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TWO APPLICATIONS OF THE INTEGRAL REGULATOR

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We review Li's refinement of the KLM regulator map, and use it to detect torsion phenomena in higher Chow groups of number fields and families of elliptic curves.

1. Introduction

Let X be a smooth projective variety defined over a subfield of $\mathbb{C}$. The KLM formula is a morphism of complexes inducing the Bloch–Beilinson regulator map

$$\mathrm{Gr}_{\gamma}^{p} K_n^{\mathrm{alg}}(X)_{\mathbb{Q}} \cong H_{\mathcal{M}}^{2p-n}(X, \mathbb{Q}(p)) \to H_{\mathscr{D}}^{2p-n}(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}(p))$$

with rational coefficients, developed by the first author together with J. Lewis and S. Müller-Stach [Kerr 2003b; Kerr et al. 2006; Kerr and Lewis 2007] (see Section 3). It works by representing motivic cohomology by cycles in Bloch's higher Chow complex (Section 2), and associating explicit currents on $X_{\mathbb{C}}^{\mathrm{an}}$ to them. The second author's refinement now enables the direct computation of the *integral* regulator

$$H_{\mathcal{M}}^{2p-n}(X, \mathbb{Z}(p)) \to H_{\mathscr{D}}^{2p-n}(X, \mathbb{Z}(p))$$

on the level of higher Chow complexes [Li 2018].

The aim of this note is to offer a brief review of Li's construction (Section 4) and give some first examples of how it may be used: First, we demonstrate how to find explicit torsion generators in higher Chow groups of number fields (Section 5); the results are summarized in Theorem 5.3. We also apply the formula (in Section 6) to *integrally* calculate a branch of the higher normal function arising from the mirror of local $\mathbb{P}^2$ (Proposition 6.4), along the way computing some integral regulator periods for curves of any genus (Theorem 6.2).

2. Higher Chow groups

Invented by Spencer Bloch [1986a; 1994] in the mid-1980s to geometrize Quillen's higher algebraic K-theory, these generalize the usual Chow groups of cycles modulo

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rational equivalence (the $n = 0$ case). In particular, for X smooth quasiprojective over a field k, they satisfy[1]

$$\mathrm{CH}^p(X, n)_{\mathbb{Q}} \cong \mathrm{Gr}^p_\gamma K_n^{\mathrm{alg}}(X)_{\mathbb{Q}}.$$

For such X (and no restriction on k), Voevodsky [2002] proved that they are *integrally* isomorphic to his motivic cohomology groups:

$$\mathrm{CH}^p(X, n) \cong H^{2p-n}_{\mathcal{M}}(X, \mathbb{Z}(p)).$$

Beyond their role in arithmetic geometry (e.g., Beilinson's conjectures [1984]), they have recently shown up in several branches of physics (e.g., quantum field theory [Bloch et al. 2015] and topological string theory [del Ángel R. et al. 2019]) and mirror symmetry [Doran and Kerr 2014; Bloch et al. 2017]. We focus on the cubical presentation of $\mathrm{CH}^p(X, n)$ as the n-th homology of a complex of *higher Chow precycles* [Levine 1994]

$$\cdots \to Z^p(X, n+1) \xrightarrow{\partial} Z^p(X, n) \xrightarrow{\partial} Z^p(X, n-1) \to \cdots$$

or its (integrally quasi-isomorphic) subcomplex of *normalized precycles* [Bloch 2004]

$$\cdots \to N^p(X, n+1) \xrightarrow{\partial} N^p(X, n) \xrightarrow{\partial} N^p(X, n-1) \to \cdots .$$

A *higher Chow cycle* is an element of $\ker(\partial)$. Roughly speaking, these are relative codimension-p cycles on

$$(X \times \mathbb{A}^n, X \times \cup \mathbb{A}^{n-1}),$$

where the $\mathbb{A}^{n-1}$'s are inserted into $\mathbb{A}^n$ as a "cubical" configuration of hyperplanes. More precisely, writing $\{z_1, \ldots, z_n\}$ for coordinates, $(z_i) = (z_i)_0 - (z_i)_\infty$ for their divisors and $|\cdot|$ for support, the *algebraic n-cube* is defined by

$$\square^n := (\mathbb{P}^1 \setminus \{1\})^n \supset \partial\square^n := \bigcup_i |(z_i)|.$$

Any component of $X \times \partial\square^n$ is called a *facet* of $X \times \square^n$, and a *face* is any intersection of facets. Now set[2]

$$c^p(X, n) := \big\{\text{cycles in } Z^p(X \times \square^n) \text{ meeting faces of } X \times \partial\square^n \text{ properly, i.e., in the expected codimension}\big\},$$

$$d^p(X, n) := \sum_i \big\{\text{cycles in } c^p(X, n) \text{ which are constant in } z_i\big\},$$

[1]The subscript $\mathbb{Q}$ denotes $\otimes\,\mathbb{Q}$.

[2]Normalized precycles may be represented (in $Z^p(X, n)$) by Z satisfying $Z \cdot \{z_i = 0\} = 0$ $(\forall i)$ and $Z \cdot \{z_i = \infty\} = 0$ $(i < n)$ simply by adding an element of $d^p(X, n)$.

$$Z^p(X,n) := c^p(X,n)/d^p(X,n),$$
$$N^p(X,n) := \{Z \mid Z\cdot(z_i)_0 = 0\ (\forall i) \text{ and } Z\cdot(z_i)_\infty = 0\ (\forall i < n)\},$$

and for $Z \in Z^p(X,n)$ or $N^p(X,n)$,

$$\partial Z := \sum_{i=1}^{n}(-1)^i\big(Z\cdot(z_i)_\infty - Z\cdot(z_i)_0\big).$$

If $X = \mathrm{Spec}(k)$, write $Z^p(k,n)$ etc. for short.

Example 2.1. For any positive integer ℓ, let ζ_ℓ denote a primitive ℓ-th root of unity. Parametrize a cycle in $N^2(\mathbb{Q}(\zeta_\ell),3)$ by $t \in \mathbb{P}^1$:

$$Z_\ell^2 := \left(1-\frac{\zeta_\ell}{t},\, 1-t,\, t^{-\ell}\right).$$

Intersections with facets $\{z_i = 0, \infty\}$ are given by $t = 0, 1, \zeta_\ell, \infty$. But all these intersections have some $z_j = 1$, so are trivial (as $1 \notin \square$). We also record the cycle

$$\mathscr{Z}_5^2 := Z_1^2 + \left(1-\frac{\zeta_5}{t},\, 1-t,\, t^{-5}\right) + \left(1-\frac{\bar{\zeta}_5}{t},\, 1-t,\, t^{-5}\right)$$

in $N^2(\mathbb{Q}(\sqrt{5}),3)$ for later reference.

3. Abel–Jacobi maps

These simultaneously generalize two classical invariants:[3]

(1) Griffiths' AJ map [1969]

$$\mathrm{CH}^p(X,0) \to H^{2p}_{\mathscr{D}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z}(p))$$

for X smooth projective over $\mathbb{C}$; and

(2) Borel's regulator map [Borel 1977; Burgos Gil 2002]

$$\mathrm{CH}^p(k, 2p-1) \to \mathbb{C}/\mathbb{Z}(p)$$

for $k \subset \mathbb{C}$ a number field.

Defined abstractly by Bloch [1986b], they map higher Chow groups to Deligne cohomology:

$$\mathrm{CH}^p(X,n) \xrightarrow{\mathrm{AJ}_X^{p,n}} H^{2p-n}_{\mathscr{D}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z}(p)).$$

[3] The versions given here are, of course, not the original ones: (1) assembles Griffiths' map (to the intermediate Jacobian $J^p(X)$) and the fundamental class map (to the Hodge classes $\mathrm{Hg}^p(X)$) into a single invariant; while (2) is an integral lift of the composition $\mathrm{CH}^p(k,2p-1) \to \mathrm{CH}^p(k,2p-1)_{\mathbb{Q}} \cong \mathrm{Gr}^p_\gamma K_{2p-1}(k)_{\mathbb{Q}} \xrightarrow{r_{\mathrm{Bo}}} \mathbb{R} \cong \mathbb{C}/\mathbb{R}(p)$.

Here we are taking X to be smooth projective, and defined over a subfield of $\mathbb{C}$. (In the smooth quasiprojective setting, the correct target is the absolute Hodge cohomology $H^{2p-n}_{\mathcal{H}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z}(p))$ [Kerr and Lewis 2007, §2].)

Kerr, Lewis, and Müller-Stach [Kerr et al. 2006] constructed a morphism of complexes, from a subcomplex of $Z^p(X, -\bullet)$ to a complex defining Deligne cohomology, which induces $\mathrm{AJ}^{p,n}_X$. Writing C^k_{sing} for singular chains of real codimension k, D^k for currents of degree k, and $F^\bullet$ for the Hodge filtration, their morphism takes the form

$$\widetilde{\mathrm{AJ}}^{p,-\bullet}_{\mathrm{KLM}} : Z^p_{\mathbb{R}}(X, -\bullet) \to C^{2p+\bullet}_{\mathscr{D}}(X, \mathbb{Z}(p)) := C^{2p+\bullet}_{\mathrm{sing}}(X; \mathbb{Z}(p)) \oplus F^p D^{2p+\bullet}(X) \oplus D^{2p-1+\bullet}(X),$$

with differential $D_{\mathscr{D}}(\alpha, \beta, \gamma) = (-\partial\alpha, -d\beta, d\gamma - \beta + \alpha)$ on the right. For $Z \in Z^p_{\mathbb{R}}(X, n)$ with projections π_1 (to $\square^n$) and π_2 (to X), they set

$$\begin{aligned}\widetilde{\mathrm{AJ}}^{p,n}_{\mathrm{KLM}}(Z) &:= (2\pi \boldsymbol{i})^{p-n}((2\pi \boldsymbol{i})^n T_Z, \Omega_Z, R_Z)\\ &:= (2\pi \boldsymbol{i})^{p-n}(\pi_2)_*(\pi_1)^*((2\pi \boldsymbol{i})^n T_n, \Omega_n, R_n),\end{aligned}$$

where $T_n := \bigcap_{i=1}^n T_{z_i} = \mathbb{R}^{\times n}_{<0}$, $\Omega_n := dz_1/z_1 \wedge \cdots \wedge dz_n/z_n$, and R_n is defined inductively by

$$R_n(z_1, \ldots, z_n) := \log(z_1)\frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n} - (-1)^n (2\pi \boldsymbol{i}) R_{n-1}(z_2, \ldots, z_n) \cdot \delta_{T_{z_1}}.$$

Here $\log(z_i)$ has a branch cut along $T_{z_i} = \{z_i \in \mathbb{R}_{<0}\}$, which is regarded as a 1-cochain on $\square^n$, oriented so that $\partial T_{z_i} = (z_i)_0 - (z_i)_\infty$; and $\delta_{T_{z_i}}$ is the current of integration along it. (Note that $\widetilde{\mathrm{AJ}}^{p,n}_{\mathrm{KLM}}$ vanishes identically on $d^p(X, n)$.)

The subcomplex $Z^p_{\mathbb{R}}(X, -\bullet) \subset Z^p(X, -\bullet)$ consists of cycles Z for which $Z^{\mathrm{an}}_{\mathbb{C}}$ properly intersects certain combinations[4] of $\{T_{z_i}\}$ and $\{|(z_j)|\}$; this is necessary in order for T_Z and R_Z to be well-defined. We call such precycles $\mathbb{R}$-*proper*. Kerr and Lewis [2007] proved the inclusion is a *rational* quasi-isomorphism, by appealing to Kleiman transversality in K-theory. Unfortunately, the claimed integral moving lemma in [Kerr et al. 2006] (which would have made this quasi-isomorphism integral) was incorrect, and [Kerr and Lewis 2007] was only written after a prolonged and unsuccessful effort to repair the integral version.

Now suppose we have a cycle $Z \in \ker(\partial) \subset Z^p_{\mathbb{R}}(X, n)$ with

$$[\widetilde{\mathrm{AJ}}^{p,n}_{\mathrm{KLM}}(Z)] \in H^{2p-n}_{\mathscr{D}}(X, \mathbb{Z}(p))$$

torsion of order exactly M. This implies $[Z] \in H_n\{Z^p_{\mathbb{R}}(X, \bullet)\}$ is at least of this order. But for $[Z] \in H_n\{Z^p(X, \bullet)\} = \mathrm{CH}^p(X, n)$, it implies no such thing: there could be

[4] Namely, for each $J \subset \{1, \ldots, n\}$ and $k \in \{0, \ldots, n\}$, set $I := \{1, \ldots, k\} \cap (\{1, \ldots, n\} \setminus J)$; then we require that $Z^{\mathrm{an}}_{\mathbb{C}}$ meet $\left(\bigcap_{i \in I} \{T_{z_i}\}\right) \cap \left(\bigcap_{j \in J} |(z_j)|\right)$ properly.

a $W \in Z^p(X, n+1) \setminus Z^p_{\mathbb{R}}(X, n+1)$ with $\partial W = Z$. So the KLM map only induces a homomorphism

$$\mathrm{AJ}^{p,n}_{\mathbb{Q}} : \mathrm{CH}^p(X, n) \to H^{2p-n}_{\mathscr{D}}(X, \mathbb{Q}(p))$$

consistent with Bloch's $\mathrm{AJ}^{p,n}$. This is frustrating, as the KLM formulas appear to be well-adapted to detecting torsion!

For $X = \mathrm{Spec}(k)$ and $(p, n) = (2, 3)$, consider the portion

$$\begin{array}{ccccccc} \cdots \longrightarrow & Z^2_{\mathbb{R}}(k,4) & \longrightarrow & Z^2_{\mathbb{R}}(k,3) & \longrightarrow & Z^2_{\mathbb{R}}(k,2) & \longrightarrow \cdots \\ & \downarrow {\scriptstyle (2\pi \boldsymbol{i})^2 T_4 \cap (\cdot)} & & \downarrow {\scriptstyle \frac{1}{2\pi \boldsymbol{i}} \int_{(\cdot)} R_{2p-1}} & & \downarrow {\scriptstyle 0} & \\ \cdots \longrightarrow & \mathbb{Z}(2) & \hookrightarrow & \mathbb{C} & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

of the KLM map of complexes. We want to use the middle map to detect torsion. Denote its image on a cycle Z by $\mathscr{R}(Z) \in \mathbb{C}/\mathbb{Z}(2)$.

Example 3.1 [Petras 2009]. We calculate

$$\begin{aligned}\mathscr{R}(Z^2_\ell) &= \frac{1}{2\pi \boldsymbol{i}} \int_{Z^2_\ell} R_3 \\ &= \frac{1}{2\pi \boldsymbol{i}} \int_{Z^2_\ell} \Big(\log(z_1) dz_2/z_2 \wedge dz_3/z_3 + (2\pi \boldsymbol{i}) \log(z_2) dz_3/z_3 \cdot \delta_{T_{z_1}} \\ &\qquad\qquad\qquad\qquad + (2\pi \boldsymbol{i})^2 \log(z_3) \delta_{T_{z_1} \cap T_{z_2}} \Big) \\ &= \int_{Z_\ell \cap T_{z_1}} \log(z_2) \frac{dz_3}{z_3} = -\int_{T_{1-\zeta_\ell/t}} \log(1-t) \frac{dt}{t} \\ &= -\int_0^{\zeta_\ell} \log(1-t) \frac{dt}{t} = \mathrm{Li}_2(\zeta_\ell).\end{aligned}$$

For $\ell = 1$, this is $\pi^2/6 \in \mathbb{C}/\mathbb{Z}(2)$, which is 24-torsion, while (for the second cycle of Example 2.1) $\mathscr{R}(\mathscr{Z}^2_5) = \mathrm{Li}_2(1) + \mathrm{Li}_2(\zeta_5) + \mathrm{Li}_2(\overline{\zeta_5}) = 7\pi^2/30$ is 120-torsion. To *deduce* that these orders of torsion exist in $\mathrm{CH}^2(\mathbb{Q}, 3)$ and $\mathrm{CH}^2(\mathbb{Q}(\sqrt{5}), 3)$, rather than just in $H_3\{Z^2_{\mathbb{R}}(k, \bullet)\}$, we need an improvement in technology.

4. The integral regulator

Turning to potential strategies for effecting this improvement, one can probably rule out:

(**1**) proving an integral moving lemma ($Z^p_{\mathbb{R}}(X, \bullet) \overset{\simeq}{\to} Z^p(X, \bullet)$),

and

(**2**) extending KLM to a map of complexes on $Z^p(X, \bullet)$,

as too naive (in view of the history recounted in Section 3); and while

(**3**) extending KLM to an infinite family of homotopic maps on nested subcomplexes with union $Z^p(X, \bullet)$,

seemed promising, what ultimately worked was

(**4**) extending KLM to an infinite family of homotopic maps on nested subcomplexes with union $N^p(X, \bullet)$.

The heuristic idea of (**3**) was to perturb the branch cuts $T_{z_i} = \{z_i \in \mathbb{R}_{<0}\}$ in $\log(z_i)$ to $T^{\epsilon}_{z_i} = \{z_i/e^{\boldsymbol{i}\epsilon} \in \mathbb{R}_{<0}\}$ (for small $\epsilon \in \mathbb{R}_{>0}$) and take a limit as $\epsilon \to 0$, an approach that had been successfully applied in [Kerr 2003a, §9]. (We write $\log^{\epsilon}(z_i)$ for the branch of logarithm along $T^{\epsilon}_{z_i}$.) Unfortunately, there are cycles in $Z^2(\mathbb{C}, 3)$ whose intersection with $T^{\epsilon}_{z_1} \cap T^{\epsilon}_{z_2} \cap T^{\epsilon}_{z_3}$ is improper for every real ϵ near 0 [Li 2018, §3]. So we need to deform the branches by distinct $\{\epsilon_i\}$ (viz., $T^{\epsilon_1}_{z_1} \cap T^{\epsilon_2}_{z_2} \cap T^{\epsilon_3}_{z_3}$); but then we cannot expect a morphism of complexes (or "limit" thereof) on $Z^p(X, \bullet)$. This forces us into strategy (**4**), and working with normalized subcomplexes.

Let $\mathcal{B}_{\varepsilon}$ denote the set of infinite sequences $\underline{\epsilon} = \{\epsilon_i\}_{i>0}$, with

$$0 < \epsilon_1 < \varepsilon, \quad 0 < \epsilon_2 < e^{-1/\epsilon_1}, \quad 0 < \epsilon_3 < e^{-1/\epsilon_2}, \quad \text{etc.},$$

so that when $\varepsilon \to 0$ its projection to any $(S^1)^n$ via $(e^{i\epsilon_1}, \dots, e^{i\epsilon_n})$ eventually avoids any given analytic subvariety. Let $N^p_{\varepsilon}(X, \bullet) \subset N^p(X, \bullet)$ denote the subcomplexes of cycles Z with Z^{an} properly intersecting (for each $\underline{\epsilon} \in \mathcal{B}_{\varepsilon}$) certain[5] combinations of $\{T^{\epsilon_i}_{z_i}\}$ and $\{(z_j)_0, (z_j)_{\infty}\}$. Since for $\varepsilon' < \varepsilon$ we have $\mathcal{B}_{\varepsilon'} \subset \mathcal{B}_{\varepsilon}$, we also have $N^p_{\varepsilon'}(X, \bullet) \supset N^p_{\varepsilon}(X, \bullet)$.

Lemma 4.1 [Li 2018, Theorems 4.2 and 7.2]. *We have*

$$\bigcup_{\varepsilon>0} N^p_{\varepsilon}(X, n) = N^p(X, n) \quad \textit{for all } n$$

and

$$\lim_{\varepsilon \to 0} H_n(N^p_{\varepsilon}(X, \bullet)) \cong H_n(N^p(X, \bullet)) \cong \mathrm{CH}^p(X, n).$$

For any $\underline{\epsilon} \in \mathcal{B}_{\varepsilon}$, replacing T_{z_i} by $T^{\epsilon_i}_{z_i}$ and $\log(z_i)$ by $\log^{\epsilon_i}(z_i)$ everywhere in the KLM formula yields a morphism of complexes

$$\widetilde{\mathrm{AJ}}^{p,-\bullet}_{\varepsilon,\underline{\epsilon}} : N^p_{\varepsilon}(X, -\bullet) \to C^{2p+\bullet}_{\mathscr{D}}(X, \mathbb{Z}(p)).$$

Lemma 4.2 [Li 2018, Theorem 6.1]. *Given* $\underline{\epsilon}, \underline{\epsilon}' \in \mathcal{B}_{\varepsilon}$, $\widetilde{\mathrm{AJ}}^p_{\varepsilon,\underline{\epsilon}}$ *and* $\widetilde{\mathrm{AJ}}^p_{\varepsilon,\underline{\epsilon}'}$ *are* ($\mathbb{Z}$-)*homotopic.*

Sketch of proof. Write $\overline{\square} := \mathbb{P}^1$ and fix an integer $N \gg 0$. For each multi-index $I \subset \{1, \dots, N\}$ and function $f : I \to \{0, \infty\}$, define $\partial^I_f \overline{\square}^N := \bigcap_{i \in I}\{z_i = f(i)\} \subset \overline{\square}^N$;

[5] as in the last footnote

and let $\rho_{i,I}^{f(i)}\colon \partial_f^I\overline{\square}^N \hookrightarrow \partial_f^{I\setminus i}\overline{\square}^N$ be the inclusions. Then we can define a double complex (zero outside $0 \le a \le N$)

$$E^{a,b} := \bigoplus_{|I|=N-a} \bigoplus_{f\colon I\to\{0,\infty\}} C_{\mathscr{D}}^{2a+b}(\partial_f^I\overline{\square}^N, \mathbb{Z}(a))$$

with $D_{\mathscr{D}}$ the vertical differential, $\delta := \sum_{I,f,i}(-1)^{\mathrm{sgn}_I(i)+\mathrm{sgn}(f(i))}(\rho_{i,I}^{f(i)})_*$ the horizontal differential, and $\mathbb{D} := D_{\mathscr{D}} + (-1)^b\delta$ the total differential.

Put $\underline{\epsilon}_I := \{\epsilon_j\}_{j\notin I}$ and

$$\mathcal{R}_{I,f}^{\underline{\epsilon}} := ((2\pi\boldsymbol{i})^a T_a^{\underline{\epsilon}_I}, \Omega_a, R_a^{\underline{\epsilon}_I}) \in C_{\mathscr{D}}^a(\partial_f^I\overline{\square}^N, \mathbb{Z}(a)) \quad (\cong C_{\mathscr{D}}^a((\mathbb{P}^1)^a, \mathbb{Z}(a))),$$

so that $\mathcal{R}_a^{\underline{\epsilon}} := \{\mathcal{R}_{I,f}^{\underline{\epsilon}}\}_{I,f}^{|I|=N-a}$ belongs to $E^{a,-a}$. Adapting [Kerr et al. 2006, (5.2)–(5.4)] to these perturbed currents, [Li 2018] deduces that $\mathcal{R}_{\square}^{\underline{\epsilon}} := \{\mathcal{R}_a^{\underline{\epsilon}}\}_{0\le a\le N}$ is a 0-cocycle in $E^{\bullet,\bullet}$. The key point of [op. cit.] is then to construct a (-1)-cochain $\mathcal{S}_{\square}^{\underline{\epsilon},\underline{\epsilon}'}$ in $E^{\bullet,\bullet}$ with $\mathbb{D}\mathcal{S}_{\square}^{\underline{\epsilon},\underline{\epsilon}'} = \mathcal{R}_{\square}^{\underline{\epsilon}} - \mathcal{R}_{\square}^{\underline{\epsilon}'}$, and with respect to which the precycles in $N_{\varepsilon}^p(X,\bullet)$ remain proper. This yields at once a homotopy formula in $C_{\mathscr{D}}^{\bullet}(X,\mathbb{Z}(\bullet))$ of the form

$$D_{\mathscr{D}}\mathcal{S}_Z^{\underline{\epsilon},\underline{\epsilon}'} - \mathcal{S}_{\partial Z}^{\underline{\epsilon},\underline{\epsilon}'} = \widetilde{\mathrm{AJ}}_{\varepsilon,\underline{\epsilon}}^p(Z) - \widetilde{\mathrm{AJ}}_{\varepsilon,\underline{\epsilon}'}^p(Z).$$

The construction of $\mathcal{S}_{\square}^{\underline{\epsilon},\underline{\epsilon}'}$ proceeds by replacing each KLM triple

$$\big((2\pi\boldsymbol{i})^a T_a^{\underline{\epsilon}}, \Omega_a, R_a^{\underline{\epsilon}}\big) = \Big(2\pi\boldsymbol{i}T_{z_1}^{\epsilon_1}, \frac{dz_1}{z_1}, \log^{\epsilon_1}(z_1)\Big) \cup\cdots\cup \Big(2\pi\boldsymbol{i}T_{z_a}^{\epsilon_a}, \frac{dz_a}{z_a}, \log^{\epsilon_a}(z_a)\Big)$$

by an alternating sum of cup-products in which $(-2\pi\boldsymbol{i}\,\delta_{\{\arg(z_i)\in(\epsilon_i,\epsilon_i')\}}, 0, 0)$ replaces the i-th term. See [op. cit.] for full details. □

We therefore have well-defined, compatible maps

$$\mathrm{AJ}_{\varepsilon}^{p,n}: H_n(N_{\varepsilon}^p(X,\bullet)) \to H_{\mathscr{D}}^{2p-n}(X,\mathbb{Z}(p))$$

for each $\varepsilon > 0$, essentially given by $\lim_{\underline{\epsilon}\to\underline{0}} \widetilde{\mathrm{AJ}}_{\underline{\epsilon}}^p$ (where the limit is taken so that $\varepsilon > \epsilon_1 \gg \epsilon_2 \gg \epsilon_3 \gg \cdots > 0$), and recovering $\widetilde{\mathrm{AJ}}_{\mathrm{KLM}}^p$ on

$$N_{\mathbb{R}}^p(X,\bullet) := Z_{\mathbb{R}}^p(X,\bullet) \cap N^p(X,\bullet).$$

More precisely:

Theorem 4.3 [Li 2018, §7]. *The* $\{\widetilde{\mathrm{AJ}}_{\varepsilon,\underline{\epsilon}}^{p,-\bullet}\}$ *induce a homomorphism*

$$\mathrm{AJ}_{\mathbb{Z}}^{p,n}: \mathrm{CH}^p(X,n) \cong \lim_{\varepsilon\to 0} H_n(N_{\varepsilon}^p(X,\bullet)) \to H_{\mathscr{D}}^{2p-n}(X,\mathbb{Z}(p))$$

factoring $\mathrm{AJ}_{\mathbb{Q}}^{p,n}$.

Corollary 4.4. *If a class* $\xi \in \mathrm{CH}^p(X,n)$ *is represented by*

$$Z \in \ker(\partial) \subset N_{\mathbb{R}}^p(X,n) \Big(\subset \bigcap_{\varepsilon>0} N_{\varepsilon}^p(X,n)\Big),$$

then

$$\mathrm{AJ}^{p,n}_{\mathbb{Z}}(\xi) = \lim_{\underline{\epsilon}\to\underline{0}} \widetilde{\mathrm{AJ}}^{p,n}_{\underline{\epsilon}}(Z) = \widetilde{\mathrm{AJ}}^{p,n}_{\mathrm{KLM}}(Z).$$

So the KLM formula holds verbatim on normalized, $\mathbb{R}$-proper representatives, validating the deductions at the end of Example 3.1. It is this statement that we (primarily) use in the applications that follow.

5. Torsion generators

Let $\mu_\infty = \bigcup_{m\in\mathbb{N}} \mu_m \subset \mathbb{C}^*$ denote the roots of unity, and $w_r(k) := |(\mu_\infty^{\otimes r})^{\mathrm{Gal}(\bar{\mathbb{Q}}/k)}|$ for any number field $k \subset \mathbb{C}$. By the universal coefficient sequence for motivic cohomology

$$H^0_{\mathcal{M}}(k, \mathbb{Z}(r)) \to H^0_{\mathcal{M}}(k, \mathbb{Z}/m\mathbb{Z}(r)) \to H^1_{\mathcal{M}}(k, \mathbb{Z}(r)) \xrightarrow{\cdot m} H^1_{\mathcal{M}}(k, \mathbb{Z}(r))$$

and vanishing of $H^0_{\mathcal{M}}(k, \mathbb{Z}(r))$,[6] we have

$$\mathrm{CH}^r(k, 2r-1)[m] \cong H^1_{\mathcal{M}}(k, \mathbb{Z}(r))[m] \cong H^0_{\mathcal{M}}(k, \mathbb{Z}/m\mathbb{Z}(r)).$$

Since the norm residue map

$$H^0_{\mathcal{M}}(k, \mathbb{Z}/m\mathbb{Z}(r)) \to H^0_{\text{ét}}(k, \mu_m^{\otimes r}) \cong (\mu_m^{\otimes r})^{\mathrm{Gal}(\bar{\mathbb{Q}}/k)}$$

is an isomorphism by a celebrated theorem of Rost-Voevodsky (see [Haesemeyer and Weibel 2019]), we conclude that $\mathrm{CH}^r(k, 2r-1)[m] \cong \mathbb{Z}/(m, w_r(k))\mathbb{Z}$, hence

$$\mathrm{CH}^r(k, 2r-1)_{\mathrm{tors}} \cong \mathbb{Z}/w_r(k)\mathbb{Z}.$$

Example 5.1. If $k = \mathbb{Q}$, one has[7] $w_{2n}(k) =$ denominator of $(|B_{2n}|)/4n$ (written in lowest terms) and $w_{2n+1} = 2$ for $n \geq 1$; so

$$\mathrm{CH}^2(\mathbb{Q}, 3) \cong \mathbb{Z}/24\mathbb{Z}, \quad \mathrm{CH}^3(\mathbb{Q}, 5)_{\mathrm{tors}} \cong \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{CH}^4(\mathbb{Q}, 7) \cong \mathbb{Z}/240\mathbb{Z}.$$

For real quadratic fields $k = \mathbb{Q}(\sqrt{d})$, the situation is more complicated (see [Weibel 2013, §VI.2]); one computes for instance

d	2	3	5	7
$w_2(\mathbb{Q}(\sqrt{d}))$	48	24	120	24
$w_4(\mathbb{Q}(\sqrt{d}))$	480	240	240	240

[6] See [Weibel 2013, Exercise VI.4.6]: since $H^{-1}_{\mathcal{M}}(k, \mathbb{Z}/m\mathbb{Z}(r)) \cong H^{-1}_{\text{ét}}(k, \mu_m^{\otimes r}) = \{0\}$, $\cdot m$ is injective on $H^0_{\mathcal{M}}(k, \mathbb{Z}(r))$ (universal coefficient sequence), which is thus torsion-free; it has rank 0 since $H^0_{\mathcal{M}}(k, \mathbb{Q}(r)) \cong \mathrm{Gr}^r_\gamma K_{2r}(k) \otimes \mathbb{Q} = \{0\}$ by Borel's theorem [1974].

[7] Bernoulli numbers: $|B_{2n}| = \frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \ldots$ for $n = 1, 2, 3, \ldots$.

so that only $\mathrm{CH}^2(\mathbb{Q}(\sqrt{2}),3)$, $\mathrm{CH}^2(\mathbb{Q}(\sqrt{5}),3)$ and $\mathrm{CH}^4(\mathbb{Q}(\sqrt{2}),7)$ are different from the $k=\mathbb{Q}$ case. Finally, for cyclotomic $k=\mathbb{Q}(\zeta_p)$ $(p\geq 5$ prime) one can show that $w_3(\mathbb{Q}(\zeta_p))=2p$, while $w_3(\mathbb{Q}(\zeta_3))=18$.

For computing torsion orders of images under

$$\mathrm{AJ}^{r,2r-1}_{\mathbb{Z}}: H_{2r-1}(N^r_{\mathbb{R}}(k,\bullet))\to \mathbb{C}/(2\pi \boldsymbol{i})^r\mathbb{Z},\quad Z\mapsto \frac{1}{(2\pi\boldsymbol{i})^{r-1}}\int_Z R_{2r-1}=:\mathscr{R}(Z)$$

we use the following basic calculation:

Proposition 5.2. *Suppose that for a given $r\in\mathbb{N}$ there exists a collection of closed precycles $Z^r_{\ell,a}\in N^r_{\mathbb{R}}(\mathbb{Q}(\zeta_\ell),2r-1)$ with*

$$\mathscr{R}(Z^r_{\ell,a})=(r-1)!\ell^{r-1}\mathrm{Li}_r(\zeta^a_\ell). \tag{5-1}$$

Then for $Z:=\sum_{a=0}^{\ell-1}f(a)Z^r_{\ell,a}$ with $f(-a)=(-1)^r f(a)$, $\mathrm{AJ}(Z)$ is torsion of order given by the denominator of

$$\tau(Z):=|(2\pi\boldsymbol{i})^{-r}\mathscr{R}(Z)|=\pm\frac{\ell^{r-1}}{2r}\sum_{a=0}^{\ell-1}f(a)B_r(a/\ell), \tag{5-2}$$

where $B_r(\cdot)$ are the Bernoulli polynomials.[8]

Proof. Writing $\hat{f}(k):=\sum_{a=0}^{\ell-1}f(a)\zeta_\ell^{-ka}$ for the finite Fourier transform (for functions on $\mathbb{Z}/\ell\mathbb{Z}$), we have

$$\begin{aligned}\sum_{a=0}^{\ell-1}f(a)\mathrm{Li}_r(\zeta^a_\ell)&=\frac{1}{2}\sum_{a=0}^{\ell-1}f(a)\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{\zeta_\ell^{ka}}{k^r}\\&=\frac{(-1)^r}{2}\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{\hat{f}(k)}{k^r}=:\frac{(-1)^r}{2}\tilde{L}(\hat{f},r)\\&=\frac{(2\pi\boldsymbol{i})^r}{2\cdot r!}\sum_{a=0}^{\ell-1}f(a)B_r(a/\ell),\end{aligned}$$

where the last step is [Kerr 2018, Theorem 3.9]. □

In practice, the $\{Z^r_{\ell,a}\}$ will be obtained from a single cycle $Z^r_\ell=Z^r_{\ell,1}$ by Galois conjugation. For $r=2$, we already have this from Examples 2.1 and 3.1: namely,

$$Z^2_{\ell,a}=\Big(1-\frac{\zeta^a_\ell}{t},1-t,t^{-\ell}\Big). \tag{5-3}$$

Now $\mathrm{CH}^r(k,2r-1)=\mathrm{CH}^r(k,2r-1)_{\mathrm{tors}}\Leftrightarrow r$ is even and k is totally real. In particular, assuming the hypothesis of Proposition 5.2, we obtain generators of

[8] $B_2(x)=x^2-x+\frac{1}{6}$, $B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x$, $B_4(x)=x^4-2x^3+x^2-\frac{1}{30}$, etc.

$\mathrm{CH}^{2n}(k, 4n-1)$ as follows:

$$k = \mathbb{Q},\ \text{any } n: \qquad \tau(Z_1^{2n}) = \frac{|B_{2n}|}{4n}\left(= \tfrac{1}{24}, \tfrac{1}{240}, \ldots\right);$$
$$k = \mathbb{Q}(\sqrt{2}),\ n = 1, 2: \qquad \tau(Z_{8,1}^{2n} + Z_{8,7}^{2n}) = \tfrac{11}{48}, \tfrac{1313}{480};$$
$$k = \mathbb{Q}(\sqrt{5}),\ n = 1: \quad \tau(Z_1^2 + Z_{5,1}^2 + Z_{5,4}^2) = \tfrac{7}{120}.$$

For $\mathrm{CH}^3(k, 5)_{\mathrm{tors}}$ one computes, for example,

$$k = \mathbb{Q}(\zeta_3): \quad \tau(Z_{3,1}^3 - Z_{3,2}^3) = \tfrac{1}{9}; \quad \text{and}$$
$$k = \mathbb{Q}(\zeta_5): \quad \tau(Z_{5,1}^3 - Z_{5,4}^3) = \tfrac{2}{5},$$

which miss only the 2-torsion element from $\mathrm{CH}^3(\mathbb{Q}, 5) \hookrightarrow \mathrm{CH}^3(k, 5)$. (So far we have no $N^3_{\mathbb{R}}(\mathbb{Q}, 5)$ representative for this element.)

It remains to construct the cycles of the Proposition for $r = 3, 4$. From [Kerr and Yang 2018, §4.2], for $r = 3$ we have[9]

$$(5\text{-}4)\quad Z_\ell^3 := -2\left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, 1-\zeta_\ell t_1 t_2, t_1^\ell, t_2^\ell\right) - \left(\frac{t}{t-1}, \frac{1}{1-\zeta_\ell t}, \frac{(u-t^\ell)(u-t^{-\ell})}{(u-1)^2}, t^\ell u, \frac{u}{t^\ell}\right)$$

which is normalized since all "boundaries" occur in the third coordinate. We have $\mathscr{R}(Z_\ell^3) = 2\mathrm{Li}_3(\zeta_\ell)$ by [op. cit., Theorem 3.6], with only the first term contributing. (This gives in particular $\mathscr{R}(Z_1^3) = 2\mathrm{Li}_3(1) = 2\zeta(3)$.)

For $r = 4$, the first construction in [Kerr and Yang 2018, §4.3] would be in $N^4_{\mathbb{R}}(\mathbb{Q}(\zeta_\ell), 7)$, but there is an error in the computation of the boundary of the last component $\mathscr{W}_2$: in fact, it is degenerate,[10] and the cycle $\tilde{\mathscr{Z}}$ is therefore not closed. A correct application of the strategy in [op. cit., §3.1], yields

$$(5\text{-}5)\quad Z_\ell^4 := 6\left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{t_3}{t_3-1}, 1-\zeta_\ell t_1 t_2 t_3, t_1^\ell, t_2^\ell, t_3^\ell\right)$$
$$+\left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\zeta_\ell t_1 t_2}, \frac{(u-t_1^\ell)(u-t_2^\ell)(u-t_1^{-\ell}t_2^{-\ell})}{(u-1)^3}, \frac{t_1^\ell}{u}, \frac{t_2^\ell}{u}, \frac{1}{u t_1^\ell t_2^\ell}\right)$$
$$+\left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\zeta_\ell t_1 t_2}, \frac{(u-t_1^\ell)(u-t_2^\ell)}{(u-t_1^\ell t_2^\ell)(u-1)}, \frac{t_1^\ell}{u}, \frac{t_2^\ell}{u}, \frac{u}{t_1^\ell t_2^\ell}\right)$$
$$+\left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\zeta_\ell t_1 t_2}, \frac{(u-t_1^\ell)(u-t_1^{-\ell}t_2^{-\ell})}{(u-t_2^{-\ell})(u-1)}, \frac{t_1^\ell}{u}, t_2^\ell u, \frac{1}{u t_1^\ell t_2^\ell}\right)$$

[9] The components are parametrized by (t_1, t_2) and (t, u) respectively.

[10] I.e., it belongs to $d^4(\mathbb{Q}(\zeta_\ell), 7)$, as can be seen by substituting $v = uw$.

$$+\left(\frac{t_1}{t_1-1},\frac{t_2}{t_2-1},\frac{1}{1-\zeta_\ell t_1t_2},\frac{(u-t_2^\ell)(u-t_1^{-\ell}t_2^{-\ell})}{(u-t_1^{-\ell})(u-1)},t_1^\ell u,\frac{t_2^\ell}{u},\frac{1}{t_1^\ell t_2^\ell u}\right)$$
$$+\left(\frac{(v-u)(v-u^{-1})}{(v-1)^2},\frac{t}{t-1},\frac{1}{1-\zeta t},\frac{(u-t^\ell)(u-t^{-\ell})}{(u-1)^2},v,\frac{t^\ell}{u},\frac{1}{ut^\ell}\right)$$
$$+\left(\frac{t}{t-1},\frac{(v-u)(v-u^{-1})}{(v-1)^2},\frac{1}{1-\zeta t},\frac{(u-t^\ell)(u-t^{-\ell})}{(u-1)^2},\frac{t^\ell}{u},v,\frac{1}{ut^\ell}\right)$$
$$+\left(\frac{t}{t-1},\frac{1}{1-\zeta t},\frac{(v-u)(v-u^{-1})}{(v-1)^2},\frac{(u-t^\ell)(u-t^{-\ell})}{(u-1)^2},\frac{t^\ell}{u},\frac{1}{ut^\ell},v\right),$$

which belongs to $\ker(\partial)\cap N^4_{\mathbb{R}}(\mathbb{Q}(\zeta_\ell),7)$. Only the first term contributes to $\mathscr{R}(Z^4_\ell)=-6\ell^3\int_{[0,1]^3}\log(1-\zeta_\ell t_1t_2t_3)\,dt_1/t_1\wedge dt_2/t_2\wedge dt_3/t_3=6\ell^3\mathrm{Li}_4(\zeta_\ell)$, see [op. cit., §3.2]. Again, cycles $Z^3_{\ell,a}$ and $Z^4_{\ell,a}$ are obtained by replacing ζ_ℓ by ζ_ℓ^a in (5-4) and (5-5), respectively.

Summarizing, we have

Theorem 5.3. *The cycles given in* (5-3), (5-4), (5-5) *generate the* (cyclic, torsion) *groups as shown in Table 1.*

Remark 5.4. An example of a cycle for which the log-branch perturbations *are* required for the integral regulator computation is

$$Z:=Z_--Z_+:=\left(\left(\frac{z-\boldsymbol{i}}{z+\boldsymbol{i}}\right)^{-2},\left(\frac{z-1}{z+1}\right)^{-2},z^{-2}\right)-\left(\left(\frac{z-\boldsymbol{i}}{z+\boldsymbol{i}}\right)^{2},\left(\frac{z-1}{z+1}\right)^{2},z^{2}\right)$$

(parametrized by $z\in\mathbb{P}^1$) in $N^2(\mathbb{Q}(\boldsymbol{i}),3)$. Indeed, T_{z^2} has support on $\boldsymbol{i}\mathbb{R}$ in the complex z-plane, and $T_{((z-1)/(z+1))^2}$ and $T_{((z-\boldsymbol{i})/(z+\boldsymbol{i}))^2}$ have support on the unit circle S^1 in the complex z-plane, so that the triple intersection (essentially $\boldsymbol{i}\mathbb{R}\cap S^1\cap S^1$) is nonempty. Though this cycle is *non*-torsion, we briefly describe the computation. After making the deformation, $T_{((z-\boldsymbol{i})/(z+\boldsymbol{i}))^2}$ and $T_{((z-1)/(z+1))^2}$ intersect twice with

Group	Cycle	Order
$\mathrm{CH}^2(\mathbb{Q},3)$	Z^2_1	24
$\mathrm{CH}^4(\mathbb{Q},7)$	Z^4_1	240
$\mathrm{CH}^2(\mathbb{Q}(\sqrt{2}),3)$	$Z^2_{8,1}+Z^2_{8,7}$	48
$\mathrm{CH}^4(\mathbb{Q}(\sqrt{2}),7)$	$Z^4_{8,1}+Z^4_{8,7}$	480
$\mathrm{CH}^2(\mathbb{Q}(\sqrt{5}),3)$	$Z^2_1+Z^2_{5,1}+Z^2_{5,4}$	120
$\dfrac{\mathrm{CH}^3(\mathbb{Q}(\zeta_3),5)_{\mathrm{tors}}}{\mathrm{CH}^3(\mathbb{Q},5)_{\mathrm{tors}}}$	$Z^3_{3,1}-Z^3_{3,2}$	9
$\dfrac{\mathrm{CH}^3(\mathbb{Q}(\zeta_5),5)_{\mathrm{tors}}}{\mathrm{CH}^3(\mathbb{Q},5)_{\mathrm{tors}}}$	$Z^3_{5,1}-Z^3_{5,4}$	5

Table 1. Some Generators (see Theorem 5.3).

opposite orientations, at points near $\boldsymbol{i}$ and $-\boldsymbol{i}$ with phase just greater than $\pi/2$ and $3\pi/2$, respectively. Since $\epsilon_3 \to 0$ much faster than ϵ_1 and ϵ_2, in the limit the

$$2\pi \boldsymbol{i} \int_{Z_+} \log^{\epsilon_3}(z^2)\, \delta_{T^{\epsilon_1}_{((z-\boldsymbol{i})/(z+\boldsymbol{i}))^2} \cap T^{\epsilon_2}_{((z-1)/(z+1))^2}}$$

term of $\frac{1}{2\pi \boldsymbol{i}} \int_{Z_+} R_3$ contributes $2\pi \boldsymbol{i}(\pi \boldsymbol{i} - \pi \boldsymbol{i}) = 0$. The remaining term yields

$$2 \int_{Z_+} \log^{\epsilon_2}\left(\left(\frac{1-z}{1+z}\right)^2\right) \frac{dz}{z}\, \delta_{T^{\epsilon_1}_{((z-\boldsymbol{i})/(z+\boldsymbol{i}))^2}}, \tag{5-6}$$

where $T^{\epsilon_1}_{((z-\boldsymbol{i})/(z+\boldsymbol{i}))^2}$ consists of two paths from $-\boldsymbol{i}$ to $\boldsymbol{i}$, along which one checks that (in the limit)

$$\log^{\epsilon_2}\left(\left(\frac{1-z}{1+z}\right)^2\right) = 2\log^{\epsilon_2}(1-z) - 2\log^{\epsilon_2}(1+z);$$

and so (5-6) becomes $8\int_{-\boldsymbol{i}}^{\boldsymbol{i}} \log(1-z)\frac{dz}{z} - 8\int_{-\boldsymbol{i}}^{\boldsymbol{i}} \log(1+z)\frac{dz}{z}$. Combining this with the portion from Z_-, we obtain

$$\mathscr{R}(Z) = 32\mathrm{Li}_2(\boldsymbol{i}) - 32\mathrm{Li}_2(-\boldsymbol{i}) = 64\boldsymbol{i}\, L(\chi_4, 2) \in \mathbb{C}/\mathbb{Z}(2).$$

6. Local $\mathbb{P}^2$ revisited

For a reflexive polytope $\Delta \subset \mathbb{R}^2$ with polar polytope Δ°, the mirror of $K_{\mathbb{P}_{\Delta^\circ}}$ ("local $\mathbb{P}_{\Delta^\circ}$") can be identified with a family of $\mathrm{CH}^2(\cdot, 2)$-elements on a family of anticanonical (elliptic) curves in $\mathbb{P}_\Delta$ [Doran and Kerr 2011, §5]. As a second application of the integral regulator, we show how to apply it to compute the correct "torsion term" in the higher normal function associated to one of these families. That is, if $\mathcal{E} \xrightarrow{\pi} \mathbb{P}^1_t$ is smooth away from $\Sigma = \{0\} \cup \Sigma^*$, with fibers $E_t = \pi^{-1}(t)$, and $\Xi \in \mathrm{CH}^2(\mathcal{E} \setminus E_0, 2)$ has fiberwise restrictions $\xi_t \in \mathrm{CH}^2(E_t, 2)$ $(t \notin \Sigma)$, we shall compute

$$\begin{aligned} \mathscr{R}_t := \mathrm{AJ}^{2,2}(\xi_t) \in H^2_{\mathscr{D}}(E_t, \mathbb{C}/\mathbb{Z}(2)) &\cong H^1(E_t, \mathbb{C}/\mathbb{Z}(2)) \\ &\cong Hom\big(H_1(E_t, \mathbb{Z}), \mathbb{C}/(2\pi \boldsymbol{i})^2\mathbb{Z}\big) \end{aligned}$$

in a neighborhood of $t = 0$. Writing $\{\omega_t\}$ for a section of $\omega_{\mathcal{E}/\mathbb{P}^1}$ vanishing at ∞, the constant term in (a branch of) the resulting truncated higher normal function

$$\nu(t) := \frac{1}{2\pi \boldsymbol{i}} \langle \omega_t, \mathscr{R}_t \rangle \tag{6-1}$$

will play a role in forthcoming work of the first author with C. Doran on quantum curves.

To begin in a somewhat more general scenario, let $\Delta \subset \mathbb{R}^2$ be any convex polytope with integer vertices $\{p_i = (a_i, b_i)\}_{i=1}^N$ and interior integer points $\{(v_j, w_j)\}_{j=1}^g$.

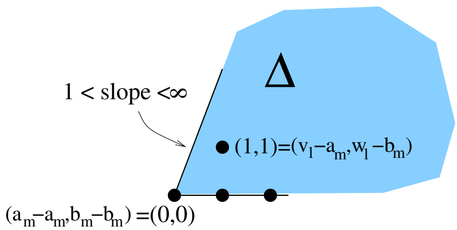

Figure 1. For the proof of Lemma 6.1.

Define a multiparameter family

$$\rho : \mathcal{C} \to \mathbb{C}^g, \quad C_{\underline{\lambda}} := \rho^{-1}(\underline{\lambda})$$

of (where smooth) genus g curves by taking the Zariski closure of

$$\mathcal{C}^* := \left\{(x, y; \lambda_1, \dots, \lambda_g) \mid 0 = \Phi_{\underline{\lambda}}(x, y) := \phi(x, y) - \Sigma_{j=1}^{g} \lambda_j x^{v_j} y^{w_j}\right\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^g$$

in $\mathbb{P}_\Delta \times \mathbb{C}^g$, where $\phi(x, y) := \sum_{(a,b)\in\partial\Delta\cap\mathbb{Z}^2} m_{a,b} x^a y^b$ is uniquely determined by requiring its edge polynomials to be powers of $(t+1)$. (If the edges of Δ have no interior points, then $\phi(x, y) = \sum_{i=1}^{N} x^{a_i} y^{b_i}$.) The symbol $\{-x, -y\}$ represents a closed precycle $\Xi^* \in Z^2(\mathcal{C}^*, 2)$ parametrized (in $\mathcal{C}^* \times \square^2$) by $(x, y, \underline{\lambda}, -x, -y)_{(x,y,\underline{\lambda})\in\mathcal{C}^*}$.

Lemma 6.1. *The class of* Ξ^* *in* $\mathrm{CH}^2(\mathcal{C}^*, 2)$ *is the restriction of a class* $\Xi \in \mathrm{CH}^2(\mathcal{C}, 2)$.

Proof. We need only check that the Tame symbol of $\{-x, -y\}|_{C^*_{\underline{\lambda}}} \in K_2^M(\mathbb{C}(C_{\underline{\lambda}}))$ is zero for general $\underline{\lambda}$. The symbol $\{-x, -y\}$ is invariant under unimodular change of toric coordinates,[11] so we may assume that (after shifting Δ by $(-a_m, -b_m)$ for some m) we have a picture (Figure 1) where the bottom edge corresponds to the toric divisor at whose intersection with $C_{\underline{\lambda}}$ we wish to compute $\mathrm{Tame}(\{-x, -y\}|_{C^*_{\underline{\lambda}}}) \in \mathbb{C}^*$. Since the edge polynomial is $(1 + x)^c$, this intersection occurs at $(-1, 0)$, so the Tame symbol is 1. □

Now set $\mathscr{R}_{\underline{\lambda}} := \mathrm{AJ}^{2,2}(\Xi|_{C_{\underline{\lambda}}}) \in \mathit{Hom}(H_1(C_{\underline{\lambda}}, \mathbb{Z}), \mathbb{C}/\mathbb{Z}(2))$. Picking any vertex p_m of Δ, we can (via unimodular coordinate change) put it in the position indicated in Figure 1. In the new coordinates (still denoted (x, y)), $C_{\underline{\lambda}}$ is cut out by an equation of the form

$$\begin{aligned} 0 = \tilde{\Phi}_{\underline{\lambda}} &:= x^{-a_m} y^{-b_m} \Phi_{\underline{\lambda}}(x, y) \\ &= (1 + x)^{\kappa_m} + y\left\{\Psi_m(x, y) - \textstyle\sum_{j=1}^{g} \lambda_j x^{v_j - a_m} y^{w_j - b_m - 1}\right\}, \end{aligned}$$

[11] That is, replacing x, y by $x^a y^b, x^c y^d$ with $ad - bc = 1$; the a_i, b_i, v_i, w_i are changed accordingly.

and acquires a node at $(0,0)$ as $\lambda_\ell \to \infty$. (Note that ℓ is determined by m.) The corresponding vanishing cycle α_m has image $|x| = |y| = \epsilon$ under

$$H_1(C_{\underline{\lambda}}), \mathbb{Z}) \xrightarrow{\text{Tube}} H_2(\mathbb{P}_\Delta \setminus C_{\underline{\lambda}})$$

for large $|\lambda_\ell|$.

Theorem 6.2. *For $\boldsymbol{i}\lambda_\ell \in \mathfrak{H}$ and $|\lambda_\ell| \gg 0$, and $\lambda_{j\neq\ell}$ sufficiently small,*[12] *we have*

$$\mathscr{R}_{\underline{\lambda}}(\alpha_m) = 2\pi\boldsymbol{i}\big(-\log(\lambda_\ell) + \textstyle\sum_{k\geq 1} \frac{1}{k}[\Psi^k_{\underline{\lambda},\ell}]_{\underline{0}}\big) \in \mathbb{C}/\mathbb{Z}(2),$$

where $\Psi_{\underline{\lambda},\ell} := (-1/\lambda_\ell)(x^{-v_\ell} y^{-w_\ell}\Phi_{\underline{\lambda}} + \lambda_\ell)$ and $[\cdot]_{\underline{0}}$ takes the constant term in a Laurent polynomial.

Proof. We use the notation

$$R\{f_1, f_2\} = \log(f_1)\frac{df_1}{f_1} - 2\pi\boldsymbol{i}\log(f_2)\delta_{T_{f_1}}, \quad \text{and}$$

$$R\{f_1, f_2, f_3\} = \log(f_1)\frac{df_2}{f_2}\wedge\frac{df_3}{f_3} + 2\pi\boldsymbol{i}\log(f_2)\frac{df_3}{f_3}\delta_{T_{f_1}} + (2\pi\boldsymbol{i})^2\log(f_3)\delta_{T_{f_1}\cap T_{f_2}}$$

for R_2 and R_3 with f_i replacing z_i. Writing D for the bottom-edge divisor in Figure 1, we have

$$\mathrm{Tame}_D\{\tilde{\Phi}_{\underline{\lambda}}, -x, -y\} = \{\tilde{\Phi}_{\underline{\lambda}}(x,0), -x\} = \{(1+x)^c, -x\} \ (=1).$$

So writing $\Gamma = \{|x| = \epsilon \geq |y|\}$ ($\Longrightarrow \alpha_m = \Gamma \cap C_{\underline{\lambda}}$) gives

$$\begin{aligned}
\mathscr{R}_t(\alpha_m) = \int_{\alpha_m} R\{-x,-y\} &= \int_\Gamma R\{-x,-y\}\cdot\delta_{C_{\underline{\lambda}}} \\
&= \frac{-1}{2\pi\boldsymbol{i}}\int_\Gamma d[R\{\tilde{\Phi}_{\underline{\lambda}},-x,-y\}] - \int_\Gamma R\{(1+x)^c,-x\}\cdot\delta_D \\
&= \frac{-1}{2\pi\boldsymbol{i}}\int_{\partial\Gamma} R\{\tilde{\Phi}_{\underline{\lambda}},-x,-y\} - \underbrace{\int_{|x|=\epsilon} R\{(1+x)^c,-x\}}_{\to 0} \\
&= \frac{-1}{2\pi\boldsymbol{i}}\int_{|x|=|y|=\epsilon} R\{x^{-v_\ell}y^{-w_\ell}\Phi_{\underline{\lambda}},-x,-y\} \\
&= \frac{-1}{2\pi\boldsymbol{i}}\int_{|x|=|y|=\epsilon} R\{\lambda(1-\Psi_{\underline{\lambda},\ell}),-x,-y\} \\
&= \frac{-1}{2\pi\boldsymbol{i}}\int_{|x|=|y|=\epsilon} \{\log(\lambda)+\log(1-\Psi_{\underline{\lambda},\ell})\}\frac{dx}{x}\wedge\frac{dy}{y} \\
&= -2\pi\boldsymbol{i}\log(\lambda) + 2\pi\boldsymbol{i}\sum_{k\geq 1}\int_{|x|=|y|=\epsilon} \Psi^k_{\underline{\lambda},\ell}\frac{dx}{x}\wedge\frac{dy}{y}
\end{aligned}$$

[12] E.g., if $c := |\Delta \cap \mathbb{Z}^2|$, then $|\lambda_\ell| > c\epsilon^{-2}$ and $|\lambda_{j\neq\ell}| < 1/(c\epsilon^3)$ will do.

modulo $\mathbb{Z}(2)$. Here only the first term of R_3 enters since $T_{\lambda(1-\Psi)} \cap |x| = |y| = \epsilon$ is empty under the given assumptions. □

Returning to the more specific scenario at the beginning of this section, if $g = 1$ and $\lambda_1 =: \lambda =: \frac{1}{t}$, then $\Phi_\lambda = \phi(x, y) - \lambda$ and $\Psi_{\lambda,1} = t\phi(x, y)$, so that (writing $\mathscr{R}_t$ instead of $\mathscr{R}_\lambda$), Theorem 6.2 yields:

Corollary 6.3. *If Δ is reflexive, then the α_m are all homologous* $(=: \alpha)$, *and*

$$\mathscr{R}_t(\alpha) \underset{\mathbb{Z}(2)}{\equiv} 2\pi \boldsymbol{i} \left(\log(t) + \sum_{k\geq 1} \frac{[\phi^k]_0}{k} \right)$$

for t small in the right-half-plane.

It remains to compute $\mathscr{R}_t(\beta)$ for a cycle β complementary to α (so that $\mathbb{Z}\langle\alpha, \beta\rangle = H_1(E_t, \mathbb{Z})$), which we shall do for the local $\mathbb{P}^2$ setting only: Δ the convex hull of $\{(1, 0), (0, 1), (-1, -1)\}$, and $\phi = x + y + x^{-1}y^{-1}$. (So $\mathbb{P}_{\Delta^\circ} \cong \mathbb{P}^2$, while the family of elliptic curves E_t lives in $\mathbb{P}_\Delta$.) Taking $t > 0$ small, write $(0 <)\, x_0(t) < x_-(t) < x_+(t) < \infty$ for the branch points of

$$E_{t\,(=\lambda^{-1})} : y^2 + (x - \lambda)y + x^{-1} = 0$$

over $\mathbb{P}^1_x$, and $y^\pm(x) = \frac{1}{2}\left\{(\lambda - x) \pm \sqrt{(x-\lambda)^2 - 4x^{-1}}\right\}$. Then β (resp. α) is given by the difference of paths (on the two branches) between $x_0(t)$ and $x_-(t)$ (resp. $x_-(t)$ and $x_+(t)$).

Now $T_{-x} = \mathbb{R}_{>0} \subset \mathbb{P}^1_x$, so taking the y^+-branch of β to run from x_0 to x_- in $\mathfrak{H}$, and the y^--branch of β to run from x_- to x_0 in $-\mathfrak{H}$, we have $\beta \cap T_{-x} = (x_0, y_0) \cup (x_-, y_-)$; moreover, $\log(-x) = \log(x) \mp \boldsymbol{i}\pi$ on the $y^\pm$-branch of β. The upshot is that

$$\begin{aligned}\int_\beta R\{-x, -y\}|_{E_t} &= \int_\beta \log(-x)\frac{dy}{y} - 2\pi\boldsymbol{i} \sum_{\beta\cap T_{-x}} \log(y) \\ &= -\int_{x_0(t)}^{x_-(t)} \log(x) \operatorname{dlog}\left(\frac{y^+(x)}{y^-(x)}\right) = \int_{x_0(t)}^{x_-(t)} \log\left(\frac{y^+(x)}{y^-(x)}\right)\frac{dx}{x} \\ &= \int_{x_0(t)}^{x_-(t)} \log\left(\frac{1+\sqrt{1-\xi}}{1-\sqrt{1-\xi}}\right)\frac{dx}{x},\end{aligned}$$

where $\xi = \dfrac{4t^2}{x(1-xt)^2}$. Writing for $\xi \in (0, 1)$

$$\log\left(\frac{1+\sqrt{1-\xi}}{1-\sqrt{1-\xi}}\right) + \log\left(\frac{\xi}{4}\right) =: -\sum_{m\geq 1} \alpha_m \xi^m,$$

the above integral decomposes into

$$-2\log(t)\int_{x_0}^{x_-} \frac{dx}{x} + \int_{x_0}^{x_-} \log(x)\frac{dx}{x} + 2\int_{x_0}^{x_-} \log(1-xt)\frac{dx}{x} - \sum_{m\geq 1} \alpha_m \int_{x_0}^{x_-} \xi^m \frac{dx}{x}.$$

Using the approximations $x_0 \simeq 4t^2(1+8t^3)$ and $x_- \simeq t^{-1}(1-2t^{3/2}-2t^3)$, a lengthy direct computation gives that

$$\mathscr{R}_t(\beta) = \tfrac{9}{2}\log^2(t) - \frac{\pi^2}{2} + \mathcal{O}(t\log(t)).$$

Let $\delta_t := t\frac{d}{dt}$. By the general result [Doran and Kerr 2011, Corollary 4.1], one knows that $\nabla_{\delta_t}\mathscr{R}_t = [\omega_t]$, where

$$\omega_t := \mathrm{Res}_{E_t}\left(\frac{\frac{dx}{x}\wedge\frac{dy}{y}}{1-t\phi(x,y)}\right)$$

has its periods $\omega_t(\gamma) := \int_\gamma \omega_t$ annihilated by the Picard–Fuchs operator

$$\mathcal{L} = \delta_t^2 - 27t^3(\delta_t+1)(\delta_t+2).$$

The regulator periods $\mathscr{R}_t(\gamma)$ are therefore killed by $\mathcal{L}\circ\delta_t$. Since $\mathcal{L}(\cdot)=0$ is known to have the basis of solutions

$$\pi_1 = \sum_{n\geq 0} a_n t^{3n},$$
$$\pi_2 = 3\log(t)\pi_1 + \sum_{n\geq 1} a_n b_n t^{3n},$$

with

$$a_n = \frac{(3n)!}{(n!)^3} \quad\text{and}\quad b_n = \sum_{k=0}^{n-1}\left(\frac{3}{3k+1}+\frac{3}{3k+2}-\frac{2}{k+1}\right),$$

it now follows that (writing $B_n = b_n - 1/n$)

$$\mathscr{R}_t(\alpha) \underset{\mathbb{Z}(2)}{\equiv} 2\pi\boldsymbol{i}\left(\log(t) + \sum_{n\geq 1}\frac{a_n}{3n}t^{3n}\right),$$
$$\mathscr{R}_t(\beta) \underset{\mathbb{Z}(2)}{\equiv} \tfrac{9}{2}\log^2(t) + 3\log(t)\sum_{n\geq 1}\frac{a_n}{n}t^{3n} + \sum_{n\geq 1}\frac{a_n B_n}{n}t^{3n} - \frac{\pi^2}{2},$$
$$\omega_t(\alpha) = 2\pi\boldsymbol{i}\sum_{n\geq 0} a_n t^{3n},$$
$$\omega_t(\beta) = 9\log(t)\sum_{n\geq 0} a_n t^{3n} + 3\sum_{n\geq 1} a_n b_n t^{3n},$$

for $0 < |t| < \frac{1}{3}$. For (6-1), this yields

Proposition 6.4. *The truncated normal function mirror to local* $\mathbb{P}^2$ *is*

$$\nu(t) = \left\langle \frac{\omega_t}{2\pi\boldsymbol{i}}, \mathscr{R}_t \right\rangle = \frac{1}{2\pi\boldsymbol{i}}\big(\mathscr{R}_t(\alpha)\omega_t(\beta) - \mathscr{R}_t(\beta)\omega_t(\alpha)\big)$$
$$\equiv \tfrac{9}{2}\log^2(t)(1+6t^3) + 3\log(t)(9t^3) + \frac{\pi^2}{2} + (3\pi^2-9)t^3 + \mathcal{O}(t^6\log^2 t),$$

modulo the period lattice $\mathbb{Z}(2)\langle\omega_t(\alpha), \omega_t(\beta)\rangle$.

Remark 6.5. This is closely related to computations in [Hosono 2006; Mohri et al. 2001]; the main difference — and the salient result here — is the identification of $\pi^2/2$ as the correct torsion offset for our motivically defined ν.

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