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SOME CLASSIFICATIONS OF BIHARMONIC HYPERSURFACES WITH CONSTANT SCALAR CURVATURE

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We give some classifications of biharmonic hypersurfaces with constant scalar curvature. These include biharmonic Einstein hypersurfaces in space forms, compact biharmonic hypersurfaces with constant scalar curvature in a sphere, and some complete biharmonic hypersurfaces of constant scalar curvature in space forms and in a nonpositively curved Einstein space. Our results provide additional cases (Theorem 2.3 and Proposition 2.8) that support the conjecture that a biharmonic submanifold in S^{m+1} has constant mean curvature, and two more cases that support Chen's conjecture on biharmonic hypersurfaces (Corollaries 2.2 and 2.7).

1. Introduction

Biharmonic maps, as a generalization of harmonic maps, are maps between Riemannian manifolds which are critical points of the bienergy functional. Biharmonic submanifolds are the images of biharmonic isometric immersions and they include minimal submanifolds as special cases. As in the study of minimal submanifolds, a fundamental problem in the study of biharmonic submanifolds is to classify nonminimal biharmonic submanifolds (called proper biharmonic submanifolds) in a model space. For example, when the ambient space is a space form, we have the following conjectures which are still far beyond our reach.

Conjecture (Chen's conjecture on biharmonic submanifolds of Euclidean space [1991; 2015]). *Any biharmonic submanifold in a Euclidean space is minimal.*

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Conjecture (Balmuş, Montaldo and Oniciuc's conjectures on biharmonic submanifolds of spheres [2008; 2012]). (1) *A proper biharmonic submanifold of a sphere has constant mean curvature.*

(2) *The only proper biharmonic hypersurface of* S^{m+1} *is a part of* $S^m(1/\sqrt{2})$ *or* $S^p(1/\sqrt{2}) \times S^q(1/\sqrt{2})$ *with* $p+q=m$ *and* $p \neq q$.

A lot of work related to these conjectures has been done since 2000; some recent developments, partial results and open problems in the topics can be found in the recent survey [Ou 2016].

For a hypersurface $\varphi : (M^m, g) \to (N^{m+1}, h)$ in a Riemannian manifold, i.e., an isometric immersion of codimension 1, we choose a local unit normal vector field ξ with respect to which, we have the second fundamental form $B(X, Y) = b(X, Y)\xi$, where $b : TM \times TM \to C^\infty(M)$ is the function-valued second fundamental form. The Gauss and the Weingarten formulae read, respectively,

$$\tilde{\nabla}_X Y = (\tilde{\nabla}_X Y)^\top + (\tilde{\nabla}_X Y)^\perp = \nabla_X Y + b(X, Y)\xi, \tag{1}$$

and

$$\tilde{\nabla}_X \xi = (\tilde{\nabla}_X \xi)^\top + (\tilde{\nabla}_X \xi)^\perp = -AX + \nabla^\perp_X \xi = -AX, \tag{2}$$

where A is the shape operator of the hypersurface with respect to the unit normal vector ξ.

We use R^M, Ric^M, and Scal^M (respectively, R^N, Ric^N, and Scal^N) to denote the Riemannian, Ricci and scalar curvatures of (M^m, g) (respectively, of (N, h)) with the following conventions:

$$\mathrm{R}^M(X, Y, Z, W) = \langle \mathrm{R}^M(Z, W)Y, X\rangle,$$
$$\mathrm{R}^M(Z, W)Y = \nabla_Z \nabla_W Y - \nabla_W \nabla_Z Y - \nabla_{[Z,W]} Y,$$
$$\mathrm{Ric}^M(X, Y) = \sum_{i=1}^m \mathrm{R}^M(X, e_i, Y, e_i) \quad \text{for an orthonormal base } \{e_i\}.$$

Using these, together with (1) and (2), we have the Gauss equation

$$\mathrm{R}^N(X, Y, Z, W) = \mathrm{R}^M(X, Y, Z, W) + b(X, W)b(Y, Z) - b(X, Z)b(Y, W), \tag{3}$$

where

$$b(X, Y) = \langle AX, Y\rangle = \langle B(X, Y), \xi\rangle, \tag{4}$$

for any $X, Y, Z, W \in TM$.

From (3) we have the relationship between the Ricci curvatures of the hypersurface and the ambient space

$$\mathrm{Ric}^N(X, Y) = \mathrm{Ric}^M(X, Y) + \langle AX, AY\rangle - mH\langle AX, Y\rangle + \mathrm{R}^N(X, \xi, Y, \xi), \tag{5}$$

and the relationship between the scalar curvatures is

$$\mathrm{Scal}^N = \mathrm{Scal}^M + |A|^2 - m^2H^2 + 2\,\mathrm{Ric}^N(\xi, \xi). \tag{6}$$

It was proved in [Ou 2010] that a hypersurface $\varphi : M^m \to N^{m+1}$ with mean curvature vector $\eta = H\xi$ is biharmonic if and only if

$$\begin{cases} \Delta H - H|A|^2 + H\mathrm{Ric}^N(\xi, \xi) = 0, \\ 2A\,(\mathrm{grad}\, H) + \frac{m}{2}\,\mathrm{grad}\; H^2 - 2\,H\,(\mathrm{Ric}^N\,(\xi))^\top = 0, \end{cases} \tag{7}$$

where $\mathrm{Ric}^N : T_qN \to T_qN$ denotes the Ricci operator of the ambient space defined by $\langle \mathrm{Ric}^N\,(Z), W\rangle = \mathrm{Ric}^N(Z, W)$.

In particular, for biharmonic hypersurfaces in an Einstein space, we have:

Corollary 1.1 [Ou 2010]. *A hypersurface $\varphi : M^m \to (N^{m+1}, h)$ of an Einstein manifold with $\mathrm{Ric}^N = \lambda h$ is biharmonic if and only if its mean curvature function H solves the equations*

$$\begin{cases} \Delta H - H(|A|^2 - \lambda) = 0, \\ A\,(\mathrm{grad}\;H) + \frac{m}{2} H\,\mathrm{grad}\;H = 0. \end{cases} \tag{8}$$

In this note, we give some classifications of biharmonic hypersurfaces with constant scalar curvature. These include biharmonic Einstein hypersurfaces in space forms, compact biharmonic hypersurfaces with constant scalar curvature in a sphere, and some complete biharmonic hypersurfaces with constant scalar curvature in space forms and in a nonpositively curved Einstein space. Our results provide a further case (Theorem 2.3) that supports Balmuş, Montaldo and Oniciuc's conjecture that a biharmonic submanifold in S^{m+1} has constant mean curvature, and two more cases that support Chen's conjecture on biharmonic hypersurfaces (Corollaries 2.2 and 2.7).

2. Biharmonic hypersurfaces with constant scalar curvature in Einstein spaces

Recall that a Riemannian manifold (M^m, g) is called an Einstein space if its Ricci curvature is proportional to the metric, i.e., $\mathrm{Ric}^M = \lambda g$. It is well known that every 2-dimensional manifold is an Einstein space; any space form is of Einstein, and any 3-dimensional Einstein space has to be a space form. One can also check that for $m \geq 3$, if (M^m, g) is an Einstein space with $\mathrm{Ric}^M = \lambda g$, then λ has to be a constant and hence an Einstein space (M^m, g) of dimension $m \geq 3$ has constant scalar curvature since $\mathrm{Scal}^M = m\lambda$.

Our first result is the following theorem which gives a classification of biharmonic Einstein hypersurfaces in a space form.

Theorem 2.1. *An Einstein hypersurface* $M^m \hookrightarrow (N^{m+1}(C), h)$ $(m \geq 3)$ *in a space form is biharmonic if and only if it is minimal or* $|A|^2 = mC$. *In the latter case, the hypersurface has positive scalar curvature, i.e.*, $\mathrm{Scal}^M = m(m-2)C + m^2H^2 > 0$.

Proof. Suppose that the mean curvature function H is not constant; then there exists an open neighborhood U of M on which $\nabla H \neq 0$. Substituting $X = Y = \nabla H$ into (5) gives

$$\mathrm{Ric}^M(\nabla H, \nabla H) = (m-1)C|\nabla H|^2 + mHg(A(\nabla H), \nabla H) - g(A(\nabla H), A(\nabla H)). \tag{9}$$

Substituting the second equation $A(\nabla H) = -\frac{m}{2}H\nabla H$ of the biharmonic equation for a hypersurface into the above equation we have

$$\mathrm{Ric}^M(\nabla H, \nabla H) = \big((m-1)C - \tfrac{3}{4}m^2H^2\big)|\nabla H|^2. \tag{10}$$

If (M^m, g) is an Einstein hypersurface with $\mathrm{Ric}^M = \mu g$ for some constant μ, then (10) becomes

$$\mu|\nabla H|^2 = \big((m-1)C - \tfrac{3}{4}m^2H^2\big)|\nabla H|^2. \tag{11}$$

Since $\nabla H \neq 0$ on $U \subset M$, we have $\mu = (m-1)C - \frac{3}{4}m^2H^2$, which implies H is a constant on $U \subset M$ since μ is a constant. This contradicts the assumption that $\nabla H \neq 0$ on $U \subset M$. The contradiction shows that the mean curvature function H has to be a constant. It follows from the first equation $\Delta H - H(|A|^2 - mC) = 0$ of (8) that either $H = 0$ and the hypersurface is minimal, or $|A|^2 = mC$, and in this case, $C > 0$. Using the scalar curvature $\mathrm{Scal}^N = (m+1)mC > 0$ of $(N^{m+1}(C), h)$, $|A|^2 = mC$ and (6) we have $\mathrm{Scal}^M = m(m-2)C + m^2H^2 > 0$.

Thus, we obtain the theorem. □

As an immediate consequence of Theorem 2.1, we have:

Corollary 2.2. *A biharmonic Einstein hypersurface of Euclidean space* $\mathbb{R}^{m+1}$ *or hyperbolic space* H^{m+1} *is minimal. A proper biharmonic Einstein hypersurface in* S^{m+1} *has constant mean curvature and* $|A|^2 = m$.

It was proved in [Chen 1993] (see also [Balmuş et al. 2012]) that for $m \geq 2$, if a compact biharmonic hypersurface in sphere $S^{m+1}(1)$ with the squared norm of the second fundamental form satisfies $|A|^2 \leq m$, then $|A|^2 = 0$, or $|A|^2 = m$ and it has constant mean curvature. Also, Fu [2015] proved that a biharmonic hypersurface with constant scalar curvature in 5-dimensional space forms $M^5(C)$ has constant mean curvature, and in [Fu and Hong 2018], it was proved that a biharmonic hypersurface with constant scalar curvature and at most six distinct principal curvatures in space forms $M^{m+1}(C)$ has constant mean curvature. Our next theorem shows that the same result as in [Fu and Hong 2018] holds when we replace the principal curvature assumption by the compactness of the hypersurface.

Theorem 2.3. *A compact hypersurface with constant scalar curvature* $(M^m, g) \hookrightarrow S^{m+1}$ *in a sphere is biharmonic if and only if it is minimal, or it has nonzero constant mean curvature, and* $|A|^2 = m$.

Proof. If the scalar curvature Scal^M of the hypersurface is constant, then, by (6), we have $|A|^2 = m^2H^2 + \text{constant}$, from which we have

$$\nabla |A|^2 = 2m^2 H \nabla H. \tag{12}$$

Using this, together with the first equation of (8) with $\lambda = m$, we have

$$\nabla \Delta H = \nabla[(|A|^2 - m)H] = (|A|^2 - m + 2m^2H^2)\nabla H. \tag{13}$$

On the other hand, we have the following estimate of the squared norm of the Hessian of H:

$$\begin{aligned} |\nabla \mathrm{d}H|^2 &= \sum_{i,j=1}^{m} [\nabla \mathrm{d}H(e_i, e_j)]^2 \geq \sum_{i=1}^{m} [\nabla \mathrm{d}H(e_i, e_i)]^2 \geq \frac{1}{m}\Big(\sum_{i=1}^{m} \nabla \mathrm{d}H(e_i, e_i)\Big)^2 \\ &= \frac{1}{m}(\Delta H)^2. \end{aligned} \tag{14}$$

Substituting (10), (13) and (14) into the Bochner formula for $|\nabla H|^2$, we have

$$\begin{aligned} \tfrac{1}{2}\Delta |\nabla H|^2 &= |\nabla \mathrm{d}H|^2 + \mathrm{Ric}^M(\nabla H, \nabla H) + \langle \nabla H, \nabla \Delta H\rangle \\ &\geq \frac{1}{m}(\Delta H)^2 + \big(|A|^2 - 1 + \tfrac{5}{4}m^2H^2\big)|\nabla H|^2. \end{aligned} \tag{15}$$

Since M is compact, we integrate both sides of (15) to get

$$\int_M \Big[\frac{1}{m}(\Delta H)^2 + \big(|A|^2 - 1 + \tfrac{5}{4}m^2H^2\big)|\nabla H|^2\Big] dv_g \leq \int_M \tfrac{1}{2}\Delta |\nabla H|^2 dv_g = 0. \tag{16}$$

Using compactness of M, (13), and the divergence theorem we have

$$\begin{aligned} \frac{1}{m}\int_M \Delta H \Delta H dv_g = \frac{1}{m}\int_M \Delta H \mathrm{div}(\nabla H) dv_g &= -\frac{1}{m}\int_M \langle \nabla \Delta H, \nabla H\rangle dv_g \\ &= -\frac{1}{m}\int_M (|A|^2 - m + 2m^2H^2)|\nabla H|^2 dv_g, \end{aligned} \tag{17}$$

Substituting this into (16) we have

$$0 \leq \int_M \Big(\Big(1 - \frac{1}{m}\Big)|A|^2 + \frac{m}{4}(5m - 8)H^2\Big)|\nabla H|^2 dv_g \leq 0. \tag{18}$$

It follows that $\big[\big(1 - \frac{1}{m}\big)|A|^2 + \frac{m}{4}(5m - 8)H^2\big]|\nabla H|^2 = 0$ from which, together with Newton's inequality $|A|^2 \geq mH^2$, we have

$$0 = \Big[\Big(1 - \frac{1}{m}\Big)|A|^2 + \frac{m}{4}(5m - 8)H^2\Big]|\nabla H|^2 \geq \tfrac{1}{16}(5m^2 - 4m - 4)|\nabla H^2|^2. \tag{19}$$

From this, we conclude that $H =$ constant. In the case where $H =$ constant $\neq 0$, we use the first equation of (8) to have $|A|^2 = m$. Thus, we obtain the theorem. □

Remark 1. (i) Notice that Theorem 2.3, together with the results in [Chen 1993] and [Balmuş et al. 2012], implies that for a compact biharmonic hypersurface in a sphere, if one of the data Scal^M, H and $|A|^2$ is constant, then so are the other two.

(ii) We would like to point out that according to a classical result of Chern, do Carmo and Kobayashi [Chern et al. 1970], any compact minimal hypersurface in S^{m+1} with $|A|^2 = m$ is locally the Clifford tori $S^p \times S^{m-p}$. On the other hand, a hypersurface of S^{m+1} with constant mean curvature and $|A|^2 = m$ is biharmonic. So it would be very interesting to classify hypersurfaces of S^{m+1} with constant mean curvature and $|A|^2 = m$. This would be an important case to solve the second of Balmuş, Montaldo and Oniciuc's conjectures.

(iii) Finally, we remark that there are infinitely many compact hypersurfaces of constant scalar curvatures in a sphere. In fact, it was proved in [Ejiri 1982] that there exist countably many isometric immersions of $S^1 \times S^{m-1}$ into a sphere S^{m+1} so that $S^1 \times S^{m-1}$ is a warped product of constant scalar curvature $m(m-1)$ with respect to the induced metric.

Theorem 2.4. *A compact Einstein hypersurface $M^m \hookrightarrow (N^{m+1}, h)$ in an Einstein manifold with $\mathrm{Ric}^N = \lambda h$ is biharmonic if and only if it is minimal or $|A|^2 = \lambda$. Furthermore, in the latter case, the hypersurface has positive scalar curvature, i.e.,*

$$\mathrm{Scal}^M = (m-2)\lambda + m^2 H^2 > 0.$$

Proof. Suppose the hypersurface M is Einstein with $\mathrm{Ric}^M = \mu g$, then, by (6),

$$(m+1)\lambda = m\mu + |A|^2 - m^2 H^2 + 2\lambda.$$

Hence,

$$\mu = \left(1 - \frac{1}{m}\right)\lambda - \frac{1}{m}|A|^2 + mH^2. \tag{20}$$

It follows that

$$\begin{aligned} \langle \nabla H, \nabla \Delta H \rangle &= \langle \nabla H, \nabla (H|A|^2 - \lambda H) \rangle \\ &= |A|^2 |\nabla H|^2 + H \langle \nabla H, \nabla |A|^2 \rangle - \lambda |\nabla H|^2 \\ &= (|A|^2 - \lambda + 2m^2 H^2) |\nabla H|^2, \end{aligned} \tag{21}$$

where the first equality was obtained by using the first equation of (8), and the third equality follows from (6) and the fact that the scalar curvature of an Einstein hypersurface is constant.

Using these and a similar computation used in obtaining (15) we have,

$$\begin{aligned}(22)\qquad \tfrac{1}{2}\Delta|\nabla H|^2 &= |\nabla \mathrm{d}H|^2+\mathrm{Ric}^M(\nabla H,\nabla H)+\langle\nabla H,\nabla\Delta H\rangle\\ &\geq \frac{1}{m}(\Delta H)^2+\mu|\nabla H|^2+(|A|^2-\lambda+2m^2H^2)|\nabla H|^2\\ &= \frac{1}{m}(\Delta H)^2+\Big(-\frac{1}{m}\lambda+\Big(1-\frac{1}{m}\Big)|A|^2+m(2m+1)H^2\Big)|\nabla H|^2.\end{aligned}$$

Similar to (17), we have

$$\begin{aligned}(23)\qquad \frac{1}{m}\int_M \Delta H\Delta H dv_g &= \frac{1}{m}\int_M \Delta H \mathrm{div}(\nabla H)dv_g = -\frac{1}{m}\int_M \langle\nabla\Delta H,\nabla H\rangle dv_g\\ &= -\frac{1}{m}\int_M (|A|^2-\lambda+2m^2H^2)|\nabla H|^2 dv_g.\end{aligned}$$

Integrating both sides of (22) and using the compactness of M and (23) we obtain

$$\begin{aligned}(24)\qquad 0 &\geq \int_M \Big(\frac{1}{m}(m-2)|A|^2+m(2m-1)H^2\Big)|\nabla H|^2 dv_g\\ &\geq \tfrac{1}{4}m(2m-1)\int_M |\nabla H^2|^2 dv_g.\end{aligned}$$

It follows that H is constant. If $H=0$, then M is minimal. If $H\neq 0$, by the first equation of (8), we have $|A|^2=\lambda$. Thus, $\mathrm{Scal}^M = m\mu = (m-2)\lambda+m^2H^2>0$. □

For the classification of complete biharmonic hypersurfaces, we notice that it was proved in [Nakauchi and Urakawa 2011], [Maeta 2014a] and [Luo 2015] that a complete hypersurface $M^m \hookrightarrow (N^{m+1},h)$ in a manifold of nonpositive Ricci curvature with mean curvature function $H\in L^p(M)$ for some $0<p<\infty$ is minimal. We will give a classification of biharmonic hypersurfaces of constant scalar curvatures in an Einstein manifold using a condition on the maximum rate of change of the mean curvature function. For that purpose, we will need the following maximum principles:

Theorem 2.5 [Yau 1976; Karp and Li 1982]. *Let M be a complete Riemannian manifold and f be a smooth function on M. If one of the following conditions is satisfied, then f is constant.*

(i) *M has Ricci curvature bounded from below and $\Delta f\geq \varepsilon f$ (for some $\varepsilon>0$) for an upper bounded function $f\geq 0$.*

(ii) *$\Delta f\geq 0$ for $f\geq 0$ and $f\in L^p(M)$ for some $1<p<\infty$.*

(iii) *$f\in L^1(M)$, $\Delta f\geq 0$, $f\geq 0$ and the Ricci curvature is bounded from below by $-c\{1+r^2(x)\}$, where $r(x)$ is the distance function on M.*

Now we are ready to give some classifications of biharmonic hypersurfaces with constant scalar curvature in a nonpositively curved Einstein manifold.

Proposition 2.6. *A complete biharmonic hypersurface $M^m \hookrightarrow (N^{m+1}, h)$ of constant scalar curvature in a nonpositively curved Einstein manifold (N^{m+1}, h) is minimal if one of the following occurs*:

(a) $|\nabla H| \in L^p$ *for some* $2 < p < \infty$.

(b) $|\nabla H| \in L^2$ *and the Ricci curvature is bounded from below by* $-c\{1 + r^2(x)\}$, *where* $r(x)$ *is the distance function on M.*

Proof. If the ambient space (N^{m+1}, h) is a nonpositively curved Einstein space with $\widetilde{\mathrm{Ric}} = \lambda h$, then from (5) we have

$$\begin{aligned} \mathrm{Ric}(\nabla H, \nabla H) &\geq \lambda|\nabla H|^2 + mH\langle A(\nabla H), \nabla H\rangle - \langle A(\nabla H), A(\nabla H)\rangle \\ &= \bigl(\lambda - \tfrac{3}{4}m^2H^2\bigr)|\nabla H|^2, \end{aligned} \tag{25}$$

where the equality was obtained by using the second equation of the biharmonic hypersurface equation (8).

Substituting (25) and (21) into the Bochner formula we have

$$\begin{aligned} \Delta|\nabla H|^2 &= 2\{|\nabla \mathrm{d}H|^2 + \mathrm{Ric}(\nabla H, \nabla H) + \langle \nabla H, \nabla \Delta H\rangle\} \\ &\geq 2\{\mathrm{Ric}(\nabla H, \nabla H) + \langle \nabla H, \nabla \Delta H\rangle\} \\ &= 2\bigl[\bigl(\lambda - \tfrac{3}{4}m^2H^2\bigr) + (|A|^2 - \lambda + 2m^2H^2)\bigr]|\nabla H|^2 \\ &= 2\bigl(\tfrac{5}{4}m^2H^2 + |A|^2\bigr)|\nabla H|^2 \geq 2\bigl(\tfrac{5}{4}m^2H^2 + mH^2\bigr)|\nabla H|^2 \\ &= \tfrac{1}{8}m(5m+4)|\nabla H^2|^2 \geq 0. \end{aligned} \tag{26}$$

Using maximum principles (ii) and (iii) in Theorem 2.5 we have that $|\nabla H|$ is constant. Using this and (26) again we conclude that $|\nabla H^2| = 0$. It follows that H is constant. If $H = \text{constant} \neq 0$, then, by the first equation of (8), we have $|A|^2 - \lambda = 0$. It follows that $|A|^2 = \lambda \leq 0$ since N is nonpositively curved. From this, we have $|A|^2 = 0$, which means that M is totally geodesic and hence minimal, which is a contradiction. So we are left with the only conclusion that $H = 0$, that is, the hypersurface is minimal. Thus, we complete the proof of the proposition. □

Remark 2. (i) For a classification of complete biharmonic submanifolds with Ricci curvature bounded from below in a nonpositively curved manifold see [Maeta 2014b].

(ii) Our Theorem 2.4 and Proposition 2.6 give some classifications of biharmonic hypersurfaces in an Einstein space. We refer to [Inoguchi and Sasahara 2016; 2017] for some examples and classifications of constant mean curvature proper biharmonic hypersurfaces in a special class of Einstein spaces — the compact Riemannian symmetric space with a G-invariant metric. Also, for some classifications of f-biharmonic hypersurfaces in an Einstein space, see [Ou 2017].

As a corollary of Proposition 2.6, we have an affirmative partial answer to Chen's conjecture:

Corollary 2.7. *Any complete biharmonic hypersurface with constant scalar curvature and $|\nabla H| \in L^p(M)$ for some $2 < p < \infty$ in a Euclidean space is minimal.*

Proposition 2.8. *Let $M^m \hookrightarrow S^{m+1}$ be a complete biharmonic hypersurface with constant scalar curvature in a sphere with $H^2 \geq (2\varepsilon + 4)/(m(5m+4))$ (for some $\varepsilon > 0$). If one of the following is satisfied, then the mean curvature is constant.*

(A) *The Ricci curvature of M is bounded from below, and $|\nabla H|$ from above.*

(B) $|\nabla H| \in L^p$ *for* $2 < p < \infty$.

(C) $|\nabla H| \in L^2$ *and the Ricci curvature is bounded from below by $-c\{1 + r^2(x)\}$, where $r(x)$ is the distance function on M.*

Proof. It follows from (15) that

$$\Delta|\nabla H|^2 \geq 2\big(-1 + \tfrac{5}{4}m^2H^2 + |A|^2\big)|\nabla H|^2.$$

Using Newton's formula $|A|^2 \geq mH^2$ for a hypersurface we have

$$\Delta|\nabla H|^2 \geq 2\Big(-1 + \frac{m}{4}(5m+4)H^2\Big)|\nabla H|^2.$$

From this, together with the assumption that $H^2 \geq (2\varepsilon + 4)/(m(5m+4))$, we have

$$\Delta|\nabla H|^2 \geq \varepsilon|\nabla H|^2. \tag{27}$$

Using the maximum principles (i), (ii), and (iii) in Theorem 2.5 with $f = |\nabla H|^2$ we have obtain the proposition. □

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