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PROJECTIVE CASES FOR THE RESTRICTION OF THE OSCILLATOR REPRESENTATION TO DUAL PAIRS OF TYPE I

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For all the irreducible dual pairs of type I (G, G'), we analyze the restriction of the oscillator representation as a $(\mathfrak{g}', K')$-module, when G' is the smaller group. For all (G, G') in the stable range, as well as one more case, the modules obtained are projective. We use the duality correspondence introduced by Howe to analyze these restrictions.

1. Introduction

A classical problem in representation theory is the understanding of the restriction of a representation Π of a group G to one of its subgroups H. This work focuses on $(\mathfrak{g}, K)$-modules, as defined by Harish-Chandra. In that setting, if Π is a $(\mathfrak{g}, K)$-module, it is useful to analyze $\mathrm{Hom}_{(\mathfrak{h}, H\cap K)}(\Pi, \pi)$, where π is an $(\mathfrak{h}, H\cap K)$-module. For this purpose, one may use the derived functors: calculating $\mathrm{Ext}^n_{(\mathfrak{h}, H\cap K)}(\Pi, \pi)$ is not necessarily easier than $\mathrm{Hom}_{(\mathfrak{h}, H\cap K)}(\Pi, \pi)$, but their Euler characteristic might be. This difficult part becomes much simpler when the restriction of Π is a projective $(\mathfrak{h}, H \cap K)$-module. In this case, $\mathrm{Ext}^n_{(\mathfrak{h}, H\cap K)}(\Pi, \pi)$ vanishes for every $n > 0$. It motivates this paper: the projectivity of a representation is an extremely powerful property. The link between Euler characteristic and projectivity is emphasized in [Adams et al. 2017], among others.

We focus on dual pairs, an approach introduced in the framework of the duality correspondence for the oscillator representation. A dual pair is a pair (G, G') of subgroups of a symplectic group $\mathrm{Sp}(V)$, such that G is the centralizer of G' in $\mathrm{Sp}(V)$, and vice versa. This work focuses on dual pairs of type I and uses the Fock model of the oscillator representation, ω. We prove:

Theorem. *Let G' be the smaller member of a dual pair (G, G') in a symplectic group $\mathrm{Sp}(V)$. Then the restriction of the Fock model of the oscillator representation ω of $\widetilde{\mathrm{Sp}}(V)$ to G' is a projective $(\mathfrak{g}', K')$-module under the condition $(*)$, as listed in Theorem 6.1. This condition includes the stable range but is slightly less restrictive.*

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It might seem unusual to focus on only one representation of one group. Due to the importance of the oscillator representation, this is however not surprising. This representation appears as (Segal–Shale)–Weil representation [Segal 1963; Shale 1962; Weil 1964] or metaplectic representation, among many other names. The theory of duality correspondence (or Theta correspondence) describes the representations that appear in the decomposition of the oscillator representation after restriction to a dual pair; see [Howe 1989] or [Kashiwara and Vergne 1978] for more details. The duality correspondence is one of the major tools used in this work.

2. Generalities

Let G be a Lie group with complexified Lie algebra $\mathfrak{g}$, and let K be a maximal compact subgroup in G, or its two-fold cover (as needed). We denote by $\mathfrak{k}$ the complexified Lie algebra of K, and we choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$.

Highest weight modules. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Let $\mathfrak{b}_{\mathfrak{k}}$ is a Borel subalgebra for $\mathfrak{k}$ containing $\mathfrak{t}$, and $\mathfrak{b}$ be a Borel subalgebra for $\mathfrak{g}$ containing $\mathfrak{b}_{\mathfrak{k}}$. We denote the set of positive roots by Δ^+, and write Δ_c for the compact roots, which are the roots coming from $\mathfrak{k}$. The set of noncompact roots is defined as $\Delta_n = \Delta - \Delta_c$. By intersecting Δ^+, we can define the positive compact roots Δ_c^+ and the positive noncompact roots Δ_n^+.

We define $\mathfrak{p}_+$ as the irreducible $\mathfrak{k}$-module spanned by Δ_n^+, and $\mathfrak{p}_-$ as the irreducible $\mathfrak{k}$-module spanned by Δ_n^-. This gives a decomposition of $\mathfrak{p}$ as $\mathfrak{p}_+ + \mathfrak{p}_-$. We then write $\mathfrak{q} = \mathfrak{q}_+ = \mathfrak{k} + \mathfrak{p}_+$ and $\mathfrak{q}_- = \mathfrak{k} + \mathfrak{p}_-$. By definition of $\mathfrak{p}_+$ and $\mathfrak{p}_-$, $\mathfrak{q}_+$ and $\mathfrak{q}_-$ are subalgebras of $\mathfrak{g}$.

Finally, we write $\mathfrak{U}(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$. For a weight λ of $\mathfrak{g}$, F_λ is the irreducible $\mathfrak{k}$-module with highest weight λ, and E_λ is the irreducible $\mathfrak{g}$-module with highest weight λ, with respect to the Borel subalgebras chosen above. We use $N(\lambda)$ to denote the generalized Verma module $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{q}_+)} F_\lambda$, which is a $\mathfrak{U}(\mathfrak{g})$-module.

Irreducibility criterion. For any $\alpha \in \Delta$ and $\lambda \in \mathfrak{t}^*$, we write $(\lambda)_\alpha = 2\langle \lambda, \alpha\rangle / \langle \alpha, \alpha\rangle$. The half sum of the positive roots is written ρ, and we use s_α for the reflection through the hyperplane determined by the root α. The following result about the irreducibility of $N(\lambda)$ appears in [Enright et al. 1983, Corollary 6.3 and Theorem 6.4], and the first part is originally due to Jantzen.

Proposition 2.1. *Assume that for any $\alpha \in \Delta_n^+$ with $(\lambda+\rho)_\alpha \in \mathbb{Z}_{>0}$, there is $\gamma \in \Delta_n$ with $(\lambda+\rho)_\gamma = 0$ and $s_\alpha(\gamma) \in \Delta_c$. Then $N(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{q})} F_\lambda$ is irreducible. Moreover, if $\mathfrak{g}$ is of type A_n, it is both a necessary and sufficient condition.*

$(\mathfrak{g}, K)$-modules. To stay in an algebraic setting, this work takes place in the category of $(\mathfrak{g}, K)$-modules, defined below. It allows us to us K to denote a maximal compact subgroup in G or its two-fold cover, as this distinction does not affect $(\mathfrak{g}, K)$-modules.

Definition. A $(\mathfrak{g}, K)$-*module* is a complex vector space V with an action of $\mathfrak{g}$ and an action of K such that

(1) for all $v \in V$, $k \in K$, $X \in \mathfrak{g}$, we have $k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$;

(2) V is K-finite, i.e., for every $v \in V$, the space generated by $K \cdot v$ is a finite-dimensional vector space;

(3) for all $v \in V$, $Y \in \mathfrak{k}$, we have $\left(\frac{d}{dt} \exp(tY) \cdot v\right)\Big|_{t=0} = Y \cdot v$.

We recall the Frobenius reciprocity, together with one important corollary.

Proposition 2.2 (Frobenius reciprocity). *Let A, B be two rings with $A \subset B$. Let M be an A-module and N be a B-module. We have a vector space isomorphism* $\mathrm{Hom}_B(B \otimes_A M, N) \cong \mathrm{Hom}_A(M, N)$.

Corollary 2.3. *Let Q be an A-module, and let $P = B \otimes_A Q$. If Q is a projective A-module, then P is a projective B-module.*

As a consequence, we get the following result for $(\mathfrak{g}, K)$-modules:

Proposition 2.4. *Let V be a $(\mathfrak{k}, K)$-module. Then $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{k})} V$ is a projective $(\mathfrak{g}, K)$-module.*

Proof. By K-finiteness, every $(\mathfrak{k}, K)$-module is projective as a $(\mathfrak{k}, K)$-module. Now the result is a direct application of Corollary 2.3 restricted to $(\mathfrak{g}, K)$-modules. □

Oscillator representation. We are interested in a particular representation $\widetilde{\omega}$ of the metaplectic group $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$, a double cover of the symplectic group. This representation, called oscillator representation, was first introduced in [Segal 1963; Shale 1962], followed by [Weil 1964]. Several constructions and different models for the oscillator representation appear in [Li 2000; Adams 2007].

For a subgroup G of $\mathrm{Sp}(2N, \mathbb{R})$, we denote by $\widetilde{G}$ its preimage in $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$. We are only interested in algebraic $\widetilde{G}$-modules; hence we consider the category of $(\mathfrak{g}, K)$-modules, for K a maximal compact subgroup of G, or its two-fold cover. Therefore, we work with the Harish-Chandra module of the oscillator representation, a realization of $\widetilde{\omega}$ as an $(\mathfrak{sp}(2N, \mathbb{C}), \widetilde{U}(N))$-module. We still call it the oscillator representation but denote it by ω.

Since most of this work is done on the Lie algebra level, double covers do not play an important role. It is therefore enough to analyze subgroups G, G' in a symplectic group, and it is not necessary to focus on their preimage $\widetilde{G}$, $\widetilde{G}'$ in $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$.

Reductive dual pairs. To decompose the oscillator representation restricted to a subgroup, we use dual pairs, following Howe's approach.

Definition. A pair (G, G') of subgroups in a symplectic group $\mathrm{Sp}(2N, \mathbb{R})$ is a *reductive dual pair* if

(1) G and G' act reductively on $\mathbb{R}^{2N}$,

(2) G and G' are centralizers of each other inside $\mathrm{Sp}(2N, \mathbb{R})$.

Moreover, if G is compact, we say that (G, G') is a *compact dual pair*.

We assume that G' is the smaller member of the pair so that the duality correspondence holds. We also consider two dual pairs with a particular relation, as introduced in [Kudla 1984]:

Definition. Two dual pairs (G, G') and (H, H') form a *seesaw dual pair* if we have the inclusions $H \subset G$ and $G' \subset H'$. We denote it by $((G, G'), (H, H'))$.

Irreducible dual pairs, i.e., pairs that cannot be decomposed as a direct sum of two dual pairs, are classified in two types. Following [Howe 1989], each pair corresponds to either a division algebra (type II) or a division algebra with an involution (type I). We focus on pairs of type I, which come in four different types:

(1) $(O(p, q), \mathrm{Sp}(2n, \mathbb{R}))$, corresponding to $\mathbb{R}$ with the identity map,

(2) $(O(p, \mathbb{C}), \mathrm{Sp}(2n, \mathbb{C}))$, corresponding to $\mathbb{C}$ with the identity map,

(3) $(U(r, s), U(p, q))$, corresponding to $\mathbb{C}$ with the conjugation map,

(4) $(\mathrm{Sp}(p, q), O^*(2n))$, corresponding to $\mathbb{H}$ with the conjugation map.

This corresponds to seven different cases, depending which group of the pair is the smallest (except for (3), which is symmetric).

Duality correspondence. For a dual pair (G, G') with G compact, we decompose the oscillator representation ω of $\mathrm{Sp}(2N, \mathbb{R})$ under the action of G. We obtain

$$\omega = \bigoplus\nolimits_{\sigma} (\mathrm{Hom}_G(\sigma, \omega) \otimes \sigma),$$

summing over all the irreducible representations σ of G. Indeed, if $T \in \mathrm{Hom}_G(\sigma, \omega)$ and $v \in \sigma$, then $T(v) \in \omega$ and we have a map $\mathrm{Hom}_G(\sigma, \omega) \times \sigma \to \omega$, $(T, v) \mapsto T(v)$. This map extends to $\mathrm{Hom}_G(\sigma, \omega) \otimes \sigma \to \omega$, which is injective when σ is irreducible. Since G is compact, ω is completely reducible, and $\omega = \bigoplus_\sigma (\mathrm{Hom}_G(\sigma, \omega) \otimes \sigma)$.

The duality correspondence gives an explicit description of $\theta(\sigma)$. By compactness of G, $\theta(\sigma)$ is a highest weight module, and we denote its highest weight by τ. We write E_τ for the irreducible $\mathfrak{g}'$-module with highest weight τ. Note that τ is also a dominant weight for $\mathfrak{k}'$, so τ is also the highest weight of a finite dimensional representation of $\mathfrak{k}'$. We use F_τ for the irreducible $\mathfrak{k}'$-module with highest weight τ. We list now the duality correspondence for the pairs of type I.

The explicit correspondence, originally due to Kashiwara and Vergne [1978], will be introduced in Section 4 when used.

3. Setup and method

This section defines the notation, for G, G' subgroups of a large symplectic group:

- G, G' are real Lie groups, with complexified Lie algebras $\mathfrak{g}$, $\mathfrak{g}'$, forming a dual pair (G, G') with G' the smaller member;
- K, K' are maximal compact subgroups of G, G' (or their two-fold covers), respectively, with complexified Lie algebras $\mathfrak{k}$, $\mathfrak{k}'$, and Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{r}'$ for $\mathfrak{g}'$;
- M' is the centralizer of K, so that $((K, M'), (G, G'))$ is a seesaw dual pair, with complexified Lie algebra $\mathfrak{m}'$;
- J' is a maximal compact subgroup of M' (or its two-fold cover) with complexified Lie algebra $\mathfrak{j}'$, and Cartan decomposition $\mathfrak{m}' = \mathfrak{j}' + \mathfrak{p}' = \mathfrak{j}' + \mathfrak{p}'_+ + \mathfrak{p}'_-$;
- $\mathfrak{t}'$ is a Cartan subalgebra of both $\mathfrak{m}'$ and $\mathfrak{j}'$;
- $\mathfrak{q}' = \mathfrak{q}'_+ = \mathfrak{j}' + \mathfrak{p}'_+$ and $\mathfrak{q}'_- = \mathfrak{j}' + \mathfrak{p}'_-$ are two parabolic subalgebras of $\mathfrak{m}'$.

To understand the restriction of ω to G', we encounter two different cases.

(1) $M' \ncong G' \times G'$: We let K act to get a decomposition $\omega = \bigoplus_\sigma (\sigma \otimes E_\tau)$, for σ an irreducible representation of K and E_τ an irreducible representation of M' with highest weight τ. We compute a condition $(*)$ so that $N(\tau)$ is irreducible, which forces $E_\tau = N(\tau) = \mathfrak{U}(\mathfrak{m}') \otimes_{\mathfrak{U}(\mathfrak{q}')} F_\tau$, a projective $(\mathfrak{m}', J')$-module. The restriction from M' to G' is computed to get

$$\omega \cong \bigoplus_\sigma \big(\sigma \otimes \big(\mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{k}')} (F_\tau|_{\mathfrak{k}'})\big)\big),$$

where each summand is a projective $(\mathfrak{g}', K')$-module, under the condition $(*)$.

(2) $(K, M') = (K_1, G') \oplus (K_2, G')$ with K_1 and K_2 of the same type: This is the case where $K = K_1 \times K_2$, which can be $K = O(p) \times O(q)$, $K = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$, or $K = U(r) \times U(s)$. First, the action of $K_1 \times K_2$ decomposes $\omega = \omega_1 \otimes \omega_2^*$, with ω_1 a highest weight module for K_1, ω_2 a lowest weight module for K_2. Each piece is decomposed as above. We compute a condition $(*)$ so that ω_1 is a projective $(\mathfrak{g}', K')$-module, and another condition $(**)$ ensuring that ω_2^* is a projective $(\mathfrak{g}', K')$-module as well. The tensor product is computed, so that when both conditions $(*)$ and $(**)$ are met, we have

$$\omega = \omega_1 \otimes \omega_2^* = \bigoplus_{\sigma, \widetilde{\sigma}} \big((\sigma \otimes \widetilde{\sigma}) \otimes \big(\mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{k}')} (F_\tau \otimes F_{\widetilde{\tau}})\big)\big),$$

and each summand is a projective $(\mathfrak{g}', K')$-module.

Section 4 explores conditions so that $N(\tau)$ is irreducible, for each compact dual pair. Section 5 analyzes the restriction from M' to G' in the first case, and the tensor product in the second case. Finally, the results are summarized in Section 6. This work follows a strategy from [Howe 1983], using seesaw pairs to reduce the problem to unitary highest weight modules. In that work, Howe gives a similar result [1983, Theorem 5.2] but from an L^2 perspective.

4. Irreducibility of $N(\tau)$

For each compact dual pair, we give a condition on the respective sizes of the groups so that the generalized Verma module $N(\tau)$ is irreducible. The stable range case is already known (see [Nishiyama et al. 2006], for example), but our results show that this irreducibility holds in one more case. We also show that our bound cannot be extended in a general case, by giving counterexamples.

Dual pair $(K, M') = (O(n,\mathbb{R}), \mathrm{Sp}(2p,\mathbb{R}))$**.** Since $O(n,\mathbb{R})$ is a disconnected group, we use the embedding

$$O(n,\mathbb{R}) = U(n) \cap \mathrm{GL}(n,\mathbb{R}).$$

Given a highest weight λ of $U(n)$ and a parameter $\epsilon = \pm 1$, the representation of $O(n,\mathbb{R})$ with highest weight $(\lambda; \epsilon)$ is the irreducible summand of the representation of $U(n)$ with highest weight λ containing the highest weight vector, tensored with the sign representation of $O(n,\mathbb{R})$ if $\epsilon = -1$.

The group $M' = \mathrm{Sp}(2p,\mathbb{R})$ has a maximal compact subgroup $J' = U(p)$. We have a correspondence between the highest weight σ for $O(n,\mathbb{R})$ and the highest weight τ for $\mathfrak{sp}(2p,\mathbb{C})$, which appears in [Kashiwara and Vergne 1978, Theorems 6.9 and 7.2, part II]. The correspondence is given by

$$\begin{aligned}\sigma &= (a_1,\dots,a_k,0,\dots,0;\epsilon)\\ &\mapsto \tau = \Bigl(-\frac{n}{2},\dots,-\frac{n}{2},\overbrace{-\frac{n}{2}-1,\dots,-\frac{n}{2}-1}^{\frac{1}{2}(1-\epsilon)(n-2k)},-a_k-\frac{n}{2},\dots,-a_1-\frac{n}{2}\Bigr),\end{aligned}$$

where σ defines an irreducible highest weight $O(n,\mathbb{R})$-module and τ defines an irreducible highest weight $\mathfrak{sp}(2p,\mathbb{C})$-module. All such weights occur, with the constraints $k \le \left[\frac{n}{2}\right]$ and $k + \frac{1}{2}(1-\epsilon)(n-2k) \le p$. Since we start with a highest weight σ for $O(n,\mathbb{R})$, we also have $a_1 \ge \cdots \ge a_k \ge 0$.

The root system occurring here is given by

- $\Delta^+ = \{e_i - e_j \mid 1 \le i < j \le p\} \cup \{e_i + e_j \mid 1 \le i < j \le p\} \cup \{2e_i \mid 1 \le i \le p\}$,
- $\Delta_n^+ = \{e_i + e_j \mid 1 \le i < j \le p\} \cup \{2e_i \mid 1 \le i \le p\}$,
- $\rho = (p,\dots,p+1-i,\dots,1)$, where $p+1-i$ is the i-th coordinate.

Case $\epsilon = 1$. The products between $\tau + \rho$ and a noncompact positive root are

$$(\tau+\rho)_{2e_i} = \begin{cases} p+1-i-\dfrac{n}{2} & \text{if } 1 \le i \le p-k, \\ p+1-i-\dfrac{n}{2}-a_{p+1-i} & \text{if } p-k < i \le p, \end{cases}$$

$$(\tau+\rho)_{e_i+e_j} = \begin{cases} 2p+2-i-j-n & \text{if } 1 \le i, j \le p-k, \\ 2p+2-i-j-n-a_{p+1-j} & \text{if } 1 \le i \le p-k < j \le p, \\ 2p+2-i-j-n-a_{p+1-i}-a_{p+1-j} & \text{if } p-k < i, j \le p. \end{cases}$$

If we take $n \ge 2p$, all these products are nonpositive, and by Proposition 2.1 $N(\tau)$ is irreducible. For $n = 2p-1$, we see that $p+1-i-\frac{n}{2} = \frac{3}{2}-i$ is not an integer. All the other products are nonpositive, so the criterion applies, and $N(\tau)$ is irreducible.

Case $\epsilon = -1$. The products of $\tau + \rho$ with noncompact positive roots are

$$(\tau+\rho)_{2e_i} = \begin{cases} p+1-i-\dfrac{n}{2} & \text{if } 1 \le i \le p+k-n, \\ p-i-\dfrac{n}{2} & \text{if } p+k-n < i \le p-k, \\ p+1-i-\dfrac{n}{2}-a_{p+1-i} & \text{if } p-k < i \le p, \end{cases}$$

$$(\tau+\rho)_{e_i+e_j} = \begin{cases} 2p+2-i-j-n & \text{if } 1 \le i, j \le p+k-n, \\ 2p-i-j-n & \text{if } p+k-n < i, j \le p-k, \\ 2p+2-i-j-n-a_{p+1-i}-a_{p+1-j} & \text{if } p-k < i, j \le p, \\ 2p+1-i-j-n & \text{if } 1 \le i \le p+k-n \\ & \quad \text{and } p+k-n < j \le p-k, \\ 2p+1-i-j-n-a_{p+1-j} & \text{if } p+k-n < i \le p-k \\ & \quad \text{and } p-k < j \le p, \\ 2p+2-i-j-n-a_{p+1-j} & \text{if } 1 \le i \le p+k-n \\ & \quad \text{and } p-k < j \le p. \end{cases}$$

For $n \ge 2p$, all these products are nonpositive; hence $N(\tau)$ is irreducible. For $n = 2p-1$, we have $p+1-i-\frac{n}{2} = \frac{3}{2}-i$, which is not an integer, and all the other products are nonpositive. We have proved:

Lemma 4.1. *If $n \ge 2p-1$, then the $(\mathfrak{sp}(2p, \mathbb{C}), \widetilde{U}(p))$-module $N(\tau)$ is irreducible for all the weights τ appearing in the restriction $\omega|_{\mathfrak{sp}(2p,\mathbb{C})}$.*

If $n = 2p-2$, we use Theorem 6.2 from [Enright and Joseph 1990]. Starting from

$$\sigma = (\overbrace{2, \dots, 2}^{p-1}, \overbrace{0, \dots, 0}^{p-1}; 1),$$

we get the highest weight $\tau = (-p+1, -p-1, \dots, -p-1)$, which we write as $\tau = (2, 0, \dots, 0)+(-p-1, \dots, -p-1) = (2, 0, \dots, 0)+(-p-1)\omega_\alpha$ following [Enright and Joseph 1990]. For the family $N(u\omega_\alpha+(2, 0, \dots, 0))$, the first reduction point of the family is given by $u = -p-1$. So $N(\tau)$ is reducible.

Dual pair $(K, M') = (U(p), U(m, n))$. The duality correspondence for this case is expressed with $U(p)$-modules and $\mathfrak{gl}(m+n, \mathbb{C})$-modules, as explained in [Kashiwara and Vergne 1978, Theorem 6.3, part III]. Explicitly, the duality correspondence is given by the map $\sigma \mapsto \tau$ described below:

$$\sigma = \Big(a_1+\frac{m-n}{2}, \dots, a_k+\frac{m-n}{2}, \frac{m-n}{2}, \dots, \frac{m-n}{2}, b_1+\frac{m-n}{2}, \dots, b_l+\frac{m-n}{2}\Big)$$
$$\mapsto \tau = \Big(-\frac{p}{2}, \dots, -\frac{p}{2}, b_1-\frac{p}{2}, \dots, b_l-\frac{p}{2}\Big) \oplus \Big(a_1+\frac{p}{2}, \dots, a_k+\frac{p}{2}, \frac{p}{2}, \dots, \frac{p}{2}\Big),$$

where σ defines an irreducible highest weight $U(p)$-module and τ defines an irreducible highest weight $\mathfrak{gl}(m+n, \mathbb{C})$-module. All such weights occur, with the constraints $k+l \le p$, $k \le m$, $l \le n$. To make notation easier for the next step, we assume that a_i and b_j can be equal to zero, and change the numbering, so we rewrite τ as

$$\tau = \Big(b_1-\frac{p}{2}, \dots, b_n-\frac{p}{2}\Big) \oplus \Big(a_{n+1}+\frac{p}{2}, \dots, a_{n+m}+\frac{p}{2}\Big)$$

with $b_n \le \dots \le b_1 \le 0$ and $0 \le a_{n+m} \le \dots \le a_{n+1}$.

We apply the irreducibility criterion given by Proposition 2.1. Our group $M' = U(n, m)$ contains a maximal compact subgroup $J' = U(n) \times U(m)$. The root system is of type A_n; hence this criterion is both necessary and sufficient for the irreducibility of $N(\tau)$. We have

- $\Delta^+ = \{e_i - e_j \mid 1 \le i < j \le n+m\}$,
- $\Delta_n^+ = \{e_i - e_j \mid 1 \le i \le n < j \le n+m\}$,
- $\rho = \Big(\frac{m+n-1}{2}, \dots, \frac{m+n-2i+1}{2}, \dots, \frac{-m-n+1}{2}\Big)$, where $\frac{m+n-2i+1}{2}$ is the i-th coordinate.

We obtain $(\tau+\rho)_{e_i-e_j} = b_i - a_j + j - i - p$ with $1 \le i \le n < j \le n+m$. Since $b_i - a_j \le 0$ for all i, j, we conclude that if $p \ge m+n-1$, then $(\tau+\rho)_{e_i-e_j}$ is nonpositive for all i, j, and $N(\tau)$ is irreducible. We deduce:

Lemma 4.2. *If $p \ge m+n-1$, then the $(\mathfrak{gl}(m+n, \mathbb{C}), \widetilde{U}(m) \times \widetilde{U}(n))$-module $N(\tau)$ is irreducible for all the weights τ appearing in the restriction $\omega|_{\mathfrak{gl}_{m+n}(\mathbb{C})}$.*

If $p = m+n-2$, with $m, n \ge 2$, and

$$\sigma = \Big(\overbrace{1+\frac{m-n}{2}, \dots, 1+\frac{m-n}{2}}^{n-1}, \overbrace{-1+\frac{m-n}{2}, \dots, -1+\frac{m-n}{2}}^{m-1}\Big),$$

we find the corresponding highest weight

$$\tau = \Big(-\frac{p}{2}, -1-\frac{p}{2}, \dots, -1-\frac{p}{2}\Big) \oplus \Big(1+\frac{p}{2}, \dots, 1+\frac{p}{2}, \frac{p}{2}\Big).$$

The products $(\tau+\rho)_{e_i-e_j}$ are strictly negative, except for $(\tau+\rho)_{e_1-e_{n+m}} = 1$. But there is no root $\gamma \in \Delta_n^+$ such that $(\tau+\rho)_\gamma = 0$. Since Proposition 2.1 becomes a necessary condition for type A_n, this $N(\tau)$ is reducible.

Dual pair $(K, M') = (\mathrm{Sp}(p), O^*(2n))$. We recall that $\mathrm{Sp}(p)$ can be seen either as the unitary quaternionic group, or as the intersection of $\mathrm{Sp}(2p, \mathbb{C})$ and $U(2p)$. Its complexified Lie algebra is given by $\mathfrak{sp}(2p, \mathbb{C})$. The group $O^*(2n) = \mathrm{SO}^*(2n)$ is the quaternionic orthogonal group. Its complexified Lie algebra is $\mathfrak{o}(2n, \mathbb{C})$.

The duality correspondence for the pair $(\mathrm{Sp}(p), O^*(2n))$ is given by the map $\sigma \mapsto \tau$ described below:

$$\sigma = (a_1, \dots, a_k, 0, \dots, 0) \mapsto \tau = (-p, \dots, -p, -p-a_k, \dots, -p-a_1),$$

where σ defines an irreducible highest weight $\mathrm{Sp}(p)$-module and τ defines an irreducible highest weight $\mathfrak{o}(2n, \mathbb{C})$-module. All such weights occur, with the constraints $k \le p$, $k \le n$. For this case, the correspondence can be deduced from [Howe 1995, Theorem 3.8.5.3]. Again, for notation purposes, we allow $a_i = 0$ and rewrite τ as

$$(-p-a_n, \dots, -p-a_{n-i+1}, \dots, -p-a_1)$$

with $a_1 \ge \cdots \ge a_n \ge 0$.

A maximal compact subgroup of M' is $J' = U(n)$. The complexified Lie algebra of M' is of type D_n. Therefore the root system of M' is given by

- $\Delta^+ = \{e_i - e_j \mid 1 \le i < j \le n\} \cup \{e_i + e_j \mid 1 \le i < j \le n\}$,
- $\Delta_n^+ = \{e_i + e_j \mid 1 \le i < j \le n\}$,
- $\rho = (n-1, \dots, n-i, \dots, 0)$, where $n-i$ is the i-th coordinate.

To apply Proposition 2.1, we calculate

$$(\tau+\rho)_{e_i+e_j} = 2n-2p-i-j-a_{n-i+1}-a_{n-j+1}$$

with $1 \le i < j \le n$. For all i, j, we know that $-a_{n-i+1}-a_{n-j+1} \le 0$. So we conclude that if $p \ge n-\frac{3}{2}$, then $(\tau+\rho)_{e_i-e_j}$ is nonpositive for all i and j, and $N(\tau)$ is irreducible. Since we only consider integral values of n and p, we rewrite the bound as $p \ge n-1$. We proved:

Lemma 4.3. *If $p \ge n-1$, then the $(\mathfrak{o}(2n, \mathbb{C}), \widetilde{U}(n))$-module $N(\tau)$ is irreducible for all the weights τ appearing in the restriction $\omega|_{\mathfrak{o}(2n,\mathbb{C})}$.*

When $p = n-2$, we use [Enright and Joseph 1990, Theorem 6.2] again to show that some modules $N(\tau)$ appearing in the restriction of ω are reducible. Choosing $\sigma = (1, \dots, 1, 0)$ gives a highest weight $\tau = (-p, -p-1, \dots, -p-1) = (-n+2, -n+1, \dots, -n+1)$, which is written as

$$(-n+1, \dots, -n+1)+(1, 0, \dots, 0) = (-n+1)\omega_\alpha+(1, 0, \dots, 0)$$

following the notation from [Enright and Joseph 1990]. The first reduction point of the family $N(u\omega_\alpha+(1, 0, \dots, 0))$ is at $u = -n+1$; hence $N(\tau)$ with τ given above is reducible.

5. Modules identifications

The results presented in this section are known; see [Loke and Ma 2015] when (G, G') is in the table range, for example. For readability and consistency of notation, we still include our approach in this paper.

Restriction from M' to G'. We start from M', with complexified Lie algebra $\mathfrak{m}'$ and maximal compact subgroup J'. We recall the Cartan decomposition $\mathfrak{m}' = \mathfrak{j}'+\mathfrak{p}'$, with $\mathfrak{p}' = \mathfrak{p}'_+ + \mathfrak{p}'_-$, and we write $\mathfrak{q}'$ for $\mathfrak{j}'+\mathfrak{p}'_+$. The group G' is a subgroup of M', with complexified Lie algebra $\mathfrak{g}'$. We have a maximal compact subgroup K' of G', and the Cartan decomposition $\mathfrak{g}' = \mathfrak{k}'+\mathfrak{r}'$.

We consider a finite-dimensional $\mathfrak{j}'$-module E, so E is a $(\mathfrak{j}', J')$-module. By letting $\mathfrak{p}'_+$ act trivially, E becomes a $\mathfrak{q}'$-module and we form $W = \mathfrak{U}(\mathfrak{m}')\otimes_{\mathfrak{U}(\mathfrak{q}')} E$, which is a $(\mathfrak{m}', J')$-module. We analyze the restriction of W as a $(\mathfrak{g}', K')$-module. As vector spaces, we have $W \cong S(\mathfrak{p}'_-)\otimes E$, with $S(\mathfrak{p}'_-)$ the symmetric algebra on $\mathfrak{p}'_-$. From E, we also create a $(\mathfrak{g}', K')$-module: by restriction, we see E as a $\mathfrak{k}'$-module $E|_{\mathfrak{k}'}$, and form the tensor product $V = \mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{k}')} (E|_{\mathfrak{k}'})$. Similarly, there is an isomorphism of vector spaces $V \cong S(\mathfrak{r}')\otimes(E|_{\mathfrak{k}'})$.

We define two filtrations, $V = \bigoplus_n V_n/V_{n-1}$ and $W = \bigoplus_n W_n/W_{n-1}$, by

$$V_n = \sum_{r\le n} S(\mathfrak{r}')[r]\mathfrak{U}(\mathfrak{k}')\otimes_{\mathfrak{U}(\mathfrak{k}')} (E|_{\mathfrak{k}'}) \quad \text{and} \quad W_n = \sum_{r\le n} S(\mathfrak{p}'_-)[r]\mathfrak{U}(\mathfrak{j}')\otimes_{\mathfrak{U}(\mathfrak{q}')} E.$$

By Frobenius reciprocity, we have a map $T : V \to W$, $1\otimes e \mapsto 1\otimes e$ for any $e \in E$. Writing $\{x_1, \dots, x_r\}$ for a basis of $\mathfrak{r}'$, $\{y_1, \dots, y_r\}$ for a basis of $\mathfrak{p}'_+$ and $\{z_1, \dots, z_r\}$ for a basis of $\mathfrak{p}'_-$ such that $x_i = y_i+z_i$ in $\mathfrak{p}'$, we extend the map T linearly so that

$$T(x_1 \cdots x_n\otimes e) = (y_1+z_1)\cdots(y_n+z_n)\otimes e$$

for any $e \in E$. The map $T : V \to W$ now preserves the filtrations, which proves:

Lemma 5.1. *The map $T : V \to W$ is an isomorphism of $\mathfrak{U}(\mathfrak{g}')$-modules, induced by an isomorphism of $S(\mathfrak{r}')$-modules on the graded spaces $T_G : \mathrm{Gr}(V) \to \mathrm{Gr}(W)$, through the isomorphism given by $\mathfrak{r}' \hookrightarrow \mathfrak{p}' \twoheadrightarrow \mathfrak{p}'_-$, as presented in* [Howe 1989].

This implies that

$$\big(\mathfrak{U}(\mathfrak{m}')\otimes_{\mathfrak{U}(\mathfrak{q}')} E\big)\Big|_{\mathfrak{g}'} \cong \mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{k}')}(E|_{\mathfrak{k}'}).$$

This is applied to $E=F_\tau$ in this work.

Tensor product for $(K, M')=(K_1, G')\oplus(K_2, G')$. We start from the Cartan decomposition $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{r}'$. We decompose $\mathfrak{r}'$ further as $\mathfrak{r}'_+ + \mathfrak{r}'_-$ and note that $\mathfrak{r}'_+$ and $\mathfrak{r}'_-$ are commutative Lie algebras. We define $\mathfrak{q}'_+=\mathfrak{k}'+\mathfrak{r}'_+$ and $\mathfrak{q}'_-=\mathfrak{k}'+\mathfrak{r}'_-$.

We consider two finite-dimensional $\mathfrak{k}'$-modules E and F; these are $(\mathfrak{k}', K')$-modules. We let $\mathfrak{r}'_+$ act on E by zero, so E becomes a $\mathfrak{q}'_+$-module. Similarly, we let $\mathfrak{r}'_-$ act on F by zero and obtain a $\mathfrak{q}'_-$-module. We define $V_E=\mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E$ and $V_F=\mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F$, which are $(\mathfrak{g}', K')$-modules, and $V=\mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{k}')}(E\otimes F)$.

By the Poincaré–Birkhoff–Witt theorem, there exists a grading on both $\mathfrak{U}(\mathfrak{r}'_+)$ and $\mathfrak{U}(\mathfrak{r}'_-)$. Since $\mathfrak{r}'_+$ and $\mathfrak{r}'_-$ are commutative, $\mathfrak{U}(\mathfrak{r}'_+)=S(\mathfrak{r}'_+)$ and $\mathfrak{U}(\mathfrak{r}'_-)=S(\mathfrak{r}'_-)$. We identify the piece $\mathfrak{U}(\mathfrak{r}'_+)[n]$ of degree n with the space $S(\mathfrak{r}'_+)[n]$ of homogeneous polynomials of degree n (same for $\mathfrak{r}'_-$). We write M_n for the subspace of elements of degree less than or equal to n in $\mathfrak{U}(\mathfrak{r}')$:

$$M_n=\sum_{r+s\le n}\big(S(\mathfrak{r}'_-)[r]\otimes S(\mathfrak{r}'_+)[s]\big)\cong\bigoplus_{i\le n} S(\mathfrak{r}')[i].$$

From this, we define a filtration $V=\bigoplus_n V_n/V_{n-1}$ by $V_n=M_n\mathfrak{U}(\mathfrak{k}')\otimes_{\mathfrak{U}(\mathfrak{k}')}(E\otimes F)$. Note that $V_0=E\otimes F$. By the description of M_n as $\bigoplus_{i\le n} S(\mathfrak{r}')[i]$, the quotient M_n/M_{n-1} is identified with $S(\mathfrak{r}')[n]$ and we get $V_n/V_{n+1}=S(\mathfrak{r}')[n]\otimes_{\mathfrak{U}(\mathfrak{k}')}(E\otimes F)$. We define a similar filtration on $V_E\otimes V_F$:

$$(V_E\otimes V_F)_n=\sum_{r+s\le n}\Big(\big(M_r\mathfrak{U}(\mathfrak{k})\otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E\big)\otimes\big(M_s\mathfrak{U}(\mathfrak{k})\otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F\big)\Big).$$

As vector spaces, this is equivalent to

$$(V_E\otimes V_F)_n=\sum_{r+s\le n}\big(S(\mathfrak{r}'_-)[r]\otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E\big)\otimes\big(S(\mathfrak{r}'_+)[s]\otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F\big).$$

Hence we have

$$(V_E\otimes V_F)_n/(V_E\otimes V_F)_{n-1}=\sum_{r+s=n}\big(S(\mathfrak{r}'_-)[r]\otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E\big)\otimes\big(S(\mathfrak{r}'_+)[s]\otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F\big).$$

Since $E\otimes F=V_0$ is naturally a subset of V, we use Frobenius reciprocity to extend this inclusion to a map

$$T: V\to V_E\otimes V_F,\quad 1\otimes(e\otimes f)\mapsto(1\otimes e)\otimes(1\otimes f),$$

for all $e\in E$ and $f\in F$. We extend this map so that it is compatible with the module structure. By using bases of $\mathfrak{r}'_+$, $\mathfrak{r}'_-$, it is a simple computation to show that T preserves the filtrations.

Lemma 5.2. *The map $T : V \to V_E \otimes V_F$, induced by an isomorphism of $S(\mathfrak{r}')$-modules on the graded spaces $T_G : \mathrm{Gr}(V) \to \mathrm{Gr}(V_E \otimes V_F)$, is an isomorphism of $\mathfrak{U}(\mathfrak{g}')$-modules.*

Proof. We know that $T(V_n) \subset (V_E \otimes V_F)_n$. Using computations with the action of basis elements of $\mathfrak{r}'_+$ and $\mathfrak{r}'_-$ and by tracking the degrees, we show that the map $T_G : \mathrm{Gr}(V) \to \mathrm{Gr}(V_E \otimes V_F)$ is surjective. Finally, as $\mathbb{C}$-vector spaces, we have

$$\begin{aligned}\dim(V_n) &= \dim(E)\dim(F)\Bigg(\sum_{r+s\le n} \dim(S(\mathfrak{p}'_-)[r])\dim(S(\mathfrak{p}'_+)[s])\Bigg)\\ &= \dim((V_E \otimes V_F)_n),\end{aligned}$$

which concludes the proof. □

Hence, we have proved that $V_E \otimes V_F \cong V$, i.e.,

$$\big(\mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E\big)\otimes\big(\mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F\big) \cong \mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{k}')}(E\otimes F).$$

6. Conclusion

Theorem 6.1. *Let G' be the smaller member of a dual pair (G, G') in a symplectic group $\mathrm{Sp}(V)$. Then the restriction of the Fock model of the oscillator representation ω of $\widetilde{\mathrm{Sp}}(V)$ to G' is a projective $(\mathfrak{g}', K')$-module under the condition $(*)$ listed in the table below*:

	(G, G')	(K, M')	$(*)$
(i)	$(\mathrm{Sp}(2n,\mathbb{R}), O(p,q))$	$(U(n), U(p,q))$	$n \ge p+q-1$
(ii)	$(O(p,q), \mathrm{Sp}(2n,\mathbb{R}))$	$(O(p), \mathrm{Sp}(2n,\mathbb{R}))$ $\oplus(O(q), \mathrm{Sp}(2n,\mathbb{R}))$	$p, q \ge 2n-1$
(iii)	$(O^*(2n), \mathrm{Sp}(p,q))$	$(U(n), U(2p,2q))$	$n \ge 2(p+q)-1$
(iv)	$(\mathrm{Sp}(p,q), O^*(2n))$	$(\mathrm{Sp}(p), O^*(2n))$ $\oplus(\mathrm{Sp}(q), O^*(2n))$	$p, q \ge n-1$
(v)	$(\mathrm{Sp}(2n,\mathbb{C}), O(p,\mathbb{C}))$	$(\mathrm{Sp}(n), O^*(2p))$	$n \ge p-1$
(vi)	$(O(p,\mathbb{C}), \mathrm{Sp}(2n,\mathbb{C}))$	$(O(p), \mathrm{Sp}(4n,\mathbb{R}))$	$p \ge 4n-1$
(vii)	$(U(r,s), U(p,q))$	$(U(r), U(p,q))$ $\oplus(U(s), U(p,q))$	$r, s \ge p+q-1$

Proof. For cases (i), (iii), (v) and (vi), ω is decomposed under the action of K as $\omega = \bigoplus_\sigma (\sigma \otimes E_\tau)$. Section 4 shows that $(*)$ is a necessary condition for the equality $N(\tau) = E_\tau$. Finally, the restriction from M' to G' is computed in Lemma 5.1.

For cases (ii), (iv) and (vii), ω is decomposed under the action of $K_1 \times K_2$ as $\omega = \omega_1 \otimes \omega_2^*$. The condition $(*)$ is computed for each case to get $N(\tau) = E_\tau$ in

Section 4. The two pieces ω_1 and ω_2^* are put back together through the tensor product described in Lemma 5.2. □

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