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η-PERIODIC MOTIVIC SPHERE SPECTRUM

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We compute the homotopy groups of the η-periodic motivic sphere spectrum over a field k of finite cohomological dimension with characteristic not 2 and in which −1 is a sum of four squares. We also study the general characteristic 0 case and show that the α_1-periodic slice spectral sequence over $\mathbb{Q}$ determines the α_1-periodic slice spectral sequence over all extensions k/$\mathbb{Q}$. This leads to a speculation on the role of a "connective Witt-theoretic J-spectrum" in η-periodic motivic homotopy theory.

1. Introduction

The motivic sphere spectrum S is the unit object in the tensor triangulated stable homotopy category of motivic spectra $(\mathrm{SH}^{\mathbb{A}^1}(k), \wedge)$ over a field k. In this category, both the simplicial circle S^1 and the punctured affine line $\mathbb{A}^1 \smallsetminus 0$ are $\wedge$-invertible, so it is crucial that we understand the bigraded homotopy groups $\pi_\star S := \bigoplus_{m,n \in \mathbb{Z}} \pi_{m+n\alpha} S$ where $\pi_{m+n\alpha} S := [(S^1)^{\wedge m} \wedge (\mathbb{A}^1 \smallsetminus 0)^{\wedge n}, S]_{\mathrm{SH}^{\mathbb{A}^1}(k)}$. See the introduction to [Ormsby et al. 2018] for a more complete discussion of the importance of this ring.

The motivic Hopf map $\eta \in \pi_\alpha S$ which is represented by the canonical $\mathbb{G}_m$-torsor $\mathbb{A}^2 \smallsetminus 0 \to \mathbb{P}^1$ plays an especially important role in $\pi_\star S$. This class is nonnilpotent over all fields [Morel 2004] and thus represents a first example of exotic behavior in $\pi_\star S$, differentiating it from the classical stable stems. (Recall that $\eta^4 = 0$ classically, and that the Nishida nilpotence theorem [1973] tells us that all classes of nonzero degree in the classical stable stems are nilpotent.) Let

$$\eta^{-1}\mathrm{S} := \mathrm{hocolim}(\mathrm{S} \xrightarrow{\eta} \Sigma^{-\alpha}\mathrm{S} \xrightarrow{\eta} \Sigma^{-2\alpha}\mathrm{S} \xrightarrow{\eta} \Sigma^{-3\alpha}\mathrm{S} \xrightarrow{\eta} \cdots)$$

denote the η-periodic sphere spectrum.[1] We have $\pi_\star \eta^{-1}\mathrm{S} \cong \eta^{-1}\pi_\star \mathrm{S}$ (where the latter term represents the localization of the ring $\pi_\star \mathrm{S}$ at the multiplicative set $\{\eta, \eta^2, \ldots\}$),

[1]Other authors have referred to this object as the η-local or η-inverted sphere. We have chosen our terminology to match the language of classical v_n-periodic homotopy theory, which seems appropriate given the emerging role of η in motivic nilpotence and periodicity [Andrews 2018; Gheorghe 2017].

so inverting η annihilates $\Gamma_\eta := \{x \in \pi_\star S \mid x\eta^N = 0 \text{ for some } N\}$ and induces an injection $\pi_\star S / \Gamma_\eta \to \pi_\star \eta^{-1}S$.

A number of authors have studied $\pi_\star \eta^{-1}S$ over particular fields, including M. Andrews and H. Miller [2017] over $\mathbb{C}$, B. Guillou and D. Isaksen [2015; 2016] over $\mathbb{C}$ and $\mathbb{R}$, and G. Wilson [2018] over finite fields, local fields, and $\mathbb{Q}$. Over $\mathbb{C}$, $\pi_{m+n\alpha}S \cong W(\mathbb{C}) \cong \mathbb{Z}/2$ for nonnegative m congruent to 0 or 3 mod 4, whereas more complicated "image of J"-style patterns occur in $\pi_\star \eta^{-1}S_2^\wedge$ (the bigraded homotopy groups of the η-periodic 2-complete sphere) over $\mathbb{R}$ and $\mathbb{Q}$. These authors work with either the motivic Adams–Novikov or motivic Adams spectral sequence in order to produce their results. In addition to these results, Röndigs [2018] has shown that $\pi_1 \eta^{-1}S = \pi_2 \eta^{-1}S = 0$ over all fields of characteristic different from 2.

In this note, we use the α_1-periodic slice spectral sequence to completely determine $\pi_\star \eta^{-1}S$ over finite-cohomological dimension fields with characteristic different from 2 in which -1 is a sum of four squares.[2] Let $W(k)$ denote the Witt ring of quadratic forms over k modulo the hyperbolic plane.

Main Theorem (see Theorem 4.8). *Let k be a field of finite cohomological dimension with characteristic not 2. If -1 is a sum of four squares in k, then*

$$\pi_\star \eta^{-1}S \cong W(k)[\eta^{\pm 1}, \sigma, \mu]/(\sigma^2),$$

where $|\sigma| = 3 + 4\alpha$ and $|\mu| = 4 + 5\alpha$. In particular, the bigraded homotopy groups of $\eta^{-1}S$ are

$$\pi_{m+n\alpha}\eta^{-1}S \cong \begin{cases} W(k) & \text{if } m \geq 0 \text{ and } m \equiv 0 \text{ or } 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

In Corollary 4.9 (see also [Ormsby et al. 2018, Theorem 5.5]), we see that for fields satisfying the same hypotheses,

$$\pi_{m+n\alpha}S \cong \pi_{m+n\alpha}\eta^{-1}S$$

for $2n \geq \max\{3m+5, 4m\}$, so we have also computed a bi-infinite range of homotopy groups of the motivic sphere spectrum.

The picture is less clear for fields in which -1 is not a sum of four squares, but we are able to produce some partial results in Section 4. Let $\alpha_1^{-1}\mathscr{S}(k)$ denote the α_1-periodic slice spectral sequence over k. In Theorem 4.5, we show that $\alpha_1^{-1}\mathscr{S}(\mathbb{Q})$ determines $\alpha_1^{-1}\mathscr{S}(k)$ for any field extension $k/\mathbb{Q}$. This leads to a conjecture on the differentials in $\alpha_1^{-1}\mathscr{S}(k)$ and some speculations regarding the structure of $\pi_\star \eta^{-1}S$ in general.

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[2]The smallest r such that -1 is a sum of r squares in k is called the *level* of k and is often denoted $s(k)$. (The s is for *Stufe*). By a theorem of Pfister [1965], $s(k)$ is always a power of 2. See [Lam 2005, Examples XI.2.4] for examples of fields of various levels.

2. The α_1-periodic slice spectral sequence

In this section, we set up the α_1-periodic slice spectral sequence and discuss its convergence properties and first two pages over a general field of characteristic different from 2.

We refer to [Röndigs et al. 2019] for the setup of the slice spectral sequence, and only briefly recall some pertinent facts here. The sphere spectrum S is effective and thus has a slice tower of the form

$$\cdots \longrightarrow f_3 S \longrightarrow f_2 S \longrightarrow f_1 S \longrightarrow f_0 S = S$$
$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
$$s_3 S \qquad \quad s_2 S \qquad \quad s_1 S \qquad \quad s_0 S$$

where $f_q S$ is the q-th effective cover of S and $s_q S$ is the q-th slice of S. Associated with this tower is the slice spectral sequence $\mathscr{S}$ with E_1-page $\mathscr{S}_1^{q,m+n\alpha} = \pi_{m+n\alpha} s_q S$. (If we need to refer to the base field, then we will denote this spectral sequence by $\mathscr{S}(k)$.) The differentials take the form $d_r : \mathscr{S}_r^{q,m+n\alpha} \to \mathscr{S}_r^{q+r,m-1+n\alpha}$.

The following theorem states the basic convergence properties of $\mathscr{S}$; it is a concatenation of [Röndigs et al. 2019, Theorem 3.50] and [Ormsby et al. 2018, Theorem 1.3]. Recall that the η-complete sphere spectrum is $S_\eta^\wedge := \text{holim } S/\eta^n$ where S/η^n is the cofiber of $\eta^n : \Sigma^{n\alpha} S \to S$.

Theorem 2.1. *The slice spectral sequence for S conditionally converges to $\pi_\star S_\eta^\wedge$. Moreover, if* $\text{cd } k < \infty$, *then* $S_\eta^\wedge \simeq S$ *and the slice spectral sequence for S converges conditionally to $\pi_\star S$.*

We also have control over $\mathscr{S}_1$ via the following slice computation. We let $\text{Ext}_{MU_* MU}^{s,t}$ denote the cohomology of the MU-Hopf algebroid in cohomological degree s and internal degree t. (Recall that this is the E_2-page of the Novikov, i.e., MU-Adams, spectral sequence from topology.)

Theorem 2.2 [Röndigs et al. 2019, Theorem 2.2]. *The q-th slice of the motivic sphere spectrum is*

$$s_q S \simeq \bigvee_{s \geq 0} \Sigma^{q-s+q\alpha} H \, \text{Ext}_{MU_* MU}^{s,2q}$$

at least after inverting the exponential characteristic of the base field.

We refer to [Ravenel 1986] for basic facts about $\text{Ext}_{MU_* MU}$, and we use its naming conventions for elements. Importantly, there is a single nonzero class $\alpha_1 \in \text{Ext}_{MU_* MU}^{1,2}$ that represents η. Multiplication by α_1 induces a map of spectral sequences $\alpha_1 : \mathscr{S}^{q,m+n\alpha} \to \mathscr{S}^{q+1,m+(n+1)\alpha}$. Taking the colimit of the tower given by iterating this map produces the α_1-periodic slice spectral sequence, $\alpha_1^{-1} \mathscr{S}$. We will analyze the target and convergence properties of $\alpha_1^{-1} \mathscr{S}$ momentarily, but it

certainly appears that this construction ought to say something about $\eta^{-1}S$ or a related object.

There is another obvious spectral sequence we could consider, namely the slice spectral sequence for $\eta^{-1}S$, but it turns out that the two spectral sequences are the same. For a motivic spectrum X, recall from [Röndigs et al. 2019, Definition 3.1] that $sc(X)$ is the *slice completion* of X.

Theorem 2.3. *The slice spectral sequence for $\eta^{-1}S$ and $\alpha_1^{-1}\mathscr{S}$ are isomorphic as spectral sequences. They both have first page additively isomorphic to*

$$\pi_\star H\mathbb{F}_2[\alpha_1^{\pm 1}, \alpha_3, \alpha_4]/(\alpha_4^2)$$

and conditionally converge to $\pi_\star sc(\eta^{-1}S)$ in the sense of [Boardman 1999]. If $cd\, k < \infty$, then $sc(\eta^{-1}S) \simeq \eta^{-1}S$ and the isomorphic spectral sequences conditionally converge to $\pi_\star \eta^{-1}S$.

Remark 2.4. In Corollary 2.9 and Section 3, we will see that convergence is in fact strong if k has odd characteristic or $cd_2\, k < \infty$. In Section 4, we will see that convergence over characteristic 0 fields to $\pi_\star sc(\eta^{-1}S)$ is strong.

Remark 2.5. As a ring object, $s_* \eta^{-1}S$ is not an $H\mathbb{F}_2$-algebra [Röndigs et al. 2019, Remark 2.33], and our identification of $\alpha_1^{-1}\mathscr{S}_1$ in Theorem 2.3 is not multiplicative. By a bidegree argument and the general properties of slice multiplicativity given in [Röndigs et al. 2019, Section 2.4], the multiplication on $\alpha_1^{-1}\mathscr{S}_1$ agrees with the "naive" multiplication up to addition of some terms involving Sq^1. Our determination of $\alpha_1^{-1}\mathscr{S}_2$ does not depend on the precise multiplicative structure, and we will see in Theorem 2.6 that the multiplication on $\alpha_1^{-1}\mathscr{S}_2$ is fairly simple.

Proof of Theorem 2.3. Let E denote the slice spectral sequence for $\eta^{-1}S$. Then

$$E_1 \cong \pi_\star H\mathbb{F}_2[\alpha_1^{\pm 1}, \alpha_3, \alpha_4]/(\alpha_4^2)$$

by [Röndigs et al. 2019, Theorem 2.35].[3] In particular, the canonical map $\mathscr{S} \to E$ takes α_1 to a unit and hence induces a map $\alpha_1^{-1}\mathscr{S} \to E$. By [Andrews and Miller 2017, Corollary 6.2.3], $\alpha_1^{-1} \mathrm{Ext}_{MU_* MU} \cong \mathbb{F}_2[\alpha_1^{\pm 1}, \alpha_3, \alpha_4]/(\alpha_4^2)$. Given this result and the form of $\mathscr{S}_1$ in Theorem 2.2, we conclude that $\alpha_1^{-1}\mathscr{S}_1 \to E_1$ is an isomorphism, and it follows that $\alpha_1^{-1}\mathscr{S} \cong E$.

The first convergence statement in Theorem 2.1 is formal given the construction of the slice spectral sequence (see [Röndigs et al. 2019, §3.1]). For the second convergence statement, Theorem 2.2 tells us that $sc(S) \simeq S$ when $cd\, k < \infty$. Given the conditional convergence conditions of [Boardman 1999, Definition 5.10], our result follows as long as the sequential colimit that inverts α_1 commutes with

---

[3]Although Theorem 2.2 holds after inverting the exponential characteristic, the slices $s_* \eta^{-1}S$ are known without inverting the exponential characteristic. The reason is that if k is a field of odd characteristic p, multiplication with p is an isomorphism on the Witt ring of k, and hence on $\eta^{-1}S$.

the limit defining slice completion. Our assumption on cohomological dimension implies a vanishing line parallel to α_1-multiplication, and hence the limit in question is finite and commutes with sequential colimits. □

This leads us to the main theorem of this section, a determination of the first slice differentials and $\alpha_1^{-1}\mathscr{S}_2$:

Theorem 2.6. *Over any field* k *of characteristic different from* 2, *the first slice differential for* $\eta^{-1}S$ *is represented by the* $(4, 4)$-*periodic matrix*

$$
\begin{pmatrix}
\mathrm{Sq}^2 & 0 & \tau & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathrm{Sq}^3\mathrm{Sq}^1 & 0 & \mathrm{Sq}^2 + \rho\mathrm{Sq}^1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \mathrm{Sq}^2 & 0 & \tau & 0 & \cdots \\
0 & 0 & \mathrm{Sq}^3\mathrm{Sq}^1 & \mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^3 & \mathrm{Sq}^2 & \rho & \tau & \cdots \\
0 & 0 & 0 & \mathrm{Sq}^3\mathrm{Sq}^1 & 0 & \mathrm{Sq}^2 + \rho\mathrm{Sq}^1 & 0 & \cdots \\
0 & 0 & 0 & 0 & \mathrm{Sq}^3\mathrm{Sq}^1 & \mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^3 & \mathrm{Sq}^2 + \rho\mathrm{Sq}^1 & \cdots \\
0 & 0 & 0 & 0 & 0 & \mathrm{Sq}^3\mathrm{Sq}^1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Sq}^3\mathrm{Sq}^1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

Here the i-th column, $i \geq 0$, gives the first slice differential restricted to the summand $\Sigma^{i+q\alpha}H\mathbb{F}_2$ of $s_q\eta^{-1}S$ (the summand is 0 if $i = 1$). The j-th row, $j \geq 0$, describes the incoming first slice differential for the summand $\Sigma^{j+1+(q+1)\alpha}H\mathbb{F}_2$ of $\Sigma s_{q+1}\eta^{-1}S$ (the summand is 0 if $j = 1$).

This results in an isomorphism of k_*^M-algebras

$$
\alpha_1^{-1}\mathscr{S}_2 \cong k_*^M[\alpha_1^{\pm 1}, \alpha_4, \alpha_5]/(\alpha_4^2),
$$

where $|\alpha_4| = (4, 3 + 4\alpha)$ and $\alpha_5 = \alpha_3^2\alpha_1^{-1}$ has degree $(5, 4 + 5\alpha)$.

Remark 2.7. The factor of α_1^{-1} in the definition of α_5 is not strictly necessary, but is there so that $\bar{\alpha}_5 \mapsto \alpha_5$ under the localization map $\mathscr{S} \to \alpha_1^{-1}\mathscr{S}$.

Remark 2.8. The determination of the first slice differential for $\eta^{-1}S$ complements the occurrences of multiplications with τ, which were used in [Ormsby et al. 2018, page 11] to deduce vanishing columns in the Andrews–Miller range of the unlocalized slice spectral sequence.

The pattern of differentials indicated by Theorem 2.6 is represented graphically in Figure 1. The form of $\alpha_1^{-1}\mathscr{S}_2$ also implies an important convergence result, which we state presently.

Corollary 2.9. *If* $\mathrm{cd}_2(k) = r < \infty$, *then* $\alpha_1^{-1}\mathscr{S}(k)$ *collapses at its $(r+1)$-th page and converges strongly to* $\pi_\star(\eta^{-1}S)$.
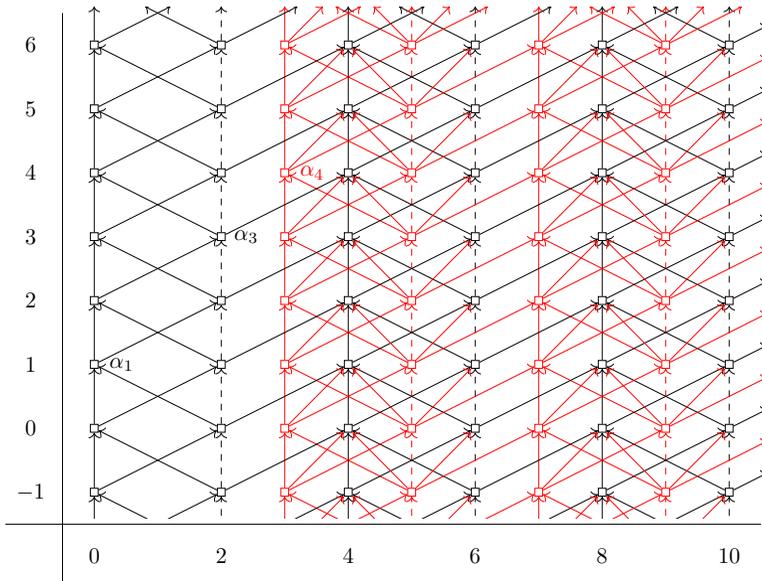
Figure 1. The first page of the α_1-periodic slice spectral sequence with its differentials. A $\square$ in position (m, n) represents a copy of $\pi_\star \Sigma^{m+n\alpha} H\mathbb{F}_2 \cong \Sigma^{m+n\alpha} k_*^M[\tau]$ in slice degree n. The black portions of the diagram are detected by the unit map $\eta^{-1}S \to$ kw, and the red portions are induced by $\sigma_\infty : \Sigma^3$kw $\to \eta^{-1}S$. Arrows with slope $-\frac{1}{2}$ represent τ, arrows with slope -1 represent ρ, solid vertical arrows represent Sq^2, dashed vertical arrows represent $\mathsf{Sq}^2 + \rho\mathsf{Sq}^1$, arrows with slope 1 represent $\mathsf{Sq}^2\mathsf{Sq}^1 + \mathsf{Sq}^3$, and arrows with slope $\frac{1}{2}$ represent $\mathsf{Sq}^3\mathsf{Sq}^1$.

Proof. The form of $\alpha_1^{-1} \mathscr{S}_2$ (which is presented graphically in Figure 2) and the fact that $k_{>r}^M(\mathsf{k}) = 0$ imply that $d_{>r} = 0$. This collapse along with the conditional convergence of Theorem 2.1 imply the strong convergence portion of the corollary. $\square$

Our proof of Theorem 2.6 requires a fascinating detour through connective Witt K-theory. Let KQ denote the motivic spectrum representing Hermitian K-theory,[4] let KW $= \eta^{-1}$KQ denote the η-periodization of KQ, which is the motivic spectrum representing Balmer's higher Witt groups, and let kw denote the connective cover of KW (in the sense of Morel's homotopy t-structure).

The slices of kw, the d_1-differentials of its slice spectral sequence, and the effect of s_* on the unit map $\eta^{-1}S \to$ kw are known.

Theorem 2.10 [Ananyevskiy et al. 2020, Theorem 19]. *Suppose the base scheme S is a scheme over* $\mathrm{Spec}(\mathbb{Z}[1/2])$. *Then*

$$s_*\mathrm{kw} \simeq H\mathbb{Z}[\eta^{\pm 1}, \sqrt{\beta}]/(2\eta, 2\sqrt{\beta}, \eta^2 \xrightarrow{\mathsf{Sq}^1} \sqrt{\beta}),$$

---

[4]The Q is for Quadratic.

where $|\eta| = \alpha$, $|\sqrt{\beta}| = 2 + 2\alpha$, *and the first slice differential takes the form*

$$
\begin{pmatrix}
\mathrm{Sq}^2 & 0 & \tau & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathrm{Sq}^3\mathrm{Sq}^1 & 0 & \mathrm{Sq}^2 + \rho\mathrm{Sq}^1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \mathrm{Sq}^3\mathrm{Sq}^1 & 0 & \mathrm{Sq}^2 & 0 & \tau & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \mathrm{Sq}^3\mathrm{Sq}^1 & 0 & \mathrm{Sq}^2 + \rho\mathrm{Sq}^1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Sq}^3\mathrm{Sq}^1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

(with the same conventions as Theorem 2.6). Moreover, there is a splitting of $s_0\eta^{-1}\mathrm{S}$ such that the unit $\eta^{-1}\mathrm{S} \to \mathrm{kw}$ induces an inclusion on every even summand, and Sq^1 on every odd summand.

Proof. The description of the slices, as well as their multiplicative structure, is given in [Ananyevskiy et al. 2020, Theorem 19]. The behavior of the unit map follows from [Röndigs et al. 2019, Lemmas 2.28 and 2.29]. ☐

Figure 2. The weight n portion of $\alpha_1^{-1}\mathscr{S}_2$ with simplicial degree on the horizontal axis and slice degree on the vertical axis. This page is 0 below slice degree n and the k_*^M-towers extend upwards. Differentials respect weight, and d_r has degree $(-1, r)$ in this representation. If the cohomological dimension of k is at most 1, then this page is concentrated in slice degrees n and $n + 1$ and there are no nonzero targets for d_r, $r \geq 2$.

Note that s_*kw and the pattern of d_1 differentials are precisely the black portion of Figure 1. The remaining portion of $s_*\eta^{-1}S$ (the red part of Figure 1) is handled by the following theorem. Following the standard convention, we write Π_n for $\bigoplus_{m\in\mathbb{Z}} \pi_{n+m\alpha}$.

Theorem 2.11. *Over* $k = \mathbb{C}$, *there is a unique homotopy class* $\sigma_\infty : \Sigma^3 kw \to \eta^{-1}S$ *inducing an isomorphism on* Π_3. *This map induces* $(1, Sq^1)$ *on every summand of a slice.*

Proof. Fix $k = \mathbb{C}$. Observe first that $kw = \eta^{-1}kq$ is a cellular spectrum, where kq is the connective cover of KQ. In fact, kq is slice-finitary by [Ananyevskiy et al. 2020, Theorem 17]; hence its slice completion sc(kq) is cellular by [Röndigs et al. 2019, Proposition 3.45]. It follows that the slice completion sc(kw) is cellular also. On the level of homotopy sheaves, [Levine 2011, Theorem 1] implies that the slice filtration on kw induces the fundamental ideal filtration on Witt sheaves. The latter is finite over the complex numbers, whence sc(kw) $\simeq$ kw is cellular. Recall from [Röndigs 2018, Section 4] that there is a cell presentation of kw over $\mathbb{C}$ of the following form. Namely, there is a sequence of cellular motivic spectra factoring the unit of kw as

$$\eta^{-1}S = D_1 \xrightarrow{i_1} D_2 \xrightarrow{i_2} \cdots \to D_n \xrightarrow{i_n} \cdots \to kw$$

such that for every n the map $D_n \to kw$ is $(4n-1)$-connective and the composition $\eta^{-1}S \to D_n \to kw$ induces isomorphisms on Π_{4k}. For every $n \geq 1$, there is a unique nontrivial class $a_n : \Sigma^{4n-1}\eta^{-1}S \to D_n$ in $\pi_{4n-1}D_n \cong \pi_{4n-1}\eta^{-1}S$ such that

$$\Sigma^{4n-1}\eta^{-1}S \xrightarrow{a_n} D_n \xrightarrow{i_n} D_{n+1} \xrightarrow{c_n} \Sigma^{4n}\eta^{-1}S$$

is a homotopy cofiber sequence with c_n inducing an isomorphism on Π_{4k+3} whenever $k \geq n$. Taking the colimit as $n \to \infty$ gives a cell presentation of kw.

We now construct σ_∞. Consider the map $\sigma_1 = \eta^{-4}\sigma : \Sigma^3\eta^{-1}S = \Sigma^3 D_1 \to \eta^{-1}S$. Assume for induction that for some $n \geq 1$ a map $\sigma_n : \Sigma^3 D_n \to \eta^{-1}S$ is given such that

(1) $\sigma_n i_{n-1} = \sigma_{n-1}$ and

(2) $[\Sigma^4 a_{n-1}, \eta^{-1}S] : [\Sigma^4 D_{n-1}, \eta^{-1}S] \to [\Sigma^{4n-1}\eta^{-1}S, \eta^{-1}S]$ is an isomorphism.

Then the cofiber sequence above induces a long exact sequence

$$[\Sigma^{4n+2}\eta^{-1}S, \eta^{-1}S] \xleftarrow{a_n} [\Sigma^3 D_n, \eta^{-1}S] \xleftarrow{i_n} [\Sigma_3 D_{n+1}, \eta^{-1}S]$$
$$\xleftarrow{c_n} [\Sigma^{4n+3}\eta^{-1}S, \eta^{-1}S] \xleftarrow{a_n} [\Sigma^4 D_n, \eta^{-1}S] \leftarrow \cdots.$$

The Andrews–Miller theorem [2017] on $\pi_\star \eta^{-1}S$ implies

$$0 = [\Sigma^{4n+2}S, \eta^{-1}S] = [\Sigma^{4n+2}\eta^{-1}S, \eta^{-1}S],$$

showing that σ_n lifts to a map σ_{n+1} such that $\sigma_{n+1}i_n = \sigma_n$.

Now note that assumption (2) implies that

$$[\Sigma^4 c_{n-1}, \eta^{-1}S] : [\Sigma^{4n}\eta^{-1}S, \eta^{-1}S] \to [\Sigma^4 D_n, \eta^{-1}S]$$

is surjective; furthermore, the composition

$$[\Sigma^{4n}\eta^{-1}S, \eta^{-1}S] \xrightarrow{[\Sigma^4 c_{n-1}, \eta^{-1}S]} [\Sigma^4 D_n, \eta^{-1}S] \xrightarrow{[\Sigma^4 a_n, \eta^{-1}S]} [\Sigma^{4n+3}\eta^{-1}S, \eta^{-1}S]$$

is the map sending $\eta^{-5n}\mu^n$ to $\eta^{-5n-4}\sigma\mu^n$, and hence an isomorphism. It follows that the map $[\Sigma^4 a_n, \eta^{-1}S]$ is an isomorphism, as desired. (In fact, we also get that there is a *unique* $\sigma_{n+1} : \Sigma^3 D_{n+1} \to \eta^{-1}S$ such that $\sigma_{n+1}i_n = \sigma_n$.)

Induction and the universal property of colimits now produce a map

$$\sigma_\infty : \Sigma^3 D_\infty \simeq \Sigma^3 kw \to \eta^{-1}S$$

sending $1 \in \pi_3\Sigma^3 kw$ to $\eta^{-4}\sigma \in \pi_3\eta^{-1}S$. The uniqueness of σ_∞ follows from the Milnor exact sequence and the vanishing of $\lim_n^1[\Sigma^4 D_n, \eta^{-1}S]$ (every group $[\Sigma^4 D_n, \eta^{-1}S]$ being finite of order 2).

Since σ_∞ is a map of $\eta^{-1}S$-modules, it induces isomorphisms on Π_{4m+3} for every integer m. The statement on slices follows from the behavior of the unit map $\eta^{-1}S \to kw$ on slices given in Theorem 2.10. $\qquad\square$

Proof of Theorem 2.6. As $\eta^{-1}S$ and $s_*\eta^{-1}S$ are invariant under base change, it suffices to determine the first slice differential d_1 over $\mathbb{Z}[1/2]$. On a summand $\Sigma^n H\mathbb{F}_2$, it is of the form

$$(x_n\tau, a_n\tau Sq^1 + \beta_n, b_n Sq^2 + \gamma_n, c_n Sq^2 Sq^1 + d_n Sq^3, e_n Sq^3 Sq^1)$$

with $x_n, a_n, b_n, c_n, d_n, e_n$ elements in $\mathbb{Z}/2$, and β_n, γ_n square classes of units in $\mathbb{Z}[1/2]$. The behavior of the unit map $\eta^{-1}S \to kw$ on slices from Theorem 2.10 provides immediate restrictions:

$$
\begin{aligned}
x_{4n} &= 0, & a_{4n}\rho + \beta_{4n} &= \gamma_{4n}, & b_{4n} &= 1, & e_{4n} &= 1 + c_{4n}, \\
x_{4n+1} + a_{4n+1} &= 1, & (a_{4n+1}+1)\rho &= \beta_{4n+1}, & c_{4n+1} &= 1, & b_{4n+1} &= d_{4n+1}, \\
x_{4n+2} &= 1, & (a_{4n+2}+1)\rho + \beta_{4n+2} &= \gamma_{4n+2}, & b_{4n+2} &= 1, & e_{4n+2} &= 1 + c_{4n+2}, \\
x_{4n+3} &= a_{4n+3}, & a_{4n+3}\rho &= \beta_{4n+3}, & c_{4n+3} &= 1, & b_{4n+3} &= d_{4n+3}.
\end{aligned}
$$

The map $\sigma_\infty : \Sigma^3 kw_\mathbb{C} \to \eta^{-1}S_\mathbb{C}$ on slices from Theorem 2.11 imposes further restrictions:

$$
\begin{aligned}
x_{4n+1} &= 1, & a_{4n+1} &= 0, & e_{4n+1} &= 1 + d_{4n+2}, \\
b_{4n+1} = c_{4n+1} &= d_{4n+1} = 1, & & & b_{4n+3} = c_{4n+3} &= d_{4n+3} = 1, \\
x_{4n+3} &= 0 = a_{4n+3}, & & & e_{4n+3} &= 1 + d_{4n+4}.
\end{aligned}
$$

Base change and the previous equations then provide the following equations:

$$\beta_{4n+1} = \rho, \quad \gamma_{4n+1} + \beta_{4n+2} = \rho, \quad \beta_{4n+3} = \gamma_{4n+3} = \gamma_{4n} = \beta_{4n} = \beta_{4n+2} = 0.$$

Since the composition $d_1 \circ d_1$ is equal to 0, Adem relations imply further coefficients. Considering the component

$$d_1^2 : \Sigma^{4n+2} H\mathbb{F}_2 \to \Sigma^{4n+1+2\alpha} H\mathbb{F}_2$$

implies that $a_{4n+2} = a_{4n} = 0$. Considering the component

$$d_1^2 : \Sigma^{4n} H\mathbb{F}_2 \to \Sigma^{4n+2+2\alpha} H\mathbb{F}_2$$

provides that $c_{4n} = d_{4n} = 0$, and hence $e_{4n} = 1 = e_{4n+3}$. The similar component

$$d_1^2 : \Sigma^{4n+2} H\mathbb{F}_2 \to \Sigma^{4n+4+2\alpha} H\mathbb{F}_2$$

gives only $c_{4n+2} = d_{4n+2}$, and hence $e_{4n+1} = e_{4n+2}$. Resorting to $s_* S$ provides the solution $c_{4n+2} = d_{4n+2} = 0$ and $e_{4n+1} = e_{4n+2} = 1$ as follows. Consider the summand $\Sigma^{2n+(2n+1)\alpha} H\mathbb{F}_2$ in $s_{2n+1} S$ generated by α_{2n+1}. The first slice differential maps it via $\mathrm{inc}^2_{y_{2n+2}} \mathrm{Sq}^2 \mathrm{Sq}^1$ to the top degree summand $\Sigma^{2n+1+(2n+2)\alpha} H\mathbb{Z}/(y_{2n+2})$ in $s_{2n+2} S$ by [Röndigs et al. 2019, Lemma 4.1]; here y_{2n+2} is the order of a cyclic group and divisible by four, and $\mathrm{inc}^2_{y_{2n+2}}$ is the map $H\mathbb{F}_2 \to H\mathbb{Z}/(y_{2n+2})$ induced by the canonical inclusion $\mathbb{Z}/(2) \to \mathbb{Z}/(y_{2n+2})$. The map

$$\Sigma^\alpha \Sigma^{2n+1+(2n+2)\alpha} H\mathbb{Z}/(y_{2n+2}) \to \Sigma^{2n+1+(2n+3)\alpha} H\mathbb{F}_2$$

induced by η is the projection $\mathrm{pr}_2^{y_{2n+2}}$. Hence already after one multiplication with η, the degree $2 + \alpha$ part of the first differential is zero on that summand. It follows that $c_{2n} = 0$.

Given the form of the differentials, the additive calculation of $\alpha_1^{-1} \mathscr{S}_2$ is nearly the same as the proof of [Röndigs and Østvær 2016, Theorem 6.3]. The exotic multiplication on $\alpha_1^{-1} \mathscr{S}_1$ mentioned in Remark 2.5 reduces to $k_*^M[\alpha_1^{\pm 1}, \alpha_4, \alpha_5]/(\alpha_4^2)$ in the subquotient $\alpha_1^{-1} \mathscr{S}_2$ since Sq^1 is only potentially nonzero on terms involving an odd power of τ, and there are no τ's in $\alpha_1^{-1} \mathscr{S}_2$. □

3. Computations for fields with odd characteristic or cohomological dimension at most 1

Given the form of $\alpha_1^{-1} \mathscr{S}_2$ and the spectral sequence's convergence properties determined in the previous section, we can now make short work of the following computations.

Proposition 3.1. *If* $\mathrm{cd}_2 \, k \le 1$, *then the* α_1-*periodic slice spectral sequence for* S *collapses with* $\alpha_1^{-1} \mathscr{S}_2 = \alpha_1^{-1} \mathscr{S}_\infty$ *and converges strongly to* $\pi_\star \eta^{-1} S$.

Proof. This is a specialization of Corollary 2.9. □

Theorem 3.2. *If* k *has odd characteristic, then the* α_1-*periodic slice spectral sequence for* S *collapses with* $\alpha_1^{-1}\mathscr{S}_2 = \alpha_1^{-1}\mathscr{S}_\infty$.

Proof. Suppose k has characteristic p and write i for the map $\operatorname{Spec} \mathsf{k} \to \operatorname{Spec} \mathbb{F}_p$. The essentially smooth base change functor i^* induces a map of spectral sequences $i^* : \alpha_1^{-1}\mathscr{S}_2(\mathbb{F}_p) \to \alpha_1^{-1}\mathscr{S}_2(\mathsf{k})$ which is given by the extension of scalars map on k_*^M and the identity on α_i for $i = 1, 4, 5$. Given the form of $\alpha_1^{-1}\mathscr{S}_2(\mathsf{k})$, it suffices to show $d_r^{\mathsf{k}}\alpha_5 = 0$ for all $r \geq 2$, but $d_r^{\mathsf{k}}\alpha_5 = i^* d_r^{\mathbb{F}_p}\alpha_5$, and $d_r^{\mathbb{F}_p}\alpha_5 = 0$ by Proposition 3.1. $\square$

At this point, we know that if k has odd characteristic or if $\operatorname{cd}_2 \mathsf{k} \leq 1$, then the α_1-periodic slice spectral sequence collapses with

$$\alpha_1^{-1}\mathscr{S}_2 = \alpha_1^{-1}\mathscr{S}_\infty \cong k_*^M[\alpha_1^{\pm 1}, \alpha_4, \alpha_5]/\alpha_4^2.$$

Paired with the conditional convergence portion of Theorem 2.3, this implies that the spectral sequence in fact converges strongly to $\pi_\star \eta^{-1}S$. In order to completely determine $\pi_\star \eta^{-1}S$ for such k, we must resolve extension problems and understand the multiplicative structure.

Suppose that $s \equiv 0$ or 3 (mod 4), and consider the short exact sequences

(3-1) $$0 \to f_{q+1}\pi_s \eta^{-1}S \to f_q \pi_s \eta^{-1}S \to k_q^M \to 0$$

obtained from the slice filtration and the determination of $\alpha_1^{-1}\mathscr{S}_\infty$ for a field of odd characteristic. Choose a lift $g_s \in f_0 \pi_s \eta^{-1}S = \pi_s \eta^{-1}S$ of the nontrivial element in k_0^M, compatible with field extensions from the prime field. If $s = 0$, $\pi_0 \eta^{-1}S$ is known to be the Witt ring by Morel's theorem, and g_0 should be chosen as the unit. The slice filtration on $\pi_0 \eta^{-1}S$ coincides with the filtration by powers of the fundamental ideal I, as one deduces for example from [Levine 2011]. The multiplicative structure on the slice filtration then supplies a natural transformation to the sequence (3-1) from the short exact sequence

$$0 \to I^{q+1} \to I^q \to h^{q,q} \to 0$$

solving Milnor's conjecture on quadratic forms [Orlov et al. 2007]. The convergence statement Corollary 2.9 shows that this natural transformation is an isomorphism for fields of finite cohomological dimension. Since the constructions involved commute with filtered colimits of fields, it is thus an isomorphism for any field of odd characteristic. In particular, the slice filtration is Hausdorff by the main result of [Arason and Pfister 1971].

Proposition 3.3. *If* k *has odd characteristic or if* $\operatorname{cd}_2 \mathsf{k} \leq 1$, *then, as a ring,*

$$\pi_\star \operatorname{sc}(\eta^{-1}S) \cong W(\mathsf{k})[\eta^{\pm 1}, \sigma, \mu]/(\sigma^2),$$

where $|\eta| = \alpha$, $|\sigma| = 3 + 4\alpha$, *and* $|\mu| = 4 + 5\alpha$. *If, additionally,* $\operatorname{cd} \mathsf{k} < \infty$,

then $\mathrm{sc}(\eta^{-1}\mathrm{S}) \simeq \eta^{-1}\mathrm{S}$ *and this is a computation of the η-periodic homotopy groups of the motivic sphere spectrum.*

Proof. The additive structure (which is simply a copy of $W(\mathrm{k})$ in nonnegative simplicial degrees congruent to 0 or 3 mod 4) follows from the above filtration considerations. There is no room for hidden extensions, so the result follows. $\square$

4. Characteristic 0 fields

We now consider the α_1-periodic slice spectral sequence over a general field k of characteristic 0. We prove that for any k this spectral sequence converges strongly to $\pi_\star \mathrm{sc}(\eta^{-1}\mathrm{S})$. Moreover, the spectral sequence over $\mathbb{Q}$ completely determines the spectral sequence over k in a manner that we make precise in Theorem 4.5. This allows us to extend the conclusion of Proposition 3.3 to fields with $\mathrm{cd}_2\, \mathrm{k} \le 2$ and to extensions of $\mathbb{Q}(\sqrt{-1})$, resulting in Theorem 4.8. We conclude with a conjectural description of the differentials which we hope will inspire further work on this problem.

The structure of our argument is somewhat surprising. After proving that $d_2^{\mathbb{Q}} = 0$, we are able to put strong restrictions on the form of the differentials which may appear in $\alpha_1^{-1}\mathscr{S}(\mathbb{Q})$. We then employ a theorem of Orlov, Vishik and Voevodsky [Orlov et al. 2007] to show that for arbitrary k/$\mathbb{Q}$, the differentials in $\alpha_1^{-1}\mathscr{S}(\mathrm{k})$ are of the same form. The proscribed form of the differentials guarantees that Boardman's $RE_\infty = 0$, whence strong convergence follows. The primary obstruction to computing the differentials seems to be the lack of a good description of $\mathrm{sc}(\eta^{-1}\mathrm{S})$.

We make some preliminary definitions in order to start our arguments. Recall that $\alpha_1^{-1}\mathscr{S}_1 \cong \pi_\star H\mathbb{F}_2[\alpha_1^{\pm 1}, \alpha_3, \alpha_4]/\alpha_4^2$. For $k \ge 0$, set $\alpha_{2k+1} := \alpha_3^k \alpha_1^{1-k}$, and for $k \ge 2$ set $\alpha_{2k} := \alpha_4 \alpha_3^{k-2} \alpha_1^{2-k}$. These classes are chosen so that $\bar{\alpha}_\ell \mapsto \alpha_\ell$ under the localization map $\mathscr{S}_1 \to \alpha_1^{-1}\mathscr{S}_1$ for all $\ell \ne 2$. Note that as a $\pi_\star H\mathbb{F}_2[\alpha_1^{\pm 1}]$-module, $\alpha_1^{-1}\mathscr{S}_1$ is generated by $1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \dots$. Also note (for the purposes of applying the Leibniz rule) that, up to multiplication by a unit, α_{4k+1} is the square of α_{2k+1}.

Lemma 4.1. *The d_2 differential in $\alpha_1^{-1}\mathscr{S}(\mathbb{Q})$ is trivial.*

Proof. It suffices to prove that $d_2^{\mathbb{Q}}\alpha_5 = 0$. We know that $d_2^{\mathbb{Q}}\alpha_5 \in k_2^M(\mathbb{Q})\{\alpha_1^3\alpha_4\}$. Base change to $\mathbb{Q}_p$ provides a comparison map $\alpha_1^{-1}\mathscr{S}(\mathbb{Q}) \to \alpha_1^{-1}\mathscr{S}(\mathbb{Q}_p)$. Since $\mathrm{cd}_2(\mathbb{Q}_p) = 2$, Corollary 2.9 implies that $\alpha_1^{-1}\mathscr{S}(\mathbb{Q}_p)$ converges strongly to $\pi_\star\eta^{-1}\mathrm{S}$. Furthermore, every class in $k_2^M(\mathbb{Q})$ is detected in some $k_2^M(\mathbb{Q}_p)$.[5] As such, the computations of Wilson [2018] over $\mathbb{Q}_p$ imply that $d_2^{\mathbb{Q}}\alpha_5 = 0$. $\square$

---

[5]Indeed, [Milnor 1970, Lemma A.1] tells us that the map $k_2^M(\mathbb{Q}) \to k_2^M(\mathbb{R}) \oplus \bigoplus_p k_2^M(\mathbb{Q}_p)$ is injective and computed on components by quadratic Hilbert symbols. Hilbert reciprocity then implies that $k_2^M(\mathbb{Q}) \to \bigoplus_p k_2^M(\mathbb{Q}_p)$ is injective as well.

Theorem 4.2. *There is a nondecreasing[6] sequence of extended integers $r_k \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$ for $k \geq 2$ such that if $r_k < \infty$ then $d_{r_k}^{\mathbb{Q}} \alpha_{2^k+1} = \rho^{r_k} \alpha_{2^k} \alpha_1^{r_k+1}$, and if $r_k = \infty$ then α_{2^k+1} is a permanent cycle in $\alpha_1^{-1} \mathscr{S}(\mathbb{Q})$. The rest of the differentials in $\alpha_1^{-1} \mathscr{S}(\mathbb{Q})$ are determined by the Leibniz rule.*

Remark 4.3. The above theorem may be thought of in the following terms. In the weight n α_1-periodic slice spectral sequence, the $4k$-column of $\alpha_1^{-1} \mathscr{S}_2$ is, up to multiplication by some power of the unit α_1, generated by α_{4k+1}, and the $4k-1$-column is generated by α_{4k} in the same sense. These columns are connected by $d_{\nu_2(4k)} = \cdot \rho^{r_{\nu_2(4k)}}$ on k_*^M, where ν_2 is 2-adic valuation.

Proof. By Lemma 4.1 and Theorem 2.6,

$$\alpha_1^{-1} \mathscr{S}_3 \cong k_*^M [\alpha_1^{\pm 1}, \alpha_4, \alpha_5]/\alpha_4^2.$$

If the spectral sequence does not collapse, then the first nonzero differential is necessarily of the form $d_r \alpha_5 = x \alpha_4 \alpha_1^{r+1}$ for some $r \geq 3$ and $x \in k_r^M(\mathbb{Q})$. Since $k_r^M(\mathbb{Q}) = \mathbb{Z}/2\{\rho^r\}$ for $r \geq 3$, we in fact have $d_r \alpha_5 = \rho^r \alpha_4 \alpha_1^{r+1}$. Set r_2 equal to this r. The $\alpha_1^{-1} \mathscr{S}_{r_2+1}$-page then has $k_*^M[\alpha_1^{\pm 1}]/\rho^{r_2}$ in positive stems congruent to 3 mod 8 and $_{\rho^{r_2}} k_*^M[\alpha_1^{\pm 1}]$ in positive stems congruent to 4 mod 8 (where $_x k_*^M = \{y \in k_*^M \mid xy = 0\}$ is the x-torsion in k_*^M); the (r_2+1)-page also continues to have $k_*^M[\alpha_1^{\pm 1}]$ in nonnegative stems congruent to 0 or 7 mod 8, and is 0 otherwise.

The potential targets of the $_{\rho^{r_2}} k_*^M[\alpha_1^{\pm 1}]$ terms are all 0; hence these classes are permanent. Thus the next nonzero differential in the spectral sequence (if one exists) is necessarily of the form $d_{r_3} \alpha_9 = \rho^{r_3} \alpha_8 \alpha_1^{r_3+1}$. The ρ^{r_3}-torsion terms in the (r_3+1)-page are again permanent, and the next differential is of the form

$$d_{r_4} \alpha_{2^4+1} = \rho^{r_4} \alpha_{2^4} \alpha_1^{r_4+1}.$$

Proceeding inductively proves the theorem. $\square$

We now abstract the behavior observed in Theorem 4.2 and show that it is in fact generic.

Definition 4.4. For a given field k, suppose that there is a nondecreasing sequence of extended integers $r_k \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$ for $k \geq 2$ such that the differentials $d_{r_k} \alpha_{2^k+1} = \rho^{r_k} \alpha_{2^k} \alpha_1^{r_k+1}$ and the Leibniz rule determine $\alpha_1^{-1} \mathscr{S}(k)$. In this case, we call $\{r_2, r_3, \ldots\}$ the *profile* of $\alpha_1^{-1} \mathscr{S}(k)$ and say that $\alpha_1^{-1} \mathscr{S}(k)$ is *determined by the profile* $\{r_k\}$.

Theorem 4.5. *Let $\{r_k\}$ denote the profile of $\alpha_1^{-1} \mathscr{S}(\mathbb{Q})$ (guaranteed to exist by Theorem 4.2). Then for any characteristic 0 field k, $\alpha_1^{-1} \mathscr{S}(k)$ is also determined by the profile $\{r_k\}$.*

---

[6]In fact, the sequence is strictly increasing unless it is eventually constant at ∞.

Proof. Consider the map of spectral sequences $i^* : \alpha_1^{-1}\mathscr{S}(\mathbb{Q}) \to \alpha_1^{-1}\mathscr{S}(k)$ induced by essentially smooth base change along $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{Q}$. We have

$$d_{r_k}^k \alpha_{2^k+1} = i^* d_{r_k}^{\mathbb{Q}} \alpha_{2^k+1} = i^* \rho^{r_k} \alpha_{2^k} \alpha_1^{r_k+1} = \rho^{r_k} \alpha_{2^k} \alpha_1^{r_k+1}.$$

It remains to show that $_{\rho^{r_k}} k_*^M[\alpha_1^{\pm 1}]\{\alpha_{2^k+1}\}$ supports no higher differentials. Invoking [Orlov et al. 2007, Theorem 3.3], we see that $_{\rho^{r_k}} k_*^M$ is generated in degree 1 as a k_*^M-module, so it suffices to show that $d_r^k[u]\alpha_{2^k+1} = 0$ for all $r > r_k$ and $u \in k^\times$ such that $[u]\rho^{r_k} = 0 \in k_{r_k+1}^M$. Fix such a u and consider the subextension $k/\mathbb{Q}(u)/\mathbb{Q}$. Let $j : \operatorname{Spec}\mathbb{Q}(u) \to \operatorname{Spec}\mathbb{Q}$ denote the corresponding map with associated map of spectral sequences $j^* : \alpha_1^{-1}\mathscr{S}(\mathbb{Q}(u)) \to \alpha_1^{-1}\mathscr{S}(k)$. Our argument now splits into two cases: u algebraic, and u transcendental.

First suppose that u is algebraic, in which case $\mathbb{Q}(u)$ is a number field. Tate's theorem [Milnor 1970, Theorem A.2] implies that $k_n^M(\mathbb{Q}(u)) = \mathbb{Z}/2\{\rho^n\}$ or 0 for $n \geq 3$ and we have already seen that $d_{r_k}^{\mathbb{Q}(u)} \alpha_{2^k+1} = \rho^{r_k} \alpha_{2^k} \alpha_1^{r_k+1}$. Recall that $r_k \geq 3$, so this differential kills k_*^M classes at and above degree r_k. In particular, for $r > r_k$ the target group for $d_r^{\mathbb{Q}(u)}[u]\alpha_{2^k+1}$ is 0 and hence the differential is 0. Finally, we see that

$$d_r^k[u]\alpha_{2^k+1} = j^* d_r^{\mathbb{Q}(u)}[u]\alpha_{2^k+1} = 0$$

as well, as desired.

Now suppose that u is transcendental, in which case [Milnor 1970, Theorem 2.3] implies that there is a split short exact sequence

$$0 \to k_*^M \mathbb{Q} \to k_*^M \mathbb{Q}(u) \xrightarrow{\oplus \partial_\pi} \bigoplus_\pi k_{*-1}^M \mathbb{Q}[u]/(\pi) \to 0,$$

where π ranges over monic irreducible polynomials in $\mathbb{Q}[u]$ and

$$\partial_\pi : k_*^M \mathbb{Q} \to k_{*-1}^M \mathbb{Q}[u]/(\pi)$$

is the residue map taking $[\pi, u_2, u_3, \ldots, u_n]$ to $[u_2, \ldots, u_n]$. In particular, for $n \geq 4$, $k_n^M \mathbb{Q}(u)$ has $\mathbb{F}_2$-basis consisting of ρ^n and $[\pi]\rho^{n-1}$ for $\pi \in \mathbb{Q}[u]$ monic irreducible. Thus the differential

$$d_{r_k}^{\mathbb{Q}(u)} \alpha_{2^k+1} = \rho^{r_k} \alpha_{2^k} \alpha_1^{r_k+1}$$

kills $k_*^M(\mathbb{Q}(u))[\alpha_1^{\pm 1}]\{\alpha_{2^k}\}$ in Milnor-degree $r_k + 1$ and above. It follows that $d_r^{\mathbb{Q}(u)}[u]\alpha_{2^k+1} = 0$ for $r > r_k$ and the same base change trick as in the previous paragraph implies that $d_r^k[u]\alpha_{2^k+1} = 0$. We conclude that $\alpha_1^{-1}\mathscr{S}(k)$ is determined by the profile $\{r_k\}$. $\square$

Theorem 4.6. *Let* k *be any field of characteristic different from* 2. *Then* $\alpha_1^{-1}\mathscr{S}(k)$ *converges strongly to* $\pi_\star \operatorname{sc}(\eta^{-1}S)$. *If* $\operatorname{cd} k < \infty$, *this target is isomorphic to* $\pi_\star \eta^{-1}S$.

Proof. We have already verified this result for odd characteristic fields and fields with finite cohomological dimension. It remains to check characteristic 0 fields of

arbitrary cohomological dimension. By Theorem 2.3, we have weak convergence to $\mathrm{sc}(\eta^{-1}\mathrm{S})$, so it suffices to check vanishing of Boardman's RE_∞ term (for $E = \alpha_1^{-1}\mathscr{S}(\mathrm{k})$). By [Boardman 1999, Remark after Theorem 7.1], it in turn suffices to check that for each tridegree $(s, m + n\alpha)$ there are at most finitely many nonzero differentials $d_r : \alpha_1^{-1}\mathscr{S}_r^{s,m+n\alpha} \to \alpha_1^{-1}\mathscr{S}_r^{s+r,m-1+n\alpha}$. By Theorem 4.5, $\alpha_1^{-1}\mathscr{S}(\mathrm{k})$ has profile $\{r_k\}$ where $\{r_k\}$ is the profile of $\alpha_1^{-1}\mathscr{S}(\mathbb{Q})$. In particular, the finiteness condition on nonzero differentials is met, and we may conclude that we indeed have strong convergence. $\qquad\square$

Theorem 4.7. *Suppose k is a field of characteristic 0 which has profile $\{r_k\}$. Let v_2 denote 2-adic valuation. If $r_k < \infty$ for all k, then*

$$\pi_m \mathrm{sc}(\eta^{-1}\mathrm{S}) \cong \begin{cases} W(\mathrm{k}) & \text{if } m = 0; \\ W(\mathrm{k})/2^{r_k} & \text{if } m > 0, m = 4\ell - 1, \text{ and } k = v_2(4\ell); \\ {}_{2^{r_k}}W(\mathrm{k}) & \text{if } m > 0, m = 4\ell, \text{ and } k = v_2(4\ell); \\ 0 & \text{otherwise.} \end{cases}$$

If $\{r_k\}$ eventually takes the value ∞ with first instance $r_K = \infty$, then

$$\pi_m \mathrm{sc}(\eta^{-1}\mathrm{S}) \cong \begin{cases} W(\mathrm{k}) & \text{if } m = 0; \\ W(\mathrm{k})/2^{r_k} & \text{if } m > 0, m = 4\ell - 1, \text{ and } k = v_2(4\ell) < K; \\ {}_{2^{r_k}}W(\mathrm{k}) & \text{if } m > 0, m = 4\ell, \text{ and } k = v_2(4\ell) < K; \\ W(\mathrm{k}) & \text{if } m > 0, m = 4\ell - 1 \text{ or } 4\ell, \text{ and } v_2(4\ell) \geq K; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This all follows from the slice filtration being the I-adic filtration, ρ representing 2 in $W(\mathrm{k})$, the structure of the differentials in Theorem 4.5, and strong convergence in Theorem 4.6. $\qquad\square$

Theorem 4.8. *Suppose that k is not of characteristic 2 and that -1 is a sum of four squares in k. Then, as a ring,*

$$\pi_\star \mathrm{sc}(\eta^{-1}\mathrm{S}) \cong W(\mathrm{k})[\eta^{\pm 1}, \sigma, \mu]/(\sigma^2)$$

where $|\eta| = \alpha$, $|\sigma| = 3 + 4\alpha$, and $|\mu| = 4 + 5\alpha$. If additionally $\mathrm{cd}\,\mathrm{k} < \infty$, then $\mathrm{sc}(\eta^{-1}\mathrm{S}) \simeq \eta^{-1}\mathrm{S}$ and this is a computation of the η-periodic homotopy groups of the motivic sphere spectrum.

Proof. By Proposition 3.3, we may additionally assume that $\mathrm{char}\,\mathrm{k} = 0$. It is standard that the latter condition is equivalent to $\rho^3 = 0 \in k_3^M(\mathrm{k})$ (see [Lam 2005, Corollary X.6.20]). By Theorem 4.5, we see that the spectral sequence collapses (regardless of the profile of $\mathbb{Q}$). By Theorem 4.6, this proves the theorem. $\qquad\square$

Corollary 4.9. *If* cd k $< \infty$ *and* $n \geq \max\{3m+5, 4m\}$, *then* $\pi_{m+n\alpha}S \cong \pi_{m+n\alpha}\eta^{-1}S$; *if, additionally,* k *is not of characteristic* 2 *and* -1 *is a sum of four squares in* k, *then these groups are* 0 *or* $W(k)$ *according to whether* $m \equiv 1, 2 \pmod 4$ *or* $m \equiv 0, 3 \pmod 4$, *respectively*.

Proof. All but the final statement was already observed in [Ormsby et al. 2018, Theorem 5.5].   □

We certainly do *not* expect that the α_1-periodic slice spectral sequence collapses at its E_2 page in general. Indeed, inspired by the computations of $\pi_\star\eta^{-1}S_2^\wedge$ by Guillou and Isaksen [2016] over $\mathbb{R}$ and Wilson [2018] over $\mathbb{Q}$, we make the following conjecture.

Conjecture 4.10. *The α_1-periodic slice spectral sequence over $\mathbb{Q}$ has profile* $\{3, 4, 5, \ldots\}$, *i.e.,* $r_k = k + 1$ *for all* k.

If Conjecture 4.10 holds, then over k of characteristic 0,

$$\pi_m \operatorname{sc}(\eta^{-1}S) \cong \begin{cases} W(k) & \text{if } m = 0, \\ W(k)/2^{v_2(4\ell)+1} & \text{if } m = 4\ell - 1 > 0, \\ {}_{2^{v_2(4\ell)+1}}W(k) & \text{if } m = 4\ell > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Curiously, this makes it appear as if $\operatorname{sc}(\eta^{-1}S)$ might fit into a "connective image of J fiber sequence" of the form $\operatorname{sc}(\eta^{-1}S) \to \mathrm{kw} \to \Sigma^4\mathrm{kw}$ where kw is the connective cover of the 2-complete Witt K-theory spectrum. Over $k = \mathbb{C}$, one may show that the cone on the map $\sigma_\infty : \Sigma^3\mathrm{kw} \to \eta^{-1}S$ of Theorem 2.11 coincides with kw. In fact, the composition $\Sigma^3\mathrm{kw} \to \eta^{-1}S \to \mathrm{kw}$ is zero, as one may deduce inductively, starting with the triviality of

$$\Sigma^3\eta^{-1}S \to \Sigma^3\mathrm{kw} \to \eta^{-1}S \to \mathrm{kw}$$

and continuing along the cell presentation of kw given in the proof of Theorem 2.11. Hence there is an induced map from the cone of $\Sigma^3\mathrm{kw} \to \eta^{-1}S$ to kw. This map induces an isomorphism on homotopy groups, and hence is an equivalence by cellularity. In particular, one may express $\eta^{-1}S$ over the complex numbers as the fiber of a map $\mathrm{kw} \to \Sigma^4\mathrm{kw}$.

The Adams operations on the 2-complete algebraic K-theory spectrum $\mathrm{KGL}_2^\wedge$ constitute an action of $\mathbb{Z}_2^\times$, the units in the 2-adic integers. When k has finite virtual cohomological dimension, the results of [Hu et al. 2011] imply that $\mathrm{KQ}_2^\wedge \simeq (\mathrm{KGL}_2^\wedge)^{h\{\pm 1\}}$ inherits an action of $\mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2$ by Adams operations. Inverting η and taking the connective cover results in Adams operations on $\mathrm{kw}_2^\wedge$. For any such ψ^g, the difference of ring maps $\psi^g - 1 : \mathrm{kw}_2^\wedge \to \mathrm{kw}_2^\wedge$ lifts to a map $\psi^g - 1 : \mathrm{kw}_2^\wedge \to \Sigma^4\mathrm{kw}_2^\wedge$. (This can be seen by observing that $\Sigma^4\mathrm{kw}_2^\wedge$ is the 4-connective

cover of $\mathrm{kw}_2^\wedge$ and the cofiber of $\Sigma^4 \mathrm{kw}_2^\wedge \to \mathrm{kw}_2^\wedge$ is the Eilenberg–MacLane spectrum associated with the homotopy module $\underline{W}_2^\wedge[\eta^{\pm 1}]$.)

The η-periodic unit $\eta^{-1}\mathrm{S}_2^\wedge \to \mathrm{kw}_2^\wedge$ factors through the fiber jw_g of $\psi^g - 1$: $\mathrm{kw}_2^\wedge \to \Sigma^4 \mathrm{kw}$ because $\psi^g - 1$ is a difference of ring maps. This leads to the following conjecture, which is similar in spirit to Mahowald's presentation of the v_1-periodic sphere in topology.

Conjecture 4.11. *The map $\eta^{-1}\mathrm{S}_2^\wedge \to \mathrm{jw}_3$ induced by the η-periodic unit $\eta^{-1}\mathrm{S}_2^\wedge \to \mathrm{kw}_2^\wedge$ is an equivalence.*

Let HW denote the Eilenberg–MacLane spectrum associated with the homotopy module $\underline{W}[\eta^{\pm 1}]$. Work in progress of Tom Bachmann and Mike Hopkins suggests that the action of ψ^3 on $HW \wedge \widehat{\mathrm{kw}_2}$ is such that the unit map smashed with $HW_2^\wedge$ induces an equivalence $HW_2^\wedge \to HW \wedge \mathrm{jw}_3$. Since $\eta^{-1}\mathrm{S}$ is HW-complete, this would immediately prove that $\eta^{-1}\mathrm{S}_2^\wedge \simeq \mathrm{jw}_3$. It is presumably also the case that $(\psi^3 - 1)(\beta^k) = (9^k - 1)\beta^k$ on $\pi_\star \mathrm{kw}_2^\wedge$, in which case a comparison of slice spectral sequences would prove Conjecture 4.10.

Remark 4.12. The equivalence $\eta^{-1}\mathrm{S}_2^\wedge \simeq \mathrm{jw}_3$ would also lead to a complete determination of the homotopy type and groups of $\eta^{-1}\mathrm{S}$. Let X_k denote the Harrison space of orderings of k. Then $\pi_\star\{\eta^{-1}, 1/2\}\mathrm{S} \cong H^0(X_k; \pi_m^{\mathrm{top}}\mathrm{S}[1/2])$, which can be seen by the results of [Bachmann 2018], a descent spectral sequence, and the fact that X_k is a Stone space. The 2-primary arithmetic fracture square would then imply that

$$\pi_m \eta^{-1}\mathrm{S} \cong$$
$$\begin{cases} W(k) & \text{if } m = 0, \\ H^0(X_k; \pi_m^{\mathrm{top}}\mathrm{S}[1/2]) & \text{if } m > 0 \text{ and } m \equiv 1 \text{ or } 2 \pmod 4, \\ W(k)/2^{v_2(4\ell)+1} \oplus H^0(X_k; \pi_m^{\mathrm{top}}\mathrm{S}[1/2]) & \text{if } m = 4\ell - 1 > 0, \\ {}_{2^{v_2(4\ell)+1}}W(k) \oplus H^0(X_k; \pi_m^{\mathrm{top}}\mathrm{S}[1/2]) & \text{if } m = 4\ell > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Acknowledgments

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