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CAIHUA LUO

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CAIHUA LUO

It is folklore that the Knapp–Stein dimension theorem should be extended word by word to general covering groups. But we note that such a proof does not exist in the literature. For completeness, we provide a proof of the classical Knapp–Stein dimension theorem for finite central covering groups. As an example, we obtain the R-group structure for Mp_{2n} based on Gan and Savin's work on the local theta correspondence for $(Mp_{2n}, \mathrm{SO}_{2n+1})$.

1. Introduction

Let G be a connected reductive group defined over a nonarchimedean local field F. By abuse of notation, we also write G for $G(F)$, and denote by $\tilde{G}$ the associated finite central covering group of $G(F)$ by μ_n, i.e.,

$$1 \to \mu_n \to \tilde{G} \xrightarrow{p} G(F) \to 1.$$

Representation-theoretically, one of the fundamental problems is to understand the classification of irreducible admissible representations of G. Notably, we have the diagram

$$\Pi_{s.c}(G) \overset{\text{(iii)}}{\subset} \Pi_{d.s}(G) \overset{\text{(ii)}}{\subset} \Pi_{\text{temp}}(G) \overset{\text{(i)}}{\subset} \Pi(G).$$

Here $\Pi(G)$, $\Pi_{\text{temp}}(G)$, $\Pi_{d.s}(G)$ and $\Pi_{s.c}(G)$ stand for the set of isomorphism classes of irreducible admissible representations, tempered representations, discrete series and supercuspidal representations of G, respectively. Recall that

- (i) is the Langlands classification [1989]. The covering case is established by Ban and Jantzen [2013],
- (ii) is the Knapp–Zuckerman classification [1982]. The covering case follows from [Waldspurger 2003, Proposition III.4.1] (see [Li 2012]).
- (iii) is the Moeglin–Tadić classification for classical groups [2002]. The covering case is unknown so far.

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Note that in (ii), as the normalized induced representation is unitary, we may investigate its finer structure, i.e., the so-called R-group theory (see [Knapp and Stein 1971; 1980; Silberger 1978]). Indeed, the R-group theory not only determines the decomposition of induced representations, but also plays an essential role in the endoscopy theory. In view of this, it is necessary to extend the R-group theory to covering groups. This is exactly what we will do in the paper.

In what follows, we give a rough outline of the main results. Exact definitions and notation are given in the body of the paper. For a standard parabolic $P = MN \subset G$ and $\sigma \in \Pi_{d.s}(M)$, we denote by $I_P^G(\sigma)$ the associated normalized induced representation of G, and by $W(M)$ the relative Weyl group of M in G. Let $W(\sigma) := \{\omega \in W(M) : \omega.\sigma = \sigma\}$. Fix a section of $W(M)$ in K_{good} which is a special compact open subgroup of G. For simplicity, we use the same letter w for the fixed lifting of $w \in W(M)$ if no confusion arises. For $\omega \in W(\sigma)$, we may define an unnormalized intertwining operator $\gamma(\omega) = A(\omega) \circ M(\omega, \sigma)$ on $I_P^G(\sigma)$ as in Lemma 2.2, and then define $W^0(\sigma) := \{\omega \in W(\sigma) : A(\omega) \circ M(\omega, \sigma) \text{ is a scalar}\}$. It is well known that $W^0(\sigma)$ is a normal subgroup of $W(\sigma)$, and the following exact sequence splits:

$$1 \to W^0(\sigma) \to W(\sigma) \to R(\sigma) \to 1.$$

Main Theorem (Theorem 2.4). Modifying the above notation for $\tilde{G}$, we have

$$\dim \operatorname{End}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})) = |R(\tilde{\sigma})|.$$

Classically, Knapp and Stein [1971; 1980] established the dimension theorem for tempered induced representations of semisimple Lie groups and later Silberger [1978] extended the Knapp–Stein dimension theorem to p-adic reductive groups. Notice that the strategies to prove the Knapp–Stein dimension theorem in [Silberger 1978] are as follows.

(i) The Harish-Chandra commuting algebra theorem; that is, $\operatorname{End}_G(I_P^G(\sigma)) = \operatorname{Span}\{\gamma(\omega) : \omega \in W(\sigma)\}$.

(ii) $\operatorname{End}_G(I_P^G(\sigma)) = \operatorname{Span}\{\gamma(\omega) : \omega \in R(\sigma)\}$.

(iii) The multiplicity of a given exponent in the tempered Jacquet module $J_{\bar{P}}^{\omega}(I_P^G(\sigma))$ is no greater than the cardinality of $W^0(\sigma)$. This in turn implies that

$$\dim \operatorname{End}_G(I_P^G(\sigma)) = [W(\sigma) : W^0(\sigma)] = |R(\sigma)|.$$

So basically, we adapt the same argument as in [Silberger 1978] to extend the Knapp–Stein dimension theorem for finite central covering groups. But instead of showing the multiplicity of a given exponent is bounded by $W^0(\sigma)$ in the weak Jacquet module $J_{\bar{P}}^{\omega}(I_P^G(\sigma))$, we follow D. Ban's argument [2004] to directly show the linear independence property of these $\gamma(\omega)$ with $\omega \in R(\sigma)$. Herein, we should mention that the Knapp–Stein dimension theorem for finite central covering groups has been announced by Wen-Wei Li [2012] and D. Szpruch [2013].

2. Knapp–Stein dimension theorem

2A. ***Notation and conventions.*** In this section, we first recall some necessary definitions and properties in [Waldspurger 2003; Li 2012] for our purpose.

Recall that F is a nonarchimedean field, G is a connected reductive group defined over F, and $\tilde{G}$ is the associated finite central covering group of $G(F)$ by μ_n, i.e.,

$$1 \to \mu_n \to \tilde{G} \xrightarrow{p} G(F) \to 1.$$

Fix a maximal split torus T of G and the associated minimal parabolic subgroup $P_0 = M_0 N_0$ with the Levi subgroup M_0 containing T. Denote by $W_G(T) := N_G(T)/C_G(T)$ the Weyl group of G with respect to T, and by $\Phi = \Phi(G, T)$ the set of relative roots of T in G. The choice of P_0 determines the set of relative simple roots Δ and the set of relative positive roots $\Phi^+ \subset \Phi$. If $\alpha \in \Phi^+$, we write $\alpha > 0$.

As is well known, the standard parabolic subgroups of G are uniquely determined by a subset Θ of Δ. For such a subset $\Theta \subset \Delta$, let $P_\Theta = M_\Theta N_\Theta$ be the associated standard parabolic subgroup of G with Levi subgroup $M_\Theta \supset M_0$, and $W(M_\Theta) := N_G(M)/M$ be the relative Weyl group with respect to M_Θ in G. Let T_{M_Θ} be the split component of the center of M_Θ, and $X(M_\Theta)_F := \mathrm{Hom}_{F-\mathrm{grp}}(M_\Theta, \mathbb{G}_m)$ be the group of all F-rational characters of M_Θ. Denote $\mathfrak{a}_{M_\Theta} := \mathrm{Hom}(X(M_\Theta)_F, \mathbb{R}) = \mathrm{Hom}(X(T_{M_\Theta})_F, \mathbb{R})$. Recall that in [Luo 2017] we have defined the Harish-Chandra homomorphism $H_{\tilde{M}_\Theta} : \tilde{M}_\Theta \to \mathfrak{a}_{M_\Theta}$ by $H_{M_\Theta} \circ p$, and $\tilde{M}^1_\Theta := \mathrm{Ker}(H_{\tilde{M}_\Theta})$. Denote $X(\tilde{M}_\Theta) := \mathrm{Hom}(\tilde{M}_\Theta/\tilde{M}^1_\Theta, \mathbb{C}^\times)$. As $X(\tilde{M}_\Theta) = X(M_\Theta)$, we may attach a complex algebraic variety structure on $X(\tilde{M}_\Theta)$ via the surjective homomorphism

$$\mathfrak{a}^*_{\tilde{M}_\Theta,\mathbb{C}} := X(M_\Theta)_F \bigotimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\iota} X(M_\Theta)$$

given by $\chi \otimes s \mapsto |\chi(\cdot)|^s$. Notice that $\mathrm{Ker}(\iota)$ is of the form $2\pi i/(\log q)L$ with L a lattice in $X(M_\Theta)_F \otimes_{\mathbb{Z}} \mathbb{Q}$; it does make sense to define the notion of real part $\mathrm{Re}(\chi)$ of $\chi \in X(M_\Theta)$. Denote $\mathrm{Im}\, X(\tilde{M}_\Theta) := \mathrm{Im}\, X(M_\Theta) = \{\chi \in X(M_\Theta) : \mathrm{Re}(\chi) = 0\}$.

Fix the canonical section of unipotent elements in G to $\tilde{G}$ as in [Mœglin and Waldspurger 1995, Appendix I] or [Li 2014, Proposition 2.2.1]. By abuse of notation, for a unipotent subgroup $N \subset G$, we also write N for its canonical section in $\tilde{G}$. For a standard parabolic subgroup $P = MN$ of G, let $\tilde{P} = \tilde{M}N$ be the preimage of P in $\tilde{G}$. Denote by $\Pi(\tilde{M})$ (resp. Π_2, $\Pi_{\mathrm{temp}}(\tilde{M})$) the set of isomorphism classes of genuine irreducible admissible (resp. discrete series, tempered) representations of $\tilde{M}$. Notice that the complex torus $X(\tilde{M})$ acts naturally on $\Pi(\tilde{M})$ by $(\chi, \tilde{\sigma}) \mapsto \tilde{\sigma} \otimes \chi$, where $\chi \in X(\tilde{M})$ and $\tilde{\sigma} \in \Pi(\tilde{M})$. The induced action of $\mathrm{Im}\, X(\tilde{M})$ preserves $\Pi_2(\tilde{M})$. For an orbit $\mathcal{O}$ in $\Pi_2(\tilde{M})$ under the action of $\mathrm{Im}(X(\tilde{M}))$, we may chose a base point $\tilde{\sigma} \in \mathcal{O}$, and then furnish the orbit with $\mathcal{O}$ a C^∞ variety structure via the isomorphism

$$\mathrm{Im}\, X(\tilde{M})/\,\mathrm{Stab}_{\mathrm{Im}\, X(\tilde{M})}(\tilde{\sigma}) \xrightarrow{\sim} \mathcal{O} : \chi \mapsto \tilde{\sigma} \otimes \chi.$$

Here $\mathrm{Stab}_{\mathrm{Im}\,X(\tilde{M})}(\tilde{\sigma})$ is a finite group. Analogously, we may attach a complex algebraic variety structure on an orbit $\mathcal{O}_{\mathbb{C}}$ for the action of $X(\tilde{M})$ on $\Pi(\tilde{M})$. In this case, we may talk about C^{∞}, regular and rational functions on $\mathcal{O}_{\mathbb{C}}$. For a genuine discrete series representation $\tilde{\sigma} \in \Pi_2(\tilde{M})$, we define the action of $W(M)$ on $\tilde{\sigma} \in \Pi_2(\tilde{M})$ and a parabolic subgroup $\tilde{P} = \tilde{M}N$ by

$$w.\tilde{\sigma}(\tilde{m}) := \tilde{\sigma}(\tilde{w}^{-1}\tilde{m}\tilde{w})$$

and $w.\tilde{P} = \tilde{w}\tilde{P}\tilde{w}^{-1}$ for $\tilde{w} \in \tilde{G}$ a representative of $w \in W(M)$ and any $\tilde{m} \in \tilde{M}$. We set $W(\tilde{\sigma}) := \{\omega \in W(M) : \tilde{w}.\tilde{\sigma} = \tilde{\sigma}\}$. In order to introduce the Knapp–Stein dimension theorem for finite central covering groups, we would like to summarize some necessary definitions and the associated properties in [Waldspurger 2003; Li 2012; Ban and Jantzen 2013] which are rather standard and will be needed later on.

Bruhat decomposition. (see [Ban and Jantzen 2013, Lemma 2.6]): For $\Theta \subset \Delta$, we attach a parabolic subgroup $P_\Theta = M_\Theta N_\Theta$ of G and the associated $\tilde{P}_\Theta = \tilde{M}_\Theta N$ of $\tilde{G}$. Denote

$${}^{P_\Theta}W^{P_\Theta} := W_{M_\Theta}(T)\backslash W_G(T)/W_{M_\Theta}(T) = \{\omega \in W_G(T) : \omega^{-1}.\Theta \subset \Phi^+,\ \omega.\Theta \subset \Phi^+\}.$$

Then we have the following decomposition of $\tilde{G}$:

$$\tilde{G} = \bigsqcup_{\omega \in {}^{P_\Theta}W^{P_\Theta}} \tilde{P}_\Theta \tilde{\omega} \tilde{P}_\Theta,$$

where $\tilde{\omega}$ is an arbitrary lifting of ω in $\tilde{G}$.

Bernstein–Zelevinsky geometric lemma (see [Ban and Jantzen 2013, Proposition 3.3] or [Waldspurger 2003, Section I.3]). Let $P = MU$ and $Q = LV$ be standard parabolic subgroups of G. For $\tilde{\sigma} \in \Pi(\tilde{M})$, we have, in the Grothendieck group,

$$J_{\tilde{Q}} \circ I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}) = \sum_{\omega \in {}^{Q}W^{P}} I^{\tilde{L}}_{\tilde{L}\cap\tilde{\omega}.\tilde{P}} \circ \tilde{\omega} \circ J_{\tilde{M}\cap\tilde{\omega}^{-1}.\tilde{Q}}(\tilde{\sigma}).$$

Here $J_{\tilde{Q}}$ stands for the normalized Jacquet functor with respect to $\tilde{Q}$, and $I_{\tilde{P}}^{\tilde{G}}$ for the normalized induced representation with respect to $\tilde{P}$. In particular, for $\tilde{\sigma} \in \Pi_2(\tilde{M})$, the geometric lemma gives rise to

$$\dim \mathrm{End}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})) \leq |W(M)|.$$

Intertwining operator. Fix two semistandard parabolic subgroups $P_1 = MU_1$ and $P_2 = MU_2$ of G, and $(\tilde{\sigma}, V) \in \Pi(\tilde{M})$. Under some continuation on $\mathcal{O}_{\mathbb{C}} \ni \tilde{\sigma}$ (see [Waldspurger 2003; Li 2012]), we may define an intertwining operator $J_{\tilde{P}_2|\tilde{P}_1}(\tilde{\sigma})$ as

$$I_{\tilde{P}_1}^{\tilde{G}}(\tilde{\sigma}) \xrightarrow{J_{\tilde{P}_2|\tilde{P}_1}(\tilde{\sigma})} I_{\tilde{P}_2}^{\tilde{G}}(\tilde{\sigma}) : f(\tilde{g}) \mapsto J_{\tilde{P}_2|\tilde{P}_1}(\tilde{\sigma})(f)(\tilde{g}) = \int_{U_1\cap U_2\backslash U_2} f(u\tilde{g})\, du.$$

For later use, we mention the following properties of $J_{\tilde{P}_2|\tilde{P}_1}$ in [Waldspurger 2003, Section IV.3]:

• (Plancherel measure) For $\tilde{\sigma} \in \Pi(\tilde{M})$, and a semistandard parabolic $P = MN$ of G, denote by $\overline{P} = M\overline{N}$ the unique opposite parabolic subgroup of P in G. By the generic irreducibility property of $\mathrm{Ind}(\tilde{\sigma})$ (see [Renard 2010, Section VI.8.5]), we may define

$$j_{\tilde{P}}(\tilde{\sigma}) := J_{\tilde{P}|\bar{\tilde{P}}}(\tilde{\sigma}) J_{\bar{\tilde{P}}|\tilde{P}}(\tilde{\sigma})$$

as a scalar which does not depend on $P \supset M$, and denoted by $j(\tilde{\sigma})$. For simplicity, for $\tilde{\sigma} \in \Pi_2(\tilde{M})$, we define the Plancherel measure attached to $\tilde{\sigma}$ as $\mu^{\tilde{G}}(\tilde{\sigma}) := j(\tilde{\sigma})^{-1}$. Then we have

- $\mu^{\tilde{G}}(\tilde{\sigma}) \geq 0$;
- $\mu^{\tilde{G}}(\tilde{\sigma}) = \prod_\alpha \mu^{\tilde{M}_\alpha}(\tilde{\sigma})$, where α runs over all the reduced roots of T_M, up to sign.
- $\mu^{\tilde{G}}(\tilde{\omega}.\tilde{\sigma}) = \mu^{\tilde{G}}(\tilde{\sigma})$, where $\omega \in W_G(T)$.
- $\mu^{\tilde{G}}(\tilde{\sigma}) = \mu^{\tilde{G}}(\check{\tilde{\sigma}})$. Here $\check{\tilde{\sigma}}$ is the contragredient of $\tilde{\sigma}$.

• (associativity) For $P_i = MU_i,\ i = 1, 2, 3$,

$$J_{\tilde{P}_3|\tilde{P}_2}(\tilde{\sigma}) \cdot J_{\tilde{P}_2|\tilde{P}_1}(\tilde{\sigma}) = \left(\prod j_\alpha(\tilde{\sigma})\right) J_{\tilde{P}_3|\tilde{P}_1}(\tilde{\sigma}),$$

where the product is taken over $\Phi_{\mathrm{red}}(P_1) \cap \Phi_{\mathrm{red}}(P_3) \cap \Phi_{\mathrm{red}}(\overline{P}_2)$. In particular, for $\omega \in W(M)$,

$$J_{\tilde{P}|\tilde{\omega}.\tilde{P}}(\tilde{\sigma}) J_{\tilde{\omega}.\tilde{P}|\tilde{P}}(\tilde{\sigma}) = \prod_{\alpha \in \Phi_{\mathrm{red}}(P) \cap \Phi_{\mathrm{red}}(\overline{\omega.P})} j_\alpha(\tilde{\sigma}),$$

where $j_\alpha(\tilde{\sigma})$ is the j-constant with respect to $(M, \tilde{\sigma})$ with M_α in place of G.

Harish-Chandra c-function. Recall that for $(\tilde{\sigma}, V) \in \mathcal{O}_{\mathbb{C}} \subset \Pi(\tilde{M})$, we denote by $(\check{\tilde{\sigma}}, \check{V})$ the contragredient representation of $(\tilde{\sigma}, V)$. For a semistandard parabolic subgroup $P = MU$ of G, we define

$$L(\tilde{\sigma}, \tilde{P}) := I_{\tilde{P}}^{\tilde{G}}(V) \otimes I_{\tilde{P}}^{\tilde{G}}(\check{V}) \hookrightarrow \mathrm{End}(I_{\tilde{P}}^{\tilde{G}}(V)),$$

and the matrix coefficient map

$$E_{\tilde{P}}^{\tilde{G}} : L(\tilde{\sigma}, \tilde{P}) \to C^\infty(\tilde{G}) : v \otimes \check{v} \mapsto \langle \tilde{\pi}(\tilde{g}) v, \check{v} \rangle,$$

where $\tilde{\pi} = I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$. For $\omega \in W(M)$, we define the Harish-Chandra c-function as

$$c_{\tilde{P}|\tilde{P}}(\omega, \tilde{\sigma}) : L(\tilde{\sigma}, \tilde{P}) \xrightarrow{\lambda(\omega)} L(\tilde{\omega}.\tilde{\sigma}, \tilde{\omega}.\tilde{P}) \xrightarrow{J_{\tilde{P}|\tilde{\omega}.\tilde{P}}(\tilde{\omega}.\tilde{\sigma}) \otimes J_{\bar{\tilde{P}}|\tilde{\omega}.\tilde{P}}(\tilde{\omega}.\tilde{\sigma})} L(\tilde{\omega}.\tilde{\sigma}, \tilde{P});$$

here $\lambda(\omega) : \phi(\,\cdot\,) \mapsto \phi(\tilde{\omega}^{-1}\,\cdot\,)$, and $\tilde{\omega} \in \tilde{K}_{\mathrm{good}}$ where K_{good} is a special compact open subgroup of G which is in good position with respect to M_0. Furthermore, we define

$${}^0c_{\tilde{P}|\tilde{P}}(\omega, \tilde{\sigma}) := c_{\tilde{P}|\tilde{P}}(1, \tilde{\omega}.\tilde{\sigma})^{-1} c_{\tilde{P}|\tilde{P}}(\omega, \tilde{\sigma}) \in \mathrm{Hom}_{\tilde{G} \times \tilde{G}}(L(\tilde{\sigma}, \tilde{P}), L(\tilde{\omega}.\tilde{\sigma}, \tilde{P})).$$

Note that these operators ${}^0c_{\tilde{P}|\tilde{P}}(\omega, \tilde{\sigma})$ for $\omega \in W(\tilde{\sigma})$ have the following properties (see [Waldspurger 2003, Section V.3] or [Li 2012, Section 2.5]).

- (regularity) They are regular on $\mathcal{O}$ and unitary operators.
- (associativity) For $\omega_1, \omega_2 \in W(M)$, $\tilde{\sigma} \in \mathcal{O} \subset \Pi_2(\tilde{M})$ and $P_i = MU_i$ with $i = 1, 2, 3$, based on the associativity of $J_{\tilde{P}'|\tilde{P}}$, we have the equality
$$ {}^0c_{\tilde{P}_3|\tilde{P}_2}(\omega_1, \tilde{\omega}_2.\tilde{\sigma})\, {}^0c_{\tilde{P}_2|\tilde{P}_1}(\omega_2, \tilde{\sigma}) = {}^0c_{\tilde{P}_3|\tilde{P}_1}(\omega_1\omega_2, \tilde{\sigma}). $$

Plancherel formula (see [Li 2012, Section 2.6]). Let $\mathcal{C}(\tilde{G})$ be the Schwartz–Harish-Chandra function space of $\tilde{G}$, and $C^\infty(\mathcal{O}, \tilde{P})$ the space of C^∞-functions on $\mathcal{O}$, i.e., $\psi[\mathcal{O}, P] : \tilde{\sigma} \to \psi[\mathcal{O}, P]_{\tilde{\sigma}} \in L(\tilde{\sigma}, \tilde{P})$, such that it is compatible with the isomorphism class of $\tilde{\sigma}$. Denote by Θ the set of pairs $(\mathcal{O}, P = MU)$, where P is a semistandard parabolic subgroup of G and $\mathcal{O} \subset \Pi_2(\tilde{M})$ an orbit under the action of $\operatorname{Im} X(\tilde{M})$. Let $C^\infty(\Theta) := \bigoplus_{(\mathcal{O},P)\in\Theta} C^\infty(\mathcal{O}, \tilde{P})$, and write an element $\psi \in C^\infty(\Theta)$ in the form of $\psi = (\psi[\mathcal{O}, P])_{\mathcal{O},P}$. We set $C^\infty(\Theta)^{\mathrm{inv}}$ to be the subspace of $C^\infty(\Theta)$ consisting of the elements ψ such that
$$ \psi[\tilde{\omega}.\mathcal{O}, P']_{\tilde{\omega}.\tilde{\sigma}} = {}^0c_{\tilde{P}'|\tilde{P}}(\omega, \tilde{\sigma})\psi[\mathcal{O}, P]_{\tilde{\sigma}}, $$
for all $(\mathcal{O}, P) \in \Theta$ and all P'. For $f \in \mathcal{C}(\tilde{G})$, we define a map $\iota : \mathcal{C}(\tilde{G}) \to C^\infty(\Theta)^{\mathrm{inv}}$ as
$$ f \mapsto (\psi_f[\mathcal{O}, P] : \tilde{\sigma} \mapsto d(\tilde{\sigma}) I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})(\check{f}))_{\mathcal{O},P}, $$
where $d(\tilde{\sigma})$ is the formal degree of $\tilde{\sigma}$, and $\check{f}(\tilde{g}) = f(\tilde{g}^{-1})$. In rough terms, given both $\mathcal{C}(\tilde{G})$ and $C^\infty(\Theta)^{\mathrm{inv}}$ the natural direct limit topology with respect to open compact subgroups, the Plancherel formula says that ι is an isomorphism of topological spaces, and the associated inverse map κ is given by
$$ \psi \mapsto \sum_{(\mathcal{O},P)\in\Theta} \gamma(G|M)|W_M(T)|\,|W_G(T)|^{-1}|\mathcal{P}(M)|^{-1} f_{\psi[\mathcal{O},P]}, $$
where $\mathcal{P}(M)$ is the set of all parabolic subgroups $P = MU$ with Levi group M,
$$ \gamma(G|M) = \int_{\bar{U}} \delta_P(m_p(\bar{u}))\, d\bar{u} \quad \text{and} \quad f_{\psi[\mathcal{O},P]}(\tilde{g}) = \int_{\mathcal{O}} \mu^{\tilde{G}}(\tilde{\sigma})(E_{\tilde{P}}^{\tilde{G}} \psi[\mathcal{O}, P]_{\tilde{\sigma}})(\tilde{g})\, d\tilde{\sigma}. $$

2B. *Knapp–Stein dimension theorem.* Now we are ready to prove the Knapp–Stein dimension theorem for finite central covering groups.

For $(\tilde{\sigma}, V) \in \mathcal{O} \subset \Pi_2(\tilde{M})$, and $\omega \in W(\tilde{\sigma})$, denote by $A(\omega)$ the unique isomorphism, up to scalars of $|\cdot| = 1$, between $\tilde{\omega}.\tilde{\sigma}$ and $\tilde{\sigma}$, and extend $A(\omega)$ to be an isomorphism between $I_{\tilde{P}}^{\tilde{G}}(\tilde{\omega}.\tilde{\sigma})$ and $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$. Note that
$$ A(\omega) \circ {}^0c_{\tilde{P}|\tilde{P}}(\omega, \tilde{\sigma}) \in \operatorname{End}_{\tilde{G}\times\tilde{G}}(L(\tilde{\sigma}, \tilde{P})) $$
is regular on $\mathcal{O}$ and unitary. Note that the $\tilde{G}\times\tilde{G}$-equivalent space $\operatorname{End}_{\tilde{G}\times\tilde{G}}(L(\tilde{\sigma}, \tilde{P}))$

is finite-dimensional. Applying the Skolem–Noether theorem, based on the associativity property of ${}^0c_{\tilde{P}|\tilde{P}}(\omega,\tilde{\sigma})$, we define a projective unitary representation $\omega\mapsto\gamma(\omega)$ of $W(\tilde{\sigma})$ on the underling vector space of the induced representation $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$, such that $A(\omega)\circ{}^0c_{\tilde{P}|\tilde{P}}(\omega,\tilde{\sigma})=\mathrm{Ad}(\gamma(\omega))$ on $L(\tilde{\sigma},\tilde{P})\hookrightarrow\mathrm{End}(I_{\tilde{P}}^{\tilde{G}}(V))$. Notice that the adjoint $\gamma(\omega)^*$ of $\gamma(\omega)$ exists and equals $\gamma(\omega^{-1})$ up to a scalar. So the vector space $\Gamma:=\mathrm{Span}\{\gamma(\omega):\omega\in W(\tilde{\sigma})\}$ is a selfadjoint algebra, and hence semisimple. Before turning to the Harish-Chandra commuting algebra theorem, we first discuss the explicit form of $\gamma(\omega)$. Denote $\Phi(\tilde{\sigma}):=\{\alpha\in\Phi_{\mathrm{red}}(P):\mu^{M_\alpha}(\tilde{\sigma})=0\}$; then $W^0(\tilde{\sigma}):=\langle S_\alpha:\alpha\in\Phi(\tilde{\sigma})\rangle\subset W(\tilde{\sigma})$, where S_α is the simple reflection associated to α (see [Waldspurger 2003, Proposition IV.2.2]).

Lemma 2.1. *Indeed, $\Phi(\tilde{\sigma})$ is a subroot system, which in turn says that $W^0(\tilde{\sigma})$ is a normal subgroup of $W(\tilde{\sigma})$.*

Proof. For the convenience of the reader, we include the argument in [Harish-Chandra 1976, Section 40] as follows. Notice that this is equivalent to showing $\Phi(\tilde{\sigma})$ is stable under the action of $W(\tilde{\sigma})$, i.e., $\omega.\alpha\in\Phi(\tilde{\sigma})$ for $\alpha\in\Phi(\tilde{\sigma})$ and $\omega\in W(\tilde{\sigma})$. The upshot is for $\lambda\in\mathrm{Im}\,\mathfrak{a}^*_{M,\mathbb{C}}$, and $\alpha\in\Phi_{\mathrm{red}}(P)$,

$$\mu^{\tilde{M}_\alpha}(\tilde{\sigma}\otimes\lambda)>0\text{ unless }\langle\alpha,\lambda\rangle=0.$$

Take $P=MU$, and let $\alpha_1,\dots,\alpha_r$ be all the distinct elements in $\Phi(\tilde{\sigma})$. Assume $\alpha=\alpha_1$. Let j be the unique index such that $\omega.\alpha=\pm\alpha_j$. Fix $\lambda\in\mathrm{Im}\,\mathfrak{a}^*_{M,\mathbb{C}}$, such that

$$\langle\alpha,\lambda\rangle=0\quad\text{and}\quad\langle\alpha_i,\lambda\rangle\neq0,\quad 2\le i\le r.$$

This implies $\mu^{\tilde{M}_\alpha}(\tilde{\sigma}\otimes\lambda)=\mu^{\tilde{M}_\alpha}(\tilde{\sigma})=0$. Thus $\mu^{\tilde{G}}(\tilde{\sigma}\otimes\lambda)=\prod_{1\le i\le r}\mu^{\tilde{M}_{\alpha_i}}(\tilde{\sigma}\otimes\lambda)=0$. Note $\tilde{\omega}.\tilde{\sigma}=\tilde{\sigma}$, so $0=\mu^{\tilde{G}}(\tilde{\sigma}\otimes\lambda)=\mu^{\tilde{G}}(\tilde{\omega}.\tilde{\sigma}\otimes\omega.\lambda)=\mu^{\tilde{G}}(\tilde{\sigma}\otimes\omega.\lambda)$. This implies $\mu^{\tilde{M}_{\alpha_i}}(\tilde{\sigma}\otimes\omega.\lambda)=0$ for some i. But it is clear that $\langle\alpha_i,\omega.\lambda\rangle\neq0$ unless $i=j$, which is to say $0=\mu^{\tilde{M}_{\alpha_j}}(\tilde{\sigma}\otimes\omega.\lambda)=\mu^{\tilde{M}_{\alpha_j}}(\tilde{\sigma})$, whence $\alpha_j\in\Phi(\tilde{\sigma})$. □

In view of this, we may define the R-group

$$R(\tilde{\sigma}):=W(\tilde{\sigma})/W^0(\tilde{\sigma})=\{\omega\in W(\tilde{\sigma}):\omega.\Delta(\tilde{\sigma})=\Delta(\tilde{\sigma})\},$$

where $\Delta(\tilde{\sigma})$ is the set of simple roots in $\Phi(\tilde{\sigma})$.

Lemma 2.2. *Keeping the same notation as above, for $\omega\in W^0(\tilde{\sigma})$, we have*

$${}^0c_{\tilde{P}|\tilde{P}}(\omega,\tilde{\sigma})=\mathrm{id},\ \textit{i.e.},\ \gamma(\omega)=\mathrm{id}.$$

On the other hand, for $\omega\in R(\tilde{\sigma})$, we have

$$\gamma(\omega)=A(\omega)\circ J_{\tilde{P}|\tilde{\omega}.\tilde{P}}(\tilde{\omega}.\tilde{\sigma})\circ\lambda(\omega),$$

where $\lambda(\omega):I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})\xrightarrow{\sim}I_{\tilde{\omega}.\tilde{P}}^{\tilde{G}}(\tilde{\omega}.\tilde{\sigma})$ is the canonical isomorphism given by $\phi(\cdot)\mapsto\phi(\tilde{\omega}^{-1}\cdot)$.

Proof. For the first part, by the associativity property of the Harish-Chandra c-function ${}^0c_{\tilde{P}|\tilde{P}}$, it suffices to prove that ${}^0c_{\tilde{P}|\tilde{P}}(S_\alpha, \tilde{\sigma}) = \mathrm{id}$ for $\alpha \in \Phi(\tilde{\sigma})$. Setting $P_\alpha = P \cap M_\alpha$, the functoriality property of ${}^0c_{\tilde{P}|\tilde{P}}$ says that

$${}^0c_{\tilde{P}|\tilde{P}}(S_\alpha, \tilde{\sigma}) = {}^0c_{\tilde{P}_\alpha|\tilde{P}_\alpha}(S_\alpha, \tilde{\sigma})|_{L(\tilde{\sigma},\tilde{P})}.$$

On the other hand, applying Savin's results on the maximal parabolic subgroup case in [Savin 2017, Proposition 2], we then have

$${}^0c_{\tilde{P}|\tilde{P}}(S_\alpha, \tilde{\sigma}) = {}^0c_{\tilde{P}_\alpha|\tilde{P}_\alpha}(S_\alpha, \tilde{\sigma})|_{L(\tilde{\sigma},\tilde{P})} = \mathrm{id}.$$

As for the second part, this results from the associativity of $J_{\tilde{P}|\tilde{\omega}.\tilde{P}}(\,\cdot\,)$, i.e.,

$$J_{\tilde{P}|\tilde{\omega}.\tilde{P}}(\,\cdot\,)J_{\tilde{\omega}.\tilde{P}|\tilde{P}}(\,\cdot\,) = \prod_{\alpha\in\Phi_{\mathrm{red}}(P)\cap\Phi_{\mathrm{red}}(\overline{\omega.P})} j_\alpha(\,\cdot\,).$$

Notice that for $\alpha \in \Phi_{\mathrm{red}}(P)\cap\Phi_{\mathrm{red}}(\overline{\omega.P})$, $\mu^{\tilde{M}_\alpha}(\tilde{\omega}.\tilde{\sigma}) = \mu^{\tilde{M}_\alpha}(\tilde{\sigma}) \neq 0$, so $J_{\tilde{P}|\tilde{\omega}.\tilde{P}}(\tilde{\omega}.\tilde{\sigma})$ and $J_{\tilde{\omega}.\tilde{P}|\tilde{P}}(\tilde{\omega}.\check{\tilde{\sigma}})$ are holomorphic at $\tilde{\sigma}$, and hence invertible. In this case, based on the general associativity property of $J_{\tilde{P}_2|\tilde{P}_1}$, we have

$$A(\omega)\circ {}^0c_{\tilde{P}|\tilde{P}}(\omega,\tilde{\sigma}) = A(\omega)\circ J_{\tilde{P}|\tilde{\omega}.\tilde{P}}(\tilde{\omega}.\tilde{\sigma})\circ\lambda(\omega) \bigotimes A(\omega)\circ J_{\tilde{\omega}.\tilde{P}|\tilde{P}}(\tilde{\omega}.\check{\tilde{\sigma}})\circ\lambda(\omega),$$

which in turn implies that

$$\gamma(\omega) = A(\omega)\circ J_{\tilde{P}|\tilde{\omega}.\tilde{P}}(\tilde{\omega}.\tilde{\sigma})\circ\lambda(\omega). \qquad \square$$

Remark. We note that [Savin 2017, Proposition 2] concerns unitary supercuspidal representations, but the argument applies to discrete series as well based on the following facts (see [Waldspurger 2003, Lemme III.3.1, Corollaire III.7.3]):

- $\mathrm{Hom}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}\tilde{\sigma}, I_{\tilde{P}}^{\tilde{G}}\tilde{\sigma}) = \mathrm{Hom}_{\tilde{M}}(J_{\tilde{M}}^{\omega}(I_{\tilde{P}}^{\tilde{G}}\tilde{\sigma}), \tilde{\sigma})$, where $J_{\tilde{M}}^{\omega}$ stands for the tempered part of the normalized Jacquet module $J_{\tilde{M}}$.
- $J_{\tilde{M}}^{\omega}(I_{\tilde{P}}^{\tilde{G}}\tilde{\sigma}) = \sum_{\omega\in W(M)} \omega.\tilde{\sigma}$ as virtual representations.

Note that $A(\omega)\circ J_{\tilde{P}|\tilde{\omega}.\tilde{P}}(\tilde{\omega}.\tilde{\sigma})\circ\lambda(\omega)$ is nothing but the well-known intertwining operator $A(\omega)\circ M(\omega,\tilde{\sigma})$ which is defined by

$$I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}) \xrightarrow{M(\omega,\tilde{\sigma})} I_{\tilde{P}}^{\tilde{G}}(\tilde{\omega}.\tilde{\sigma}) : f \mapsto \int_{U\cap\tilde{\omega}U\tilde{\omega}^{-1}\backslash U} f(\tilde{\omega}^{-1}u\tilde{g})\,du.$$

In the following lemma, we show the linear independence of $\{\gamma(\omega) : \omega \in R(\tilde{\sigma})\}$ adapting the argument in [Ban 2004].

Lemma 2.3. *For $\omega \in R(\tilde{\sigma})$, $\gamma(\omega)$ are linearly independent on $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$.*

Proof. For the convenience of the reader, we recall the argument in [Ban 2004, Theorem 4.3] as follows. The upshot is to construct a function $f \in I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$ with the following "separation" property for the nontrivial $\omega \in W(M)$:

- $f(1) = 0$.

- $M(\omega, \tilde{\sigma}) f(1)$ is absolutely convergent and nonzero.
- For any $\omega_1 \in W(M)$ with $\omega_1 \not\geq \omega$, $M(\omega_1, \tilde{\sigma}) f(1) = 0$.

Such a function is constructed as follows. Fix a nonzero element ν in $(\tilde{\sigma}, V)$. Let K be a compact open subgroup of G splitting in $\tilde{G}$, such that $\delta^{1/2}(k)\tilde{k}\nu = \nu$ for all $k \in K \cap \tilde{M}$. In addition, we may assume K is invariant under conjugation by $\omega \in W(M)$. Notice that $\tilde{P}(\overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega} \cap K)$ is open in $\tilde{P}(\overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega})$. Hence we may choose a compact subgroup $K_0 \subset K$ which is invariant under conjugation by $\omega \in W(M)$, such that

$$K_0 \cap \tilde{P}(\overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega}) \subset \tilde{P}(\overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega} \cap K) \quad \text{and} \quad \tilde{\omega}^{-1} K_0 \subset \tilde{G}_{\omega^{-1}},$$

where $\tilde{G}_{\omega^{-1}} := \bigcup_{\omega' \geq \omega^{-1}} \tilde{P} \tilde{\omega}' \tilde{P}$. Then we may define the "separation" function f as

$$f(\tilde{g}) = \begin{cases} \delta^{1/2}(m)\tilde{\sigma}(\tilde{m})\nu & \text{if } \tilde{g} = \tilde{m} u \tilde{\omega}^{-1} k \in \tilde{P}\tilde{\omega}^{-1} K_0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see such an f is well defined and belongs to $I_{\tilde{P}}^{\tilde{G}}(V)$: If $\tilde{p}_1 \tilde{\omega}^{-1} k_1 = \tilde{p}_2 \tilde{\omega}^{-1} k_2$, for $\tilde{p}_1, \tilde{p}_2 \in \tilde{P}$ and $k_1, k_2 \in K_0$, we then have

$$\tilde{p}_2^{-1} \tilde{p}_1 = \tilde{\omega}^{-1} k_2 k_1^{-1} \tilde{\omega} \in K_0,$$

which in turn implies that, as $\delta^{1/2}(k)\tilde{\sigma}(k)\nu = \nu$ for $k \in K$,

$$\delta^{1/2}(\tilde{p}_1)\tilde{\sigma}(\tilde{p}_1)\nu = \delta^{1/2}(\tilde{p}_2)\tilde{\sigma}(\tilde{p}_2)\nu.$$

On the other hand, we have

$$\operatorname{supp}(f) \subset \tilde{P}\tilde{\omega}^{-1} K_0 \subset \tilde{G}_{\omega^{-1}} := \bigcup_{\omega' \geq \omega^{-1}} \tilde{P}\tilde{\omega}'\tilde{P},$$

and $\operatorname{supp}(f) \cap \tilde{P} = \varnothing$. Observe that

$$\begin{aligned} M(\omega, \tilde{\sigma}) f(1) &= \int_{U \cap \tilde{\omega} U \tilde{\omega}^{-1} \backslash U} f(\tilde{\omega}^{-1} u)\, du \\ &= \int_{\overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega}} f(u \tilde{\omega}^{-1})\, du = \int_{\tilde{P} K_0 \cap \overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega}} f(u \tilde{\omega}^{-1})\, du. \end{aligned}$$

Notice that for $u \in \overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega} \cap \tilde{P} K_0$, we write $u = \tilde{p} k_0$; then

$$\tilde{p}^{-1} u = k_0 \in K_0 \cap \tilde{P}(\overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega}) \subset \tilde{P}(\overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega} \cap K),$$

thus $u \in \overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega} \cap K \subset K$ and $\tilde{p} \in K$, therefore

$$f(u \tilde{\omega}^{-1}) = f(\tilde{p} K_0 \tilde{\omega}^{-1}) = \delta^{1/2}(\tilde{p})\tilde{\sigma}(\tilde{p})\nu = \nu,$$

which in turn says that

$$M(\omega, \tilde{\sigma}) f(1) = \operatorname{mes}(\overline{U} \cap \tilde{\omega}^{-1} U \tilde{\omega} \cap P K_0)\nu \neq 0.$$

The remaining vanishing statement is easy. □

To finish the proof of Knapp–Stein dimension theorem for finite central covering groups, it remains to prove Harish-Chandra commuting algebra theorem. For simplicity, we state Harish-Chandra commuting algebra theorem and Knapp–Stein dimension theorem together as follows.

Theorem 2.4. *Keeping the same notation as before*, *we have*

$$\mathrm{End}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})) = \mathrm{Span}\{\gamma(\omega) : \omega \in W(\tilde{\sigma})\} = \mathrm{Span}\{\gamma(\omega) : \omega \in R(\tilde{\sigma})\}.$$

Therefore, $\dim \mathrm{End}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})) = |R(\tilde{\sigma})|$.

Proof. This follows from the Plancherel formula stated in Section 2A using the same argument as in [Silberger 1979, Theorem 5.5.3.2]. For the convenience of the reader, we sketch the main ideas as follows. Denote

$$L^0(\tilde{\sigma}, \tilde{P}) := \{\phi \in L(\tilde{\sigma}, \tilde{P}) : {}^0c_{\tilde{P}|\tilde{P}}(\omega, \tilde{\sigma})\phi = \phi \text{ for all } \omega \in W(\tilde{\sigma})\}.$$

Recall $\Gamma = \mathrm{Span}\{\gamma(\omega) : \omega \in W(\tilde{\sigma})\} \subset \mathrm{End}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}))$.

Step 1. The centralizer $C_{L(\tilde{\sigma},P)}(\Gamma)$ of Γ in $L(\tilde{\sigma}, P)$ satisfies $C_{L(\tilde{\sigma},P)}(\Gamma) = L^0(\tilde{\sigma}, \tilde{P})$, and $L^0(\tilde{\sigma}, \tilde{P}) = I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})(\mathcal{C}(\tilde{G}))$. The latter plays the key role which follows from the Plancherel formula, i.e., the isomorphism

$$\mathcal{C}(\tilde{G}) \xrightarrow{\sim} C^\infty(\Theta)^{\mathrm{inv}}.$$

Step 2. $C_{L(\tilde{\sigma},P)}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})(\mathcal{C}(\tilde{G}))) = \mathrm{End}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}))$. This follows from the definition.

Step 3. The Wedderburn double centralizer theorem says that

$$\Gamma = C_{L(\tilde{\sigma},P)}(C_{L(\tilde{\sigma},P)}(\Gamma)) = \mathrm{End}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})).$$

In order to apply the Wedderburn double centralizer theorem, one has to consider some finite-dimensional subspaces $L(\sigma, P)^{K\times K}$ of $L(\sigma, P)$ consisting of $K \times K$-invariant vectors, where K is an open-compact subgroup of G which splits in $\tilde{G}$ and is small enough, as in [Silberger 1979].

To be precise, let K be a sufficiently small open-compact subgroup of G which splits in $\tilde{G}$, then we have

$$\mathrm{End}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})) = \mathrm{End}_{\mathcal{C}(\tilde{G})}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})) = \mathrm{End}_{\mathcal{C}_K(\tilde{G})}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})^K),$$

where $\mathcal{C}_K(\tilde{G})$ is the subspace of double K-invariant functions in $\mathcal{C}(\tilde{G})$, and $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})^K$ is the subspace of K-invariant vectors in $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$.

Denoting by Γ_K the restriction of the action of Γ to $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})^K$, we have $\dim \Gamma_K = \dim \Gamma$. As $\Gamma \subset L(\tilde{\sigma}, P)$ is semisimple, thus

$$\Gamma_K = C_{L(\tilde{\sigma},P)^{K\times K}}(C_{L(\tilde{\sigma},P)^{K\times K}}(\Gamma_K)) = C_{L(\tilde{\sigma},P)^{K\times K}}(L^0(\tilde{\sigma}, P)^{K\times K}).$$

On the other hand, Harish-Chandra's Plancherel formula implies that

$$L^0(\tilde{\sigma}, P)^{K\times K} = I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})(\mathcal{C}_K(\tilde{G})),$$

which in turn implies that

$$\begin{aligned}\Gamma_K &= C_{L(\tilde{\sigma},P)^{K\times K}}(L^0(\tilde{\sigma}, P)^{K\times K})\\ &= C_{L(\tilde{\sigma},P)^{K\times K}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})(\mathcal{C}_K(\tilde{G}))) = \mathrm{End}_{\mathcal{C}_K(\tilde{G})}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})^K),\end{aligned}$$

whence

$$\Gamma = \Gamma_K = \mathrm{End}_{\mathcal{C}_K(\tilde{G})}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})^K) = \mathrm{End}_{\tilde{G}}(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})). \qquad \square$$

2C. ***An example: R-group for genuine unramified principal series.*** The decomposition of tempered induced representation $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$ is determined by our $R(\tilde{\sigma})$-group, especially intertwining operators. Recall that the associativity property of intertwining operators says that

$$J_{\tilde{P}_3|\tilde{P}_2}(\tilde{\sigma}) J_{\tilde{P}_2|\tilde{P}_1}(\tilde{\sigma}) = \left(\prod j_\alpha(\tilde{\sigma})\right) J_{\tilde{P}_3|\tilde{P}_1}(\tilde{\sigma}).$$

We normalize those intertwining operators $J_{P'|P}(\tilde{\sigma})$ for covering groups by a factor $r_{P'|P}(\tilde{\sigma})$ as done by Arthur for linear groups (please refer to [Li 2012] for the details) so that our normalized intertwining operators

$$R_{P'|P}(\tilde{\sigma}) := r_{P'|P}(\tilde{\sigma})^{-1} J_{P'|P}(\tilde{\sigma})$$

satisfy some natural properties, for example

$$R_{\tilde{P}_3|\tilde{P}_2}(\tilde{\sigma}) R_{\tilde{P}_2|\tilde{P}_1}(\tilde{\sigma}) = R_{\tilde{P}_3|\tilde{P}_1}(\tilde{\sigma}).$$

As in [Arthur 1993], we then define the normalized intertwining operators

$$R(w, \tilde{\sigma}) := A(w) \circ \lambda(w) \circ R_{w^{-1}.P|P}(\tilde{\sigma}) : I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}) \to I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$$

which satisfy

$$\Gamma := \mathrm{Span}\{\gamma(w) : w \in R(\tilde{\sigma})\} = \mathrm{Span}\{R(w, \tilde{\sigma})\}.$$

On the other hand, the definition of $A(w) \circ \lambda(w)$ depends on the lift of w in $\tilde{K}_{\text{good}}$. For simplicity, we use the same letter w to be the fixed lifting of $w \in W_G(T)$ if no confusion arises. In general, $w \to R(w, \tilde{\sigma})$ is not a homomorphism, but we have the formula

$$R(w_1 w_2, \tilde{\sigma}) = \eta_{\tilde{\sigma}}(w_1, w_2) R(w_1, \tilde{\sigma}) R(w_2, \tilde{\sigma}), \quad w_1, w_2 \in R(\tilde{\sigma}),$$

where

$$\eta_{\tilde{\sigma}}(w_1, w_2) = A(w_1 w_2) \circ \lambda(w_1 w_2) \circ \lambda(w_2)^{-1} \circ A(w_2)^{-1} \circ \lambda(w_1)^{-1} \circ A(w_1)^{-1}$$

is a 2-cocycle for $R(\tilde{\sigma})$ with values in $\mathbb{C}^\times$. Thus the image $\bar{\eta}_{\tilde{\sigma}}$ of $\eta_{\tilde{\sigma}}$ in $H^2(R(\tilde{\sigma}), \mathbb{C}^\times)$

gives the obstruction of extending the representation $\tilde{\sigma}$ to the groups generated by $\tilde{M}$ and $\{\tilde{w} \in \tilde{K}_{\text{good}} : w \in R(\tilde{\sigma})\}$.

As in [Arthur 1993, page 87], we deal with the problem by fixing a finite central extension

$$1 \to Z_{\tilde{\sigma}} \to \bar{R}_{\tilde{\sigma}} \to R(\tilde{\sigma}) \to 1$$

over which $\eta_{\tilde{\sigma}}$ splits. We then choose a function $\kappa_{\tilde{\sigma}} : \bar{R}_{\tilde{\sigma}} \to \mathbb{C}^\times$ such that $\eta_{\tilde{\sigma}}$ splits, i.e.,

$$\eta_{\tilde{\sigma}}(r_1, r_2) = \kappa_{\tilde{\sigma}}(r_1 r_2)\kappa_{\tilde{\sigma}}(r_2)^{-1}\kappa_{\tilde{\sigma}}(r_1)^{-1}, \qquad r_1, r_2 \in \bar{R}_{\tilde{\sigma}},$$

where $\eta_{\tilde{\sigma}}$ is identified with its pullback to $\bar{R}_{\tilde{\sigma}} \times \bar{R}_{\tilde{\sigma}}$. It follows that

$$\kappa_{\tilde{\sigma}}(zr) = \chi_{\tilde{\sigma}}(z)\kappa_{\tilde{\sigma}}(r), \quad z \in Z_{\tilde{\sigma}},\ r \in \bar{R}_{\tilde{\sigma}},$$

where $\chi_{\tilde{\sigma}}$ is a character on the central subgroup $Z_{\tilde{\sigma}}$. We twist our intertwining operators by $\kappa_{\tilde{\sigma}}$, i.e.,

$$\bar{R}(r, \tilde{\sigma}) := \kappa_{\tilde{\sigma}}(r)^{-1} R(r, \tilde{\sigma}), \quad r \in \bar{R}_{\tilde{\sigma}}$$

which gives rise to a homomorphism of $\bar{R}_{\tilde{\sigma}}$ to the group of unitary intertwining operators for $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$ satisfying

$$\bar{R}(zr, \tilde{\sigma}) = \chi_{\tilde{\sigma}}(z)^{-1} \bar{R}(r, \tilde{\sigma}), \quad z \in Z_{\tilde{\sigma}},\ r \in \bar{R}_{\tilde{\sigma}}.$$

Therefore we obtain a representation R of $\bar{R}_{\tilde{\sigma}} \times \tilde{G}$ on the underling vector space $\mathcal{H}_{\tilde{\sigma}}$ of $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$, i.e.,

$$R(r, g) := \bar{R}(r, \tilde{\sigma}) I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}, g), \quad r \in \bar{R}_{\tilde{\sigma}},\ g \in \tilde{G}.$$

Thus our Knapp-Stein dimension theorem, i.e., Theorem 2.4, implies that

$$R = \bigoplus_{\rho} \check{\rho} \otimes \pi_\rho,$$

where ρ runs over the set $\Pi(\bar{R}_{\tilde{\sigma}})_{\chi_{\tilde{\sigma}}}$ of irreducible representations of $\bar{R}_{\tilde{\sigma}}$ with $Z_{\tilde{\sigma}}$-central character $\chi_{\tilde{\sigma}}$, and $\check{\rho}$ is the contragredient representation of ρ, while $\pi_\rho \in JH(I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}))$.

It is well known that such a 2-cocycle $\bar{\eta}_{\tilde{\sigma}}$ is trivial if $\Pi(\bar{R}_{\tilde{\sigma}})_{\chi_{\tilde{\sigma}}}$ contains a one-dimensional representation, thus giving:

Lemma 2.5 (D. Keys). *Keep the notions as above. If the tempered induction $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$ contains a constituent which is of multiplicity one, then the 2-cocycle $\bar{\eta}_{\tilde{\sigma}}$ is trivial.*

Proof. Since $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$ contains a constituent which is of multiplicity one, the decomposition

$$R = \bigoplus_{\rho \in \Pi(\bar{R}_{\tilde{\sigma}})_{\chi_{\tilde{\sigma}}}} \check{\rho} \otimes \pi_\rho$$

implies $\Pi(\bar{R}_{\tilde{\sigma}})_{\chi_{\tilde{\sigma}}}$ contains a one-dimensional representation, so our claim holds. □

A typical example of such a situation is when $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$ is an unramified genuine unitary principal series, that is:

Corollary 2.6. *Keep the notions as above. For genuine unramified unitary principal series $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$, the representation R of $R_{\tilde{\sigma}} \times \tilde{G}$ on the underling vector space $\mathcal{H}_{\tilde{\sigma}}$ of $I_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma})$ decomposes as*

$$R = \bigoplus_{\rho \in \Pi(R_{\tilde{\sigma}})} \check{\rho} \otimes \pi_\rho.$$

In what follows, we would like to investigate R-groups for genuine unramified unitary principal series of the tame Brylinski–Deligne n-fold covering group $\tilde{G}$ of a split simply connected group G defined over the nonarchimedean local field F, where tame means that n and p are coprime. Under this setting, those R-groups are isomorphic to the associated R-groups of the incarnation split linear group G_n which gives rise to the same Langlands dual group $\tilde{G}^\vee$; moreover G_n is an isogeny to G' which has the same or dual root system of G depending on the cover (see [Savin 2004; Gan and Gao 2018; Weissman 2018]). Note that R-groups for G are well known (see [Keys 1982]). So it reduces to investigating the relation of R-groups under isogeny. Let $p : G \to G_n$ be the isogeny map. Restricting to their maximal torus gives $p : T \to T_n$ and $p* : \Pi(T_n) \to \Pi(T)$, i.e., $\chi_n \mapsto \chi := \chi_n \circ p$. Therefore $W_{\chi_n} := \{w \in W_G(T) : w.\chi_n = \chi_n\} < W_\chi := \{w \in W_G(T) : w.\chi = \chi\}$ as the map p is $W_G(T)$-equivalent. On the other hand, for $\chi \in \Pi_2(T)$ and a root $\alpha \in \Phi := \Phi(G, T)$, it is well known that the corank one Plancherel measure $\mu_\alpha(\chi) := \mu^{M_\alpha}(\chi)$ is equal to 0 if and only if $\chi_\alpha := \chi \circ H_{\alpha^\vee} = 1$ (see [Winarsky 1978]), where $H_{\alpha^\vee}$ is the one-parameter subgroup given by α under Harish-Chandra homomorphism (see [Waldspurger 2003]). A similar criterion holds for covering groups (please refer to [Goldberg and Szpruch 2016] for the details). Note that $\chi_\alpha = 1$ says that $(\chi_n)_\alpha = 1$, which implies that $S_\alpha.\chi_n = \chi_n$. In either PGL_2 or SL_2, such a corank one unitary induction is always irreducible, thus $\mu_\alpha(\chi_n) = 0$, i.e., $W^0_{\chi_n} := \langle S_\alpha : \mu_\alpha(\chi_n) = 0, \alpha \in \Phi(G, T)\rangle = W^0_\chi := \langle S_\alpha : \mu_\alpha(\chi) = 0, \alpha \in \Phi(G, T)\rangle$.

Lemma 2.7. *Retain the notions as above. We have*

$$R_{\chi_n} := W_{\chi_n} / W^0_{\chi_n} \lhd R_\chi := W_\chi / W^0_\chi.$$

Proof. This is equivalent to showing that $W_{\chi_n} \lhd W_\chi$, i.e., for any $w \in W_\chi$ and $w_n \in W_{\chi_n}$, $w_n.(w.\chi_n) = (w.\chi_n)$. But $w \in W_\chi$ implies that $w.\chi_n = \chi_n \chi_c$ for some $\chi_c \in \Pi_2(T_n/p(T))$. Note that any $\chi_c \in \Pi_2(T_n/p(T))$ is $W_G(T)$-invariant which follows from the fact that $S_\alpha.y - y$ lies in the coroot lattice of T for any coroot y of T_n and $\alpha \in \Phi(G, T)$. Thus $w_n.(w.\chi_n) = w_n(\chi_n \chi_c) = w_n.\chi_n \chi_c = (w.\chi_n)$. □

It is also well known that R-groups for unitary unramified principal series of adjoint groups are trivial (see [Li 1992, Corollary 2.6]). Thus the nontrivial R-

groups for split semisimple groups which have not been discussed in [Keys 1982] and [Goldberg 1994] are as follows:

Corollary 2.8 (Keys). *$G_n^\vee = SL_\iota(\mathbb{C})/\mu_m : R \simeq \mathbb{Z}/d\mathbb{Z}$ with $d|m|\iota$.*

2D. *An example: R-group of Mp_{2n}.* In what follows, we discuss some properties of R-groups for Mp_{2n}. Let us first introduce a simple fact. Recall that we have the following decomposition of $\tilde{R}\times G$ acting on $I_P^G(\sigma)$:

$$I_P^G(\sigma) = \bigoplus_{\rho\in\Pi(\tilde{R})_{\chi_\sigma}} \check{\rho} \bigotimes \pi_\rho.$$

As an easy corollary, we have the following criterion on the abelian property of R.

Corollary 2.9. *If $\Pi_{\chi_\sigma}(\tilde{R})$ consists of one-dimensional representations, then $R \simeq \tilde{R}/Z$ is abelian.*

Proof. This results from the following fact: For a finite group G and a subgroup $H < Z(G)$, fix a character χ of H, if as G-modules

$$(\star) \qquad \mathrm{Ind}_H^G(\chi) = \bigoplus_{i=1}^{|G/H|} \chi_i;$$

then G/H is abelian.

Note that if χ is trivial, then this is quite obvious. As for χ nontrivial, we may consider a new set of characters $S := \{\chi_1^{-1}\cdot\chi_i : i = 1, \dots, |G/H|\}$. It is easy to see $\chi_1^{-1}\chi_i \neq \chi_1^{-1}\chi_j$ for $i \neq j$, and these are the characters of G which are trivial on H, which in turn says

$$\mathrm{Ind}_H^G(1) = \bigoplus_{i=1}^{|G/H|} \chi_1^{-1}\chi_i,$$

whence $(\star)$ holds. □

In view of the above corollary, based on Gan and Savin's work on local theta correspondence [2012], we have:

Corollary 2.10. *Keeping the notation as before, $R(\tilde{\sigma})$ is abelian, and*

$$R(\tilde{\sigma}) = R(\Theta(\tilde{\sigma})).$$

Here $(\tilde{\sigma}, \Theta(\tilde{\sigma}))$ is a Howe duality pair under the local theta correspondence for (Mp_{2n}, SO_{2n+1}).

Proof. The first part follows from the preservation of multiplicities in tempered inductions under the local theta correspondence. The second part follows from the preservation of Plancherel measures under the local theta correspondence. □

Remark. We recently learned that M. Hanzer [2019] had described the R-group for Mp_{2n} using the local theta correspondence.

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CAIHUA LUO
DEPARTMENT OF MATHEMATICAL SCIENCES
CHALMERS UNIVERSITY OF TECHNOLOGY
GOTEBORG
SWEDEN
chluo@amss.ac.cn

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