposition 1.10.

Proof. Obviously $\operatorname{gr}\mathcal{A}^\flat \subset \operatorname{gr}\mathcal{A}$ consists of flat objects and is closed under kernels. It remains to check the claim on the tensor functor with $\mathcal{T} = D^b(\mathcal{A})$. This amounts to the naive Künneth formula for these objects. The subcategory $\mathcal{T}^\flat$ is stable under the canonical truncation functor and shift. Hence it suffices to check the formula for objects of $\mathcal{A}^\flat \subset \mathcal{T}^\flat$. They are flat; hence the derived tensor product agrees with the tensor product in $\mathcal{A}^\flat$. As a byproduct of the formula we see that $\mathcal{T}^\flat$ is stable under the derived tensor product. □

We now replace $\mathcal{A}(H_R^*)$ by $\mathcal{A}(H_R^*|_{\mathcal{T}^\flat})$ and set as before $\mathcal{A}_0(H_R^*|_{\mathcal{T}^\flat})$ to be the subcategory of objects concentrated in degree 0.

Corollary 3.12. *Let $\mathcal{T}$ be a tensor triangulated category. Let $\mathcal{A}$ be an abelian tensor category with a right-exact tensor product. Let $\mathcal{A}^\flat \subset \mathcal{A}$ be a $\flat$-subcategory and assume that the derived tensor product exists on $D^b(\mathcal{A})$. Let $R : \mathcal{T} \to D^b(\mathcal{A})$ be a tensor triangulated functor. Let $\mathcal{T}^\flat$ and $\mathcal{A}_0(H_R^*|_{\mathcal{T}^\flat})$ be as above.*

Assume in addition, that R factors via $D^b(\mathcal{A}_0(H_R^*|_{\mathcal{T}^\flat}))$. Then $\mathcal{A}(H_R^0)$ carries a natural tensor structure such that $\mathcal{A}(H_R^0) \to \mathcal{A}$ is a tensor functor.

4. Nori motives

Recall the original definition of Nori. Let k be a field and $\sigma : k \to \mathbb{C}$ be an embedding. Let Sch_k be the category of schemes which are separated and of finite type over the field k. Let D^{Nori} be Nori's quiver on Sch_k having vertices (X, Y, n) where $Y \subseteq X$ is a closed subscheme and $n \in \mathbb{Z}$ and edges $(X', Y', n) \to (X, Y, n)$ for each morphism $f : X \to X'$ in Sch_k such that $f(Y) \subseteq Y'$, and an additional edge $(Y, Z, n) \to (X, Y, n+1)$ for $Z \subseteq Y \subseteq X$ closed subschemes. Let

$$H_B : D^{\mathrm{Nori}} \to \mathbb{Z}\text{-mod}$$

be the representation given by $(X, Y, n) \rightsquigarrow H_B^n(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$, the relative singular cohomology group after base change to the complex numbers.

Definition 4.1 (Nori; see also [Huber and Müller-Stach 2017, §9]). The abelian category

$$\mathsf{ECM}_k := \mathcal{A}(H_B)$$

is the category of effective cohomological Nori motives. There is a noneffective version that we shall denote NM_k.

Remark 4.2. The diagram D^{Nori} above agrees with the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ of [Huber and Müller-Stach 2017, Definition 9.1.1]. In loc. cit., the abelian categories are denoted by $\mathsf{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(k)$ and $\mathsf{MM}_{\mathrm{Nori}}(k)$, respectively. Noneffective motives are obtained either by localisation of the diagram or of the category with respect to the Lefschetz motive $\mathbf{1}(-1) = (\mathbb{G}_m, \{1\}, 1)$. This is somewhat premature at this point as it involves the tensor structure. We are going to concentrate on the effective case.

Tensor product via graded ⊗-quivers. Let $D^{\mathrm{Nori},\otimes}$ be the same quiver with, in addition, the following structure of a graded ⊗-quiver in the sense of Definition 2.13. The grading is given by

$$(X, Y, n) \mapsto \bar{n} \in \mathbb{Z}/2\mathbb{Z}.$$

For vertices (X, Y, n), (X', Y, n') we put

$$(X, Y, n) \otimes (X', Y', n') := (X \times_k X', X \times_k Y' \cup Y \times_k X', n + n'),$$

making use of the product in Sch_k. We choose the edges id, α, β, β', u, u' and the vertex $\mathbf{1}$ in the canonical way, e.g., the unit $\mathbf{1} = (\mathrm{Spec}(k), \varnothing, 0)$,

$$u : (X, Y, n) \to (\mathrm{Spec}(k), \varnothing, 0) \otimes (X, Y, n)$$

and $u' : (\mathrm{Spec}(k), \varnothing, 0) \otimes (X, Y, n) \to (X, Y, n)$ are the canonical maps. As relations we use the relations required by Definition 2.13. All this is completely parallel to [Huber and Müller-Stach 2017, §9.3]. By construction we obtain a graded ⊗-quiver.

Recall that the singular cohomology H_B^* is provided with a natural cross or external product

$$\kappa^B_{n,n'} : H^n_B(X, Y) \otimes H^{n'}_B(X', Y') \to H^{n+n'}_B(X \times_k X', X \times_k Y' \cup Y \times_k X').$$

Note that the representation H_B is *not* a ⊗-representation since $\kappa^B_{n,n'}$ fails to be an isomorphism, in general.

Following Nori, we set $D^{\mathrm{good},\otimes}$ for the full sub-⊗-quiver of vertices (X, Y, n) such that $H_B^*(X, Y)$ is concentrated in degree n and free as a $\mathbb{Z}$-module.

Lemma 4.3. *The Betti cohomology*

$$H_B^{\mathrm{good}} := H_B|_{D^{\mathrm{good}}} : D^{\mathrm{good},\otimes} \to \mathbb{Z}\text{-}\mathit{mod}$$

is a graded ⊗-representation with values in the subcategory $(\mathbb{Z}\text{-}\mathit{mod})^\flat$ *of free* $\mathbb{Z}$-*modules of finite type.*

Proof. On good pairs, the map $\kappa^B_{n,n'}$ is indeed an isomorphism by the Künneth formula. The relations of the tensor quiver are all mapped to equalities in $\mathbb{Z}$-mod by the standard properties of singular cohomology. Most are checked explicitly in [Huber and Müller-Stach 2017, Proposition 9.3.1]. The remaining ones (e.g., concerning the inverse u' of u) are obvious. □

Indeed, our definition of a graded ⊗-quiver was modelled on this case.

Corollary 4.4. *The abelian category* $\mathcal{A}(H_B^{\mathrm{good}})$ *carries a natural ⊗-structure compatible with the forgetful functor to* $\mathbb{Z}$-*mod.*

Proof. See Theorem 2.20. □

Nori's basic lemma comes into play in comparing the universal categories for the two diagrams.

Theorem 4.5 (Nori; see [Huber and Müller-Stach 2017, Theorem 9.2.22]). *The quiver D^{Nori} can be represented in $\mathcal{A}(H_B^{\mathrm{good}})$ in a compatible way with H_B. In particular,*

$$\mathsf{ECM}_k \cong \mathcal{A}(H_B^{\mathrm{good}})$$

carries a natural tensor structure.

In the above, we are copying Nori's approach, but replace his approach to the universal abelian category and its tensor product with the one developed in this paper. In [Barbieri-Viale and Prest 2020] we go further, providing a general axiomatic framework for tensor motivic categories associated to a cohomological functor on a suitable base category; the tensor structure is induced, using our main theorem, by the cartesian tensor structure on the base category via a cohomological Künneth formula.

We now turn to a different approach which does not mention D^{Nori} and D^{good} (at least not obviously so).

Tensor product via triangulated motives. Let $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ be Voevodsky's category of geometric motives over k with rational coefficients. Let

$$R_B : \mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \to D^b(\mathbb{Q}\text{-vsp.})$$

be the Betti-realisation. It maps the motive of an algebraic variety to its singular cochain complex.

Remark 4.6. The existence of the Betti-realisation is completely straightforward. The first reference with rational coefficients is [Huber 2000] and the correction thereto, where they appear as a byproduct of a functor into mixed realisations. For integral coefficients it is formulated in [Harrer 2016]. In the original literature on motives, realisation functors were usually contravariant. This is also the viewpoint taken in the above references.

More recently, Voevodsky and then Ayoub who, in [Ayoub 2010] constructs Betti-realisations for motives over any base, use the covariant point of view.

For our application, it does not matter which point of view is taken. We fix on the contravariant one because we want to refer to [Harrer 2016] later on.

Definition 4.7. Let $\mathsf{MM}_k := \mathcal{A}(H_B^0)$ be the universal abelian category defined by the Betti-realisation.

Based on a sketch of Nori, Harrer [2016] showed the following:

Theorem 4.8 [Harrer 2016, Theorem 7.3.1]. *The Betti-realisation factors naturally via the bounded derived category of* MM_k *and even that of* $\mathcal{A}_0(H_B^*)$.

Remark 4.9. The proof is based on Nori's basic lemma: for every affine variety X and subvariety Y, there is a subvariety

$$X \supset Z \supset Y$$

such that the singular cohomology of the pair (X, Z) is concentrated in the degree equal to the dimension of X, i.e., $(X, Z, \dim X)$ is a good pair. As pointed out by Nori, this can be used in order to construct, for every affine variety X, a natural complex of motives. Using Čech-complexes, this extends to all varieties. Harrer's main effort was to establish functoriality of the construction with respect to finite correspondences. When working with rational coefficients (as we do), functoriality with respect to morphisms is enough; see [Ivorra 2016; Huber and Müller-Stach 2017]. Harrer's result is formulated for NM_k, but actually proved for $\mathcal{A}(H_B^{\mathrm{good}})$. The same proof also works without change for $\mathsf{MM}_k = \mathcal{A}(H_B^0)$ and even the refinement $\mathcal{A}_0(H_B^*)$.

Theorem 4.10. *The category* MM_k *carries a natural tensor structure such that* $\mathsf{MM}_k \to \mathbb{Q}$*-vsp. is a tensor functor and* $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \to D^b(\mathsf{MM}_k)$ *is a triangulated tensor functor.*

In particular, MM_k *is Tannakian.*

Proof. We apply Corollary 3.9 to the rigid tensor category $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ and the Betti-realisation. The assumption is satisfied by Theorem 4.8. This makes MM_k a rigid tensor category; the Betti-realisation $\mathsf{MM}_k \to \mathbb{Q}$-vsp. is a fibre functor. □

Definition 4.11. The *motivic Galois group* of k is defined as the Tannakian dual of the MM_k.

Proposition 4.12. MM_k *is naturally equivalent to Nori's original category*, *i.e.*, $\mathsf{NM}_k \cong \mathsf{MM}_k$. *The motivic Galois group is naturally isomorphic to Nori's original motivic Galois group.*

Proof. For the abelian category, this is already shown in [Huber and Müller-Stach 2017]. The tensor structures are based on the Künneth formula. In each case it is uniquely determined by its value for very good pairs, hence they are the same. The statement about the motivic Galois group follows. □

Remark 4.13. The whole argument also works for motives with coefficients in any field (including finite fields) or Dedekind domain (in particular the integers). Corollary 3.12 can be used instead of the more straightforward Corollary 3.9. The integral case is handled in [Harrer 2016].

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