Pacific Journal of Mathematics

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Volume 306 No. 2 June 2020

PACIFIC JOURNAL OF MATHEMATICS
Vol. 306, No. 2, 2020
https://doi.org/10.2140/pjm.2020.306.699

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We consider the Noether Problem for stable and retract rationality for the sequence of d-torsion subgroups $T[d]$ of a torus T, $d \geq 1$. We show that the answer to these questions only depends on $d \pmod{e(T)}$, where $e(T)$ is the period of the generic T-torsor. When T is the norm one torus associated to a finite Galois extension, we find all d such that the Noether Problem for retract rationality has a positive solution for $T[d]$. We also give an application to the Grothendieck ring of stacks.

1. Introduction

Let k be a field, G be a finite group (i.e., a finite constant group scheme over k) and V be a faithful k-linear G-representation. The Noether Problem asks whether the quotient variety V/G is rational, that is, birational to some affine space over k. This question originated in Noether's work [1917] on the inverse Galois problem. If $k = \mathbb{Q}$, and V/G is rational over G for some V, then Hilbert's irreducibility theorem implies that G arises as a Galois group over $\mathbb{Q}$; see [Jensen et al. 2002, §3.3, §5.1].

It turns out that the Noether Problem does not always have a positive solution. Swan [1969] and Voskresenski [1970] gave counterexamples to the Noether Problem over $\mathbb{Q}$ for some cyclic groups of prime order ($\mathbb{Z}/47\mathbb{Z}$, $\mathbb{Z}/113\mathbb{Z}$, $\mathbb{Z}/223\mathbb{Z}, \dots$).

Subsequent work naturally led to several variants of the original Noether Problem. For example, one may ask if the variety V/G is stably rational, or retract rational; see [Colliot-Thélène and Sansuc 2007, §1] for the definitions. The examples of Swan and Voskresenski give a negative answer to the Noether Problem for stable rationality and, by a result of Saltman [1984, Theorem 4.12], a positive answer for retract rationality.

One may also consider the Noether Problem for an arbitrary linear algebraic group scheme G over k (not necessarily smooth). Let V be a finite-dimensional generically free representation of G. By [Berhuy and Favi 2003, Theorem 4.7], there exists a dense G-invariant open subset of V such that a geometric quotient U/G

MSC2010: 14L15, 14M20.
Keywords: Noether problem, algebraic tori, retract rationality.

exists and $U \to U/G$ is a G-torsor. The variety U/G may be regarded as a birational approximation of the classifying stack BG. Indeed, by the no-name lemma [Reichstein and Vistoli 2006, Lemma 2.1], the stable (retract) rationality of U/G does not depend on the representation V but only on G. We say that BG is stably (retract) rational if U/G is, that is, if the Noether Problem for stable (retract) rationality for G has an affirmative answer; see also [Florence and Reichstein 2018, §3].

The stable (retract) rationality of BG in the case when G is a group of multiplicative type is an interesting problem. Lenstra [1974] gave a complete classification of abelian groups for which the Noether Problem fails. In [Lenstra 1974, (7.2)], he showed that the smallest group for which the Noether Problem over $\mathbb{Q}$ fails is the cyclic group $\mathbb{Z}/8\mathbb{Z}$.

If $G = T$ is a k-torus then, by an observation of Colliot-Thélène and Merkurjev [Scavia 2020, Proposition 6.1], BT is stably (retract) rational if and only if the dual torus of T is. If the group of multiplicative type G is not necessarily a torus, we embed G in a quasisplit torus S. By construction, S is a G-stable open subset of a generically free G-representation. The quotient S/G is a torus, and BG is stably (retract) rational if and only if S/G is.

The stable (retract) rationality of a k-torus T may be understood in terms of the action of the absolute Galois group of k on the character lattice of T; see [Voskresenski 1998, §4.7] and [Colliot-Thélène and Sansuc 1977]. A complete classification of stably (retract) rational k-tori, in the spirit of [Lenstra 1974], does not exist. The birational geometry of tori is an interesting and subtle topic, and has been studied in [Voskresenski 1965; 1971; 1973; Miyata 1971; Endô and Miyata 1973; 1975; Colliot-Thélène and Sansuc 1977; 1987; Kunyavski 1978; 1987], among many others.

Of particular interest are the birational properties of norm one tori. Let L/k be a finite separable field extension, and let $T = R^{(1)}_{L/k}(\mathbb{G}_{\mathrm{m}})$ be the associated norm one torus. It fits into a short exact sequence

$$1 \to T \to R_{L/k}(\mathbb{G}_{\mathrm{m}}) \xrightarrow{N} \mathbb{G}_{\mathrm{m}} \to 1, \tag{1-1}$$

where $R_{L/k}$ denotes Weil restriction and N is the norm map. For the case when L/k is Galois, the retract rationality of T has been studied in [Colliot-Thélène and Sansuc 1977]. For the case when L/k is not necessarily Galois, the rationality problem for norm one tori has been considered in special cases from a computational perspective by Hoshi and Yamasaki [2017; 2018] and by Hasegawa, Hoshi and Yamasaki [Hasegawa et al. 2020].

Since the Noether Problem for groups of multiplicative type in full generality is out of the reach of currently known techniques, we restrict ourselves to the following class of groups. Let k be a field and T a k-torus. For every $d \geq 1$, we consider the Noether Problem for stable rationality and retract rationality for the torsion subgroup $T[d]$. Note that these questions are not covered by the aforementioned results of Lenstra [1974]. Lenstra solved the Noether Problem for finite abelian constant groups. The torsion subgroup schemes $T[d]$ we will consider are finite and abelian, but not necessarily constant.

In Proposition 3.3 we show that the answer to each of these versions of the Noether Problem for the torsion subgroups $T[d]$ (of a particular torus T) is periodic in d. We investigate more closely the retract rationality of $BT[d]$ in the case where $T = R^{(1)}_{L/k}(\mathbb{G}_{\mathrm{m}})$ is a norm one torus and L/k is a finite Galois extension. Note that sequence (1-1) implies that BT is stably rational, but as we will see, this need not be true for $BT[d]$. By Lemma 5.2 it is enough to consider the case when G is a p-group. For every L/k and every $d \geq 1$, we determine whether $BT[d]$ is retract rational.

Theorem 1.1. *Let L/k be a finite Galois extension such that $G := \mathrm{Gal}(L/k)$ has order p^n for some prime number p, and let $T := R^{(1)}_{L/k}(\mathbb{G}_{\mathrm{m}})$. Assume that p is odd.*

(a) *If G is cyclic, then $BT[d]$ is retract rational for every d.*

(b) *If G is not cyclic, then $BT[d]$ is retract rational if and only if $p \nmid d$.*

Assume that $p = 2$.

(a') *If G is cyclic, then $BT[d]$ is retract rational for every d.*

(b') *If G is a dihedral group, then $BT[d]$ is retract rational if and only if $4 \nmid d$.*

(c') *Otherwise, $BT[d]$ is retract rational if and only if d is odd.*

Assume that $\operatorname{char} k = p$ is positive. Then it is well known that for every finite constant p-group Γ, $B\Gamma$ is stably rational; see [Jensen et al. 2002, §5.6]. By contrast, Theorem 1.1 gives examples of finite group schemes A of order a power of p for which BA is not even retract rational. Note that such A are nonreduced.

In Section 8 we give an application of Proposition 3.3 to the Grothendieck ring of k-stacks $K_0(\mathrm{Stacks}_k)$.

2. Preliminaries

Let k be a field, and let G be a linear algebraic group over k. If V is a generically free representation of G, there exists a nonempty open G-invariant subset $U \subseteq V$ together with a G-torsor $U \to Z$, where Z is a k-variety. The G-torsor $U \to Z$ is called a standard G-torsor in [Merkurjev 2018]. We say that BG is stably birational to a variety X if Z is birational to X. We say that BG is stably rational or retract rational if Z is. Different choices of V and U yield stably birational Z, hence the definitions are independent of the choice of V and U; see [Merkurjev 2018, §5].

Let T be a k-torus. We will denote by $\hat{T}$ the character lattice of T. Recall that this is $\mathrm{Hom}_{k_s}(T_{k_s}, \mathbb{G}_{\mathrm{m},k_s})$, where k_s denotes a separable closure of k. The association $T \mapsto \hat{T}$ establishes an antiequivalence between the categories of k-tori and the category of $\mathbb{Z}$-free continuous $\mathrm{Gal}(k_s/k)$-modules of finite rank; see [Voskresenski 1998, §3.4]. We refer the reader to [Voskresenski 1998, p. 27] for the definitions of splitting group and minimal splitting field of T. We write T° for the dual torus of T, that is, the k-torus whose character lattice is dual to $\hat{T}$.

Consider a short exact sequence

$$1 \to T \to S \xrightarrow{\varphi} Q \to 1$$

where S is quasisplit. Then S is an open subset of an affine space V, and the multiplication action of T on S extends to a linear action on V. It follows that BT is stably birational to Q.

The generic fiber of φ is a T-torsor over $k(Q)$, and is called a generic T-torsor. If $E \to \operatorname{Spec} K$ is a T-torsor, the period of $E \to \operatorname{Spec} K$ is by definition its order in the group $H^1(K, T_K)$. We denote by $e(T)$ the period of a generic T-torsor. By [Merkurjev 2010, Proposition 1.1], $e(T)$ is divisible by the period of every T-torsor. In particular, it does not depend on the choice of the resolution (3-1).

Lemma 2.1. *Let T be a k-torus, with splitting group G. Then*:

(a) *$e(T)$ divides the order of G.*

(b) $e(T) = e(T^\circ)$.

Proof. (a) Let l/k be the minimal splitting field of T. The extension l/k is Galois, with Galois group G. Let $E \to \operatorname{Spec} K$ be a generic T-torsor, where K is a field containing k, and denote by L the compositum of K and l inside some fixed algebraic closure of K containing l. Then L splits T_K and so $H^1(L, T_L) = H^1(L, \mathbb{G}_{\mathrm{m}}^{\operatorname{rk} T}) = 0$. The restriction-corestriction sequence

$$H^1(K, T_K) \to H^1(L, T_L) \to H^1(K, T_K)$$

shows that $H^1(K, T_K)$ is $[L : K]$-torsion, hence $e(T) \mid [L : K]$. By basic Galois theory $[L : K] \mid [l : k] = |G|$, so $e(T) \mid |G|$, as desired.

(b) Set $M := \hat{T}$ and fix a coflasque resolution $0 \to N \to U \to M \to 0$, corresponding to a class $\alpha \in \operatorname{Ext}^1_G(M, N)$. Let M' and N' be the dual lattices of M and N, respectively. There is a G-equivariant isomorphism

$$\operatorname{Hom}_{\mathbb{Z}}(M, N) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(N', M')$$

given by taking the transpose. By [Brown 1982, §III, Proposition 2.2] we obtain a commutative diagram of group isomorphisms

$$\begin{array}{ccc} H^1(G, \operatorname{Hom}_{\mathbb{Z}}(M, N)) & \xrightarrow{\sim} & \operatorname{Ext}^1_G(M, N) \\ \downarrow \wr & & \downarrow \wr\, \eta \\ H^1(G, \operatorname{Hom}_{\mathbb{Z}}(N', M')) & \xrightarrow{\sim} & \operatorname{Ext}^1_G(N', M'), \end{array} \tag{2-1}$$

where η is defined by sending an extension $0 \to N \to P \to M \to 0$ to its dual $0 \to M' \to P' \to N' \to 0$. By [Merkurjev 2010, Theorem 3.1], $e(T)$ is the order of α in $\mathrm{Ext}^1_G(M, N)$. By [Merkurjev 2010, Theorem 3.2], $e(T^\circ)$ is equal to the order of the class α' of the dual sequence $0 \to M' \to U' \to N' \to 0$ in $\mathrm{Ext}^1_G(N', M')$. Now (2-1) shows α and α' have the same order, hence $e(T) = e(T^\circ)$. □

3. Periodicity

The purpose of this section is to prove Proposition 3.3. Let T be a k-torus. Fix a coflasque resolution of T,

$$1 \to T \to S \xrightarrow{\varphi} Q \to 1, \tag{3-1}$$

and the corresponding short exact sequence of character lattices

$$0 \to \hat{Q} \to \hat{S} \to \hat{T} \to 0. \tag{3-2}$$

For every $d \geq 1$, we may consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & 1 & & & & T & & \\
 & & \downarrow & & & & \downarrow & & \\
1 & \longrightarrow & T[d] & \longrightarrow & S & \xrightarrow{\psi} & T_d & \longrightarrow & 1 \\
 & & \downarrow & & \| & & \downarrow & & \\
1 & \longrightarrow & T & \longrightarrow & S & \xrightarrow{\varphi} & Q & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle d} & & & & \downarrow & & \\
 & & T & & & & 1 & & \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array} \tag{3-3}$$

Here, the second row is (3-1), the torus T_d is defined so as to make the first row exact, and the copy of T in the upper right corner is identified with the copy of T on the lower left corner via the connecting homomorphism given by the snake lemma. Since S is quasisplit, we see that $BT[d]$ is stably birational to T_d.

The main ingredient for the proof of Proposition 3.3 is the following observation.

Proposition 3.1. *Let $\alpha \in \mathrm{Ext}^1_G(\hat{T}, \hat{Q})$ be the class of* (3-2), *and α_d be the class of the sequence*

$$0 \to \hat{Q} \to \hat{T}_d \to \hat{T} \to 0$$

in $\mathrm{Ext}^1_G(\hat{T}, \hat{Q})$, for every $d \geq 1$. Then $\alpha_d = d \cdot \alpha$.

In particular, $T_m \cong T_n$ when $m \equiv \pm n \pmod{e(T)}$, and if $e(T) \mid d$, then $T_d \cong T \times Q$.

Proof. From (3-3), we obtain the following diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T & \longrightarrow & S & \xrightarrow{\varphi} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \psi} & & \| & & \\ 1 & \longrightarrow & T & \longrightarrow & T_d & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

Passing to character lattices, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{Q} & \longrightarrow & \hat{T}_d & \longrightarrow & \hat{T} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle d} & & \\ 0 & \longrightarrow & \hat{Q} & \longrightarrow & \hat{S} & \longrightarrow & \hat{T} & \longrightarrow & 0 \end{array} \tag{3-4}$$

By [Merkurjev 2010, Theorem 3.1], $e(T)$ is equal to the order of the class $\alpha \in \mathrm{Ext}^1_G(\hat{T}, \hat{Q})$ of the sequence

$$0 \to \hat{Q} \to \hat{S} \to \hat{T} \to 0.$$

The long exact sequence for the functor $\mathrm{Hom}_G(\hat{T}, -)$ associated to (3-4) gives a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_G(\hat{T}, \hat{T}) & \xrightarrow{\partial_1} & \mathrm{Ext}^1_G(\hat{T}, \hat{Q}) \\ \downarrow{\scriptstyle d} & & \| \\ \mathrm{Hom}_G(\hat{T}, \hat{T}) & \xrightarrow{\partial_2} & \mathrm{Ext}^1_G(\hat{T}, \hat{Q}). \end{array}$$

We have $\alpha_d = \partial_1(\mathrm{id}_{\hat{T}})$ and $\alpha = \partial_2(\mathrm{id}_{\hat{T}})$, hence $\alpha_d = d \cdot \alpha$.

Since α has order $e(T)$, if $m \equiv n \pmod{e(T)}$ then $T_m \cong T_n$. Recall that if M, N are G-lattices and $\gamma \in \mathrm{Ext}^1_G(M, N)$ is the class of $0 \to N \to P \xrightarrow{\eta} M \to 0$, then $-\gamma$ is the class of $0 \to N \to P \xrightarrow{-\eta} M \to 0$. Hence $T_m \cong T_n$ when $m \equiv \pm n \pmod{e(T)}$. Finally, if $e(T) \mid d$, then $\alpha_d = 0$, so $\hat{T}_d \cong \hat{T} \oplus \hat{Q}$. □

Before proving Proposition 3.3, we need the following lemma.

Lemma 3.2. *Let T_1 and T_2 be two k-tori split by a finite Galois extension with Galois group G. Let $f : T_1 \to T_2$ be a surjective homomorphism of tori of finite degree prime to $|G|$. Then T_1 is retract rational if and only if T_2 is retract rational.*

Proof. Let

$$1 \to F_i \to S_i \to T_i \to 1$$

be a flasque resolution of T_i, for $i = 1, 2$. By [Saltman 1984, Theorem 3.14(a)], T_i is retract rational if and only if $\hat{F}_i$ is invertible, that is, $\hat{F}_i$ is a direct summand of a permutation G-lattice. It thus suffices to show that $\hat{F}_1$ is invertible if and only if $\hat{F}_2$ is.

Set $a := \deg(f)$. If p is a prime that does not divide a, the map $f : T_1 \to T_2$

induces an isomorphism $(\hat{T}_1)_{(p)} \cong (\hat{T}_2)_{(p)}$ of $\mathbb{Z}_{(p)}[G]$-modules. Construct the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & (\hat{F}_2)_{(p)} & = & (\hat{F}_2)_{(p)} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & (\hat{S}_2)_{(p)} & \longrightarrow & X_p & \longrightarrow & (\hat{F}_1)_{(p)} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & (\hat{T}_1)_{(p)} & \longrightarrow & (\hat{S}_1)_{(p)} & \longrightarrow & (\hat{F}_1)_{(p)} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3-5}$$

Since the $\hat{F}_i$ are flasque and the $\hat{S}_i$ are permutation lattices for $i = 1, 2$, by [Colliot-Thélène and Sansuc 1977, Lemme 1(vii)] we have

$$\mathrm{Ext}^1_{\mathbb{Z}_{(p)}[G]}((\hat{F}_i)_{(p)}, (\hat{S}_{3-i})_{(p)}) = \mathrm{Ext}^1_G(\hat{F}_i, \hat{S}_{3-i})_{(p)} = 0.$$

It follows from the diagram that $(\hat{F}_1 \oplus \hat{S}_2)_{(p)} \cong X_p \cong (\hat{F}_2 \oplus \hat{S}_1)_{(p)}$ for every p not dividing a. Since $|G|$ and a are coprime, this holds for every p which divides $|G|$. In the terminology of [Scavia 2020, Definition 2.1], this means that $\hat{F}_1$ is p-invertible if and only if $\hat{F}_2$ is, for every prime p which divides $|G|$. By [Scavia 2020, Lemma 2.3], it follows that $\hat{F}_1$ is invertible if and only if $\hat{F}_2$ is, as desired. □

Proposition 3.3. *Let T be a k-torus and let $m, n \geq 1$.*

(a) *If $m \equiv \pm n \pmod{e(T)}$, then $BT[m]$ and $BT[n]$ are stably birational.*

(b) *If $e(T) \mid n$, then $BT[n]$ is stably birational to $T \times BT$.*

(c) *If $n \equiv \pm 1 \pmod{e(T)}$, then $BT[n]$ is stably rational.*

(d) *If $\langle m \rangle = \langle n \rangle$ in $\mathbb{Z}/e(T)\mathbb{Z}$, then $BT[m]$ is retract rational if and only if $BT[n]$ is. In particular, $BT[m]$ is retract rational when m is invertible in $\mathbb{Z}/e(T)\mathbb{Z}$.*

Proof of Proposition 3.3. (a) By construction $BT[d]$ is stably birational to T_d for every $d \geq 1$, so the conclusion follows from Proposition 3.1.

(b) If $e(T) \mid n$, by Proposition 3.1 the sequence

$$1 \to T \to T_n \to Q \to 1$$

splits, so $T_n \cong T \times Q$. Since BT is stably birational to Q and $BT[n]$ is stably birational to T_n, we conclude that $BT[n]$ is stably birational to $T \times BT$.

(c) Since $T[1]$ is trivial, $T_1 = S$ is rational. Now (c) follows from (a).

(d) Let $m, n \geq 1$ be such that $\langle m \rangle = \langle n \rangle$ in $\mathbb{Z}/e(T)\mathbb{Z}$. There exists $a \geq 1$ invertible modulo $e(T)$ such that $m \equiv an \pmod{e(T)}$. By Proposition 3.1, $T_m \cong T_{an}$. By Lemma 3.2, T_m is retract rational if and only if T_{am} is. The second statement is now a consequence of (b) and (c). □

Remark 3.4. Using Lemma 2.1(a), we see that Proposition 3.3 remains valid if we replace $e(T)$ by $|G|$.

It would be interesting to know if the hypotheses of Proposition 3.3(d) imply that $BT[m]$ and $BT[n]$ are stably birational to each other. Statements of this sort are in the spirit of Amitsur's conjecture; see [Amitsur 1955] or [Auel et al. 2011, Conjecture 10.2]. We thank the referee for this comment.

4. Norm one tori

Let $T := R^{(1)}_{L/k}(\mathbb{G}_{\mathrm{m}})$, where L/k is a finite Galois extension with Galois group G. By [Merkurjev 2010, Example 4.1], $e(T^\circ) = [L : k] = |G|$. Using Lemma 2.1(b), we deduce that $e(T) = |G|$.

Define T_d via the diagram (3-3), where $S = R_{L/k}(\mathbb{G}_{\mathrm{m}})$ and T is embedded in S as the kernel of the norm map, so that $Q = \mathbb{G}_{\mathrm{m}}$. By construction we have a short exact sequence

$$1 \to T[d] \to R_{L/k}(\mathbb{G}_{\mathrm{m}}) \to T_d \to 1.$$

Let $\{e_g\}_{g \in G}$ be the standard basis of $\mathbb{Z}[G]$, and for every subgroup G' of G let $\sigma_{G'} := \sum_{g \in G'} e_g \in \mathbb{Z}[G]$. We have $\hat{T} = \mathbb{Z}[G]/\langle \sigma_G \rangle$, so the sequence

$$1 \to T[d] \to T \xrightarrow{d} T \to 1$$

shows that the character module of $T[d]$ is $(\mathbb{Z}/d\mathbb{Z}[G])/\langle \sigma_G \rangle$. It follows that

$$\hat{T}_d = \langle d\mathbb{Z}[G], \sigma_G \rangle = \langle dg, \sigma_G \rangle_{g \in G}.$$

We may thus construct the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \oplus \mathbb{Z} \to \hat{T}_d \to 0, \tag{4-1}$$

where the first homomorphism sends 1 to $(1, \dots, 1; -d)$ and the second map sends $(g, 0) \mapsto dg$ and $(0, 1) \mapsto \sigma_G$.

If G is a finite group, M is a G-module and $i \in \mathbb{Z}$, we write $H^i(G, M)$ for the i-th group of Tate cohomology, group denoted by $\hat{H}^0(G, M)$ if i is explicitly set equal to 0.

Lemma 4.1. *Let T and G be as above. For every $i \in \mathbb{Z}$ and $d \geq 1$, there is an isomorphism*

$$H^i(G, \hat{T}_d) \xrightarrow{\sim} H^i(G, \mathbb{Z}/d\mathbb{Z}),$$

which is compatible with restriction to subgroups of G.

Proof. Let $d \geq 1$, and let U_d be the character module of $T[d]$. We have a short exact sequence of G-modules

$$0 \to \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}[G] \to U_d \to 0,$$

where the first homomorphism is induced by $1 \mapsto \sigma_G$. We also have the sequence

$$0 \to \hat{T}_d \to \mathbb{Z}[G] \to U_d \to 0,$$

corresponding to the first row of (3-3). By Shapiro's lemma $H^i(G, \mathbb{Z}[G]) = 0$ and $H^i(G, \mathbb{Z}/d\mathbb{Z}[G]) = 0$ for every integer i. Looking at the associated cohomology long exact sequences, we deduce

$$H^i(G, \hat{T}_d) \cong H^{i-1}(G, U_d) \cong H^i(G, \mathbb{Z}/d\mathbb{Z}).$$

The compatibility of this isomorphism with restrictions is clear from the construction. □

Recall that if M is a G-module and $i \in \mathbb{Z}$, then

$$Ш^i(G, M) := \mathrm{Ker}(H^i(G, M) \to \oplus_{g \in G} H^i(\langle g \rangle, M)).$$

If M is a G-lattice, by [Lorenz 2005, Proposition 2.9.2(a)] we have

$$Ш^2(G, M) \cong H^1(G, [M]^{\mathrm{fl}}). \tag{4-2}$$

Lemma 4.2. *Let T and G be as above, and $d \geq 1$. Let G act trivially on $\mathbb{Z}/d\mathbb{Z}$, and assume that $Ш^2(G, \mathbb{Z}/d\mathbb{Z}) \neq 0$. Then $BT[d]$ is not retract rational.*

Proof. By Lemma 4.1 we have a commutative square with horizontal isomorphisms,

$$\begin{array}{ccc} H^2(G, \hat{T}_d) & \xrightarrow{\sim} & H^2(G, \mathbb{Z}/d\mathbb{Z}) \\ \downarrow & & \downarrow \\ \oplus_{g \in G} H^2(\langle g \rangle, \hat{T}_d) & \xrightarrow{\sim} & \oplus_{g \in G} H^2(\langle g \rangle, \mathbb{Z}/d\mathbb{Z}). \end{array} \tag{4-3}$$

The diagram shows that $Ш^2(G, \hat{T}_d) \cong Ш^2(G, \mathbb{Z}/d\mathbb{Z})$, so $Ш^2(G, \hat{T}_d) \neq 0$. By (4-2), we deduce that $H^1(G, [\hat{T}_d]^{\mathrm{fl}}) \neq 0$, hence $[\hat{T}_d]^{\mathrm{fl}}$ is not invertible. It follows that T_d is not retract rational, so $BT[d]$ is not retract rational. □

The following group-theoretic lemma will be used in the proof of Theorem 1.1.

Lemma 4.3. *Let p be a prime, and let G be a finite p-group.*

(a) *Assume G does not contain a proper subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Then either G is cyclic, or $p = 2$ and $G = Q_{2^n}$ for some $n \geq 3$.*

(b) *Assume that every proper subgroup of G is cyclic. Then either G is cyclic or isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, or $p = 2$ and $G = Q_8$.*

(c) *Assume that $p = 2$ and that every proper subgroup of G is cyclic or dihedral. Then G is cyclic, dihedral, or $|G| = 8$.*

Proof. (a) Assume that G contains at least two distinct subgroups of order p. Since the center $Z(G)$ of G is not trivial, there exists a subgroup H of $Z(G)$ order p. If H' is another subgroup of order p, then $\langle H, H'\rangle \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. It follows from the assumption that $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Assume now that G contains exactly one subgroup of order p. By [Berkovich 2008, Proposition 1.3], either G is cyclic or $p = 2$ and G is a generalized quaternion group:

$$Q_{2^{n+1}} := \langle a, b \mid a^{2^{n-1}} = b^2, a^{2^n} = e, b^{-1}ab = a^{-1}\rangle.$$

(b) If p is odd, the statement immediately follows from (a), so assume $p = 2$. If $n \geq 4$, Q_{2^n} contains Q_8 as a proper subgroup. It follows from (a) that if $p = 2$ and G is not cyclic, then $G = Q_8$.

(c) The conclusion is obvious when G is abelian, so assume that G is not abelian. We may also suppose that $|G| = 2^n$ for some $n \geq 4$. If G contains at least one nonabelian proper subgroup, it is dihedral [Miller 1907]. Assume now that every proper subgroup of G is abelian. By [Rédei 1947] or [Kang and Reichstein 2002, Lemma 4.5], G is isomorphic to

$$A_{u,v} := \langle a, b \mid a^{2^u} = b^{2^v} = 1, ba = a^{2^{u-1}+1}b\rangle \cong \mathbb{Z}/2^u\mathbb{Z} \rtimes \mathbb{Z}/2^v\mathbb{Z},$$

where $u \geq 2$, $v \geq 1$ and $n = u + v$, or to

$$B_{u,v} := \langle a, b, c \mid a^{2^u} = b^{2^v} = c^2 = 1, ba = cab\rangle \cong (\mathbb{Z}/2^u\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/2^v\mathbb{Z},$$

where $u, v \geq 1$ and $n = u + v + 1$. Since $n \geq 4$, both groups contain a subgroup of the form $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which is neither cyclic nor dihedral, hence no such G exists. □

5. Proof of Theorem 1.1 for p odd

Lemma 5.1. *Let G be a finite group, H be a subgroup of G and $d \geq 1$ be an integer. Let L/k be a Galois extension with Galois group G, $k' := L^H$, $T := R^{(1)}_{L/k}(\mathbb{G}_m)$ and $T' := R^{(1)}_{L/k'}(\mathbb{G}_m)$. Then $BT_{k'}[d]$ is stably birational to $BT'[d]$ over k'.*

In particular, if $BT'[d]$ is not retract rational over k', then $BT[d]$ is not retract rational over k.

Proof. Let $r := [k' : k]$. We have an isomorphism

$$T_{k'} \cong T' \times R_{L/k'}(\mathbb{G}_m)^{r-1},$$

and so $T_{k'}[d]$ is isomorphic to $T'[d] \times R_{L/k'}(\mathbb{G}_m)[d]^{r-1}$. This shows that

$$BT_{k'}[d] \cong BT'[d] \times BR_{L/k'}(\mathbb{G}_m)[d]^{r-1}.$$

Since $BR_{L/k'}(\mathbb{G}_m)[d]$ is stably rational over k', we conclude that $BT_{k'}[d]$ is stably birational to $BT'[d]$ over k'. □

The next lemma allows us to reduce the problem of the retract rationality of $BT[d]$ to the case when G is a p-group.

Lemma 5.2. *Let T be a k-torus split by a group G. Assume that $d \mid |G|$ and let $d = p_1^{a_1} \cdots p_r^{a_r}$ be the prime factorization of d. For every $i = 1, \dots, r$, let G_i be a p_i-Sylow subgroup of G, set $k_i := L^{G_i}$, and let $T_i := R^{(1)}_{L/k_i}(\mathbb{G}_{\mathrm{m}})$. Then $BT[d]$ is retract rational if and only if $BT_i[p_i^{a_i}]$ is retract rational over k_i for every $i = 1, \dots, r$.*

Proof. If k is finite, every k-torus is retract rational, and there is nothing to prove. Assume now that k is infinite. We have $BT[d] \cong BT[p_1^{a_1}] \times \cdots \times BT[p_r^{a_r}]$, so $BT[d]$ is retract rational if and only if $BT[p_i^{a_i}]$ is retract rational for $i = 1, \dots, r$, that is, if and only if $T_{p_i^{a_i}}$ is retract rational for every i. By [Colliot-Thélène and Sansuc 1977, Lemme 9], $T_{p_i^{a_i}}$ is retract rational if and only if $(T_{k_j})_{p_i^{a_i}} = (T_{p_i^{a_i}})_{k_j}$ is retract rational over k_j for every $j = 1, \dots, r$. Thus $BT[d]$ is retract rational if and only if $BT_{k_j}[p_i^{a_i}]$ is retract rational over k_j for every i, j.

If $i \neq j$, $p_i^{a_i}$ is invertible modulo $|G_j|$, so by Proposition 3.3(d) $BT_{k_j}[p_i^{a_i}]$ is retract rational over k_j. It follows that $BT[d]$ is retract rational if and only if $BT_{k_i}[p_i^{a_i}]$ is retract rational over k_i for every i. The conclusion now follows from Lemma 5.1. □

Let L/k be a finite Galois extension with Galois group G, and let $T := R^{(1)}_{L/k}(\mathbb{G}_{\mathrm{m}})$.

Lemma 5.3. *Let p be an odd prime, $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. If p is odd, $BT[p]$ is not retract rational.*

Proof. By Lemma 4.2 it suffices to prove that $\text{Ш}^2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \neq 0$. Let H_p be the Heisenberg group of order p^3. The group H_p fits in a central short exact sequence,

$$1 \to \mathbb{Z}/p\mathbb{Z} \to H_p \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to 1.$$

Let $\alpha \in H^2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ be the class corresponding to this extension. Since p is odd, H_p is not commutative, so $\alpha \neq 0$. If $g \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, the image of α in $H^2(\langle g \rangle, \mathbb{Z}/p\mathbb{Z})$ is the class of the extension

$$1 \to \mathbb{Z}/p\mathbb{Z} \to \pi^{-1}(\langle g \rangle) \to \langle g \rangle \to 1.$$

Since every nontrivial element $g \in H_p$ has order p, $\pi^{-1}(\langle g \rangle) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, hence α restricts to the trivial class in $H^2(\langle g \rangle, \mathbb{Z}/p\mathbb{Z})$. Therefore we have that $0 \neq \alpha \in \text{Ш}^2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$, as desired. □

Note that Lemma 5.3 fails when $p = 2$; see Lemma 7.1(a) below.

Proof of Theorem 1.1 for p odd. Note that if $p \nmid d$, then $BT[d]$ is retract rational by Proposition 3.3(d). From now on, we assume that $p \mid d$.

(a) When G is cyclic, every torus split by G is retract rational by [Colliot-Thélène and Sansuc 1977, Proposition 2], hence T_d is retract rational. Since $BT[d]$ is stably birational to T_d, it follows that $BT[d]$ is also retract rational.

(b) If G is not cyclic, by Lemma 4.3(a) it contains a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Since $p \mid d$, the conclusion follows from Lemmas 5.3 and 5.1. □

6. A flasque resolution of $\hat{T}'_d$

In order to complete the proof of Theorem 1.1, it remains to consider the case $p = 2$. In this section we construct an explicit flasque resolution of $\hat{T}'_d$, when G is assumed to be a p-group. The construction works for any prime p, but we will use it only in the case when $p = 2$.

Assume that G is a p-group, and let d be a divisor of $|G|$. Let M_d be the character lattice of the torus dual to T_d. Consider the sequence

$$0 \to M_d \to \mathbb{Z}[G] \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0, \tag{6-1}$$

obtained by dualizing (4-1). The map α is defined by $\alpha(e_g, 0) = 1$ for every $g \in G$ and $\alpha(0, 1) = -d$, so

$$M_d = \left\{((a_g)_{g\in G}, a) \in \mathbb{Z}[G] \oplus \mathbb{Z} : \sum_{g\in G} a_g - da = 0\right\}.$$

We need to construct an explicit coflasque resolution of M_d. The standard coflasque resolution (see, e.g., [Colliot-Thélène and Sansuc 1977, Lemme 3]) applied to M_d is too unwieldy for the computations that we want to perform. Thus we produce an ad hoc coflasque resolution of M_d. Consider the short exact sequence

$$0 \to N_d \to P_d \xrightarrow{\beta} M_d \to 0. \tag{6-2}$$

Here $P_d := \mathbb{Z}[G \times G] \oplus R_d$, where $R_d := \oplus_{G'<G}\mathbb{Z}[G/G']$, the sum being over all subgroups G' of G; in particular P_d is a permutation lattice. We denote by (g, g'), $g, g' \in G$, the elements of the standard basis of $\mathbb{Z}[G \times G]$. For every subgroup G' of G, denote by $e_{G'}$ the element of P_d supported on the summand $\mathbb{Z}[G/G']$ and having a 1 in the $\mathbb{Z}$-component corresponding to the coset G' in G/G', and 0 everywhere else. Recall that $\mathrm{Hom}_G(\mathbb{Z}[G/G'], M_d) \cong \mathrm{Hom}_{G'}(\mathbb{Z}, M_d) \cong M_d^{G'}$, the isomorphism being given by $\varphi \mapsto \varphi(e_{G'}) \in M_d^{G'}$. It follows that a map $R_d \to M_d$ is specified by the images of all the $e_{G'}$, and for every subgroup G' the image of $e_{G'}$ must be G'-invariant. Then β, as a map $P_d \to \mathbb{Z}[G] \oplus \mathbb{Z}$, is defined by $\beta((g, g'), 0) := (e_g - e_{g'}, 0)$ and

$$\beta(e_{G'}) := \begin{cases} (\sigma_{G'}, |G'|/d) & \text{if } d \mid |G'|, \\ ((d/|G'|)\sigma_{G'}, 1) & \text{if } d \nmid |G'|. \end{cases}$$

Note that since G is a p-group and $d \mid |G|$, either $d \mid |G'|$ or $|G'| \mid d$. To see that β maps P_d to M_d, observe that clearly $\alpha(\beta((g_1, g_2), 0)) = 0$ and that

$$\alpha(\beta(e_{G'})) = \begin{cases} |G'| - d \cdot |G'|/d = 0 & \text{if } d \mid |G'|, \\ (d/|G'|) \cdot |G'| - d = 0 & \text{if } d \nmid |G'|. \end{cases}$$

Moreover, $\beta(e_{G'}) \in M_d^{G'}$, so β is G-equivariant. If we fix a subgroup $G_0 < G$ of order d, then M_d is generated as a G-module by elements of the form $(e_g - e_{g'}, 0)$, $g, g' \in G$, and $(\sigma_{G_0}, 1)$. Now, $e_g - e_{g'} = \beta((g, g'), 0)$, and $(\sigma_{G_0}, 1) = \beta(e_{G_0})$, so β is surjective. We define $N_d := \operatorname{Ker} \beta$.

Lemma 6.1. *The G-lattice N_d is coflasque, therefore* (6-2) *is a coflasque resolution of M_d.*

Proof. Let G' be a subgroup of G. Since P_d is a permutation lattice, we have $H^1(G', P_d) = 0$. Recall that we denote degree 0 Tate cohomology by $\hat{H}^0$. The cohomology long exact sequence associated to (6-2) then yields

$$\hat{H}^0(G', P_d) \xrightarrow{\hat{H}^0(\beta)} \hat{H}^0(G', M_d) \to H^1(G', N_d) \to 0,$$

so to prove that $H^1(G', N_d) = 0$ it suffices to show that $\hat{H}^0(\beta)$ is surjective. Since $H^{-1}(G', \mathbb{Z}) = 0$, passing to cohomology in (6-1) gives

$$0 \to \hat{H}^0(G', M_d) \to \hat{H}^0(G', \mathbb{Z}[G] \oplus \mathbb{Z}) \to \hat{H}^0(G', \mathbb{Z}),$$

which can be rewritten as

$$0 \to \hat{H}^0(G', M_d) \to \mathbb{Z}/|G'|\mathbb{Z} \xrightarrow{-d} \mathbb{Z}/|G'|\mathbb{Z}. \tag{6-3}$$

We have $\hat{H}^0(G', P_d) = \hat{H}^0(G', R_d) = \oplus_{G'' < G} \hat{H}^0(G', \mathbb{Z}[G/G''])$. To prove that $\hat{H}^0(\beta)$ is surjective it suffices to show that the map

$$\gamma : \hat{H}^0(G', \mathbb{Z}[G/G']) \to \hat{H}^0(G', M_d)$$

given by the summand relative to $G'' = G'$ is surjective. If we view $\hat{H}^0(G', M_d)$ as the d-torsion of $\hat{H}^0(G', \mathbb{Z}[G] \oplus \mathbb{Z})$ via (6-3), then $\gamma(e_{G'})$ coincides with the $\mathbb{Z}$-component of $\beta(e_{G'}) \in \mathbb{Z}[G] \oplus \mathbb{Z}$ modulo $|G'|$, that is:

$$\gamma(e_{G'}) = \begin{cases} |G'|/d & \text{if } d \mid |G'|, \\ 1 & \text{if } d \nmid |G'|. \end{cases}$$

In both cases $\gamma(e_{G'})$ generates $\hat{H}^0(G', M_d)$. It follows that γ is surjective, as desired. □

Proposition 6.2. *Let L/k be a Galois extension whose Galois group G is a p-group, and set $T := R^{(1)}_{L/k}(\mathbb{G}_m)$. If $BT[d]$ is retract rational for some $d \mid |G|$, then any subgroup of G of order d is cyclic.*

Proof. Assume that $BT[d]$ is retract rational. Since $BT[d]$ is stably birational to T_d, T_d is also retract rational. By Lemma 6.1, the dual of (6-2) is a flasque resolution of $\hat{T}_d$. By [Saltman 1984, Theorem 3.14(a)], the dual of N_d is invertible, hence so is N_d. This means that there exists a G-lattice U such that $N_d \oplus U = P$ for some permutation G-lattice P. In particular, for every subgroup G' of G, we have an embedding $H^2(G', N_d) \hookrightarrow H^2(G', P)$. Since $H^{-1}(G', P_d) = 0$, the long exact sequence associated to (6-2) shows that $H^1(G', M_d)$ embeds in $H^2(G', N_d)$, and hence in $H^2(G', P)$. The long exact sequence associated to (6-1) gives

$$H^0(G', \mathbb{Z}[G] \oplus \mathbb{Z}) \to H^0(G', \mathbb{Z}) \to H^1(G', M_d) \to 0.$$

This shows that if $d \mid |G'|$, then $H^1(G', M_d) \cong \mathbb{Z}/d\mathbb{Z}$. If $|G'| = d$, it follows that $H^2(G', P)$ contains an element of order $|G'|$. Now, P is a permutation G'-module, that is, a direct sum of G'-modules of the form $\mathbb{Z}[G'/G'']$, where G'' is a subgroup of G'. We have

$$H^2(G', \mathbb{Z}[G'/G'']) = H^2(G'', \mathbb{Z}) = H^1(G'', \mathbb{Q}/\mathbb{Z}) \cong (G'')_{ab}$$

for every subgroup G'' of G', therefore the existence of an element of order $|G'|$ in $H^2(G', P)$ implies that G' must be cyclic. □

Note that, when p is odd, Proposition 6.2 is weaker than Theorem 1.1(b), which we have already proved. We will use Proposition 6.2 to prove Theorem 1.1 when $p = 2$.

7. Proof of Theorem 1.1 for $p = 2$

For every even integer $r \geq 4$, we denote by D_r the dihedral group of r elements. In particular, $D_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Lemma 7.1. *Let L/k be a finite Galois extension with Galois group G, and let $T := R^{(1)}_{L/k}(\mathbb{G}_{\mathrm{m}})$.*

(a) *Assume that $G = D_{2^n}$ for some $n \geq 1$. Then $BT[2]$ is retract rational.*

(b) *Let $G = Q_8$ be the quaternion group. Then $BT[2]$ is not retract rational.*

(c) *Let $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $BT[2]$ is not retract rational.*

Proof. (a) We start by showing that $Ш^2(D_{2^n}, \mathbb{Z}/2\mathbb{Z}) = 0$. Recall that the nontrivial element α of $H^2(D_{2^n}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is given by the central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Q_{2^{n+1}} \to D_{2^n} \to 1.$$

Let $g \in Q_{2^{n+1}}$ be an element of order 2^n, and let $\bar{g} \in D_{2^n}$ be the image of g. Then $\bar{g}$ has order 2^{n-1} and the following sequence is exact:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \langle g \rangle \to \langle \bar{g} \rangle \to 1.$$

This shows that the restriction map $H^2(D_{2^n}, \mathbb{Z}/2\mathbb{Z}) \to H^2(\langle \bar{g} \rangle, \mathbb{Z}/2\mathbb{Z})$ sends α to a nonzero class. This means that $Ш^2(D_{2^n}, \mathbb{Z}/2\mathbb{Z}) = 0$, as claimed.

The conclusion now follows from known results. If G' is a dihedral subgroup of D_{2^n}, then we have just shown that $Ш^2(G', \mathbb{Z}/2\mathbb{Z}) = 0$. By (4-2), we obtain $H^1(G', [\hat{T}_2]^{\text{fl}}) = 0$. If G' is a cyclic subgroup of G, then $H^1(G', [\hat{T}_2]^{\text{fl}}) = 0$ by [Colliot-Thélène and Sansuc 1977, Proposition 2]. Since every subgroup of G is either cyclic or dihedral, it follows that $[\hat{T}_2]^{\text{fl}}$ is coflasque. By [Voskresenski 1998, §4, Theorem 5], $[\hat{T}_2]^{\text{fl}}$ is invertible, so by [Saltman 1984, Proposition 3.14(a)], the torus T_2 is retract rational. Since $BT[2]$ is stably birational to T_2, we conclude that $BT[2]$ is retract rational.

(b) By Lemma 4.2 it suffices to show that $Ш^2(Q_8, \mathbb{Z}/2\mathbb{Z}) \neq 0$. Recall that $H^2(Q_8, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the three nontrivial classes correspond to central extensions of the form

$$1 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \Gamma \xrightarrow{\pi} Q_8 \to 1, \tag{7-1}$$

where

$$\Gamma := \langle a, b \mid a^4 = b^4 = e, bab^{-1} = a^{-1} \rangle \cong \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$$

is the unique nontrivial semidirect product of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$. Note that $Z(\Gamma) = \{e, a^2, b^2, a^2b^2\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $\iota(1) = a^2b^2$. Let $\alpha \in H^2(Q_8, \mathbb{Z}/2\mathbb{Z})$ be the class of (7-1), in the case when $\pi(a) = i$ and $\pi(b) = j$. Since

$$\pi^{-1}(\langle -1 \rangle) = \langle a^2, b^2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

the sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \pi^{-1}(\langle -1 \rangle) \to \langle -1 \rangle \to 1$$

splits, so α restricts to 0 in $H^1(\langle -1 \rangle, \mathbb{Z}/2\mathbb{Z})$. Moreover, the subgroups

$$\pi^{-1}(\langle i \rangle) = \langle a, b^2 \rangle, \quad \pi^{-1}(\langle j \rangle) = \langle a, b^2 \rangle, \quad \pi^{-1}(\langle i \rangle) = \langle ab, a^2b^2 \rangle$$

all admit a set of two commuting generators of orders 4 and 2, hence they are isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It follows that α restricts to 0 in $H^2(G', \mathbb{Z}/2\mathbb{Z})$ for every proper subgroup G' of Q_8, so $0 \neq \alpha \in Ш^2(Q_8, \mathbb{Z}/2\mathbb{Z})$.

(c) Let $a := (1, 0)$ and $b := (0, 1)$ be two generators of $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $\Gamma := G \rtimes \mathbb{Z}/2\mathbb{Z}$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on G by sending $a \mapsto ab$ and $b \mapsto b$. The center of Γ is $\langle a^2, b \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and we have a short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \to 1, \tag{7-2}$$

where $\iota(1) := b$. Let $\gamma \in \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and consider the exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma' \to \langle \gamma \rangle \to 1,$$

where $\Gamma' := \pi^{-1}(\langle \gamma \rangle)$. It is easy to check that $(b, 0)$ is not the square of any element of Γ, hence $\pi^{-1}(\langle\gamma\rangle)$ is a noncyclic extension of $\langle\gamma\rangle$ by $\mathbb{Z}/2\mathbb{Z}$. If $\langle\gamma\rangle \cong \mathbb{Z}/2\mathbb{Z}$, then necessarily $\Gamma' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and if $\langle\gamma\rangle \cong \mathbb{Z}/4\mathbb{Z}$, then $\Gamma' \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This shows that the class of (7-2) in $H^2(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is a nontrivial element of $Ш^2(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$. □

In the next proposition we collect the two pieces of information needed to complete the proof of Theorem 1.1 that we obtained via a computer calculation. We use Algorithm F2 of [Hoshi and Yamasaki 2017, §5.2], implemented in the computer algebra system GAP. I thank Thomas Rüd for helping me understand the code in [Hoshi and Yamasaki 2017].

Proposition 7.2. *If $G = Q_8$, then $BT[4]$ is not retract rational. If $G = (\mathbb{Z}/2\mathbb{Z})^3$, then $BT[2]$ is not retract rational.*

Proof. If (G, d) equals $(Q_8, 4)$ or $((\mathbb{Z}/2\mathbb{Z})^3, 2)$, it is enough to check that T_d is not retract rational, that is, that $[T_d]^{\text{fl}}$ is not invertible. Using the presentation (6-1), this can be done using Algorithm F2 of [Hoshi and Yamasaki 2017, §5.2]. □

Remark 7.3. Much as in the proof of Lemma 7.1(a) (or indeed by another GAP computation), one may show that $Ш^2(Q_8, \mathbb{Z}/4\mathbb{Z}) = 0$ and $Ш^2((\mathbb{Z}/2\mathbb{Z})^3, \mathbb{Z}/2\mathbb{Z}) = 0$. In other words, in the two cases considered in Proposition 7.2, T_d is coflasque but not retract rational; cf. [Voskresenski 1998, §4 Example 2, Theorem 5]. We will not make use of this in the sequel.

Proof of Theorem 1.1 for $p = 2$. If d is odd, the conclusion of the theorem follows from Proposition 3.3(d). From now on, we assume that d is even.

(a') The proof is the same as that of (a). When G is cyclic, every torus split by G is retract rational by [Colliot-Thélène and Sansuc 1977, Proposition 2], hence T_d is retract rational, and so $BT[d]$ is also retract rational.

(b') When $4 \nmid d$, $BT[d]$ is retract rational by Lemma 7.1(a) and Proposition 3.3(d), so we may assume that $4 \mid d$. If $G = D_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $e(T) = 4$; see the first paragraph of Section 4. Since BT is rational, by Proposition 3.3(b) we deduce that $BT[d]$ is stably birational to T. The torus T is not retract rational by [Colliot-Thélène and Sansuc 1977, Proposition 2], hence $BT[d]$ is not retract rational in this case. If $G = D_{2^n}$ for some $n \geq 3$, then G contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the claim follows from Lemma 5.1.

(c') Let $|G| = 2^n$. Assume first that $BT[d]$ is retract rational for some d divisible by 4. We prove by induction on n that G is cyclic. If $n = 1$, there is nothing to prove. If $n = 2$, the claim follows from (b'). When $n = 3$, by the case $n = 2$ and Lemma 5.1, G does not contain a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, hence $G = \mathbb{Z}/8\mathbb{Z}$ or Q_8. When $G = Q_8$, $BT[d]$ is not retract rational by Proposition 7.2, hence $G = \mathbb{Z}/8\mathbb{Z}$. If $n \geq 4$, by Lemma 5.1 and the inductive assumption, every subgroup of G is either cyclic or dihedral, hence by Lemma 4.3(c), G is also cyclic or dihedral.

Assume now that $BT[d]$ is retract rational for some $d \equiv 2 \pmod 4$. We prove that G is either cyclic or dihedral by induction on n. If $n = 1, 2$, there is nothing to prove. If $n = 3$, then by Lemma 7.1, the group G is not isomorphic to Q_8 or $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and by Proposition 7.2, it is not isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, hence it is either $\mathbb{Z}/8\mathbb{Z}$ or D_8, as desired. If $n \geq 4$, by inductive assumption, every proper subgroup of G is either cyclic or dihedral. By Lemma 4.3(c), G is either cyclic or dihedral. □

Example 7.4. Let L/k be a Galois extension with Galois group a symmetric group S_n, $d \geq 1$, be an integer, and p be an odd prime. Recall that if $n/2 < p \leq n$, so that $p \mid n!$ but $p^2 \nmid n!$, then any p-Sylow subgroup of S_n is cyclic of order p, and that if $p \leq n/2$ then any p-Sylow subgroup of S_n contains a subgroup of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. The 2-Sylow subgroups of S_n are cyclic for $n = 2, 3$, dihedral for $n = 4, 5$, and neither cyclic nor dihedral for $n \geq 6$ (they contain subgroups of the form $D_8 \times \mathbb{Z}/2\mathbb{Z}$). Applying Theorem 1.1 and Lemma 5.2, we obtain:

- If there exists an odd prime p such that $2p \leq n$ and $p \mid d$, then $BT[d]$ is not retract rational.
- If $n = 4, 5$, and $4 \mid d$, then $BT[d]$ is not retract rational.
- If $n \geq 6$, and d is even, then $BT[d]$ is not retract rational.

In all other cases, $BT[d]$ is retract rational.

Assume further that $d \mid |S_n| = n!$; by Proposition 3.3(d) one can reduce to this case by multiplying by some invertible element in $\mathbb{Z}/e(T)\mathbb{Z}$. Define $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$\sigma(1) = 0, \quad \sigma(2) = 2, \quad \sigma(3) = 6, \quad \sigma(4) = 6, \quad \sigma(5) = 30$$

and

$$\sigma(n) = \prod_{n/2 < p \leq n} p, \quad n \geq 6.$$

Then we have just shown that $BT[d]$ is p-retract rational if and only if $d \mid \sigma(n)$.

8. An application to the Grothendieck ring of stacks

The ring $K_0(\mathrm{Stacks}_k)$ was introduced by Ekedahl [2009b]. By definition, it is the quotient of the free abelian group on the equivalence classes $\{\mathcal{X}\}$ of all algebraic stacks $\mathcal{X}$ of finite type over k with affine stabilizers, by the scissor relations $\{\mathcal{X}\} = \{\mathcal{Y}\} + \{\mathcal{X} \setminus \mathcal{Y}\}$ for every closed substack $\mathcal{Y} \subseteq \mathcal{X}$ and the relations $\{\mathcal{E}\} = \{\mathcal{X} \times_k \mathbb{A}^n_k\}$ for every vector bundle $\mathcal{E}$ of rank n over $\mathcal{X}$. The product is defined by $\{\mathcal{X}\} \cdot \{\mathcal{Y}\} := \{\mathcal{X} \times_k \mathcal{Y}\}$ on the generators and extended by linearity; the multiplicative identity is $1 = \{\operatorname{Spec} k\}$.

If G is a finite or connected group, there appears to be a connection between properties of $\{BG\}$ and the Noether Problem for G; as of now, the link between the two is largely unexplained in either direction. Consider the following equations in $K_0(\mathrm{Stacks}_k)$, called expected class formulas:

$$\begin{aligned} &\{BG\} = 1, && G \text{ a finite group;} \\ &\{BG\} = \{G\}^{-1}, && G \text{ a connected linear group.} \end{aligned}$$

As shown in [Ekedahl 2009a], it frequently happens that $\{BG\} = 1$ for a finite group G. This is true, for example, if G is a symmetric group; see [Ekedahl 2009a, Theorem 4.3]. It is striking that all the known counterexamples G to $\{BG\} = 1$ are also counterexamples to the Noether Problem.

If G is a connected group and k is algebraically closed, no counterexample to $\{BG\}\{G\} = 1$ is known; this is again in line with the Noether Problem, for which no negative answer is known among connected groups. In [Scavia 2018, Theorem 1.5], we exhibited the first connected counterexample T to the expected class formula, in the case when k admits a biquadratic field extension and $\mathrm{char}\, k = 0$. More precisely, $T := R^{(1)}_{E_1 \times E_2/k}(\mathbb{G}_\mathrm{m})$, where E_1 and E_2 are distinct quadratic extensions of k; see [Scavia 2018, §3]. Using this result, we showed that $\{BT[2]\} = \{BT\}\{T\} \neq 1$ in [Scavia 2018, Theorem 1.6]. The same argument also shows that $\{BT[2d]\} = \{BT\}\{T\} \neq 1$ for every $d \geq 1$. On the other hand, one has $\{BT[2d-1]\} = 1$ for every $d \geq 1$. Proposition 3.3 provides a conceptual explanation for this periodicity. Moreover, it allows us to compute $\{BT[n]\}$ for every $n \geq 1$.

Proposition 8.1. *Let T be a k-torus, and $m, n \geq 1$ be integers.*

(a) *If $n \equiv \pm m \pmod{e(T)}$, then $\{BT[n]\} = \{BT[m]\}$.*

(b) *If $n \equiv \pm 1 \pmod{e(T)}$, then $\{BT[n]\} = 1$.*

(c) *If $e(T) \mid n$, then $\{BT[n]\} = \{BT\}\{T\}$.*

Proof. Fix an embedding of T into a quasisplit torus S, and consider the corresponding diagram (3-3) for some $d \geq 1$. Since the quasisplit torus S is special, by [Bergh 2016, Corollary 2.4], we have $\{BS\}\{S\} = 1$ in $K_0(\mathrm{Stacks}_k)$, hence $\{S\}$ is invertible. By [Bergh 2016, Proposition 2.9], we have

$$\{BT\} = \{Q\}/\{S\} \tag{8-1}$$

and

$$\{BT[d]\} = \{T_d\}/\{S\} \tag{8-2}$$

in $K_0(\mathrm{Stacks}_k)$.

(a) If $n \equiv m \pmod{e(T)}$, by Proposition 3.1 we have $T_n \cong T_m$, hence the claim follows from (8-2).

(b) If $n \equiv \pm 1 \pmod{e(T)}$ then by (a) we have $\{BT[n]\} = \{BT[1]\} = 1$.

(c) If $e(T) \mid n$, by Proposition 3.1 we have $T_n \cong T \times Q$, so $\{T_n\} = \{T\}\{Q\}$. Combining this with (8-1) and (8-2) (for $n = d$), we obtain

$$\{BT[n]\} = \{T_n\}/\{S\} = \{T\}\{Q\}/\{S\} = \{T\}\{BT\}. \qquad \square$$

Proposition 8.2. *Let $T := R^{(1)}_{E/k}(\mathbb{G}_{\mathrm{m}})$, where E is a product of two distinct separable quadratic extensions of k. For every $d \geq 1$, we have*

$$\{BT[2d-1]\} = 1, \quad \{BT[2d]\} = \{BT\}\{T\}$$

in $K_0(\mathrm{Stacks}_k)$.

Proof. Using the norm exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[C_2] \oplus \mathbb{Z}[C_2] \to \hat{T} \to 0,$$

the argument of [Merkurjev 2010, Example 5.1] adapts verbatim to a proof that $e(T) = 2$. The conclusion now follows from Proposition 8.1. $\square$

Since $\{BT\}\{T\} \neq 1$ by [Scavia 2018, Theorem 1.5], the previous proposition gives a more conceptual proof of [Scavia 2018, Theorem 1.6].

Acknowledgments

I would like to thank my advisor Zinovy Reichstein for useful comments on this work, and Thomas Rüd for helping me with GAP. I am grateful to the anonymous referee for a number of useful suggestions.

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Received May 1, 2019. Revised December 25, 2019.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 306 No. 2 June 2020

0030-8730(202006)306:2;1-J