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PLANAR HAMILTONIAN KOLMOGOROV CUBIC  
POLYNOMIAL DIFFERENTIAL SYSTEMS**

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## ON THE CONFIGURATIONS OF CENTERS OF PLANAR HAMILTONIAN KOLMOGOROV CUBIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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**We study the kind of centers that Hamiltonian Kolmogorov cubic polynomial differential systems can exhibit. Moreover, we analyze the possible configurations of these centers with respect to the invariant coordinate axes, and obtain that the real algebraic curve  $xy(a + bx + cy + dx^2 + exy + fy^2) = h$  has at most four families of level ovals in  $\mathbb{R}^2$  for all real parameters  $a, b, c, d, e, f$  and  $h$ .**

### 1. Introduction and statement of the main results

An equilibrium  $p$  of a differential system in  $\mathbb{R}^2$  is a *center* if there is a neighborhood  $U$  of  $p$  such that  $p$  is the unique equilibrium in  $U$  and  $U \setminus \{p\}$  is filled by periodic orbits surrounding  $p$ . The equilibrium  $p$  is a *focus* if there is a neighborhood  $U$  of  $p$  such that all the orbits in  $U \setminus \{p\}$  spiral tending to  $p$  either in forward, or in backward time. These notions of center and focus go back to Poincaré [1881] and Dulac [1908].

In the qualitative theory of planar polynomial differential systems, the problem of distinguishing between a focus or a center (known simply as the *center-focus problem*), and the problem of knowing the possible configurations of centers are two very important topics, which are related to the Hilbert's 16th problem; see [Hilbert 1900; Ilyashenko 2002; Li 2003].

For the quadratic polynomial differential systems, the center-focus problem and the possible configurations of their centers were solved by Bautin [1952], Kapteyn [1911; 1912], Schlomiuk [1993], Vulpe [1983] and Żołądek [1994c]. However, these two problems are unsolved for cubic polynomial differential systems. There are many works on the centers for some different subclasses of cubic differential systems. For example, the centers of the cubic polynomial differential systems without quadratic terms have been determined by Malkin [1964], Vulpe and Sibirskii [1988],

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and Żołądek [1994b]. The classification of reversible cubic polynomial differential systems with a center has been done by Żołądek [1994a], and Buzzi et al. [2009]. The Hamiltonian linear type centers and the nilpotent ones of cubic polynomial differential systems without quadratic terms have been classified by Colak et al. [2014a; 2014b; 2015; 2017].

In this paper we study the centers and their configurations for the Hamiltonian cubic polynomial differential systems having two invariant straight lines which intersect. Since with an affine transformation these two invariant straight lines go through the axes of coordinates, these systems become Kolmogorov systems [Kolmogorov 1936; Llibre and Xiao 2017] which allows us to study dynamics of species in competition [Hofbauer and Sigmund 1988; Smale 1976]. So it is a very interesting topic to understand their global dynamics from the view of mathematician and biologist.

Hamiltonian systems come originally from physics, but also play an important role in dynamical system theory [Arnold 1978]. Hamiltonian systems in the plane are the easiest differential systems having a first integral. The Kolmogorov polynomial differential systems [1936] in the plane generalize the Lotka–Volterra polynomial differential systems [Lotka 1925; Volterra 1926] of degree two to higher degree. In this paper we focus our attention on subclasses of cubic polynomial differential systems which are Hamiltonian and Kolmogorov systems, and we shall read them as a Hamiltonian–Kolmogorov system.

Let  $(x, y)$  be the coordinates of  $\mathbb{R}^2$ . We consider Kolmogorov systems of the form

$$(1) \quad \begin{aligned} \dot{x} &= x P_1(x, y), \\ \dot{y} &= y P_2(x, y), \end{aligned}$$

in  $\mathbb{R}^2$ , where the dot throughout this paper means the derivative with respect to the independent variable  $t$ , and the  $P_i(x, y)$  for  $i = 1, 2$  are polynomials in the variables  $x$  and  $y$  such that the maximum of their degrees is two. So system (1) is a cubic polynomial differential system. Next, we characterize the Kolmogorov systems (1) which are Hamiltonian.

**Theorem 1.** *Assume that the maximum degree of the polynomials  $P_i(x, y)$  is two for  $i = 1, 2$ . Then system (1) is a Hamiltonian system if and only if its Hamiltonian is of the form*

$$(2) \quad H(x, y) = xy(a + bx + cy + dx^2 + exy + fy^2),$$

where  $a, b, c, d, e$  and  $f$  are real parameters.

Theorem 1 is proved in Section 2. Therefore the Hamiltonian–Kolmogorov polynomial systems of degree three are

$$(3) \quad \begin{aligned} \dot{x} &= -x(a + bx + 2cy + dx^2 + 2exy + 3fy^2), \\ \dot{y} &= y(a + 2bx + cy + 3dx^2 + 2exy + fy^2), \end{aligned}$$

where  $a, b, c, d, e$  and  $f$  are real parameters. For brevity we denote these systems as *HK-systems* in what follows.

A polynomial differential system in the plane having a center at the origin of coordinates, after a linear change of variables and a rescaling of the time variable (if necessary), can be written in one of the following three forms:

$$(4) \quad \begin{aligned} \dot{x} &= -y + X_2(x, y), \\ \dot{y} &= x + Y_2(x, y), \end{aligned}$$

called a *linear type center*;

$$(5) \quad \begin{aligned} \dot{x} &= y + X_2(x, y), \\ \dot{y} &= Y_2(x, y), \end{aligned}$$

called a *nilpotent center*;

$$(6) \quad \begin{aligned} \dot{x} &= X_2(x, y), \\ \dot{y} &= Y_2(x, y), \end{aligned}$$

called a *degenerate center*, where  $X_2(x, y)$  and  $Y_2(x, y)$  are polynomials starting at least with terms of second order. For additional information about these three classes of centers, see, for instance, [Llibre 2016].

The objective of this paper is to study the centers and their configurations of the HK-systems (3). In other words, we study the existence and configurations of level ovals for planar real algebraic curves  $H(x, y) = h$ ,  $h \in \mathbb{R}$ .

Our main result is the following.

**Theorem 2.** *Consider the HK-systems (3) defined by the Hamiltonian (2). Then a system (3) has at most four centers and the following statements hold.*

(i) *If a HK-system (3) has nonisolated equilibria, i.e., there is an infinite number of equilibria, then this system has at most one center, which is of linear type. Hence there exist values of  $a, b, c, d, e$  and  $f$  for which  $H(x, y) = h$  has no level ovals for all real  $h$ , and there exist other values of  $a, b, c, d, e$  and  $f$  for which  $H(x, y) = h$  has only one family of level ovals for all real  $h$ .*

(ii) *A HK-system (3) has neither degenerate centers nor two nilpotent centers for any real  $a, b, c, d, e$  and  $f$ .*

(iii) *If a HK-system (3) has a unique center, then this center can be of linear type or nilpotent. That is, there is only a family of level ovals of  $H(x, y) = h$  for all real  $h$ ; see Figure 1.*

(iv) *If a HK-system (3) has two centers, then these two centers are either two linear type centers, or one linear type center and one nilpotent center. And there are HK-systems with two centers realizing all possible configurations with respect to the*

coordinate axes; i.e., two centers in the same quadrant (see Figure 2), two centers in different opposite quadrants (see Figure 3), and two centers in different consecutive quadrants (see Figure 4). Hence, the level ovals of  $H(x, y) = h$  have two families for all real  $h$ .

(v) If a HK-system (3) has three centers, then they are of linear type and there are HK-systems with three centers realizing all possible configurations with respect to the coordinate axes; i.e., three centers in the same quadrant (see Figure 5), two centers in one quadrant and the other in a consecutive quadrant (see Figure 6), two centers in one quadrant and the other in the opposite quadrant (see Figure 7), and three centers in three different quadrants (see Figure 8). Moreover in this case, the level ovals of  $H(x, y) = h$  have three families for all real  $h$ .

(vi) If a HK-system (3) has four centers, then they are of linear type. The level ovals of the corresponding Hamiltonian function,  $H(x, y) = h$ , have exactly four families for all real numbers  $h$ . Moreover, there are HK-systems with four centers in the four different quadrants (see Figure 9), and with four centers with two in the same quadrant, and each one of the other two centers is located in a different adjacent quadrant (see Figure 10).

Theorem 2 is proved in Section 3. As a direct application of Theorem 2, we have:

**Corollary 3.** *The real algebraic curve*

$$xy(a + bx + cy + dx^2 + exy + fy^2) = h$$

*has at most four families of level ovals in  $\mathbb{R}^2$  for all real parameters  $a, b, c, d, e, f$  and  $h$ .*

## 2. Preliminaries

In this section we first give the proof of Theorem 1, then discuss the equilibria of the systems (3) and recall some definitions of equilibria and related results.

*Proof of Theorem 1.* Consider an arbitrary polynomial in the variables  $x$  and  $y$  of degree four,

$$H = \sum_{i+j=1}^4 a_{ij}x^i y^j.$$

We let  $H$  define a Hamiltonian system of the form (1). Then,

$$(7) \quad \dot{x} = -H_y, \quad \dot{y} = H_x.$$

In order for  $H_y$  to appear  $x$  as a common factor we take  $a_{01} = a_{02} = a_{03} = a_{04} = 0$ , and  $y$  appears as a common factor in  $H_x$  if and only if  $a_{10} = a_{20} = a_{30} = a_{40} = 0$ .

Therefore the Hamiltonian system (7) becomes a Kolmogorov system (1) if and only if

$$H = xy(a_{11} + a_{21}x + a_{12}y + a_{31}x^2 + a_{22}xy + a_{13}y^2).$$

Renaming the coefficients of  $H$ , the theorem follows.  $\square$

We now discuss the equilibria of a system (3) in  $\mathbb{R}^2$ . It is clear that the number of equilibria of system (3) is finite in  $\mathbb{R}^2$  if and only if this polynomial vector field

$$(p_1(x, y), p_2(x, y)) = (-x(a + bx + 2cy + dx^2 + 2exy + 3fy^2), \\ y(a + 2bx + cy + 3dx^2 + 2exy + fy^2))$$

does not have a common nonconstant component by Bézout's theorem (see for instance [Fulton 1969]). Hence, if the two polynomials  $p_1(x, y)$  and  $p_2(x, y)$  have a nonconstant common component (or *common divisor*), then system (3) has nonisolated equilibria in  $\mathbb{R}^2$ . The following lemma gives the necessary and sufficient conditions for the existence of a nonconstant common divisor of  $p_1(x, y)$  and  $p_2(x, y)$ .

**Lemma 4.** *Polynomials  $p_1(x, y)$  and  $p_2(x, y)$  have a nonconstant common divisor if and only if one of the conditions in Table 1 holds, where conditions C9 and C12 are equivalent.*

*Proof.* It is clear that polynomials  $p_1(x, y)$  and  $p_2(x, y)$  do not have a common divisor which is a polynomial of  $(x, y)$  with degree three. Hence, we only look for the common divisor which is a polynomial of  $(x, y)$  with degree two or one.

By straightforward calculation, polynomials  $p_1(x, y)$  and  $p_2(x, y)$  have a common divisor with degree two if and only if the common divisor has one of the four forms  $x^2$ ,  $y^2$ ,  $xy$  and  $a + 2exy$  for any real parameters  $a, b, c, d, e$  and  $f$ . Hence, the conditions C1–C4 for the corresponding common divisor with degree two can be given in the first four rows in Table 1.

We now look for the common divisor of polynomials  $p_1(x, y)$  and  $p_2(x, y)$  with degree only one. Note that

$$p_1(x, y) = -x(a + bx + 2cy + dx^2 + 2exy + 3fy^2), \\ p_2(x, y) = y(a + 2bx + cy + 3dx^2 + 2exy + fy^2).$$

Suppose that  $y - s - tx$  is a common divisor of  $a + bx + 2cy + dx^2 + 2exy + 3fy^2$  and  $a + 2bx + cy + 3dx^2 + 2exy + fy^2$ , where  $s$  and  $t$  are undetermined real parameters. Then for any real  $x$ ,

$$(8) \quad \begin{aligned} a + bx + 2c(s + tx) + dx^2 + 2ex(s + tx) + 3f(s + tx)^2 &\equiv 0, \\ a + 2bx + c(s + tx) + 3dx^2 + 2ex(s + tx) + f(s + tx)^2 &\equiv 0. \end{aligned}$$

<i>list</i>	<i>conditions</i>	<i>nonconstant common divisors</i>
C1	$a = b = c = e = f = 0, d \neq 0$	$x^2$
C2	$a = b = c = d = e = 0, f \neq 0$	$y^2$
C3	$a = b = c = d = f = 0, e \neq 0$	$xy$
C4	$b = c = d = f = 0, ae \neq 0$	$a + 2exy$
C5	$a = c = f = 0, b \neq 0$ (or $de \neq 0$ )	$x$
C6	$a = b = d = 0, c \neq 0$ (or $ef \neq 0$ )	$y$
C7	$c = e = f = 0, bd \neq 0, a = \frac{b^2}{4d}$	$x + \frac{b}{2d}$
C8	$a = b = c = 0, ef \neq 0, d = \frac{e^2}{4f}$	$x + \frac{2f}{e}y$
C9	$bef \neq 0, a = \frac{b^2f}{e^2}, c = \frac{2bf}{e}, d = \frac{e^2}{4f}$	$x + \frac{2f}{e}y + \frac{2bf}{e^2}$
C10	$b = d = e = 0, ac \neq 0, f = \frac{c^2}{4a}$	$y + \frac{2a}{c}$
C11	$a = b = c = 0, de \neq 0, f = \frac{e^2}{4d}$	$y + \frac{2d}{e}x$
C12	$ace \neq 0, b = \frac{2ae}{c}, d = \frac{ae^2}{c^2}, f = \frac{c^2}{4a}$	$y + \frac{2ae}{c^2}x + \frac{2a}{c}$

**Table 1.** The necessary and sufficient conditions for nonconstant common divisor.

Thus,

$$\begin{aligned}
 (9) \quad & a + 2cs + 3fs^2 = 0, & a + cs + fs^2 &= 0, \\
 & b + 2ct + 2es + 6fst = 0, & 2b + ct + 2es + 2fst &= 0, \\
 & d + 2et + 3ft^2 = 0, & 3d + 2et + ft^2 &= 0.
 \end{aligned}$$

We divide four cases of  $(s, t)$  to study the solutions of system (9).

**Case 1:**  $s = t = 0$ . From (9),  $a = b = d = 0$ . Hence,  $a + bx + 2cy + dx^2 + 2exy + 3fy^2$  and  $a + 2bx + cy + 3dx^2 + 2exy + fy^2$  have a common divisor  $y$  if  $a = b = d = 0$  and  $c^2 + e^2 + f^2 \neq 0$ . However, if we require this common divisor to be the only polynomial with degree one, then  $a = b = d = 0$  and  $c \neq 0$ , or  $a = b = d = 0$  and  $ef \neq 0$ , which is the condition C6 in Table 1.

**Case 2:**  $s \neq 0$  and  $t = 0$ . The system (9) becomes

$$\begin{aligned}
 & a + 2cs + 3fs^2 = 0, \\
 & a + cs + fs^2 = 0, \\
 & b = d = e = 0.
 \end{aligned}$$

Therefore, when  $b = d = e = 0, ac \neq 0$  and  $f = c^2/(4a)$ , we have that  $y + 2a/c$  is the common divisor of

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 \quad \text{and} \quad a + 2bx + cy + 3dx^2 + 2exy + fy^2.$$

In fact, now,

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 = \frac{c^2}{4a} \left( y + \frac{2a}{c} \right) \left( 3y + \frac{2a}{c} \right)$$

and

$$a + 2bx + cy + 3dx^2 + 2exy + fy^2 = \frac{c^2}{4a} \left( y + \frac{2a}{c} \right)^2.$$

**Case 3:**  $s = 0$  and  $t \neq 0$ . The system (9) becomes

$$\begin{aligned} a &= b = c = 0, \\ d + 2et + 3ft^2 &= 0, \\ 3d + 2et + ft^2 &= 0. \end{aligned}$$

Therefore, when  $a = b = c = 0$ ,  $de \neq 0$  and  $f = e^2/(4d)$ , we have that  $y + 2d/ex$  is the common divisor of

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 \quad \text{and} \quad a + 2bx + cy + 3dx^2 + 2exy + fy^2.$$

In fact, now we have

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 = \frac{3e^2}{4d} \left( y + \frac{2d}{e}x \right) \left( y + \frac{2d}{3e}x \right)$$

and

$$a + 2bx + cy + 3dx^2 + 2exy + fy^2 = \frac{3e^2}{4d} \left( y + \frac{2d}{e}x \right) \left( y + \frac{6d}{e}x \right).$$

**Case 4:**  $s \neq 0$  and  $t \neq 0$ . From the first two equations in (9) it follows that

$$(10) \quad s = -\frac{2a}{c} \quad \text{with } ac \neq 0.$$

From the last two equations in (9) it follows that

$$(11) \quad t = -\frac{2d}{e} \quad \text{with } de \neq 0.$$

Plugging (10) and (11) into (9), we obtain

$$b = \frac{2ae}{c}, \quad d = \frac{ae^2}{c^2}, \quad f = \frac{c^2}{4a}.$$

Hence when  $ace \neq 0$ ,  $b = 2ae/c$ ,  $d = ae^2/c^2$  and  $f = c^2/(4a)$ , we have that  $y + 2ae/c^2x + 2a/c$  is a common divisor of  $a + bx + 2cy + dx^2 + 2exy + 3fy^2$  and  $a + 2bx + cy + 3dx^2 + 2exy + fy^2$ .

In fact, now we have

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 = \frac{c^2}{4a} \left( y + \frac{2ae}{c^2}x + \frac{2a}{c} \right) \left( 3y + \frac{2ae}{c^2}x + \frac{2a}{c} \right),$$

and

$$a + 2bx + cy + 3dx^2 + 2exy + fy^2 = \frac{c^2}{4a} \left( y + \frac{2ae}{c^2}x + \frac{2a}{c} \right) \left( y + \frac{6ae}{c^2}x + \frac{2a}{c} \right).$$

Summarizing the above analysis, we obtain the necessary and sufficient conditions that  $p_1(x, y)$  and  $p_2(x, y)$  have common divisors  $y$ ,  $y - s$ ,  $y - tx$  and  $y - s - tx$ , respectively. This leads to the conditions C6 and C10–C12 in Table 1.

Using similar arguments, one can discuss the common divisor of

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 \quad \text{and} \quad a + 2bx + cy + 3dx^2 + 2exy + fy^2$$

with form  $x - s - ty$ . Hence, we obtain the conditions shown in the condition C5 and C7–C9 in Table 1.  $\square$

**Remark 5.** The eleven conditions from C1 to C11 in Table 1 give the necessary and sufficient conditions so that the polynomial  $H(x, y)$  can be factorized in real number field. This gives all possible real branch curves of the real algebraic curve  $H(x, y) = 0$ .

From Bézout's theorem and a straightforward calculation, we have:

**Theorem 6.** *The following statements hold on the equilibria of HK-system (3).*

(I) *If polynomials  $p_1(x, y)$  and  $p_2(x, y)$  have a nonconstant common divisor  $g(x, y)$ , then a HK-system (3) has nonisolated equilibria, and each point on the curve  $g(x, y) = 0$  is an equilibrium of the HK-system (3). Hence a HK-system (3) has infinitely many equilibria.*

(II) *If polynomials  $p_1(x, y)$  and  $p_2(x, y)$  do not have nonconstant common divisors, then a HK-system (3) has at most nine finite equilibria with five of them located at the coordinate axes and the other four are in the interior of the quadrants in  $\mathbb{R}^2$ .*

In the two-dimensional phase plane  $\mathbb{R}^2$ , we say that an equilibrium is *nonelementary* if both of the eigenvalues of the linear part of the vector field at that point are zero, and *elementary* otherwise. A nonelementary equilibrium is called *degenerate* if the linear part is identically zero, otherwise it is called *nilpotent*. An elementary equilibrium is *hyperbolic* if both of the eigenvalues of the linear part at the equilibrium have nonzero real part, and it is called *semihyperbolic* if one and only one of the two eigenvalues is zero. The unique elementary equilibrium which is neither hyperbolic nor semihyperbolic is the one with purely imaginary eigenvalues, which only can be the focus or center; for more details, see [Dumortier et al. 2006; Zhang et al. 1992].

The local phase portraits at the nilpotent equilibrium points were classified by Andreev [1958]; see also Theorem 3.5 of [Dumortier et al. 2006]. We summarize the conditions which characterize whether a nilpotent equilibrium is either a focus or a center as follows.

**Lemma 7.** *Let  $(0, 0)$  be an isolated equilibrium point of the analytic differential system*

$$(12) \quad \dot{x} = y + A(x, y), \quad \dot{y} = B(x, y),$$

*defined in a neighborhood of the point  $(0, 0)$ , and assume that  $A(x, y)$  and  $B(x, y)$  start at least with terms of degree two in  $x$  and  $y$ . Let  $y = f(x)$  be the solution of the equation  $y + A(x, y) = 0$  in a neighborhood of the point  $(0, 0)$ , and consider  $F(x) = B(x, f(x))$  and  $G(x) = (\partial A/\partial x + \partial B/\partial y)(x, f(x))$ . Then the origin of system (12) is a center or a focus if and only if one of the following conditions holds.*

- (i)  $G(x) \equiv 0$ ,  $F(x) = ax^m + o(x^m)$ ,  $m > 1$  is an odd integer and  $a < 0$ .
- (ii)  $G(x) = bx^n + o(x^n)$ ,  $F(x) = ax^m + o(x^m)$ ,  $m > 1$  is an odd integer,  $a < 0$ , and either  $m < 2n + 1$  or both  $m = 2n + 1$  and  $b^2 + 4a(n + 1) < 0$ .

For convenience, prior to applying Lemma 7, we recall some characteristics of a nilpotent equilibrium. As it is known that a planar analytic differential system

$$(13) \quad \dot{u} = f(u, v), \quad \dot{v} = g(u, v),$$

has a nilpotent equilibrium at  $(u_0, v_0)$ , the Jacobian matrix  $M$  of system (13) at  $(u_0, v_0)$  must be

$$M = \begin{pmatrix} \frac{\partial f(u_0, v_0)}{\partial u} & \frac{\partial f(u_0, v_0)}{\partial v} \\ \frac{\partial g(u_0, v_0)}{\partial u} & \frac{\partial g(u_0, v_0)}{\partial v} \end{pmatrix} \triangleq \begin{pmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{pmatrix},$$

where  $m_1^2 + m_2m_3 = 0$  and  $m_1^2 + m_2^2 + m_3^2 \neq 0$ .

By an affine transformation

$$(14) \quad \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix},$$

system (13) can be transformed into system (12), where the matrix  $P$  depends on  $m_1$ ,  $m_2$  and  $m_3$ , and has one of the following three forms:

- (i) If  $m_1 \neq 0$ , then  $m_2 \neq 0$  and  $m_3 \neq 0$ . Hence,

$$P = \begin{pmatrix} 0 & -\frac{m_2}{m_1^2} \\ 1 & \frac{m_2}{m_1} \end{pmatrix}.$$

(ii) If  $m_1 = 0$  and  $m_2 \neq 0$ , then  $m_3 = 0$ . Hence,

$$P = \begin{pmatrix} \frac{1}{m_2} & 0 \\ 0 & 1 \end{pmatrix}.$$

(iii) If  $m_1 = 0$  and  $m_3 \neq 0$ , then  $m_2 = 0$ . Hence,

$$P = \begin{pmatrix} 0 & -\frac{1}{m_3} \\ -1 & 0 \end{pmatrix}.$$

**Theorem 8.** *Suppose that system (13) is a Hamiltonian system with a nilpotent equilibrium at  $(u_0, v_0)$ , then this nilpotent equilibrium is a center if and only if the case (i) in Lemma 7 holds, that is,  $G(x) \equiv 0$  and  $F(x) = ax^m + o(x^m)$ , where  $m > 1$  is an odd integer and  $a < 0$ .*

*Proof.* Since system (13) is a Hamiltonian system, there exists an analytic function  $H(u, v)$  such that

$$\frac{\partial H(u, v)}{\partial v} \triangleq H_v(u, v) = f(u, v), \quad \frac{\partial H(u, v)}{\partial u} \triangleq H_u(u, v) = -g(u, v).$$

Thus, system (13) can be written as

$$(15) \quad \dot{u} = H_v(u, v), \quad \dot{v} = -H_u(u, v)$$

with  $H_v(u_0, v_0) = -H_u(u_0, v_0) = 0$ .

Using the affine transformation (14) and its inverse transformation

$$(16) \quad \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

system (15) can be transformed to

$$(17) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = P \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P^T \begin{pmatrix} \tilde{H}_x(x, y) \\ \tilde{H}_y(x, y) \end{pmatrix},$$

where  $\tilde{H}(x, y) = H(u(x, y), v(x, y))$  and  $P^T$  is the transpose of matrix  $P$ .

Let

$$Q = P \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P^T.$$

Depending on the Jacobian matrix  $M$  of system (13) at nilpotent equilibrium  $(u_0, v_0)$ , by direct computation we obtain three forms of  $Q$  as follows.

(i) If  $m_1 \neq 0$ ,

$$Q = \begin{pmatrix} 0 & \frac{m_2}{m_1^2} \\ -\frac{m_2}{m_1^2} & 0 \end{pmatrix}.$$

(ii) If  $m_1 = 0$  and  $m_2 \neq 0$ ,

$$Q = \begin{pmatrix} 0 & \frac{1}{m_2} \\ -\frac{1}{m_2} & 0 \end{pmatrix}.$$

(iii) If  $m_1 = 0$  and  $m_3 \neq 0$ ,

$$Q = \begin{pmatrix} 0 & -\frac{1}{m_3} \\ \frac{1}{m_3} & 0 \end{pmatrix}.$$

This gives that system (17) is a Hamiltonian system even though  $P$  is not a symplectic matrix if nonzero  $m_i \neq 1$  for some  $i \in \{1, 2, 3\}$ . Following the notation in Lemma 7, we have

$$G(x) = c \left( \frac{\partial^2 \tilde{H}(x, y)}{\partial x \partial y} - \frac{\partial^2 \tilde{H}(x, y)}{\partial y \partial x} \right) \equiv 0,$$

where  $c$  is one of  $m_2/m_1^2$ ,  $1/m_2$  and  $-1/m_3$ .

Therefore, we obtain the conclusion by Lemma 7. □

### 3. Proof of Theorem 2

We shall prove Theorem 2 statement by statement in this section.

*Proof of statement (i) of Theorem 2.* Since the  $x$ -axis and  $y$ -axis are invariant by the flow of a HK-system (3), either the equilibrium on the coordinate axes or the nonisolated equilibrium cannot become a center by the definition of center. If a HK-system (3) does not have a center, then the algebraic curve  $H(x, y) = h$  does not have level ovals in  $\mathbb{R}^2$ .

From Theorem 6, a HK-system (3) has at most four isolated equilibria outside the coordinate axes. Hence, a HK-system (3) has at most four centers in the interior of quadrants of  $\mathbb{R}^2$ .

We now prove statement (i) of Theorem 2: *if a HK-system (3) has nonisolated equilibria, then this system has at most one center, which is of linear type.*

From Lemma 4, we know that a HK-system (3) has nonisolated equilibria if and only if one of the conditions listed in Table 1 holds. By straightforward calculation, a HK-system (3) has no isolated equilibrium outside the coordinate axes for conditions C1–C4, C7, C8, C10 and C11 in Table 1. Hence, under these conditions a HK-system (3) has no center equilibrium, which implies that  $H(x, y) = h$  has no level ovals for all real  $h$ .

When condition C5 holds (i.e.,  $a = c = f = 0$ ,  $b \neq 0$  (or  $de \neq 0$ )), a HK-system (3) has an isolated equilibrium outside the coordinate axes if and only if  $bde \neq 0$ . In the case  $a = c = f = 0$ ,  $bde \neq 0$ , a HK-system (3) has a unique isolated equilibrium at  $(-b/(2d), -b/(4e))$  with eigenvalues  $\pm b^2/(2\sqrt{2}|d|)i$ , which is a linear type center.

<i>conditions</i>	<i>center</i>	<i>level ovals of <math>H(x, y) = h</math></i>
C1	none	$dx^3y = h$ , none
C2	none	$fx^3y = h$ , none
C3	none	$ex^2y^2 = h$ , none
C4	none	$xy(a+exy) = h$ , none
C5	$b = 0, de \neq 0$	$x^2y(dx+ey) = h$ , none
	$bde \neq 0$	$(-\frac{b}{2d}, -\frac{b}{4e})$ $x^2y(b+dx+ey) = h$ , a family
C6	$c = 0, ef \neq 0$	$xy^2(ex+fy) = h$ , none
	$cef \neq 0$	$(-\frac{c}{4e}, -\frac{c}{2f})$ $xy^2(c+ex+fy) = h$ , a family
C7	none	$xy(b+2dx)^2/(4d) = h$ , none
C8	none	$xy(ex+2fy)^2/(4f) = h$ , none
C9	$(-\frac{bf}{2e^2}, -\frac{b}{4e})$	$xy(2bf+e^2x+2efy)^2/(4e^2f) = h$ , a family
C10	none	$xy(2a+cy)^2/(4a) = h$ , none
C11	none	$xy(2dx+ey)^2/(4d) = h$ , none

**Table 2.** Center of a HK-system (3) with nonisolated equilibria.

So there is only one family of level ovals for  $H(x, y) = x^2y(b+dx+ey) = h, h \in \mathbb{R}$ .

Similarly, we can discuss the cases C6 and C9 (or C12). For clarity, we list them in Table 2. Thus, statement (i) is proved. □

It is clear that a HK-system (3) has no center if it does not have any isolated equilibria outside the coordinate axes. In the proof of the remaining statements of Theorem 2, we always assume that a HK-system (3) has an isolated equilibrium outside the coordinate axes. Without loss of generality, it is in the positive quadrant, and consequently we can assume that it is  $(x, y) = (\alpha, \beta)$  with  $\alpha$  and  $\beta$  positive real numbers. Then, scaling the variables  $x$  and  $y$  if necessary, we can suppose that  $\alpha = \beta = 1$ . In short, in the rest of the proofs of the statements of Theorem 2 we always assume that the point  $(1, 1)$  will be an equilibrium of a HK-system (3). Hence, two parameters of  $a, b, c, d, e$  and  $f$  can be determined by the other four parameters, e.g.,

$$e = -\frac{1}{4}(2a + 5b + c + 8d), \quad f = \frac{1}{2}(b - c + 2d).$$

Therefore a HK-system (3) with equilibrium  $(1, 1)$  becomes

$$(18) \quad \begin{aligned} \dot{x} &= -x(a+bx+dx^2+2cy-axy-\frac{5}{2}bxy-\frac{1}{2}cxy-4dxy+\frac{3}{2}by^2-\frac{3}{2}cy^2+3dy^2), \\ \dot{y} &= y(a+2bx+3dx^2+cy-axy-\frac{5}{2}bxy-\frac{1}{2}cxy-4dxy+by^2/2-cy^2/2+dy^2). \end{aligned}$$

The Jacobi matrix of a HK-system (18) at  $(1, 1)$  is

$$M = \begin{pmatrix} (2a + 3b + c + 4d)/2 & (2a - b + 3c - 4d)/2 \\ -(2a + b + c - 4d)/2 & -(2a + 3b + c + 4d)/2 \end{pmatrix},$$

and the determinant of  $M$  is denoted by  $\Delta$ :

$$\Delta = ac - bc - 3ab - \frac{5b^2 - c^2}{2} - 2(4a + 3b + 3c)d.$$

*Proof of statement (ii) of Theorem 2.* To prove that a HK-system (3) has neither degenerate centers nor two nilpotent centers for any real  $a, b, c, d, e$  and  $f$ , we first prove that a HK-system (3) does not have a degenerate center.

Assume that a HK-system (3) has a degenerate center at  $(1, 1)$  without loss of generality. Then the HK-system (3) is system (18),  $\Delta = 0$  and the matrix  $M$  must be identically zero, that is, the parameters  $a, b, c$  and  $d$  must satisfy the equations

$$(19) \quad \begin{aligned} 2a + 3b + c + 4d &= 0, \\ 2a - b + 3c - 4d &= 0, \\ 2a + b + c - 4d &= 0. \end{aligned}$$

The solutions of (19) are

$$b = c = -4d, \quad a = 6d.$$

In the following we claim that the equilibrium  $(1, 1)$  is not a degenerate center of system (18) if  $b = c = -4d$  and  $a = 6d$ .

When  $b = c = -4d$  and  $a = 6d$ , system (18) becomes

$$(20) \quad \begin{aligned} \dot{x} &= -dx(6 - 4x + x^2 - 8y + 2xy + 3y^2), \\ \dot{y} &= dy(6 - 8x + 3x^2 - 4y + 2xy + y^2). \end{aligned}$$

Of course,  $d \neq 0$ , otherwise this system has no isolated equilibria.

Moving the equilibrium  $(1, 1)$  of system (20) to the origin of the coordinates, we obtain

$$(21) \quad \begin{aligned} \dot{x} &= -d(1+x)(x^2 + 2xy + 3y^2), \\ \dot{y} &= d(1+y)(3x^2 + 2xy + y^2). \end{aligned}$$

Doing blow ups of the origin of system (21), letting  $x = r \cos \theta$ ,  $y = r \sin \theta$ , it can be seen that the local phase portrait of the origin is formed by two hyperbolic sectors, which implies that the origin of system (21) is not a center. So the equilibrium  $(1, 1)$  of system (20) is not a center. For more information on the changes of variables called blow ups for studying the local phase portraits of the equilibrium points of a planar analytic differential system, see, for instance, Section 3.1 of [Dumortier et al. 2006].

Therefore, we obtain that a HK-system (3) does not have a degenerate center.

To complete the proof of statement (ii) of Theorem 2, it only remains to prove that there do not exist values of the parameters  $a, b, c, d, e$  and  $f$  such that a HK-system (3) has two nilpotent centers.

It is well known that if an equilibrium is a nilpotent center, then the two eigenvalues of this equilibrium are zero, but the Jacobi matrix at this equilibrium is not identically zero.

Assume that  $(1, 1)$  is a nilpotent center of HK-system (3), then eigenvalues of the Jacobi matrix  $M$  at equilibrium  $(1, 1)$  are

$$(22) \quad \pm \sqrt{\frac{1}{2}(5b^2 + 2cb + 12db - c^2 + 12cd + 2a(3b - c + 8d))},$$

which must be zero, but  $M \neq 0$ . We separate the rest of the proof into two cases.

Case I :  $3b - c + 8d = 0$ .

Case II :  $3b - c + 8d \neq 0$ .

First, we consider Case I. Here,  $c = 3b + 8d$ , which leads to the eigenvalues being  $\pm|b + 4d|i$ . Therefore  $b = -4d$ . Plugging the expressions of  $c$  and  $b$  into  $M$ , the matrix  $M$  becomes

$$\begin{pmatrix} a - 6d & a - 6d \\ -a + 6d & -a + 6d \end{pmatrix}.$$

Since  $M \neq 0$ ,  $a - 6d \neq 0$ .

Now system (18) becomes

$$(23) \quad \begin{aligned} \dot{x} &= -ax + 4dx^2 + 8dxy - dx^3 + (a - 8d)x^2y - 3dxy^2, \\ \dot{y} &= ay - 8dxy - 4dy^2 + 3dx^2y - (a - 8d)xy^2 + dy^3. \end{aligned}$$

If  $a - 12d \neq 0$ , then system (23) has only two equilibria outside the coordinate axes, one is  $(1, 1)$  and the other is  $P = (-a/(a - 12d), -a/(a - 12d))$ .

The determinant of the Jacobi matrix  $M_p$  of system (23) at equilibrium  $P$  is denoted by  $\Delta_p$ :

$$\Delta_p = -\frac{16a^2(a - 6d)^2d}{(a - 12d)^3}.$$

We distinguish three subcases to discuss  $\Delta_p$ :

(I.1)  $ad \neq 0$ . Here,  $\Delta_p \neq 0$ , which implies that equilibrium  $P$  is either a linear type center or a saddle depending on  $\Delta_p > 0$  or  $\Delta_p < 0$ , respectively, i.e.,  $P$  is not a nilpotent center. Hence, system (23) cannot have two nilpotent centers as  $c = 3b + 8d$ ,  $a - 12d \neq 0$  and  $ad \neq 0$ .

(I.2)  $a = 0$  and  $d \neq 0$ . Here,  $\Delta_p = 0$  and  $M_p = 0$ . This gives that  $P$  is a degenerated equilibrium. We have proved that the degenerated equilibrium of a

HK-system (3) cannot be a center; hence, system (23) cannot have two nilpotent centers as  $c = 3b + 8d$ ,  $a = 0$  and  $d \neq 0$ .

(I.3)  $d = 0$  and  $a \neq 0$ . Here,  $\Delta_p = 0$  and system (23) becomes

$$\begin{aligned} \dot{x} &= -ax(1 - xy), \\ \dot{y} &= ay(1 - xy), \end{aligned}$$

which has nonisolated equilibria. Statement (i) (see C4 in Table 2) has shown that this system has no center. Therefore, system (23) cannot have two nilpotent centers if  $c = 3b + 8d$  and  $a - 12d \neq 0$ .

If  $a - 12d = 0$ , then system (23) has a unique equilibrium at  $(1, 1)$  outside the coordinate axes. Hence, system (23) cannot have two nilpotent centers as  $c = 3b + 8d$  and  $a - 12d = 0$ .

In summary, in Case I we prove that if a HK-system (3) has a nilpotent equilibrium at  $(1, 1)$ , then this system does not have another nilpotent center which is different from  $(1, 1)$ . Hence a HK-system (3) cannot have two nilpotent centers if  $3b - c + 8d = 0$ .

We now discuss whether a HK-system (3) has two nilpotent equilibria in Case II:  $3b - c + 8d \neq 0$ .

Since  $3b - c + 8d \neq 0$  and the eigenvalues (22) must be zero in order for equilibrium  $(1, 1)$  to be a nilpotent center,

$$a = \frac{c^2 - 5b^2 - 2bc - 12bd - 12cd}{2(3b - c + 8d)}.$$

Then the Jacobi matrix of system (18) at the equilibrium  $(1, 1)$  is

$$\begin{pmatrix} \frac{(b+4d)(2b-c+4d)}{3b-c+8d} & -\frac{(2b-c+4d)^2}{3b-c+8d} \\ \frac{(b+4d)^2}{3b-c+8d} & -\frac{(b+4d)(2b-c+4d)}{3b-c+8d} \end{pmatrix},$$

and this equilibrium is nilpotent if and only if  $(b + 4d)^2 + (2b - c + 4d)^2 \neq 0$ .

We distinguish three subcases to study whether the equilibrium  $(1, 1)$  is a nilpotent center: (II.1)  $b + 4d = 0$  and  $2b - c + 4d \neq 0$ , (II.2)  $b + 4d \neq 0$  and  $2b - c + 4d = 0$ , (II.3)  $(b + 4d)(2b - c + 4d) \neq 0$ .

(II.1) If  $b + 4d = 0$  and  $2b - c + 4d \neq 0$ , then  $b = -4d$ . By moving the equilibrium  $(1, 1)$  to the origin, system (18) becomes

$$(24) \quad \begin{aligned} \dot{x} &= (c+4d)y - dx^2 + (c+2d)xy + \frac{3}{2}(c+2d)y^2 - dx^3 - 2dx^2y + \frac{3}{2}(c+2d)xy^2, \\ \dot{y} &= 3dx^2 + 2dxy - \frac{1}{2}(c+2d)y^2 + 3dx^2y + 2dxy^2 - \frac{1}{2}(c+2d)y^3. \end{aligned}$$

We must assume that  $c + 4d \neq 0$ , otherwise the equilibrium  $(0, 0)$  of system (24)

is not nilpotent. Rescaling the independent variable by  $c + 4d$  we have system (24) in the normal form for applying Lemma 7.

$$(25) \quad \begin{aligned} \dot{x} &= y - \frac{d}{c+4d}x^2 + \frac{c+2d}{c+4d}xy + \frac{3(c+2d)}{2(c+4d)}y^2 \\ &\quad - \frac{d}{c+4d}x^3 - \frac{2d}{c+4d}x^2y + \frac{3(c+2d)}{2(c+4d)}xy^2, \\ \dot{y} &= \frac{3d}{c+4d}x^2 + \frac{2d}{c+4d}xy - \frac{c+2d}{2(c+4d)}y^2 \\ &\quad + \frac{3d}{c+4d}x^2y + \frac{2d}{c+4d}xy^2 - \frac{c+2d}{2(c+4d)}y^3. \end{aligned}$$

Using the notation of Lemma 7,

$$\begin{aligned} G(x) &\equiv 0, \\ F(x) &= \frac{3d}{c+4d}x^2 + \frac{2d^2}{(c+4d)^2}x^3 + O(x^4). \end{aligned}$$

Therefore, we must take  $d = 0$  if we want the equilibrium  $(0, 0)$  to be a nilpotent center by Theorem 8. However, when  $d = 0$ , system (24) becomes

$$(26) \quad \begin{aligned} \dot{x} &= \frac{1}{2}cx(y-1)(-1+3y), \\ \dot{y} &= -\frac{1}{2}cy(y-1)^2, \end{aligned}$$

which has nonisolated equilibria in the line  $y = 1$  and the  $x$ -axis is invariant for the flow of this system. Hence, the origin  $(0, 0)$  cannot become center of system (26), which implies that equilibrium  $(1, 1)$  is not a nilpotent center of system (18) as  $b + 4d = 0$  and  $2b - c + 4d \neq 0$ .

(II.2) If  $b + 4d \neq 0$  and  $2b - c + 4d = 0$ , then  $c = 2b + 4d$ .

Let  $x = Y + 1$ ,  $y = X + 1$ . Then system (18) becomes

$$(27) \quad \begin{aligned} \dot{X} &= (b+4d)Y + 3dY^2 + 2dXY - \frac{1}{2}(b+2d)X^2 + 3dXY^2 \\ &\quad - (b+2d)X^2Y - \frac{1}{2}(b+2d)X^3, \\ \dot{Y} &= -dY^2 + (b+2d)XY + \frac{3}{2}(b+2d)X^2 - dY^3 \\ &\quad + (b+2d)XY^2 + \frac{3}{2}(b+2d)X^2Y. \end{aligned}$$

Since  $b + 4d \neq 0$ , we have system (27) in the normal form for applying Lemma 7 by rescaling the independent variable  $b + 4d$ . Again using the notation of that lemma, we have  $G(X) \equiv 0$  and

$$F(X) = \frac{3(b+2d)}{2(b+4d)}X^2 + \frac{(b+2d)^2}{2(b+4d)^2}X^3 + O(X^4).$$

Therefore we must take  $b = -2d$  if the origin of system (27) is a nilpotent center. However, when  $b = -2d$ , system (27) becomes

$$(28) \quad \begin{aligned} \dot{X} &= dY(1+X)(2+3Y), \\ \dot{Y} &= -dY^2(1+Y), \end{aligned}$$

which has nonisolated equilibria which fill up the  $X$ -axis. Thus, the equilibrium  $(0, 0)$  cannot be a center of system (28), which gives that the equilibrium  $(1, 1)$  of system (18) is not a nilpotent center as  $b + 4d \neq 0$  and  $2b - c + 4d = 0$ .

(II.3) If  $(b + 4d)(2b - c + 4d) \neq 0$ , we shall apply Theorem 8 to prove that the nilpotent equilibrium  $(1, 1)$  is not a center.

To apply this theorem, we first need to translate the equilibrium  $(1, 1)$  of system (18) to the origin of coordinates. For this, we make the change of variables  $x = X + 1$  and  $y = Y + 1$  in system (18), and then write the matrix of the linear part at the origin of this system into its real Jordan normal form with the change of variables

$$X = -\frac{(c - 2b - 4d)^2 u}{3b - c + 8d} - \frac{(2b - c + 4d)v}{b + 4d}, \quad Y = -\frac{(b + 4d)(2b - c + 4d)u}{3b - c + 8d}.$$

Finally, a HK-system (3) in the variables  $(u, v)$  can be written as

$$(29) \quad \begin{aligned} \dot{u} &= v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 + a_{21}u^2v + a_{12}uv^2, \\ \dot{v} &= b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3, \end{aligned}$$

where

$$\begin{aligned} a_{20} &= \frac{(2b - c + 4d)(b^2 + 2(b + c)d)(17b^2 - 14cb + 80db + 3c^2 + 96d^2 - 32cd)}{2(-3b + c - 8d)^2(b + 4d)}, \\ a_{11} &= \frac{2(2b - c + 4d)(2b^3 - (c + d)b^2 + 8(c - 5d)db + d(-3c^2 + 24dc - 64d^2))}{(b + 4d)^2(3b - c + 8d)}, \\ a_{02} &= \frac{3d(2b - c + 4d)(-3b + c - 8d)}{(b + 4d)^3}, \\ a_{30} &= -\frac{(-2b + c - 4d)^2(b^2 + 2(b + c)d)(17b^2 - 14cb + 80db + 3c^2 + 96d^2 - 32cd)}{2(3b - c + 8d)^3}, \\ a_{21} &= -\frac{(-2b + c - 4d)^2(5b^3 - 2(c - 5d)b^2 + 16(c - 2d)db - 2d(3c^2 - 24dc + 32d^2))}{(-3b + c - 8d)^2(b + 4d)}, \\ a_{12} &= \frac{3d(-2b + c - 4d)^2}{(b + 4d)^2}, \end{aligned}$$

$$\begin{aligned}
b_{20} &= -\frac{3(2b-c+4d)(b^2+2(b+c)d)(5b^2-4(c-6d)b+c^2+32d^2-8cd)}{2(-3b+c-8d)^2}, \\
b_{11} &= -\frac{(2b-c+4d)(b^2+2(b+c)d)(17b^2-14cb+80db+3c^2+96d^2-32cd)}{(-3b+c-8d)^2(b+4d)}, \\
b_{02} &= -\frac{(2b-c+4d)(2b^3-(c+d)b^2+8(c-5d)db+d(-3c^2+24dc-64d^2))}{(b+4d)^2(3b-c+8d)}, \\
b_{30} &= \frac{2(b+4d)(2b-c+4d)^3(b^2+2(b+c)d)(7b^2-5cb+36db+c^2+48d^2-12cd)}{(-3b+c-8d)^4}, \\
b_{21} &= \frac{3(-2b+c-4d)^2(b^2+2(b+c)d)(17b^2-14cb+80db+3c^2+96d^2-32cd)}{2(3b-c+8d)^3}, \\
b_{12} &= \frac{(-2b+c-4d)^2(5b^3-2(c-5d)b^2+16(c-2d)db-2d(3c^2-24dc+32d^2))}{(-3b+c-8d)^2(b+4d)}, \\
b_{03} &= -\frac{(-2b+c-4d)^2d}{(b+4d)^2}.
\end{aligned}$$

Using the notation introduced in the statement of Lemma 7, we have

$$\begin{aligned}
v &= f(u) = -a_{20}u^2 + (a_{11}a_{20} - a_{30})u^3 + O(u^4), \\
G(u) &\equiv 0, \\
F(u) &= b_{20}u^2 + (b_{30} - a_{20}b_{11})u^3 + O(u^4).
\end{aligned}$$

In order for the equilibrium  $(0, 0)$  of system (29) to be a nilpotent center, from Theorem 8 we require

$$b_{20} = -\frac{3(2b-c+4d)(b^2+2(b+c)d)(5b^2-4(c-6d)b+c^2+32d^2-8cd)}{2(-3b+c-8d)^2} = 0.$$

Since  $2b - c + 4d \neq 0$ , we must have

$$(b^2 + 2(b+c)d)(5b^2 - 4(c-6d)b + c^2 + 32d^2 - 8cd) = 0.$$

Then either  $b^2 + 2(b+c)d = 0$  or  $5b^2 - 4(c-6d)b + c^2 + 32d^2 - 8cd = 0$ .

We now divide into three subcases to discuss whether the equilibrium  $(1, 1)$  of system (18) is a nilpotent center:

(II.a)  $b^2 + 2(b+c)d = 0$  and  $b+c \neq 0$ .

(II.b)  $b^2 + 2(b+c)d = 0$  and  $b+c = 0$ .

(II.c)  $b^2 + 2(b+c)d \neq 0$  and  $5b^2 - 4(c-6d)b + c^2 + 32d^2 - 8cd = 0$ .

For (II.a),  $d = -b^2/(2(b+c))$ . Plugging in the expressions of  $a$  and  $d$ , system (18) becomes

$$\begin{aligned}\dot{x} &= \frac{x(bx + cy - b - c)(bx + 3cy - b - c)}{2(b+c)}, \\ \dot{y} &= -\frac{y(bx + cy - b - c)(3bx + cy - b - c)}{2(b+c)},\end{aligned}$$

which has a nonisolated equilibrium  $(1, 1)$  on the straight line  $bx + cy - b - c = 0$  filled with equilibria, consequently it cannot be a nilpotent center.

For (II.b),  $b = c = 0$  and system (18) becomes

$$\begin{aligned}\dot{x} &= -dx(x - 3y)(x - y), \\ \dot{y} &= -dy(-3x + y)(x - y),\end{aligned}$$

which again has a nonisolated equilibrium  $(1, 1)$  on a straight line  $x - y = 0$  filled with equilibria.

For (II.c),  $d = (-3b + c \pm \sqrt{-(b-c)^2})/8$ . Since  $d$  is real number, we must have  $b = c$  and  $d = (-3b + c)/8$ , which implies  $3b - c + 8d = 0$ . This is not possible because we are in Case II:  $3b - c + 8d \neq 0$ .

In summary, in Case II we prove that a HK-system (3) cannot have any a nilpotent equilibrium. Hence a HK-system (3) cannot have two nilpotent centers. This completes the proof of statement (ii) of Theorem 2.  $\square$

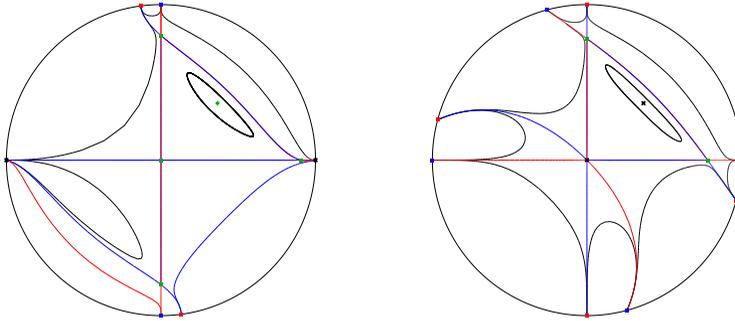
From the proof of nonexistence of two nilpotent centers, we can see that a HK-system (3) can have a nilpotent center. And a HK-system (3) has at most two isolated equilibria outside the coordinate axes if there is a nilpotent center. Therefore when a HK-system (3) has three or four centers, all of them must be linear type centers. Moreover, if a HK-system (3) has only two isolated equilibria outside the coordinate axes, both of them are centers: one is nilpotent and the other is linear type, and they are either in the same quadrant or in the opposite quadrant. Hence, if one of two centers is nilpotent, they cannot be in consecutive quadrants.

In the proofs of statements (iii)–(vi) of Theorem 2, we will give examples showing the existence of HK-systems with centers realizing all possible configurations with respect to the coordinate axes.

*Proof of statement (iii) of Theorem 2.* The eigenvalues of the linear part of a HK-system (3) at the equilibrium  $(1, 1)$  are  $\pm\sqrt{\Delta/2}$  where

$$\Delta = 6ab + 5b^2 - 2ac + 2bc - c^2 + 16ad + 12bd + 12cd.$$

So if  $\Delta < 0$  the equilibrium  $(1, 1)$  is a linear type center, because the HK-system is a Hamiltonian system and it cannot have a focus at the equilibrium  $(1, 1)$ . Hence we have proved that there are HK-systems having at least one linear type center.



**Figure 1.** Left: A HK-system (30) with a unique linear type center. Right: A HK-system (31) with a unique nilpotent center.

Now we shall show that there are HK-systems with a unique center of linear type. Consider the HK-system defined by the Hamiltonian

$$(30) \quad H = xy(-40 + 4x + 15xy + 2y^2).$$

The equilibria of this HK-system are

$$(0, 0), \quad (0, \pm 2\sqrt{5}), \quad (10, 0), \quad (1, 1).$$

So the only one of these equilibria which can be a center is the equilibrium  $(1, 1)$ , which is a linear type center because the eigenvalues of its linear part are  $\pm\sqrt{11/40}i$ . See its global phase portrait in the Poincaré disc in Figure 1(left). For a definition of the Poincaré disc, see, for instance, Chapter 5 of [Dumortier et al. 2006].

The real algebraic curve

$$xy(-40 + 4x + 15xy + 2y^2) = h$$

is a family of level ovals in  $\mathbb{R}^2$  if and only if  $-19 < h < 0$ .

Next, we provide a HK-system with a unique nilpotent center. Consider the HK-system defined by the Hamiltonian

$$(31) \quad H = xy\left(x + y - \frac{x^2}{4} - xy - \frac{y^2}{4}\right),$$

i.e., the HK-system is

$$(32) \quad \begin{aligned} \dot{x} &= -x\left(x + 2y - \frac{x^2}{4} - 2xy - \frac{3y^2}{4}\right), \\ \dot{y} &= y\left(2x + y - \frac{3x^2}{4} - 2xy - \frac{y^2}{4}\right). \end{aligned}$$

The equilibria of this HK-system are

$$(0, 0), \quad (4, 0), \quad (0, 4), \quad (1, 1).$$

Therefore again the only one of these equilibria which can be a center is the equilibrium  $(1, 1)$ . The two eigenvalues of this equilibrium are zero, and since its linear part is not identically zero, it is a nilpotent equilibrium. We shall use Lemma 7 for proving that it is a nilpotent center.

First, using the translations  $x = X + 1$  and  $y = Y + 1$  we translate the equilibrium  $(1, 1)$  to the origin of coordinates, and we get the differential system

$$(33) \quad \begin{aligned} \dot{X} &= \frac{1}{4}(6X + 6Y + 7X^2 + 14XY + 3Y^2 + X^3 + 8X^2Y + 3XY^2), \\ \dot{Y} &= -\frac{1}{4}(6X + 6Y + 3X^2 + 14XY + 7Y^2 + 3X^2Y + 8XY^2 + Y^3). \end{aligned}$$

In order to apply Lemma 7 we shall write the linear part at the origin of this system in the real Jordan normal form. So we use the change of variables  $X = 3u/2$  and  $Y = v - 3u/2$ , and in the new variables  $u$  and  $v$ , the differential system (33) becomes

$$(34) \quad \begin{aligned} \dot{u} &= v - \frac{3}{2}u^2 + 2uv + \frac{1}{2}v^2 - \frac{9}{4}u^3 + \frac{3}{4}u^2v + \frac{3}{4}uv^2, \\ \dot{v} &= -v^2 + 3uv - \frac{27}{4}u^3 + \frac{27}{4}u^2v - \frac{3}{4}uv^2 - \frac{1}{4}v^3. \end{aligned}$$

Now we can apply Lemma 7 to system (34), and using its notation we have that  $v = f(u) = 3u^2/2 + O(u^3)$  and  $F(u) = -9u^3/4 + O(u^4)$ . Then,  $G(u) \equiv 0$ ,  $a = -9/3 < 0$  and  $m = 3$ , so the origin of system (34) is a center by Theorem 8. Consequently the equilibrium  $(1, 1)$  of system (32) is a nilpotent center. See its global phase portrait in the Poincaré disc in Figure 1(right).

The real algebraic curve

$$xy \left( x + y - \frac{x^2}{4} - xy - \frac{y^2}{4} \right) = h$$

is a family of level ovals in  $\mathbb{R}^2$  if and only if  $0 < h < \frac{1}{2}$ .

□

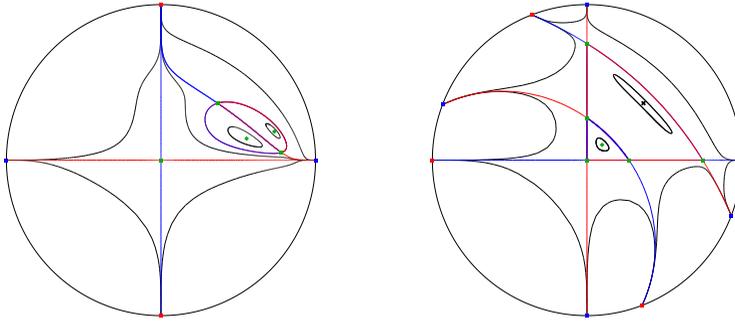
*Proof of statement (iv) of Theorem 2.* First we prove the existence of a HK-system having two linear type centers in the positive quadrant, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(35) \quad H = xy(200 - 60x - 240y + 6x^2 + 23xy + 96y^2).$$

The corresponding HK-system has four equilibria outside the axes of coordinates, namely,

$$p_1 = (1, 1), \quad p_2 = \left(4, \frac{1}{4}\right), \quad p_{3,4} = \left(\frac{20(9 \pm \sqrt{10})}{71}, \frac{5(9 \pm \sqrt{10})}{71}\right).$$

The equilibria  $p_1$  and  $p_2$  are hyperbolic saddles because the determinant of their linear parts in both is  $-1350$ , and the equilibria  $p_{3,4}$  are linear type centers because



**Figure 2.** Left: A HK-system (35) with two linear type centers in the same quadrant. Right: A HK-system (36) with one linear type center and one nilpotent center, both in the same quadrant.

the determinant of their linear parts is

$$\frac{120000(7460 \pm 1531\sqrt{10})}{357911},$$

and the system is Hamiltonian. The phase portrait of the HK-system defined by the Hamiltonian (35) is given in Figure 2(left). The real algebraic curve

$$xy(200 - 60x - 240y + 6x^2 + 23xy + 96y^2) = h$$

is a family of level ovals in  $\mathbb{R}^2$  if and only if either  $10000(467 - 240\sqrt{10})/357911 < h < 25$  or  $25 < h < 10000(467 + 240\sqrt{10})/357911$ . Taken together, this real algebraic curve has two families of level ovals in  $\mathbb{R}^2$  for all real  $h$ .

Now we provide a HK-system having one linear type center and one nilpotent center in the positive quadrant, and no other centers. The HK-system defined by the Hamiltonian

$$(36) \quad H = xy(2 - 4x + x^2 - 4y + 3xy + y^2)$$

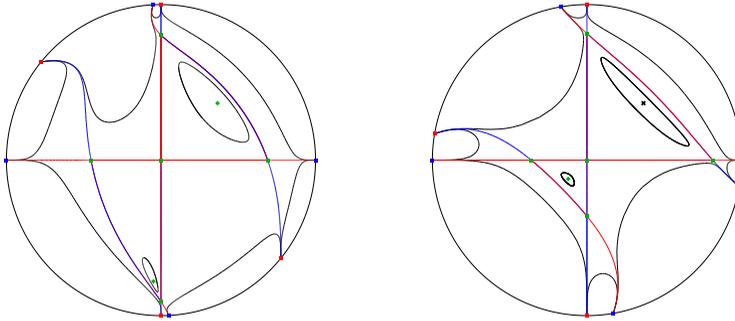
has only the following two equilibria outside the axes of coordinates:

$$p_1 = (1, 1) \quad p_2 = \left(\frac{1}{5}, \frac{1}{5}\right).$$

The equilibrium  $p_2$  has eigenvalues  $\pm\sqrt{128/25}i$ , so it is a linear type center.

We shall show that the equilibrium  $p_1$  is a nilpotent center. The two eigenvalues of this equilibrium are zero, and since its linear part is not identically zero, it is a nilpotent equilibrium. In order to study its local phase portrait we translate the equilibrium  $p_1$  to the origin of coordinates using the translations  $x = X + 1$  and  $y = Y + 1$ , and we obtain the differential system

$$(37) \quad \begin{aligned} \dot{X} &= 4X - 4Y - 5X^2 - 10XY - 3Y^2 - X^3 - 6X^2Y - 3XY^2, \\ \dot{Y} &= 4X + 4Y + 3X^2 + 10XY + 5Y^2 + 3X^2Y + 6XY^2 + Y^3. \end{aligned}$$



**Figure 3.** Left: A HK-system (39) with two linear type centers in different opposite quadrants. Right: A HK-system (40) with one linear type center and one nilpotent center in different opposite quadrants.

Now we shall write the linear part at the origin of this system in its real Jordan normal form; for this we use the change of variables  $X = -4u$  and  $Y = 4u + v$ . In the new variables  $u$  and  $v$  the differential system (37) becomes

$$(38) \quad \begin{aligned} \dot{u} &= v - 8u^2 - 4uv + \frac{3}{4}v^2 + 32u^3 - 3uv^2, \\ \dot{v} &= 16uv + 2v^2 - 256u^3 - 96u^2v + v^3. \end{aligned}$$

We can apply Lemma 7 to system (38), and using the notation of that lemma, we have  $v = f(u) = 8u^2 + O(u^3)$  and  $F(u) = -128u^3 + O(u^4)$ . Then,  $G(u) \equiv 0$ ,  $a = -128 < 0$  and  $m = 3$ , so the origin of system (38) is a center by Theorem 8. Consequently the equilibrium  $p_1$  of the Hamiltonian system defined by (36) is a nilpotent center. See its global phase portrait in the Poincaré disc in Figure 2(right). The real algebraic curve

$$xy(2 - 4x + x^2 - 4y + 3xy + y^2) = h$$

is a family of level ovals in  $\mathbb{R}^2$  if and only if either  $-1 < h < 0$  or  $0 < h < \frac{3}{125}$ . In total, this real algebraic curve has two families of level ovals in  $\mathbb{R}^2$  for all real  $h$ .

Now we prove the existence of a HK-system having two linear type centers in different opposite quadrants, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(39) \quad H = xy(96 + 48x - 12y - 32x^2 - 41xy - 2y^2).$$

The corresponding HK-system has four equilibria outside the axes of coordinates, namely,

$$(1, 1), \quad p_2 = \left(-\frac{1}{4}, -4\right).$$

These two equilibria are linear type centers because the determinant of their linear parts is  $475/384$  and the system is Hamiltonian. The phase portrait of the HK-system defined by the Hamiltonian (39) is given in Figure 3(left). The real algebraic curve

$$xy(96 + 48x - 12y - 32x^2 - 41xy - 2y^2) = h$$

has two families of level ovals in  $\mathbb{R}^2$  if and only if  $0 < h < 57$ .

Now we exhibit a HK-system having one linear type center and one nilpotent center in different opposite quadrants, and no other centers. Thus the HK-system defined by the Hamiltonian

$$(40) \quad H = xy(4 + 4x + 4y - x^2 - 6xy - y^2),$$

has only the following two equilibria outside the axes of coordinates:

$$p_1 = (1, 1) \quad p_2 = \left(-\frac{1}{4}, -\frac{1}{4}\right).$$

The linear part of the equilibrium  $p_2$  has determinant equal to  $225/64$ , so it is a linear type center.

We shall prove that the equilibrium  $p_1$  is a nilpotent center. Since the two eigenvalues of this equilibrium are zero, and its linear part is not identically zero, it is a nilpotent equilibrium. For studying its local phase portrait we translate the equilibrium  $p_1$  to the origin of coordinates using the translations  $x = X + 1$  and  $y = Y + 1$ , and we obtain the differential system

$$(41) \quad \begin{aligned} \dot{X} &= \frac{3}{4}(10X + 10Y + 11X^2 + 22XY + 3Y^2 + X^3 + 12X^2Y + 3XY^2), \\ \dot{Y} &= -\frac{3}{4}(10X + 10Y + 3X^2 + 22XY + 11Y^2 + 3X^2Y + 12XY^2 + Y^3). \end{aligned}$$

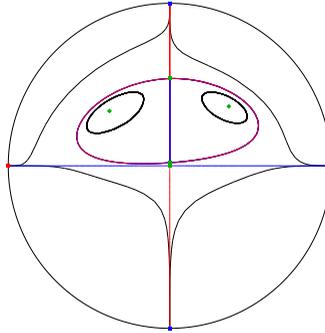
We write the linear part at the origin of this system in its real Jordan normal form using the change of variables  $X = 15u/2$  and  $Y = v - 15u/2$ . In the new variables  $u$  and  $v$ , the differential system (41) becomes

$$(42) \quad \begin{aligned} \dot{u} &= v + \frac{1}{20}(-900u^2 + 240uv + 6v^2 - 6750u^3 + 675u^2v + 45uv^2), \\ \dot{v} &= -\frac{3}{4}(-120uv + 8v^2 + 6750u^3 - 1350u^2v + 45uv^2 + v^3). \end{aligned}$$

Applying Lemma 7 to system (42), and using the notation of that theorem, we have  $v = f(u) = 45u^2 + O(u^3)$  and  $F(u) = -101258u^3/2 + O(u^4)$ . Then,  $G(u) \equiv 0$ ,  $a = -10125/2 < 0$  and  $m = 3$ , so the origin of system (42) is a center by Theorem 8. Consequently the equilibrium  $p_1$  of the Hamiltonian system defined by (40) is a nilpotent center. See its global phase portrait in the Poincaré disc in Figure 3(right). The real algebraic curve

$$xy(4 + 4x + 4y - x^2 - 6xy - y^2) = h$$

has two families of level ovals in  $\mathbb{R}^2$  if and only if  $0 < h < \frac{3}{32}$ . And when  $\frac{3}{32} \leq h < 4$ ,



**Figure 4.** HK-systems with two centers in different consecutive quadrants.

then  $xy(4 + 4x + 4y - x^2 - 6xy - y^2) = h$  has a family of level ovals in  $\mathbb{R}^2$ . In summary, for all  $h \in \mathbb{R}$ , the real algebraic curve  $xy(4 + 4x + 4y - x^2 - 6xy - y^2) = h$  has two families of level ovals in  $\mathbb{R}^2$ .

We provide a HK-system having two linear type centers in different consecutive quadrants, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(43) \quad H = xy(178074 + 356148x - 4989116y + 558568x^2 - 404079xy + 3231200y^2).$$

The corresponding HK-system has two equilibria outside the axes of coordinates, namely,

$$(1, 1), \quad \left(-1, \frac{361}{400}\right).$$

These two equilibria are linear type centers because the determinants of their linear parts are

$$6889484999096 \quad \text{and} \quad \frac{461859088701689761}{80000},$$

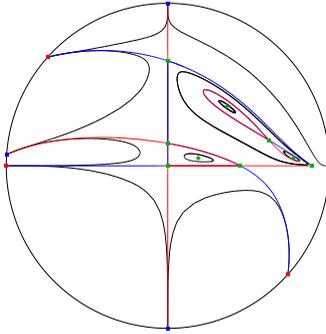
respectively. The phase portrait of the HK-system defined by the Hamiltonian (43) is given in Figure 4. This completes the proof of statement (iv) of Theorem 2. The real algebraic curve

$$xy(178074 + 356148x - 4989116y + 558568x^2 - 404079xy + 3231200y^2) = h$$

has a family of level ovals in  $\mathbb{R}^2$  if and only if either  $-1069205 < h < 0$  or  $0 < h < 162546646559/160000$ . In summary, for all  $h \in \mathbb{R}$ , the real algebraic curve  $xy(178074 + 356148x - 4989116y + 558568x^2 - 404079xy + 3231200y^2) = h$  has two families of level ovals in  $\mathbb{R}^2$ .  $\square$

*Proof of statement (v) of Theorem 2.* We show a HK-system having three linear type centers in the same quadrant, and no other centers. Let the HK-system be defined by the Hamiltonian

$$(44) \quad H = xy(20 - 20x - 80y + 2x^2 + 31xy + 32y^2).$$



**Figure 5.** HK-systems with three linear type centers in the same quadrant.

The associated HK-system has four equilibria outside the axes of coordinates, namely,

$$(45) \quad (1, 1), \quad (4, \frac{1}{4}), \quad (\frac{2}{47}(30 \mp \sqrt{430}), \frac{1}{94}(30 \mp \sqrt{430})).$$

The determinants of their linear parts are

$$\frac{27}{10}, \quad \frac{27}{10} \quad \text{and} \quad \frac{4(\pm 995\sqrt{430} - 13588)}{103823},$$

respectively. Therefore the three first equilibria of (45) are linear type centers and the fourth one is a hyperbolic saddle.

The real algebraic curve

$$xy(20 - 20x - 80y + 2x^2 + 31xy + 32y^2) = h$$

has two families of level ovals in  $\mathbb{R}^2$  if and only if

$$-15 < h < -\frac{100(6889 + 344\sqrt{430})}{103823},$$

and it has a family of level ovals in  $\mathbb{R}^2$  if and only if either

$$-\frac{100(6889 + 344\sqrt{430})}{103823} \leq h < 0 \quad \text{or} \quad 0 < h < \frac{100(-6889 + 344\sqrt{430})}{103823}.$$

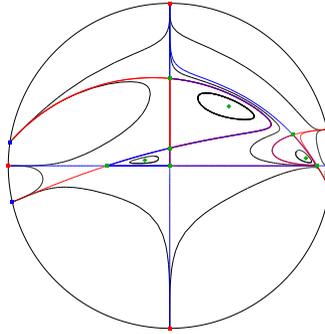
In summary, for all  $h \in \mathbb{R}$ , the real algebraic curve

$$xy(20 - 20x - 80y + 2x^2 + 31xy + 32y^2) = h$$

has four families of level ovals in  $\mathbb{R}^2$ . In Figure 5 we show the phase portrait of the HK-system defined by the Hamiltonian (44).

Now we provide a HK-system having two linear type centers in the same quadrant and the other in a consecutive quadrant, and no other centers. Let the HK-system be defined by the Hamiltonian

$$(46) \quad H = xy(-20 - 20x + 104y + 2x^2 + 5xy - 60y^2).$$



**Figure 6.** HK-systems with two linear type centers in the same quadrant and the other in a consecutive quadrant.

The associated HK-system has four equilibria outside the axes of coordinates, namely,

$$(47) \quad (4, 1), \quad (1, 1), \quad \left( \frac{2(56071 \mp 829\sqrt{6157})}{101(485 \pm \sqrt{6157})}, \frac{1}{606}(121 \mp \sqrt{6157}) \right).$$

The determinants of their linear parts are

$$-\frac{108}{5}, \quad \frac{63}{10} \quad \text{and} \quad \frac{658(731635 \pm 9034\sqrt{6157})}{46363545},$$

respectively. Therefore the first equilibrium of (47) is a hyperbolic saddle and the other three equilibria are linear type centers. In Figure 6 we show the phase portrait of the HK-system defined by the Hamiltonian (46). The phase portrait shows that the real algebraic curve

$$xy(-20 - 20x + 104y + 2x^2 + 5xy - 60y^2) = h$$

has three families of level ovals in  $\mathbb{R}^2$  for all  $h \in \mathbb{R}$ , in which this real algebraic curve has two families of level ovals if and only if

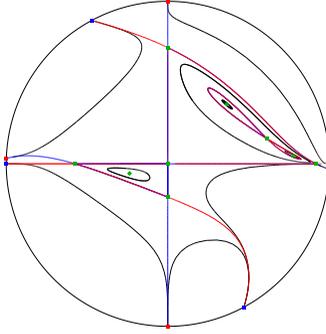
$$0 < h < \frac{94(-2956253 + 37859\sqrt{6157})}{9272709},$$

and it has a family of level ovals if and only if either

$$\frac{94(-2956253 + 37859\sqrt{6157})}{9272709} \leq h < 11 \quad \text{or} \quad -\frac{94(2956253 + 37859\sqrt{6157})}{9272709} < h < -16.$$

We give a HK-system having three linear type centers; two in the same quadrant and the other in the opposite quadrant, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(48) \quad H = xy(-40 - 20x - 80y + 2x^2 + 61xy + 32y^2).$$



**Figure 7.** HK-systems with three linear type centers; two in the same quadrant and the other in the opposite quadrant.

The corresponding HK-system has four equilibria outside the axes of coordinates, namely,

$$(49) \quad (1, 1), \quad (4, \frac{1}{4}), \quad (\frac{4}{77}(15 \mp \sqrt{610}), \frac{1}{77}(15 \mp \sqrt{610})).$$

The determinants of their linear parts are

$$\frac{81}{40}, \quad \frac{81}{40} \quad \text{and} \quad \frac{4(\pm 2995\sqrt{610} - 30988)}{456533},$$

respectively. Therefore the three first equilibria of (49) are linear type centers and the fourth one is a hyperbolic saddle. The phase portrait of the HK-system defined by the Hamiltonian (48) is given in Figure 7. The phase portrait shows that the real algebraic curve

$$xy(-40 - 20x - 80y + 2x^2 + 61xy + 32y^2) = h$$

has four families of level ovals in  $\mathbb{R}^2$  for all  $h \in \mathbb{R}$ , in which this real algebraic curve has two families of level ovals if and only if either

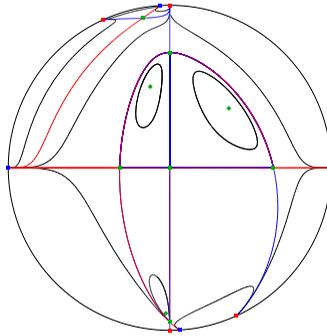
$$-45 < h < -\frac{400(25189+976\sqrt{610})}{456533} \quad \text{or} \quad \frac{400(-25189+976\sqrt{610})}{456533} < h < 0,$$

and it has a family of level ovals if and only if

$$-\frac{400(25189+976\sqrt{610})}{456533} \leq h \leq \frac{400(-25189+976\sqrt{610})}{456533}.$$

We show a HK-system having three linear type centers in three different quadrants, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(50) \quad H = xy(-108 - 108x + 32y + 72x^2 + 37xy + 2y^2).$$



**Figure 8.** HK-systems with three linear type centers in three different quadrants.

The corresponding HK-system has four equilibria outside the axes of coordinates, namely,

$$(51) \quad (1, 1), \quad \left(-\frac{1}{4}, -9\right), \quad \left(\frac{2}{793}(-606 \pm 275\sqrt{3}), \frac{27}{793}(239 \mp 115\sqrt{3})\right).$$

The determinants of their linear parts are

$$\frac{2425}{288}, \quad \frac{3925}{1458} \quad \text{and} \quad -\frac{47375215450}{498677257} \pm \frac{27501846750\sqrt{3}}{498677257},$$

respectively. Therefore the three first equilibria of (51) are linear type centers and the fourth one is a hyperbolic saddle. In Figure 8 we show the phase portrait of the HK-system defined by the Hamiltonian (50). The real algebraic curve

$$xy(-108 - 108x + 32y + 72x^2 + 37xy + 2y^2) = h$$

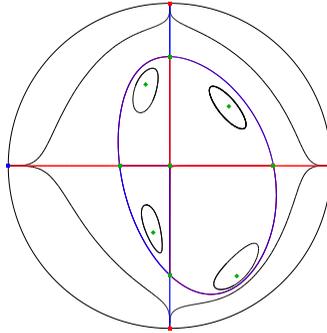
has three families of level ovals in  $\mathbb{R}^2$  for all  $h \in \mathbb{R}$ , in which this real algebraic curve has two families of level ovals if and only if  $-73 < h < 0$ , and it has a family of level ovals if and only if either

$$-\frac{4293}{16} < h \leq -73 \quad \text{or} \quad 0 < h < \frac{5832(-241463148 + 140160125\sqrt{3})}{498677257}.$$

This completes the proof of statement (v) of Theorem 2.  $\square$

*Proof of statement (vi) of Theorem 2.* From the proof of the nonexistence of two nilpotent centers, we can see that the four centers must be linear if a HK-system (3) has four centers. And we claim that this system has exactly four families of periodic orbits, one of which contains a linear type center in its interior.

In fact, if a HK-system (3) has four centers, then this system has four families of periodic orbits, one of which contains a linear type center in its interior by the definition of center.



**Figure 9.** HK-systems with four linear type centers in four different quadrants.

On the other hand, if a HK-system (3) has four centers, then all other equilibria of this system should be on the  $x$ -axis or the  $y$ -axis, i.e., this system cannot have the other equilibria in the interior of quadrants of  $\mathbb{R}^2$  by Theorem 6 and statement (i) of Theorem 2. Note that the  $x$ -axis and  $y$ -axis are invariant for the flow of a HK-system (3). The periodic orbit of a HK-system (3) cannot intersect with the  $x$ -axis or the  $y$ -axis. Hence, there does not exist a periodic orbit of this system such that the periodic orbit contains more than two equilibria in its interior due to the sum of indices of these equilibria being one and the index of a linear type center being one. Therefore a HK-system having four linear type centers has exactly four families of periodic orbits, one of which contains a linear type center in its interior. This gives that the level set of the corresponding Hamiltonian

$$xy(a + bx + cy + dx^2 + exy + fy^2) = h$$

has four families of level ovals in  $\mathbb{R}^2$ .

We now face an interesting problem: what are the configurations of four centers with respect to the coordinate axes if a HK-system (3) has four centers? In the following, we provide a HK-system having four linear type centers in four different quadrants. Let the HK-system be defined by the Hamiltonian

$$(52) \quad H = xy(-12 - 12x + 8x^2 + 5xy + 2y^2).$$

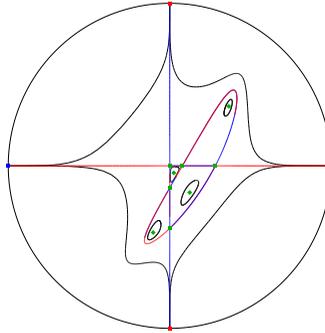
The associated HK-system has four equilibria outside the axes of coordinates, namely,

$$(53) \quad (1, 1), \quad \left(-\frac{1}{4}, -1\right), \quad \left(\frac{2}{13}(6 \pm 5\sqrt{3}), \frac{1}{13}(-15 \mp 19\sqrt{3})\right),$$

with one equilibrium in each quadrant. The determinants of their linear parts are

$$\frac{23}{6}, \quad \frac{83}{96} \quad \text{and} \quad \frac{2(30191 \pm 16525\sqrt{3})}{2197},$$

respectively. All these determinants are positive. Hence the four equilibria of (53)



**Figure 10.** HK-systems with four linear type centers; two are in the same quadrant, and each one of the other two centers is located in a different adjacent quadrant.

are linear type centers. In Figure 9 there is the phase portrait of the HK-system defined by the Hamiltonian (52). The real algebraic curve

$$xy(-12 - 12x + 8x^2 + 5xy + 2y^2) = h$$

has four families of level ovals in  $\mathbb{R}^2$  for all  $h \in \mathbb{R}$ , in which this real algebraic curve has two families of level ovals if and only if either

$$-\frac{21}{16} < h < 0 \quad \text{or} \quad 0 < h < \frac{72(1660 - 911\sqrt{3})}{2197},$$

and it has a family of level ovals if and only if either

$$-9 < h \leq -\frac{21}{16} \quad \text{or} \quad \frac{72(1660 - 911\sqrt{3})}{2197} \leq h < \frac{72(1660 + 911\sqrt{3})}{2197}.$$

We show a HK-system having four linear type centers, two in the same quadrant, and each one of the other two centers located in a different adjacent quadrant. Let the HK-system be defined by the Hamiltonian

$$(54) \quad H = xy(57 - 456x + 270y + 592x^2 - 710xy + 229y^2).$$

The associated HK-system has four equilibria outside the axes of coordinates, namely,

$$(55) \quad (1, 1), \quad \left(-\frac{1}{4}, -1\right), \quad \left(\frac{72732 \mp 5\sqrt{95939265}}{470788}, \frac{19(-16455 \pm \sqrt{95939265})}{1456898}\right).$$

The determinants of their linear parts are

$$\frac{5159}{8664}, \quad \frac{367}{4332} \quad \text{and} \quad \frac{5(109357753909491323 \mp 8468817930191\sqrt{95939265})}{36973060615773331024},$$

respectively. All these determinants are positive. Therefore the four equilibria

of (55) are linear type centers. In Figure 10 there is the phase portrait of the HK-system defined by the Hamiltonian (54). The real algebraic curve

$$xy(57 - 456x + 270y + 592x^2 - 710xy + 229y^2) = h$$

has four families of level ovals in  $\mathbb{R}^2$  for all  $h \in \mathbb{R}$ , in which this real algebraic curve has three families of level ovals if and only if

$$\frac{16245(2340735907022315 - 256172312597\sqrt{95939265})}{36973060615773331024} < h < 0,$$

it has two families of level ovals if and only if

$$-\frac{21}{8} < h \leq \frac{16245(2340735907022315 - 256172312597\sqrt{95939265})}{36973060615773331024},$$

and it has a family of level ovals if and only if either  $-18 < h \leq -\frac{21}{8}$  or

$$0 < h < \frac{16245(2340735907022315 + 256172312597\sqrt{95939265})}{36973060615773331024}. \quad \square$$

From our results remains the following open question.

**Open question.** Are the two configurations of four centers with respect to the coordinate axes described in statement (vi) of Theorem 2 the only ones which are realizable with four centers by the HK-systems?

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