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A CONICAL APPROACH TO LAURENT EXPANSIONS FOR MULTIVARIATE MEROMORPHIC GERMS WITH LINEAR POLES

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We develop a geometric approach using convex polyhedral cones to build Laurent expansions for multivariate meromorphic germs with linear poles, which naturally arise in various contexts in mathematics and physics. We express such a germ as a sum of a holomorphic germ and a linear combination of special nonholomorphic germs called polar germs. In analyzing the supporting cones — cones that reflect the pole structure of the polar germs — we obtain a geometric criterion for the nonholomorphicity of linear combinations of polar germs. For any given germ, the above decomposition yields a Laurent expansion which is unique up to suitable subdivisions of the supporting cones. These Laurent expansions lead to new concepts on the space of meromorphic germs, such as a generalization of Jeffrey–Kirwan's residue and a filtered residue, all of which are independent of the choice of the specific Laurent expansion.

1. Introduction

Our study aims at an extension of the classical Laurent theory in one variable to multivariate meromorphic germs.

The classical Laurent theory assigns to a meromorphic germ at zero in one variable a unique Laurent expansion. So the space $\mathcal{M}_{\mathbb{C}}(\mathbb{C})$ of meromorphic germs

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at zero (say in the variable ε) splits into the space $\mathcal{M}_{\mathbb{C},+}(\mathbb{C})$ of holomorphic germs at zero and the space

$$\mathbb{C}[\![\varepsilon^{-1}]\!] = \bigoplus_{k\leq -1} \mathbb{C}\varepsilon^k$$

consisting of the polar part of meromorphic germs

$$\mathcal{M}_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}[\![\varepsilon^{-1}]\!] \oplus \mathcal{M}_{\mathbb{C},+}(\mathbb{C}), \tag{1}$$

the direct sum on the right-hand side hosting Laurent expansions. We note that the direct sum decomposition comes as a consequence of the Laurent expansions.

One can raise the same questions for multivariate germs, namely how to

(a) define a "polar part" as a linear complement of the space of holomorphic germs;

(b) construct a basis of this linear complement from which to assign Laurent expansions to meromorphic germs.

We provide an answer to both questions for a class of multivariate meromorphic germs, called multivariate meromorphic germs at zero with linear poles, i.e., whose poles lie on unions of hyperplanes. Such meromorphic germs arise in various areas of mathematics and in physics as "regularized" integrals or sums with linear or conical constraints in the variables, and more specifically in

- perturbative quantum field theory when computing Feynman integrals by means of analytic regularization à la Speer [1974; 1975], where the linear constraints in the integration variables correspond to conservation of momentum;
- number theory with multiple zeta functions [Hoffman 1992; Zagier 1994] (see also [Guo and Zhang 2008; Manchon and Paycha 2010; Matsumoto 2003; Zhao 2000]) and their generalizations such as cyclotomic [Terasoma 2004] and Witten multiple zeta functions [Komori et al. 2015] — the conical constraint $n_1 > \cdots > n_k$ on the variables in the summand used to define multiple zeta functions is responsible for the linearity of the poles;
- the combinatorics on cones when evaluating exponential integrals or sums on cones following Berline and Vergne [2007] (see also [Garoufalidis and Pommersheim 2012]) in the context of the Euler–Maclaurin formula;
- algebraic geometry, in particular with the celebrated Jeffrey–Kirwan residue [Jeffrey and Kirwan 1995; 1997]; see also [De Concini and Procesi 2011] for a review.

We develop a geometric approach of multivariate meromorphic germs with linear poles, by which the classical Laurent theory generalizes to a *local Laurent theory* in the sense that one can cover the space of meromorphic germs at zero with linear poles by what we call Laurent subspaces. To define these Laurent subspaces, we use the geometry of cones. Laurent subspaces are indexed by properly positioned families of simplicial cones, a family of cones being properly positioned when the cones meet along faces and their union does not contain any nontrivial subspace. So a meromorphic germ in a Laurent subspace has a Laurent expansion *supported* by the corresponding properly positioned family of simplicial cones, called *supporting cones*. In the framework of certain generalized subdivisions we call pan-subdivisions, one can build a direct system of properly positioned families of simplicial cones. Thus the corresponding set of Laurent subspaces inherits a direct system structure whose direct limit is the whole space of meromorphic germs at zero with linear poles.

We explain below our geometric approach and at the same time give an outline of the paper.

To distinguish the polar part from the holomorphic part, we fix an inner product and use the orthogonal complement to define the key concept of polar germs, which serve as building blocks for the polar part. A polar germ is a nonholomorphic germ represented by a fraction $h(\ell_1, \dots, \ell_m)/(L_1^{s_1} \cdots L_n^{s_n})$, with h a holomorphic germ at zero in variables $\ell_1, \dots, \ell_m$ orthogonal to the linear forms $L_1, \dots, L_n$ in the pole with multiplicities $s_1, \dots, s_n$ (Definition 2.4). We then decompose a meromorphic germ as a sum of polar germs and a holomorphic germ (Theorem 2.11) by operations on fractions.

However such a decomposition is not unique, as indicated by the equation

$$\frac{1}{L_1 L_2} = \frac{1}{L_1(L_1 + L_2)} + \frac{1}{L_2(L_1 + L_2)}.$$

We use the geometry underlying meromorphic germs to address the uniqueness of the decompositions. To a polar germ $h(\ell_1, \dots, \ell_m)/(L_1^{s_1} \cdots L_n^{s_n})$, we assign a supporting cone (Definition 3.1) $\langle L_1, \dots, L_n \rangle$ generated by the vectors $L_1, \dots, L_n$.

The notion of supporting cones provides a geometric criterion for the non-holomorphicity, in particular the linear independence, of a sum of polar germs in Theorem 3.7. More precisely, if the family of supporting cones of a linear combination of polar germs is "properly positioned" (Definition 3.2), then this linear combination cannot be holomorphic. Properly positioned supporting cones are essential to assign Laurent expansions to meromorphic germs (Definition 4.5) with the help of a surjective "forgetful map" (17) from the space of formal Laurent expansions to the space of meromorphic germs. The very fact that this definition makes sense — namely that there is a well-defined notion of Laurent expansion for multivariate meromorphic germs with linear poles — is a central result of this paper.

To achieve this goal, we first identify the Laurent subspaces in Proposition 4.3 and build from there the Laurent expansion supported on an appropriate properly positioned family of cones. With the notation of the previous example, $1/(L_1 L_2)$

is the Laurent expansion supported on the properly positioned family $\{\langle L_1, L_2\rangle\}$ while $1/(L_1(L_1+L_2)) + 1/(L_2(L_1+L_2))$ is the Laurent expansion supported on the properly positioned family $\{\langle L_1, L_1+L_2\rangle, \langle L_2, L_1+L_2\rangle\}$.

From this Laurent theory for multivariate meromorphic germs with linear poles, we easily derive various other results, some of which are already known. As with the one variable case, an immediate consequence of the Laurent expansions is a splitting of the space of meromorphic germs with linear poles into a direct sum of the space of holomorphic ones and the space spanned by the polar germs (Corollary 4.16). This direct sum defines in turn a projection onto the holomorphic part along the polar part, a "multivariate subtraction", which is multiplicative on orthogonally variate germs (Corollary 4.18), which generalizes the minimal subtraction projection operator for meromorphic germs in one variable. This multivariate projection is a key ingredient for the multivariate algebraic Birkhoff factorization [Guo et al. 2017].

On the grounds of the Laurent expansions and homogeneity properties of the kernel of the forgetful map (Theorem 4.20), we equip the space of meromorphic germs with multiple gradings, given by total orders of the poles — called the p-order — of the polar germs arising in the expansion, the spaces spanned by the supporting cones, as well as the dimensions of the supporting cones (Theorem 5.3). These gradings yield several further applications by means of a uniformized approach. More precisely:

(a) We generalize the Brion–Vergne decomposition [Brion and Vergne 1999] $R_\Delta = G_\Delta \oplus NG_\Delta$ of rational germs at zero with poles lying in unions of hyperplanes in a given hyperplane arrangement Δ. See Corollary 5.5.

(b) Through a projection to a suitable component from one of the gradings, we define a generalized Jeffrey–Kirwan residue [Brion and Vergne 1999; Jeffrey and Kirwan 1995] valid for all meromorphic germs at zero with linear poles amongst which lie rational functions in R_Δ. See Corollary 5.7 and Definition 5.8. This generalized residue compares with Grothendieck's residue symbol (see, e.g., [Hopkins and Lipman 1979]), also called multidimensional residue, in that it coincides with the latter for functions with linear poles. Similarly to our residue, the Grothendieck residue arises in the combinatorics on polytopes as a tool to count integer points on certain dilated polytopes [Beck 2000], leading to a derivation of Erhardt's polynomial. It further arises together with its integral representation in the context of queuing networks [Bertozzi and McKenna 1993].

The multivariate Laurent expansions we build here have further interesting consequences leading to new results. The "p-order" generalized to meromorphic germs at zero gives a filtration of the meromorphic germs which generalizes the filtration by the order of the poles on meromorphic germs at zero in one variable and defines a valuation on the division ring of Laurent series [Efrat 2006, Example 4.2.2]. We further introduce a filtered residue, the "p-residue" (Definition 6.1) which, for Laurent expansions $\sum_{n=-N}^{\infty} a_n\, x^n$ in one variable filtered by the valuation given by the order $N \geq 0$ of the poles, corresponds to a_N/x^N. Composed with the exponential sum S [Barvinok 2008] on a lattice cone p-res $\circ\, S$, the p-residue turns out to be compatible with subdivisions (Proposition 6.7), as a result of which (see Corollary 6.8), the p-residue yields back the corresponding exponential integral on the lattice cone, related to the former by the Euler–Maclaurin formula studied in [Guo et al. 2017]. The multiplicativity of the holomorphic projection on orthogonally variate germs stated in Corollary 4.18 turns out to be of central importance in the context of renormalization and serves as a starting point for the more general notion of locality and locality maps introduced in subsequent work [Clavier et al. 2019]. Composing the holomorphic projection with the evaluation at zero gives a locality character in the terminology of [Clavier et al. 2019], namely a multivariate renormalized evaluator at zero.

From the perspective of renormalization, widely studied in the mathematical community since the pioneering work of Connes and Kreimer [1998], in a multivariate approach à la Speer [1974; 1975], Laurent expansions are expected to play a central role. Indeed, they serve to classify multivariate renormalized evaluators on meromorphic germs with linear poles at zero. This, together with the study of related group actions on evaluators reminiscent of the renormalization group in quantum field theory, is the purpose of separate ongoing work.

Throughout the paper, we work with a fixed subfield $\mathbb{F}$ of $\mathbb{R}$. The rational case $\mathbb{F} = \mathbb{Q}$ of particular interest in applications gives rational Laurent expansions for rational meromorphic germs.

2. A decomposition of meromorphic germs

In order to show the existence of a decomposition of the space of meromorphic germs, we first introduce the concept of a polar germ, which later serves as the building blocks of the linear complement of holomorphic germs.

2A. ***Polar germs.*** We begin with some necessary preliminary concepts.

Definition 2.1. (a) A *lattice* (*vector*) *space* is a pair (V, Λ_V), where V is a finite-dimensional real vector space and Λ_V is a lattice in V, that is, a finitely generated abelian subgroup of V whose $\mathbb{R}$-linear span is V.

(b) An $\mathbb{F}$-*inner product* on a lattice space (V, Λ_V) is an inner product Q on V such that the restriction of Q to $\Lambda_V \otimes \mathbb{F} \subseteq \Lambda_V \otimes \mathbb{R} = V$ and hence to Λ_V takes values in $\mathbb{F}$.

(c) A lattice space with an $\mathbb{F}$-inner product is called an $\mathbb{F}$-*Euclidean lattice space*.

(d) A *filtered space* is a real vector space V with a filtration $V_1 \subset V_2 \subset \cdots$ of finite-dimensional real vector subspaces such that $V = \bigcup_{k\geq 1} V_k$. Let $j_k : V_k \to V_{k+1}$ denote the inclusion.

(e) A *filtered lattice space* is a filtered space $V = \bigcup_{k\geq 1} V_k$ with lattices $\Lambda_k := \Lambda_{V_k}$ of V_k such that $\Lambda_{k+1}|_{V_k} = \Lambda_k$, $k \geq 1$. Then we denote the filtered lattice space by $(V, \Lambda_V) = \bigcup_{k\geq 1}(V_k, \Lambda_{V_k})$, where $\Lambda_V = \bigcup_{k\geq 1} \Lambda_{V_k}$.

(f) An *inner product* Q on a filtered space $V = \bigcup_{k\geq 1} V_k$ is a sequence of inner products

$$Q_k(\cdot\,,\cdot) = (\cdot\,,\cdot)_k : V_k \otimes V_k \to \mathbb{R}, \quad k \geq 1,$$

that is compatible with the inclusions j_k, $k \geq 1$.

(g) An $\mathbb{F}$-*inner product* on a filtered lattice space (V, Λ_V) is an inner product $Q := \{Q_k\}_{k\geq 1}$ on the filtered space $V = \bigcup_{k\geq 1} V_k$ such that Q_k is an $\mathbb{F}$-inner product for each $k \geq 1$. A filtered lattice space together with an $\mathbb{F}$-inner product is called a *filtered* $\mathbb{F}$-*Euclidean lattice space*.

We now assume that $(V, \Lambda_V) = \bigcup_{k\geq 1}(V_k, \Lambda_k)$ is a filtered $\mathbb{F}$-Euclidean lattice space. Let $V_k^* := \mathrm{Hom}(V_k, \mathbb{R})$ be the dual space of V_k. The $\mathbb{F}$-inner product $Q_k : V_k \otimes V_k \to \mathbb{R}$ induces an isomorphism $Q_k^* : V_k \to V_k^*$. This yields an embedding $V_k^* \hookrightarrow V_{k+1}^*$ induced from $j_k : V_k \to V_{k+1}$. The direct limit

$$V^{\circledast} := \varinjlim V_k^* = \bigcup_{k=0}^{\infty} V_k^* \tag{2}$$

is called the *filtered dual space* of $V = \bigcup_{k=0}^{\infty} V_k$. Notice that $V^{\circledast}$ is a proper subspace of the usual dual space V^* unless V is finite-dimensional.

Definition 2.2. Let $\bigcup_{k\geq 1}(V_k, \Lambda_k)$ be a filtered lattice space.

(a) A meromorphic germ $f(\vec{\varepsilon})$ on $V_k^* \otimes \mathbb{C}$ is said to have *linear poles at zero with coefficients in* $\mathbb{F}$ if there exist vectors $L_1, \dots, L_n \in \Lambda_{V_k} \otimes \mathbb{F}$ (possibly with repetitions) such that $f \prod_{i=1}^n L_i$ is a holomorphic germ at zero whose Taylor expansion for coordinates in the dual basis $\{e_1^*, \dots, e_k^*\}$ of a given (and hence every) basis $\{e_1, \dots, e_k\}$ of Λ_k has coefficients in $\mathbb{F}$.

(b) Let $\mathcal{M}_{\mathbb{F}}(V_k^* \otimes \mathbb{C})$ denote the set of germs of meromorphic functions on $V_k^* \otimes \mathbb{C}$ with linear poles at zero and with coefficients in $\mathbb{F}$. It is a linear space over $\mathbb{F}$.

(c) A germ of meromorphic functions of the form $1/(L_1^{s_1} \cdots L_n^{s_n})$ with linearly independent vectors $L_1, \dots, L_n$ in $\Lambda_k \otimes \mathbb{F}$ and $s_1, \dots, s_n \geq 1$ is called a *simplicial fraction* with coefficients in $\mathbb{F}$ (or *simplicial* $\mathbb{F}$-*fraction*). Such a fraction is called *simple* if $s_1 = \cdots = s_n = 1$.

Remark 2.3. The case $\mathbb{F} = \mathbb{Q}$ is of special interest in applications; we call meromorphic germs with linear poles at zero *rational* if they have rational coefficients.

Since a set of vectors in $\Lambda_k \otimes \mathbb{F}$ is $\mathbb{F}$-linearly independent if and only if it is $\mathbb{R}$-linearly independent in V_k, from now on we just call it linearly independent without specifying the type of coefficients.

Composing with the map $j_k^* : V_{k+1}^* \to V_k^*$ dual to $j_k : V_k \to V_{k+1}$ yields the embedding

$$\mathcal{M}_{\mathbb{F}}(V_k^* \otimes \mathbb{C}) \hookrightarrow \mathcal{M}_{\mathbb{F}}(V_{k+1}^* \otimes \mathbb{C}),$$

giving rise to the direct limit

$$\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) := \varinjlim \mathcal{M}_{\mathbb{F}}(V_k^* \otimes \mathbb{C}) = \bigcup_{k=1}^{\infty} \mathcal{M}_{\mathbb{F}}(V_k^* \otimes \mathbb{C}).$$

Let $\mathcal{M}_{\mathbb{F},+}(V_k^* \otimes \mathbb{C})$ denote the space of germs of holomorphic functions at zero in $V_k^* \otimes \mathbb{C}$ whose Taylor expansions at zero under the dual basis of a basis of Λ_k have coefficients in $\mathbb{F}$. We set

$$\mathcal{M}_{\mathbb{F},+}(V^{\circledast} \otimes \mathbb{C}) := \bigcup_{k=1}^{\infty} \mathcal{M}_{\mathbb{F},+}(V_k^* \otimes \mathbb{C}).$$

When $\mathbb{F} = \mathbb{R}$, we usually drop the subscript $\mathbb{F}$ from the notation.

When V_k is taken to be $\mathbb{R}^k$ and is equipped with its standard lattice $\mathbb{Z}^k$, the dual space V_k^* is identified with $\mathbb{R}^k$ equipped with the standard lattice. Then the space $\mathcal{M}_{\mathbb{F},+}(\mathbb{C}^k) = \mathcal{M}_{\mathbb{F},+}(V_k^* \otimes \mathbb{C})$ corresponds to the space of germs of holomorphic functions at zero in $\mathbb{C}^k$ whose Taylor expansions at zero have coefficients in $\mathbb{F}$ with respect to the canonical basis of $\mathbb{R}^k$.

We next identify a linear complement of $\mathcal{M}_{\mathbb{F},+}(V_k^* \otimes \mathbb{C})$ which is canonical upon fixing an inner product on V_k. It is spanned by a class of germs which then can be regarded as purely nonholomorphic germs. More importantly, they also serve as the building blocks for our Laurent expansions of meromorphic germs in multiple variables with linear poles. See Section 4. For notational simplicity, we call them polar germs.

Definition 2.4. Let $(V, \Lambda_V) = \bigcup_k (V_k, \Lambda_k)$ be a filtered $\mathbb{F}$-Euclidean lattice space with its $\mathbb{F}$-inner product Q. A *polar germ with* $\mathbb{F}$*-coefficients* or simply an $\mathbb{F}$*-polar germ* in $V_k^* \otimes \mathbb{C}$ is a germ of meromorphic functions at zero of the form

$$\frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}}, \tag{3}$$

where

(a) h lies in $\mathcal{M}_{\mathbb{F},+}(\mathbb{C}^m)$,

(b) $\ell_1, \dots, \ell_m, L_1, \dots, L_n$ lie in $\Lambda_k \otimes \mathbb{F}$, with $\ell_1, \dots, \ell_m$ and $L_1, \dots, L_n$ linearly independent, such that, setting $[k] := \{1, \dots, k\}$ for positive integers k,

$$Q(\ell_i, L_j) = 0 \quad \text{for all } (i, j) \in [m] \times [n],$$

(c) $s_1, \dots, s_n$ are positive integers.

For notational simplicity, we also set $\vec{L}^{\vec{s}} := L_1^{s_1} \cdots L_n^{s_n}$ and write

$$\frac{h(\vec{\ell})}{\vec{L}^{\vec{s}}} := \frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}}. \tag{4}$$

Definition 2.5. We let $\mathcal{M}^Q_{\mathbb{F},-}(V_k^* \otimes \mathbb{C})$ denote the linear subspace of $\mathcal{M}_{\mathbb{F}}(V_k^* \otimes \mathbb{C})$ spanned by $\mathbb{F}$-polar germs and set

$$\mathcal{M}^Q_{\mathbb{F},-}(V^{\circledast} \otimes \mathbb{C}) := \bigcup_{k=1}^{\infty} \mathcal{M}^Q_{\mathbb{F},-}(V_k^* \otimes \mathbb{C}) = \varinjlim \mathcal{M}^Q_{\mathbb{F},-}(V_k^* \otimes \mathbb{C}),$$

regarding $\{\mathcal{M}^Q_{\mathbb{F},-}(V_k^* \otimes \mathbb{C})\}_k$ as a subdirect system of $\{\mathcal{M}_{\mathbb{F}}(V_k^* \otimes \mathbb{C})\}_k$.

Remark 2.6. The space $\mathcal{M}_{\mathbb{F},-}(V^{\circledast} \otimes \mathbb{C})$ is not closed under the function product. As a simple example, for the canonical inner product on $\mathbb{R}^2$, both $f(\varepsilon_1, \varepsilon_2) := \varepsilon_1/\varepsilon_2$ and $g(\varepsilon_1, \varepsilon_2) := \varepsilon_2/\varepsilon_1$ are polar germs. But their product 1 is not.

Example 2.7. (a) For linearly independent vectors $L_1, \dots, L_k \in \Lambda_k \otimes \mathbb{F}$ and $s_1, \dots, s_k > 0$, $1/(L_1^{s_1} \cdots L_k^{s_k})$ lies in $\mathcal{M}^Q_{\mathbb{F},-}(V_k^* \otimes \mathbb{C})$ for any inner product Q.

(b) For the canonical Euclidean inner product on $\mathbb{R}^2$, the functions $f(\varepsilon_1 e_1^* + \varepsilon_2 e_2^*) = (\varepsilon_1 - \varepsilon_2)^t/(\varepsilon_1 + \varepsilon_2)^s$, $s > 0$, $t \geq 0$, lie in $\mathcal{M}^Q_{\mathbb{Q},-}((\mathbb{R}^2)^* \otimes \mathbb{C})$.

Remark 2.8. We mostly work with a filtered $\mathbb{F}$-Euclidean lattice space given by a fixed $\mathbb{F}$-inner product Q. Thus we often drop the superscript Q to simplify notation.

The following lemma shows the uniqueness of the expression of polar germs.

Lemma 2.9. *If a polar germ can be written as* $h(\ell_1, \dots, \ell_m)/(L_1^{s_1} \cdots L_n^{s_n})$ *and* $g(\ell'_1, \dots, \ell'_j)/((L'_1)^{t_1} \cdots (L'_\ell)^{t_\ell})$, *both satisfying the conditions in Definition 2.4, then* $n = \ell$ *and* $L'_1, \dots, L'_\ell$ *can be rearranged in such a way that* L_i *is a multiple of* L'_i *and* $s_i = t_i$ *for* $1 \leq i \leq n$.

Proof. We implement an induction on $M := \max(s_1 + \cdots + s_n, t_1 + \cdots + t_\ell)$.

To deal with the case when $M = 1$, assume that $h(\ell_1, \dots, \ell_m)/L$ equals either $g(\ell'_1, \dots, \ell'_j)/L'$ or $g(\ell'_1, \dots, \ell'_j)$, with h and g holomorphic. Extend $\{L, \ell_1, \dots, \ell_m\}$ to a basis $\{z_1, \dots, z_{m+1}, \dots, z_k\}$, with $z_1 = L$, $z_2 = \ell_1, \dots, z_{m+1} = \ell_m$. If we have $h(\ell_1, \dots, \ell_m)/L = g(\ell'_1, \dots, \ell'_j)/L'$, and L' is not a multiple of L, then

$$L'(z_1, z_2, \dots, z_k) = L''(z_2, \dots, z_k) + c z_1,$$

with $L''(z_2, \dots, z_k)$ not identically zero, so we can pick values $z_2^0, \dots, z_k^0$ such that $h(z_2^0, \dots, z_{m+1}^0) \neq 0$ and $L''(z_2^0, \dots, z_k^0) \neq 0$. Consider the restriction to $(z_1, z_2^0, \dots, z_k^0)$ of the equality $h(\ell_1, \dots, \ell_m)/L = g(\ell'_1, \dots, \ell'_j)/L'$. The left-hand side of the equality is singular in z_1 while the right-hand side is holomorphic in z_1.

This is a contradiction, showing that L must be a multiple of L'. The same argument shows that $h(\ell_1, \dots, \ell_m)/L = g(\ell'_1, \dots, \ell'_j)$ is impossible.

For the inductive step, suppose that none of the linear forms $L'_1, \dots, L'_\ell$ is a multiple of L_1. As in the case for $M = 1$, when

$$\frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1}} = \frac{g(\ell'_1, \dots, \ell'_k) L_2^{s_2} \cdots L_n^{s_n}}{(L'_1)^{t_1} \cdots (L'_\ell)^{t_\ell}}$$

is restricted to a proper choice of $z_2^0, \dots, z_k^0$, the left-hand side of the equality has a nontrivial singular part in z_1, while the right-hand side is holomorphic in z_1, which leads to a contradiction. Therefore we can rearrange $L'_1, \dots, L'_\ell$ so that $L_1 = cL'_1$ for some constant $c \neq 0$. Thus, from

$$\frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}} = \frac{g(\ell'_1, \dots, \ell'_j)}{(L'_1)^{t_1} \cdots (L'_\ell)^{t_\ell}}$$

we obtain

$$\frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1 - 1} \cdots L_n^{s_n}} = \frac{c g(\ell'_1, \dots, \ell'_j)}{(L'_1)^{t_1 - 1} \cdots (L'_\ell)^{t_\ell}}.$$

By the inductive hypothesis, the conclusion holds for the two sides of the equation. This completes the induction. □

2B. *Decomposition of a meromorphic germ into polar germs.* In this subsection we consider any lattice space (V, Λ), which can be taken to be (V_k, Λ_k) from a filtered lattice space. The notions for V_k such as $\mathcal{M}_{\mathbb{F}}(V_k^* \otimes \mathbb{C})$ and $\mathcal{M}_{\mathbb{F},\pm}(V_k^* \otimes \mathbb{C})$ can be defined in the same way for V.

Before giving the decomposition, we first provide some preliminary results.

Lemma 2.10. *Let (V, Λ) be a lattice space. Let $L_1, \dots, L_n$, $n \geq 2$, be vectors in $\Lambda \otimes \mathbb{F}$ and let $s_1, \dots, s_n$ be positive integers.*

(a) *If $L_1, \dots, L_n$ are $\mathbb{F}$-linearly independent and $L_{n+1} = \sum_{i=1}^n c_i L_i$ with nonzero $c_i \in \mathbb{F}$, $1 \leq i \leq n$, then*

$$\frac{1}{L_1^{s_1} \cdots L_{n+1}^{s_{n+1}}} = \sum_j \frac{b_j}{N_{j1}^{t_{j1}} \cdots N_{jn}^{t_{jn}}}. \tag{5}$$

For each j in the above sum, b_j is in $\mathbb{F}$ and $\{N_{j1}, \dots, N_{jn}\}$ is one of the sets $\{L_1, \dots, \widehat{L_i}, \dots, L_{n+1}\}$, $1 \leq i \leq n$ (where $\widehat{L_i}$ means that the factor L_i is omitted) and hence is a basis of the linear span $\operatorname{lin}\{L_1, \dots, L_{n+1}\}$ *of $L_1, \dots, L_{n+1}$.*

(b) *In general, the fraction $1/(L_1^{s_1} \cdots L_n^{s_n})$ can be rewritten as a linear combination*

$$\sum_i \frac{a_i}{M_{i1}^{t_{i1}} \cdots M_{in_i}^{t_{in_i}}},$$

with $a_i \in \mathbb{F}$ and linear independent subsets $\{M_{i1}, \dots, M_{in_i}\}$ of $\{L_1, \dots, L_n\}$.

Proof. (a) The statement easily follows from the straightforward identity

$$\frac{1}{L_1 \cdots L_r} = \sum_{i=1}^{n} \frac{c_i}{L_1 \cdots \widehat{L_i} \cdots L_r L_{r+1}},$$

by induction on the sum $m := \sum_{j=1}^{n} s_j$.

(b) Combining factors of linear forms that are multiples of each other if necessary, we can assume that the L_i are not multiples of each other. The statement then follows from an induction on the difference $d := n - \dim(\mathrm{lin}\{L_1, \ldots, L_n\})$ using (5) applied to a subset $L_{i_1}, \ldots, L_{i_r}$ of linearly independent forms such that $L_{i_r+1} = \sum_{j=1}^{r} c_j L_{i_j}$ for some $2 \le r \le n$. □

We are now ready to prove the existence of a decomposition of meromorphic germs at zero into a sum of holomorphic germs and polar germs. See [Berline and Vergne 2016] for a related result.

Theorem 2.11. *Let (V, Λ) be an $\mathbb{F}$-Euclidean lattice space with an $\mathbb{F}$-inner product Q. For any $f \in \mathcal{M}_{\mathbb{F}}(V^* \otimes \mathbb{C})$, there exists a finite set of $\mathbb{F}$-polar germs $\{S_j\}_{j \in J}$ and a holomorphic germ h in $\mathcal{M}_{\mathbb{F}}(V^* \otimes \mathbb{C})$ such that*

$$f = \left(\sum_{j \in J} S_j \right) + h. \tag{6}$$

Furthermore, the $\mathbb{F}$-polar germs S_j can be chosen to satisfy the following properties.

- *Their linear poles are taken from the linear poles of f.*
- *If the germ f can be written in the form $\tilde{f}(\ell_1, \ldots, \ell_n)$ for some function $\tilde{f}$ on $\mathbb{C}^n$ and linearly independent linear forms $\ell_1, \ldots, \ell_n$ on $(\Lambda \otimes \mathbb{F})^*$, then the polar germs S_j and the holomorphic germ h can be written as compositions of functions on $\mathbb{C}^n$ and linearly independent linear forms in* $\mathrm{span}(\ell_1, \ldots, \ell_n)$.

Remark 2.12. Whereas the holomorphic part turns out to be uniquely defined, the individual polar germs arising in this decomposition are not. In Corollary 3.8 we provide geometric conditions under which the polar germs can be uniquely determined, leading to Laurent expansions.

Proof. Thanks to Lemma 2.10(b), without loss of generality we can reduce the proof to meromorphic germs at zero of the form

$$f = \frac{h}{L_1^{s_1} \cdots L_m^{s_m}}$$

with $h \in \mathcal{M}_{\mathbb{F},+}(V^* \otimes \mathbb{C})$, $L_1, \ldots, L_m \in \Lambda \otimes \mathbb{F}$ linearly independent and $s_1, \ldots, s_m$ positive integers. Then we extend $\{L_1, \ldots, L_m\}$ to a basis $\{L_1, \ldots, L_m, \ell_1, \ldots, \ell_{k-m}\}$

of $\Lambda \otimes \mathbb{F}$ satisfying

$$Q(L_i, \ell_j) = 0, \quad 1 \le i \le m, \ 1 \le j \le k - m.$$

We proceed by induction on the sum $s := s_1 + \cdots + s_m$. If $s = 1$, then $m = 1$ and $s_1 = 1$. Under these conditions we have

$$\begin{aligned} f &= \frac{h(L_1, \ell_1, \dots, \ell_{k-1})}{L_1} \\ &= \frac{h(0, \ell_1, \dots, \ell_{k-1})}{L_1} + \frac{h(L_1, \ell_1, \dots, \ell_{k-1}) - h(0, \ell_1, \dots, \ell_{k-1})}{L_1}. \end{aligned}$$

The first term lies in $\mathcal{M}_{\mathbb{F},-}(V^* \otimes \mathbb{C})$ as a consequence of the orthogonality of L_1 with the ℓ_i. The second term is holomorphic at 0. This yields the required decomposition.

For $t \ge 1$, assume that the decomposition exists when $s \le t$ and consider $f = h(L_1, \dots, L_m, \ell_1, \dots, \ell_{k-m})/(L_1^{s_1} \cdots L_m^{s_m})$ with $s = s_1 + \cdots + s_m = t + 1$. We note that by [Grauert and Fritzsche 1976] the Taylor expansion of h gives

$$h(L_1, \dots, L_m, \ell_1, \dots, \ell_{k-m}) := h(0, \dots, 0, \ell_1, \dots, \ell_{k-m}) + \sum_{i=1}^{m} L_i g_i,$$

where the g_i are holomorphic. Thus $S_0 := h(0, \dots, 0, \ell_1, \dots, \ell_{k-m})/(L_1^{s_1} \cdots L_m^{s_m})$ is in $\mathcal{M}_{\mathbb{F},-}(V^* \otimes \mathbb{C})$, while by the induction hypothesis,

$$\frac{L_i g_i}{L_1^{s_1} \cdots L_m^{s_m}} = h_i + \sum_{j_i \in J_i} S_{j_i}$$

with h_i a holomorphic germ at zero and S_{j_i} polar germs in $\mathcal{M}_{\mathbb{F}}(V^* \otimes \mathbb{C})$. Hence $f = S_0 + \sum_{i=1}^{m} h_i + \sum_{j \in J_1 \cup \cdots \cup J_m} S_j$ is the sum of a holomorphic germ $\sum_{i=1}^{m} h_i$ and finitely many polar germs S_j.

Now for a germ f expressed in the form $\tilde{f}(\ell_1, \dots, \ell_n)$ as given in the theorem, replace the lattice space (V, Λ) by its lattice subspace $(W, \Lambda \cap W)$, where $W := \mathrm{span}(\ell_1, \dots, \ell_n)$. Then f is in $\mathcal{M}_{\mathbb{F}}(W^* \otimes \mathcal{C})$ and applying the first part of the theorem yields the second part of the theorem. □

3. A geometric criterion for nonholomorphicity

In this section, we pursue our geometric approach initiated in [Guo et al. 2014] to study meromorphic germs at zero through the cones associated to the germs. By means of the supporting cone of a polar germ, we first give a geometric criterion for the linear independence of simplicial fractions in Section 3A. We then obtain the main nonholomorphicity theorem in Section 3B.

3A. ***A geometric criterion for the linear independence of simplicial fractions.*** We briefly recall the notations and terminology of [Guo et al. 2014] and use the results obtained there on the geometry of cones underlying the decomposition of fractions, further refined to require that the coefficients lie in the subfield $\mathbb{F}$.

As in [Guo et al. 2014] we consider *closed convex polyhedral cones*, henceforth simply called *cones*, in a filtered lattice space $(V, \Lambda_V) = \bigcup_{k\geq 1}(V_k, \Lambda_{V_k})$. We call $\mathbb{F}$-*cones* the ones whose generators lie in $\Lambda_k \otimes \mathbb{F}$. A $\mathbb{Q}$-cone is called *rational*.

We recall that a *subdivision* of a cone C is a set $\{C_1, \dots, C_r\}$ of cones which have the same dimension as C, whose union is C, and which intersect along their faces, i.e., $C_i \cap C_j$ is a face of both C_i and C_j. Such a subdivision is *simplicial* (resp. *smooth*, in the case when C is rational) if all the C_i are simplicial (resp. smooth). An $\mathbb{F}$-*subdivision* of an $\mathbb{F}$-cone is a subdivision such that every C_i is an $\mathbb{F}$-cone.

On the grounds of Lemma 2.9, we can assign a simplicial cone to a polar germ.

Definition 3.1. Let
$$f := \frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}}$$
be a polar germ, as defined by the conditions in Definition 2.4. We say that the cone $\langle L_1, \dots, L_n\rangle$ inside V *supports* the germ; it is a *supporting cone of the germ*.

By Lemma 2.9, the supporting cones of a polar germ are defined up to the choice of a sign of each of the vectors $L_1, \dots, L_n$. Indeed, any cone $\langle \pm L_1, \dots, \pm L_n\rangle$ is also a supporting cone.

We now introduce the key concepts concerning families of cones.

Definition 3.2. (a) A family of cones is said to be *properly positioned* if the cones meet along faces and the union does not contain any nonzero linear subspace (equivalently, any line).

(b) A family of polar germs is called *properly positioned* if there is a choice of a supporting cone for each of the polar germs such that the resulting family of cones is properly positioned.

(c) A family of polar germs is called *projectively properly positioned* if it is properly positioned and none of the denominators of the polar germs is proportional to another.

So a family of polar germs is projectively properly positioned if it is properly positioned when viewed in the projective space of polar germs (that is, modulo scalar multiples), whence the terminology.

Example 3.3. Let $\{e_1, e_2\}$ be the canonical basis in $\mathbb{R}^2$.

(a) The family of cones $\{C_1 := \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, (e_1+e_2);\ C_2 := \mathbb{R}_{\geq 0}\, (e_1+e_2) + \mathbb{R}_{\geq 0}\, e_2\}$ is properly positioned, whereas the family of cones $\{C_1, C_2, C_3\}$ with $C_3 := \mathbb{R}_{\geq 0}\,(2e_1 + e_2) + \mathbb{R}_{\geq 0}\,(e_1 + 2e_2)$ is not.

(b) It follows that the family of polar germs

$$\Big\{f_1(z_1,z_2):=\frac{1}{z_1(z_1+z_2)};\ f_2(z_1,z_2):=\frac{1}{z_2(z_1+z_2)}\Big\}$$

is properly positioned whereas the family of polar germs

$$\Big\{f_1(z_1,z_2),\ f_2(z_1,z_2),\ f_3(z_1,z_2):=\frac{1}{(2z_1+z_2)(z_1+2z_2)}\Big\}$$

is not.

(c) The two germs in one variable $f(z_1)=1/z_1$ and $g(z_1):=1/(2z_1)$ have the same supporting cones $C:=\mathbb{R}_{\geq 0}\, e_1$ and are proportional (yet not equal) as germs.

We next give a reinterpretation of a related result in [Guo et al. 2014]; see also [Berline and Vergne 2007; Garoufalidis and Pommersheim 2012; Lawrence 1991].

Let C be a simplicial cone in V_k with $\mathbb{R}$-linearly independent generators $v_1,\dots,v_n$ expressed in a fixed basis $\{e_1,\dots,e_k\}$ as $v_i=\sum_{j=1}^k a_{ji}e_j$, for $1\leq i\leq n$. Define linear functions L_i, $1\leq i\leq n$, on $V_k^*\otimes\mathbb{C}$ by $L_i(\vec{\varepsilon}):=L_{v_i}(\vec{\varepsilon}):=\sum_{j=1}^k a_{ji}\varepsilon_j$, where $\vec{\varepsilon}:=\sum_{i=1}^k \varepsilon_j e_j^*\in V_k^*\otimes\mathbb{C}$ and $\{e_1^*,\dots,e_k^*\}$ is the dual basis in V_k^*. Let $A_C=[a_{ij}]$ denote the associated matrix in $M_{k\times n}(\mathbb{R})$. Let $w(v_1,\dots,v_n)$ or $w(C)$ denote the sum of absolute values of the determinants of all minors of A_C of rank n. As in [Guo et al. 2014] (except with a different notation Φ instead of I and a sign convention), define

$$I(C):=(-1)^n\frac{w(v_1,\dots,v_n)}{L_1\cdots L_n}. \tag{7}$$

We note that $I(C)$ is independent of the choice of the spanning vectors $v_1,\dots,v_n$, since a second choice of the spanning vectors amounts to rescaling the original spanning vectors and this does not change the fraction in $I(C)$.

Let C be a cone in V_k and let $\{C_i\}$ be a simplicial subdivision of C. By [Guo et al. 2014, Lemma 3.3], the sum

$$I(C):=\sum_i I(C_i) \tag{8}$$

is well-defined, independent of the choice of simplicial subdivisions, hence yielding a linear map

$$I:\mathbb{R}\mathcal{C}(\mathbb{R})\to\mathcal{M}_{\mathbb{F},-}(V^{\circledast}\otimes\mathbb{C}), \tag{9}$$

where $\mathbb{R}\mathcal{C}(\mathbb{R})$ is the $\mathbb{R}$-linear space spanned by the set $\mathcal{C}(\mathbb{R})$ of cones in V.

Now we are ready for our first geometric criterion for the linear independence of fractions.

Lemma 3.4. *A projectively properly positioned family of simple $\mathbb{F}$-fractions whose supporting cones span the same linear subspace is $\mathbb{F}$-linearly independent.*

Proof. We choose the supporting cones $\{C_i\}$ in such a way that the family is properly positioned. Since the simple $\mathbb{F}$-fractions are not pairwise proportional, these supporting cones are distinct. Since the cones are properly positioned, their union does not contain any nonzero linear subspace. Thus the union of the cones has a topological boundary in the sense that the union is not the whole space, that is, not all points of the union are interior points. Thus by [Guo et al. 2014, Lemma 3.5], the set $\{I(C_i)\}$ is linearly independent. But each $I(C_i)$ is a nonzero multiple of the original fraction. Thus the original family of simple fractions is linearly independent. □

Before the treatment of more general fractions, we give the following "locality" lemma.

Lemma 3.5. *Let $h_i/\vec{L}_i^{\vec{s}_i}$, $i = 1, \ldots, r$, be $\mathbb{F}$-polar germs and h_0 a holomorphic germ at zero satisfying*

$$\sum_{i=1}^{r} a_i \frac{h_i}{\vec{L}_i^{\vec{s}_i}} = h_0 \tag{10}$$

with $a_1, \ldots, a_r \in \mathbb{F}$. For any linear $\mathbb{F}$-subspace W of V and $N \in \mathbb{Z}_{>0}$, denote

$$I(W, N) := \{i \in [r] \mid \operatorname{span}(L_{i1}, \ldots, L_{in_i}) = W,\ |s_i| := s_{i1} + \cdots + s_{in_i} = N\}.$$

Then

$$\sum_{i \in I(W,N)} a_i \frac{h_i}{\vec{L}_i^{\vec{s}_i}} = 0,$$

with the convention that the sum over an empty set is zero.

Proof. For distinct pairs (W, N) and (W', N') arising in the expression (10) we have $I(W, N) \cap I(W', N') = \varnothing$. Thus $[r]$ is partitioned into finitely many nonempty and disjoint subsets $I(W_1, N_1), \ldots, I(W_p, N_p)$. Then

$$\sum_{j=1}^{p} \sum_{i \in I(W_j,N_j)} a_i \frac{h_i}{\vec{L}_i^{\vec{s}_i}} = \sum_{i=1}^{r} a_i \frac{h_i}{\vec{L}_i^{\vec{s}_i}} = h_0.$$

Suppose that an expression in (10) is a counterexample to the lemma. Then

$$\sum_{i \in I(W_j,N_j)} a_i \frac{h_i}{\vec{L}_i^{\vec{s}_i}} \neq 0 \tag{11}$$

for some $j \in [p]$. By dropping those $j \in [p]$ with $\sum_{i \in I(W_j,N_j)} a_i (h_i/\vec{L}_i^{\vec{s}_i}) = 0$ if necessary, we can assume that (11) holds for all $j \in [p]$.

Let $N := \max\{|s_i| = \sum_j s_{ij} \mid i \in [r]\}$ and let W be one of those $\operatorname{span}(L_{i1}, \ldots, L_{in_i})$ with $|s_i| = N$ whose dimension is minimal. Reordering the terms of the sum in (10) if necessary, we can assume that $I(W, N) = [t]$ for some $t \geq 1$. Thus $|s_i| = N$ and $\operatorname{span}(L_{i1}, \ldots, L_{in_i}) = W$ precisely for $i \in [t]$. Then (11) implies

$$\sum_{i=1}^{t} a_i \frac{h_i}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} \neq 0. \tag{12}$$

We extend the linearly independent linear forms $L_{11}, \ldots, L_{1n_1}$ to a basis $e_1, \ldots, e_k$ of $\Lambda_k \otimes \mathbb{F}$ with $e_i = L_{1i}$ for $i \in [n_1]$ such that $Q(e_j, e_\ell) = 0$ for $1 \leq j \leq n_1$, $n_1 + 1 \leq \ell \leq k$. Write the polar germs

$$\frac{h_i}{\vec{L}_i^{\vec{s}_i}} = \frac{h_i(\ell_{i1}, \ldots, \ell_{im_i})}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}$$

as in Definition 2.4. Since $e_1, \ldots, e_{n_1}$ is also a basis of $\operatorname{span}(L_{i1}, \ldots, L_{in_i})$ for $i \in [t]$, we have

$$Q(e_j, \ell_{ij}) = 0, \quad 1 \leq j \leq n_i,\ 1 \leq \ell \leq m_i,\ 1 \leq i \leq t.$$

So the linear forms $\ell_{i1}, \ldots, \ell_{im_i}$ lie in $\operatorname{span}(e_{n_1+1}, \ldots, e_k)$. Thus with respect to the dual basis $\{e_1^*, \ldots, e_k^*\}$ of the basis $\{e_1, \ldots, e_k\}$ of $V_k \otimes \mathbb{C}$, the functions $h_1(\ell_{11}, \ldots, \ell_{1m_1}), \ldots, h_t(\ell_{t1}, \ldots, \ell_{tm_t})$ as functions in the variables $\vec{\varepsilon} = \sum \varepsilon_i e_i^*$ are in fact germs at zero of holomorphic functions depending only on the variables $\varepsilon_{n_1+1}, \ldots, \varepsilon_k$, which we write as $h_1(\varepsilon_{n_1+1}, \ldots, \varepsilon_k), \ldots, h_t(\varepsilon_{n_1+1}, \ldots, \varepsilon_k)$.

Fix $i > t$. For any $j \in [n_i]$, write $L_{ij} = L'_{ij} + L''_{ij}$, where L'_{ij} is a linear combination of $e_1, \ldots, e_{n_1}$ and L''_{ij} is a linear combinations of $e_{n_1+1}, \ldots, e_k$. Thus $L''_{ij}(\vec{\varepsilon})$ is a linear function in $\varepsilon_{n_1+1}, \ldots, \varepsilon_k$. We note that $i > t$ if and only if either $\sum_{j=1}^{n_i} s_{ij} < N$, or there is an index j such that $L''_{ij} \neq 0$ as a result of the fact that $\{L_{i1}, \ldots, L_{in_i}\}$ and $\{L_{11}, \ldots, L_{1n_1}\}$ do not span the same linear space.

Since $h_i(\varepsilon_{n_1+1}, \ldots, \varepsilon_k)$, $1 \leq i \leq t$, are not identically zero, there are fixed values $\varepsilon_{n_1+1}^0, \ldots, \varepsilon_k^0$ of $\varepsilon_{n_1+1}, \ldots, \varepsilon_k$ for which $h_i(\varepsilon_{n_1+1}^0, \ldots, \varepsilon_k^0) \neq 0$, $1 \leq i \leq t$, and $L''_{ij}(\varepsilon_{n_1+1}^0, \ldots \varepsilon_k^0) \neq 0$, $i > t$ for those $L''_{ij} \neq 0$. These values form a nonempty open subset.

We next introduce a new set of variables r_m, $1 \leq m \leq n_1$, and ε, and apply the substitution $(\varepsilon_1, \ldots, \varepsilon_k) = (r_1\varepsilon, \ldots, r_{n_1}\varepsilon, \varepsilon_{n_1+1}^0, \ldots, \varepsilon_k^0)$ in (10). This gives rise to a series in ε that is holomorphic at zero by the choice of the germ. Thus the coefficient of every given negative power of ε is 0. In particular the coefficient of the least possible power ε^{-N} is zero. In order for a term $h_i/(L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}})$ in the sum to contribute to this coefficient, we must have $\sum_j s_{ij} = N$ and $L''_{ij} = 0$, that is, $1 \leq i \leq t$ as a result of the definition of t. On the other hand, for $1 \leq i \leq t$, $L_{i1}, \ldots, L_{in_i}$ are linear homogeneous in $L_{11}(\vec{\varepsilon}) = \varepsilon_1, \ldots, L_{1n_1}(\vec{\varepsilon}) = \varepsilon_{n_1}$. Hence under the above substitution, they give $\varepsilon L_{i1}, \ldots, \varepsilon L_{in_i}$ in the variables $r_1, \ldots, r_{n_1}$

and the coefficient of ε^{-N} in the Laurent series reads

$$\sum_{i=1}^{t} a_i \frac{h_i(\varepsilon^0_{n_1+1}, \dots, \varepsilon^0_k)}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}.$$

Hence it is zero as a sum of fractions in variables $r_1, \dots, r_{n_1}$. Thus

$$\sum_{i=1}^{t} a_i \frac{h_i(\varepsilon^0_{n_1+1}, \dots, \varepsilon^0_k)}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} = 0 \tag{13}$$

for any generic point $(\varepsilon^0_{n_1+1}, \dots, \varepsilon^0_k)$. Comparing with (12) gives the desired contradiction. □

Based on this lemma, Lemma 3.4 can be generalized to the following statement.

Proposition 3.6. *A projectively properly positioned family of simplicial* $\mathbb{F}$*-fractions is* $\mathbb{F}$*-linearly independent.*

Proof. We only need to prove that a contradiction follows from any linear relation

$$\sum_{i=1}^{r} a_i \frac{1}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} = 0, \quad 0 \neq a_i \in \mathbb{F}, \tag{14}$$

of a projectively properly positioned family of $\mathbb{F}$-fractions $G_i := 1/(L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}})$.

By Lemma 3.5, we can assume that, for each $1 \le i \le r$, the weight $|s_i| = s_{i1} + \cdots + s_{in_i}$ is the same and the linear forms in the denominators span the same space. In particular, $n_1 = \cdots = n_r = n$.

We next proceed by induction on $s := |s_i|$. So $s \ge n$. If $s = n$, then the powers of all the linear forms are equal to 1. It then follows from Lemma 3.4 that $a_i = 0$ for all indices i, leading to the expected contradiction. Assume that a contradiction arises for any relation in (14) with $s = N \ge n$ and consider such a relation with $s = N + 1$. In this case, at least one linear form, say L_1, has exponent greater than one.

Let r_1 be the maximal power of L_1 in all the simplicial fractions G_i, $1 \le i \le r$. We split these fractions into three disjoint sets. Let $G_1, \dots, G_m$ be all the simplicial fractions with L_1 raised to the power of r_1. Let $G_{m+1}, \dots, G_{m+\ell}$ be all the simplicial fractions, if any, with L_1 raised to a positive power less than r_1. Let $G_{m+\ell+1}, \dots, G_r$ be all the simplicial fractions, if any, that do not contain L_1 in their denominator. Thus

$$0 = L_1 \sum_{i=1}^{r} a_i G_i = \sum_{i=1}^{m} a_i L_1 G_i + \sum_{i=m+1}^{m+\ell} a_i L_1 G_i + \sum_{i=m+\ell+1}^{r} a_i L_1 G_i.$$

For any $m + 1 \le i \le m + \ell$, the power of $1/L_1$ in $L_1 G_i$ is less than $r_1 - 1$. Since the linear forms in the denominators are assumed to span the same spaces, we write L_1

as a linear combination of the linear forms $L_{i1}, \dots, L_{in_i}$ of G_i for $m+\ell+1 \le i \le r$:

$$L_1 = a_{i1}L_{i1} + \cdots + a_{in_i}L_{in_i}.$$

Thus each

$$L_1 G_i = \sum_{j_i=1}^{k} \frac{a_{ij_i}}{L_{i1}^{s_{i1}} \cdots L_{ij_i}^{s_{ij_i}-1} \cdots L_{in_i}^{s_{in_i}}}, \quad m+\ell+1 \le i \le r$$

is a linear combination of fractions that do not contain L_1 as a linear form in the denominator. In summary, each fraction in $\sum_{i=m+1}^{r} a_i L_1 G_i$ has its power of $1/L_1$ less than $r_1 - 1$, so that no such monomial can cancel with any fraction in $\sum_{i=1}^{m} a_i L_1 G_i$.

With the notation of Lemma 3.5, in $\sum_{i=1}^{r} a_i L_1 G_i = 0$ we can use the space spanned by $L_{11}, \dots, L_{1n_1}$ to single out an equation

$$\sum_{i=1}^{m} a_i L_1 G_i + \sum_{i=m+1}^{m+\ell} a_i' L_1 G_i + \cdots = 0.$$

In this equality, simplicial fractions $L_1G_1, \dots, L_1G_m$ have nonzero coefficients $a_1, \dots, a_m$, and the weight of each term in the sum is N. Thus by the induction hypothesis we must have $a_i = 0$, $i = 1, \dots m$, which yields the expected contradiction. □

3B. ***The nonholomorphicity of polar germs.*** Based on Proposition 3.6, we prove the following nonholomorphicity of polar germs at zero, a central result of this paper.

Theorem 3.7 (nonholomorphicity theorem). *A projectively properly positioned family*

$$\left\{ \frac{h_i}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} \;\middle|\; 1 \le i \le p \right\}$$

of $\mathbb{F}$-polar germs at zero is nonholomorphic in the sense that, if a linear combination

$$\sum_{i=1}^{p} a_i \frac{h_i}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}, \quad a_i \in \mathbb{F},\ 1 \le i \le p, \tag{15}$$

is holomorphic, then $a_i = 0$ for $1 \le i \le p$. In particular, this family of polar germs is linearly independent and the holomorphic function for the linear combination is identically zero.

Proof. It suffices to show the theorem for $\mathbb{F} = \mathbb{R}$ since the result then follows for any subfield $\mathbb{F}$ of $\mathbb{R}$.

By Lemma 3.5, we can assume that, for each $1 \le i \le r$, the weight $|s_i| = s_{i1} + \cdots + s_{in_i}$ is the same and the linear forms in the denominators span the same space. As in the proof of Lemma 3.5, we can pick values $\ell^0_{n_1+1}, \dots, \ell^0_k$ of $\ell_{n_1+1}, \dots, \ell_k$ such that $h_i(\ell^0_{n_1+1}, \dots, \ell^0_k) \neq 0$, $1 \le i \le t$, and

$$\sum_{i=1}^{t} a_i \frac{h_i(\ell^0_{n_1+1}, \dots, \ell^0_k)}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} = 0.$$

But the set of fractions $1/(L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}})$, $1 \le i \le t$, is projectively properly positioned. Thus, by Proposition 3.6, the coefficients $a_i h_i(\ell^0_{n_1+1}, \dots, \ell^0_k)$ and hence the coefficients a_i, $1 \le i \le t$, are zero, which leads to a contradiction. □

A direct consequence of Theorem 3.7 is the following uniqueness result.

Corollary 3.8. *Let $\{S_i\}_{1\le i\le \ell}$ and $\{T_j\}_{1\le j\le m}$ be projectively properly positioned families of $\mathbb{F}$-polar germs at zero sharing the same properly positioned family of supporting cones (upon a suitable choice of signs of the linear forms). If*

$$\sum_{i=1}^{\ell} S_i + g_0 = \sum_{j=1}^{m} T_j + h_0 \tag{16}$$

for holomorphic germs g_0 and h_0, then $\{S_i\}_{1\le i\le \ell} = \{T_j\}_{1\le j\le m}$ and $g_0 = h_0$.

Proof. Let $\{C_1, \dots, C_r\}$ be a properly positioned family of supporting cones of $\{S_i\}_{1\le i\le \ell}$ and of $\{T_j\}_{1\le j\le m}$. For $1 \le i \le r$, let $L_{i1}, \dots, L_{in}$ be fixed generators of the $\mathbb{F}$-cones C_i. Let N be the largest sum of powers in the denominators of $\{S_i\}_{1\le i\le \ell}$ and $\{T_j\}_{1\le j\le m}$ and denote

$$\{M_1, \dots, M_t\} = \left\{ L_{i1}^{s_1} \cdots L_{in}^{s_n} \,\middle|\, i = 1, \dots, r,\ |\vec{s}| := \sum_j s_j \le N \right\}.$$

Then we have

$$\sum_{i=1}^{\ell} S_i = \sum_{k=1}^{t} \frac{g_k}{M_k} \quad \text{and} \quad \sum_{j=1}^{m} T_j = \sum_{k=1}^{t} \frac{h_k}{M_k},$$

where for $1 \le k \le t$, the g_k and h_k, some of which can be zero, are holomorphic in some linear forms orthogonal to the linear forms in M_k with respect to the given inner product. Thus (16) gives

$$\sum_{k=1}^{t} \frac{g_k - h_k}{M_k} = h_0 - g_0.$$

But the terms in the sum satisfy the conditions of Theorem 3.7. Thus we have $g_k = h_k$ for $0 \le k \le t$, which implies that the S_i match with the T_j, giving the identification we want. □

4. Laurent expansions of meromorphic germs at zero with linear poles

In this section, we apply the nonholomorphicity theorem (Theorem 3.7) and Corollary 3.8 to develop a notion of Laurent expansions for multivariate meromorphic germs at zero with linear poles.

Central to the notion of Laurent expansions is the forgetful map in Definition 4.2, which gives formal expansions of meromorphic germs at zero. Taking local cross sections of this map, we first identify the Laurent subspaces in Proposition 4.3, formally defined in Definition 4.5. We then show in Theorem 4.13 that these Laurent subspaces cover the whole space $\mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$ with the help of the surjectivity of φ proved in the preparation Theorem 2.11. We then establish a consistency property of these Laurent subspaces in Proposition 4.15. Finally, we determine the kernel of the forgetful map in Theorem 4.20.

4A. ***The space of formal expansions.*** We first generalize the concept of decorated smooth cones in [Guo et al. 2014].

Definition 4.1. A *decorated simplicial* $\mathbb{F}$*-cone* is a formal monomial

$$\underline{C} := \langle v_1\rangle^{s_1}\dots\langle v_n\rangle^{s_n},$$

where $v_1,\dots,v_n$ are linearly independent $\mathbb{F}$-vectors and $s_1,\dots,s_n$ are in $\mathbb{Z}_{\geq 1}$. The simplicial $\mathbb{F}$-cone $\langle v_1,\dots,v_n\rangle$ generated by $v_1,\dots,v_n$ is called the *geometric cone* of the decorated cone $\underline{C}$, and is denoted by $G(\underline{C})$.

As before, these generators define linear functions $L_1,\dots,L_n$ on $V_k^*\otimes\mathbb{C}$. For a different choice of the spanning vectors $v_1,\dots,v_n$, the function $\vec{L}_{\underline{C}} := L_1^{s_1}\cdots L_n^{s_n}$ differs by a constant. So for any subspace U of $V\otimes\mathbb{C}$, the subspace $(1/\vec{L}_{\underline{C}})\mathcal{M}_{\mathbb{F}}(U^*)$ does not depend on the choice of the spanning vectors $v_1,\dots,v_n$. In particular this holds for $U=\mathrm{lin}^{\perp}(G(\underline{C}))$, the orthogonal complement of the linear span of $G(\underline{C})$ in $V\otimes\mathbb{C}$ with respect to the given inner product Q. The space

$$\mathcal{M}_{\underline{C}} := \mathcal{M}_{\mathbb{F},\underline{C}} := \frac{1}{\vec{L}_{\underline{C}}}\mathcal{M}_{\mathbb{F},+}\big((\mathrm{lin}^{\perp}(G(\underline{C})))^*\big) \subseteq \mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$$

is precisely the space spanned by polar germs whose support is $G(\underline{C})$ and with the fixed denominator $\vec{L}_{\underline{C}}$.

Definition 4.2. (a) Define the *space of formal expansions in polar germs* to be

$$\begin{aligned}\mathbb{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C}) &:= \Big(\bigoplus_{\underline{C}}\mathcal{M}_{\mathbb{F},\underline{C}}\Big)\bigoplus\mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C})\\ &= \Big(\bigoplus_{\underline{C}}\frac{1}{\vec{L}_{\underline{C}}}\mathcal{M}_{\mathbb{F},+}\big((\mathrm{lin}^{\perp}(G(\underline{C})))^*\big)\Big)\bigoplus\mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C}),\end{aligned}$$

where the direct sum is taken over decorated simplicial $\mathbb{F}$-cones $\underline{C}$.

(b) Define the *forgetful map*

$$\varphi : \mathbb{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C}) \to \mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C}),$$

(17) $$\bigoplus_{\underline{C}} S_{\underline{C}} \oplus h \mapsto \sum_{\underline{C}} S_{\underline{C}} + h, \quad S_{\underline{C}} \in \mathcal{M}_{\mathbb{F},\underline{C}},\ h \in \mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C}),$$

sending direct sums in $\mathbb{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$ to sums of functions in $\mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$.

Note that different decorated simplicial $\mathbb{F}$-cones might give the same space $\mathcal{M}_{\mathbb{F},\underline{C}}$, for example when the generators of a cone change signs, giving multiple copies of identical summand in $\mathbb{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$. For instance, $\mathcal{M}_{\langle e_1\rangle} = \mathcal{M}_{\langle -e_1\rangle}$ but they give distinct summands in $\mathbb{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$.

By definition, the restriction of φ to $(1/\vec{L}_{\underline{C}})\mathcal{M}_{\mathbb{F},+}\big((\mathrm{lin}^{\perp}(G(\underline{C})))^*\big)$ for each decorated simplicial cone $\underline{C}$, as well as to $\mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C})$, is injective. The following direct consequence of Corollary 3.8 (which follows from the nonholomorphicity result in Theorem 3.7) shows that this injectivity of φ holds for much larger subspaces of $\mathbb{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$.

Proposition 4.3. *Let $\mathfrak{C}$ be a properly positioned family of cones in V. Denote*

(18) $$\mathbb{M}_{\mathbb{F},\mathfrak{C},-}(V^{\circledast}\otimes\mathbb{C}) := \bigoplus_{G(\underline{C})\in\mathfrak{C}} \mathcal{M}_{\mathbb{F},\underline{C}} = \bigoplus_{G(\underline{C})\in\mathfrak{C}} \frac{1}{\vec{L}_{\underline{C}}}\mathcal{M}_{\mathbb{F},+}\big((\mathrm{lin}^{\perp}(G(\underline{C})))^*\big).$$

The restriction of φ to

(19) $$\mathbb{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast}\otimes\mathbb{C}) := \mathbb{M}_{\mathbb{F},\mathfrak{C},-}(V^{\circledast}\otimes\mathbb{C}) \oplus \mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C}) \subseteq \mathbb{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$$

is injective.

Remark 4.4. As a consequence of the proposition, we have

(20) $$\mathcal{M}_{\mathbb{F},\mathfrak{C},-}(V^{\circledast}\otimes\mathbb{C}) := \varphi(\mathbb{M}_{\mathbb{F},\mathfrak{C},-}(V^{\circledast}\otimes\mathbb{C})) \cong \mathbb{M}_{\mathbb{F},\mathfrak{C},-}(V^{\circledast}\otimes\mathbb{C}).$$

Note that $\mathcal{M}_{\mathbb{F},\mathfrak{C},-}(V^{\circledast}\otimes\mathbb{C})$ is the space spanned by polar germs whose supporting cone is contained in $\mathfrak{C}$.

Definition 4.5. Let $\mathfrak{C}$ be a properly positioned family of simplicial cones. A meromorphic germ $f \in \mathcal{M}_{\mathbb{F}}(V_k^*\otimes\mathbb{C})$ is said to *admit a Laurent expansion supported on* $\mathfrak{C}$ if it is contained in $\varphi(\mathbb{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast}\otimes\mathbb{C}))$ or, more precisely, if there exists a projectively properly positioned family $\{S_j\}_{j\in J}$ of polar germs whose supporting cones are contained in $\mathfrak{C}$, together with a holomorphic germ h, all with coefficients in $\mathbb{F}$, such that

$$f = \varphi\Big(\bigoplus_{j\in J} S_j \oplus h\Big).$$

The element $\bigoplus_{j\in J} S_j \oplus h \in \mathbb{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$ with this property, unique by the injectivity in Proposition 4.3, is called the $\mathfrak{C}$-supported *Laurent expansion* of f, denoted by $\mathfrak{L}_{\mathfrak{C}}(f)$, that is,

$$\mathfrak{L}_{\mathfrak{C}}(f) := \bigoplus_{j\in J} S_j \oplus h. \tag{21}$$

The subspace $\varphi(\mathbb{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast}\otimes\mathbb{C}))$ of $\mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$ is called the *Laurent subspace* supported by $\mathfrak{C}$.

Remark 4.6. (a) Clearly, for a polar germ $f = h(\ell_1, \dots, \ell_m)/(L_1^{s_1}\cdots L_n^{s_n})$ with supporting cone C and a properly positioned family $\mathfrak{C}$ of simplicial cones containing C, we have $\mathfrak{L}_{\mathfrak{C}}(f) = f$.

(b) For any $f \in \mathcal{M}_{\mathbb{F}}(V_k^*\otimes\mathbb{C})$ which admits a $\mathfrak{C}$-supported Laurent expansion, we have

$$\varphi\circ\mathfrak{L}_{\mathfrak{C}}(f) = f.$$

(c) Let us note that any Laurent expansion of a rational meromorphic germ has rational coefficients, as can be seen from the construction in the case $\mathbb{F}=\mathbb{Q}$.

Example 4.7. Take $\mathfrak{C} = \{\langle e_1\rangle\}$ in the standard Euclidean space. Then polar germs at the variables given by $z = \sum \varepsilon_i e_i^*$ and supported on $\mathfrak{C}$ are h_i/ε_1^i, $i \geq 0$, for holomorphic functions h_i in variables other than ε_1. Thus the Laurent subspace supported by $\mathfrak{C}$, restricted to $V_1 := \mathbb{R}e_1$, is precisely

$$\mathcal{M}_{\mathbb{C},\mathfrak{C}}(V_1\otimes\mathbb{C}) := \mathcal{M}_{\mathbb{C},\mathfrak{C}}(V^{\circledast}\otimes\mathbb{C}) \cap \mathcal{M}_{\mathbb{C}}(V_1\otimes\mathbb{C}) = \bigoplus_{i\geq 1}\mathbb{C}\frac{1}{\varepsilon^i}\oplus\mathbb{C}\{\{\varepsilon\}\},$$

recovering the classical Laurent series expansions.

4B. ***The subdivision operators.*** In order to show that every element in $\mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$ admits a Laurent expansion, we want to cover $\mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$ with Laurent subspaces which are compatible on overlaps. This is achieved by the subdivision operators, built on the following notion generalizing the concept of a subdivision of a cone.

Definition 4.8. A *subdivision of a family of cones* $\mathfrak{C}=\{C_i\}$ is a set $\mathfrak{D}=\{D_1, \dots, D_r\}$ of cones such that

(a) $D_1, \dots, D_r$ intersect along their faces,

(b) for any i, there is $I_i \subset [r]$ such that $\{D_\ell\}_{\ell\in I_i}$ is a subdivision of C_i, and

(c) $\bigcup_i I_i = [r]$.

If only conditions (a) and (b) are satisfied, then $\mathfrak{D}$ is called a *pan-subdivision* of $\mathfrak{C}$.

We introduce a notion which will be convenient for later discussions.

Definition 4.9. Fix an ordered basis $\{e_i\}$ of a filtered space $V = \bigcup_{k\geq 1} V_k$ such that $\{e_i\}\cap V_k$ is a basis of V_k. A nonzero vector $v = \sum_i c_i e_i$ is called *pseudopositive* if the leading coefficient of v, namely the nonzero coefficient of v with the largest subscript i, is positive. By convention, 0 is taken to be a pseudopositive vector. Let $\boldsymbol{P}$ denote the set of pseudopositive vectors.

As can be easily checked, the set $\boldsymbol{P}$ is the union of the increasing filtration consisting of the strictly convex sets $\boldsymbol{P}_n \subseteq \mathbb{R}\{e_1, \ldots, e_n\}$, $n \geq 0$, where by convention $\boldsymbol{P}_0 := \{0\}$, and recursively,

$$\boldsymbol{P}_{n+1} := \boldsymbol{P}_n \cup (\mathbb{R}\{e_1, \ldots, e_n\} \times \mathbb{R}_{>0}\, e_{n+1}), \quad n \geq 0.$$

Consequently, $\boldsymbol{P}$ is a strictly convex set.

Lemma 4.10. *Any finite family of cones whose union does not contain a nonzero linear subspace has a properly positioned family of cones as a subdivision. The union of the family of the cones does not change with the subdivision. In particular, if a finite family of cones is in $\boldsymbol{P}$, then so is any of its properly positioned families of subdivisions.*

Proof. The existence of a subdivision follows from the proof of Lemma 2.3(a) in [Guo et al. 2014], noting that the assumption made there, namely that the cones span the same linear subspace, is redundant. Then the assumption that the union of the family does not contain a nonzero linear subspace guarantees that the resulting family is properly positioned. The second statement also follows from the proof of [Guo et al. 2014, Lemma 2.3(a)]. □

Lemma 4.11. *Given any finite family of polar germs, there is a choice of the family of supporting cones whose union does not contain a nonzero linear subspace.*

Proof. Fix an ordered basis of V. By rescaling if necessary, we can assume that all linear forms in the denominators of the polar germs are pseudopositive. The corresponding supporting cones of the polar germs are therefore contained in the strictly convex set $\boldsymbol{P}$ and hence do not contain any nonzero linear subspace. □

Let $\mathfrak{C}$ be a properly positioned family of cones and $\mathfrak{D}$ a simplicial pan-subdivision of $\mathfrak{C}$. We next define a subdivision operator

$$\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})} : \mathbb{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast} \otimes \mathbb{C}) \to \mathbb{M}_{\mathbb{F},\mathfrak{D}}(V^{\circledast} \otimes \mathbb{C}).$$

Since $\mathbb{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast} \otimes \mathbb{C}) := \bigoplus_{G(\underline{C})\in\mathfrak{C}} \mathcal{M}_{\mathbb{F},\underline{C}}$, we only need to define its action on $\mathcal{M}_{\mathbb{F},\underline{C}}$ for a decorated cone $\underline{C} := \langle v_1\rangle^{s_1} \ldots \langle v_n\rangle^{s_n}$ as in Definition 4.1, with $G(\underline{C})$ in $\mathfrak{C}$.

We first consider the action when $s_1 = \cdots = s_n = 1$. Then a polar germ in $\mathcal{M}_{\mathbb{F},\underline{C}}$ is of the form $g/(L_1 \cdots L_n)$ for a simple fraction $1/(L_1 \cdots L_n)$ and a holomorphic germ g in a set of variables orthogonal to the linear span of $L_1, \ldots, L_n$. Let $G(\underline{C}) = C$. There is a unique subset $\{D_\mu\}_{\mu\in J}$ of $\mathfrak{D}$ that gives a subdivision of C. As in (7), we have

$$I(C) = (-1)^n \frac{a}{L_1 \cdots L_n}, \qquad I(D_\mu) = (-1)^n \frac{b_\mu}{M_{\mu 1} \cdots M_{\mu n}},$$

where a, b_μ are constants in $\mathbb{F}$. By (8),

$$\frac{1}{L_1 \cdots L_n} = \frac{(-1)^n}{a} I(C) = \frac{(-1)^n}{a} \sum_{\mu \in J} I(D_\mu) = \sum_{\mu \in J} \frac{b_\mu}{a} \frac{1}{M_{\mu 1} \cdots M_{\mu n}}. \tag{22}$$

Note that $I(D_\mu)$ is supported on D_μ, and by Lemma 3.4, such a decomposition with support on $\mathfrak{D}$ is unique. Thus we can define

$$\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}\Big(\frac{g}{L_1 \cdots L_n}\Big) := \bigoplus_{\mu \in J} \frac{b_\mu}{a} \frac{g}{M_{\mu 1} \cdots M_{\mu n}} \in \bigoplus_{\mu \in J} \mathcal{M}_{\mathbb{F}, D_\mu} \subseteq \mathbb{M}_{\mathbb{F},\mathfrak{D}}(V^{\circledast} \otimes \mathbb{C}). \tag{23}$$

We next introduce a class of differential operators in order to treat general decorated cones. Let $\{e_1, e_2, \dots\}$ be a basis of the filtered space V and let $\{e_1^*, e_2^*, \dots\}$ be the dual basis. Let $\vec{\varepsilon} = \sum \varepsilon_i e_i^*$ be a generic vector in $V^{\circledast} \otimes \mathbb{C}$. Then with respect to the variables ε_i, we have differential operators

$$\partial_i := -\frac{\partial}{\partial \varepsilon_i} : \mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) \to \mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}).$$

For a fixed vector $v^* = \sum_i c_i e_i^* \in V^{\circledast}$, denote $\partial_{v^*} := \sum_i c_i \partial_i$, the negative of the directional derivation. Then for any function f in linear independent linear forms $K_1, \dots, K_m$, the chain rule gives

$$\partial_{v^*} f(K_1, \dots, K_m) = -\sum_{i=1}^{m} (v^*, K_m) \frac{\partial f}{\partial K_m}. \tag{24}$$

Now for any given decorated cone $\underline{C}$ with $G(\underline{C}) \in \mathfrak{C}$ and polar germ $g/(L_1^{s_1} \cdots L_n^{s_n})$ in $\mathcal{M}_{\mathbb{F},\underline{C}}$, let $\big\{L_i^* = \sum_j c_{ij} e_j^*\big\}_i$ be dual to the linear forms $\{L_i\}_i$ in the sense that $(L_i, L_j^*) = \delta_{ij}$, $1 \le i, j \le n$. By (24) we obtain

$$\partial_{L_i^*} \frac{1}{M_1^{r_1} \cdots M_n^{r_n}} = \sum_{j=1}^{n} \frac{c_{ij}}{M_1^{r_1} \cdots M_j^{r_j+1} \cdots M_n^{r_n}},$$

for some constants $c_{i1}, \dots, c_{in}$ depending on the poles $M_1, \dots, M_n$ and on L_i. Since g is in a set of variables orthogonal to $M_1, \dots, M_n$, we further obtain

$$\partial_{L_i^*} \frac{g}{M_1^{r_1} \cdots M_n^{r_n}} = \sum_{j=1}^{n} \frac{c_{ij}\, g}{M_1^{r_1} \cdots M_j^{r_j+1} \cdots M_n^{r_n}} = g\, \partial_{L_i^*} \frac{1}{M_1^{r_1} \cdots M_n^{r_n}}. \tag{25}$$

We then define

$$\delta_{L_i^*} : \mathcal{M}_{\mathbb{F},\underline{D}} \to \bigoplus_{\mathbb{F}, G(\underline{E}) \in \mathfrak{D}} \mathcal{M}_{\mathbb{F},\underline{E}} \subseteq \mathbb{M}_{\mathbb{F},\mathfrak{D}}(V^{\circledast} \otimes \mathbb{C}),$$

$$\frac{g}{M_1^{r_1} \cdots M_n^{r_n}} \mapsto \bigoplus_{j=1}^{n} \frac{c_{ij}\, g}{M_1^{r_1} \cdots M_j^{r_j+1} \cdots M_n^{r_n}},$$

which, by acting componentwise in $\mathbb{M}_{\mathbb{F},\mathfrak{D}}(V^{\circledast}\otimes\mathbb{C}) := \bigoplus_{G(\underline{D})\in\mathfrak{D}} \mathcal{M}_{\mathbb{F},\underline{D}}$, gives rise to an operator

$$\delta_{L_i^*} : \mathbb{M}_{\mathbb{F},\mathfrak{D}}(V^{\circledast}\otimes\mathbb{C}) \to \mathbb{M}_{\mathbb{F},\mathfrak{D}}(V^{\circledast}\otimes\mathbb{C}). \tag{26}$$

By [Guo et al. 2014, Proposition 4.8(b)], we have

$$\frac{1}{L_1^{s_1}\cdots L_n^{s_n}} = \frac{1}{(s_1-1)!\cdots(s_n-1)!}\partial_{L_1^*}^{s_1-1}\cdots\partial_{L_n^*}^{s_n-1}\frac{1}{L_1\cdots L_n}. \tag{27}$$

We accordingly apply (23) and (26) to define

$$\begin{aligned}\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}\Big(\frac{g}{L_1^{s_1}\cdots L_n^{s_n}}\Big) \\ := \frac{1}{(s_1-1)!\cdots(s_n-1)!}\delta_{L_1^*}^{s_1-1}\cdots\delta_{L_n^*}^{s_n-1}\Big(\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}\Big(\frac{g}{L_1\cdots L_n}\Big)\Big),\end{aligned} \tag{28}$$

completing the definition of the *subdivision operator*

$$\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})} : \mathbb{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast}\otimes\mathbb{C}) \to \mathbb{M}_{\mathbb{F},\mathfrak{D}}(V^{\circledast}\otimes\mathbb{C}). \tag{29}$$

For example, take $\mathfrak{C} := \{\langle e_1, e_2\rangle\}$ and $\mathfrak{D} := \{\langle e_1, e_1+e_2\rangle, \langle e_1+e_2, e_2\rangle\}$ in the standard Euclidean space with the standard basis $\{e_i\}_{i\geq 1}$ and let $\varepsilon_1, \varepsilon_2$ be (part of) the coordinates with respect to the dual basis $\{e_i^*\}_{i\geq 1}$. Then we have

$$\begin{aligned}\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}\Big(\frac{\varepsilon_3}{\varepsilon_1\varepsilon_2}\Big) &= \frac{\varepsilon_3}{\varepsilon_1(\varepsilon_1+\varepsilon_2)} + \frac{\varepsilon_3}{\varepsilon_2(\varepsilon_1+\varepsilon_2)},\\ \mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}\Big(\frac{\varepsilon_3}{\varepsilon_1^2\varepsilon_2}\Big) &= \frac{\varepsilon_3}{\varepsilon_1^2(\varepsilon_1+\varepsilon_2)} + \frac{\varepsilon_3}{\varepsilon_1(\varepsilon_1+\varepsilon_2)^2} + \frac{\varepsilon_3}{\varepsilon_2(\varepsilon_1+\varepsilon_2)^2}.\end{aligned}$$

Notice that by (25),

$$\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}\Big(\frac{g}{L_1^{s_1}\cdots L_n^{s_n}}\Big) = g\,\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}\Big(\frac{1}{L_1^{s_1}\cdots L_n^{s_n}}\Big). \tag{30}$$

In the definition of the subdivision operator, we choose a basis of V and a dual of the linear forms in the polar germs. The following proposition shows that this operator does not actually depend on such choices.

Proposition 4.12. (a) *The subdivision operator $\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}$ is compatible with the forgetful map φ, i.e., $\varphi\circ\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})} = \varphi$.*

(b) *The subdivision operator $\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}$ does not depend on the choice of the basis of V.*

Proof. (a) By (22) and (23), the desired equation holds for polar germs with $s_1 = \cdots = s_n = 1$. Since $\varphi\circ\delta_{L_j^*} = \partial_{L_j^*}\circ\varphi$ by construction, the desired equation follows from (27) and (28).

(b) For a polar germ f supported on $\mathfrak{C}$, and for any choice of the basis of V, $\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}(f)$ is a sum of polar germs supported on $\mathfrak{D}$, which equals f as a function by part (a), so that by Corollary 3.8, $\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}(f)$ is unique. □

Furthermore, for a simplicial $\mathbb{F}$-pan-subdivision $\mathfrak{E}$ of $\mathfrak{D}$, by the transitivity of pan-subdivisions, we obtain

$$\mathfrak{S}_{(\mathfrak{D},\mathfrak{E})} \circ \mathfrak{S}_{(\mathfrak{C},\mathfrak{D})} = \mathfrak{S}_{(\mathfrak{C},\mathfrak{E})}. \tag{31}$$

Now we show that the Laurent subspaces cover the whole space $\mathcal{M}_{\mathbb{F}}(V_k^* \otimes \mathbb{C})$, proving the existence of a Laurent expansion for any meromorphic germ.

Theorem 4.13. *Let f be an element in $\mathcal{M}_{\mathbb{F}}(V_k^* \otimes \mathbb{C})$. There exists a properly positioned family of simplicial cones $\mathfrak{C}$ such that f has a Laurent expansion supported on $\mathfrak{C}$. In other words, there is a projectively properly positioned family of polar germs $\{S_j\}_{j\in J}$ supported on $\mathfrak{C}$, together with a holomorphic germ h, all with coefficients in $\mathbb{F}$, such that*

$$f = \varphi\Big(\bigoplus_{j\in J} S_j \oplus h\Big), \tag{32}$$

or as function decomposition,

$$f = \sum_{j\in J} S_j + h. \tag{33}$$

In fact, the family $\mathfrak{C}$ can be taken to be in $\boldsymbol{P}$.

Proof. Take any decomposition of f as in Theorem 2.11, $f = \sum_{i\in I} g_i + h$, where $\{g_i\}$ is a finite set of polar germs and h is holomorphic at zero. By Lemma 4.11, there is a choice $\mathfrak{C}$ of the family of the supporting cones of the polar germs such that the union of the cones does not contain any nonzero linear subspace. In fact, the proof of Lemma 4.11 shows that $\mathfrak{C}$ can be chosen to be contained in $\boldsymbol{P}$. By combining colinear terms, we can assume that the decorated cones of these polar germs are distinct. Hence we can write $f = \varphi\big(\bigoplus_{i\in I} g_i \oplus h\big)$.

By Lemma 4.10, the family $\mathfrak{C}$ has a pan-subdivision $\mathfrak{D}$ that is properly positioned. Then through the subdivision operator $\mathfrak{S}(\mathfrak{C},\mathfrak{D})$, the sum $\bigoplus_{i\in I} \mathfrak{S}(\mathfrak{C},\mathfrak{D})(g_i) \oplus h$ is a desired Laurent expansion of f supported on $\mathfrak{D}$. □

Example 4.14. In the standard Euclidean space, we have

$$\frac{\varepsilon_1+2\varepsilon_2}{\varepsilon_1(\varepsilon_1+\varepsilon_2)\varepsilon_2} = \frac{1}{\varepsilon_1\varepsilon_2} + \frac{1}{\varepsilon_1(\varepsilon_1+\varepsilon_2)} = \frac{2}{\varepsilon_1(\varepsilon_1+\varepsilon_2)} + \frac{1}{(\varepsilon_1+\varepsilon_2)\varepsilon_2}.$$

Here the first equation expressed the meromorphic germ as a sum of polar germs as in Theorem 2.11. The second equation expresses the sum of polar germs as a sum of projectively properly positioned family of polar germs, as in Theorem 4.13, thus providing the Laurent expansion of the meromorphic germ of the left-hand side supported on the family $\mathfrak{C} = \{\langle e_1, e_1+e_2\rangle, \langle e_2, e_1+e_2\rangle\}$.

We finally prove the coherence of Laurent expansions arising from different properly positioned family of cones, namely their compatibility with the subdivision operators.

Proposition 4.15. (a) *Assume that* $f \in \mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$ *admits a* $\mathfrak{C}$*-supported Laurent expansion and let* $\mathfrak{D}$ *be a simplicial* $\mathbb{F}$*-pan-subdivision of* $\mathfrak{C}$*. Then* $\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}\mathfrak{L}_{\mathfrak{C}}(f)$ *is the* $\mathfrak{D}$*-supported Laurent expansion of* f.

(b) *With respect to the inclusion operators, the set of Laurent subspaces supported on cones in the set* $\boldsymbol{P}$ *forms a direct system. Its direct limit is* $\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$.

Proof. (a) This follows in a straightforward manner from Proposition 4.12(a).

(b) For two properly positioned families of simplicial cones in $\boldsymbol{P}$, their union is contained in $\boldsymbol{P}$. Thus by Lemma 4.10, their union has a properly positioned subdivision of simplicial cones, giving a common pan-subdivision of the two families. Thus the set of properly positioned families of simplicial cones in $\boldsymbol{P}$ is direct with respect to pan-subdivisions. Through the subdivision operators, the set

$$\{\mathbb{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast} \otimes \mathbb{C}) \mid \text{properly positioned families } \mathfrak{C} \text{ in } \boldsymbol{P}\}$$

is a direct system. Then the set of Laurent subspaces

$$\{\mathcal{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast} \otimes \mathbb{C}) \mid \text{properly positioned families } \mathfrak{C} \text{ in } \boldsymbol{P}\}$$

is a direct system with respect to the inclusion maps by Proposition 4.12(a). Its direct limit is $\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$ since the union of Laurent subspaces supported in $\boldsymbol{P}$ is $\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$ by Theorem 4.13. □

As an immediate consequence, we obtain the following Rota–Baxter type decomposition utilized in [Guo et al. 2017].

Corollary 4.16. *Let* (V, Λ_V) *be a filtered* $\mathbb{F}$*-Euclidean lattice space. There is a direct sum decomposition*

$$\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) = \mathcal{M}_{\mathbb{F},-}(V^{\circledast} \otimes \mathbb{C}) \oplus \mathcal{M}_{\mathbb{F},+}(V^{\circledast} \otimes \mathbb{C}).$$

In particular, the holomorphic part h *and the polar part* $\sum_j S_j$ *in* (6) *are uniquely determined by the germ* f.

Proof. For each properly positioned family $\mathfrak{C}$ of simplicial cones, Proposition 4.3 gives the direct sum decomposition

$$\mathcal{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast} \otimes \mathbb{C}) = \mathcal{M}_{\mathbb{F},\mathfrak{C},-}(V^{\circledast} \otimes \mathbb{C}) \oplus \mathcal{M}_{\mathbb{F},+}(V^{\circledast} \otimes \mathbb{C}).$$

By Proposition 4.15, we obtain

$$\begin{aligned}\mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C}) &= \varinjlim \mathcal{M}_{\mathbb{F},\mathfrak{C}}(V^{\circledast}\otimes\mathbb{C})\\ &= \varinjlim \mathcal{M}_{\mathbb{F},\mathfrak{C},-}(V^{\circledast}\otimes\mathbb{C})\oplus\mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C})\\ &= \mathcal{M}_{\mathbb{F},-}(V^{\circledast}\otimes\mathbb{C})\oplus\mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C}),\end{aligned}$$

where the direct limits are taken over those $\mathfrak{C}$ in $\boldsymbol{P}$. □

We introduce a notation before stating the next result.

Definition 4.17. Germs $f, g \in \mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C})$ are said to be *orthogonally variate germs* if there are germs $\tilde{f}$ on $\mathbb{C}^n$ and $\tilde{g}$ on $\mathbb{C}^m$ such that $f = \tilde{f}(L_1, \dots, L_m)$ and $g = \tilde{g}(M_1, \dots, M_n)$ for linear independent linear forms $\{L_1, \dots, L_m\}$ and $\{M_1, \dots, M_n\}$ on $V^{\circledast}\otimes\mathbb{C}$ with $Q(L_i, M_j) = 0$ for $(i, j) \in [1, m] \times [1, n]$.

Corollary 4.18 (multiplicativity of π_+ on orthogonally variate germs). *Let*

$$\pi_+ : \mathcal{M}_{\mathbb{F}}(V^{\circledast}\otimes\mathbb{C}) \to \mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C}) \tag{34}$$

denote the projection map onto $\mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C})$ *along* $\mathcal{M}_{\mathbb{F},-}(V^{\circledast}\otimes\mathbb{C})$. *For orthogonally variate germs* f *and* g, *we have*

$$\pi_+(fg) = \pi_+(f)\pi_+(g). \tag{35}$$

Proof. Let f and g be in $\mathcal{M}_{\mathbb{F},+}(V^{\circledast}\otimes\mathbb{C})$. Using (6), we decompose $f = h + \sum_{i=1}^m S_i$ and $g = k + \sum_{j=1}^n T_j$ with h, k holomorphic germs and S_i, T_j polar germs. Further, by Theorem 2.11, with the notations in Definition 4.17, h and S_i (resp. g and T_j) can be written as functions in linear forms in $\operatorname{span}(L_1, \dots, L_m)$ (resp. $\operatorname{span}(M_1, \dots, M_n)$). Now

$$fg = hk + h\biggl(\sum_{j=1}^n T_j\biggr) + k\biggl(\sum_{i=1}^m S_i\biggr) + \sum_{i,j} S_i T_j.$$

By the orthogonality of $\operatorname{span}(L_1, \dots, L_m)$ and $\operatorname{span}(M_1, \dots, M_n)$, the germs hT_j, kS_i and S_iT_j are all polar germs. Thus this is a decomposition of fg into the sum of a holomorphic germ hk and a linear combination of polar germs. Thus by Corollary 4.16, $\pi_+(fg) = hk = \pi_+(f)\pi_+(g)$. □

Remark 4.19. The projection π_+ is a multivariate generalization of the minimal subtraction operator in one variable. The multiplicativity on orthogonally variate germs stated in Corollary 4.18 is closely related to locality in quantum field theory and central in renormalization issues.

4C. ***The kernel of the forgetful map.*** We finally determine the kernel of the forgetful map $\varphi : \mathbb{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) \to \mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$ introduced in (17).

Theorem 4.20. *The kernel of the map φ is the subspace of $\mathbb{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$ spanned by elements of the following forms*:

$$\text{(I)} \qquad \frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}} \oplus \left((-1)^{s_1+1} \frac{h(\ell_1, \dots, \ell_m)}{(-L_1)^{s_1} \cdots L_n^{s_n}} \right),$$

for all polar germs of the form $h(\ell_1, \dots, \ell_m)/(L_1^{s_1} \cdots L_n^{s_n})$;

$$\text{(II)} \qquad \frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}} \oplus \mathfrak{S}_{(\mathfrak{C},\mathfrak{D})} \left(-\frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}} \right),$$

for all polar germs of the form $h(\ell_1, \dots, \ell_m)/(L_1^{s_1} \cdots L_n^{s_n})$, $\mathfrak{C} := \{\langle L_1, \dots, L_n\rangle\}$ and $\mathfrak{D}$ a simplicial subdivision of $\langle L_1, \dots, L_n\rangle$.

Thus, modulo changing of signs, relations among polar germs amount to subdivision relations.

Proof. Clearly, the subspace W of $\mathbb{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$ generated by elements of the forms (I) and (II) is a subspace of $\ker \varphi$. So we only need to prove that if $G \oplus H$ is in $\ker \varphi$ with $G = \bigoplus_j S_j$ a sum of polar germs S_j and H a holomorphic germ at zero as in Theorem 2.11, then G lies in W and H vanishes.

By Lemma 4.11, modulo elements of form (I), we can assume that the union of the supporting cones of S_j does not contain any nonzero subspace. Let $\mathfrak{C} := \{C_j \mid j \in J\}$ be the family of supporting cones, and let $\mathfrak{D}$ be a simplicial subdivision of $\mathfrak{C}$. Then $G + \mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}(-G) = \sum_j (S_j + \mathfrak{S}_{(\mathfrak{C}_j,\mathfrak{D}_j)}(-S_j))$ — where $\mathfrak{C}_j$ is the singleton $\{C_j\}$ and $\mathfrak{D}_j$ is the subdivision of C_j induced by $\mathfrak{D}$ — is a sum of elements of type (II) and hence lies in W. Since $\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}(-G) - H = -G - H \in \ker \varphi$, we have

$$\varphi(\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}(-G)) - \varphi(H) = 0.$$

Theorem 3.7 and Proposition 4.3 then yield $\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}(-G) = 0$ and $H = 0$. Therefore,

$$G + H = G + \mathfrak{S}_{\mathfrak{C},\mathfrak{D}}(-G) - \mathfrak{S}_{\mathfrak{C},\mathfrak{D}}(-G) + H$$

is in W. □

5. Refined gradings and applications

Laurent expansions have many useful applications, such as providing much finer decompositions of $\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$ than the one in Corollary 4.16. As applications, we obtain the Brion–Vergne decomposition and the Jeffrey–Kirwan residue of a class of meromorphic germs.

5A. ***Decompositions of meromorphic germs at zero.*** Let (V, Λ_V) be a filtered lattice space.

Definition 5.1. (a) For a polar germ $S_j := h(\ell_1, \ldots, \ell_m)/(L_1^{s_1} \cdots L_n^{s_n})$, we call the sum $s_1 + \cdots + s_n$ the *p-order* of the polar germ S_j, which we denote by $\text{p-ord}(S_j)$.

(b) We call the subspace spanned by the supporting cone of the polar germ the *supporting subspace* of a polar germ.

(c) For $p \in \mathbb{Z}_{\geq 0}$, let $\mathcal{M}_{\mathbb{F}}^p(V^{\circledast} \otimes \mathbb{C})$ denote the linear span of $\mathbb{F}$-polar germs with p-order p.

(d) For any $\mathbb{F}$-subspace $U \subset V$, let $\mathcal{M}_{\mathbb{F},U}(V^{\circledast} \otimes \mathbb{C})$ denote the linear span of $\mathbb{F}$-polar germs with supporting subspace U.

(e) For $d \in \mathbb{Z}_{\geq 0}$, let $\mathcal{M}_{\mathbb{F},d}(V^{\circledast} \otimes \mathbb{C})$ denote the linear span of $\mathbb{F}$-polar germs whose supporting subspaces have dimension d.

(f) For any $\mathbb{F}$-subspace $U \subset V$ and $p \in \mathbb{Z}_{\geq 0}$, let $\mathcal{M}_{\mathbb{F},U}^p(V^{\circledast} \otimes \mathbb{C})$ denote the linear span of $\mathbb{F}$-polar germs with supporting subspace U and p-order p.

Remark 5.2. With this notation, we have $\mathcal{M}_{\mathbb{F},\{0\}}(V^{\circledast} \otimes \mathbb{C}) = \mathcal{M}_{\mathbb{F},0}(V^{\circledast} \otimes \mathbb{C}) = \mathcal{M}_{\mathbb{F},+}(V^{\circledast} \otimes \mathbb{C})$ for the trivial cone $\{0\}$ and integer $d = 0$.

Theorem 5.3. *We have the decompositions*

$$\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) = \bigoplus_{p \geq 0} \mathcal{M}_{\mathbb{F}}^p(V^{\circledast} \otimes \mathbb{C}), \tag{36}$$

$$\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) = \bigoplus_{U \subset V} \mathcal{M}_{\mathbb{F},U}(V^{\circledast} \otimes \mathbb{C}), \tag{37}$$

$$\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) = \bigoplus_{d \geq 0} \mathcal{M}_{\mathbb{F},d}(V^{\circledast} \otimes \mathbb{C}), \tag{38}$$

$$\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) = \bigoplus_{\substack{U \subset V \\ p \in \mathbb{Z}_{\geq 0}}} \mathcal{M}_{\mathbb{F},U}^p(V^{\circledast} \otimes \mathbb{C}). \tag{39}$$

Proof. By Theorem 4.20, the kernel of the surjective linear map

$$\varphi : \mathbb{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) \to \mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$$

is linearly spanned by elements each of which is a linear combination of polar germs with the same p-order, the same supporting subspace and the same dimension of the supporting subspace. Then the equations follow. □

Theorem 5.3 is a refinement of a decomposition derived simultaneously and independently by Berline and Vergne [2016, Theorem 7.3]. In contrast to their approach, which applies to a meromorphic germ with a prescribed set of poles determined by a given hyperplane arrangement Δ, here we consider the whole class of meromorphic germs at zero with linear poles. The equation (37) yields the decomposition in [Berline and Vergne 2016, Theorem 7.3] corresponding to a sum running over the set of subspaces spanned by elements of the hyperplane arrangements corresponding to the poles.

On the grounds of Theorem 5.3, we can give the following definitions.

Definition 5.4. Let U be an $\mathbb{F}$-subspace of (V, Λ_V) and $p \in \mathbb{Z}_{\geq 0}$. Define

$$P_U^p : \mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) \to \mathcal{M}_{\mathbb{F},U}^p(V^{\circledast} \otimes \mathbb{C}) \subset \mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}),$$
$$P_U : \mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) \to \mathcal{M}_{\mathbb{F},U}(V^{\circledast} \otimes \mathbb{C}) \subset \mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C})$$

to be the projections, called the *projection of f onto the space U of p-order p* and *projection of f onto the space U*, respectively.

For $d \in \mathbb{Z}_{\geq 0}$, setting

$$\mathcal{M}_{\mathbb{F},\leq d}(V^{\circledast}\otimes\mathbb{C}) := \bigoplus_{0\leq k\leq d} \mathcal{M}_{\mathbb{F},k}(V^{\circledast}\otimes\mathbb{C}), \quad \mathcal{M}_{\mathbb{F},>d}(V^{\circledast}\otimes\mathbb{C}) := \bigoplus_{k>d} \mathcal{M}_{\mathbb{F},k}(V^{\circledast}\otimes\mathbb{C}),$$

then we have

$$\mathcal{M}_{\mathbb{F}}(V^{\circledast} \otimes \mathbb{C}) = \mathcal{M}_{\mathbb{F},\leq d}(V^{\circledast} \otimes \mathbb{C}) \oplus \mathcal{M}_{\mathbb{F},>d}(V^{\circledast} \otimes \mathbb{C}). \tag{40}$$

This yields back the decomposition of Corollary 4.16 if we take $d = 0$.

The decomposition in (40) also yields back Brion and Vergne's decomposition [1999, Theorem 1] as follows. Let Δ be a finite subset of lattice vectors in some V with coefficients in $\mathbb{F}$. Let

$$U := \mathrm{span}(\Delta), \quad r := \dim(U). \tag{41}$$

The symmetric algebra $S(U)$ (over $\mathbb{C}$) can be viewed as the algebra of polynomial functions on U^*. Following the notation of [Brion and Vergne 1999], let us denote by

$$R_\Delta := \Delta^{-1} S(U)$$

the localization of $S(U)$ with respect to Δ, which is naturally regarded as a subset of $S(U)$. It corresponds to the algebra of rational functions with linear poles in Δ. A subset $\kappa \subset \Delta$ is called *generating* if the linear span of κ is U, and it is called *a basis* if it is a basis of U. Consider the following subspaces of R_Δ:

$$S_\Delta := \mathrm{span}\left\{ \frac{1}{\prod_{\alpha\in\kappa} \alpha} \,\middle|\, \kappa \subseteq \Delta \text{ bases of } U \right\},$$
$$G_\Delta := \mathrm{span}\left\{ \frac{1}{\prod_{\alpha\in\kappa} \alpha^{n_\alpha}} \,\middle|\, \kappa \subseteq \Delta \text{ generating subsets of } U, n_\alpha \in \mathbb{Z}_{>0} \right\},$$
$$NG_\Delta := \mathrm{span}\left\{ \frac{h}{\prod_{\alpha\in\kappa} \alpha^{n_\alpha}} \,\middle|\, \kappa \subseteq \Delta \text{ nongenerating subsets of } U, n_\alpha \in \mathbb{Z}_{\geq 0}, h \in S(U) \right\}.$$

Clearly,

$$\mathcal{M}_{\mathbb{F},>r-1}(V^{\circledast}\otimes\mathbb{C})\cap R_\Delta = G_\Delta, \quad \mathcal{M}_{\mathbb{F},\le r-1}(V^{\circledast}\otimes\mathbb{C})\cap R_\Delta = NG_\Delta.$$

Thus (40) recovers the following decomposition of R_Δ obtained by Brion–Vergne.

Corollary 5.5 [Brion and Vergne 1999, Theorem 1]. *There is a direct sum decomposition*

$$R_\Delta = G_\Delta \oplus NG_\Delta.$$

5B. ***The generalized Jeffrey–Kirwan residue.*** The Jeffrey–Kirwan residue, introduced in [Jeffrey and Kirwan 1995] (see also [Jeffrey and Kirwan 1997]) in the study of localization for nonabelian compact group actions, is a powerful tool to compute intersection numbers for symplectic quotients.

There are several ways to define the Jeffrey–Kirwan residue, namely using iterated residues, inverse Laplace transforms or nested sets [Brion and Vergne 1999; De Concini and Procesi 2005; Jeffrey and Kirwan 1995; 1997; Jeffrey and Kogan 2005; Szenes and Vergne 2004]. We use Brion–Vergne's presentation, which we briefly recall here.

Taking total degrees gives a grading on the space $R_\Delta = \bigoplus_{j\in\mathbb{Z}} R_\Delta[j]$ and G_Δ is contained in $R_\Delta[\le -r] := \bigoplus_{j\le -r} R_\Delta[j]$. Thus from Corollary 5.5 we obtain

$$R_\Delta[\le -r] = G_\Delta \oplus (NG_\Delta \cap R_\Delta[\le -r]), \tag{42}$$

where $r = \dim(\operatorname{span}(\Delta))$ as in (41). Furthermore, $S_\Delta = G_\Delta[-r]$ is the highest degree part of G_Δ, giving the decomposition

$$G_\Delta = G_\Delta[< -r] \oplus S_\Delta. \tag{43}$$

Consider the localization

$$\hat{R}_\Delta := \Delta^{-1}\hat{S}(U)$$

of the ring $\hat{S}(U)$ of formal power series by inverting the linear functions $\alpha \in \Delta$ and the natural decomposition

$$\hat{R}_\Delta = \hat{R}_\Delta[> -r] \oplus R_\Delta[\le -r]. \tag{44}$$

Putting equations (42)–(44) together yields the decomposition

$$\hat{R}_\Delta = \hat{R}_\Delta[> -r] \oplus (NG_\Delta \cap R_\Delta[\le -r]) \oplus G_\Delta[< -r] \oplus S_\Delta. \tag{45}$$

Definition 5.6. The *Jeffrey–Kirwan residue map*

$$\operatorname{Res}_\Delta : \hat{R}_\Delta \to S_\Delta$$

is defined to be the projection to the direct summand S_Δ in (45).

Since the Jeffrey–Kirwan residue of a Laurent power series is defined by that of the corresponding truncated Laurent polynomial, for the sake of simplicity, we focus here on R_Δ, which is a subspace of $\mathcal{M}_{\mathbb{F}}(U^* \otimes \mathbb{C})$, and the decomposition

$$R_\Delta = R_\Delta[> -r] \oplus (NG_\Delta \cap R_\Delta[\leq -r]) \oplus G_\Delta[< -r] \oplus S_\Delta, \tag{46}$$

analogous to (45).

Corollary 5.7. *Let $U = \operatorname{span}(\Delta)$ and $r = \dim U$. Then for any $f \in R_\Delta$, the projection $P_U^r(f)$ from Definition 5.4 is the Jeffrey–Kirwan residue of f.*

Proof. Let $1/\prod_{\alpha\in\kappa} \alpha$ be a spanning fraction of S_Δ. Then κ is a basis of U and $\prod_{\alpha\in\kappa} \alpha$ has degree r. Thus the fraction is in $\mathcal{M}_U^r(V^\circledast \otimes \mathbb{C})$ and hence is fixed by P_U^r. On the other hand, the supporting cones for polar germs in $R_\Delta[> -r]$ or $(NG_\Delta \cap R_\Delta[\leq -r])$ do not span U, while the polar germs in $G_\Delta[< -r]$ do not have p-order r. Hence the polar germs are annihilated by P_U^r. □

Motivated by this fact, we set the following definition.

Definition 5.8. Given a meromorphic germ f and an $\mathbb{F}$-subspace U of V, let $d = \dim(U)$. Then $P_U^d(f)$ is called the *generalized Jeffrey–Kirwan residue* of f supported on U.

Example 5.9. Let $\{e_1, e_2, e_3, e_n\}$ denote the canonical basis in $\mathbb{R}^4$ and let U be the space spanned by e_1, e_2 so that $d = 2$. The meromorphic germ

$$f(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) := \frac{\varepsilon_4}{\varepsilon_1\varepsilon_2\varepsilon_3} + \frac{\varepsilon_2}{\varepsilon_1(\varepsilon_1+\varepsilon_2)} + \frac{\varepsilon_4}{\varepsilon_2(\varepsilon_1+\varepsilon_2)} + \frac{\varepsilon_1}{\varepsilon_2^2(\varepsilon_1+2\varepsilon_2)}$$

reads

$$f(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \frac{\varepsilon_4}{\varepsilon_1\varepsilon_2\varepsilon_3} + \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_1+\varepsilon_2} + \frac{\varepsilon_4}{\varepsilon_2(\varepsilon_1+\varepsilon_2)} + \frac{1}{\varepsilon_2^2} - \frac{2}{\varepsilon_2(\varepsilon_1+2\varepsilon_2)},$$

so

$$P_U^2(f) = \frac{\varepsilon_4}{\varepsilon_2(\varepsilon_1+\varepsilon_2)} - \frac{2}{\varepsilon_2(\varepsilon_1+2\varepsilon_2)}.$$

6. A filtered residue

In this part, we give two further applications of our Laurent theory developed in Section 4. We study the p-order of a meromorphic germ at zero, and define an invariant, called p-residue for the germ. We show that for exponential sums, taking p-residue amounts to the exponential integrals.

6A. *The p-order and p-residue.* The grading in (36) by p-orders of polar germs gives a p-order for any element in $\mathcal{M}(V^\circledast \otimes \mathbb{C})$.

Definition 6.1. Let $f \in \mathcal{M}(V^\circledast \otimes \mathbb{C})$. Let

$$\mathfrak{L}_{\mathfrak{C}}(f) = \bigoplus_{j\in J} S_j \oplus h \tag{47}$$

be a $\mathfrak{C}$-supported Laurent expansion of f for some appropriate family of supporting cones $\mathfrak{C}$ as in Definition 4.5.

(a) Define the *polar order*, or *p-order* for short, of f to be

$$\text{p-ord}(f) := \max_j(\text{p-ord}(S_j)),$$

where p-ord(S_j) is as in Definition 5.1.

(b) Let $S_i = h_i/(L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}})$, $1 \le i \le t$, be the polar germs in (47). We define the *highest polar order residue*, or the *p-residue* for short, of f to be

$$\text{p-res}(f) := \sum_{i=1}^{t} \frac{h_i(0)}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}.$$

These notions are well-defined thanks to the following property.

Proposition 6.2. *The p-order and p-residue of a meromorphic germ with linear poles depend neither on the choice of a Laurent expansion nor on the choice of the inner product used in the decomposition of* $\mathcal{M}(V^{\circledast} \otimes \mathbb{C})$ *in Theorem 4.13.*

Furthermore, for orthogonally variate f and g in the sense of Definition 4.17, we have

$$\text{p-res}(fg) = \text{p-res}(f)\,\text{p-res}(g).$$

Proof. The independence of the p-order on the choice of a Laurent expansion follows from the grading in (36). From (30), the numerator of a polar germ and hence the p-residue of f does not change under the subdivision map $\mathfrak{S}_{(\mathfrak{C},\mathfrak{D})}$. Then the independence of the p-residue on the choice of a Laurent expansion is a consequence of the following elementary yet useful result:

Let $\mathcal{I}$ be a direct system and let φ be a function on $\mathcal{I}$. If $\varphi(i) = \varphi(j)$ for all $i \le j$ in $\mathcal{I}$, then φ is a constant.

We next prove the independence of the p-order on the inner product. For an inner product Q in V and $f \in \mathcal{M}(V^{\circledast} \otimes \mathbb{C})$ with p-ord$(f) = p$, following (47) we write the Laurent expansion of f supported by $\mathfrak{C}$ as

$$\mathfrak{L}_{\mathfrak{C}}(f) = \sum_{i=1}^{r} S_i + \sum_{j=r+1}^{n} S_j + h, \tag{48}$$

with the polar germs sharing the largest p-order p grouped in the first sum and those with lesser p-order in the second sum.

Relative to a different inner product R on V, an S_i might not be a polar germ any longer. Set

$$S_i = \frac{h_i(\ell_{i1}, \ldots, \ell_{im_i})}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}$$

with $Q(\ell_{ip}, L_{iq}) = 0$. For $j = 1, \ldots, m_i$, there are coefficients a_{ij}^k such that

$$\ell_{ij} = \ell'_{ij} - \sum_{k=1}^{n_i} a_{ij}^k L_{ik},$$

where $R(\ell'_{ij}, L_{ik}) = 0$ for $k = 1, \ldots, n_i$. Then

$$S_i = \frac{h_i(\ell'_{i1}, \ldots, \ell'_{im_i})}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} + \text{terms of lower denominator degrees.}$$

Thus

$$f = \sum_{i=1}^{r} \frac{h_i(\ell'_{i1}, \ldots, \ell'_{im_i})}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} + \text{terms of lower denominator degrees.} \tag{49}$$

This gives a decomposition of f as a linear combination of polar germs for the inner product R.

The supporting cones from the right-hand side of the above equation are easily seen to be faces of the supporting cones in the decomposition of f under the inner product Q arising in (48). So they remain properly positioned. Since $h_i(\ell_{i1}, \ldots, \ell_{im_i}) \neq 0$, we also have $h_i(\ell'_{i1}, \ldots, \ell'_{im_i}) \neq 0$. Therefore, under the inner product R, the p-order of f is again p.

Furthermore, (49) shows how the polar germs of p-order p-ord(f) change for a different inner product. In particular, the constant terms of the numerators remain the same. Thus the p-residue does not depend on the choice of inner products.

The second statement follows from the fact that the product of a polar germ with either a polar germ or a holomorphic germ which are orthogonally variate is again a polar germ. Thus the highest polar order part of fg is the product of the highest polar order part of f and that of g. □

Remark 6.3. The proof of this proposition actually shows how the terms of p-order p change as the inner products change.

To simplify the notation, for a polar germ $S = h(\ell_1, \ldots, \ell_m)/(L_1^{s_1} \cdots L_k^{s_k})$, we set

$$S(0_{\vec{\ell}}) := \frac{h(0_{\vec{\ell}})}{L_1^{s_1} \cdots L_k^{s_k}}.$$

Proposition 6.4. *Let $f = \sum_i S_i + \sum_j T_j + h$, with S_i, T_j polar germs at zero, h a holomorphic germ at zero, the* p-ord(S_i) *all equal to r, $\sum_i S_i \neq 0$ and* p-ord$(T_j) < k$. *Then* p-ord$(f) = r$ *and* p-res$(f) = \sum_i S_i(0_{\vec{\ell}})$.

Proof. Taking a subdivision of the set of supporting cones of the germs S_i and T_j, we have $S_i = \sum_{i\ell} S_{i\ell}$ and $T_j = \sum_{jm} T_{jm}$. Then $f = \sum_{i\ell} S_{i\ell} + \sum_{jm} T_{jm} + h$. Combining terms that are proportional to one another, we can assume that this decomposition satisfies the conditions in Theorem 4.13. In this decomposition there are no terms of p-order greater than r and the sum of all the terms of p-order r is $\sum_{i,\ell} S_{i\ell} = \sum_i S_i \neq 0$. So $\text{p-ord}(f) = k$ and $\text{p-res}(f) = \sum_{i\ell} S_{i\ell}(0_{\vec{\ell}}) = \sum_i S_i(0_{\vec{\ell}})$. □

6B. ***The p-residue of the exponential sum on a lattice cone.*** As in [Guo et al. 2017], we can reinterpret the constructions of [Berline and Vergne 2007; Garoufalidis and Pommersheim 2012; Lawrence 1991] in terms of lattice cones.

We recall from [Guo et al. 2017] that a *lattice cone* in V_k is a pair (C, Λ_C) with C a cone in V_k and Λ_C a lattice in $\text{lin}(C)$ generated by lattice vectors. A lattice cone (C, Λ_C) is called *strongly convex* (resp. *simplicial*) if C is. A lattice cone (C, Λ_C) is called *smooth* if the additive monoid $\Lambda_C \cap C$ has a monoid basis. In other words, there are linearly independent lattice vectors $v_1, \dots, v_\ell$ such that $\Lambda_C \cap C = \mathbb{Z}_{\geq 0}\{v_1, \dots, v_\ell\}$.

To a lattice cone (C, Λ_C) we can assign two meromorphic functions. One is the exponential sum $S(C, \Lambda_C)$ [Barvinok 2008] (corresponding to $S^c(C, \Lambda_C)$ in [Guo et al. 2017]), given in the strongly convex case by $S(C, \Lambda_C)(\vec{\varepsilon}) := \sum_{\vec{n} \in C \cap \Lambda_C} e^{\langle \vec{n}, \vec{\varepsilon} \rangle}$. The other is the exponential integral $I(C, \Lambda_C)$ [Guo et al. 2017], which is a generalization of (7), where the matrix A_C is with respect to a basis of Λ_C.

Lemma 6.5. *For a smooth lattice cone* (C, Λ_C), *we have*

$$\text{p-ord}(S(C, \Lambda_C)) = \text{p-ord}(I(C, \Lambda_C)) = \dim(C), \quad \text{p-res}(S(C, \Lambda_C)) = I(C, \Lambda_C).$$

In fact, *we have*

$$S(C, \Lambda_C) = I(C, \Lambda_C) + (\text{terms of p-order} < \dim(C)).$$

Proof. Let $v_1, \dots, v_d$ (where $d = \dim C$) be a basis of Λ_C that generates C as a cone. Then

$$S(C, \Lambda_C)(\vec{\varepsilon}) = \prod_{i=1}^{d} \frac{1}{1 - e^{\langle v_i, \vec{\varepsilon} \rangle}} = \prod_{i=1}^{d} \left(-\frac{1}{\langle v_i, \vec{\varepsilon} \rangle} + h(\langle v_i, \vec{\varepsilon} \rangle) \right),$$

where h is holomorphic. So the highest p-order term is $\prod_{i=1}^{d} (-1/\langle v_i, \vec{\varepsilon} \rangle)$, which is $I(C, \Lambda_C)$ and has p-order d. □

Lemma 6.6. *For a lattice cone* (C, Λ_C),

$$I(C, \Lambda_C) \neq 0 \iff S(C, \Lambda_C) \neq 0 \iff C \textit{ is strongly convex.}$$

Proof. We already know that $I(C, \Lambda_C) = 0$ and $S(C, \Lambda_C) = 0$ if C is not strongly convex [Guo et al. 2017]. So we only need to prove that if C is strongly convex, then $I(C, \Lambda_C) \neq 0$ and $S(C, \Lambda_C) \neq 0$.

Take a smooth subdivision $\{C_i\}$ of C. Since C is strongly convex, $\{C_i\}$ is properly positioned. So the $I(C_i, \Lambda_{C_i})$ are linearly independent by the nonholomorphicity theorem (Theorem 3.7). Then $I(C, \Lambda_C)$, as their sum, cannot be 0.

Further, note that

$$\begin{aligned} S(C,\Lambda_C) &= \sum_i S(C_i,\Lambda_C) + (\text{terms with p-order} < \dim(C)) \\ &= \sum_i S(C_i,\Lambda_C) = I(C_i,\Lambda_C) + (\text{terms with p-order} < \dim(C)). \end{aligned}$$

By Proposition 6.4, p-ord$(S(C,\Lambda_C)) < \dim(C)$ implies $\sum_i I(C_i,\Lambda_C) = 0$, which is a contradiction. Then p-ord$(S(C,\Lambda_C)) = \dim(C)$ and so $S(C,\Lambda_C) \neq 0$. □

Proposition 6.7. *For any subdivision* $\{(C_i,\Lambda_C)\}$ *of a lattice cone* (C,Λ_C), *we have*

$$\text{p-res}(S(C,\Lambda_C)) = \sum_i \text{p-res}(S(C_i,\Lambda_{C_i})).$$

So the map p-res $\circ\, S$ *is compatible with subdivisions.*

Proof. It is sufficient to consider a subdivision $\{(C_i,\Lambda_C)\}$ of (C,Λ_C) with smooth cones C_i, since any other subdivision can be further subdivided into one containing only smooth cones. It follows from the definition of $S(C,\Lambda_C)$ that

$$S(C,\Lambda_C) = \sum_{I\subseteq[r]} (-1)^{|I|+1} S(C_I,\Lambda_I) = \sum_i S(C_i,\Lambda_C) + (\text{terms of p-order} < \dim(C)),$$

where $C_I := \bigcap_{i\in I} C_i$. Also, by Lemma 6.5,

$$S(C_i,\Lambda_C) = \sum_j T_{ij} + (\text{terms of p-order} < \dim(C)),$$

where T_{ij} are polar germs at zero with p-ord$(T_{ij}) = \dim(C)$. Thus

$$S(C,\Lambda_C) = \sum_{i,j} T_{ij} + (\text{terms of p-order} < \dim(C)).$$

If C is strongly convex, then p-ord$(S(C,\Lambda_C)) = \dim(C)$, and by Proposition 6.4,

$$\text{p-res}(S(C,\Lambda_C)) = \sum_{i,j} T_{ij}(0) = \sum_i \text{p-res}(S(C_i,\Lambda_C)).$$

If C is not strongly convex, then $S(C,\Lambda_C) = 0$, while by Proposition 6.4, this means $\sum_{i,j} T_{ij} = 0$, that is, $\sum_i$ p-res$(S(C_i,\Lambda_C)) = 0$. So the equality in the theorem holds in either case. □

Note that the operator I is also compatible with subdivisions [Guo et al. 2017]. Thus, as a consequence of Proposition 6.7, we obtain the following statement.

Corollary 6.8. *For a lattice cone* (C,Λ_C), *we have* p-res$(S(C,\Lambda_C)) = I(C,\Lambda_C)$.

Example 6.9. Take $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$ and $C = \langle e_1, e_1+e_2\rangle$ with (e_1,e_2) the canonical orthonormal basis in $\mathbb{R}^2$. Then $S^c(C,\Lambda_C) = 1/((1-e^{\varepsilon_1})(1-e^{\varepsilon_1+\varepsilon_2}))$ has p-order 2 and p-residue $I(C,\Lambda_C) = 1/(\varepsilon_1(\varepsilon_1+\varepsilon_2))$.

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