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GENERALIZED MULLINEUX INVOLUTION AND PERVERSE EQUIVALENCES

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We define a generalization of the Mullineux involution on multipartitions using the theory of crystals for higher-level Fock spaces. Our generalized Mullineux involution turns up in representation theory via two important derived functors on cyclotomic Cherednik category $\mathcal{O}$: Losev's "$\kappa = 0$" wall-crossing, and Ringel duality.

Introduction

It has been known since the foundational work of Frobenius that partitions of n naturally label the complex irreducible representations of the symmetric group $\mathfrak{S}_n$. If we take an irreducible representation labeled by a partition λ and tensor it with the sign representation, we obtain an irreducible representation labeled by the transpose of λ. The story in positive characteristic is more subtle: the irreducible representations of $\mathfrak{S}_n$ over a field of characteristic $p > 0$ are labeled by the p-regular partitions (partitions in which each nonzero part occurs at most $p-1$ times). Tensoring such a representation with the sign representation still yields an irreducible representation, but the resulting involution on p-regular partitions lacks such a simple description as taking the transpose. Mullineux [1979] defined a combinatorial algorithm producing an involution on p-regular partitions (now called the Mullineux involution), and he conjectured that this involution describes the result of tensoring an irreducible representation with the sign representation in characteristic p.

Kleshchev [1996] came up with a surprising algorithm to compute the Mullineux involution. In fact, whereas Mullineux's algorithm involved repeated operations with strips of boxes in the rim of the Young diagram, it was later understood that Kleshchev's algorithm could be interpreted in terms of the Kashiwara crystal of an irreducible highest weight module of level 1 for the quantum group of affine type A_{p-1} [Lascoux et al. 1996]: the Mullineux involution is the automorphism of oriented $\mathbb{Z}/p\mathbb{Z}$-colored graphs which switches the sign of each arrow. This algorithm led to Ford and Kleshchev's proof of the Mullineux conjecture [1997]; a different proof was given later by Bessenrodt and Olsson [1998].

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The Mullineux involution can be generalized to various extents. First, one can look at the Hecke algebra of $\mathfrak{S}_n$ (which can be seen as a deformation of the group algebra) with parameter specialized to a primitive e-th root of 1, $e \in \mathbb{Z}_{\geq 2}$. An involution on the set of e-regular partitions (which parametrize the associated irreducible representations) can then be defined using crystals as above; see [Lascoux et al. 1996, Section 7]. Fayers [2008] defined a Mullineux involution for the Hecke algebra of the complex reflection group $G(\ell, 1, n)$ (the Ariki–Koike algebra). Fayers' involution can also be computed using crystal graphs (now for irreducible highest weight modules of level ℓ) or via a combinatorial algorithm generalizing Mullineux's original procedure [Jacon and Lecouvey 2009]. The Ariki–Koike algebra has cell modules labeled by all ℓ-partitions, but simples labeled only by Uglov ℓ-partitions (which coincide with e-regular partitions for $\ell = 1$). However, its module category is a quotient of a highest weight category $\mathcal{O}_{\kappa,\mathbf{s}}$ where every ℓ-partition labels a simple module, raising the question whether the Mullineux involution admits a further meaningful extension to that bigger category.

Namely, consider the category $\mathcal{O}_{\kappa,\mathbf{s}}(n)$ of the Cherednik algebra of $G(\ell, 1, n)$. This category depends on parameters $\kappa \in \mathbb{Q}^\times$ and $\mathbf{s} \in \mathbb{Q}^\ell$ [Ginzburg et al. 2003; Rouquier 2008; Losev 2017b] and its Grothendieck group has a basis consisting of ℓ-partitions of n. In order to relate categories depending on different parameters, Losev [2015] introduced derived equivalences called wall-crossing functors. Each wall-crossing can be thought of as a partial version of a duality functor called Ringel duality. The wall-crossing functors and Ringel duality are examples of a special kind of derived equivalence called a perverse equivalence [Chuang and Rouquier 2008; Losev 2017a], and consequently they effect a permutation of the set of simple objects, that is, a permutation of ℓ-partitions. It is natural to ask for an explicit formula for these combinatorial maps.

We now summarize the main results of this paper. In Theorem 2.10 we define a generalization of the Mullineux involution on all multipartitions. The proof uses the result of [Gerber 2018] that the $\widehat{\mathfrak{sl}_e}$-, $\mathfrak{sl}_\infty$-, and $\widehat{\mathfrak{sl}_\ell}$-crystals on the level ℓ Fock space all commute. Our involution Φ is compatible with both Fayers' and Losev's involutions, recovering Fayers' in the case of Uglov multipartitions. The next question is the representation-theoretic meaning of Φ. In Section 3 we study the combinatorics of perverse equivalences on module categories of Cherednik algebras. Theorems 3.5 and 3.7 give some formulas for the $\kappa = 0$ wall-crossing in terms of ℓ copies of the level 1 Mullineux involution; we recover [Losev 2015, Corollary 5.7] when $\ell = 1$. Next, we look for a duality functor which produces the involution Φ, and we find in Theorem 3.14 that Φ arises from Ringel duality. Here the perspective of diagrammatic Cherednik algebras [Webster 2017] is crucial, especially [Webster 2017, Corollary 5.11]. In Section 4, we define a refinement of Φ with a speculative eye towards the Alvis–Curtis duality, a perverse equivalence for finite groups of Lie type which still lacks a combinatorial description outside type A. This generalizes Dudas and Jacon's [2018] definition of a generalized Mullineux involution in the case $\ell = 1$ by refining the $\mathfrak{sl}_\infty$-crystal with respect to an integer parameter d.

1. The Mullineux involution for cyclotomic Hecke algebras

We here give a quick review of the definition of the Mullineux involution for cyclotomic Hecke algebras and its crystal interpretation [Fayers 2008; Jacon and Lecouvey 2009]. This generalizes the usual notion of Mullineux involution.

1A. ***Definition.*** Let $\ell \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 1}$. Denote by $W_{\ell,n}$ the complex reflection group $G(\ell, 1, n) = \mathfrak{S}_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$. Let R be a field of arbitrary characteristic and let $v \in R^\times$ and let $(s_1, s_2, \dots, s_\ell)$ be an ℓ-tuple of integers.

The cyclotomic Hecke algebra (also called the Ariki–Koike algebra) $\mathcal{H}^{\mathbf{s}}_{R,n} = \mathcal{H}(v; s_1, \dots, s_\ell)$ over R is the unital associative R-algebra with a presentation by

- generators: $T_0,\ T_1, \dots,\ T_{n-1}$,
- relations:

$$\begin{aligned} T_0T_1T_0T_1 &= T_1T_0T_1T_0, && \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (i = 1, \dots, n-2), \\ T_iT_j &= T_jT_i && (|j-i| > 1), \\ (T_0 - v^{s_1})(T_0 - v^{s_2}) \cdots (T_0 - v^{s_\ell}) &= 0, && \\ (T_i - v)(T_i + 1) &= 0, && (i = 1, \dots, n-1). \end{aligned}$$

It can be seen as a deformation of the group algebra of $W_{\ell,n}$. In particular, if $\ell = 1$, it is the usual Hecke algebra of type A and if moreover $v = 1$, we obtain the group algebra $R\mathfrak{S}_n$ of the symmetric group. We denote by:

- Π^ℓ the set of all ℓ-partitions, that is, the set of all ℓ-tuples $(\lambda^1, \dots, \lambda^\ell)$ of partitions.
- $\Pi = \Pi^1$ the set of all partitions.

The unique ℓ-partition of size 0 is denoted by $\boldsymbol{\emptyset}$. For any subset $\mathcal{E}$ of Π^ℓ and any $n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{E}(n)$ the set of ℓ-partitions in $\mathcal{E}$ of total size $|\lambda^1| + \cdots + |\lambda^\ell| = n$.

Let e be the multiplicative order of v in R. We assume that $v \neq 1$ so that we have $e \in \{2, 3, \dots\} \sqcup \{\infty\}$. We now recall several facts about the representation theory of cyclotomic Hecke algebras. We refer to [Geck and Jacon 2011, Chapter 5] for details. For each $\boldsymbol{\lambda} \in \Pi^\ell(n)$, there is an $\mathcal{H}^{\mathbf{s}}_{R,n}$-module $S^{\boldsymbol{\lambda}}$ which is the Specht module associated to $\boldsymbol{\lambda}$. There exists a natural bilinear form, $\mathcal{H}^{\mathbf{s}}_{R,n}$-invariant, on each of these modules and an associated radical such that the quotients $D^{\boldsymbol{\lambda}} := S^{\boldsymbol{\lambda}} / \operatorname{rad}(S^{\boldsymbol{\lambda}})$ are either 0 or irreducible. The nonzero $D^{\boldsymbol{\lambda}}$ then give a complete set of nonisomorphic simple $\mathcal{H}^{\mathbf{s}}_{R,n}$-modules.

The set $\{\boldsymbol{\lambda} \in \Pi^{\ell}(n) \mid D^{\boldsymbol{\lambda}} \neq 0\}$ depends only on e and $\mathbf{s}$ and is known as the set of Kleshchev ℓ-partitions, denoted $\mathrm{Kl}_{e,\mathbf{s}}(n)$. It was originally defined using the notion of crystal (see [Geck and Jacon 2011, Section 6.2.10]), but there is another independent description; see [Jacon 2018].

Remark 1.1. Let $e \geq 2$, $\mathbf{s} = (s_1, \dots, s_\ell)$ and $\mathbf{t} = (t_1, \dots, t_\ell)$ such that $t_i = s_i \mod e$ for all $i = 1, \dots, \ell$. Observe that for any $n \in \mathbb{Z}_{\geq 0}$, $\mathcal{H}^{\mathbf{s}}_{R,n} = \mathcal{H}^{\mathbf{t}}_{R,n}$, and the definition of Kleshchev ℓ-partitions gives that

$$\mathrm{Kl}_{e,\mathbf{s}}(n) = \mathrm{Kl}_{e,\mathbf{t}}(n).$$

Now if there exists $\sigma \in \mathfrak{S}_\ell$ such that $t_i = s_{\sigma(i)} \mod e$ for all $i = 1, \dots, \ell$ then we still have that for any $n \in \mathbb{Z}_{\geq 0}$, $\mathcal{H}^{\mathbf{s}}_{R,n} = \mathcal{H}^{\mathbf{t}}_{R,n}$ but $\mathrm{Kl}_{e,\mathbf{s}}(n)$ is different from $\mathrm{Kl}_{e,\mathbf{t}}(n)$ in general.

Set $\widetilde{\mathcal{H}}^{\mathbf{s}}_{R,n} := \mathcal{H}_{R,n}(v^{-1}; s_\ell, \dots, s_1)$ and denote by $\widetilde{T}_0, \dots, \widetilde{T}_{\ell-1}$ the associated standard generators. For each $\boldsymbol{\lambda} \in \Pi^{\ell}(n)$, denote by $\widetilde{S}^{\boldsymbol{\lambda}}$ the associated Specht module of $\widetilde{\mathcal{H}}^{\mathbf{s}}_{R,n}$. By [Fayers 2008], the simple modules of $\widetilde{\mathcal{H}}^{\mathbf{s}}_{R,n}$ are labeled by the set $\mathrm{Kl}_{e,-\mathbf{s}_{\mathrm{rev}}}(n)$ where $-\mathbf{s}_{\mathrm{rev}} = (-s_\ell, \dots, -s_1) \in (\mathbb{Z}/e\mathbb{Z})^{\ell}$. Thus, for each $\boldsymbol{\lambda} \in \mathrm{Kl}_{e,-\mathbf{s}_{\mathrm{rev}}}(n)$, we have an associated simple $\widetilde{\mathcal{H}}^{\mathbf{s}}_{R,n}$-module $\widetilde{D}^{\boldsymbol{\lambda}}$. We have an involutive isomorphism $\theta : \mathcal{H}^{\mathbf{s}}_{R,n} \to \widetilde{\mathcal{H}}^{\mathbf{s}}_{R,n}$ given by

$$T_0 \mapsto \widetilde{T}_0, \qquad T_i \mapsto -v\widetilde{T}_i \ (i = 1, \dots, n-1).$$

Then, θ induces a functor F from the category of $\widetilde{\mathcal{H}}^{\mathbf{s}}_{R,n}$-modules to the category of $\mathcal{H}^{\mathbf{s}}_{R,n}$-modules. As a consequence, we obtain a bijective map

$$\mathfrak{m}_{e,\mathbf{s}} : \mathrm{Kl}_{e,\mathbf{s}}(n) \to \mathrm{Kl}_{e,-\mathbf{s}_{\mathrm{rev}}}(n),$$

satisfying

$$F(\widetilde{D}^{\mathfrak{m}_{e,\mathbf{s}}(\boldsymbol{\lambda})}) \simeq D^{\boldsymbol{\lambda}},$$

for all $\lambda \in \mathrm{Kl}_{e,\mathbf{s}}$.

Remark 1.2. (1) By definition of θ, $\mathfrak{m}_{e,-\mathbf{s}_{\mathrm{rev}}} \circ \mathfrak{m}_{e,\mathbf{s}} = \mathrm{Id}_{\mathrm{Kl}_{e,\mathbf{s}}}$ and $\mathfrak{m}_{e,\mathbf{s}} \circ \mathfrak{m}_{e,-\mathbf{s}_{\mathrm{rev}}} = \mathrm{Id}_{\mathrm{Kl}_{e,-\mathbf{s}_{\mathrm{rev}}}}$.

(2) Assume that $\ell = 1$. Then the map $\mathfrak{m}_{e,\mathbf{s}}$ is an involution and it does not depend on the choice of $\mathbf{s}$. In fact, we have $\mathfrak{m}_{e,\mathbf{s}} = m_e$, where m_e is the usual Mullineux involution defined in the introduction.

1B. ***The quantum algebra*** $\mathcal{U}_t(\widehat{\mathfrak{sl}_e})$. We denote by $\Lambda_0, \dots, \Lambda_{e-1}$ (where the subscripts are understood modulo e) the fundamental weights attached to the Kac–Moody algebra $\widehat{\mathfrak{sl}_e}$. The simple roots are denoted by $\alpha_0, \dots, \alpha_{e-1}$ and $\delta := \alpha_0 + \cdots + \alpha_{e-1}$ is the null root. The fundamental weights and the simple roots are related by the formula

$$\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} + \delta_{i,0}\delta \quad \text{ for all } 0 \leq i \leq e-1$$

(where δ_{ij} denotes the Kronecker symbol). We denote by $\mathcal{P} = \bigoplus_{0\le i\le e-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ the weight lattice and by $\mathcal{U}_t(\widehat{\mathfrak{sl}_e})$ the quantum algebra associated to $\widehat{\mathfrak{sl}_e}$, where t is an indeterminate. This is an algebra over $\mathbb{C}(t)$ with generators e_i, f_i, $t_i^{\pm 1}$ $(0 \le i \le e-1)$ and $\partial^{\pm 1}$ subject to standard relations which we do not recall. We refer to [Geck and Jacon 2011, Chapter 6] for details on this algebra and its representation theory.

1C. ***The level ℓ Fock space.*** Let us fix some notation. Fix $e, \ell \ge 2$ and $s \in \mathbb{Z}$. For $\mathbb{K} = \mathbb{Z}$ or $\mathbb{Q}$, we denote

$$\mathbb{K}^\ell(s) = \left\{ (s_1, \dots, s_\ell) \in \mathbb{K}^\ell \;\middle|\; \sum_{i=1}^{\ell} s_i = s \right\}.$$

For $\mathbf{s} \in \mathbb{Z}^\ell(s)$, we denote by $\Pi^\ell_{\mathbf{s}}$ the set of all symbols of the form $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ with $\boldsymbol{\lambda} \in \Pi^\ell$. Further, denote by Π^ℓ_s the sets of all elements in $\Pi^\ell_{\mathbf{s}}$ where $\mathbf{s} \in \mathbb{Z}^\ell(s)$. Let $\mathcal{F}_{e,\mathbf{s}}$ be the $\mathbb{C}(t)$-vector space with standard basis $\Pi^\ell_{\mathbf{s}}$, i.e., $\mathcal{F}_{e,\mathbf{s}} = \bigoplus_{\boldsymbol{\lambda}\in\Pi^\ell} \mathbb{C}(t)|\boldsymbol{\lambda}, \mathbf{s}\rangle$, called the Fock space of level ℓ and rank e (associated to the charge $\mathbf{s}$). This space can be endowed with a structure of an integrable $\mathcal{U}_t(\widehat{\mathfrak{sl}_e})$-module; see [Geck and Jacon 2011, Section 6.2].

One can decompose this module as a direct sum of remarkable vector spaces. Indeed, if $w := \sum_{0\le i\le e-1} a_i\Lambda_i + d\delta \in \mathcal{P}$, define

$$\mathcal{F}_{e,\mathbf{s}}[w] := \{ m \in \mathcal{F}_{e,\mathbf{s}} \mid \partial m = t^d m,\ t_i m = t^{a_i} m \text{ for all } i \in [0, e-1] \}.$$

If this space is nonzero, we say that w is a weight for $\mathcal{F}_{e,\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{s}}[w]$ is called the w-weight space. The elements of $\mathcal{F}_{e,\mathbf{s}}[w]$ are called weight vectors. Importantly, each element of the standard basis $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is a weight vector and the associated weight may be easily computed (see, for example, [Yvonne 2007, Corollary 2.5]). In particular, one can always write it as

$$d\delta + \sum_{0\le i\le e-1} \Lambda_{s_i} - \sum_{0\le i\le e-1} m_i\alpha_i$$

and the number $\sum_{0\le i\le e-1} m_i$ corresponds to the size of $\boldsymbol{\lambda}$. In particular, the weight of $|\boldsymbol{\emptyset}, \mathbf{s}\rangle$ is $\sum_{0\le i\le e-1} \Lambda_{s_i}$. Thus $\mathcal{F}_{e,\mathbf{s}}$ is the direct sum of its weight spaces.

1D. ***The $\widehat{\mathfrak{sl}_e}$-crystal of the Fock space.*** As mentioned in the introduction, an important part of the representation theory of cyclotomic Hecke algebras is controlled by the theory of crystals for Fock spaces. The $\widehat{\mathfrak{sl}_e}$-crystal of the Fock space $\mathcal{F}_{e,\mathbf{s}}$ is a combinatorial construction arising from the action of $\mathcal{U}_t(\widehat{\mathfrak{sl}_e})$ on the Fock space (see the general definition in [Kashiwara 1991; Hong and Kang 2002]). Concretely, the $\widehat{\mathfrak{sl}_e}$-crystal is a graph with

- vertices: the elements of $\Pi^\ell_{\mathbf{s}}$,

- arrows: $|\boldsymbol{\lambda}, \mathbf{s}\rangle \xrightarrow{i} |\boldsymbol{\mu}, \mathbf{s}\rangle$ for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Pi^\ell$, $i \in \{0, \dots, e-1\}$ if and only if $|\boldsymbol{\mu}, \mathbf{s}\rangle = \widetilde{f}_i |\boldsymbol{\lambda}, \mathbf{s}\rangle$, where $\widetilde{f}_i$ is the i-th lowering Kashiwara operator of $\mathcal{U}_t(\widehat{\mathfrak{sl}_e})$.

An explicit recursive formula for computing the $\widehat{\mathfrak{sl}_e}$-crystal is given in [Jimbo et al. 1991] in terms of adding good boxes; see also [Foda et al. 1999]. It has infinitely many connected components, each of which is parametrized by its unique source vertex, called a highest weight vertex. We denote by $\mathrm{Ug}_{e,\mathbf{s}}$ the ℓ-partitions appearing in the connected component parametrized by the highest weight vertex $\boldsymbol{\emptyset} = (\emptyset, \dots, \emptyset)$, and call them the Uglov ℓ-partitions. When $\ell = 1$, this set is nothing but the set of e-regular partitions. The following is an easy consequence of the definition of Kleshchev and Uglov ℓ-partitions (see [Geck and Jacon 2011, Example 6.2.16]).

Proposition 1.3. *Fix $n \in \mathbb{Z}_{\geq 0}$. Let $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$ and $\mathbf{t} = (t_1, \dots, t_\ell) \in \mathbb{Z}^\ell$ be such that $s_i = t_i \mod e$ for all $i = 1, \dots, \ell$, and $t_i - t_{i-1} > n - 1$ for all $i = 2, \dots, \ell$. Then*

$$\mathrm{Ug}_{e,\mathbf{t}}(n) = \mathrm{Kl}_{e,\mathbf{t}}(n) = \mathrm{Kl}_{e,\mathbf{s}}(n).$$

In other words, Kleshchev ℓ-partitions are a particular case of Uglov ℓ-partitions, i.e., we can index irreducible modules of cyclotomic Hecke algebras by certain vertices of Fock space crystals. The following result is due to Fayers [2008, Section 2] in the case of Kleshchev multipartitions (that is, under the condition of Proposition 1.3) and to [Jacon and Lecouvey 2009, Section 4] in general.

Theorem 1.4. *Let $n \in \mathbb{Z}_{\geq 0}$, $\mathbf{s} \in \mathbb{Z}^\ell(s)$ and $e \geq 2$. There exists a unique bijection*

$$\boldsymbol{\Phi}_{e,\mathbf{s}} : \mathrm{Ug}_{e,\mathbf{s}}(n) \to \mathrm{Ug}_{e,-\mathbf{s}_{\mathrm{rev}}}(n), \qquad \boldsymbol{\lambda} \mapsto \boldsymbol{\Phi}_{e,\mathbf{s}}(\boldsymbol{\lambda})$$

such that

- $\boldsymbol{\Phi}_{e,\mathbf{s}}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$,
- *for all $0 \leq i \leq e-1$, we have $\boldsymbol{\Phi}_{e,\mathbf{s}} \circ \widetilde{f}_i = \widetilde{f}_{-i} \circ \boldsymbol{\Phi}_{e,\mathbf{s}}$.*

This means that for all paths

$$|\boldsymbol{\emptyset}, \mathbf{s}\rangle \xrightarrow{i_1} \cdot \xrightarrow{i_2} \cdot \xrightarrow{i_3} \cdots \xrightarrow{i_n} |\boldsymbol{\lambda}, \mathbf{s}\rangle$$

in the $\widehat{\mathfrak{sl}_e}$-crystal on the Fock space $\mathcal{F}_{e,\mathbf{s}}$, there exists a corresponding path

$$|\boldsymbol{\emptyset}, -\mathbf{s}_{\mathrm{rev}}\rangle \xrightarrow{-i_1} \cdot \xrightarrow{-i_2} \cdot \xrightarrow{-i_3} \cdots \xrightarrow{-i_n} |\boldsymbol{\mu}, -\mathbf{s}_{\mathrm{rev}}\rangle$$

in the $\widehat{\mathfrak{sl}_e}$-crystal on the Fock space $\mathcal{F}_{e,-\mathbf{s}_{\mathrm{rev}}}$ from the empty ℓ-partition to an ℓ-partition $\boldsymbol{\mu} \in \mathrm{Ug}_{e,-\mathbf{s}_{\mathrm{rev}}}$. Then $\boldsymbol{\Phi}_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \boldsymbol{\mu}$. In [Jacon and Lecouvey 2009], it is explained how the map $\boldsymbol{\Phi}_{e,\mathbf{s}}$ can be explicitly computed without constructing the $\widehat{\mathfrak{sl}_e}$-crystal.

Example 1.5. Take $s=4,\ e=4,\ \ell=3,\ \mathbf{s}=(5,-1,0)$ (so that $-\mathbf{s}_{\mathrm{rev}}=(0,1,-5)$) and $\boldsymbol{\lambda}=(1,3.2,\emptyset)$.[1] One can write for instance $\boldsymbol{\lambda}=\tilde{f}_1\tilde{f}_1\tilde{f}_3\tilde{f}_0\tilde{f}_2\tilde{f}_3\,\boldsymbol{\emptyset}$, so that $\boldsymbol{\lambda}\in \mathrm{Ug}_{e,\mathbf{s}}(6)$. Therefore, in the crystal of the Fock space $\mathcal{F}_{e,-\mathbf{s}_{\mathrm{rev}}}$, we get

$$\begin{aligned}\boldsymbol{\Phi}_{e,\mathbf{s}}&=\tilde{f}_{-1}\tilde{f}_{-1}\tilde{f}_{-3}\tilde{f}_0\tilde{f}_{-2}\tilde{f}_{-3}\,\boldsymbol{\emptyset}\\&=\tilde{f}_3\tilde{f}_3\tilde{f}_1\tilde{f}_0\tilde{f}_2\tilde{f}_1\,\boldsymbol{\emptyset}\\&=(2.1,3,\emptyset).\end{aligned}$$

The following result by Fayers [2008] gives the desired crystal interpretation of the Mullineux involution for cyclotomic Hecke algebras.

Theorem 1.6 (Fayers). *Fix $n\in\mathbb{Z}_{\geq 0}$, $\mathbf{s}\in\mathbb{Z}^\ell$ and $e\geq 2$. For all $\boldsymbol{\lambda}\in \mathrm{Kl}_{e,\mathbf{s}}(n)$, we have*

$$\mathfrak{m}_{e,\mathbf{s}}(\boldsymbol{\lambda})=\boldsymbol{\Phi}_{e,\mathbf{s}}(\boldsymbol{\lambda}).$$

To summarize, starting with the usual Mullineux involution m_e for the symmetric group, we obtain

- a generalization of m_e: the involution $\mathfrak{m}_{e,\mathbf{s}}$ on the set of Kleshchev ℓ-partitions which label the irreducible representations of cyclotomic Hecke algebras. If $\ell=1$, we have $\mathfrak{m}_{e,\mathbf{s}}=m_e$.
- a generalization of $\mathfrak{m}_{e,\mathbf{s}}$: the involution $\boldsymbol{\Phi}_{e,\mathbf{s}}$ on the set of Uglov ℓ-partitions. If $\mathbf{s}$ is such that $s_i-s_{i-1}>n-1$ for all $i=2,\dots,\ell$, we have $\boldsymbol{\Phi}_{e,\mathbf{s}}=\mathfrak{m}_{e,\mathbf{s}}$.

2. The generalized Mullineux involution

Recall from Section 1C that we have fixed $e,\ell\geq 2$ and $s\in\mathbb{Z}$. Let us denote $\mathcal{F}_{e,s}=\bigoplus_{\mathbf{s}\in\mathbb{Z}^\ell(s)}\mathcal{F}_{e,\mathbf{s}}$ and $\mathcal{F}_e=\bigoplus_{s\in\mathbb{Z}}\mathcal{F}_{e,s}$.

2A. ***Triple crystal structure.*** By Section 1C, the space $\mathcal{F}_{e,s}$ has a structure of integrable $\mathcal{U}_t(\widehat{\mathfrak{sl}_e})$-module of level ℓ. This space can also be endowed with a structure of $\mathcal{U}_{-1/t}(\widehat{\mathfrak{sl}_\ell})$-module of level e. Denote by $\dot{\Lambda}_0,\dots,\dot{\Lambda}_{\ell-1}$ the fundamental weights attached to the Kac–Moody algebra $\widehat{\mathfrak{sl}_\ell}$. The simple roots are denoted by $\dot{\alpha}_0,\dots,\dot{\alpha}_{\ell-1}$ and $\dot{\delta}$ is the null root. We denote by $\dot{\mathcal{P}}=\bigoplus_{0\leq i\leq \ell-1}\mathbb{Z}\dot{\Lambda}_i\oplus\mathbb{Z}\dot{\delta}$ the corresponding weight lattice.

Following [Gerber 2018], there is a *level-rank duality* between ℓ-partitions and e-partitions. This is a map

$$\mathsf{k}_s^{\ell,e}:\Pi_s^\ell\to\Pi_{-s}^e$$

inducing a linear map between the Fock spaces $\mathcal{F}_{e,s}\to\mathcal{F}_{\ell,-s}$. To avoid cumbersome

[1]In the examples, we use the multiplicative notation for partitions and we forget the brackets around components of a multipartition.

notation, write k for $\mathsf{k}_s^{\ell,e}$ and $\dot{\mathsf{k}}$ for $\mathsf{k}_{-s}^{e,\ell}$. From [Gerber 2018, Formula (3.8)], it is straightforward that $\dot{\mathsf{k}} \circ \mathsf{k} = \mathrm{Id}_{\Pi_s^\ell}$ and $\mathsf{k} \circ \dot{\mathsf{k}} = \mathrm{Id}_{\Pi_{-s}^e}$.

We can extend k linearly to $\mathcal{F}_{e,s}$, which endows it with the structure of a $\mathcal{U}_{-1/t}(\widehat{\mathfrak{sl}_\ell})$-module, by considering the natural action on $\mathcal{F}_{\ell,-s}$ and composing with $\dot{\mathsf{k}}$. This yields an $\widehat{\mathfrak{sl}_\ell}$-crystal structure on $\mathcal{F}_{e,s}$. More precisely, if we denote by $\dot{\tilde{f}}_j$, $j = 0, \dots, \ell-1$, the lowering $\widehat{\mathfrak{sl}_\ell}$-crystal operators, the action of $\dot{\tilde{f}}_j$ on an ℓ-partition is defined by $\dot{\mathsf{k}} \circ \dot{\tilde{f}}_j \circ \mathsf{k}$, as indicated by the following diagram:

$$\begin{array}{ccc} \Pi_s^\ell & \xrightarrow{\ \mathsf{k}\ } & \Pi_{-s}^e \\ \downarrow & & \downarrow \dot{\tilde{f}}_j \\ \Pi_s^\ell & \xleftarrow[\ \dot{\mathsf{k}}\ ]{} & \Pi_{-s}^e \end{array} \tag{2.1}$$

Remark 2.2. (1) As explained in [Gerber 2018, Section 7.1], the map k is, up to a twist by conjugation, categorified by *Koszul duality* between the corresponding Cherednik categories $\mathcal{O}$. This justifies the notation.

(2) The level-rank duality k used in our paper is not the same as the one used in Yvonne and Uglov's paper. However, our map can be recovered from Uglov and Yvonne's ones by composing with the map $|\boldsymbol{\lambda}, \mathbf{s}\rangle \mapsto |\boldsymbol{\lambda}_{\mathrm{rev}}^{\mathrm{tr}}, -\mathbf{s}_{\mathrm{rev}}\rangle$, where for $\boldsymbol{\lambda} := (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ we have

$$\boldsymbol{\lambda}_{\mathrm{rev}}^{\mathrm{tr}} = ((\lambda^{(\ell)})^{\mathrm{tr}}, \dots, (\lambda^{(1)})^{\mathrm{tr}})$$

and λ^{tr} is the transpose of λ.

For $s \in \mathbb{Z}$, we denote

$$A(s) = \{(s_1, \dots, s_\ell) \in \mathbb{Z}^\ell(s) \mid s_1 \leq \cdots \leq s_\ell \leq s_1 + e\}$$

and, in a dual fashion,

$$\dot{A}(s) = \{(t_1, \dots, t_e) \in \mathbb{Z}^e(s) \mid t_1 \leq \cdots \leq t_e \leq t_1 + \ell\}.$$

Write $\dot{\boldsymbol{\emptyset}} = (\emptyset, \dots, \emptyset) \in \Pi^e$. Note that for $\mathbf{s} \in A(s)$, the set $\mathrm{Ug}_{e,\mathbf{s}}$ has a convenient nonrecursive definition; see [Foda et al. 1999, Theorem 2.10]. By [Gerber 2018, Formula (3.8)], if $\mathbf{s} \in A(s)$, then $\mathsf{k}|\boldsymbol{\emptyset}, \mathbf{s}\rangle = |\dot{\boldsymbol{\emptyset}}, \dot{\mathbf{s}}\rangle$ for some $\dot{\mathbf{s}} \in \dot{A}(-s)$.

Finally, there is an $\mathfrak{sl}_\infty$-crystal structure on Π_s^ℓ arising from the action of a Heisenberg algebra [Shan and Vasserot 2012; Losev 2015; Gerber 2019]. Its connected components are all isomorphic to the branching graph of the symmetric group in characteristic 0 and thus have vertices in bijection with Π. If $\boldsymbol{\lambda}_0$ is a highest weight vertex for the $\mathfrak{sl}_\infty$-crystal, then any ℓ-partition in the same crystal component as $\boldsymbol{\lambda}_0$ is obtained as $\tilde{a}_\sigma(\boldsymbol{\lambda}_0)$ for a unique $\sigma \in \Pi$, where $\tilde{a}_\sigma$ denotes the Heisenberg crystal operator associated to σ; see [Losev 2015; Gerber 2019].

We will make repeated use of the following important theorem, proved in [Gerber 2018, Theorems 6.17 and 6.19], and its corollary.

Theorem 2.3. (1) *The three crystals pairwise commute.*

(2) *Every* $|\boldsymbol{\lambda}, \mathbf{s}\rangle \in \Pi^\ell_{\mathbf{s}}$ *decomposes as*

$$|\boldsymbol{\lambda}, \mathbf{s}\rangle = \tilde{\dot{f}}_{j_r} \cdots \tilde{\dot{f}}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle$$

for some $\mathbf{r} \in A(s)$, $\sigma \in \Pi$, $p, r \in \mathbb{Z}_{\geq 0}$ *and some* $i_p, \dots, i_1 \in \{0, 1, \dots, e-1\}$ *and* $j_r, \dots, j_1 \in \{0, 1, \dots, \ell-1\}$.

Corollary 2.4. *The elements* $\mathbf{r}$, σ, p *and* r *of Theorem 2.3 are uniquely determined by* $|\boldsymbol{\lambda}, \mathbf{s}\rangle$. *This yields a bijection*

$$\beta : \Pi^\ell_s \to \bigsqcup_{\mathbf{r} \in A(s)} \mathrm{Ug}_{e,\mathbf{r}} \times \Pi \times \mathrm{Ug}_{\ell,\dot{\mathbf{r}}},$$

$$|\boldsymbol{\lambda}, \mathbf{s}\rangle \mapsto (\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle, \sigma, \tilde{\dot{f}}_{j_r} \cdots \tilde{\dot{f}}_{j_1} |\dot{\boldsymbol{\emptyset}}, \dot{\mathbf{r}}\rangle).$$

Proof. Let $\boldsymbol{\lambda} \in \Pi^\ell$. By Theorem 2.3(2), there exist $\mathbf{r} \in A(s)$, $\sigma \in \Pi$, $p, r \in \mathbb{Z}_{\geq 0}$ and elements $i_p, \dots, i_1 \in \{0, 1, \dots, e-1\}$ and $j_r, \dots, j_1 \in \{0, 1, \dots, \ell-1\}$ such that

$$|\boldsymbol{\lambda}, \mathbf{s}\rangle = \tilde{\dot{f}}_{j_r} \cdots \tilde{\dot{f}}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle.$$

Assume that we have

$$\tilde{\dot{f}}_{j'_{r'}} \cdots \tilde{\dot{f}}_{j'_1} \tilde{a}_{\sigma'} \tilde{f}_{i'_{p'}} \cdots \tilde{f}_{i'_1} |\boldsymbol{\emptyset}, \mathbf{r}'\rangle = \tilde{\dot{f}}_{j_r} \cdots \tilde{\dot{f}}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle$$

for $\mathbf{r}' \in A(s)$, $\sigma' \in \Pi$, $p', r' \in \mathbb{Z}_{\geq 0}$ and indices $i'_{p'}, \dots, i'_1 \in \{0, 1, \dots, e-1\}$ and $j'_{r'}, \dots, j'_1 \in \{0, 1, \dots, \ell-1\}$. Then the elements

$$|\boldsymbol{\mu}', \mathbf{t}'\rangle := \tilde{\dot{f}}_{j'_{r'}} \cdots \tilde{\dot{f}}_{j'_1} \tilde{a}_{\sigma'} |\boldsymbol{\emptyset}, \mathbf{r}'\rangle \quad \text{and} \quad |\boldsymbol{\mu}, \mathbf{t}\rangle := \tilde{\dot{f}}_{j_r} \cdots \tilde{\dot{f}}_{j_1} \tilde{a}_\sigma |\boldsymbol{\emptyset}, \mathbf{r}\rangle$$

are both highest weight vertices in the $\widehat{\mathfrak{sl}_e}$-crystal. As we have $\tilde{f}_{i'_{p'}} \cdots \tilde{f}_{i'_1} |\boldsymbol{\mu}', \mathbf{t}'\rangle = \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\mu}, \mathbf{t}\rangle$, these two elements are in the same connected component of the $\widehat{\mathfrak{sl}_e}$-crystal so they must be equal. From this equality, we deduce in the same way that the two $\widehat{\mathfrak{sl}_e}$-highest weight vertices must be equal: $\tilde{a}_{\sigma'} |\boldsymbol{\emptyset}, \mathbf{r}'\rangle = \tilde{a}_\sigma |\boldsymbol{\emptyset}, \mathbf{r}\rangle$. By the description of the $\mathfrak{sl}_\infty$-crystal operators [Losev 2015; Gerber 2018], we obtain $\sigma = \sigma'$ and $\mathbf{r}' = \mathbf{r}$. We deduce that $\tilde{\dot{f}}_{j'_{r'}} \cdots \tilde{\dot{f}}_{j'_1} |\dot{\boldsymbol{\emptyset}}, \dot{\mathbf{r}}\rangle = \tilde{\dot{f}}_{j_r} \cdots \tilde{\dot{f}}_{j_1} |\dot{\boldsymbol{\emptyset}}, \dot{\mathbf{r}}\rangle$, where $|\dot{\boldsymbol{\emptyset}}, \dot{\mathbf{r}}\rangle = \mathsf{k} |\boldsymbol{\emptyset}, \mathbf{r}\rangle$. In particular, we have $r' = r$. Using the same argument but exchanging the roles of e and ℓ, we also get $\tilde{f}_{i'_{p'}} \cdots \tilde{f}_{i'_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle = \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle$ and $p' = p$. This proves uniqueness, and therefore β is well defined.

For $\mathbf{r} \in A(s)$ and $(\boldsymbol{\nu}, \boldsymbol{\pi}) \in \mathrm{Ug}_{e,\mathbf{r}} \times \mathrm{Ug}_{\ell,\dot{\mathbf{r}}}$, by [Foda et al. 1999, Theorem 2.10], there exist indices $i_1, \dots, i_p \in \{0, 1, \dots, e-1\}$ and $j_1, \dots, j_r \in \{0, 1, \dots, \ell-1\}$ such that

$$\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle = |\boldsymbol{\nu}, \mathbf{r}\rangle \quad \text{and} \quad \tilde{\dot{f}}_{j_r} \cdots \tilde{\dot{f}}_{j_1} |\dot{\boldsymbol{\emptyset}}, \dot{\mathbf{r}}\rangle = |\boldsymbol{\pi}, \dot{\mathbf{r}}\rangle.$$

The map $\delta : \bigsqcup_{\mathbf{r}\in A(s)} \mathrm{Ug}_{e,\mathbf{r}} \times \Pi \times \mathrm{Ug}_{\ell,\dot{\mathbf{r}}} \to \Pi^\ell_s, (\boldsymbol{\nu}, \sigma, \boldsymbol{\pi}) \mapsto \dot{\tilde{f}}_{j_r} \cdots \dot{\tilde{f}}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle$ is well defined, since it does not depend on the choice of the indices and the three crystals commute. It is straightforward that β and δ are inverse to each other, which concludes the proof. □

Example 2.5. Take $s = 1$, $e = 3$, $\ell = 4$, $\mathbf{s} = (-3, 2, 1, 1)$ and $\boldsymbol{\lambda} = (\emptyset, 3.2^2, \emptyset, 3)$. Then

$$\begin{aligned}\beta(|\boldsymbol{\lambda}, \mathbf{s}\rangle) &= (|(\emptyset, \emptyset, \emptyset, 3), (-1, 0, 0, 2)\rangle, (2), |(2^2, 2.1, \emptyset), (-1, -1, 1)\rangle) \\ &= (\tilde{f}_1 \tilde{f}_0 \tilde{f}_2 |(\emptyset, \emptyset, \emptyset, \emptyset), (-1, 0, 0, 2)\rangle, (2), \\ &\qquad \dot{\tilde{f}}_0 \dot{\tilde{f}}_2 \dot{\tilde{f}}_3 \dot{\tilde{f}}_3 \dot{\tilde{f}}_0 \dot{\tilde{f}}_2 \dot{\tilde{f}}_3 |(\emptyset, \emptyset, \emptyset), (-1, -1, 1)\rangle).\end{aligned}$$

Remark 2.6. Note that for $\ell = 1$, Corollary 2.4 reduces to a very simple bijection. Indeed, there is no $\widehat{\mathfrak{sl}_\ell}$-crystal (and no level-rank duality) in this case, and the bijection associates to any partition λ a pair of partitions (ρ, σ) determined by the "euclidean division" of λ by e, as follows. Given two partitions μ and μ', let $\mu \sqcup \mu'$ be the partition obtained by concatenating the two partitions and then reordering the parts to obtain a partition (see for instance [Dudas and Jacon 2018, Section 3.1]). Then we can uniquely write

$$\lambda = (\sigma)^e \sqcup \rho,$$

where ρ is an e-regular partition and $\sigma \in \Pi$.

Example 2.7. Choose $e = 3$ and $\lambda = (4^4.3^2.2.1^8)$. Then $\lambda = (4.1^2)^3 \sqcup (4.3^2.2.1^2)$.

2B. ***The generalized Mullineux map.*** In the following, we will need to go from one indexation by ℓ-partitions to the other by e-partitions using the map k. We will use the relationship between the weight spaces for the action of $\widehat{\mathfrak{sl}_e}$ and the weight spaces for the action of $\widehat{\mathfrak{sl}_\ell}$ of $\mathcal{F}_{s,e}$.

We start by defining a map $\theta_{\ell,e,s}$ by setting

$$\theta_{\ell,e,s} : \mathbb{Q}^\ell(s) \to \mathbb{Q}^\ell(e), \qquad (s_1, \dots, s_\ell) \mapsto (e - s_1 + s_\ell, s_1 - s_2, \dots, s_{\ell-1} - s_\ell).$$

This is a bijection with inverse map

$$\theta^{-1}_{\ell,e,s} : \mathbb{Q}^\ell(e) \to \mathbb{Q}^\ell(s), \qquad (a_1, \dots, a_\ell) \mapsto (s_1, \dots, s_\ell),$$

where we have for all $1 \le i \le \ell$,

$$s_i = \frac{1}{\ell}\Bigg(s - \sum_{1\le j\le \ell-1} j a_{j+1}\Bigg) + \sum_{i+1\le j\le \ell} a_j.$$

Lemma 2.8. *Keeping the above notation, assume that* $\mathbf{s} = \theta^{-1}_{\ell,e,s}(a_1, \dots, a_\ell)$*; then we have* $-\mathbf{s}_{\mathrm{rev}} = \theta^{-1}_{\ell,e,-s}(a_1, a_\ell, \dots, a_2)$.

Proof. Write $\mathbf{s} = (s_1, \dots, s_\ell)$ and $\mathbf{v} = \theta^{-1}_{\ell,e,-s}(a_1, a_\ell, \dots, a_2)$. On the one hand, we have for all $i = 1, \dots, \ell$

$$s_i = \frac{1}{\ell}\Bigl(s - \sum_{1\le j\le \ell-1} j a_{j+1}\Bigr) + \sum_{i+1\le j\le \ell} a_j,$$

and on the other hand

$$\begin{aligned} v_{\ell-i+1} &= \frac{1}{\ell}\Bigl(-s - \sum_{1\le j\le \ell-1} j a_{\ell-j+1}\Bigr) + \sum_{\ell-i+2\le j\le \ell} a_{\ell-j+2} \\ &= \frac{1}{\ell}\Bigl(-s - \sum_{1\le j\le \ell-1} (\ell-k) a_{k+1}\Bigr) + \sum_{2\le k\le i} a_k. \end{aligned}$$

We obtain

$$s_j + v_{\ell-j+1} = 0$$

and the result follows. □

We have the following result whose proof can be found in [Yvonne 2007, Proposition 2.12], taking into account Remark 2.2(2).

Proposition 2.9. *Let $\dot{\mathbf{s}} = (\dot{s}_1, \dots, \dot{s}_e) \in \mathbb{Z}^e(s)$ and let $\dot{w} \in \dot{\mathcal{P}}$ be a weight for $\mathcal{F}_{\ell,\dot{\mathbf{s}}}$. Then there exist a unique $\mathbf{s} \in \mathbb{Z}^\ell(s)$ and a unique $w \in \mathcal{P}$ such that $\mathsf{k}(\mathcal{F}_{e,\mathbf{s}}[w]) = \mathcal{F}_{\ell,\dot{\mathbf{s}}}[\dot{w}]$. If we write $\dot{w} = d\dot{\delta} + \sum_{0\le i\le \ell-1} a_i \dot{\Lambda}_{i-1}$ with $(a_1, \dots, a_\ell) \in \mathbb{Z}^\ell$ we have*

$$\mathbf{s} = \theta^{-1}_{\ell,e,s}(a_1, a_\ell, \dots, a_2).$$

Moreover the associated weight is

$$w = d\delta + \sum_{0\le i\le e-1} (\dot{s}_i - \dot{s}_{i+1})\Lambda_i,$$

where $\dot{s}_0 = \ell + \dot{s}_e$.

We are now ready to prove the first main result of this paper. Recall the generalized Mullineux map $\mathbf{\Phi}_{e,\mathbf{s}}$ on Uglov ℓ-partitions of Theorem 1.4. By level-rank duality, we have a dual Mullineux map $\mathbf{\Phi}_{\ell,\mathbf{t}}$ for all $\mathbf{t} \in \mathbb{Z}^e(-s)$ which acts on e-partitions.

Theorem 2.10.

(1) *There exists a unique bijection*

$$\Phi : \Pi^\ell_s \to \Pi^\ell_{-s}, \qquad |\boldsymbol{\lambda}, \mathbf{s}\rangle \mapsto |\boldsymbol{\mu}, -\mathbf{s}_{\mathrm{rev}}\rangle$$

such that for all $0 \le i \le e-1$, $\sigma \in \Pi$ and $0 \le j \le \ell-1$,

(a) $\Phi \circ \tilde{f}_i = \tilde{f}_{-i} \circ \Phi$,

(b) $\Phi \circ \tilde{a}_\sigma = \tilde{a}_{\sigma^{\mathrm{tr}}} \circ \Phi$,

(c) $\Phi \circ \dot{\tilde{f}}_j = \dot{\tilde{f}}_{-j} \circ \Phi$,
(d) $\Phi(|\boldsymbol{\emptyset}, \mathbf{s}\rangle) = |\boldsymbol{\emptyset}, -\mathbf{s}_{\mathrm{rev}}\rangle$.

(2) *Using the notation of Corollary 2.4, we have*

$$\Phi = \beta^{-1} \circ (\boldsymbol{\Phi}_{e,\mathbf{r}}, (\cdot)^{\mathrm{tr}}, \boldsymbol{\Phi}_{\ell,\dot{\mathbf{r}}}) \circ \beta.$$

In other words, writing $|\boldsymbol{\lambda}, \mathbf{s}\rangle = \dot{\tilde{f}}_{j_r} \cdots \dot{\tilde{f}}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle$ *with* $\mathbf{r} \in A(s)$, *we have* $\Phi|\boldsymbol{\lambda}, \mathbf{s}\rangle = \dot{\tilde{f}}_{-j_r} \cdots \dot{\tilde{f}}_{-j_1} \tilde{a}_{\sigma^{\mathrm{tr}}} \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} |\boldsymbol{\emptyset}, -\mathbf{r}_{\mathrm{rev}}\rangle$.

(3) *We have* $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}|$ *if* $|\boldsymbol{\mu}, -\mathbf{s}_{\mathrm{rev}}\rangle = \Phi(|\boldsymbol{\lambda}, \mathbf{s}\rangle)$.

Proof. Let $\boldsymbol{\lambda} \in \Pi^\ell$, $\mathbf{s} \in \mathbb{Z}^\ell(s)$ and write $|\boldsymbol{\lambda}, \mathbf{s}\rangle = \dot{\tilde{f}}_{j_r} \cdots \dot{\tilde{f}}_{j_r} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{r}\rangle$ with $\mathbf{r} \in A(s)$ as in Theorem 2.3. If Φ satisfies the four assumptions of the theorem, we have that

$$\Phi(|\boldsymbol{\lambda}, \mathbf{s}\rangle) = \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \tilde{a}_{\sigma^{\mathrm{tr}}} \dot{\tilde{f}}_{-j_r} \cdots \dot{\tilde{f}}_{-j_1} |\boldsymbol{\emptyset}, -\mathbf{r}_{\mathrm{rev}}\rangle$$

and this shows uniqueness. Now, to prove (1) and (2), we need to show that there exists an ℓ-partition $\boldsymbol{\mu}$ such that

$$|\boldsymbol{\mu}, -\mathbf{s}_{\mathrm{rev}}\rangle = \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \tilde{a}_{\sigma^{\mathrm{tr}}} \dot{\tilde{f}}_{-j_r} \cdots \dot{\tilde{f}}_{-j_1} |\boldsymbol{\emptyset}, -\mathbf{r}_{\mathrm{rev}}\rangle.$$

First, note that $\mathsf{k}|\boldsymbol{\emptyset}, -\mathbf{r}_{\mathrm{rev}}\rangle = |\dot{\boldsymbol{\emptyset}}, -\dot{\mathbf{r}}_{\mathrm{rev}}\rangle$ by [Gerber 2018, Formula (3.8)]. Consider the e-partition $\boldsymbol{\lambda}_2$ such that

$$|\boldsymbol{\lambda}_2, \dot{\mathbf{r}}\rangle = \dot{\tilde{f}}_{j_r} \cdots \dot{\tilde{f}}_{j_1} |\dot{\boldsymbol{\emptyset}}, \dot{\mathbf{r}}\rangle$$

and the e-partition $\boldsymbol{\mu}_2$ such that

$$|\boldsymbol{\mu}_2, -\dot{\mathbf{r}}_{\mathrm{rev}}\rangle = \dot{\tilde{f}}_{-j_r} \cdots \dot{\tilde{f}}_{-j_1} |\dot{\boldsymbol{\emptyset}}, -\dot{\mathbf{r}}_{\mathrm{rev}}\rangle$$

defined thanks to Theorem 1.4, i.e.,

$$|\boldsymbol{\mu}_2, -\dot{\mathbf{r}}_{\mathrm{rev}}\rangle = \boldsymbol{\Phi}_{\ell,\dot{\mathbf{r}}}(|\boldsymbol{\lambda}_2, \dot{\mathbf{r}}\rangle).$$

Let $|\boldsymbol{\lambda}_1, \mathbf{s}\rangle = \dot{\mathsf{k}}(|\boldsymbol{\lambda}_2, \dot{\mathbf{r}}\rangle)$, so that $|\boldsymbol{\lambda}, \mathbf{s}\rangle = \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma |\boldsymbol{\lambda}_1, \dot{\mathbf{s}}\rangle$. Let $\mathbf{v} \in \mathbb{Z}^\ell(-s)$ and μ_1 be the ℓ-partition such that $|\boldsymbol{\mu}_1, \mathbf{v}\rangle = \dot{\mathsf{k}}(|\boldsymbol{\mu}_2, -\dot{\mathbf{r}}_{\mathrm{rev}}\rangle)$. Let us show that $\mathbf{v} = -\mathbf{s}_{\mathrm{rev}}$. By hypothesis, the $\widehat{\mathfrak{sl}_\ell}$-weight $\dot{w}$ of $|\boldsymbol{\lambda}_2, \dot{\mathbf{r}}\rangle$ can be written

$$\dot{w} = d\dot{\delta} + \sum_{1 \le i \le \ell} \dot{\Lambda}_{r_i} - \sum_{1 \le j \le t} \alpha_{i_j}$$

and there exists a sequence of nonnegative integers $(a_1, \ldots, a_\ell)$ such that

$$\dot{w} = d\dot{\delta} + \sum_{1 \le i \le \ell} a_i \dot{\Lambda}_{i-1}.$$

By Proposition 2.9, with this notation, we have $\mathbf{s} = \theta^{-1}_{\ell,e,s}(a_1, a_\ell, \dots a_2)$. By hypothesis, the $\widehat{\mathfrak{sl}_e}$-weight $\dot{w}'$ of $|\boldsymbol{\mu}_2, -\dot{\mathbf{r}}_{\mathrm{rev}}\rangle$ can be written

$$\dot{w}' = d'\dot{\delta} + \sum_{1\le i\le \ell} \dot{\Lambda}_{-r_i} - \sum_{1\le j\le t} \dot{\alpha}_{-i_j}$$

and thus we have

$$\dot{w}' = d'\dot{\delta} + \sum_{1\le i\le \ell} a_i \dot{\Lambda}_{1-i}.$$

We thus have $\mathbf{v} = \theta^{-1}_{\ell,e,-s}(a_1, a_2, \dots, a_\ell)$. We conclude that $\mathbf{v} = -\mathbf{s}_{\mathrm{rev}}$ using Lemma 2.8. Therefore, we can set $\boldsymbol{\mu}$ to be the ℓ-partition such that $|\boldsymbol{\mu}, -\mathbf{s}_{\mathrm{rev}}\rangle = \beta^{-1}(\boldsymbol{\mu}_1, \sigma, \boldsymbol{\mu}_2)$, see Corollary 2.4, and this concludes the proof of (1) and (2).

It remains to prove (3). To do this, let us study the $\widehat{\mathfrak{sl}_e}$-weight of $|\boldsymbol{\lambda}_1, \mathbf{s}\rangle$ and $|\boldsymbol{\mu}_1, -\mathbf{s}_{\mathrm{rev}}\rangle$. Again by Proposition 2.9, the weight w of $|\boldsymbol{\lambda}_1, \mathbf{s}\rangle$ is

$$w = d\delta + (e - r_1 + r_e)\Lambda_0 + (r_1 - r_2)\Lambda_1 + \cdots + (r_{e-1} - r_e)\Lambda_{e-1}$$

which can be written as

$$w = d\delta + \sum_{1\le i\le \ell} \Lambda_{t_i} - \sum_{0\le i\le e-1} m_i \alpha_i$$

for a sequence of nonnegative integers $(m_i)_{i=0,\dots,e-1}$. Note that, with this notation, the number $N := \sum_{0\le i\le e-1} m_i$ corresponds to the size of $\boldsymbol{\lambda}_1$ (see Section 1C). Now again, the weight w' of $|\boldsymbol{\mu}_1, -\mathbf{s}_{\mathrm{rev}}\rangle$ is

$$w' = d\delta + (e - r_1 + r_e)\Lambda_0 + (r_{e-1} - r_e)\Lambda_1 + \cdots + (r_1 - r_2)\Lambda_{e-1}$$

which thus can be written as

$$w' = d\delta + \sum_{1\le i\le \ell} \Lambda_{-t_i} - \sum_{0\le i\le e-1} (m_{-i})\alpha_i.$$

We conclude that $|\boldsymbol{\lambda}_1| = |\boldsymbol{\mu}_1|$, that is, $N := \sum_{0\le i\le e-1} m_i$. It follows that $|\boldsymbol{\mu}| = |\boldsymbol{\lambda}| = N + |\sigma| e + r$. □

We may write $\Phi(\boldsymbol{\lambda})$ instead of $\Phi(|\boldsymbol{\lambda}, \mathbf{s}\rangle)$ when the charge $\mathbf{s}$ is understood.

Example 2.11. Take the same values as in Example 2.5. Denote $\mathbf{r} = (-1, 0, 0, 2)$, so that $\dot{\mathbf{r}} = (-1, -1, 1)$. Then we have

$$\begin{aligned}\Phi(|\boldsymbol{\lambda}, \mathbf{s}\rangle) &= \beta^{-1}(\boldsymbol{\Phi}_{e,\mathbf{r}}(\tilde{f}_1\tilde{f}_0\tilde{f}_2|\boldsymbol{\emptyset}, \mathbf{r}\rangle), (2)^{\mathrm{tr}}, \boldsymbol{\Phi}_{\ell,\dot{\mathbf{r}}}(\dot{\tilde{f}}_0\dot{\tilde{f}}_2\dot{\tilde{f}}_3\dot{\tilde{f}}_3\dot{\tilde{f}}_0\dot{\tilde{f}}_2\dot{\tilde{f}}_3|\dot{\boldsymbol{\emptyset}}, \dot{\mathbf{r}}\rangle))\\ &= \beta^{-1}(\tilde{f}_2\tilde{f}_0\tilde{f}_1|(\emptyset, \emptyset, \emptyset, \emptyset), (-2, 0, 0, 1)\rangle, (1^2),\\ &\qquad\qquad \dot{\tilde{f}}_0\dot{\tilde{f}}_2\dot{\tilde{f}}_1\dot{\tilde{f}}_1\dot{\tilde{f}}_0\dot{\tilde{f}}_2\dot{\tilde{f}}_1|(\emptyset, \emptyset, \emptyset), (-1, 1, 1)\rangle)\\ &= \beta^{-1}(|(\emptyset, 1, \emptyset, 2), (-2, 0, 0, 1)\rangle, (1^2), |(\emptyset, 2^2, 2.1), (-1, 1, 1)\rangle)\\ &= |(1, 2.1, \emptyset, 2^3), (-1, -1, -2, 3)\rangle.\end{aligned}$$

The following corollary shows that Φ generalizes the map $\mathbf{\Phi}_{e,\mathbf{s}}$ of Section 1.

Corollary 2.12. *Let* $e \geq 2$, $\mathbf{s} \in \mathbb{Z}^\ell(s)$, *and* $n \geq 0$. *For all* $\boldsymbol{\lambda} \in \mathrm{Ug}_{e,\mathbf{s}}(n)$, *we have*

$$\Phi(\boldsymbol{\lambda}) = \mathbf{\Phi}_{e,\mathbf{s}}(\boldsymbol{\lambda}).$$

Proof. Let $\mathbf{s} \in \mathbb{Z}^\ell(s)$. By Property (3) of Theorem 2.10, we have

$$\Phi(|\boldsymbol{\emptyset}, \mathbf{s}\rangle) = |\boldsymbol{\emptyset}, -\mathbf{s}_{\mathrm{rev}}\rangle.$$

Now if $\boldsymbol{\lambda} \in \mathrm{Ug}_{e,\mathbf{s}}(n)$ there exists a sequence of Kashiwara operators such that

$$\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{s}\rangle = |\boldsymbol{\lambda}, \mathbf{s}\rangle.$$

So we can use Property (1) of Theorem 2.10 to see that

$$\Phi(\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\boldsymbol{\emptyset}, \mathbf{s}\rangle) = \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} |\boldsymbol{\emptyset}, -\mathbf{s}_{\mathrm{rev}}\rangle.$$

By definition of $\mathbf{\Phi}_{e,\mathbf{s}}$ in Theorem 1.4, we get the result. □

2C. ***More on crystal isomorphisms.*** Let $P_\ell := \mathbb{Z}^\ell$ be the $\mathbb{Z}$-module with standard basis $\{z_i \mid i = 1, \dots, \ell\}$. For $k = 1, \dots, \ell - 1$, we denote by σ_k the transposition $(k, k+1)$ of $\mathfrak{S}_\ell$. The extended affine symmetric group $\widehat{\mathfrak{S}}_\ell$ is the semidirect product $P_\ell \rtimes \mathfrak{S}_\ell$ with the relations given by $\sigma_i z_j = z_j \sigma_i$ for $j \neq i, i+1$ and $\sigma_i z_i \sigma_i = z_{i+1}$ for $i = 1, \dots, \ell - 1$ and $j = 1, \dots, \ell$. It acts faithfully on $\mathbb{Z}^\ell$ as follows: for any $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$,

$$\sigma_c.\mathbf{s} = (s_1, \dots, s_{c-1}, s_{c+1}, s_c, s_{c+2}, \dots, s_\ell) \quad \text{for } c = 1, \dots, \ell - 1 \text{ and}$$
$$z_i.\mathbf{s} = (s_1, s_2, \dots, s_i + e, \dots, s_\ell) \quad \text{for } i = 1, \dots, \ell.$$

If $\mathbf{s}$ and $\mathbf{s}'$ are in the same orbit modulo the action of $\widehat{\mathfrak{S}}_\ell$, then there is an $\widehat{\mathfrak{sl}_e}$-crystal isomorphism $\Psi_{\mathbf{s}\to\mathbf{s}'}$ between the Fock spaces $\mathcal{F}_{e,\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{s}'}$; that is, a map

$$\Psi_{\mathbf{s}\to\mathbf{s}'} : \Pi^\ell_{\mathbf{s}} \to \Pi^\ell_{\mathbf{s}'}$$

such that

- $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is a highest weight vertex in $\mathcal{F}_{e,\mathbf{s}}$ if and only if $|\Psi_{\mathbf{s}\to\mathbf{s}'}(\boldsymbol{\lambda}), \mathbf{s}'\rangle$ is a highest weight vertex in $\mathcal{F}_{e,\mathbf{s}'}$,
- for all $\boldsymbol{\lambda} \in \Pi^\ell$, we have $\Psi_{\mathbf{s}\to\mathbf{s}'}(\tilde{f}_i |\boldsymbol{\lambda}, \mathbf{s}\rangle) = \tilde{f}_i \Psi_{\mathbf{s}\to\mathbf{s}'}(|\boldsymbol{\lambda}, \mathbf{s}\rangle)$.

These crystal isomorphisms have been explicitly described in [Jacon and Lecouvey 2009]. Let us now come back to our situation. Assume that $\mathbf{s} \in \mathbb{Z}^\ell$ and choose any $\mathbf{s}' \in \mathbb{Z}^\ell$ in the orbit of $-\mathbf{s}$ modulo the action of the extended affine symmetric group. This is in particular the case for $-\mathbf{s}_{\mathrm{rev}}$. There is an $\widehat{\mathfrak{sl}_e}$-crystal isomorphism between the Fock spaces $\mathcal{F}_{e,-\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{s}'}$. Composing this map with Φ thus gives an isomorphism between $\mathcal{F}_{e,\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{s}'}$.

3. Combinatorics of perverse equivalences for cyclotomic Cherednik category $\mathcal{O}$

In Section 1, we studied the Mullineux involution in the context of representations of cyclotomic Hecke algebras. In this section, we use the results of Section 2 to study the Mullineux involution in the context of representations of cyclotomic rational Cherednik algebras. The goal is to realize the generalized Mullineux involution as the permutation of Π^ℓ induced by certain perverse equivalences. We follow Losev's approach [2015; 2017b].

3A. ***Representations of cyclotomic rational Cherednik algebras.*** We can deform $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rtimes \mathbb{C}W_{\ell,n}$ to obtain an algebra called the cyclotomic rational Cherednik algebra [Etingof and Ginzburg 2002]. This deformation depends on parameters $\kappa \in \mathbb{Q}^\times$ and $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{Q}^\ell$ (the charge). We denote it $\mathsf{H}_{\kappa,\mathbf{s}}(n)$. The charge $\mathbf{s}$ is identified with $\mathbf{s}+\alpha(1, 1, \dots, 1)$ for any scalar α, thus the parameter space is ℓ-dimensional. As a $\mathbb{C}$-vector space,

$$\mathsf{H}_{\kappa,\mathbf{s}}(n) = \mathbb{C}[y_1, \dots, y_n] \otimes \mathbb{C}[W_{\ell,n}] \otimes \mathbb{C}[x_1, \dots, x_n].$$

This makes it possible to define a category $\mathcal{O}$ for $\mathsf{H}_{\kappa,\mathbf{s}}(n)$ as the full subcategory of $\mathsf{H}_{\kappa,\mathbf{s}}(n)$-mod consisting of finitely generated $\mathsf{H}_{\kappa,\mathbf{s}}(n)$-modules which are locally nilpotent for the action of $\mathbb{C}[y_1, \dots, y_n]$ [Ginzburg et al. 2003]. Let $\mathcal{O}_{\kappa,\mathbf{s}}(n)$ denote the category $\mathcal{O}$ of $\mathsf{H}_{\kappa,\mathbf{s}}(n)$, and $\mathcal{O}_{\kappa,\mathbf{s}} = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{O}_{\kappa,\mathbf{s}}(n)$. $\mathcal{O}_{\kappa,\mathbf{s}}(n)$ is a highest weight category whose simple objects are indexed by $\mathrm{Irr}_{\mathbb{C}} W_{\ell,n}$, the irreducible representations of the underlying group algebra $\mathbb{C}W_{\ell,n}$ [Ginzburg et al. 2003], and thus by $\Pi^\ell(n)$. Furthermore, the highest weight structure of $\mathcal{O}_{\kappa,\mathbf{s}}(n)$ depends on a partial order on $\Pi^\ell(n)$ which is determined by the parameter $(\kappa, \mathbf{s})$.

3B. ***Branching rules and crystals.*** In this subsection we restrict to the case of *integral parameters*, that is

$$\kappa = \pm 1/e \quad \text{for some } e \in \mathbb{Z}_{\geq 2} \quad \text{and} \quad \mathbf{s} \in \mathbb{Z}^\ell.$$

The (complexified) Grothendieck group of $\mathcal{O}_{\kappa,\mathbf{s}}$ is a level ℓ Fock space. Recall from Section 1C that the parameters e, $\mathbf{s}$ for the Fock space come from its $\mathcal{U}_t(\widehat{\mathfrak{sl}_e})$-module structure. Shan [2011] has shown that if $\kappa = 1/e$, there is a notion of branching rule arising from Bezrukavnikov and Etingof's parabolic induction functors [2009], which categorifies the $\widehat{\mathfrak{sl}_e}$-crystal of the Fock space $\mathcal{F}_{e,\mathbf{s}}$ [Shan 2011, Theorem 6.3]. Moreover, there is a categorical Heisenberg action on $\mathcal{O}_{\kappa,\mathbf{s}}$ giving rise to the $\mathfrak{sl}_\infty$-crystal on $\mathcal{F}_{e,\mathbf{s}}$ [Shan and Vasserot 2012; Losev 2015]. We have:

Theorem 3.1 [Shan and Vasserot 2012, Proposition 5.18]. *The following are equivalent*:

(1) $\boldsymbol{\lambda}$ *is a highest weight vertex for both the* $\widehat{\mathfrak{sl}_e}$*- and* $\mathfrak{sl}_\infty$*-crystals on* $\mathcal{F}_{e,\mathbf{s}}$.

(2) $L(\boldsymbol{\lambda})$ *is killed by the categorical Heisenberg and* $\widehat{\mathfrak{sl}_e}$ *annihilation operators.*

(3) $L(\boldsymbol{\lambda})$ *is finite-dimensional.*

Thus finite-dimensional simples are labeled by the source vertices of the $\widehat{\mathfrak{sl}_e}$- and $\mathfrak{sl}_\infty$-crystals.

3C. ***Perverse equivalences.*** Perverse equivalences are a special kind of derived equivalence introduced by Chuang and Rouquier [2008; 2017] which are well-suited for combinatorial applications. Let $\mathcal{A}$, $\mathcal{A}'$ be abelian categories with finitely many simple objects and in which every object has finite length. Let S, S' be the sets of isomorphism classes of simple objects of $\mathcal{A}$, $\mathcal{A}'$ respectively. Let

$$0 \subset S_0 \subset S_1 \subset \cdots \subset S_r = S$$

be a filtration of S and $0 \subset S'_0 \subset S'_1 \subset \cdots \subset S'_r = S'$ a filtration of S'. Let $\mathcal{A}_i \subset \mathcal{A}$ and $\mathcal{A}'_i \subset \mathcal{A}'$ be the Serre subcategories generated by S_i and S'_i, respectively, and let $\pi : \{0, 1, \ldots, r\} \to \mathbb{Z}$ be a function.

Definition 3.2 (Chuang–Rouquier). A derived equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$ is *perverse* if for all $i \geq 0$ and all $s \in S_i \setminus S_{i-1}$, the complex $F(s)$ is such that

(1) for $j \neq \pi(i)$, all composition factors of $H^j(F(s))$ are in S'_{i-1},

(2) all the composition factors of $H^{\pi(i)}(F(s))$ are in S'_{i-1} except for a unique one in $S'_i \setminus S'_{i-1}$.

A perverse equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$ therefore gives rise to a canonical bijection $\mathfrak{f} : S \to S'$ sending $s \in S_i \setminus S_{i-1}$ to $\mathfrak{f}(s) := H^{\pi(i)}(F(s)) \bmod \mathcal{A}'_{i-1}$. In Sections 3D and 3E, we show how the generalized Mullineux involution arises from the perverse equivalences given by wall-crossing functors and Ringel duality.

3D. ***Wall-crossing functors.*** Let us recall Losev's [2015] construction of wall-crossing functors. These are derived equivalences between category $\mathcal{O}_{?,\dagger}(n)$'s for parameters which differ by a perturbation of the partial order on $\Pi^\ell(n)$. Denote by $\mathfrak{c}_\mathbb{Z} \subset \mathbb{Q}^\times \times \mathbb{Q}^\ell$ the ℓ-dimensional lattice in the parameter space $\mathbb{C}^\ell$ consisting of those parameters $c' = (\kappa', \mathbf{s}')$ that have *integral difference* with a fixed parameter $c = (\kappa, \mathbf{s})$, i.e., such that $\kappa' - \kappa \in \mathbb{Z}$ and $\kappa'(s'_i - s'_j) - \kappa(s_i - s_j) \in \mathbb{Z}$ for all $1 \leq i < j \leq \ell$. Note that replacing the ℓ-tuple $\mathbf{s} = (s_1, \ldots, s_\ell)$ with $\mathbf{s} + (\alpha, \ldots, \alpha)$ for $\alpha \in \mathbb{Z}$ does not affect the definition of the corresponding rational Cherednik algebra; see the formulas in [Shan and Vasserot 2012, §3.3, Theorem 6.10]. The lattice $\mathfrak{c}_\mathbb{Z}$ is the shift by c of the dual lattice to the lattice spanned by certain hyperplane elements; see [Losev 2015, Sections 2.1.4, 2.7, 4.1.3]. It is isomorphic to $\mathbb{Z}^\ell$. There is a finite set of hyperplanes in $\mathbb{C}^\ell$ called walls dividing $\mathfrak{c}_\mathbb{Z}$ into open cones called chambers. These hyperplanes are defined as follows. For each $\boldsymbol{\lambda} = ((\lambda^1_1, \lambda^1_2, \ldots), \ldots (\lambda^\ell_1, \lambda^\ell_2, \ldots)) \in \Pi^\ell(n)$, let

$$[\boldsymbol{\lambda}] := \{(a, b, j) \mid 1 \leq a,\ 1 \leq b \leq \lambda^j_a,\ 1 \leq j \leq \ell\}$$

be the Young diagram of $\boldsymbol{\lambda}$. For each $(a, b, j) \in [\boldsymbol{\lambda}]$, we define

$$co_c(\gamma) = \kappa\ell(b-a) + \ell h_j,$$

where $h_j = \kappa s_j - j/\ell$ and

$$c_{\boldsymbol{\lambda}} := \sum_{\gamma \in [\boldsymbol{\lambda}]} co_c(\gamma)$$

is called the c-function; the formula in our cyclotomic case was given in [Gordon and Losev 2014]. By definition, the walls are the hyperplanes $\Pi_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ given by $c_{\boldsymbol{\lambda}} = c_{\boldsymbol{\lambda}'}$. Among these walls, we will be interested in the so-called "essential walls" which are the following ones:

(1) The wall $\kappa = 0$ between chambers containing parameters $(\kappa, \mathbf{s})$ such that the denominator e of κ satisfies $2 \le e \le n$, in terms of the above definition. It is of the form $\Pi_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ for some $[\boldsymbol{\lambda}] = [\boldsymbol{\mu}] \sqcup \{\gamma\}$ and $[\boldsymbol{\lambda}'] = [\boldsymbol{\mu}] \sqcup \{\gamma'\}$, for some multipartition $\boldsymbol{\mu}$ and two boxes $\gamma = (a, b, j)$ and $\gamma' = (a', b', j')$ of $\boldsymbol{\mu}$ with $j = j'$.

(2) The walls $h_i - h_j = \kappa m$ with $i \neq j$, $m \in \mathbb{Z}$ and $|m| < n$ between chambers containing parameters such that $s_i - s_j - m \in \kappa^{-1}\mathbb{Z}$. In terms of the above definition they are of the form $\Pi_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ where

$$[\boldsymbol{\lambda}] = [\boldsymbol{\mu}] \sqcup \{\gamma\} \quad \text{and} \quad [\boldsymbol{\lambda}'] = [\boldsymbol{\mu}] \sqcup \{\gamma'\}$$

for a multipartition $\boldsymbol{\mu}$ and two boxes $\gamma = (a, b, i)$ and $\gamma' = (a', b', j)$.

Two categories whose parameters lie in the same chamber are equivalent as highest weight categories [Losev 2015, Proposition 2.8]. On the other hand, the bounded derived categories of $\mathcal{O}_c(n)$ and $\mathcal{O}_{c'}(n)$ are derived equivalent when $c := (\kappa, \mathbf{s})$ is obtained from $c' := (\kappa', \mathbf{s}')$ by crossing a wall to an adjacent chamber. Two chambers are separated by the wall $\Pi_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ if and only if the sign of $c_{\boldsymbol{\lambda}} - c_{\boldsymbol{\lambda}'}$ in a chamber is opposite to $c_{\boldsymbol{\lambda}} - c_{\boldsymbol{\lambda}'}$ in the adjacent one. The derived equivalences $\mathsf{WC}_{c \leftarrow c'} : D^b(\mathcal{O}_c) \to D^b(\mathcal{O}_{c'})$ are called wall-crossing functors. They are defined by taking the derived tensor product with a Harish-Chandra bimodule; see [Losev 2015, Section 2.8] or [Losev 2017a] for the construction of these functors.

Losev [2015, Proposition 2.12] proved that $\mathsf{WC}_{c \leftarrow c'} : D^b(\mathcal{O}_c) \to D^b(\mathcal{O}_{c'})$ is a perverse equivalence with respect to the filtration of simple modules by their supports (the function π then picks out the dimension of the support). Therefore $\mathsf{WC}_{c \leftarrow c'} : D^b(\mathcal{O}_c) \to D^b(\mathcal{O}_{c'})$ induces a canonical bijection $\mathsf{wc}_{c \leftarrow c'}$ on Π^ℓ, called the combinatorial wall-crossing. For walls of type (b), the combinatorial wall-crossings have been studied in [Losev 2015; Jacon and Lecouvey 2018]. In particular, they are given by the crystal isomorphisms of Section 2C for the appropriate parameters [Jacon and Lecouvey 2018, Theorem 11]. We are here interested in the walls of type (a). First we need to see for which types of parameters they are defined.

We will denote by $\mathsf{wc}_{-\leftarrow+}$ the combinatorial wall-crossing from $\kappa > 0$ to $\kappa' < 0$ corresponding to the wall of type (a).

For the next proposition, we will use the following notation. For $c := (\kappa, \mathbf{s})$ and $\pi \in \mathfrak{S}_\ell$, we say that c is *π-asymptotic* if, for all $i = 1, \dots, \ell$,

$$\left(s_{\pi(i)} - \frac{\pi(i)}{\kappa\ell}\right) - \left(s_{\pi(i+1)} - \frac{\pi(i+1)}{\kappa\ell}\right) > n-1.$$

Note that this is slightly more general than Losev's definition [2015], and that both agree for integral parameters. We say that c is *asymptotic* if it is π-asymptotic for some π.

Proposition 3.3. (1) *Let $c = (\kappa, \mathbf{s})$ and $c' = (\kappa', \mathbf{s}')$ in $\mathfrak{c}_\mathbb{Z}$ be such that κ and κ' have the same sign and such that c and c' are both π-asymptotic for some $\pi \in \mathfrak{S}_\ell$. Then there are no essential walls between c and c'.*

(2) *Assume that $c := (\kappa, \mathbf{s})$ and $c' := (\kappa - 1, \mathbf{s}')$ are two parameters with integral difference, with $\kappa > 0$ and $\kappa' := \kappa - 1 < 0$ lying in two different chambers separated by a unique wall of type (a) (and no wall of type (b)). Then there exists $\pi \in \mathfrak{S}_\ell$ such that c is π-asymptotic and c' is π'-asymptotic where $\pi' \in \mathfrak{S}_\ell$ is defined by $\pi'(i) = \pi(\ell - i + 1)$ for all $i = 1, \dots, \ell$. Conversely, if c and c' are as above, they are separated by a unique wall of type (a) and no wall of type (b).*

Proof. Let $(i_1, i_2) \in \{1, \dots, \ell\}^2$ be such that $i_1 \neq i_2$ and assume that

$$\left(s_{i_1} - \frac{i_1}{\kappa\ell}\right) - \left(s_{i_2} - \frac{i_2}{\kappa\ell}\right) > n-1.$$

Let $\boldsymbol{\lambda}$ be an arbitrary ℓ-partition of n and let $\gamma_1 = (a_1, b_1, i_1)$ and $\gamma_2 = (a_2, b_2, i_2)$ be two boxes of the Young diagram. By [Geck and Jacon 2011, Example 5.5.18], we have

$$b_1 - a_1 - (b_2 - a_2) \geq 1 - n.$$

Thus we deduce

$$b_1 - a_1 + s_{i_1} - \frac{i_1}{\kappa\ell} > b_2 - a_2 + s_{i_2} - \frac{i_2}{\kappa\ell},$$

which implies that $co_c(\gamma_1) > co_c(\gamma_2)$. The same calculation holds for c' and we get $co_{c'}(\gamma_1) > co_{c'}(\gamma_2)$. Thus the order on boxes induced by the c-function is the same if we choose it with respect to c or to c', and therefore there is no wall of type (b) between c and c'. Thus we cannot have any walls of type $\Pi_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ between the two parameters. This directly implies (1).

Now we prove (2). Assume thus that $c := (\kappa, \mathbf{s})$ and $c' := (\kappa - 1, \mathbf{s}')$ have integral difference, with $\kappa > 0$ and $\kappa' := \kappa - 1 < 0$ lying in two different chambers separated by a unique wall of type (a) (and no wall of type (b)). Let $(i_1, i_2) \in \{1, \dots, \ell\}^2$

be such that $i_1 \neq i_2$. Consider the ℓ-partition $\boldsymbol{\lambda}$ of $n-1$ such that $\lambda^k = \emptyset$ for all $k \in \{1, \dots, \ell\} \setminus \{i_1\}$ and $\lambda^{i_1} = (n-1)$. We then consider the ℓ-partition $\boldsymbol{\mu}$ of n obtained from $\boldsymbol{\lambda}$ by adding $\gamma_1 := (1, n, i_1)$ and the ℓ-partition $\boldsymbol{\nu}$ of n obtained from $\boldsymbol{\lambda}$ by adding $\gamma_2 := (1, 1, i_2)$. If $c := (\kappa, \mathbf{s})$ does not lie in an essential wall, we have two cases to consider:

- If $c_{\boldsymbol{\mu}} < c_{\boldsymbol{\nu}}$ then we have $co_c(\gamma_1) < co_c(\gamma_2)$. We thus obtain

$$\kappa\ell\left(n-1+s_{i_1}-\frac{i_1}{\kappa\ell}\right) < \kappa\ell\left(s_{i_2}-\frac{i_2}{\kappa\ell}\right)$$

which implies that

$$\left(s_{i_2}-\frac{i_2}{\kappa\ell}\right)-\left(s_{i_1}-\frac{i_1}{\kappa\ell}\right) > n-1.$$

Now consider the ℓ-partition $\boldsymbol{\lambda}'$ of $n-1$ such that $\lambda'^k = \emptyset$ for all $k \in \{1, \dots, \ell\} \setminus \{i_1\}$ and $\lambda'^{i_1} = (1^{n-1})$. We then consider the ℓ-partition $\boldsymbol{\mu}'$ of n obtained from $\boldsymbol{\lambda}$ by adding $\gamma_1 := (n, 1, i_1)$ and the ℓ-partition $\boldsymbol{\nu}'$ of n obtained from $\boldsymbol{\lambda}$ by adding $\gamma_2 := (1, 1, i_2)$. We must have

$$\kappa\ell\left(1-n+s_{i_1}-\frac{i_1}{\kappa\ell}\right) < \kappa\ell\left(s_{i_2}-\frac{i_2}{\kappa\ell}\right)$$

but now as no walls of type (b) must be crossed to go to c', we must also have

$$\kappa'\ell\left(1-n+s'_{i_1}-\frac{i_1}{\kappa\ell}\right) < \kappa'\ell\left(s'_{i_2}-\frac{i_2}{\kappa\ell}\right).$$

This implies that

$$\left(s'_{i_1}-\frac{i_1}{\kappa'\ell}\right)-\left(s'_{i_2}-\frac{i_2}{\kappa'\ell}\right) > n-1$$

because κ' is negative.

- If $c_{\boldsymbol{\mu}} > c_{\boldsymbol{\nu}}$, we now have $co_c(\gamma_1) > co_c(\gamma_2)$ and

$$\kappa\ell\left(n-1+s_{i_1}-\frac{i_1}{\kappa\ell}\right) > \kappa\ell\left(s_{i_2}-\frac{i_2}{\kappa\ell}\right).$$

This implies that

$$\left(s'_{i_2}-\frac{i_2}{\kappa'\ell}\right)-\left(s'_{i_1}-\frac{i_1}{\kappa'\ell}\right) > n-1$$

and by considering the same $\boldsymbol{\lambda}'$ as above we now obtain

$$\left(s_{i_1}-\frac{i_1}{\kappa'\ell}\right)-\left(s_{i_2}-\frac{i_2}{\kappa'\ell}\right) > n-1.$$

We can thus conclude that c is π-asymptotic and c' is π'-asymptotic for a certain $\pi \in \mathfrak{S}_\ell$. Now let us take such a pair (c, c') and let $(i_1, i_2) \in \{1, \dots, \ell\}^2$ such that $i_1 \neq i_2$ and

$$\left(s_{i_1} - \frac{i_1}{\kappa\ell}\right) - \left(s_{i_2} - \frac{i_2}{\kappa\ell}\right) > n-1$$

so that

$$\left(s'_{i_2} - \frac{i_2}{\kappa'\ell}\right) - \left(s'_{i_1} - \frac{i_1}{\kappa'\ell}\right) > n-1.$$

We have already seen that for any ℓ-partition $\boldsymbol{\lambda}$, and $\gamma_1 = (a_1, b_1, i_1)$ and $\gamma_2 = (a_2, b_2, i_2)$ two boxes of the Young diagram, we have $co_c(\gamma_1) > co_c(\gamma_2)$. But, the hypothesis also shows that $co_{c'}(\gamma_1) > co_{c'}(\gamma_2)$. This implies that no wall of type (b) is crossed between c and c'. If we now take an arbitrary ℓ-partition of $n-1$ and consider two addable boxes γ_1 and γ_2 in the same component such that $co_c(\gamma_1) < co_c(\gamma_2)$, then we have $co_{c'}(\gamma_1) > co_{c'}(\gamma_2)$. This implies that a wall of type (a) is crossed. This concludes the proof. □

Assume that $\mathbf{s}$ is π-asymptotic for some $\pi \in \mathfrak{S}_\ell$ and assume moreover that we are in the integral parameter case. We denote $\mathbf{s}_{\text{opp}} := \mathbf{s}'$ an associated multicharge which is π'-asymptotic with respect to the notation of the proposition. This is a slight abuse of notation because $\mathbf{s}'$ is of course not unique in general. However, two π-asymptotic parameters lie in the same chamber. Thus, the associated categories are equivalent as highest weight categories, and we can identify these two parameters. Moreover, note that one can assume that $\mathbf{s}'$ and $\mathbf{s}$ live in the same orbit modulo the action of the affine symmetric group. Indeed, if we denote $\mathbf{s} := (s_1, \dots, s_\ell)$. One can choose $(k_1, \dots, k_\ell) \in \mathbb{Z}^\ell$ so that $\mathbf{s}'' := (s_1 + k_1 e, \dots, s_\ell + k_\ell e)$ is π'-asymptotic. In this case, $c'' = (\kappa', \mathbf{s}'')$ and c are in $\mathfrak{c}_\mathbb{Z}$ and again c'' and c' lie in the same chamber.

Proposition 3.4 [Losev 2015, Proposition 5.6]. (1) *The $\widehat{\mathfrak{sl}_e}$-crystal commutes with* $\mathsf{wc}_{-\leftarrow+}$.

(2) *The $\mathfrak{sl}_\infty$-crystal commutes with* $\mathsf{wc}_{-\leftarrow+}$ *up to taking the transpose*, *that is*, $\mathsf{wc}_{-\leftarrow+} \circ \tilde{a}_\sigma = \tilde{a}_{\sigma^{\text{tr}}} \circ \mathsf{wc}_{-\leftarrow+}$ *for all* $\sigma \in \Pi$.

For the next definition, if $\boldsymbol{\lambda}$ is an arbitrary ℓ-partition and $\mathbf{s}$ an arbitrary ℓ-charge, we denote $|\boldsymbol{\lambda}^{\text{tr}}, -\mathbf{s}\rangle$ by $(|\boldsymbol{\lambda}, \mathbf{s}\rangle)^{\text{tr}}$, where $\boldsymbol{\lambda}^{\text{tr}}$ is the ℓ-partition $((\lambda^1)^{\text{tr}}, \dots, (\lambda^\ell)^{\text{tr}})$ and $-\mathbf{s} := (-s_1, \dots, -s_\ell)$. Now it follows from the definition of the crystal operators and [Losev 2015, Section 4.1.4] that when changing κ to $-\kappa$, $()^{\text{tr}}$ commutes with the $\widehat{\mathfrak{sl}_e}$-crystal up to changing $\tilde{f}_i$ to $\tilde{f}_{-i}$ (see [Jacon and Lecouvey 2010, Section 3.2.3]), and commutes with the $\mathfrak{sl}_\infty$-crystal up to taking the transpose (when $\kappa < 0$, the $\mathfrak{sl}_\infty$-crystal adds horizontal e-strips instead of vertical e-strips; see [Losev 2015, Section 4.2.3]). We deduce the following result.

Theorem 3.5. *Assume that we are in the integral case, and that moreover* $(\kappa, \mathbf{s})$ *is asymptotic. Assume that* $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^\ell)$ *is in the connected component of the empty multipartition for the* $\widehat{\mathfrak{sl}_e}$*-crystal and the* $\mathfrak{sl}_\infty$*-crystal. Then we have*

$$\Psi_{-\mathbf{s}_{\mathrm{rev}} \to -\mathbf{s}_{\mathrm{opp}}} \circ \Phi(\boldsymbol{\lambda}) = \mathsf{wc}_{-\leftarrow+}(\boldsymbol{\lambda})^{\mathrm{tr}}.$$

Proof. By hypothesis, there exist $\sigma \in \Pi$ and $(i_1, \dots, i_p) \in (\mathbb{Z}/e\mathbb{Z})^p$ such that

$$\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma |\boldsymbol{\emptyset}, \mathbf{s}\rangle = |\boldsymbol{\lambda}, \mathbf{s}\rangle.$$

From Theorem 2.10(1) together with the definition of the crystal isomorphism in Section 2C, we have

$$\begin{aligned}\Psi_{-\mathbf{s}_{\mathrm{rev}} \to -\mathbf{s}_{\mathrm{opp}}} \circ \Phi(\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma |\boldsymbol{\emptyset}, \mathbf{s}\rangle) &= \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \Psi_{-\mathbf{s}_{\mathrm{rev}} \to -\mathbf{s}_{\mathrm{opp}}} \circ \Phi(\tilde{a}_\sigma |\boldsymbol{\emptyset}, \mathbf{s}\rangle) \\ &= \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \tilde{a}_{\sigma^{\mathrm{tr}}} |\boldsymbol{\emptyset}, -\mathbf{s}_{\mathrm{opp}}\rangle\end{aligned}$$

using the explicit formulae of the crystal isomorphism together with the explicit formula of the action of $\tilde{a}_\sigma$ [Gerber 2019, Section 5]. On the other hand, by Proposition 3.4, we also get

$$\mathsf{wc}_{-\leftarrow+}(\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma |\boldsymbol{\emptyset}, \mathbf{s}\rangle) = \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_{\sigma^{\mathrm{tr}}} |\boldsymbol{\emptyset}, \mathbf{s}_{\mathrm{opp}}\rangle,$$

where the right-hand side is computed with respect to $\kappa' < 0$. The result then follows by sending κ' to $-\kappa'$ (the latter is positive) and applying our operator $()^{\mathrm{tr}}$. □

This Theorem is in fact a generalization of a result by Losev in level 1 [2015, Corollary 5.7], which we recover below. Using the notation of Remark 2.6, denote

$$M_e : \Pi \to \Pi, \qquad \lambda = (\sigma^e) \sqcup \rho \mapsto (\sigma^{\mathrm{tr}})^e \sqcup m_e(\rho).$$

Corollary 3.6 (Losev). *Suppose* $\ell = 1$. *Then for all* $\lambda \in \Pi$,

$$\mathsf{wc}_{-\leftarrow+}(\lambda) = (M_e(\lambda))^{\mathrm{tr}}.$$

Proof. In the case $\ell = 1$, the charge is irrelevant (see also Remark 1.2), thus so is $\Psi_{-\mathbf{s}_{\mathrm{rev}} \to -\mathbf{s}_{\mathrm{opp}}}$. By Remark 2.6, $\lambda = (\sigma^e) \sqcup \rho$ is the level 1 analogue of the decomposition of Corollary 2.4 used to define Φ, and we can identify $M_e(\lambda)$ with $(m_e(\rho), (\sigma^{\mathrm{tr}})^e)$. Thus by Theorem 2.10(2), we have $\Phi(\lambda) = (m_e(\rho), (\sigma^{\mathrm{tr}})^e) = M_e(\lambda)$, and we conclude using Theorem 3.5. □

We are able to partially generalize the level 1 statement to level ℓ:

Theorem 3.7. *Assume that we are in the integral case, and that moreover* $(\kappa, \mathbf{s})$ *is asymptotic. Let* $k \in \{1, \dots, \ell\}$ *be such that*

$$s_k = \max\{s_i ; i = 1, \dots, \ell\}.$$

Assume that $\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^\ell)$ *is such that* λ^i *is e-regular for all* $i\in\{1,\dots,\ell-1\}\setminus\{k\}$ *and* λ^k *is arbitrary. The combinatorial wall-crossing* $\mathsf{wc}_{-\leftarrow+}(\boldsymbol{\lambda})$ *is then given by the formula*

$$\mathsf{wc}_{-\leftarrow+}(\boldsymbol{\lambda})=(m_e(\lambda^1),\dots,M_e(\lambda^k),\dots,m_e(\lambda^\ell))^{\mathrm{tr}}.$$

Proof. Let us first assume that λ^j is e-regular for all $j=1,\dots,\ell$. By [Losev 2015, Proposition 3.1], we know that the wall-crossing is independent of the choice of a Weil generic parameter. In our situation this means that we can assume that for each $\boldsymbol{\lambda}\in\Pi^\ell(n)$, if two boxes have the same residues then they are in the same component. We thus have an action of a Kac–Moody algebra as a tensor product of ℓ copies of $\widehat{\mathfrak{sl}_e}$, one for each component of the multipartition. By Proposition 3.4, the associated Kashiwara operators commute with $\mathsf{wc}_{-\leftarrow+}$. Moreover, we know that $\mathsf{wc}_{-\leftarrow+}$ sends the empty multipartition to the empty multipartition and that for each e-regular partition λ, there exists a sequence of Kashiwara operators sending $\emptyset$ to λ. The result follows.

Next, assume that $\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^\ell)$ is such that λ^j is e-regular for all $j\in\{1,\dots,\ell\}\setminus\{k\}$ and λ^k is arbitrary. By our assumption on k and the definition of the action of the Heisenberg crystal operators $\tilde{a}_\sigma$ in [Losev 2015, Proposition 5.3], we know that there exist $\sigma\in\Pi$ and $\boldsymbol{\mu}'=(\mu^1,\dots,\mu^\ell)$ such that $\mu^i=\lambda^i$ if $i\neq k$, μ^k is e-regular, and $\tilde{a}_\sigma\boldsymbol{\mu}=\boldsymbol{\lambda}$. The result then follows from Proposition 3.4. □

We obtain the following interesting corollary which was not immediate from the crystal graph perspective.

Corollary 3.8. *Assume that we are in the integral case and that* $(\kappa,\mathbf{s})$ *is an asymptotic parameter and assume that* $\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^\ell)$ *is a highest weight vertex such that each* λ^j *is e-regular. Then*

$$\mathsf{wc}_{-\leftarrow+}(\boldsymbol{\lambda})=(m_e(\lambda^1),\dots,m_e(\lambda^\ell))^{\mathrm{tr}}$$

is a highest weight vertex for the opposite asymptotic parameter.

Example 3.9. Take $\ell=2$ and $s\in\mathbb{Z}$ such that $s>n-1$ so that $\mathbf{s}=(0,s)$ is an asymptotic 2-charge. The bipartitions (λ^1,λ^2) which are both highest weight vertex for the $\mathfrak{sl}_\infty$-crystal and the $\widehat{\mathfrak{sl}_e}$-crystal are exactly the ones satisfying $\lambda^2=\emptyset$ and one of the following conditions:

- $\lambda^1=\mu^e$ for a partition μ.
- λ^1 has exactly one good removable box of residue $s \bmod e$.

Let us consider the second case and assume that λ^1 is e-regular. Then our theorem asserts that $\mathsf{wc}_{-\leftarrow+}(\boldsymbol{\lambda})=(m_e(\lambda^1)^{\mathrm{tr}},\emptyset)$. This is consistent with the fact that it must be both a highest weight vertex for the $\mathfrak{sl}_\infty$-crystal and the $\widehat{\mathfrak{sl}_e}$-crystal because $m_e(\lambda^1)^{\mathrm{tr}}$ has exactly one removable box of residue $-s \bmod e$.

3E. ***Ringel duality for $\mathcal{O}_{\kappa,\mathbf{s}}$.*** Consider $\mathcal{O}_{\kappa,\mathbf{s}}(n)$, the cyclotomic Cherednik category $\mathcal{O}$ of the complex reflection group $G(\ell, 1, n)$ for parameters $\kappa = r/e$ such that $e \geq 2$ and $\gcd(r, e) = 1$, and $\mathbf{s} \in \mathbb{Q}^\ell$ satisfying $\kappa e s_i \in \mathbb{Z}$ for $i = 1, \dots, \ell$. The category $\mathcal{O}_{\kappa,\mathbf{s}}(n)$ is a highest weight category with standard objects $\Delta(\boldsymbol{\lambda})$, costandard objects $\nabla(\boldsymbol{\lambda})$, simple objects $L(\boldsymbol{\lambda})$, etc., indexed by $\{\boldsymbol{\lambda} \in \Pi^\ell(n)\}$ [Ginzburg et al. 2003]. By general theory [Donkin 1998, Appendix] there is a corresponding Ringel duality functor

$$\mathsf{R} : D^b(\mathcal{O}_{\kappa,\mathbf{s}}(n)) \xrightarrow{\sim} D^b({}^\vee\mathcal{O}_{\kappa,\mathbf{s}}(n))$$

induced by the derived Hom functor with respect to a full tilting module, and which was realized explicitly in [Ginzburg et al. 2003]. The category ${}^\vee\mathcal{O}_{\kappa,\mathbf{s}}(n)$ is called the Ringel dual of $\mathcal{O}_{\kappa,\mathbf{s}}(n)$. It is shown in [Ginzburg et al. 2003] that ${}^\vee\mathcal{O}_{\kappa,\mathbf{s}}(n)$ is again a cyclotomic Cherednik category $\mathcal{O}$. In particular, it holds that ${}^\vee\mathcal{O}_{\kappa,\mathbf{s}}(n) \simeq \mathcal{O}_{\kappa',\mathbf{s}'}(n)$ for some parameters κ', $\mathbf{s}'$ of the same rank and level, so that the Grothendieck group of $\bigoplus_n \mathcal{O}_{\kappa',\mathbf{s}'}(n)$ is again a level ℓ and rank e Fock space. Losev [2017a, Lemma 2.5] proved that Ringel duality is perverse with respect to the filtration by support of simple modules. As discussed in Section 3C, this means that we can pick out a unique composition factor $L(\boldsymbol{\mu})$ in the homology of the complex $\mathsf{R}(L(\boldsymbol{\lambda}))$ such that $\boldsymbol{\mu}$ has maximal bidepth in the $\widehat{\mathfrak{sl}_e}$ and $\mathfrak{sl}_\infty$-crystals. Set $\mathsf{r}(\boldsymbol{\lambda}) = \boldsymbol{\mu}$. For the rest of this section, we assume n is fixed and write $\mathcal{O}_{\kappa,\mathbf{s}}$ as shorthand for $\mathcal{O}_{\kappa,\mathbf{s}}(n)$.

Note that in Section 3D, we have defined wall-crossing functors as crossing a single essential wall. However, there is a derived equivalence

$$\mathsf{WC}_{c' \leftarrow c} : D^b(\mathcal{O}_c) \xrightarrow{\sim} D^b(\mathcal{O}_{c'})$$

for any c' in the lattice $\mathfrak{c}_\mathbb{Z}$ of parameters having integral difference with c, defined by taking the derived tensor product with a Harish-Chandra bimodule [Losev 2017a]. By abuse of language let us refer to such an equivalence as a wall-crossing functor whenever c and c' do not lie in the same chamber.

We are now going to recall Losev's comparison of the Ringel duality with the wall-crossing functor to a maximally distant chamber. The first part of the following proposition is from immediately after Theorem 6.1 in [Losev 2017a]. We add some details to the proof by justifying the assumption of the existence of a parameter c' in the same lattice as c such that the order on category $\mathcal{O}_{c'}$ is opposite to that on $\mathcal{O}_c$.

Lemma 3.10. *Assume the parameter $c = (\kappa, \mathbf{s})$ is as in the assumptions of* [*Losev 2017a, Theorem 4.1*]. *The following statements hold.*

1. *The Ringel duality $\mathsf{R} : D^b(\mathcal{O}_{\kappa,\mathbf{s}}) \to D^b({}^\vee\mathcal{O}_{\kappa,\mathbf{s}})$ can be realized by the* (*inverse*) *wall-crossing functor to the opposite chamber.*
2. *The combinatorial bijection r induced by Ringel duality commutes with the $\widehat{\mathfrak{sl}_e}$-crystal.*

Proof. (1) This is a special case of [Losev 2017a, Theorem 6.1], which follows from [Losev 2017a, Lemma 2.5 and Theorem 4.1]. We justify the existence of an opposite chamber and its intersection with the $\mathbb{Z}$-lattice of parameters $\mathfrak{c}_{\mathbb{Z}}$ in the case $W = G(\ell, 1, n)$.

The category $\mathcal{O}_{\kappa,\mathbf{s}}$ is a highest weight category with respect to the order on $\operatorname{Irr} G(\ell, 1, n) = \Pi^\ell(n)$ given by the c-function $c_{\boldsymbol{\lambda}}$ (see Section 3D). It is a general fact about Ringel duality for highest weight categories that the Ringel dual category has the same poset with the opposite partial order [Donkin 1998, Appendix]. Thus the c-order on $\Pi^\ell(n)$ with respect to the parameters $(\kappa', \mathbf{s}')$ is opposite to the c-order on $\Pi^\ell(n)$ with respect to the parameters $(\kappa, \mathbf{s})$. Recall from [Losev 2015] that the c-order is constant on the open chambers given by the complement of the hyperplane arrangement of essential walls.

We claim that we can find an example of a parameter $c' = (\kappa', \mathbf{s}')$ with the property that $\mathcal{O}_{\kappa',\mathbf{s}'}$ realizes the Ringel dual of $\mathcal{O}_{\kappa,\mathbf{s}}$ and $(\kappa', \mathbf{s}')$ has integral difference with $(\kappa, \mathbf{s})$. Recall the definitions of walls and chambers from Section 3D. First, assume that $\kappa > 0$, and let $t \in \mathbb{Z}_{>0}$ such that $t\mathbf{s} \in \mathbb{Z}^\ell$ and $n \in \mathbb{Z}$ be such that $\kappa - n < 0$, t divides n and $n/(\ell\kappa) \in \mathbb{Z}$. Set $c' := (\kappa', \mathbf{s}') \in \mathbb{Q}^\times_{<0} \times \mathbb{Q}^\ell$ such that $\kappa' := \kappa - n$ and for all $j = 1, \dots, \ell$, we have $s'_j = s_j + jn/(\ell\kappa\kappa')$. Then for all $j = 1, \dots, \ell$, we have

$$\kappa' s'_j - \kappa s_j = (\kappa - n)\left(s_j + \frac{jn}{\ell\kappa(\kappa - n)}\right) - \kappa s_j = -ns_j + \frac{jn}{\ell\kappa} \in \mathbb{Z}.$$

Since we have in addition $\kappa - \kappa' \in \mathbb{Z}$, we deduce that $(\kappa, \mathbf{s})$ and $(\kappa', \mathbf{s}')$ are in the same lattice. Now, let $\gamma = (a, b, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0} \times \{1, \dots, \ell\}$ be a box of an ℓ-partition. We have

$$co_c(\gamma) = \kappa\ell\left(b - a + s_j - \frac{j}{\ell\kappa}\right) \quad \text{and} \quad co_{c'}(\gamma) = \kappa'\ell\left(b - a + s'_j - \frac{j}{\ell\kappa'}\right)$$

and for $j = 1, \dots, \ell$,

$$s'_j - \frac{j}{\ell\kappa'} - \left(s_j - \frac{j}{\ell\kappa}\right) = \frac{jn}{\ell\kappa\kappa'} - \frac{j}{\ell\kappa'} + \frac{j}{\ell\kappa} = \frac{jn - j\kappa + j(\kappa - n)}{\ell\kappa(\kappa - n)} = 0.$$

As κ and κ' have opposite sign, we conclude that c and c' induce opposite orders with respect to the c-function.

(2) The proof follows by exactly the same argument as the proof of [Losev 2015, Proposition 5.6(1)]. Indeed, the assumption of [Losev 2015, Proposition 5.6(1)] that the wall-crossing is through a single essential wall is only used in reference to [Losev 2015, Proposition 3.8], which states that wall-crossing functors across essential walls intertwine the parabolic restriction functors. The analogous statement that (inverse) Ringel duality intertwines parabolic induction and restriction functors is proved in [Losev 2017a, Lemma 4.7]. □

The symmetrical statement to Lemma 3.10(2) for the $\widehat{\mathfrak{sl}_\ell}$-crystal follows by level-rank duality which switches the roles of e and ℓ and commutes with Ringel duality. For this, we need to be in the integral parameter case again.

Lemma 3.11. *The map* r *commutes with the* $\widehat{\mathfrak{sl}_\ell}$*-crystal.*

Proof. By [Rouquier et al. 2016, Theorem 7.4] and [Webster 2017], the category $\mathcal{O}_{\kappa,\mathbf{s}}$ is standard Koszul, which by [Mazorchuk 2010] implies that Ringel duality commutes with the Koszul duality. The Koszul duality K lifts the level-rank duality k, by which the $\widehat{\mathfrak{sl}_\ell}$-crystal is defined, to the categorical level [Rouquier et al. 2016, Theorem 7.4]. Since we need to compare Ringel duality for level e with Ringel duality for level ℓ, write R for the former and $\dot{\mathsf{R}}$ for the latter; likewise, write r for the former and $\dot{\mathsf{r}}$ for the latter on the level of Grothendieck groups. We have $\dot{\mathsf{R}}\mathsf{K} = \mathsf{K}\mathsf{R}$ and thus $\dot{\mathsf{r}}\mathsf{k} = \mathsf{k}\mathsf{r}$ and $\mathsf{r}\dot{\mathsf{k}} = \dot{\mathsf{k}}\dot{\mathsf{r}}$. Let $\dot{\tilde{f}}_j$ be an $\widehat{\mathfrak{sl}_\ell}$ crystal operator. Recall from (2.1) that the action of $\dot{\tilde{f}}_j$ in level ℓ and rank e is defined by $\dot{\mathsf{k}}\tilde{f}_j\mathsf{k}$. By the preceding lemma, in level e and rank ℓ we have $\tilde{f}_j\dot{\mathsf{r}} = \dot{\mathsf{r}}\tilde{f}_j$. Thus, $\mathsf{r}(\dot{\mathsf{k}}\tilde{f}_j\mathsf{k}) = \dot{\mathsf{k}}\dot{\mathsf{r}}\tilde{f}_j\mathsf{k} = \dot{\mathsf{k}}\tilde{f}_j\dot{\mathsf{r}}\mathsf{k} = (\dot{\mathsf{k}}\tilde{f}_j\mathsf{k})\mathsf{r}$. □

3F. *The combinatorial Ringel duality.* For this section, we again take parameters to be integral. The category $\mathcal{O}_{\kappa,\mathbf{s}}$ is equivalent as a highest weight category to the category $\mathcal{S}_{\kappa,\mathbf{s}}$ of finite-dimensional modules over the diagrammatic Cherednik algebra with Uglov weighting defined by Webster [2017, Theorem 4.8]. Webster [2017, Corollary 5.11] proved that for the category $\mathcal{S}_{\kappa,\mathbf{s}}$, taking the Ringel dual corresponds to sending κ to $-\kappa$ and keeping the charge $\mathbf{s}$ fixed. Moreover, there is a natural isomorphism of diagram algebras (which is given by replacing the label $i \in \mathbb{Z}/e\mathbb{Z}$ with $-i$ on black strands and the label $j \in \mathbb{Z}/\ell\mathbb{Z}$ with $-j$ on red strands; see [Webster 2017, Proposition 4.5]). This isomorphism induces an equivalence

$$* : \mathcal{S}_{\kappa,\mathbf{s}} \to \mathcal{S}_{-\kappa,-\mathbf{s}_{\mathrm{rev}}}$$

which evidently satisfies $\tilde{f}_{-i} \circ * = * \circ \tilde{f}_i$ and $\dot{\tilde{f}}_{-j} \circ * = * \circ \dot{\tilde{f}}_j$. Composing $*$ with Ringel duality for the diagrammatic Cherednik algebra then gives an equivalence (via identification of $\mathcal{O}_{\kappa,\mathbf{s}}$ with $\mathcal{S}_{\kappa,\mathbf{s}}$),

$$* \circ \mathsf{R} : D^b(\mathcal{O}_{\kappa,\mathbf{s}}) \simeq D^b(\mathcal{S}_{\kappa,\mathbf{s}}) \xrightarrow{\sim} D^b(\mathcal{S}_{\kappa,-\mathbf{s}_{\mathrm{rev}}}) \simeq D^b(\mathcal{O}_{\kappa,-\mathbf{s}})$$

which is perverse, being the composition of abelian equivalences with a perverse equivalence [Chuang and Rouquier 2017].

Set $\mathsf{D} = * \circ \mathsf{R} : D^b(\mathcal{O}_{\kappa,\mathbf{s}}) \xrightarrow{\sim} D^b(\mathcal{O}_{\kappa,-\mathbf{s}_{\mathrm{rev}}})$ (see [Ginzburg et al. 2003, Proposition 4.10]). On the level of Grothendieck groups, D yields an involutive isomorphism of Fock spaces,

$$\mathsf{d} : \mathcal{F}_{e,\mathbf{s}} \to \mathcal{F}_{e,-\mathbf{s}_{\mathrm{rev}}}.$$

We call d the *combinatorial Ringel duality*.

Corollary 3.12. *For all* $i = 0, \dots, e-1$ *and all* $j = 0, \dots, \ell-1$ *we have*

$$\mathsf{d}\tilde{f}_i = \tilde{f}_{-i}\mathsf{d} \quad \textit{and} \quad \mathsf{d}\dot{\tilde{f}}_j = \dot{\tilde{f}}_{-j}\mathsf{d}.$$

Proof. This is straightforward from Lemmas 3.10 and 3.11, because $\mathsf{d} = * \circ \mathsf{r}$ and $*$ verifies $\tilde{f}_{-i} \circ * = * \circ \tilde{f}_i$ and $\dot{\tilde{f}}_{-j} \circ * = * \circ \dot{\tilde{f}}_j$, as explained above. □

Lemma 3.13. *For all* $\sigma \in \Pi$, *we have* $\mathsf{d}\tilde{a}_\sigma = \tilde{a}_{\sigma^{\mathrm{tr}}}\mathsf{d}$.

Proof. Since D is perverse with respect to supports of simple modules, i.e., cuspidal depth, d is an involutive isomorphism of $\mathfrak{sl}_\infty$-crystals $\mathcal{F}_{e,\mathbf{s}} \to \mathcal{F}_{e,-\mathbf{s}_{\mathrm{rev}}}$. Thus either $\mathsf{d} \circ \tilde{a}_\sigma = \tilde{a}_\sigma \circ \mathsf{d}$ or $\mathsf{d} \circ \tilde{a}_\sigma = \tilde{a}_{\sigma^{\mathrm{tr}}} \circ \mathsf{d}$. By Corollary 3.12, d commutes with the two kinds of Kashiwara crystal operators up to a change of sign in the indices. Therefore, by Theorem 2.3(2), it is enough to prove that the two maps agree on multipartitions of the form $|\boldsymbol{\emptyset}, \mathbf{s}\rangle$. Further, by [Jacon and Lecouvey 2018], $|\boldsymbol{\emptyset}, \mathbf{s}\rangle$ can be obtained by applying a sequence of combinatorial wall-crossings of type (b) to $|\boldsymbol{\emptyset}, \mathbf{s}'\rangle$ for some asymptotic parameter $(\kappa, \mathbf{s}')$. By [Losev 2015, Proposition 5.6 (1) and (2)], the combinatorial wall-crossings of type (b) commute with the three kinds of crystal operators. Therefore, we can reduce the proof to the case where $(\kappa, \mathbf{s})$ is asymptotic. Now, we have already seen that in this case, $\tilde{a}_\sigma$ acts only on one component of the empty multipartition, so we can reduce the proof to the case $\ell = 1$.

For the rest of the proof, consider the functors D and $* \circ \mathsf{WC}_{-\leftarrow+}$ when $\ell = 1$ so that we are considering functors on the category $\mathcal{O}_\kappa(\mathfrak{S}_n)$ (as $G(1,1,n) = \mathfrak{S}_n$). By [Rouquier 2008, Corollary 5.13], if $r > 0$ is coprime to e then $\mathcal{O}_{-r/e}(\mathfrak{S}_n) \simeq \mathcal{O}_{-1/e}(\mathfrak{S}_n)$ are equivalent as highest weight categories. Moreover, the equivalence sends $\Delta(\lambda)$ to $\Delta(\lambda)$, as follows from [Ginzburg et al. 2003, Corollary 6.10] together with the fact that the Hecke algebra at a primitive e'-th root of unity depends only on e, not on the root of unity [Brundan and Kleshchev 2009]. Thus, though abusing notation, the composition $* \circ \mathsf{WC}_{-\leftarrow+}$ is well defined.

We claim that for $\ell = 1$, the combinatorial Ringel duality d coincides with the transpose of the combinatorial wall-crossing. If $\kappa > 0$ and $\kappa' < 0$ and $\mathsf{WC}_{-\leftarrow+} : D^b(\mathcal{O}_\kappa(\mathfrak{S}_n)) \xrightarrow{\sim} D^b(\mathcal{O}_{\kappa'}(\mathfrak{S}_n))$ is the wall-crossing functor across the (unique) essential wall defined by the equation $\kappa = 0$, then the combinatorial wall-crossing $\mathsf{wc}_{-\leftarrow+}$ is given by $\mathsf{wc}_{-\leftarrow+}(\lambda) = (M_e(\lambda))^{\mathrm{tr}}$ for all $\lambda \in \Pi(n)$ [Losev 2015, Corollary 5.7]. The functors R and $\mathsf{WC}_{-\leftarrow+}$ are both perverse equivalences from $\mathcal{O}_\kappa(\mathfrak{S}_n)$ with the same perversity function, namely the cuspidal depth of $L(\lambda)$. Thus $\mathsf{D} = * \circ \mathsf{R}$ and $* \circ \mathsf{WC}_{-\leftarrow+}$ are perverse self-equivalences of $\mathcal{O}_\kappa(\mathfrak{S}_n)$ with the same perversity function [Losev 2017a, Lemma 2.5 and Theorem 6.1]. By [Chuang and Rouquier 2017, Proposition 4.17], if $\mathsf{F} : D^b(\mathcal{C}) \xrightarrow{\sim} D^b(\mathcal{D})$ and $\mathsf{G} : D^b(\mathcal{C}) \xrightarrow{\sim} D^b(\mathcal{E})$ are two perverse equivalences with the same perversity function, then $\mathsf{G} \circ \mathsf{F}^{-1} : \mathcal{D} \xrightarrow{\sim} \mathcal{E}$ is an equivalence of abelian categories. In particular, this implies that $\mathsf{R}(\mathcal{O}_\kappa(\mathfrak{S}_n)) \simeq \mathsf{WC}_{-\leftarrow+}(\mathcal{O}_\kappa(\mathfrak{S}_n))$. Set $\mathsf{A} := (* \circ \mathsf{WC}_{-\leftarrow+}) \circ \mathsf{D}^{-1} : \mathcal{O}_\kappa(\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{O}_\kappa(\mathfrak{S}_n)$. On K_0, it holds that $[\mathsf{D}(\Delta(\lambda))] = [(* \circ \mathsf{R})(\Delta(\lambda))] = [(* \circ \mathsf{WC}_{-\leftarrow+})(\Delta(\lambda))] = [\Delta(\lambda^{\mathrm{tr}})]$ for any $\lambda \in \Pi(n)$ by [Ginzburg et al. 2003, Proposition 3.3, Corollary 4.8 and Proposition 4.10] and [Losev 2015, Proposition 3.2]. Thus $[\mathsf{A}(\Delta(\lambda))] = [\Delta(\lambda)]$ for all $\lambda \in \Pi(n)$. Moreover, it follows from the definitions of R and $\mathsf{WC}_{-\leftarrow+}$ that A sends standardly filtered objects to standardly filtered objects. Therefore $\mathsf{A}(\Delta(\lambda)) = \Delta(\lambda)$, from which it follows that $\mathsf{A}(L(\lambda)) = L(\lambda)$. We conclude that $\mathsf{d} = (-)^{\mathrm{tr}} \circ \mathsf{wc}_{-\leftarrow+} = M_e$. This concludes the proof. □

Now recall the generalized Mullineux involution $\Phi : \mathcal{F}_{e,\mathbf{s}} \to \mathcal{F}_{e,-\mathbf{s}_{\mathrm{rev}}}$ from Theorem 2.10.

Theorem 3.14. *The map* d *is equal to* Φ.

Proof. Corollary 3.12 and Lemma 3.13 combined imply that d and Φ satisfy the same commutation relations with the operators for the $\mathfrak{sl}_\infty$-, $\widehat{\mathfrak{sl}_e}$-, and $\widehat{\mathfrak{sl}_\ell}$-crystals. Moreover, as seen in the proof of Lemma 3.13, d maps $|\boldsymbol{\emptyset}, \mathbf{s}\rangle$ to $|\boldsymbol{\emptyset}, -\mathbf{s}_{\mathrm{rev}}\rangle$. By Theorem 2.10(1), Φ is the unique involution with these properties, so $\mathsf{d} = \Phi$. □

We recall for the reader the fact that the finite-dimensional modules of a rational Cherednik algebra $H_c(W)$ coincide with the cuspidal modules, i.e., those sent to 0 by every parabolic restriction functor with respect to a proper parabolic subgroup of W [Bezrukavnikov and Etingof 2009].

Corollary 3.15. *Suppose* $L(\boldsymbol{\lambda})$ *is a finite-dimensional irreducible representation of the rational Cherednik algebra of type* $B_n = G(2, 1, n)$ *for parameters corresponding to Fock space charge* $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$. *Then* $\Phi(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$, *and* $\mathsf{D}(L(\boldsymbol{\lambda})) = L(\boldsymbol{\lambda})$, *where we identify* $L(\boldsymbol{\lambda})$ *with the complex concentrated in degree* 0.

Proof. First, we make some remarks which hold for arbitrary ℓ. By Theorem 3.1, $L(\boldsymbol{\lambda})$ is finite-dimensional if and only if $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ has depth 0 in both the $\mathfrak{sl}_\infty$- and $\widehat{\mathfrak{sl}_e}$-crystals. Recall that D is perverse with respect to filtration by the depth in the $\widehat{\mathfrak{sl}_e}$- and $\mathfrak{sl}_\infty$-crystals, i.e., cuspidal depth. Since D is not just a derived equivalence but a perverse equivalence, by Definition 3.2 it holds that D induces an abelian equivalence on each associated graded layer of $\mathcal{O}_{\kappa,\mathbf{s}}(n)$ with respect to the filtration by supports of simple modules. In particular, D restricts to an equivalence of abelian categories on the bottom filtration layer of $\mathcal{O}_{\kappa,\mathbf{s}}(n)$, and this subcategory coincides with the subcategory of cuspidal, that is finite-dimensional, modules. Thus if $\boldsymbol{\lambda}$ has depth 0 in both the $\widehat{\mathfrak{sl}_e}$- and $\mathfrak{sl}_\infty$-crystals, D sends $L(\boldsymbol{\lambda})$ to some $L(\boldsymbol{\mu})$ where $\boldsymbol{\mu}$ again has depth 0 in both the $\widehat{\mathfrak{sl}_e}$- and $\mathfrak{sl}_\infty$-crystals. Here, we identify $L(\boldsymbol{\lambda})$ with a complex concentrated in degree 0, and likewise for $L(\boldsymbol{\mu})$. In this case, $\mathsf{D}(L(\boldsymbol{\lambda})) = L(\Phi(\boldsymbol{\lambda}))$ by Theorem 3.14.

Now we specify to the case $\ell = 2$. When $\ell = 2$, the map Φ fixes the $\widehat{\mathfrak{sl}_\ell}$-crystal operators $\tilde{\dot{f}}_j$ since $j = -j \bmod 2$. Moreover,

$$-\mathbf{s}_{\mathrm{rev}} = (-s_2, -s_1) = (s_1, s_2) - (s_2 + s_1, s_2 + s_1),$$

i.e., $-\mathbf{s}_{\mathrm{rev}}$ is just an integer shift of $\mathbf{s}$. Recall the remark at the beginning of Section 3A that charges differing by an integer shift define the same rational Cherednik algebra. Thus we may identify the Cherednik algebras defined by $(\kappa, \mathbf{s})$ and $(\kappa, -\mathbf{s}_{\mathrm{rev}})$, and hence we may identify their categories $\mathcal{O}$. Now, it is straightforward to see that the action of $\tilde{\dot{f}}_j$ on an element of $\Pi^\ell_{\mathbf{s}}$ is not affected by an integer shift of the charge; more precisely,

$$\tilde{\dot{f}}_j|\boldsymbol{\nu}, \mathbf{t}\rangle =: |\boldsymbol{\nu}', \mathbf{t}'\rangle \Rightarrow \tilde{\dot{f}}_j|\boldsymbol{\nu}, \mathbf{t}+k\rangle = |\boldsymbol{\nu}', \mathbf{t}'+k\rangle$$

for all $\boldsymbol{\nu} \in \Pi^\ell$, $\mathbf{t} \in \mathbb{Z}^\ell(s)$ and $k \in \mathbb{Z}$. In particular, we get $\Phi(|\boldsymbol{\lambda}, \mathbf{s}\rangle) = |\boldsymbol{\lambda}, -\mathbf{s}_{\mathrm{rev}}\rangle$, which proves the claim. □

Remark 3.16. Corollary 3.15 implies that Ringel duality can be seen as a self-equivalence of $\mathcal{O}_{e,\mathbf{s}}(B_n)$ which fixes cuspidal (i.e., finite-dimensional) modules, and we recover [Ginzburg et al. 2003, Remark 4.12].

Example 3.17. Take $e = 3$, $\mathbf{s} = (-1, 3)$ and $\boldsymbol{\lambda} = (3.3, \emptyset)$. Then

$$\begin{aligned}\beta(|\boldsymbol{\lambda}, \mathbf{s}\rangle) &= (|(\emptyset, \emptyset), (0, 2)\rangle, \emptyset, |(2, 2, 1), (-1, -1, 0)\rangle) \\ &= (|(\emptyset, \emptyset), (0, 2)\rangle, \emptyset, \tilde{\dot{f}}_0\tilde{\dot{f}}_0\tilde{\dot{f}}_0\tilde{\dot{f}}_1\tilde{\dot{f}}_1|(\emptyset, \emptyset, \emptyset), (-1, -1, 0)\rangle),\end{aligned}$$

so that $L(\boldsymbol{\lambda})$ is finite-dimensional in $\mathcal{O}_{1/e,\mathbf{s}}(6)$. We have

$$\begin{aligned}\Phi(|\boldsymbol{\lambda}, \mathbf{s}\rangle) &= \beta^{-1}(|(\emptyset, \emptyset), (-2, 0)\rangle, \emptyset, \tilde{\dot{f}}_0\tilde{\dot{f}}_0\tilde{\dot{f}}_0\tilde{\dot{f}}_1\tilde{\dot{f}}_1|(\emptyset, \emptyset, \emptyset), (0, 1, 1)\rangle) \\ &= \beta^{-1}(|(\emptyset, \emptyset), (-2, 0)\rangle, \emptyset, |(1, 2, 2), (0, 1, 1)\rangle) \\ &= |(3.3, \emptyset), (-3, 1)\rangle = |\boldsymbol{\lambda}, -\mathbf{s}_{\mathrm{rev}}\rangle.\end{aligned}$$

4. Other crystal structures and generalization of Mullineux involutions

We now explain a possible generalization of the Mullineux involution defined in [Dudas and Jacon 2018] that uses the results of Section 2. We already know that we have three different crystal structures on level ℓ Fock spaces which are pairwise commuting, namely the $\widehat{\mathfrak{sl}_e}$-, $\mathfrak{sl}_\infty$- and $\widehat{\mathfrak{sl}_\ell}$-crystals. We now slightly generalize this by introducing an additional parameter $d \in \mathbb{Z}_{\geq 0}$ (which plays the role of the parameter ℓ in [Dudas and Jacon 2018] in level one — that is, the type A situation). To do this, recall that if $\lambda \in \Pi$, we can uniquely write the decomposition

$$\lambda = \lambda_{(0)} + \lambda_{(1)}^{d} + \cdots + \lambda_{(n)}^{d^n}$$

for $n \in \mathbb{Z}_{\geq 0}$ and where each $\lambda_{(i)}$ is d-regular.

Let $j \in \mathbb{Z}_{>0}$. One can define an action of a Kashiwara operator $\tilde{f}_{k,j}$ (for $k = 0, \ldots, d-1$) as

$$\tilde{f}_{k,j}.\lambda = \mu \quad \text{for all } \lambda \in \Pi,$$

where $\mu_{(t)} = \lambda_{(t)}$ when $t \neq d$ and $\mu_{(d)} = \tilde{f}_k\lambda_{(d)}$ (where $\tilde{f}_k$ is denoting the usual

Kashiwara operator acting on d-regular partitions).

Using the decomposition in Theorem 2.3, we get an action of Kashiwara operators on the whole Fock space as follows. Let $|\boldsymbol{\lambda}, \mathbf{s}\rangle \in \Pi_s^\ell$ and write

$$\beta(|\boldsymbol{\lambda}, \mathbf{s}\rangle) = (|\boldsymbol{\lambda}_1, \mathbf{r}\rangle, \sigma, |\boldsymbol{\lambda}_2, \dot{\mathbf{r}}\rangle).$$

Then, for all $j \in \mathbb{Z}_{>0}$,

$$\tilde{f}_{k,j}.|\boldsymbol{\lambda}, \mathbf{s}\rangle = \beta^{-1}(|\boldsymbol{\lambda}_1, \mathbf{r}\rangle, \tilde{f}_{k,j}.\sigma, |\boldsymbol{\lambda}_2, \dot{\mathbf{r}}\rangle).$$

For each $j \in \mathbb{Z}_{\geq 0}$, we thus get an $\widehat{\mathfrak{sl}_d}$-crystal.

It is immediate to see that these actions also commute with the $\widehat{\mathfrak{sl}_e}$-crystal and the $\widehat{\mathfrak{sl}_\ell}$-crystal (it just follows from the existence of the bijection β). Finally, there is an obvious analogue of the Mullineux involution for this decomposition, which depends on d. Namely, for $|\boldsymbol{\lambda}, \mathbf{s}\rangle \in \Pi_s^\ell$ and $\beta(|\boldsymbol{\lambda}, \mathbf{s}\rangle) = (|\boldsymbol{\lambda}_1, \mathbf{r}\rangle, \sigma, |\boldsymbol{\lambda}_2, \mathbf{r}\rangle)$, we define

$$\Phi^{(d)}(|\boldsymbol{\lambda}, \mathbf{s}\rangle) = \beta^{-1}(m_{e,\mathbf{r}}(\boldsymbol{\lambda}_1), m_e(\sigma_{(0)}) + m_e(\sigma_{(1)})^d + \cdots + m_e(\sigma_{(n)})^{d^n}, m_{\ell,\dot{\mathbf{r}}}(\boldsymbol{\lambda}_2)),$$

so that $\Phi^{(d)}$ simultaneously generalizes Φ and the version of the Mullineux involution of [Dudas and Jacon 2018] (which we recover by taking $\ell = 1$). As in [Dudas and Jacon 2018], we believe it would be interesting to look at the case $\ell = 2$ and investigate the relationship between $\Phi^{(d)}$ on the one hand, and the Alvis–Curtis duality for unipotent representations of finite unitary groups in transverse characteristic d on the other hand (or more generally of finite groups of Lie types B and C).

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