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**SCHUR ALGEBRAS FOR THE ALTERNATING GROUP
AND KOSZUL DUALITY**

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We introduce the alternating Schur algebra $AS_F(n, d)$ as the commutant of the action of the alternating group A_d on the d -fold tensor power of an n -dimensional F -vector space. When F has characteristic different from 2, we give a basis of $AS_F(n, d)$ in terms of bipartite graphs, and a graphical interpretation of the structure constants. We introduce the abstract Koszul duality functor on modules for the even part of any $Z/2Z$ -graded algebra. The algebra $AS_F(n, d)$ is $Z/2Z$ -graded, having the classical Schur algebra $S_F(n, d)$ as its even part. This leads to an approach to Koszul duality for $S_F(n, d)$ -modules that is amenable to combinatorial methods. We characterize the category of $AS_F(n, d)$ -modules in terms of $S_F(n, d)$ -modules and their Koszul duals. We use the graphical basis of $AS_F(n, d)$ to study the dependence of the behavior of derived Koszul duality on n and d .

1. Introduction

1A. Schur–Weyl duality and its variants. Frobenius [1900] determined the irreducible characters of the symmetric group S_d over \mathbb{C} , the field of complex numbers. Building on this, Schur classified the irreducible polynomial representations of $GL_n(\mathbb{C})$ and computed their characters in his PhD thesis [Schur 1901]. The group $GL_n(\mathbb{C})$ acts on the factors of $(\mathbb{C}^n)^{\otimes d}$, while S_d permutes the tensor factors. Schur [1927] used these commuting actions to reprove the results of his dissertation. Following Weyl’s expositions of this method [Weyl 1931; Weyl 1939], it is known as Schur–Weyl duality.

Over the years, several variants of Schur–Weyl duality have emerged. Shrinking $GL_n(\mathbb{C})$ to the orthogonal group $O_n(\mathbb{C})$, Brauer [1937] obtained the duality between Brauer algebras $Br_d(n)$ and $O_n(\mathbb{C})$. Motivated by the Potts model in statistical mechanics, Jones [1994] and Martin [1991] further shrunk $O_n(\mathbb{C})$ down to S_n , obtaining the partition algebras $P_d(n)$. Bloss [2005] reduced S_n to A_n to obtain an algebra $AP_d(n)$ which coincides with the partition algebra when $n \geq 2d+2$. We take the smallest possible step in the opposite direction: we reveal what takes the place

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$F^{n \otimes d}$		
??	this article	A_d
\cup		\cap
$GL_n(F)$	Schur–Weyl	S_d
\cup		\cap
$O_n(F)$	Brauer	$Br_d(n)$
\cup		\cap
S_n	Martin and Jones	$P_d(n)$
\cup		\cap
A_n	Bloss	$AP_d(n)$

Table 1. Dualities arising from tensor space.

of the polynomial representations of $GL_n(\mathbb{C})$ when the action of the symmetric group S_d is restricted to the alternating group A_d . The situation is summarized in Table 1. The significance of this investigation lies in its connection with the Koszul duality functor on the category of homogeneous polynomial representations of $GL_n(\mathbb{C})$ of degree d .

1B. Schur algebras for the alternating group. Motivated by Green [2007, Theorem 2.6c], define the Schur algebra as

$$S_F(n, d) = \text{End}_{S_d}((F^n)^{\otimes d})$$

for any field F , and positive integers n and d . When F is infinite, then $S_F(n, d)$ -modules are the same as homogeneous polynomial representations of $GL_n(F)$ of degree d (see [Green 2007, Section 2.4] and [Prasad 2015, Section 6.2]). Define the alternating Schur algebra $AS_F(n, d)$ by replacing S_d by A_d in the definition above:

$$AS_F(n, d) = \text{End}_{A_d}((F^n)^{\otimes d}).$$

When F has characteristic different from 2, this algebra has a decomposition (see Lemma 2.1)

$$(1) \quad AS_F(n, d) = S_F(n, d) \oplus S_F^-(n, d)$$

as a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra. Here $S_F^-(n, d) = \text{Hom}_{S_d}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn})$, where sgn is used to denote the sign character of S_d . The subspace $S_F^-(n, d)$ is an $(S_F(n, d), S_F(n, d))$ -bimodule.

When $n^2 < d$, then $S_F^-(n, d) = 0$, and $AS_F(n, d) = S_F(n, d)$, as observed by Regev [2002, Theorem 1]. But when $n^2 \geq d$, $S_F^-(n, d) \neq 0$, and in Lemma 2.2, we note that $S_F^-(n, d)$ is a full tilting left $S_F(n, d)$ -module as studied by Donkin [1993, Section 3].

1C. Bases and structure constants. Schur [1927] gave a combinatorial description of a basis and the corresponding structure constants of the Schur algebra (see also [Green 2007, Section 2.3]). By indexing Schur's basis of $S_F(n, d)$ by bipartite multigraphs with $n + n$ vertices and d edges, Méndez [2001] (see also [Geetha and Prasad 2014]) gave a graphic interpretation of the structure constants. We describe a basis of $S_F^-(n, d)$ in terms of bipartite simple graphs in Theorem 2.24. So from the decomposition (1), a basis of $AS_F(n, d)$ is obtained. A graphic interpretation of the structure constants of $AS_F(n, d)$ is given in Theorems 2.14 and 2.25. This will be used to derive properties of $AS_F(n, d)$, its bimodule $S_F^-(n, d)$, and Koszul duality.

1D. Koszul duality and modules. The term *Koszul duality* is used for several constructions which interchange the roles of exterior and symmetric powers.

The earliest notion of Koszul duality was introduced by Priddy [1970]. It applies to *pre-Koszul algebras*, which are also called *quadratic algebras*. A pre-Koszul algebra is a quotient of a tensor algebra

$$T(V) = \bigoplus_{n \geq 0} \otimes^n V$$

by a two-sided ideal I that is generated in degree two. Its Koszul dual is the algebra

$$T(V^*)/(I \cap (V \otimes V))^\perp;$$

the quotient of the dual tensor algebra by the annihilator in degree two of I . In this setting the Koszul dual of the symmetric algebra of V is the exterior algebra of V^* .

Bernstein, Gelfand, and Gelfand [Bernstein et al. 1978, Theorem 3] introduced an equivalence between the bounded derived categories of graded modules over symmetric and exterior algebras, which was called the Koszul duality functor by Beilinson, Ginsburg, and Schectman [Beilinson et al. 1988].

Friedlander and Suslin [1997] introduced the category of *strict polynomial functors* of degree d as the representations of the Schur category of degree d , for each nonnegative integer d (see Section 4A). The category of strict polynomial functors of degree d unifies the categories of homogeneous polynomial representations of $GL_n(F)$ of degree d across all n . Standard examples of strict polynomial functors of degree d are the d -th tensor power functor \otimes^d , the d -th symmetric power functor Sym^d , and the d -th exterior power functor \wedge^d . Evaluating a strict polynomial functor of degree d at F^n gives an $S_F(n, d)$ -module for each n . Friedlander and Suslin showed that this evaluation functor is an equivalence of categories when $n \geq d$. Chałupnik [2008] and Touzé [2014] used the term Koszul duality to refer to a functor on the category of strict polynomial functors of degree d which takes the Schur functor associated to the partition λ of d to the Weyl functor associated with the partition λ' conjugate to λ . Krause [2013] discovered an internal tensor product on the category of strict polynomial functors of fixed degree d . Given such a tensor

product it was then natural for him to define Koszul duality in this category as the tensor product with \wedge^d . This definition is different from the Koszul duality functors defined earlier by Chalušnik and Touzé. Those coincide with a duality defined by Ringel [1991] using tilting modules for quasihereditary algebras. This tilting module was described by Donkin [1993] in the case of Schur algebras.

We introduce the term *abstract Koszul duality* to refer to a very simple functor which makes sense for any $\mathbb{Z}/2\mathbb{Z}$ -graded algebra $AS = S \oplus S^-$. The abstract Koszul dual of an S -module V is defined as

$$D(V) = S^- \otimes_S V.$$

The multiplication operation on AS gives rise to an (S, S) -bimodule homomorphism $\phi : S^- \otimes_S S^- \rightarrow S$ and hence a natural transformation from $D \circ D$ to the identity functor on the category of S -modules. We prove (Theorem 3.4) that the category of AS -modules is the same as the category of pairs (M, θ_M) where M is an S -module and $\theta_M : D(M) \rightarrow M$ is compatible with ϕ in the sense of (16).

In Section 4, we specialize to the case $AS_F(n, d) = S_F(n, d) \oplus S_F^-(n, d)$ to obtain a Koszul duality functor D on the category of $S_F(n, d)$ -modules. In Theorem 4.5, we show that the evaluation at F^n of the Koszul duality functor of Krause is naturally isomorphic to our Koszul duality functor when $n \geq d$. In this sense, our abstract Koszul duality functor on Schur algebras coincides with Krause's Koszul duality.

Our description of the structure constants of $AS_F(n, d)$ allows us to give a direct combinatorial proof of the well-known fact that, when $n \geq d$ and when the characteristic of F is 0 or greater than d , then abstract Koszul duality is an equivalence (Theorem 4.2).

Krause [2013] showed that the derived Koszul duality functor is an autoequivalence of the unbounded derived category of strict polynomial functors. Since the evaluation functor is an equivalence, this implies that derived Koszul duality is an autoequivalence at the level of the unbounded derived category of $S_F(n, d)$ -modules when $n \geq d$. However, this does not address the case where $n < d$. Using our combinatorial methods, we show that derived Koszul duality is not an equivalence when $n < d$ (Theorem 4.9). This proof uses a criterion of Happel [1987] for a tensor functor to be a derived equivalence. In the context of derived Koszul duality, this criterion requires that the canonical algebra homomorphism $S_F(n, d) \rightarrow \text{End}_{S_F(n, d)}(S_F^-(n, d))$ is an isomorphism. Donkin [1993, Proposition 3.7] proved this for $n \geq d$. For the case when the characteristic of F is not 2 we give a combinatorial proof of Donkin's result, and also show that it fails when $n < d$ (Theorem 4.7). Figure 1 on page 181 describes the behavior of Koszul duality for all values of the parameters n and d .

We conclude this paper by discussing a possible application of our techniques to Bloss's alternating partition algebra, and a diagrammatic interpretation of the Schur category (Section 5).

2. The alternating Schur algebra

Let F be a field of characteristic different from 2, and n and d be positive integers. The symmetric group S_d acts on the tensor space $(F^n)^{\otimes d}$ by permuting the tensor factors. The Schur algebra can be defined as

$$S_F(n, d) := \text{End}_{S_d}((F^n)^{\otimes d}).$$

By restricting the action of S_d to the alternating group A_d , define the *alternating Schur algebra* as

$$AS_F(n, d) := \text{End}_{A_d}((F^n)^{\otimes d}).$$

Clearly, $S_F(n, d)$ is a subalgebra of $AS_F(n, d)$.

Lemma 2.1. *For any representations V and W of S_d ,*

$$(2) \quad \text{Hom}_{A_d}(V, W) = \text{Hom}_{S_d}(V, W) \oplus \text{Hom}_{S_d}(V, W \otimes \text{sgn}).$$

Here $W \otimes \text{sgn}$ denotes the twist of W by the sign character $\text{sgn} : S_d \rightarrow \{\pm 1\}$.

Define

$$(3) \quad S_F^-(n, d) := \text{Hom}_{S_d}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn}).$$

Lemma 2.1 gives a $\mathbf{Z}/2\mathbf{Z}$ -grading of $AS_F(n, d)$ in the sense of Bourbaki [1974, Chapter III, Section 3.1]:

$$(4) \quad AS_F(n, d) = S_F(n, d) \oplus S_F^-(n, d).$$

The summand $S_F^-(n, d)$ is an $(S_F(n, d), S_F(n, d))$ -bimodule. Recall that a *weak composition* of d with n parts is a vector $\lambda = (\lambda_1, \dots, \lambda_n)$ of nonnegative integers summing to d . Let $\Lambda(n, d)$ denote the set of weak compositions of d with n parts. For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n, d)$, define

$$\wedge^\lambda F^n = \wedge^{\lambda_1} F^n \otimes \dots \otimes \wedge^{\lambda_n} F^n,$$

where, for a nonnegative integer s , $\wedge^s F^n$ is the s -th exterior power of F^n . As a left $S_F(n, d)$ -module,

$$(5) \quad S_F^-(n, d) = \bigoplus_{\lambda \in \Lambda(n, d)} \wedge^\lambda F^n.$$

For each partition λ of d with at most n parts, let $\Delta(\lambda)$ denote the $S_F(n, d)$ -module known as the *Weyl module* with highest weight λ as in [Donkin 1993, Section 1]. A *tilting module* is an $S_F(n, d)$ -module V such that both V and its dual V^* have filtrations by the Weyl modules $\Delta(\lambda)$. Ringel [1991] showed that, for every such λ , there exists an indecomposable tilting module $M(\lambda)$ with unique highest weight λ . A *full tilting module* is a tilting module that contains $M(\lambda)$ as

a direct summand for every partition λ of d with at most n parts [Donkin 1993, Section 3]. By Donkin [1993, Lemma 3.4], (5) implies:

Lemma 2.2. *The left module $S_F^-(n, d)$ is a full tilting module of $S_F(n, d)$.*

2A. Twisted permutation representations. Let X be a finite set on which a group G acts on the right (henceforth called a G -set). The space $F[X]$ of F -valued functions on X may be regarded as a representation of G :

$$(6) \quad \rho_X(g)f(x) = f(x \cdot g) \quad \text{for } x \in X, g \in G, \text{ and } f \in F[X].$$

Let χ be a multiplicative character $G \rightarrow F^*$. We may twist the representation (6) by χ :

$$(7) \quad \rho_X^\chi(g)f(x) = \chi(g)f(x \cdot g).$$

Denote the representation space of this twisted action as $F[X] \otimes \chi$.

Suppose that X and Y are finite G -sets. Given a function $\kappa : X \times Y \rightarrow F$, the *integral operator* $\xi_\kappa : F[Y] \rightarrow F[X]$ associated to κ is defined as

$$(8) \quad \xi_\kappa f(x) = \sum_{y \in Y} \kappa(x, y)f(y) \quad \text{for } f \in F[Y].$$

The function κ is known as the *integral kernel* of ξ_κ .

If Z is another finite G -set, $\kappa' : X \times Y \rightarrow F$ and $\kappa'' : Y \times Z \rightarrow F$ are functions. Then,

$$\xi_{\kappa'} \circ \xi_{\kappa''} = \xi_{\kappa' * \kappa''},$$

where $\kappa' * \kappa'' : X \times Z \rightarrow F$ is the *convolution product*

$$(9) \quad \kappa' * \kappa''(x, z) = \sum_{y \in Y} \kappa'(x, y)\kappa''(y, z).$$

We have (see [Prasad 2015, Section 4.2]):

Theorem 2.3. *For any finite G -spaces X and Y , and any multiplicative character $\chi : G \rightarrow F^*$,*

$$(10) \quad \begin{aligned} \text{Hom}_G(F[Y], F[X] \otimes \chi) \\ = \{ \xi_\kappa \mid \kappa : X \times Y \rightarrow F \text{ such that } \kappa(x \cdot g, y \cdot g) = \chi(g)\kappa(x, y) \}. \end{aligned}$$

The identity (10) implies that

$$\dim \text{Hom}_G(F[Y], F[X] \otimes \chi) \leq |(X \times Y)/G|,$$

with equality holding if χ is the trivial character. However, if $g \in G$, and $(x, y) \in X \times Y$ are such that $(x \cdot g, y \cdot g) = (x, y)$, then if $\xi_\kappa \in \text{Hom}_G(F[Y], F[X] \otimes \chi)$,

$$\kappa(x, y) = \kappa(x \cdot g, y \cdot g) = \chi(g)\kappa(x, y),$$

so that either $\chi(g) = 1$ or κ vanishes on the G -orbit of (x, y) .

For each element $x \in X$, let $G_x = \{g \in G \mid g \cdot x = x\}$ be the stabilizer of x in G .

Definition 2.4 (transverse pair). A pair $(x, y) \in X \times Y$ is said to be *transverse* with respect to χ if $G_x \cap G_y \subset \ker \chi$. If (x, y) is a transverse pair with respect to χ , we write $x \pitchfork y$.

If (x, y) is a transverse pair, then

$$\kappa(x \cdot g, y \cdot g) := \chi(g)\kappa(x, y)$$

is a well-defined nonzero function on the G -orbit of (x, y) . Let

$$X \pitchfork Y = \{(x, y) \in X \times Y \mid x \pitchfork y\}.$$

Then $X \pitchfork Y$ is stable under the diagonal action of G on $X \times Y$. We have (see [Prasad 2015, Theorem 4.2.3]):

Theorem 2.5. *Let X and Y be finite G -sets, and $\chi : G \rightarrow F^*$ be a multiplicative character. For each orbit $O \in (X \pitchfork Y)/G$, choose a base point $(x_O, y_O) \in O$. Define*

$$\kappa_O(x, y) = \begin{cases} \chi(g) & \text{if } x = x_O \cdot g \text{ and } y = y_O \cdot g \text{ for some } g \in G, \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity, write ξ_O for ξ_{κ_O} . Then the set

$$\{\xi_O \mid O \in (X \pitchfork Y)/G\}$$

is a basis for $\text{Hom}_G(F[Y], F[X] \otimes \chi)$. Consequently,

$$\dim \text{Hom}_G(F[Y], F[X] \otimes \chi) = |(X \pitchfork Y)/G|.$$

In the special case where χ is the trivial character, we get:

Corollary 2.6. *Let X and Y be finite G -sets. For each orbit O in $(X \times Y)/G$ define*

$$\kappa_O(x, y) = \begin{cases} 1 & \text{if } (x, y) \in O, \\ 0 & \text{otherwise.} \end{cases}$$

Write $\xi_O = \xi_{\kappa_O}$. Then the set

$$\{\xi_O \mid O \in (X \times Y)/G\}$$

is a basis for $\text{Hom}_G(F[Y], F[X])$. Consequently,

$$\dim \text{Hom}_G(F[Y], F[X]) = |(X \times Y)/G|.$$

Given a function $\kappa : X \times Y \rightarrow F$, define

$$\kappa^*(y, x) = \kappa(x, y) \quad \text{for } x \in X, y \in Y.$$

The following is easy to see:

Lemma 2.7. *For any G -set X , the map $\xi_\kappa \mapsto \xi_{\kappa^*}$ is an anti-involution on the algebra $\text{End}_G(F[X])$.*

2B. Structure constants of the Schur algebra. We recall the combinatorial interpretation of structure constants of the Schur algebra from [Geetha and Prasad 2014]. Let $[n] = \{1, \dots, n\}$ and

$$I(n, d) = \{\underline{i} := (i_1, \dots, i_d) \mid i_s \in [n]\}.$$

An element $w \in S_d$ acts on $I(n, d)$ by permuting the coordinates:

$$(i_1, \dots, i_d) \cdot w = (i_{w(1)}, \dots, i_{w(d)}).$$

For $\underline{i} = (i_1, \dots, i_d) \in I(n, d)$, define

$$e_{\underline{i}} = e_{i_1} \otimes \dots \otimes e_{i_d},$$

where e_i is the i -th coordinate vector in F^n . The vector space $(F^n)^{\otimes d}$ has a basis

$$\{e_{\underline{i}} \mid \underline{i} \in I(n, d)\}$$

and $w \in S_d$ acts on a basis vector $e_{\underline{i}}$ as follows:

$$w \cdot e_{\underline{i}} = w \cdot (e_{i_1} \otimes \dots \otimes e_{i_d}) = e_{i_{w^{-1}(1)}} \otimes \dots \otimes e_{i_{w^{-1}(d)}}.$$

Let $F[I(n, d)]$ denote the space of all F -valued functions on $I(n, d)$. Mapping $e_{\underline{i}}$ to the indicator function of $\underline{i} \in I(n, d)$ defines an isomorphism of $(F^n)^{\otimes d}$ onto $F[I(n, d)]$. Thus $(F^n)^{\otimes d}$ can be regarded as a permutation representation of S_d .

Let $B(n, d)$ denote the set of all configurations of d distinguishable balls, numbered $1, \dots, d$ in n boxes, numbered $1, \dots, n$. The symmetric group S_d acts on such configurations by permuting the d balls. An element of $B(n, d)$ is a set partition

$$\{1, \dots, d\} = S_1 \coprod \dots \coprod S_n,$$

where S_i is the set of balls in the i -th box.

Lemma 2.8. *Given $\underline{i} \in I(n, d)$, let $b(\underline{i})$ denote the balls-in-boxes configuration in $B(n, d)$ where the i -th box contains the balls $\{s \mid i_s = i\}$. Then*

$$b : I(n, d) \rightarrow B(n, d)$$

is an S_d -equivariant bijection of $I(n, d)$ onto $B(n, d)$.

By Corollary 2.6, a basis for $S_F(n, d)$ is indexed by orbits for the diagonal action of S_d on $B(n, d) \times B(n, d)$.

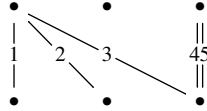
Definition 2.9 (labelled bipartite multigraph). Let $[n] = \{1, \dots, n\}$ (as before) and $[n'] = \{1', \dots, n'\}$. A labelling of a bipartite multigraph with vertex set $[n'] \coprod [n]$ and d edges is a function $l : [d] \rightarrow [n'] \times [n]$ such that, for each $(i', j) \in [n'] \times [n]$, the cardinality of $l^{-1}(i', j)$ is the number of edges joining i' and j . In other words, labels are assigned to edges without distinguishing between edges joining the same pair of vertices.

Given a pair $S = (S_1, \dots, S_n)$ and $T = (T_1, \dots, T_n)$ in $B(n, d)$, define a labelled bipartite graph $\gamma_{S,T}$ with multiple edges on the vertex set $[n'] \coprod [n]$ as follows:

There are $|S_j \cap T_i|$ edges between i' and j , labelled by the numbers of the balls in $S_j \cap T_i$.

The bipartite multigraph is always drawn in two rows, with the vertices from $[n']$ in the upper row and vertices from $[n]$ in the lower row, numbered from left to right. Since the vertices are always labelled in this manner, the vertex labels can be omitted in the drawing.

Example 2.10. When $S = (\{1\}, \{2\}, \{3, 4, 5\})$ and $T = (\{1, 2, 3\}, \emptyset, \{4, 5\})$, the associated labelled multigraph is



Clearly, $(S, T) \mapsto \gamma_{S,T}$ is a bijection from $B(n, d) \times B(n, d)$ onto the set of labelled bipartite multigraphs with vertex set $[n'] \coprod [n]$ and d edges. The symmetric group S_d acts on $B(n, d) \times B(n, d)$ by permuting labels. Therefore the S_d orbits in $B(n, d) \times B(n, d)$ are obtained by simply forgetting the labels, leaving only the underlying bipartite multigraph. We write $\Gamma_{S,T}$ for the bipartite multigraph underlying $\gamma_{S,T}$. Such a graph can also be represented by its *adjacency matrix* (whose (i, j) -th entry is the number of edges joining i' and j), which is a matrix of nonnegative integers that sum to d .

In view of Corollary 2.6, we recover a result of [Geetha and Prasad 2014; Méndez 2001]:

Theorem 2.11. *Let $M(n, d)$ denote the set of all bipartite multigraphs with vertex set $[n'] \coprod [n]$ and d edges. For each $\Gamma \in M(n, d)$, define $\xi_\Gamma \in S_F(n, d) = \text{End}_{S_d}(F[B(n, d)])$ by*

$$\xi_\Gamma f(S) = \sum_{\{T \mid \Gamma_{S,T} = \Gamma\}} f(T).$$

Then

$$\{\xi_\Gamma \mid \Gamma \in M(n, d)\}$$

is a basis for $S_F(n, d)$.

Remark 2.12. If $(\underline{i}, \underline{j})$ has image Γ under the composition $I(n, d)^2 \rightarrow B(n, d)^2 \rightarrow M(n, d)$, then the basis element $\xi_{\underline{i}, \underline{j}}$ of [Green 2007, Section 2.6] coincides with the basis element ξ_Γ of Theorem 2.11.

The structure constants $c_{\Gamma_1 \Gamma_2}^\Gamma$ are defined by

$$\xi_{\Gamma_1} \xi_{\Gamma_2} = \sum_{\Gamma \in M(n, d)} c_{\Gamma_1 \Gamma_2}^\Gamma \xi_\Gamma.$$

Definition 2.13. Let l, l_1 and l_2 be labellings of graphs Γ, Γ_1 and Γ_2 in $M(n, d)$, respectively. We say that (l_1, l_2) is compatible with l if, for all $s = 1, \dots, d$, if we write $l_1(s) = (i'_1, j_1)$ and $l_2(s) = (i'_2, j_2)$, then

$$(2.13.a) \quad i'_2 = j_1, \text{ and}$$

$$(2.13.b) \quad l(s) = (i'_1, j_2).$$

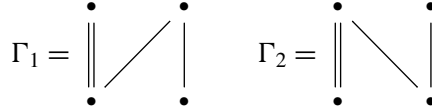
We obtain yet another enumerative description of the structure constants of the Schur algebra (see also [Green 2007, 2.3(b)] and [Geetha and Prasad 2014; Méndez 2001]).

Theorem 2.14. Let l be any labelling of Γ . The structure constant $c_{\Gamma, \Gamma''}^\Gamma$ is the number of pairs (l_1, l_2) of labellings of Γ_1 and Γ_2 that are compatible with l .

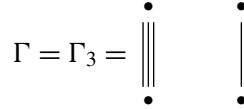
Before giving a proof, we illustrate the theorem with a few examples.

Example 2.15. Let $w \in S_d$ be a permutation, and assume that $n \geq d$. Let $\Gamma(w)$ denote the bipartite graph where (i', j) is an edge if and only if $1 \leq i \leq d$ and $w(i) = j$. Then, for all $w_1, w_2 \in S_d$, $\xi_{\Gamma(w_1)} \xi_{\Gamma(w_2)} = \xi_{\Gamma(w_1 w_2)}$.

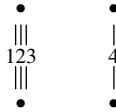
Example 2.16. Consider



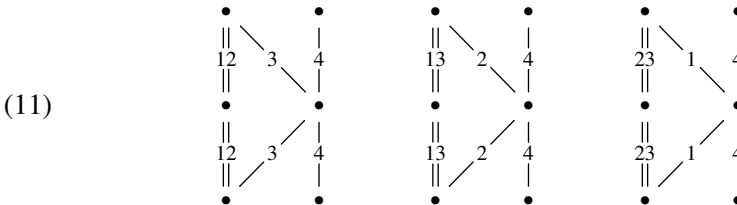
To find $c_{\Gamma_1 \Gamma_2}^\Gamma$, with



choose any labelling of Γ , such as



For this there are clearly three pairs of compatible labellings of Γ_1 and Γ_2 , namely, we can choose which of the first three balls ends up in the second box of the middle row:



On the other hand, if

$$\Gamma = \Gamma_4 = \begin{array}{cc} \bullet & \bullet \\ \parallel & \diagdown \\ \bullet & \bullet \end{array}$$

we may take the labelling

$$(12) \quad \begin{array}{cc} \bullet & \bullet \\ \parallel & \diagdown \\ 12 & 3 \\ \parallel & \diagup \\ \bullet & \bullet \end{array}$$

For this labelling, the only compatible labellings of Γ_1 and Γ_2 are

$$\begin{array}{cc} \bullet & \bullet \\ \parallel & \diagdown \\ 12 & 3 \\ \bullet & \bullet \\ \parallel & \diagup \\ 12 & 4 \\ \bullet & \bullet \end{array}$$

It turns out that for no other $\Gamma \in \Gamma(n, d)$ is it possible to find even one compatible way of labelling Γ_1 and Γ_2 , so we have

$$\xi_{\Gamma_1} \xi_{\Gamma_2} = 3\xi_{\Gamma_3} + \xi_{\Gamma_4}.$$

Example 2.17. Let $F_{n,n}$ denote the complete bipartite graph which has vertex set $[n'] \sqcup [n]$, where every vertex in $[n']$ is connected to every vertex in $[n]$. Then the coefficient of $\xi_{F_{n,n}}$ in $\xi_{F_{n,n}} \xi_{F_{n,n}}$ is the number of Latin squares of order n [OEIS, Sequence A002860].

To see this, let l be any labelling of the edges of $F_{n,n}$. Given labellings l_1 and l_2 of $F_{n,n}$ that are compatible with l , define the (i, j) -th entry of the Latin square associated to (l_1, l_2) to be k if $l^{-1}(i', j) = l_2^{-1}(k', j) = l_1^{-1}(i', k)$. Remarkably, the number of Latin squares of order n is known only for $n = 1, \dots, 11$.

Proof of Theorem 2.14. Given a labelling l of Γ , define:

$$S_j = \cup_{i=1}^n l^{-1}(i', j) \quad \text{and} \quad U_i = \cup_{j=1}^n l^{-1}(i', j).$$

Then $S = (S_1, \dots, S_n)$, and $U = (U_1, \dots, U_n)$ are elements of $B(n, d)$, and by construction $\Gamma_{S,U} = \Gamma$. Now (9) implies that

$$(13) \quad c_{\Gamma_1 \Gamma_2}^\Gamma = \#\{T \in B(n, d) \mid \Gamma_{S,T} = \Gamma_1 \text{ and } \Gamma_{T,U} = \Gamma_2\}.$$

Given $T \in B(n, d)$ contributing to the above count, define labellings l_1 and l_2 of Γ_1 and Γ_2 by:

$$l_1^{-1}(i', j) = S_j \cap T_i \quad \text{and} \quad l_2^{-1}(i', j) = T_j \cap U_i.$$

Then (l_1, l_2) is compatible with l . Conversely, for every pair (l_1, l_2) compatible with l , take $T = (T_1, \dots, T_n)$ where

$$T_k = \bigcup_{i'=1}^n l_1^{-1}(i', k) = \bigcup_{j=1}^n l_2^{-1}(k', j).$$

Then T contributes to the count in (13). \square

Example 2.18. In Example 2.16, the three compatible pairs of labels in (11) correspond to taking T as $(\{1, 2\}, \{3, 4\})$, $(\{1, 3\}, \{2, 4\})$, and $(\{2, 3\}, \{1, 4\})$, respectively, and the compatible pair of labels in (12) corresponds to $T = (\{1, 2\}, \{3, 4\})$.

2C. A basis for $S_F^-(n, d)$. By Theorem 2.5, a basis of $S_F^-(n, d)$ is indexed by orbits in $B(n, d) \bowtie B(n, d)/S_d$. Here \bowtie denotes transversality with respect to the sign character $\text{sgn} : S_n \rightarrow \{\pm 1\}$ (see Definition 2.4).

Lemma 2.19. A pair $(S, T) \in B(n, d)^2$ lies in $B(n, d) \bowtie B(n, d)$ if and only if $\gamma_{S,T}$ is a simple bipartite graph.

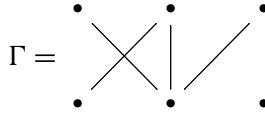
Proof. Let $S = (S_1, \dots, S_n)$ and $T = (T_1, \dots, T_n)$. If $\gamma_{S,T}$ is not simple, then there exist indices i and j such that $S_j \cap T_i$ contains at least two elements, say k and l . The transposition $(kl) \in S_d$ stabilizes (S, T) but has $\text{sgn}((kl)) = -1$, so $(S, T) \notin B(n, d) \bowtie B(n, d)$.

However, if $\gamma_{S,T}$ is simple, then the simultaneous stabilizer of S and T in S_d is trivial, so $(S, T) \in B(n, d) \bowtie B(n, d)$. \square

In order to specify a basis for $S_F^-(n, d)$ using Theorem 2.5, we need to choose a base point for each S_d -orbit in $B(n, d) \bowtie B(n, d)$. We do this using Definition 2.20.

Definition 2.20 (standard labelling of a bipartite simple graph). Given a bipartite simple graph Γ with vertex set $[n'] \coprod [n]$, label each edge by its index when the edges (i', j) are arranged in increasing lexicographic order, with priority given to the upper index, i.e., $(i', j) < (r', s)$ if either $i' < r'$ or $i' = r'$ and $j < s$.

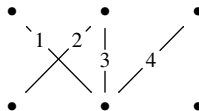
Example 2.21. Take



The edges, written in lexicographic order, are

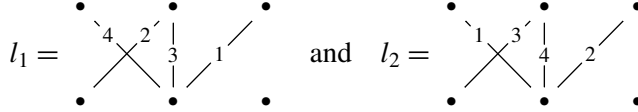
$$(1', 2), (2', 1), (2', 2), (3', 2).$$

Therefore the standard labelling is



Definition 2.22 (sign of a labelling of a bipartite simple graph). Let l_0 denote the standard labelling of a simple bipartite graph Γ on $[n'] \coprod [n]$. Let $l : [d] \rightarrow [n'] \times [n]$ be a labelling of Γ (see Definition 2.9). The sign $\epsilon(\Gamma, l)$ of l is the sign of the permutation on $[d]$ which takes $l_0(i)$ to $l(i)$ for each i .

Example 2.23. For the graph from Example 2.21, the labellings



give rise to permutations 4231 and 1342 respectively, so that $\epsilon(\Gamma, l_1) = -1$, and $\epsilon(\Gamma, l_2) = +1$.

Recall, from Section 2B, that $\gamma_{S,T}$ is a labelled bipartite graph associated to $(S, T) \in B(n, d) \times B(n, d)$, whose underlying unlabelled graph is denoted by $\Gamma_{S,T}$. Let $l_{S,T} : [d] \rightarrow [n] \times [n']$ denote the labelling of $\gamma_{S,T}$, and write $\epsilon(\gamma_{S,T})$ for $\epsilon(\Gamma_{S,T}, l_{S,T})$.

Theorem 2.24. Let $N(n, d)$ denote the set of all bipartite simple graphs with vertex set $[n'] \coprod [n]$ and d edges. For each $\Gamma \in N(n, d)$, define $\zeta_\Gamma \in S_F^-(n, d) = \text{Hom}_{S_d}(F[B(n, d)], F[B(n, d)] \otimes \text{sgn})$ by

$$\zeta_\Gamma f(S) = \sum_{\{T \mid \Gamma_{S,T} = \Gamma\}} \epsilon(\gamma_{S,T}) f(T).$$

The set

$$\{\zeta_\Gamma \mid \Gamma \in N(n, d)\}$$

forms a basis of $S_F^-(n, d)$.

Proof. Recall that we choose the pair (S_0, T_0) corresponding to the standard labelling l_0 of Γ as the base point of the orbit associated to Γ . A pair (S, T) is in the orbit of (S_0, T_0) if and only if $\Gamma_{S,T} = \Gamma$. And the sign of the permutation $w \in S_d$ such that $S = S_0.w$ and $T = T_0.w$ is the sign of the labelled bipartite graph $\gamma_{S,T}$. So the integral kernel κ_Γ of the operator ζ_Γ is

$$\kappa_\Gamma(S, T) = \begin{cases} \epsilon(\gamma_{S,T}) & \text{if } \Gamma_{S,T} = \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

So the theorem follows from Theorem 2.5. □

Theorem 2.14 tells us how to multiply two elements of the subalgebra $S_F(n, d)$ of $AS_F(n, d)$. The remaining structure constants are given by the following theorem.

Theorem 2.25. The remaining structure constants are given as follows:

(2.25.a) Given $\Gamma_1 \in M(n, d)$ and $\Gamma_2 \in N(n, d)$,

$$\xi_{\Gamma_1} \zeta_{\Gamma_2} = \sum_{\Gamma \in N(n, d)} c_{\Gamma_1 \Gamma_2}^{\Gamma} \zeta_{\Gamma},$$

where

$$c_{\Gamma_1 \Gamma_2}^{\Gamma} = \sum_{l_1, l_2} \epsilon(\Gamma_2, l_2),$$

and the sum runs over all labellings l_1 and l_2 of Γ_1 and Γ_2 , respectively, that are compatible with the standard labelling l of Γ .

(2.25.b) Given $\Gamma_1 \in N(n, d)$ and $\Gamma_2 \in M(n, d)$,

$$\zeta_{\Gamma_1} \xi_{\Gamma_2} = \sum_{\Gamma \in N(n, d)} c_{\Gamma_1 \Gamma_2}^{\Gamma} \zeta_{\Gamma},$$

where

$$c_{\Gamma_1 \Gamma_2}^{\Gamma} = \sum_{l_1, l_2} \epsilon(\Gamma_1, l_1),$$

and the sum runs over all labellings l_1 and l_2 of Γ_1 and Γ_2 , respectively, that are compatible with the standard labelling l of Γ .

(2.25.c) Given $\Gamma_1 \in N(n, d)$ and $\Gamma_2 \in N(n, d)$,

$$\zeta_{\Gamma_1} \zeta_{\Gamma_2} = \sum_{\Gamma \in M(n, d)} c_{\Gamma_1 \Gamma_2}^{\Gamma} \xi_{\Gamma},$$

where

$$c_{\Gamma_1 \Gamma_2}^{\Gamma} = \sum_{l_1, l_2} \epsilon(\Gamma_1, l_1) \epsilon(\Gamma_2, l_2),$$

and the sum runs over all labellings l_1 and l_2 of Γ_1 and Γ_2 , respectively, that are compatible with a fixed labelling l of Γ .

Proof. Given a labelling l of Γ , construct S and U in $B(n, d)$ as in the proof of Theorem 2.14. Define $\kappa_{\Gamma_2} : B(n, d) \times B(n, d) \rightarrow F$ by

$$\kappa_{\Gamma_2}(T, U) = \epsilon(\gamma_{T, U}).$$

Then ζ_{Γ_2} is the integral operator $\xi_{\kappa_{\Gamma_2}}$, as in (8). Then, by (9), the structure constant in (2.25.a) of the theorem is given by

$$c_{\Gamma_1 \Gamma_2}^{\Gamma} = \sum_T \zeta_{\Gamma_2}(T, U),$$

where the sum runs over all $T \in B(n, d)$ such that $\Gamma_{S, T} = \Gamma_1$ and $\Gamma_{T, U} = \Gamma_2$. Defining labellings l_1 and l_2 of Γ_1 and Γ_2 as in the proof of Theorem 2.14, we find that $\zeta_{\Gamma_2}(T, U) = \epsilon(\Gamma_2, l_2)$, proving (2.25.a). The proofs of the remaining assertions are similar. \square

Definition 2.26. Given $\Gamma \in M(n, d) \sqcup N(n, d)$, we define Γ^* to be the horizontal reflection of Γ , i.e., i' is connected to j in Γ^* if and only if j' is connected to i in Γ . The operation $*$ on the set $M(n, d) \sqcup N(n, d)$ is an involution.

Lemma 2.27. For every $\Gamma \in N(n, d)$, let l_0 denote its standard labelling. Let l_0^* denote the labelling of Γ^* given by $l_0^*(i', j) = l_0(j', i)$. Then the linear map $AS_F(n, d) \rightarrow AS_F(n, d)$ defined by

$$\begin{aligned}\xi_\Gamma &\mapsto \xi_{\Gamma^*} \text{ for } \Gamma \in M(n, d), \\ \zeta_\Gamma &\mapsto \epsilon(\Gamma^*, l_0^*)\zeta_{\Gamma^*} \text{ for } \Gamma \in N(n, d)\end{aligned}$$

is an anti-involution of $AS_F(n, d)$.

Remark 2.28. The above involution, when restricted to the Schur algebra, is the same as the one described by Green [2007, Section 2.7].

Proof. We show that the linear map in Lemma 2.27 is the same as the anti-involution in Lemma 2.7 with $X = B(n, d)$ and $G = A_d$.

For $\Gamma \in M(n, d)$, ξ_Γ is the integral operator with kernel

$$\kappa_\Gamma(S, T) = \begin{cases} 1 & \text{if } \Gamma_{S,T} = \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Gamma_{T,S} = \Gamma_{S,T}^*$, $\kappa_\Gamma^* = \kappa_{\Gamma^*}$.

For $\Gamma \in N(n, d)$, ζ_Γ is the integral operator with kernel

$$\kappa_\Gamma(S, T) = \begin{cases} \epsilon(\gamma_{S,T}) & \text{if } \Gamma_{S,T} = \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $\gamma_{S,T} = (\Gamma, l_0)$, then $\gamma_{T,S} = (\Gamma^*, l_0^*)$. Therefore,

$$\begin{aligned}\kappa_{\Gamma^*}(T, S) &= \epsilon(\gamma_{T,S}) \\ &= \epsilon(\Gamma^*, l_0^*)\kappa_\Gamma(S, T).\end{aligned}$$

So the kernels κ_{Γ^*} and $\epsilon(\Gamma^*, l_0^*)\kappa_\Gamma^*$ coincide at (T, S) , and hence on its entire S_d -orbit in $B(n, d)$. \square

We illustrate the above results with an example that will be used in the proof of Lemma 4.1.

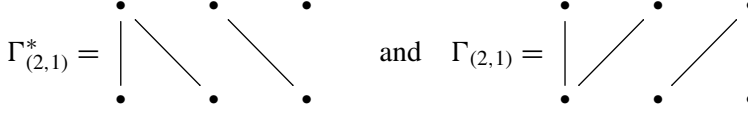
Example 2.29. Recall that $\Lambda(n, d)$ denotes the set of all weak compositions of d with at most n parts. For $\lambda \in \Lambda(n, d)$ with $n \geq d$, let $\Gamma_\lambda \in N(n, d)$ denote the bipartite graph where i' is connected to j if

$$\lambda_1 + \cdots + \lambda_{i'-1} < j \leq \lambda_1 + \cdots + \lambda_{i'}.$$

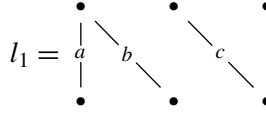
Then we have

$$\zeta_{\Gamma_\lambda} \zeta_{\Gamma_\lambda^*} = \lambda_1! \cdots \lambda_n! \xi_{\Gamma_\lambda^0},$$

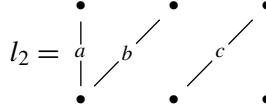
where $\Gamma_\lambda^0 \in M(n, d)$ is the bipartite multigraph where i' is connected to i by λ_i edges. For example,



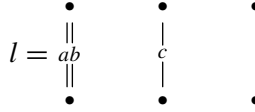
For a labelling



of $\Gamma_{(2,1)}^*$ only the labelling



of $\Gamma_{(2,1)}$, and the labelling



of $\Gamma_{(2,1)}^0$ are such that (l_1, l_2) are compatible with l . Moreover, $\epsilon(\Gamma_{(2,1)}^*, l_1) = \epsilon(\Gamma_{(2,1)}, l_2)$. Interchanging the labels a and b in l_1 and l_2 , respectively, gives another pair of labels compatible with l , so that $\zeta_{\Gamma_\lambda} \zeta_{\Gamma_\lambda^*} = 2\xi_{\Gamma_\lambda^0}$.

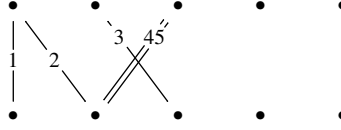
The remaining results in this section help us understand the structure of $S_F^-(n, d)$ as an $S_F(n, d)$ -module. Some of them will play an important role in understanding Koszul duality (Section 4).

The notion of standard labelling (Definition 2.20) of graphs in $N(n, d)$ can be extended to graphs in $M(n, d)$ as follows: when an edge (i', j) occurs with multiplicity m , it is simply listed m times when the edges are arranged in lexicographic order with priority given to the upper index. Example 2.10 is the standard labelling of its underlying graph.

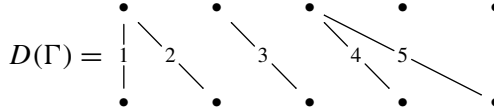
Definition 2.30. For $n \geq d$, define the following simple bipartite graphs associated to $\Gamma \in M(n, d)$:

- (2.30.a) Let $D(\Gamma) \in N(n, d)$ be the graph with edges (i', s) for every edge (i', j) with label s under the standard labelling of Γ .
- (2.30.b) Let $U(\Gamma) \in N(n, d)$ be the graph with edges (s', j) for every edge (i', j) with label s under the standard labelling of Γ .

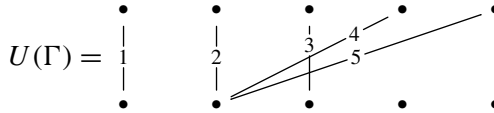
Example 2.31. Let $n = 5$, $d = 5$, and Γ (with its standard labelling) be given by



Then



and



The significance of the elements $U(\Gamma)$ and $D(\Gamma)$, for $\Gamma \in M(n, d)$, is elaborated in the following lemmas.

Lemma 2.32. Let $n \geq d$ and $\Gamma \in N(n, d)$. Then $\zeta_\Gamma = \xi_{U(\Gamma)} \zeta_{\Gamma_{\lambda_0}} \xi_{D(\Gamma)}$, where $\lambda_0 = (1^d, 0^{n-d}) \in \Lambda(n, d)$. Consequently, $S_F^-(n, d)$ is a cyclic $(S_F(n, d), S_F(n, d))$ -bimodule.

Proof. This can be done in two steps. Firstly, $\zeta_{\Gamma_{\lambda_0}} \xi_{D(\Gamma)} = \zeta_{D(\Gamma)}$, and secondly $\xi_{U(\Gamma)} \zeta_{D(\Gamma)} = \zeta_\Gamma$. We indicate the proof of the second identity (the first is similar): Let l_0 , l_1 , and l_2 be the standard labellings of Γ , $U(\Gamma)$ and $D(\Gamma)$, respectively. The labellings (l_1, l_2) of $D(\Gamma)$ and $U(\Gamma)$ are the only ones that are compatible with l_0 . This is because, for the edge (i', j) of Γ labelled s , s is the unique vertex such that i' is connected to s in $D(\Gamma)$ and s' is connected to j in $U(\Gamma)$. The identity now follows from (2.25.a). \square

Similarly, we get Lemma 2.33:

Lemma 2.33. For $n \geq d$ and $\Gamma \in N(n, d)$, we have $\zeta_\Gamma = \zeta_{U(\Gamma)} \xi_{D(\Gamma)}$ and $\zeta_\Gamma = \xi_{U(\Gamma)} \zeta_{D(\Gamma)}$.

Corollary 2.34. As a left $S_F(n, d)$ -module, $S_F^-(n, d)$ is generated by

$$\{\zeta_{\Gamma_\lambda^*} \mid \lambda \in \Lambda(n, d)\},$$

and as a right $S_F(n, d)$ -module, it is generated by $\{\zeta_{\Gamma_\lambda} \mid \lambda \in \Lambda(n, d)\}$. Here Γ_λ is the graph associated to λ in Example 2.29.

Proof. For any $\Gamma \in M(n, d)$, $D(\Gamma)$ is of the form Γ_λ^* for some $\lambda \in \Lambda(n, d)$, so the statement for left modules follows from the second identity in Lemma 2.33. The statement for right modules follows by applying Lemma 2.27 to the first identity in Lemma 2.33. \square

Lemma 2.35. *Let $n \geq d$ and $\Gamma \in M(n, d)$. Then, for $\Gamma' \in M(n, d)$, the structure constant of $\zeta_{U(\Gamma)}$ in the product $\xi_{\Gamma'} \zeta_{D(\Gamma)^*}$ is $\delta_{\Gamma', \Gamma}$.*

Proof. The edge (i', j) with label s in the standard labelling of Γ gives rise to an edge (s', j) with standard label s in $U(\Gamma)$. The graph $D(\Gamma)^*$ has only one edge originating at s' , namely (s', i) . Therefore, for any compatible pair (l_1, l_2) of labellings of $D(\Gamma)^*$ and Γ' , this edge must have label s . Thus Γ' must have an edge (i', j) labelled s . In other words, $\Gamma' = \Gamma$ and l_2 is its standard labelling. \square

3. Abstract Koszul duality

3A. The algebra. Recall [Bourbaki 1974, Chapter III, Section 3.1] that a $\mathbf{Z}/2\mathbf{Z}$ grading on a ring AS is a decomposition $AS = S \oplus S^-$ into additive subgroups such that S is a subring, S^- is closed under left and right multiplication by elements of S , and for any $\alpha, \beta \in S^-$, $\alpha\beta \in S$. This $\mathbf{Z}/2\mathbf{Z}$ -grading gives rise to

- (a) an (S, S) -bimodule structure on S^- ,
- (b) and an (S, S) -bimodule homomorphism $\phi : S^- \otimes_S S^- \rightarrow S$ (induced by the S -balanced bilinear map $(\alpha, \beta) \mapsto \alpha\beta$ for $\alpha, \beta \in S^-$).

Example 3.1. We may take $AS = AS_F(n, d) = \text{End}_{A_d}((F^n)^{\otimes d})$ and $S = S_F(n, d) = \text{End}_{S_d}((F^n)^{\otimes d})$, for any field F with characteristic different from 2.

3B. Modules. Let M be an AS -module. The AS -module structure can be viewed as a linear map:

$$AS \otimes_{\mathbf{Z}} M = (S \oplus S^-) \otimes_{\mathbf{Z}} M = (S \otimes_{\mathbf{Z}} M) \oplus (S^- \otimes_{\mathbf{Z}} M) \rightarrow M.$$

So M is an S -module, and restriction of the module action $AS \otimes_{\mathbf{Z}} M \rightarrow M$ to $S^- \otimes_{\mathbf{Z}} M$ induces an S -module homomorphism

$$(14) \quad \theta_M : S^- \otimes_S M \rightarrow M.$$

Furthermore, this homomorphism θ_M has the property that the diagram

$$(15) \quad \begin{array}{ccc} S^- \otimes_S S^- \otimes_S M & \xrightarrow{\phi \otimes \text{id}_M} & S \otimes_S M \\ \text{id}_{S^-} \otimes \theta_M \downarrow & & \parallel \\ S^- \otimes_S M & \xrightarrow{\theta_M} & M \end{array}$$

commutes.

Definition 3.2. Given an (S, S) -bimodule S^- and an (S, S) -bimodule homomorphism $\phi : S^- \otimes_S S^- \rightarrow S$, for an S -module N , an S -module homomorphism

$\theta : S^- \otimes_S N \rightarrow N$ is said to be compatible with ϕ if the diagram

$$(16) \quad \begin{array}{ccc} S^- \otimes_S S^- \otimes_S N & \xrightarrow{\phi \otimes \text{id}_N} & S \otimes_S N \\ \text{id}_{S^-} \otimes \theta \downarrow & & \parallel \\ S^- \otimes_S N & \xrightarrow{\theta} & N \end{array}$$

commutes.

3C. Duality. Let $AS = S \oplus S^-$ be as before. Define a functor $D : S\text{-Mod} \rightarrow S\text{-Mod}$ by

$$D(M) = S^- \otimes_S M,$$

for every S -module M . Given S -modules M and N , and an S -module homomorphism $f : M \rightarrow N$, let $D(f) = \text{id}_{S^-} \otimes f : D(M) \rightarrow D(N)$. We call the resulting functor $D : S\text{-Mod} \rightarrow S\text{-Mod}$ an *abstract Koszul duality functor*. In Section 4 it will be shown that, in the setting of Example 3.1 (the alternating Schur algebra), abstract Koszul duality is essentially the Koszul duality functor of Krause [2013].

The commutative diagram (16), defining the compatibility of θ with ϕ , can be rewritten in terms of abstract Koszul duality as

$$(17) \quad \begin{array}{ccc} D^2(N) & \xrightarrow{\phi \otimes \text{id}_N} & N \\ D(\theta) \downarrow & \nearrow \theta & \\ D(N) & & \end{array}$$

Definition 3.3. Given an (S, S) -bimodule S^- and an (S, S) -bimodule homomorphism $\phi : S^- \otimes_S S^- \rightarrow S$, let $(S, \phi)\text{-Mod}$ denote the category whose objects are pairs (N, θ) , where N is an S -module, and $\theta : D(N) \rightarrow N$ is compatible with ϕ . A morphism $(N, \theta) \rightarrow (N', \theta')$ is an S -module homomorphism $f : N \rightarrow N'$ such that the diagram

$$\begin{array}{ccc} D(N) & \xrightarrow{\theta} & N \\ D(f) \downarrow & & \downarrow f \\ D(N') & \xrightarrow{\theta'} & N' \end{array}$$

commutes.

Theorem 3.4. Given an AS -module M , let θ_M be as in (14). Then $M \mapsto (M, \theta_M)$ is an isomorphism of categories $AS\text{-Mod} \rightarrow (S, \phi)\text{-Mod}$.

Proof. Given an object (N, θ) in $(S, \phi)\text{-Mod}$, the compatibility of θ with ϕ allows the S -module structure on N to be extended to an AS -module structure. This constructs the inverse of the functor in the theorem. \square

Given an S -module N , the morphism $\phi : S^- \otimes_S S^- \rightarrow S$ gives rise to a natural transformation $\eta : D^2 \rightarrow \text{id}_{S\text{-Mod}}$, defined as the composition

$$(18) \quad \begin{array}{ccc} D^2(N) & \xrightarrow{\phi \otimes \text{id}_N} & S \otimes_S N \\ & \searrow \eta_N & \parallel \\ & & N \end{array}$$

Theorem 3.5. *Let AS , S , S^- and ϕ be as in Section 3A. The following are equivalent:*

- (3.5.a) *The map $\phi : S^- \otimes_S S^- \rightarrow S$ is an isomorphism.*
- (3.5.b) *The natural transformation $\eta : D^2 \rightarrow \text{id}_{S\text{-Mod}}$ is a natural isomorphism.*
- (3.5.c) *For every object (N, θ) in $(S, \phi)\text{-Mod}$, $\theta : S^- \otimes_S N \rightarrow N$ is an isomorphism of S -modules.*

Proof. To see that (3.5.a) implies (3.5.b), observe from the diagram (18) that if ϕ is an isomorphism, then η_N is an isomorphism for every N . It follows that η is a natural isomorphism. For the converse, taking $N = S$, the commutativity of (18) shows that ϕ is an isomorphism.

To see that (3.5.a) implies (3.5.c), note that the commutativity of (16) implies that, if ϕ is an isomorphism, then θ is an epimorphism, and $\text{id}_{S^-} \otimes \theta$ is a monomorphism. Since tensoring is a right-exact functor, it follows that $\text{id}_{S^-} \otimes \theta$ is also an epimorphism, and hence an isomorphism. Since $\phi \otimes \text{id}_N$ is also an isomorphism, the inverse of θ can be constructed by reversing the arrows in (16). For the converse, just take $N = S$ in (3.5.c). \square

3D. Abstract Ringel duality. Let S^- be an (S, S) -bimodule. Denote the left S -module S^- by ${}_S S^-$. For a left S -module M , the homomorphism space $\text{Hom}_S({}_S S^-, M)$ inherits the structure of a left S -module from the right S -module structure on S^- . Motivated by [Ringel 1991, Section 6], we call the functor

$$\text{Hom}_S({}_S S^-, -) : S\text{-Mod} \rightarrow S\text{-Mod}$$

the abstract Ringel duality functor on $S\text{-Mod}$. It is clear that the abstract Koszul duality functor is the left adjoint of the abstract Ringel duality functor.

3E. Abstract simple modules. In general, it is not clear how simple AS -modules can be classified using simple S -modules and Koszul duality. In this section, we give some results in this direction. These are enough to give a complete solution in the semisimple case.

Let M be a simple S -module. We consider the following cases:

3E1. DM is isomorphic to M . If $\eta_M : D^2M \rightarrow M$ is zero, then $(M, 0)$ (where 0 is the zero map from $DM \rightarrow M$) is the unique ϕ -compatible morphism. Otherwise, any nonzero morphism $\theta : DM \rightarrow M$ is an isomorphism. Schur's lemma implies that $\theta \circ D\theta = a\eta_M$ for some $a \in (\text{End}_S M)^*$ (the multiplicative group of nonzero elements in the division algebra $\text{End}_S M$). If a has a square root in $(\text{End}_S M)^*$, then θ can be normalized to make it a ϕ -compatible morphism. Moreover, after normalization, $\pm\theta$ are two ϕ -compatible morphisms, leading to two nonisomorphic AS-modules. Also, in this case, $(M, \pm\theta)$ are simple, because their restrictions to S are simple. On the other hand, if a does not have a square root in $\text{End}_S(M)$, then there is no simple AS-module whose restriction to S is isomorphic to M .

3E2. $DM = 0$. In this case $(M, 0)$ is the unique AS-module whose restriction to S is isomorphic to M .

3E3. DM is simple, but not isomorphic to M , $\eta_M \neq 0$. Let $\tilde{M} = DM \oplus M$. We have $D\tilde{M} = D^2M \oplus DM$. Any morphism $\theta : D\tilde{M} \rightarrow \tilde{M}$ can be written in matrix form as

$$\theta = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where $X : D^2M \rightarrow DM$, $Y : DM \rightarrow DM$, $Z : D^2M \rightarrow M$, and $W : DM \rightarrow M$. By Schur's lemma, $W = 0$. The compatibility of θ with ϕ becomes

$$\begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix} \begin{pmatrix} DX & DY \\ DZ & 0 \end{pmatrix} = \begin{pmatrix} \eta_{DM} & 0 \\ 0 & \eta_M \end{pmatrix}.$$

Multiplying out the left-hand side gives

$$\begin{pmatrix} XDX + YDZ & XDY \\ ZDX & ZDY \end{pmatrix} = \begin{pmatrix} \eta_{DM} & 0 \\ 0 & \eta_M \end{pmatrix}.$$

Since $\eta_M \neq 0$, $DY \neq 0$, and so $Y \neq 0$. Since DM is simple, by Schur's lemma, Y is invertible. Since D is a functor, DY is also invertible. Hence, equality of top right entries implies that $X = 0$. Moreover, $Z = \eta_M DY^{-1}$. In other words, θ is of the form

$$\theta_Y = \begin{pmatrix} 0 & Y \\ \eta_M DY^{-1} & 0 \end{pmatrix}.$$

Lemma 3.6. For all $Y, Y' \in (\text{End}_S M)^*$, $\text{Hom}_{\text{AS}}((\tilde{M}, \theta_Y), (\tilde{M}, \theta_{Y'}))$ is nonzero, and $\text{End}_{\text{AS}}(\tilde{M}, \theta_Y)$ is a division ring.

Proof. Any AS-module morphism $(\tilde{M}, \theta_Y) \rightarrow (\tilde{M}, \theta_{Y'})$ can be written in matrix form as

$$\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix}, \text{ where } X \in \text{End}_S DM \text{ and } W \in \text{End}_S M,$$

and must satisfy

$$\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} 0 & Y \\ \eta_M DY^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & Y' \\ \eta_M DY'^{-1} & 0 \end{pmatrix} \begin{pmatrix} DX & 0 \\ 0 & DW \end{pmatrix}.$$

We get $XY = Y'DW$ and $W\eta_M DY^{-1} = \eta_M DY'^{-1}DX$. Taking $W = \text{id}_M$ and $X = Y'Y^{-1}$ gives a nonzero element of $\text{Hom}_{\text{AS}}((\tilde{M}, \theta_Y), (\tilde{M}, \theta_{Y'}))$. When $Y = Y'$, then we have $X = YDWY^{-1}$, so that X is nonzero (and hence invertible) if and only if W is. It follows that every nonzero element of $\text{End}_{\text{AS}}(\tilde{M}, \theta_Y)$ is invertible. \square

Lemma 3.7. *Let M be an S -module. Then the AS-module $\text{AS} \otimes_S M$ is isomorphic to $(DM \oplus M, \theta)$, where θ is given by the matrix*

$$\theta = \begin{pmatrix} 0 & \text{id}_{DM} \\ \eta_M & 0 \end{pmatrix}.$$

Proof. Note that

$$\text{AS} \otimes_S M = (S^- \otimes_S M) \oplus (S \otimes_S M) = DM \oplus M.$$

The map θ comes from the action of S^- on this AS-module, which gives $\eta_M : D^2M \rightarrow M$ on the first summand, and $\text{id}_{DM} : DM \rightarrow DM$ on the second. \square

Theorem 3.8. *The AS-module $(DM \oplus M, \theta_Y)$ defined above is isomorphic to $\text{AS} \otimes_S M$ for every $Y \in (\text{End}_S DM)^*$. Consequently, whenever M and DM are simple, nonisomorphic S -modules, and $\eta_M \neq 0$, then $\text{AS} \otimes_S M$ is, up to isomorphism, the unique simple AS-module whose restriction to S contains M .*

Proof. To see that $\text{AS} \otimes_S M = DM \oplus M$ is simple, note that its only proper nontrivial S -submodules are M and DM . But M is not AS-invariant because S^- maps M onto DM . Also, DM is not AS-invariant, because S^- maps DM onto D^2M . Since $\eta_M \neq 0$, D^2M cannot be contained in DM . The theorem now follows from Lemma 3.6. \square

3E4. *The case where ϕ is an isomorphism.* When $\phi : S^- \otimes_S S^- \rightarrow S$ is an isomorphism, the preceding results, using Theorem 3.5, can be summarized in the following form:

Theorem 3.9. *Suppose that AS is endowed with a $\mathbb{Z}/2\mathbb{Z}$ -grading $\text{AS} = S \oplus S^-$, and $\phi : S^- \otimes_S S^- \rightarrow S$ (as defined in Section 3A) is an isomorphism. Let M be a simple S -module. Then:*

- (3.9.a) *Suppose there exists an isomorphism $\theta : DM \rightarrow M$. Then θ can be scaled to become compatible with ϕ . There exist at most two isomorphism classes of simple AS-modules $(M, \pm\theta)$ whose restrictions to S are isomorphic to M . If $(\text{End}_S M)^*$ is a 2-divisible group, then these two classes always exist.*

(3.9.b) *Otherwise, up to isomorphism, $\text{AS} \otimes_S M$ is the unique simple AS-module whose restriction to S contains M as a submodule. Also, $\text{AS} \otimes_S M$ and $\text{AS} \otimes_S DM$ are isomorphic as AS-modules.*

Corollary 3.10. *Suppose F is an algebraically closed field of characteristic different from 2. Let $\text{AS} = S \oplus S^-$ be a $\mathbb{Z}/2\mathbb{Z}$ -graded F -algebra. A complete set of isomorphism classes of simple AS-modules is given by*

(3.10.c) $(M, \pm\theta)$ (defined in Section 3E1), as M runs over isomorphism classes of simple S -modules such that DM is isomorphic to M ,

(3.10.d) $\text{AS} \otimes_S M$, as M runs over isomorphism classes of all unordered pairs $\{M, M'\}$ of nonisomorphic mutually dual simple S -modules.

4. Koszul duality for modules over Schur algebra

In this section, let S denote the Schur algebra $S_F(n, d)$, and let S^- denote the (S, S) -bimodule $S_F^-(n, d)$. We now use our combinatorial methods from Section 2 to determine when abstract Koszul duality is an equivalence.

Lemma 4.1. *When the characteristic of F is 0 or greater than d , and $n \geq d$, the map $\phi : S^- \otimes_S S^- \rightarrow S$ is an isomorphism.*

Proof. For each $\lambda \in \Lambda(n, d)$, let $\Gamma_\lambda^0 \in M(n, d)$ be the bipartite multigraph with λ_i edges from i' to i (and no other edges), as in Example 2.29. Then

$$(19) \quad \text{id}_S = \sum_{\lambda \in \Lambda(n, d)} \xi_{\Gamma_\lambda^0}.$$

Therefore by Example 2.29,

$$(20) \quad \text{id}_S = \sum_{\lambda \in \Lambda(n, d)} \frac{1}{\lambda_1! \cdots \lambda_n!} \zeta_{\Gamma_\lambda} \zeta_{\Gamma_\lambda}^*.$$

Therefore the image of ϕ , which is a two-sided ideal of S , contains the identity element, and therefore is all of S .

The injectivity of ϕ can be proved using a dimension count. Let N^d (resp., N_d) denote the graphs in $N(n, d)$ with upper (resp., lower) degree sequence $(1^d, 0^{n-d})$. Let $\Gamma(w) \in N(n, d)$ be as in Example 2.15. For $\Gamma \in N^d$, define $\Gamma \cdot w \in N^d$ by $\xi_\Gamma \xi_{\Gamma(w)} = \xi_{\Gamma \cdot w}$. Similarly, for $\Gamma \in N_d$, define $w \cdot \Gamma \in N_d$ by $\xi_{\Gamma(w)} \xi_\Gamma = \xi_{w \cdot \Gamma}$. Consider the equivalence relation on $N^d \times N_d$ where $(\Gamma, \Gamma') \sim (\Gamma \cdot w^{-1}, w \cdot \Gamma')$ for $w \in S_d$. Let $N^d \times_{S_d} N_d$ denote the set of equivalence classes.

Now, given $(\Gamma', \Gamma'') \in N^d \times N_d$, define $\Gamma = \Phi(\Gamma', \Gamma'') \in M(n, d)$ to be the graph for which the number of edges joining (i', j) is the number of indices $1 \leq k \leq n$ such that (i', k) is an edge of Γ'' and (k', j) is an edge of Γ' . This map induces an

injective function $\bar{\Phi} : N^d \times_{S_d} N_d \rightarrow M(n, d)$. Moreover, $\Gamma = \Phi(U(\Gamma), D(\Gamma))$, so $\bar{\Phi} : N^d \times_{S_d} N_d \rightarrow M(n, d)$ is a bijection.

The elements $\zeta_{\Gamma'} \otimes \zeta_{\Gamma''}$ as Γ' and Γ'' run over $N(n, d)$, span $S^- \otimes_S S^-$. We have

$$\begin{aligned} \zeta_{\Gamma} \otimes \zeta_{\Gamma'} &= \zeta_{U(\Gamma)} \xi_{D(\Gamma)} \otimes \xi_{U(\Gamma')} \zeta_{D(\Gamma')} \quad (\text{from Lemma 2.33}) \\ &= \zeta_{U(\Gamma)} \otimes \xi_{D(\Gamma)} \xi_{U(\Gamma')} \zeta_{D(\Gamma')}. \end{aligned}$$

Now $U(\Gamma) \in N^d$ and $\xi_{D(\Gamma)} \xi_{U(\Gamma')} \zeta_{D(\Gamma')}$ lies in the span of $\zeta_{\Gamma''}$, for $\Gamma'' \in N_d$. Therefore $\zeta_{\Gamma'} \otimes \zeta_{\Gamma''}$ span $S^- \otimes_S S^-$ as $(\Gamma', \Gamma'') \in N^d \times N_d$. Moreover, $\zeta_{\Gamma' \cdot w} \otimes \zeta_{w^{-1} \cdot \Gamma''} = \zeta_{\Gamma'} \otimes \zeta_{\Gamma''}$, so $\dim S^- \otimes_S S^- \leq |N^d \times_{S_d} N_d| = |M(n, d)|$. \square

Now, using Theorem 3.5 we have established a direct combinatorial proof of the following theorem:

Theorem 4.2. *For a field F of characteristic 0 or greater than d , and $n \geq d$, the Koszul duality functor $D : \mathbf{S}\text{-Mod} \rightarrow \mathbf{S}\text{-Mod}$ is an equivalence of categories.*

4A. Strict polynomial functor. Friedlander and Suslin [1997] introduced strict polynomial functors in order to establish the finite generation of the full cohomology ring of a finite group scheme. They also showed that the strict polynomial functors of degree d unify modules over the Schur algebras $S_F(n, d)$ across all n . In this section, we briefly recall the definition of strict polynomial functors and some useful functors on the category of strict polynomial functors.

Following [Krause 2013; van der Kallen 2015], define the *Schur category* (also known as the *divided power category*) $\mathbf{\Gamma}_F^d$ as the category whose objects are finite-dimensional vector spaces over F . For objects V and W , the morphism space is

$$\text{Hom}_{\mathbf{\Gamma}_F^d}(V, W) := \text{Hom}_{S_d}(V^{\otimes d}, W^{\otimes d}).$$

The category $\text{Rep } \mathbf{\Gamma}_F^d$ of strict polynomial functors is the functor category

$$\text{Func}(\mathbf{\Gamma}_F^d, F\text{-Mod}).$$

Thus it is an abelian, complete, and cocomplete category.

Example 4.3. Let V and W be objects of $\mathbf{\Gamma}_F^d$. Some examples of strict polynomial functors are:

(4.3.a) The d -th tensor power functor $\otimes^d : \mathbf{\Gamma}_F^d \rightarrow F\text{-Mod}$. On objects, $\otimes^d(V) = V^{\otimes d}$. On the morphism space, the map

$$\text{Hom}_{S_d}(V^{\otimes d}, W^{\otimes d}) \rightarrow \text{Hom}_{S_d}(V^{\otimes d}, W^{\otimes d})$$

is the identity map.

(4.3.b) The d -th divided power functor $\Gamma^d : \Gamma_F^d \rightarrow F\text{-Mod}$. On objects $\Gamma^d(V) = (V^{\otimes d})^{S_d}$ and on the morphism space, the map

$$\text{Hom}_{\Gamma_F^d}(V, W) \rightarrow \text{Hom}_{S_d}((V^{\otimes d})^{S_d}, (W^{\otimes d})^{S_d})$$

is given by the restriction.

(4.3.c) Similarly, the d -th exterior power functor $\wedge^d : \Gamma_F^d \rightarrow F\text{-Mod}$ and the d -th symmetric power functor $\text{Sym}^d : \Gamma_F^d \rightarrow F\text{-Mod}$ are strict polynomial functors of degree d .

(4.3.d) Let U be an object in Γ_F^d . Then define $\mathbf{h}_U : \Gamma_F^d \rightarrow F\text{-Mod}$ as follows:

$$\mathbf{h}_U(W) = \text{Hom}_{\Gamma_F^d}(U, W) = \text{Hom}_{S_d}(U^{\otimes d}, W^{\otimes d}).$$

The functor $\mathbf{h}_U \in \text{Rep } \Gamma_F^d$ is called a *representable functor*. The functor $\mathbf{h} : U \mapsto \mathbf{h}_U$ is the contravariant Yoneda embedding.

(4.3.e) For any object U of Γ_F^d and any $X \in \text{Rep } \Gamma_F^d$, we define a functor $X^U : \Gamma_F^d \rightarrow F\text{-Mod}$ by

$$(21) \quad X^U(W) = X(\text{Hom}_F(U, W)).$$

When $X = \Gamma^d$, $X^U = \mathbf{h}_U$.

Given a strict polynomial functor X , $X(F^n)$ inherits the structure of an $S_F(n, d)$ -module. For every nonnegative integer n , we have the evaluation functor $\text{ev}_n : \text{Rep } \Gamma_F^d \rightarrow S_F(n, d)\text{-Mod}$ as

$$\text{ev}_n(X) = X(F^n) \quad \text{for } X \in \text{Rep } \Gamma_F^d.$$

Theorem 4.4 [Friedlander and Suslin 1997, Theorem 3.2]. *The functor $\text{ev}_n : \text{Rep } \Gamma_F^d \rightarrow S_F(n, d)\text{-Mod}$ is an equivalence of categories whenever $n \geq d$.*

4B. Koszul duality of strict polynomial functors: Krause [2013] defined an internal tensor product ($\underline{\otimes}$) on the category of strict polynomial functors of a fixed degree d . Kulkarni, Srivastava, and Subrahmanyam [Kulkarni et al. 2018], and independently, Aquilino and Reischuk [2017] showed that this internal tensor product, via the Schur functor, is related to the Kronecker tensor product of representations of the symmetric group S_d . Krause used this internal tensor product to introduce Koszul duality as the functor $(\wedge^d \underline{\otimes} -) : \text{Rep } \Gamma_F^d \rightarrow \text{Rep } \Gamma_F^d$. We can think about this functor as follows: for the representable functor $\mathbf{h}_V \in \text{Rep } \Gamma_F^d$, we have, using the notation of (21),

$$(22) \quad \wedge^d \underline{\otimes} \mathbf{h}_V = \wedge^{d, V}.$$

For arbitrary $X \in \text{Rep } \mathbf{\Gamma}_F^d$, following [Krause 2013], we exploit a theorem of Mac Lane [1998, III.7, Theorem 1], namely:

$$(23) \quad X = \text{colim}_{\mathbf{h}_V \rightarrow X} \mathbf{h}_V.$$

Using this we have

$$(24) \quad \wedge^d \underline{\otimes} X = \text{colim}_{\mathbf{h}_V \rightarrow X} \wedge^d \underline{\otimes} \mathbf{h}_V = \text{colim}_{\mathbf{h}_V \rightarrow X} \wedge^{d,V}.$$

In the following theorem, we relate the abstract Koszul duality of Schur algebra with the Koszul duality of strict polynomial functors.

Theorem 4.5. *Consider the functors*

$$\begin{aligned} (S^- \otimes_S \text{ev}_n(-)) : \text{Rep } \mathbf{\Gamma}_F^d &\rightarrow \text{S-Mod}, \\ \text{ev}_n(\wedge^d \underline{\otimes} -) : \text{Rep } \mathbf{\Gamma}_F^d &\rightarrow \text{S-Mod}. \end{aligned}$$

Then there exists a natural transformation

$$\eta : (S^- \otimes_S \text{ev}_n(-)) \rightarrow \text{ev}_n(\wedge^d \underline{\otimes} -),$$

which is an isomorphism when $n \geq d$.

Proof. Let $X = \mathbf{h}_V$. Then,

$$\begin{aligned} \text{ev}_n(\wedge^d \underline{\otimes} \mathbf{h}_V) &= \text{ev}_n(\wedge^{d,V}) && \text{(by (22))} \\ &= \wedge^{d,V}(F^n) \\ &= \wedge^d \text{Hom}_F(V, F^n) && \text{(by (21))} \\ &\simeq \text{Hom}_{S_d}(V^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn}). \end{aligned}$$

On the other hand,

$$\begin{aligned} S^- \otimes_S \text{ev}_n(\mathbf{h}_V) &= S^- \otimes_S \mathbf{h}_V(F^n) \\ &= \text{Hom}_{S_d}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn}) \otimes_S \text{Hom}_{S_d}(V^{\otimes d}, (F^n)^{\otimes d}). \end{aligned}$$

Using these identifications, $\eta_{\mathbf{h}_V}(g_1 \otimes g_2) = g_1 \circ g_2$, for $g_1 \in S^-$ and $g_2 \in \text{ev}_n(\mathbf{h}_V)$, defines an S -linear map

$$(25) \quad \eta_{\mathbf{h}_V} : S^- \otimes_S \text{ev}_n(\mathbf{h}_V) \rightarrow \text{ev}_n(\wedge^d \underline{\otimes} \mathbf{h}_V).$$

For arbitrary $X \in \text{Rep } \mathbf{\Gamma}_F^d$, we construct η_X using (23):

$$\eta_X = \text{colim}_{\mathbf{h}_V \rightarrow X} \eta_{\mathbf{h}_V}.$$

From the Yoneda lemma [Mac Lane 1998, page 59], every morphism $\mathbf{h}_V \rightarrow \mathbf{h}_W$ between the representable functors is of the form \mathbf{h}_f for a unique morphism

$f \in \text{Hom}_{F^d}(W, V)$. The following diagram commutes:

$$\begin{array}{ccc} S^- \otimes_S \text{ev}_n(\mathbf{h}_V) & \xrightarrow{\eta_{h_V}} & \text{ev}_n(\wedge^d \underline{\otimes} \mathbf{h}_V) \\ \text{id}_{S^-} \otimes \text{ev}_n(\mathbf{h}_f) \downarrow & & \downarrow \text{ev}_n(\text{id}_{\wedge^d} \underline{\otimes} \mathbf{h}_f) \\ S^- \otimes_S \text{ev}_n(\mathbf{h}_W) & \xrightarrow{\eta_W} & \text{ev}_n(\wedge^d \underline{\otimes} \mathbf{h}_V) \end{array}$$

Taking colimits then gives the naturality of η .

If $n \geq d$, \mathbf{h}_{F^n} is a small projective generator of $\text{Rep } \Gamma_F^d$, i.e., every object has a presentation by \mathbf{h}_{F^n} (see [Krause 2013]). Note that the map $\eta_{\mathbf{h}_{F^n}}$ (25) is surjective because $\eta_{\mathbf{h}_{F^n}}(f \otimes \text{id}_S) = f$ for $f \in S^-$ and hence an isomorphism because $\text{ev}_n(\wedge^d \underline{\otimes} \mathbf{h}_{F^n})$ is isomorphic to S^- . By the construction of η_X , this implies that each η_X is an isomorphism for $X \in \text{Rep } \Gamma_F^d$. \square

4C. Derived abstract Koszul duality. For a finite-dimensional associative algebra A , let $\mathcal{D}(A\text{-Mod})$ be the unbounded derived category of $A\text{-Mod}$. For an (A, A) -bimodule M , the functor $(M \otimes_A -)$ is a right exact functor so the total left derived functor $(M \otimes_A^L -) : \mathcal{D}(A\text{-Mod}) \rightarrow \mathcal{D}(A\text{-Mod})$ exists. From Happel [1987], we recall necessary and sufficient conditions for the functor $(M \otimes_A^L -)$ to be an equivalence of categories.

For each $x \in A$, let $\psi_x \in \text{End}_A(M_A)$ be defined by

$$\psi_x(y) = xy.$$

Taking x to ψ_x gives rise to a homomorphism of algebras:

$$(26) \quad \psi : A \rightarrow \text{End}_A(M_A).$$

Theorem 4.6 [Happel 1987]. *For a finite-dimensional algebra A and an (A, A) -bimodule M , the functor $(M \otimes_A^L -) : \mathcal{D}(A\text{-Mod}) \rightarrow \mathcal{D}(A\text{-Mod})$ is an equivalence of categories if and only if:*

- (4.6.a) *The module M_A admits a finite resolution by finitely generated projective right modules over A .*
- (4.6.b) *The canonical map $\psi : A \rightarrow \text{End}_A(M_A)$ is an isomorphism, and for $i \geq 1$, $\text{Ext}_A^i(M, M) = 0$.*
- (4.6.c) *There exists an exact sequence consisting of right A -modules:*

$$0 \rightarrow A \rightarrow M_1 \rightarrow \cdots \rightarrow M_l \rightarrow 0,$$

where for $1 \leq i \leq l$, M_i is a direct summand of finite direct sum of copies of M .

Theorem 4.7. *Let $A = S$ and M be the (S, S) -bimodule S^- . Then the map ψ in (26) is an isomorphism if and only if $n \geq d$.*

Remark 4.8. When $n \geq d$, it is known that ψ is an isomorphism, even for q -Schur algebras (see Donkin [1998, page 82]).

Proof. Suppose $n < d$. Consider the labelled bipartite multigraph

$$\Gamma_1 = \begin{array}{cccc} \bullet & \bullet & \dots & \bullet \\ ||| & & & \\ 12 \dots d & & & \\ ||| & & & \\ \bullet & \bullet & \dots & \bullet \end{array}$$

Let $\Gamma_2 \in N(n, d)$. Since Γ_2 is a simple bipartite graph, any labelling of Γ_2 which satisfies Definition 2.13(a) requires d balls to place into d distinct boxes out of n . This is not possible as $n < d$. Thus we get $\xi_{\Gamma_1} \zeta_{\Gamma_2} = 0$. Since ζ_{Γ} for $\Gamma \in N(n, d)$ forms a basis of S^- we get that ξ_{Γ_1} is in the kernel of ψ , and so ψ is not injective.

For the converse, suppose $n \geq d$. That the map ψ is an isomorphism is known from [Donkin 1993, Proposition 3.7], but we give a combinatorial proof here. For $\theta \in \text{End}_S(S_S^-)$, we denote the coefficient of ζ_{Γ_1} in $\theta(\zeta_{\Gamma_2})$ by $\langle \theta(\zeta_{\Gamma_2}), \zeta_{\Gamma_1} \rangle$.

To see that ψ is injective, note that any element of S is of the form $s = \sum_{\Gamma \in M(n, d)} \alpha_{\Gamma} \xi_{\Gamma}$. Now $\psi(s) = 0$ if and only if

$$\sum_{\Gamma \in M(n, d)} \alpha_{\Gamma} \xi_{\Gamma} \zeta_{\Gamma'} = 0,$$

for every $\Gamma' \in N(n, d)$. Fix $\Gamma_1 \in M(n, d)$ and let $\Gamma' = D(\Gamma_1)^*$. Then by Lemma 2.35,

$$\alpha_{\Gamma_1} = \left\langle \sum_{\Gamma \in M(n, d)} \alpha_{\Gamma} \xi_{\Gamma} \zeta_{D(\Gamma_1)^*}, \zeta_{U(\Gamma_1)} \right\rangle.$$

Thus $\alpha_{\Gamma_1} = 0$.

To see that ψ is surjective, we will show that $\dim_F \text{End}_S(S_S^-) \leq |M(n, d)|$. Firstly, by Lemma 2.33, S_S^- is generated by $G = \{\zeta_{\Gamma} \mid \Gamma \in N^d\}$. Recall that N^d denotes the set of graphs in $N(n, d)$ with upper degree sequence $(1^d, 0^{n-d})$. Therefore any $\theta \in \text{End}_S(S_S^-)$ is determined by its values on this set. Since θ is an S -module homomorphism, $\theta(\zeta_{\Gamma})$ again lies in the span of G . Therefore θ is completely determined by the values

$$\{\langle \theta(\zeta_{\Gamma}), \zeta_{\Gamma'} \rangle \mid \Gamma, \Gamma' \in N^d\}.$$

Moreover, for any $w \in S_d$,

$$\langle \theta(\zeta_{\Gamma}), \zeta_{\Gamma'} \rangle = \langle \theta(\zeta_{\Gamma \cdot w}), \zeta_{\Gamma' \cdot w} \rangle.$$

Therefore $\dim_F \text{End}_S(S_S^-) \leq |(N^d \times N^d)/S_d| = |M(n, d)|$. □

Theorem 4.9. *Let F be any field of characteristic different from 2. The functor*

$$(27) \quad (S^- \otimes_S^L -) : \mathcal{D}(S\text{-Mod}) \rightarrow \mathcal{D}(S\text{-Mod})$$

is an equivalence of categories if and only if $n \geq d$.

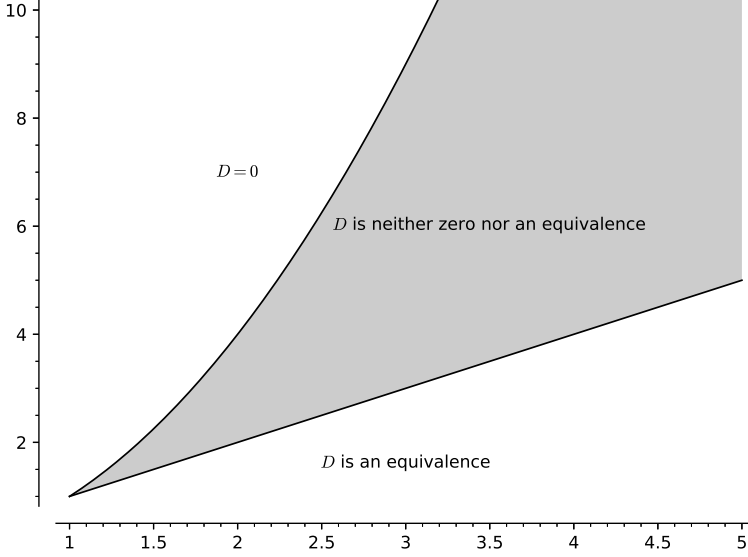


Figure 1. Dependence of derived Koszul duality on n and d .

Proof. If $n \geq d$ then from Theorem 4.5, $S^- \otimes_S \text{ev}_n(-)$ is isomorphic to $\text{ev}_n(\wedge^d \underline{\otimes} -)$. Since the functor ev_n is exact, the total left derived functors of $S^- \otimes_S \text{ev}_n(-)$ and $\text{ev}_n(\wedge^d \underline{\otimes} -)$ are isomorphic to $S^- \otimes_S^L \text{ev}_n(-)$ and $\text{ev}_n(\wedge^d \underline{\otimes}^L -)$, respectively. From [Krause 2013, Theorem 4.9], $(\wedge^d \underline{\otimes}^L -)$ is an equivalence. From Theorem 4.4 $\text{ev}_n(-)$ is an equivalence. Therefore $S^- \otimes_S^L \text{ev}_n(-)$ is an equivalence. So $\text{ev}_n(-)$ being an equivalence forces $(S^- \otimes_S^L -)$ to be an equivalence.

For the converse, suppose $(S^- \otimes_S^L -)$ is an equivalence. Then from Theorem 4.6 the map $\psi : S \rightarrow \text{End}_S(S^-)$ is an isomorphism. Theorem 4.7 now implies $n \geq d$. \square

The behavior of derived Koszul duality for different values of n and d is summarized in Figure 1.

5. Concluding remarks

5A. Toward alternating partition algebras. The centralizer algebra $\text{End}_{S_n}((F^n)^{\otimes d})$ is a quotient of the partition algebra $P_d(n)$ of [Jones 1994] and [Martin 1991]. Further restricting the action of S_n to the alternating group A_n , we get the *alternating partition algebra* $\text{AP}_d(n) = \text{End}_{A_n}((F^n)^{\otimes d})$, which from the isomorphism (2) decomposes as follows:

$$(28) \quad \text{End}_{A_n}((F^n)^{\otimes d}) = \text{End}_{S_n}((F^n)^{\otimes d}) \oplus \text{Hom}_{S_n}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn})$$

Letting $P_d^-(n) = \text{Hom}_{S_n}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn})$, $P_d^-(n)$ becomes a $(P_d(n), P_d(n))$ -bimodule by inflation. Bloss [2005] showed $P_d^-(n)$ is nonzero if and only if

$n < 2d + 2$. So we get an abstract Koszul duality

$$(\mathcal{P}_d^-(n) \otimes_{\mathcal{P}_d(n)} -)$$

on the category of modules over the partition algebra $\mathcal{P}_d(n)$ when $n < 2d + 2$. Many of the ideas and techniques in this article can be used to study bases, structure constants, and abstract Koszul duality for partition algebras. For $F = \mathbb{C}$, the dimensions of simple modules of $\mathcal{AP}_d(n)$ are given combinatorially by Benkart, Halverson, and Harman; see [Benkart et al. 2017].

5B. A diagrammatic interpretation of the Schur category. The following is one possible way to define the notion of a diagram category in the spirit of [Martin 2008].

Definition 5.1 (diagram category). A category \mathcal{C} is called a *diagram category* if there exists a sequence $\{V_n\}_{n \geq 0}$ of objects which constitute a skeleton of \mathcal{C} , and for each pair (m, n) of nonnegative integers, a class of “diagrams” $M(m, n)$, a basis

$$\mathcal{B}_{m,n} = \{\xi_\Gamma \mid \Gamma \in M(m, n)\}$$

of $\text{Hom}_{\mathcal{C}}(V_n, V_m)$, and a combinatorial rule for computing the structure constants $c_{\Gamma'\Gamma''}^\Gamma$ that are defined by

$$\xi_{\Gamma'} \circ \xi_{\Gamma''} = \sum_{\Gamma} c_{\Gamma'\Gamma''}^\Gamma \xi_\Gamma$$

for $\Gamma' \in M(l, m)$, $\Gamma'' \in M(m, n)$ and $\Gamma \in M(l, n)$.

Remark 5.2. In the examples discussed by Martin [2008], given diagrams Γ' and Γ'' , there can exist more than one diagram Γ such that $c_{\Gamma'\Gamma''}^\Gamma > 0$. This is not a requirement in the above definition.

Consider the Schur category $\mathbf{\Gamma}_F^d$ defined in Section 4A. Take $V_n = F^n$. Define $M_d(m, n)$ to be the set of all bipartite multigraphs with vertex set $[n'] \coprod [m]$ with d edges. Mimicking the discussion in Section 2B, one may endow the Schur category with the structure of a diagram category in the sense of Definition 5.1.

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
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