**MEAN CURVATURE FLOW IN A RIEMANNIAN MANIFOLD
ENDOWED WITH A KILLING VECTOR FIELD**

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We consider the Killing graphs over a bounded regular domain M in an integral distribution orthogonal to a Killing vector field with prescribed variable contact angle. Under some appropriate condition between the geometry of the domain and the contact angle, based on the maximum principle and the approximation method, we show that the solutions to the mean curvature flow of Killing graphs with capillarity type boundary condition converge to a translating solution.

1. Introduction

In this paper, we are interested in the study of the evolution of graphs defined over a Riemannian manifold by the nonparametric mean curvature flow, whose speed in the direction of their normal is equal to their mean curvature and with a prescribed variable contact angle at the boundary. Throughout this paper, let (N^{n+1}, g) be an $(n+1)$-dimensional Riemannian manifold endowed with a Killing vector field $\mathcal{V}$. Suppose that the distribution orthogonal to $\mathcal{V}$ is of constant rank and integrable. Given an integral leaf $\mathbb{L}^n$ of that distribution, let $M \subset \mathbb{L}^n$ be a bounded domain with boundary $\partial M \in C^3$. We suppose for simplicity that $\mathbb{L}^n$ is complete. In this case, let $\zeta : \overline{M} \times \mathbb{R} \to N$ be the flow generated by $\mathcal{V}$ with initial values in N. In geometric terms, the ambient manifold N is a warped product manifold as

$$N = \mathbb{L}^n \times_{\frac{1}{\sqrt{\gamma}}} \mathbb{R},$$

with the metric given by $g = \sigma + \frac{1}{\gamma} ds^2$, where $\gamma := 1/g(\mathcal{V}, \mathcal{V})^2$ and σ is the metric on $\mathbb{L}^n$. Since we have $\mathcal{V}(\gamma) = 0$ by the Killing equation, γ can be viewed as a function on $\mathbb{L}^n$.

The Killing graph of a differentiable function $u : \overline{M} \to \mathbb{R}$ is the hypersurface $\Sigma_u \subset N$ parametrized by the map $X(x) = \zeta(x, u(x))$, $x \in \overline{M}$. The Killing cylinder

$\mathcal{K}$ over ∂M is defined by

$$\mathcal{K} := \{\zeta(x, s) : x \in \partial M, \, s \in \mathbb{R}\}.$$

We want to study the asymptotic behavior of solutions $u(x, t)$ to the Killing graphs dominated by its mean curvature in the direction of the unit normal μ with prescribed variable contact angle along the Killing cylinder, that is,

(1-1)
$$\begin{cases} \frac{\partial X}{\partial t} = \mathcal{H}\mu & \text{in } M^n \times (0, \infty), \\ \langle \mu \circ X, \nu \circ X \rangle = -\varphi & \text{on } \partial M \times (0, \infty), \\ X(\cdot, 0) = \zeta(\cdot, u_0(\cdot)) & \text{on } M, \end{cases}$$

where $M \subset \mathbb{L}^n$, $n \geq 2$, is a compact domain with smooth boundary ∂M, μ is the upward normal for the Killing graph $\mathcal{K}$, which is given by

$$\mu := \frac{\gamma \mathcal{V} - \zeta_*(\nabla u)}{v},$$

with $v := \sqrt{e^{2\rho} + |\nabla u|^2}$ and $\rho := \log \sqrt{\gamma}$ throughout this paper for convenience. And φ is the cosine of the contact angle, that is, $\varphi := \cos \theta$ for some $\theta : \partial M \to \mathbb{R}$ being the variable contact angle on the intersection of the Killing cylinder and the graph, which is given by $\langle \mu, \nu \rangle = -\cos \theta$. Then the boundary value condition in (1-1) is equivalent to $\nabla_\nu u = \cos \theta \sqrt{\gamma + |\nabla u|^2}$, where ν is the unit inner normal of ∂M, one may extend φ to $\overline{M}$ with $\varphi \in C^2(\overline{M})$. And $u_0(x)$ is a smooth function and satisfying the compatible condition

$$\nabla_\nu u_0 = \varphi \sqrt{\gamma + |\nabla u_0|^2} \quad \text{on } \partial M,$$

while $\mathcal{H}$ in (1-1) is the Killing mean curvature operator with the expression (see [Dajczer et al. 2008]) that

$$\mathcal{H} := \mathrm{div}\left(\frac{\nabla u}{\sqrt{\gamma + |\nabla u|^2}}\right) - \left\langle \frac{\nabla \gamma}{2\gamma}, \frac{\nabla u}{v} \right\rangle,$$

where the differential operators div and ∇ are respectively the divergence and gradient in $\mathbb{L}$ with respect to the metric σ.

For the prescribed contact angle boundary value problem, a more general type problem with an extra term is to study the following equation,

(1-2) $$u_t = \sqrt{e^{2\rho} + |\nabla u|^2}(\mathcal{H} - \psi) \quad \text{in } M \times [0, T].$$

Such types of evolution problem were studied by de Lira and Wanderley [2015] previously, where they proved the long time existence for $\psi : \overline{M} \to \mathbb{R}$ and prescribed contact angle function $\varphi : \partial M \to \mathbb{R}$ with $|\varphi| \leq b_0 < 1$. Nevertheless, they obtained the convergence results only for the vertical angle case, which corresponds to $\varphi = 0$ on the boundary. One could also refer to [Huisken 1989] or [Zhou 2018] for similar results in the vertical angle case. Besides, when $\mathcal{V} := \frac{\partial}{\partial s}$, the long time existence of

flow (1-2) was obtained for product ambient manifolds by Zhou [2018] recently, which is a generalization of the previous result in Euclidean space by Guan [1996].

However, when it goes to study the asymptotic behavior of $u(x, t)$ in (1-2) for the not perpendicular contact angle case, to the author's best knowledge, only a few results are known. For instance, when $n = 2$, Altschuler and Wu [1994] showed that $u(x, t)$ of Euclidean nonparametric mean curvature flow with contact angle boundary condition will converge to the translating surface. For $n \geq 2$, recently, [Gao et al. ≥ 2020] proved the same results under the condition that the contact angle is the small perturbation of $\frac{\pi}{2}$. See also [Zhou 2018] for the same convergence conclusion under the condition that the ambient Riemannian surface requires carrying the metric with nonnegative Gauss curvature. One could also refer to [Ma et al. 2018; Oliker and Uraltseva 1993; Schnürer 2002] for various studies on the asymptotic behavior of geometric curvature flow with Dirichlet, Neumann and second boundary value condition problems.

We only focus on the capillarity type boundary value condition in this paper. Based on above discussions, we may rewrite (1-1) in the following expression,

$$(1\text{-}3) \quad \begin{cases} u_t = \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} u - \frac{1}{2}(1/\gamma + 1/v^2)\langle \nabla \gamma, \nabla u \rangle & \text{in } M \times [0, +\infty), \\ \nabla_v u = \varphi(x)\sqrt{\gamma + |\nabla u|^2} & \text{on } \partial M \times [0, +\infty), \\ u(x, 0) = u_0(x) & \text{on } \overline{M} \times \{0\}, \end{cases}$$

where

$$a^{ij} := \sigma^{ij} - \frac{\nabla^i u \cdot \nabla^j u}{\gamma + |\nabla u|^2}$$

and v is the unit inner normal of ∂M.

Our first main result about the asymptotic behavior of a solution to (1-3) can be stated as follows.

Theorem 1.1. *Let $M \subset \mathbb{L}^n$ be a bounded strictly convex domain with $\partial M \in C^3$. If $\mathrm{Ric}_{\mathbb{L}} + \nabla^2 \rho \geq k_0 \sigma$ for some positive constant k_0, where $\mathrm{Ric}_{\mathbb{L}}$ denotes the Ricci curvature tensor of $\mathbb{L}$, then there exists $\delta_0 > 0$ such that*

$$(1\text{-}4) \qquad\qquad \|\varphi\|_{C^2(\overline{M})} \leq \delta_0,$$

and it holds that the unique smooth solution $u(x, t)$ of (1-3) uniformly converges to $\hat{u}(x) + \tau t$ as $t \to \infty$, which means that

$$\lim_{t \to +\infty} \|u(\cdot, t) - (\hat{u}(\cdot) + \tau t)\|_{C^0(\overline{M})} = 0,$$

where $(\tau, \hat{u})$ is a solution satisfying

$$(1\text{-}5) \quad \begin{cases} \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} u - \frac{1}{2}(1/\gamma + 1/v^2)\langle \nabla \gamma, \nabla u \rangle = \tau & \text{in } M, \\ \nabla_v u = \varphi\sqrt{\gamma + |\nabla u|^2} & \text{on } \partial M. \end{cases}$$

In particular, if $\int_{\partial M} \varphi/\sqrt{\gamma} d\sigma = 0$, then $\tau = 0$, hence $\Sigma_{\hat{u}}$ is a minimal hypersurface in (N^{n+1}, g).

The main ingredients and approach to get such convergence results are as follows. Firstly, by using the standard comparison principle, one shows that

$$\frac{u(x, t)}{t} \to \tau \quad \text{uniformly as } t \to +\infty,$$

which indicates that the so called ergodic constant τ governs the asymptotic behavior of the evolution equation. This part is somewhat relatively easy. Secondly, by establishing the uniform gradient estimate, we get the existence of solutions to the stationary equations (1-5) and show the more precise asymptotic behavior

$$u(x, t) - \tau t \to \hat{u}(x) \quad \text{uniformly as } t \to +\infty.$$

We will achieve this by firstly deriving an a priori estimate for the C^1-norm of $u(x, t)$ to (1-3), which is time-independent. This will be achieved by choosing an appropriate auxiliary function and combining with the maximum principle.

We mainly use the methods in [Altschuler and Wu 1994; Ma and Xu 2016; Schnürer 2002], but with the necessary technical tricks for choosing the right functions to get estimates, which take control of the complications introduced by the terms containing the warped function γ and the curvature tensor of $\mathbb{L}$. We want to point out that the existence of solutions to stationary equations (1-5) are closely related to the "ergodic control problems," which consist in solving the following type of fully nonlinear elliptic equations associated with nonlinear oblique type boundary conditions:

$$\begin{cases} F(x, \nabla u, \nabla^2 u) = \mu_1 & \text{in } M, \\ L(x, \nabla u) = \mu_2 & \text{on } \partial M. \end{cases}$$

Such types of problem not only have a close relation with the asymptotic behavior of solutions to parabolic equations, which are the case we study here, but also have a strong impact and application in ergodic control problems, homogenization of elliptic and parabolic PDEs, etc. We recommend [Barles 1993; Barles and Da Lio 2005; Barles and Souganidis 2001; Ishii 2013] for more details and interesting results.

Thus, we adapted the idea used there to prove the existence of a translating solution to our flow equation. That is, we have the following existence results for stationary equations (1-5) (called the solvability problem of the translating soliton equation with capillary boundary condition in some literature).

Theorem 1.2. *Under the assumption of Theorem 1.1, there exist a unique $\tau \in \mathbb{R}$ and a solution $\hat{u} \in C^{2,\alpha}(\overline{M})$ satisfying (1-5), $0 < \alpha < 1$. In particular, the solution $\hat{u}$ is unique up to an additive constant.*

Moreover, for the dimension $n = 2$, we could release the condition on the range of the variable contact angle and the compatible relation for the Ricci curvature on leafs with the product function γ in Theorem 1.1. Indeed, we have the following theorem.

Theorem 1.3. *Let M be a strictly convex domain in $\mathbb{L}^2$ with κ the geodesic curvature of ∂M, satisfying*

$$\kappa - \left(\frac{|\nabla^T \varphi|}{\sqrt{1 - \varphi^2}} + |\varphi| \cdot |\nabla \rho| \right) \geq \delta_1, \quad on \ \partial M,$$

for some positive constant $\delta_1 > 0$, where $\nabla^T \varphi$ is the tangential part of $\nabla \varphi$ restricted on the boundary. If γ satisfies the compatible condition

$$K + \Delta \log \sqrt{\gamma} \geq 0 \quad and \quad \Delta \gamma - \frac{|\nabla \gamma|^2}{2\gamma} \geq 0, \qquad in \ M,$$

where K is the Gauss curvature of M. Then the unique smooth solution $u(x, t)$ of (1-3) *uniformly converges to $\hat{u}(x) + \tau t$ as $t \to \infty$, where $(\tau, \hat{u})$ is a solution satisfying* (1-5).

In particular, if $\int_{\partial M} \varphi / \sqrt{\gamma} d\sigma = 0$, then $\tau = 0$, hence $\Sigma_{\hat{u}}$ is a minimal surface in (N^3, g).

We want to emphasize the fact that the above conditions are very well adapted for applications to some special cases, say [Altschuler and Wu 1994] or [Zhou 2018] for example. Let us recall that for the Euclidean graph case, such results were first proved in [Altschuler and Wu 1994], later for the product Riemannian manifold case $\mathbb{L} \times \mathbb{R}$, see [Zhou 2018], which both correspond to the special Killing vector field $\mathcal{V} = \frac{\partial}{\partial s}$ in our setting.

Outline of the paper. The article is organized as follows. In Section 2, by using the maximum principle, the key uniform gradient estimate is established for flow (1-3). Theorem 1.2 is proved in Section 3, which is complied with the approach that has been used in [Altschuler and Wu 1994] or [Ma et al. 2018], once one gets the uniform gradient estimate. In Section 4, we turn to discuss the $n = 2$ case and prove the uniform gradient estimate being stated as Theorem 4.2. The last section is devoted to showing the asymptotic behavior of solutions to (1-3) and proving Theorem 1.1, which used the method in [Altschuler and Wu 1994; Schnürer 2002]. Also, the same idea can be utilized to verify Theorem 1.3, after obtaining the key uniform gradient estimate for a Riemann surface in Section 4.

2. Priori estimates

In this section, for studying the asymptotic behavior of the nonparametric mean curvature with prescribed contact angle condition, we establish the uniformly gradient estimate for (1-3) under condition (1-4).

Firstly, we describe the evolution problem in local coordinates and compute some geometric quantities induced by embedding of the hypersurface into (N, g). One could also find those results in references [Dajczer et al. 2008], [Impera et al. 2018] or [de Lira and Wanderley 2015]. Assume that $\{e_i\}_{i=1}^n$ is the local frame on $\mathbb{L}^n$, and s is the flow parameter of the Killing vector field $\mathcal{V}$. We use the notation $\sigma_{ij} = \sigma(e_i, e_j) := \langle e_i, e_j \rangle$ and $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j$. Then the tangent vector of Σ_u at the point $X(x)$ are

$$X_*(e_j) = \zeta_*(e_j) + \zeta_*\left(\frac{\partial}{\partial s}\right)\nabla_j u := e_j|_X + \partial_s|_X \nabla_j u,$$

then the induced metric $\hat{g}$ on Σ_u is given by

$$\hat{g}_{ij} := X^*g(e_i, e_j) = \sigma_{ij} + \frac{1}{\gamma}\nabla_i u \nabla_j u,$$

where we note that $\gamma := 1/g(\mathcal{V}, \mathcal{V})^2$ and σ is the metric on $\mathbb{L}$. For a differentiable function u defined on M^n. We lifted the indices with respect to the metric on $\mathbb{L}$, i.e., $\nabla^i u := \sigma^{ij}\nabla_j u$ and $|\nabla u|^2 := \sigma(\nabla u, \nabla u) = \sigma^{ij}\nabla_i u \nabla_j u$, where ∇ denotes the Riemannian connection in $\mathbb{L}$ and $\nabla u = \sigma^{ij}\nabla_i u e_j$ is the gradient relatively to $\mathbb{L}$. And $\nabla^2 u$ is the Hessian which is given by

$$\nabla_{ij} u = \nabla_i(\nabla_j u) - (\nabla_i e_j)u.$$

Recall that $\nabla_{ij} u = \nabla_{ji} u$ and by using the Ricci identities for the third covariant derivative of u, we have

$$\nabla_{kji} u = \nabla_{jik} u + \sum_{l=1}^n \nabla_l u R^l_{ijk},$$

where R is the Riemannian curvature tensor

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

with $R_{ijkl} := \sigma(R(e_i, e_j)e_l, e_k)$ and $R^i_{jkl} := \sum_{s=1}^n \sigma^{is}R_{sjkl}$.

We will use the distance function to construct the auxiliary function to get the uniform gradient estimate, this kind of idea has been widely used previously, see for instance [Guan 1996; Korevaar 1988; de Lira and Wanderley 2015; Ma and Xu 2016; Xu 2014]. We note that $d(x) := \text{dist}(x, \partial M)$ is a smooth function for x close to the boundary, then extend it to the whole manifold M, and $d = 0$, $\nabla_\nu d = 1$ on ∂M. For convenience, we denote $L_1 := \sup_{\overline{M}} |\nabla^2 d|$, $L_2 := \sup_{\overline{M}} |\nabla^3 d|$, and

define the big O notation $O(s)$, which means that there exists a constant $C > 0$, such that $|O(s)| \leq Cs$. In particular, we have the positive constant C only depending on (M, σ), γ, L_1, L_2 and n throughout this paper.

The following result has been proved previous by de Lira and Wanderley [2015, Proposition 1], which established an a priori bound for $\dot{u} := \partial_t u$ for a general domain $M \subset \mathbb{L}$ and $\varphi \in C^1$ with $\|\varphi\|_{C^0} < 1$.

Lemma 2.1. *If $u(x, t)$ is a smooth solution to (1-3), then*

$$\sup_{M \times [0,T]} \dot{u}^2 = \sup_M \dot{u}^2 \big|_{t=0},$$

that is, there exists $C = C(u_0) > 0$ such that

$$(2\text{-}1) \qquad \sup_{M \times [0,T]} |\dot{u}| \leq C.$$

Next we obtain the uniform gradient estimate for problem (1-3), which turns the quasilinear evolution equation (1-3) into a uniformly parabolic equation and the infinite time existence of smooth solutions follows by standard regularity theory.

Theorem 2.2. *Under the assumption of Theorem 1.1, there exists a positive constant C depending on n, M, u_0, φ, γ such that*

$$\sup_{M \times [0,T]} |\nabla u| \leq C.$$

Remark 2.3. The constant δ_0 in (1-4) depends only on the geometry of domain M. In fact, even for 2-dimensional Euclidean space, see [Altschuler and Wu 1994], under the condition that $M \subset \mathbb{R}^2$ is strictly convex and $\kappa - |\nabla^T \varphi| / \sqrt{1 - \varphi^2} \geq c_0 > 0$, where κ is the geodesic curvature of the curve ∂M, then they deduce that the solutions to the nonparametric mean curvature flow with capillarity type boundary condition converge to translating solution. This means that the contact angle will be affected by the geometry of domain along the flow.

Proof. Choosing the auxiliary function

$$w(x, t) := v - \sigma(\nabla u, \nabla d)\varphi,$$

where $v := \sqrt{\gamma + |\nabla u|^2}$. We want to get the uniform bound of $|\nabla u|$ in $M_{T'} := M \times [0, T']$, which is independent of T' ($0 < T' < T$).

Assume that $w(x, t)$ attains its maximum value at $(x_0, t_0) \in \overline{M}_{T'}$. We split it into three cases to discuss. And all the computation below are done at this maximum value point.

<u>Case 1</u>: $(x_0, t_0) \in \partial M \times [0, T']$. In order to do the calculation, we choose an orthonormal frame $\{e_i\}_{i=1}^n$ at x_0 such that e_n be the inner normal vector field of ∂M, which is exactly equal to ν.

First we notice that $w = v - \varphi\sigma(\nabla u, \nabla d) = v - \varphi u_n$ on the boundary ∂M. Denote $\nabla' u$ and u_n be the tangential and normal part of ∇u on the boundary by our choice of frame above. From the boundary condition $u_n = \varphi v$, we deduce that

$$u_n^2 = \varphi^2 v^2 = \varphi^2(\gamma + |\nabla' u|^2 + u_n^2),$$

so it directly follows that

(2-2)
$$u_n^2 = \frac{\varphi^2}{1 - \varphi^2}(\gamma + |\nabla' u|^2),$$

and in particular, we have

$$w = (1 - \varphi^2)v = \sqrt{(\gamma + |\nabla' u|^2) \cdot (1 - \varphi^2)}.$$

By using the Gauss–Weingarten equation, we get

$$\nabla_n v = \frac{1}{2v}\left(\nabla_n \gamma + 2\sum_{\alpha=1}^{n-1} u_\alpha \nabla_{n\alpha} u + 2u_n \nabla_{nn} u\right)$$

$$= \frac{\nabla_n \gamma}{2v} + \frac{1}{v}\sum_{\alpha=1}^{n-1} u_\alpha \cdot \left(u_{n\alpha} + \sum_{\beta=1}^{n-1} b_{\alpha\beta} u_\beta\right) + \varphi\nabla_{nn} u,$$

where $(b_{\alpha\beta})$ is the second fundamental form of ∂M with respect to the inner normal ν and satisfies $(b_{\alpha\beta}) \geq \kappa(\delta_{\alpha\beta})$ for some $\kappa > 0$, if ∂M is strictly convex. Using the Hopf lemma, it gives us that

(2-3) $0 \geq \nabla_n w(x_0, t_0) = \nabla_n v - \nabla_n(\langle \nabla u, \nabla d\rangle)\varphi$

$$= \nabla_n v - \nabla_{nn} u\varphi - \sum_{k=1}^{n} \nabla_k u \nabla_{kn} d\varphi$$

$$= \frac{\nabla_n \gamma}{2v} + \frac{1}{v}\sum_{\alpha=1}^{n-1}\left(u_\alpha u_{n\alpha} + \sum_{\beta=1}^{n-1} u_\alpha b_{\alpha\beta} u_\beta\right) - \sum_{k=1}^{n} \nabla_k u \nabla_{kn} d\varphi.$$

Since $\{e_\alpha\}$ are the tangential vector fields for all $1 \leq \alpha \leq n - 1$, we obtain

(2-4) $0 = \nabla'_\alpha w(x_0, t_0) = v_\alpha - u_{n\alpha}\varphi - \varphi\varphi_\alpha v.$

On the other hand, by taking the tangential derivative to the boundary value condition in (1-3) and combining with (2-4), it yields that

$$u_{n\alpha} = \nabla'_\alpha(\varphi v) = \varphi v_\alpha + \varphi_\alpha v = \varphi^2 u_{n\alpha} + \varphi^2 \varphi_\alpha v + \varphi_\alpha v,$$

that is,

(2-5)
$$u_{n\alpha} = \frac{1 + \varphi^2}{1 - \varphi^2}\varphi_\alpha v.$$

Substituting (2-5) into (2-3), and using the assumption in Theorem 1.1, it follows that

$$0 \geq \nabla_n w(x_0, t_0) = \frac{\nabla_n \gamma}{2v} + \frac{1}{v} \sum_{\alpha=1}^{n-1} \left(u_\alpha u_{n\alpha} + \sum_{\beta=1}^{n-1} u_\alpha b_{\alpha\beta} u_\beta \right) - \sum_{k=1}^{n} u_k \nabla_{kn} d\varphi$$

$$= \frac{\nabla_n \gamma}{2v} + \frac{1+\varphi^2}{1-\varphi^2} \sigma(\nabla'\varphi, \nabla'u) + \sum_{\alpha,\beta=1}^{n-1} \frac{u_\alpha u_\beta b_{\alpha\beta}}{v} - \varphi^2 v \nabla_{nn} d - \sum_{\alpha=1}^{n-1} u_\alpha \nabla_{\alpha n} d\varphi$$

$$\geq \frac{\kappa |\nabla'u|^2}{v} - 2L_1 |\varphi| v - \frac{2|\nabla'\varphi|}{1-\varphi^2} |\nabla'u| - \frac{C}{v}$$

$$\geq \frac{1}{v} \left(\kappa - \frac{2L_1 \delta_0}{1-\delta_0^2} - \frac{2\delta_0}{(1-\delta_0^2)^{3/2}} \right) |\nabla'u|^2 - \frac{C}{v},$$

where C is a positive constant, only depending on $\|\nabla\gamma\|_{C^0(M)}$. By choosing δ_0 with $\|\varphi\|_{C^1} \leq \delta_0$ such that

$$(2\text{-}6) \qquad 0 < \delta_0 \leq \min\left\{ \frac{\sqrt{3}}{2}, \frac{\kappa}{16L_1 + 32} \right\},$$

we obtain from above inequality that $|\nabla'u| \leq C$, which also gives us $|\nabla u| \leq C$.

Case 2: $(x_0, t_0) \in \overline{M} \times \{t = 0\}$. Then it directly yields that

$$(2\text{-}7) \qquad \begin{aligned} w(x, t) &\leq w(x_0, 0) = \sqrt{\gamma + |\nabla u_0|^2} - \sigma(\nabla u_0, \nabla d)\varphi \\ &\leq C(u_0, \varphi, d, \gamma). \end{aligned}$$

Note that $|\varphi| \leq b_0 < 1$, thus it follows from above that

$$(2\text{-}8) \qquad \sup_{M \times [0, T']} |\nabla u| \leq C(u_0, \gamma, d).$$

Case 3: $(x_0, t_0) \in M \times [0, T']$. Firstly, by choosing the local orthonormal frame $\{e_i\}_{i=1}^n$ on M such that at x_0, it holds that

$$(2\text{-}9) \qquad e_1(x_0) = \frac{\nabla u}{|\nabla u|}(x_0), \quad \text{and} \quad \{\nabla_{\alpha\beta} u\}_{2 \leq \alpha, \beta \leq n} \quad \text{is diagonal.}$$

Then it follows that,

$$(2\text{-}10) \qquad a^{ij}|_{(x_0, t_0)} = \begin{cases} \dfrac{\gamma}{\gamma + (\nabla_1 u)^2}, & \text{for } i = j = 1, \\ 1, & \text{for } i = j \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

We always assume $\nabla_1 u(x_0, t_0)$ large enough in the below computation, such that $\nabla_1 u$, $v = \sqrt{\gamma + (\nabla_1 u)^2}$, and $w = v - \nabla_1 u \cdot \nabla_1 d\varphi$ (since we assume $|\varphi| \leq b_0 < 1$) are equivalent to each other at (x_0, t_0). Otherwise, we have completed the proof.

Due to the direct computation, it gives us that

(2-11) $0 = \nabla_i w(x_0, t_0)$

$$= \nabla_i v - \sum_{k=1}^{n} \nabla_{ik} u \nabla_k d\varphi - \sum_{k=1}^{n} \nabla_k u \nabla_{ik} d\varphi - \sigma(\nabla u, \nabla d)\nabla_i \varphi$$

for all $1 \leq i \leq n$. Equivalently, we have

(2-12) $$\sum_{l=1}^{n}\left(\frac{\nabla_l u}{v} - \nabla_l d\varphi\right)\nabla_{il} u = \sum_{l=1}^{n} \nabla_l u \nabla_{il} d\varphi + \sigma(\nabla u, \nabla d)\nabla_i \varphi - \frac{\nabla_i \gamma}{2v},$$

Let $S := \nabla_1 u / v - \nabla_1 d\varphi$, then we obtain

(2-13) $$\nabla_{11} u = \sum_{\alpha=2}^{n} \frac{\nabla_\alpha d\varphi}{S}\nabla_{1\alpha} u + \frac{\nabla_{11} d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d\nabla_1 \varphi}{S}\nabla_1 u - \frac{\nabla_1 \gamma}{2vS},$$

and for $2 \leq \alpha \leq n$, it holds that

(2-14) $$\nabla_{1\alpha} u = \frac{\nabla_\alpha d\varphi}{S}\nabla_{\alpha\alpha} u + \frac{\nabla_{1\alpha} d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d\nabla_\alpha \varphi}{S}\nabla_1 u - \frac{\nabla_\alpha \gamma}{2vS}.$$

Substituting (2-14) into (2-13), we conclude that

$$\nabla_{11} u = \sum_{\alpha=2}^{n} \frac{\nabla_\alpha d\varphi}{S}\left(\frac{\nabla_\alpha d\varphi}{S}\nabla_{\alpha\alpha} u + \frac{\nabla_1 u \nabla_{1\alpha} d\varphi}{S} + \frac{\nabla_1 u \nabla_1 d\nabla_\alpha \varphi}{S} - \frac{\nabla_\alpha \gamma}{2vS}\right)$$

$$+ \frac{\nabla_{11} d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d\nabla_1 \varphi}{S}\nabla_1 u - \frac{\nabla_1 \gamma}{2vS}$$

$$= \sum_{\alpha=2}^{n}\left[\frac{(\nabla_\alpha d)^2 \varphi^2}{S^2}\nabla_{\alpha\alpha} u + \frac{\nabla_\alpha d\nabla_{1\alpha} d\varphi^2}{S^2}\nabla_1 u + \frac{\nabla_1 d\nabla_\alpha d\varphi \nabla_\alpha \varphi}{S^2}\nabla_1 u - \frac{\nabla_\alpha d\nabla_\alpha \gamma \varphi}{2vS^2}\right]$$

$$+ \frac{\nabla_{11} d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d\nabla_1 \varphi}{S}\nabla_1 u - \frac{\nabla_1 \gamma}{2vS},$$

then it follows that

(2-15) $$\nabla_{11} u = \frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2 \nabla_{\alpha\alpha} u + \left[O(|\varphi|) + O(|\nabla\varphi|)\right]\nabla_1 u.$$

For convenience, we denote

$$F(x, \nabla u) := -\tfrac{1}{2}\left(\frac{1}{\gamma} + \frac{1}{v^2}\right)\langle \nabla\gamma, \nabla u\rangle.$$

By taking the covariant derivative to the first equation in (1-3), for $1 \leq k \leq n$, we get

$$(2\text{-}16) \quad \nabla_k u_t = \sum_{i,j=1}^{n} \nabla_k a^{ij} \cdot \nabla_{ij} u + \sum_{i,j=1}^{n} a^{ij} \nabla_{kij} u + \sum_{i=1}^{n} F_{p_i} \nabla_{ki} u + F_k$$

$$= \sum_{i,j=1}^{n} a^{ij} \nabla_{kij} u + \sum_{i=1}^{n} F_{p_i} \nabla_{ik} u - \sum_{i,j=1}^{n} \frac{2 \nabla_{ij} u \nabla_{ik} u \nabla_j u}{v^2}$$

$$+ \sum_{i,j=1}^{n} \frac{\nabla_{ij} u \nabla_i u \nabla_j u}{v^4} \left(\nabla_k \gamma + \sum_{l=1}^{n} 2 \nabla_l u \nabla_{lk} u \right)$$

$$+ \frac{1}{2} \left(\frac{\nabla_k \gamma}{\gamma^2} + \frac{\nabla_k \gamma}{v^4} \right) \langle \nabla \gamma, \nabla u \rangle - \frac{1}{2} \left(\frac{1}{\gamma} + \frac{1}{v^2} \right) \sum_{i=1}^{n} \nabla_{ik} \gamma \nabla_i u,$$

where

$$F_{p_i} := -\frac{\nabla_i \gamma}{2} \left(\frac{1}{\gamma} + \frac{1}{v^2} \right) + \langle \nabla \gamma, \nabla u \rangle \frac{\nabla_i u}{v^4},$$

and

$$F_k := -\sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2\gamma} + \frac{\nabla_k \gamma}{2\gamma^2} \langle \nabla \gamma, \nabla u \rangle - \sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2v^2} + \langle \nabla \gamma, \nabla u \rangle \frac{\nabla_k \gamma}{2v^4}.$$

Therefore, by substituting above equations into (2-16) and rearranging them, we obtain

$$(2\text{-}17) \quad \nabla_k u_t - \sum_{i,j=1}^{n} a^{ij} \nabla_{kij} u - \sum_{i=1}^{n} F_{p_i} \nabla_{ki} u$$

$$= -\frac{2}{v^4} \left(\nabla_1 u \nabla_{11} u \nabla_{1k} u \gamma + \sum_{\alpha=2}^{n} \nabla_{1\alpha} u \nabla_{\alpha k} u \nabla_1 u v^2 \right) + \frac{\nabla_{11} u (\nabla_1 u)^2 \nabla_k \gamma}{v^4}$$

$$- \sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2\gamma} + \frac{\nabla_k \gamma}{2\gamma^2} \langle \nabla \gamma, \nabla u \rangle - \sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2v^2} + \langle \nabla \gamma, \nabla u \rangle \frac{\nabla_k \gamma}{2v^4}$$

On the other hand, by direct computation, we see that

$$0 \leq \left(\partial_t - \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} - \sum_{i=1}^{n} F_{p_i} \nabla_i \right) w$$

$$= \left(\partial_t v - \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} v - \sum_{i=1}^{n} F_{p_i} \nabla_i v \right)$$

$$+ \left(\sum_{i,j,k=1}^{n} a^{ij} \nabla_{jik} u \nabla_k d\varphi - \sum_{k=1}^{n} \nabla_k u_t \nabla_k d\varphi + \sum_{i,k=1}^{n} F_{p_i} \nabla_{ki} u \nabla_k d\varphi \right)$$

$$+ 2 \sum_{i,j,k=1}^{n} \left(\varphi a^{ij} \nabla_{ki} u \nabla_{kj} d + a^{ij} \nabla_{ki} u \nabla_k d \nabla_j \varphi \right)$$

$$+ \sum_{i,j,k=1}^{n} \left(2 a^{ij} \nabla_{ki} d \nabla_j \varphi \nabla_k u + \nabla_k u a^{ij} \nabla_{jik} d \varphi + a^{ij} \nabla_{ij} \varphi \nabla_k u \nabla_k d \right)$$

$$+ \sum_{i,k=1}^{n} F_{p_i} \left(\nabla_k u \nabla_{ki} d \varphi + \nabla_k u \nabla_k d \nabla_i \varphi \right)$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5.$$

Next we handle the above five terms one by one. Firstly, by using the Ricci identities for the third covariant derivative of u, it follows that

$$I_1 := \partial_t v - \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} v - \sum_{i=1}^{n} F_{p_i} \nabla_i v$$

$$= \sum_{k=1}^{n} \frac{\nabla_k u \nabla_k u_t}{v} - \frac{1}{2v} \sum_{i,j=1}^{n} \left(a^{ij} \nabla_{ij} \gamma + 2 a^{ij} \sum_{k=1}^{n} \nabla_{ki} u \nabla_{kj} u + 2 \sum_{k=1}^{n} a^{ij} \nabla_k u \nabla_{jik} u \right)$$

$$+ \frac{1}{4v^3} \sum_{i,j=1}^{n} a^{ij} \left(\nabla_i \gamma + 2 \sum_{k=1}^{n} \nabla_k u \nabla_{ki} u \right) \left(\nabla_j \gamma + 2 \sum_{l=1}^{n} \nabla_l u \nabla_{lj} u \right)$$

$$- \sum_{i=1}^{n} F_{p_i} \frac{1}{2v} \left(\nabla_i \gamma + 2 \sum_{k=1}^{n} \nabla_k u \nabla_{ki} u \right)$$

$$= \sum_{k=1}^{n} \frac{\nabla_k u}{v} \left(\nabla_k u_t - \sum_{i,j=1}^{n} a^{ij} \nabla_{kji} u - \sum_{i=1}^{n} F_{p_i} \nabla_{ki} u \right)$$

$$+ \sum_{i,j,k=1}^{n} \frac{1}{v} \left(\frac{1}{v^2} \sum_{l=1}^{n} a^{ij} \nabla_k u \nabla_l u \nabla_{ki} u \nabla_{lj} u - a^{ij} \nabla_{ki} u \nabla_{kj} u \right)$$

$$+ \frac{1}{v^3} \sum_{i,j,l=1}^{n} a^{ij} \nabla_i \gamma \nabla_l u \nabla_{lj} u$$

$$+ \left(\frac{1}{4v^3} \sum_{i,j=1}^{n} a^{ij} \nabla_i \gamma \nabla_j \gamma - \frac{1}{2v} \sum_{i=1}^{n} F_{p_i} \nabla_i \gamma - \frac{1}{2v} \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} \gamma \right)$$

$$- \sum_{i,j,k,l=1}^{n} \frac{a^{ij} \nabla_k u \nabla_l u R_{likj}}{v}$$

$$:= I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.$$

Now we switch to handle those terms one by one. By substituting (2-17) into term I_{11}, we have

$$I_{11} := \sum_{k=1}^{n} \frac{\nabla_k u}{v} \left(\nabla_k u_t - \sum_{i,j=1}^{n} a^{ij} \nabla_{kji} u - \sum_{i=1}^{n} F_{p_i} \nabla_{ki} u \right)$$

$$= \sum_{k=1}^{n} \frac{\nabla_k u}{v} \left[-\frac{2}{v^4} \left(\nabla_1 u \nabla_{11} u \nabla_{1k} u \gamma + \sum_{\alpha=2}^{n} \nabla_{1\alpha} u \nabla_{\alpha k} u \nabla_1 u v^2 \right) + \frac{\nabla_{11} u (\nabla_1 u)^2 \nabla_k \gamma}{v^4} \right.$$

$$\left. - \sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2\gamma} + \frac{\nabla_k \gamma}{2\gamma^2} \langle \nabla \gamma, \nabla u \rangle - \sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2v^2} + \langle \nabla \gamma, \nabla u \rangle \frac{\nabla_k \gamma}{2v^4} \right]$$

$$= \left[-\frac{2(\nabla_1 u)^2 \gamma}{v^5} (\nabla_{11} u)^2 - \frac{2(\nabla_1 u)^2}{v^3} \sum_{\alpha=2}^{n} (\nabla_{1\alpha} u)^2 + \frac{(\nabla_1 u)^3 \gamma_1}{v^5} \nabla_{11} u \right]$$

$$+ \left[\frac{\langle \nabla \gamma, \nabla u \rangle^2}{2\gamma^2 v} - \frac{\nabla^2 \gamma (\nabla u, \nabla u)}{2\gamma v} - \frac{\nabla^2 \gamma (\nabla u, \nabla u)}{2v^3} + \frac{(\nabla_1 u)^2 (\nabla_1 \gamma)^2}{2v^5} \right]$$

$$:= I_{111} + I_{112}.$$

We note that

$$I_{112} := \frac{\langle \nabla \gamma, \nabla u \rangle^2}{2\gamma^2 v} - \frac{\nabla^2 \gamma (\nabla u, \nabla u)}{2\gamma v} - \frac{\nabla^2 \gamma (\nabla u, \nabla u)}{2v^3} + \frac{(\nabla_1 u)^2 (\nabla_1 \gamma)^2}{2v^5}$$

$$= \frac{\langle \nabla \gamma, \nabla u \rangle^2}{2\gamma^2 v} - \frac{\nabla^2 \gamma (\nabla u, \nabla u)}{2\gamma v} + O\left(\frac{1}{v} \right).$$

Using (2-10), we have

$$I_{12} := \frac{1}{v^3} \sum_{i,j,k,l=1}^{n} a^{ij} \nabla_k u \nabla_l u \nabla_{ki} u \nabla_{lj} u - \frac{1}{v} \sum_{i,j,k=1}^{n} a^{ij} \nabla_{ki} u \nabla_{kj} u$$

$$= \frac{\gamma (\nabla_1 u)^2}{v^5} (\nabla_{11} u)^2 + \frac{(\nabla_1 u)^2}{v^3} \sum_{\alpha=2}^{n} (\nabla_{1\alpha} u)^2 - \frac{\gamma}{v^3} \sum_{k=1}^{n} (\nabla_{k1} u)^2 - \frac{1}{v} \sum_{k=1}^{n} \sum_{\alpha=2}^{n} (\nabla_{k\alpha} u)^2$$

$$= -\frac{\gamma^2}{v^5} (\nabla_{11} u)^2 - \frac{2\gamma}{v^3} \sum_{\alpha=2}^{n} (\nabla_{1\alpha} u)^2 - \frac{1}{v} \sum_{\alpha=2}^{n} (\nabla_{\alpha\alpha} u)^2,$$

and similarly, combining with (2-14) and (2-15), we obtain

$$I_{13} := -\frac{1}{v^3} \sum_{i,j,l=1}^{n} a^{ij} \nabla_i \gamma \nabla_l u \nabla_{lj} u = -\frac{\gamma \nabla_1 \gamma \nabla_1 u}{v^5} \nabla_{11} u - \sum_{\alpha=2}^{n} \frac{\nabla_\alpha \gamma \nabla_1 u}{v^3} \nabla_{1\alpha} u$$

$$= -\frac{\gamma \nabla_1 \gamma \nabla_1 u}{v^5} \left[\frac{\varphi^2}{S^2} \sum_{\alpha=2}^{n} (\nabla_\alpha d)^2 \nabla_{\alpha\alpha} u + \left(O(|\varphi|) + O(|\nabla \varphi|) \right) \nabla_1 u \right]$$

$$- \sum_{\alpha=2}^{n} \frac{\nabla_\alpha \gamma \nabla_1 u}{v^3} \left(\frac{\nabla_\alpha d \varphi}{S} \nabla_{\alpha\alpha} u + \frac{\nabla_{1\alpha} d \varphi}{S} \nabla_1 u + \frac{\nabla_1 d \nabla_\alpha \varphi}{S} \nabla_1 u - \frac{\nabla_\alpha \gamma}{2vS} \right)$$

$$= -\frac{\gamma \nabla_1 \gamma \nabla_1 u}{v^5} \frac{\varphi^2}{S^2} \sum_{\alpha=2}^{n} (\nabla_\alpha d)^2 \nabla_{\alpha\alpha} u - \frac{\nabla_1 u \varphi}{v^3 S} \sum_{\alpha=2}^{n} \nabla_\alpha \gamma \nabla_\alpha d \nabla_{\alpha\alpha} u + O\left(\frac{1}{v} \right)$$

$$= O\left(\frac{1}{v^2}\right) \sum_{\alpha=2}^{n} |\nabla_{\alpha\alpha} u| + O\left(\frac{1}{v}\right),$$

also

$$I_{14} := \frac{1}{4v^3} \sum_{i,j=1}^{n} a^{ij} \nabla_i \gamma \nabla_j \gamma - \frac{1}{2v} \sum_{i=1}^{n} F_{p_i} \nabla_i \gamma - \frac{1}{2v} \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} \gamma$$

$$= \frac{|\nabla\gamma|^2}{4v}\left(\frac{1}{\gamma} + \frac{1}{v^2}\right) - \frac{\langle \nabla\gamma, \nabla u\rangle^2}{2v^5} - \frac{\gamma \nabla_{11}\gamma}{2v^3} - \frac{\nabla_{\alpha\alpha}\gamma}{2v} + \frac{\gamma(\nabla_1\gamma)^2}{4v^5} + \frac{(\nabla_\alpha\gamma)^2}{4v^3}$$

$$= O\left(\frac{1}{v}\right).$$

And the term I_{15}, which relates to the curvature, is

$$I_{15} := - \sum_{i,j,k,l=1}^{n} \frac{a^{ij} \nabla_k u \nabla_l u R_{likj}}{v} = -\frac{1}{v}\,\mathrm{Ric}_{\mathbb{L}}(\nabla u, \nabla u).$$

Secondly, we are going to handle term I_2, by using (2-17) again, it yields that

$$I_2 := \sum_{i,j,k=1}^{n} a^{ij} \nabla_{jik} u \nabla_k d\varphi - \sum_{k=1}^{n} \nabla_k u_t \nabla_k d\varphi + \sum_{i,k=1}^{n} F_{p_i} \nabla_{ki} u \nabla_k d\varphi$$

$$= \sum_{k=1}^{n} \nabla_k d\varphi \cdot \left(\sum_{i,j=1}^{n} a^{ij} \nabla_{kji} u - \nabla_k u_t + \sum_{i=1}^{n} F_{p_i} \nabla_{ki} u + \sum_{i,j,l=1}^{n} a^{ij} \nabla_l u R^l_{ikj} \right)$$

$$= \sum_{k=1}^{n} \nabla_k d\varphi \left[-\frac{2}{v^4}\left(\nabla_1 u \nabla_{11} u \nabla_{1k} u \gamma + \sum_{\alpha=2}^{n} \nabla_{1\alpha} u \nabla_{\alpha k} u \nabla_1 u v^2 \right) \right.$$

$$+ \frac{\nabla_{11} u (\nabla_1 u)^2 \nabla_k \gamma}{v^4} - \sum_{i=1}^{n} \frac{\nabla_{ik}\gamma \nabla_i u}{2\gamma} + \frac{\nabla_k \gamma}{2\gamma^2}\langle \nabla\gamma, \nabla u\rangle$$

$$\left. - \sum_{i=1}^{n} \frac{\nabla_{ik}\gamma \nabla_i u}{2v^2} + \langle \nabla\gamma, \nabla u\rangle \frac{\nabla_k \gamma}{2v^4} \right] + \mathrm{Ric}_{\mathbb{L}}(\nabla u, \nabla d)\varphi$$

$$= \sum_{k=1}^{n} \left(\frac{2\nabla_k d\gamma \nabla_1 u}{v^4} \nabla_{11} u \nabla_{1k} u \varphi - \sum_{\alpha=2}^{n} \frac{2\nabla_k d\varphi \nabla_1 u}{v^2} \nabla_{1\alpha} u \nabla_{k\alpha} u \right.$$

$$- \frac{\nabla_k d \nabla_k \gamma (\nabla_1 u)^2}{v^4} \nabla_{11} u - \frac{\nabla_k d \nabla_{1k}\gamma \nabla_1 u}{2\gamma}\varphi + \frac{\nabla_k d \nabla_k \gamma \nabla_1 \gamma}{2\gamma^2} \nabla_1 u \varphi$$

$$\left. - \frac{\nabla_k d \nabla_{1k}\gamma \nabla_1 u \varphi}{2v^2} + \frac{\nabla_k d \nabla_k \gamma \nabla_1 \gamma \nabla_1 u}{2v^4}\varphi \right) + \mathrm{Ric}_{\mathbb{L}}(\nabla u, \nabla d)\varphi$$

$$= \left(\frac{2\nabla_1 d\gamma \nabla_1 u}{v^4}(\nabla_{11} u)^2 \varphi - \frac{2\nabla_1 d\varphi \nabla_1 u}{v^2} \sum_{\alpha=2}^{n}(\nabla_{1\alpha} u)^2 \right)$$

$$+\left(\frac{2\gamma\nabla_1 u}{v^4}\nabla_{11}u\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{1\alpha}u\varphi - \frac{2\varphi\nabla_1 u}{v^2}\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{1\alpha}u\nabla_{\alpha\alpha}u\right.$$
$$\left. - \frac{(\nabla_1 u)^2}{v^4}\nabla_{11}u\langle\nabla d,\nabla\gamma\rangle\right)$$
$$+\left[-\sum_{k=1}^{n}\frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u}{2\gamma}\varphi + \frac{\varphi\nabla_1\gamma}{2\gamma^2}\nabla_1 u\langle\nabla d,\nabla\gamma\rangle\right.$$
$$\left. - \sum_{k=1}^{n}\frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u\varphi}{2v^2} + \frac{\varphi\nabla_1\gamma\nabla_1 u}{2v^4}\langle\nabla d,\nabla\gamma\rangle + \mathrm{Ric}_{\mathbb{L}}(\nabla u,\nabla d)\varphi\right]$$
$$:= I_{21} + I_{22} + I_{23}.$$

Combining terms I_{111}, I_{12} and I_{21} together, and using (2-15), it yields that

$$I_{111} + I_{12} + I_{21}$$
$$:= -\frac{2(\nabla_1 u)^2\gamma}{v^5}(\nabla_{11}u)^2 - \frac{2(\nabla_1 u)^2}{v^3}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2 + \frac{(\nabla_1 u)^3\nabla_1\gamma}{v^5}\nabla_{11}u - \frac{\gamma^2}{v^5}(\nabla_{11}u)^2$$
$$- \frac{2\gamma}{v^3}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2 - \frac{1}{v}\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2$$
$$+ \left[\frac{2\nabla_1 d\gamma\nabla_1 u}{v^4}(\nabla_{11}u)^2\varphi - \frac{2\nabla_1 d\varphi\nabla_1 u}{v^2}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2\right]$$
$$= -\frac{2(\nabla_1 u)^2\gamma}{v^5}\left(1 + \frac{v}{\nabla_1 u}\nabla_1 d\varphi\right)(\nabla_{11}u)^2 - \frac{\gamma^2}{v^5}(\nabla_{11}u)^2 - \frac{1}{v}\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2$$
$$- \frac{2}{v}\left(1 + \frac{\nabla_1 u}{v}\nabla_1 d\varphi\right)\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2$$
$$+ \frac{(\nabla_1 u)^3\nabla_1\gamma}{v^5}\left[\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u + \left(O(|\varphi|) + O(|\nabla\varphi|)\right)\nabla_1 u\right].$$

By choosing δ_0 small enough with $\|\varphi\|_{C^0(M)} \le \delta_0$, say $\delta_0 \le \frac{1}{2}$, such that

$$1 + \frac{v}{\nabla_1 u}d_1\varphi < 0 \quad \text{and} \quad 1 + \frac{\nabla_1 u}{v}d_1\varphi < 0.$$

Then it follows that

$$I_{111} + I_{12} + I_{21}$$
$$\le -\frac{1}{v}\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2 + \frac{(\nabla_1 u)^3\nabla_1\gamma}{v^5}\left[\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u + \left(O(|\varphi|) + O(|\nabla\varphi|)\right)\nabla_1 u\right]$$

$$= -\frac{1}{v} \sum_{\alpha=2}^{n} (\nabla_{\alpha\alpha}u)^2 + O\left(\frac{1}{v^2}\right) \sum_{\alpha=2}^{n} |\nabla_{\alpha\alpha}u| + \left[O(|\varphi|) + O(|\nabla\varphi|)\right]\nabla_1 u.$$

Secondly, we are going to handle term I_{22}, by substituting (2-14) and (2-15) into term I_{22}; we obtain that

$$I_{22} := \frac{2\gamma \nabla_1 u}{v^4} \varphi \nabla_{11} u \sum_{\alpha=2}^{n} \nabla_\alpha d \nabla_{1\alpha} u - \frac{2\varphi \nabla_1 u}{v^2} \sum_{\alpha=2}^{n} \nabla_\alpha d \nabla_{1\alpha} u \nabla_{\alpha\alpha} u$$
$$- \frac{(\nabla_1 u)^2}{v^4} \nabla_{11} u \langle \nabla d, \nabla \gamma \rangle$$

$$= \frac{2\gamma \nabla_1 u}{v^4} \varphi \left[\frac{\varphi^2}{S^2} \sum_{\alpha=2}^{n} (\nabla_\alpha d)^2 \nabla_{\alpha\alpha} u + \left(O(|\varphi|) + O(|\nabla\varphi|)\right) \nabla_1 u \right]$$
$$\cdot \sum_{\alpha=2}^{n} \nabla_\alpha d \cdot \left[\frac{\nabla_\alpha d\varphi}{S} \nabla_{\alpha\alpha} u + \frac{\nabla_{1\alpha} d\varphi}{S} \nabla_1 u + \frac{\nabla_1 d \nabla_\alpha \varphi}{S} \nabla_1 u - \frac{\nabla_\alpha \gamma}{2vS} \right]$$

$$- \frac{2\varphi \nabla_1 u}{v^2} \sum_{\alpha=2}^{n} \nabla_\alpha d \cdot \left[\frac{\nabla_\alpha d\varphi}{S} \nabla_{\alpha\alpha} u + \frac{\nabla_{1\alpha} d\varphi}{S} \nabla_1 u + \frac{\nabla_1 d \nabla_\alpha \varphi}{S} \nabla_1 u - \frac{\nabla_\alpha \gamma}{2vS} \right] \cdot \nabla_{\alpha\alpha} u$$

$$- \frac{(\nabla_1 u)^2}{v^4} \cdot \left[\frac{\varphi^2}{S^2} \sum_{\alpha=2}^{n} (\nabla_\alpha d)^2 \nabla_{\alpha\alpha} u + \left(O(|\varphi|) + O(|\nabla\varphi|)\right) \nabla_1 u \right] \langle \nabla d, \nabla \gamma \rangle$$

$$= O\left(\frac{1}{v^3}\right) \sum_{\alpha=2}^{n} (\nabla_{\alpha\alpha}u)^2 + O\left(\frac{1}{v}\right) \sum_{\alpha=2}^{n} |\nabla_{\alpha\alpha}u| + O\left(\frac{1}{v}\right).$$

And we notice that for term I_{23}, we have

$$I_{23} := -\sum_{k=1}^{n} \frac{\nabla_k d \nabla_{1k} \gamma \nabla_1 u}{2\gamma} \varphi + \frac{\varphi \nabla_1 \gamma}{2\gamma^2} \nabla_1 u \langle \nabla d, \nabla \gamma \rangle - \sum_{k=1}^{n} \frac{\nabla_k d \nabla_{1k} \gamma \nabla_1 u \varphi}{2v^2}$$
$$+ \frac{\varphi \nabla_1 \gamma \nabla_1 u}{2v^4} \langle \nabla d, \nabla \gamma \rangle + \text{Ric}_{\mathbb{L}}(\nabla u, \nabla d)\varphi$$

$$= O(|\varphi|)\nabla_1 u.$$

Substituting (2-14) and (2-15) into term I_3, we deduce that

$$I_3 := \sum_{i,j,k=1}^{n} \left(2\varphi a^{ij} \nabla_{ki} u \nabla_{kj} d + 2a^{ij} \nabla_{ki} u \nabla_k d \nabla_j \varphi\right)$$

$$= 2\sum_{k=1}^{n} \left(\sum_{\alpha=2}^{n} \nabla_{k\alpha} u \nabla_{k\alpha} d\varphi + \sum_{\alpha=2}^{n} \nabla_{k\alpha} u \nabla_k d \nabla_\alpha \varphi + \frac{\nabla_{k1} u \nabla_{k1} d}{v^2} \gamma\varphi + \frac{\nabla_{k1} u \nabla_k d \nabla_1 \varphi}{v^2} \gamma\right)$$

$$= 2\left[\frac{\nabla_{11} u}{v^2} \gamma \left(\nabla_{11} d\varphi + \nabla_1 d \nabla_1 \varphi\right) + \sum_{\alpha=2}^{n} \nabla_{1\alpha} u \left(\nabla_{1\alpha} d\varphi + \nabla_1 d \nabla_\alpha \varphi\right) \right.$$

$$+ \sum_{\alpha=2}^{n} \frac{\nabla_{1\alpha} u}{v^2} \gamma \left(\nabla_{1\alpha} d\varphi + \nabla_\alpha d\nabla_1 \varphi\right) + \sum_{\alpha=2}^{n} \nabla_{\alpha\alpha} u \left(\nabla_{\alpha\alpha} d\varphi + \nabla_\alpha d\nabla_\alpha \varphi\right)\Bigg]$$

$$= \frac{2\gamma}{v^2} \Bigg[\frac{\varphi^2}{S^2} \sum_{\alpha=2}^{n} (\nabla_\alpha d)^2 \nabla_{\alpha\alpha} u + \left(O(|\varphi|) + O(|\nabla\varphi|)\right) \nabla_1 u \Bigg] \cdot \left(\nabla_{11} d\varphi + \nabla_1 d\nabla_1 \varphi\right)$$

$$+ 2 \sum_{\alpha=2}^{n} \left(\frac{\nabla_\alpha d\varphi}{S} \nabla_{\alpha\alpha} u + \frac{\nabla_{1\alpha} d\varphi}{S} \nabla_1 u + \frac{\nabla_1 d\nabla_\alpha \varphi}{S} \nabla_1 u - \frac{\nabla_\alpha \gamma}{2vS} \right) \cdot \left(\nabla_{1\alpha} d\varphi + \nabla_1 d\nabla_\alpha \varphi\right)$$

$$+ \sum_{\alpha=2}^{n} \frac{4\gamma}{v^2} \left(\frac{\nabla_\alpha d\varphi}{S} \nabla_{\alpha\alpha} u + \frac{\nabla_{1\alpha} d\varphi}{S} \nabla_1 u + \frac{\nabla_1 d\nabla_\alpha \varphi}{S} \nabla_1 u - \frac{\nabla_\alpha \gamma}{2vS} \right) \cdot \left(\nabla_{1\alpha} d\varphi + \nabla_\alpha d\nabla_1 \varphi\right)$$

$$+ 2 \sum_{\alpha=2}^{n} \nabla_{\alpha\alpha} u \left(\nabla_{\alpha\alpha} d\varphi + \nabla_\alpha d\nabla_\alpha \varphi\right)$$

$$= \frac{2\gamma}{v^2} \frac{\varphi^2}{S^2} \sum_{\alpha=2}^{n} (\nabla_\alpha d)^2 \nabla_{\alpha\alpha} u \left(\nabla_{11} d\varphi + \nabla_1 d\nabla_1 \varphi\right)$$

$$+ \frac{2\varphi}{S} \sum_{\alpha=2}^{n} \nabla_{\alpha\alpha} u \nabla_\alpha d \left(\nabla_{1\alpha} d\varphi + \nabla_1 d\nabla_\alpha \varphi\right) + \frac{4\gamma\varphi}{v^2 S} \sum_{\alpha=2}^{n} \nabla_\alpha d\nabla_{\alpha\alpha} u \left(\nabla_{1\alpha} d\varphi + \nabla_\alpha d\nabla_1 \varphi\right)$$

$$+ 2 \sum_{\alpha=2}^{n} \nabla_{\alpha\alpha} u \left(\nabla_{\alpha\alpha} d\varphi + \nabla_\alpha d\nabla_\alpha \varphi\right) + \left(O(|\varphi|) + O(|\nabla\varphi|)\right) \nabla_1 u$$

$$= O(|\varphi| + |\nabla\varphi|) \sum_{\alpha=2}^{n} |\nabla_{\alpha\alpha} u| + \left(O(|\varphi|) + O(|\nabla\varphi|)\right) \nabla_1 u.$$

Moreover, we get

$$\mathrm{I}_4 := \sum_{i,j,k=1}^{n} \left(2a^{ij} \nabla_{ki} d\nabla_j \varphi \nabla_k u + a^{ij} \nabla_k u \nabla_{jik} d\varphi + a^{ij} \nabla_{ij} \varphi \nabla_k u \nabla_k d\right)$$

$$= \sum_{\alpha=2}^{n} \left(\nabla_1 u \nabla_{\alpha\alpha} \varphi \nabla_1 d + 2\nabla_1 u \nabla_\alpha \varphi \nabla_{1\alpha} d + \nabla_1 u \nabla_{\alpha\alpha 1} d\varphi\right)$$

$$+ \gamma \left(\frac{\nabla_1 u}{v^2} \nabla_{111} d\varphi + \frac{\nabla_1 u}{v^2} \nabla_{11} \varphi \nabla_1 d + \frac{2\nabla_1 u}{v^2} \nabla_{11} d\nabla_1 \varphi \right)$$

$$= \left[O(|\varphi|) + O(|\nabla\varphi|) + O(|\nabla^2 \varphi|)\right] \nabla_1 u + O\left(\frac{1}{v}\right),$$

and

$$\mathrm{I}_5 := \sum_{i,k=1}^{n} F_{p_i} \left(\nabla_k u \nabla_{ik} d\varphi + \nabla_k u \nabla_k d\nabla_i \varphi\right)$$

$$= \nabla_1 u \cdot \sum_{i=1}^{n} \left[-\frac{\nabla_i \gamma}{2} \left(\frac{1}{\gamma} + \frac{1}{v^2}\right) + \langle \nabla\gamma, \nabla u \rangle \frac{\nabla_i u}{v^4} \right] \cdot \left(\nabla_{1i} d\varphi + \nabla_1 d\nabla_i \varphi\right)$$

$$= \left[O(|\varphi|) + O(|\nabla\varphi|)\right] \nabla_1 u + O\left(\frac{1}{v}\right).$$

By adding terms I_3, I_4, and I_5 together, we obtain

$I_3+I_4+I_5$

$$:= \frac{2\gamma}{v^2} \frac{\varphi^2}{S^2} \sum_{\alpha=2}^{n} (\nabla_\alpha d)^2 \nabla_{\alpha\alpha} u \left(\nabla_{11} d\varphi + \nabla_1 d \nabla_1 \varphi \right)$$

$$+ \frac{2\varphi}{S} \sum_{\alpha=2}^{n} \nabla_{\alpha\alpha} u \nabla_\alpha d \left(\nabla_{1\alpha} d\varphi + \nabla_1 d \nabla_\alpha \varphi \right) + \frac{4\gamma\varphi}{v^2 S} \sum_{\alpha=2}^{n} \nabla_\alpha d \nabla_{\alpha\alpha} u \left(\nabla_{1\alpha} d\varphi + \nabla_\alpha d \nabla_1 \varphi \right)$$

$$+ 2 \sum_{\alpha=2}^{n} \nabla_{\alpha\alpha} u \left(\nabla_{\alpha\alpha} d\varphi + \nabla_\alpha d \nabla_\alpha \varphi \right) + \left[O(|\varphi|) + O(|\nabla\varphi|) + O(|\nabla^2\varphi|) \right] \nabla_1 u + O\left(\frac{1}{v}\right)$$

$$= \left[O(|\varphi|) + O(|\nabla\varphi|) \right] \sum_{\alpha=2}^{n} |\nabla_{\alpha\alpha} u| + O\left(|\varphi| + |\nabla\varphi| + |\nabla^2\varphi|\right) \nabla_1 u + O\left(\frac{1}{v}\right).$$

Finally, by adding all above terms together and using the assumption in Theorem 1.1, we get

$$0 \le \left(\partial_t - \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} - \sum_{i=1}^{n} F_{p_i} \nabla_i \right) w = I_1 + I_2 + I_3 + I_4 + I_5$$

$$\le \frac{\langle \nabla\gamma, \nabla u \rangle^2}{2\gamma^2 v} - \frac{\nabla^2 \gamma (\nabla u, \nabla u)}{2\gamma v} - \frac{1}{v} \operatorname{Ric}_{\mathbb{L}}(\nabla u, \nabla u)$$

$$- \frac{1}{v} \sum_{\alpha=2}^{n} (\nabla_{\alpha\alpha} u)^2 + \left[O(|\varphi|) + O(|\nabla\varphi|) \right] \sum_{\alpha=2}^{n} |\nabla_{\alpha\alpha} u|$$

$$+ O\left(|\varphi| + |\nabla\varphi| + |\nabla^2\varphi|\right) \nabla_1 u + O\left(\frac{1}{v}\right)$$

$$\le -k_0 \frac{|\nabla u|^2}{v} + C_1 |\varphi\|_{C^2} \nabla_1 u + \frac{C_2}{v},$$

where we have used the inequality that $-\alpha s^2 + \beta s \le \beta^2/(4\alpha)$ holds for any $\alpha > 0$ in the last equality above. And C_1, C_2 are the positive constants which only depend on the L_1, L_2, n, and M.

Hence, by choosing $\delta_0 \le k_0/(2C_1+1)$ for $\|\varphi\|_{C^2} \le \delta_0 < 1$, we obtain the gradient estimate

$$v(x_0, t_0) \le C.$$

Therefore, combining all above three cases together, by choosing

$$(2\text{-}18) \qquad\qquad 0 < \delta_0 \le \min\left\{ \frac{1}{2}, \frac{\kappa}{16L_1 + 32}, \frac{k_0}{2C_1 + 1} \right\},$$

we conclude that $v(x_0, t_0) \le C$, where C is a positive constant which is independent of T. This finishes the proof of Theorem 2.2. $\qquad\qquad\qquad\qquad\qquad\square$

We conclude this section by giving some particular examples to illustrate the Ricci compatible condition in Theorem 1.1. In particular, if the induced metric on $\mathbb{L}$ is rotational invariant. That is,

$$\sigma = dr^2 + h^2(r)g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard metric on $\mathbb{S}^{n-1}$. We can write the Ricci curvature condition with respect to γ explicitly. In fact, let $\{\hat{\omega}_\alpha\}_{\alpha=2}^n$ be an orthonormal coframe on $\mathbb{S}^{n-1}$ with respect to $g_{\mathbb{S}^{n-1}}$, then we define $\omega_1 = dr$ and $\omega_\alpha = h(r)\hat{\omega}_\alpha$ for $2 \le \alpha \le n$. Then the coframe $\{\omega_i\}_{i=1}^n$ forms an orthonormal coframe of $\mathbb{L}$ with respect to σ. The Ricci curvature of σ is then given by (see for example the Appendix A in the monograph [Li 2012])

$$R_{1j} = -(n-1)\big\{(\log h(r))'' + [(\log h(r))']^2\big\}\delta_{1j},$$

and

$$\begin{aligned} R_{\alpha\beta} &= h^{-2}(r)R_{\alpha\beta}^{\mathbb{S}^{n-1}} - \big\{(\log h(r))'' + (n-1)[(\log h)']^2\big\}\delta_{\alpha\beta} \\ &= \big\{(n-1)(n-2)h^{-2}(r) - (\log h(r))'' + (n-1)[(\log h)']^2\big\}\delta_{\alpha\beta}. \end{aligned}$$

We also assume that $\gamma = \gamma(r)$, which means the norm of the Killing vector field only depends on r. Thus the ambient space metric can be written as

$$g = \varrho^2(r)ds^2 + dr^2 + h^2(r)g_{\mathbb{S}^{n-1}},$$

where $\varrho(r) := 1/\sqrt{\gamma(r)} = e^{-\rho}$. In this case, the ambient space N can be viewed as a doubly warped product manifold with the warping functions depending only on r. Therefore, the Ricci curvature compatible condition

$$\mathrm{Ric}_{\mathbb{L}} + \nabla^2\rho = \mathrm{Ric}_{\mathbb{L}} + \nabla^2\log\varrho^{-1}(r) \ge k_0\sigma,$$

is corresponding to that h and ϱ satisfy

$$-(n-1)\big\{(\log h(r))'' + [(\log h(r))']^2\big\} - \frac{\varrho''}{\varrho} + \frac{(\varrho')^2}{\varrho^2} \ge k_0,$$

and

$$(n-1)(n-2)h^{-2}(r) - (\log h(r))'' + (n-1)[(\log h)']^2 \ge k_0,$$

for any positive constant k_0. One can check directly that specific examples like

(1) $\qquad h(r) := r, \qquad \varrho(r) := e^{-(c/2)r^2} \quad$ for any $c > 0$,

(2) $\qquad h(r) := \sin r, \quad \varrho(r) := e^{-(c/2)r^2} \quad$ for any $c > 1 - n$,

(3) $\qquad h(r) := \sinh r, \quad \varrho(r) := e^{-(c/2)r^2} \quad$ for any $c > n - 1$

are included in the above conditions.

3. Existence for the approximating problems

The aim of this section is to show the existence of Equations (1-5), and then prove Theorem 1.2. One can see that if $\hat{u}$ is a solution of (1-5), then $\hat{u} + c$ is also a solution of (1-5). Hence one can not expect to obtain the C^0 estimate of $\hat{u}$. We use the approximation scheme to handle this problem. The main idea is that the limit of $\hat{u}_\varepsilon$ to the approximating problems (3-1) will solve problem (1-5). Firstly, we need to get the uniform gradient estimate of Equations (3-1), which does not depend on the C^0 norm of the solution and ε. To be precise, we get the following results.

Lemma 3.1. *Under the assumption of Theorem 1.1, if u solves the equations*

$$
(3\text{-}1) \quad \begin{cases} \sum_{i,j=1}^n a^{ij}\nabla_{ij}u - \frac{1}{2}(1/\gamma + 1/v^2)\langle\nabla\gamma,\nabla u\rangle = \varepsilon u & \text{in } M, \\ \nabla_\nu u = \varphi\sqrt{\gamma + |\nabla u|^2} & \text{on } \partial M. \end{cases}
$$

Then we have

$$
\sup_{\overline{M}} |\nabla u| \leq C,
$$

where C is a positive constant depending only on n, M, and φ, but not on ε and $\|u\|_{C^0}$.

Proof. As before, we use the same auxiliary function

$$
w(x) := v - \sigma(\nabla u, \nabla d)\varphi,
$$

where $v := \sqrt{\gamma + |\nabla u|^2}$. We want to get the uniform bound of $|\nabla u|$ in $\overline{M}$, which is independent of ε and $\|u\|_{C^0}$.

Assume $w(x)$ attains its maximum value at $x_0 \in \overline{M}$. We split it into two cases to discuss and finish the proof.

Case 1: If $x_0 \in \partial M$. This case is the same as in Case 1 in Theorem 2.2, since we retain the same boundary value condition. By choosing δ_0 as in (2-6), we obtain the estimate for $|\nabla u|$.

Case 2: If $x_0 \in M$. As the idea is same as in Case 3 in Theorem 2.2, we mainly focus only on the difference when we replace u_t there with εu here. Firstly, we have

$$
\varepsilon\nabla_k u = \sum_{i,j=1}^n a^{ij}\nabla_{kij}u + \sum_{i=1}^n F_{p_i}\nabla_{ki}u - \sum_{i,j=1}^n \frac{2\nabla_{ij}u\nabla_{ik}u\nabla_j u}{v^2}
$$

$$
+ \sum_{i,j=1}^n \frac{\nabla_{ij}u\nabla_i u\nabla_j u}{v^4} \cdot \left(\nabla_k\gamma + 2\sum_{l=1}^n \nabla_l u\nabla_{kl}u\right)
$$

$$
+ \frac{1}{2}\left(\frac{\nabla_k\gamma}{\gamma^2} + \frac{\nabla_k\gamma}{v^4}\right)\langle\nabla\gamma,\nabla u\rangle - \frac{1}{2}\left(\frac{1}{\gamma} + \frac{1}{v^2}\right)\sum_{i=1}^n \nabla_{ik}\gamma\nabla_i u.
$$

By choosing the local orthonormal frame $\{e_i\}_{i=1}^n$ on M such that at x_0, it holds that

$$(3\text{-}2) \qquad e_1(x_0) = \frac{\nabla u}{|\nabla u|}(x_0), \quad \text{and} \quad \{\nabla_{\alpha\beta}u\}_{2\le\alpha,\beta\le n} \quad \text{is diagonal.}$$

Then it follows from the above equation that at x_0,

$$(3\text{-}3) \quad -\left(\sum_{i,j=1}^n a^{ij}\nabla_{kji}u + \sum_{i=1}^n F_{p_i}\nabla_{ki}u\right)$$

$$= -\frac{2}{v^4}\left(\nabla_1 u \nabla_{11}u\nabla_{1k}u\gamma + \sum_{\alpha=2}^n \nabla_{1\alpha}u\nabla_{\alpha k}u\nabla_1 uv^2\right) + \frac{\nabla_{11}u(\nabla_1 u)^2\nabla_k\gamma}{v^4}$$

$$-\sum_{i=1}^n \frac{\nabla_{ik}\gamma\nabla_i u}{2\gamma} + \frac{\nabla_k\gamma}{2\gamma^2}\langle\nabla\gamma,\nabla u\rangle - \sum_{i=1}^n \frac{\nabla_{ik}\gamma\nabla_i u}{2v^2} + \langle\nabla\gamma,\nabla u\rangle\frac{\nabla_k\gamma}{2v^4} - \varepsilon\nabla_k u.$$

On the other hand, we have

$$0 \le -\left(\sum_{i,j=1}^n a^{ij}\nabla_{ij} + \sum_{i=1}^n F_{p_i}\nabla_i\right)w(x_0)$$

$$= -\left(\sum_{i,j=1}^n a^{ij}\nabla_{ij}v + \sum_{i=1}^n F_{p_i}\nabla_i v\right) + \varphi\sum_{k=1}^n \nabla_k d\cdot\left(\sum_{i,j=1}^n a^{ij}\nabla_{jik}u + \sum_{i=1}^n F_{p_i}\nabla_{ik}u\right)$$

$$+ \sum_{i,j,k=1}^n \left(2\varphi a^{ij}\nabla_{ki}u\nabla_{kj}d + 2a^{ij}\nabla_{ki}u\nabla_k d\nabla_j\varphi\right)$$

$$+ \sum_{i,j,k=1}^n \left(2a^{ij}\nabla_{ki}d\nabla_j\varphi\nabla_k u + \nabla_k ua^{ij}\nabla_{jik}d\varphi + a^{ij}\nabla_{ij}\varphi\nabla_k u\nabla_k d\right)$$

$$+ \sum_{i,k=1}^n F_{p_i}\left(\nabla_k u\nabla_{ki}d\varphi + \nabla_k u\nabla_k d\nabla_i\varphi\right)$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5.$$

From direct computation as before, we have

$$I_1 := -\left(\sum_{i,j=1}^n a^{ij}\nabla_{ij}v + \sum_{i=1}^n F_{p_i}\nabla_i v\right)$$

$$= -\frac{1}{2v}\sum_{i,j=1}^n \left(a^{ij}\nabla_{ij}\gamma + 2a^{ij}\sum_{k=1}^n \nabla_{ki}u\nabla_{kj}u + 2\sum_{k=1}^n a^{ij}\nabla_k u\nabla_{jik}u\right)$$

$$+ \frac{1}{4v^3}\sum_{i,j=1}^n a^{ij}\left(\nabla_i\gamma + 2\sum_{k=1}^n \nabla_k u\nabla_{ki}u\right)\left(\nabla_j\gamma + 2\sum_{l=1}^n \nabla_l u\nabla_{lj}u\right)$$

$$-\sum_{i=1}^{n} F_{p_i} \frac{1}{2v}\left(\nabla_i \gamma + 2\sum_{k=1}^{n} \nabla_k u \nabla_{ki} u\right)$$

$$= -\sum_{k=1}^{n} \frac{\nabla_k u}{v} \cdot \left(\sum_{i,j=1}^{n} a^{ij}\nabla_{kji}u + \sum_{i=1}^{n} F_{p_i}\nabla_{ik}u\right)$$

$$+ \sum_{i,j,k=1}^{n} \frac{1}{v}\left(\frac{1}{v^2}\sum_{l=1}^{n} a^{ij}\nabla_k u \nabla_l u \nabla_{ki} u \nabla_{lj} u - a^{ij}\nabla_{ki}u\nabla_{kj}u\right)$$

$$+ \frac{1}{v^3}\sum_{i,j,l=1}^{n} a^{ij}\nabla_i\gamma\nabla_l u\nabla_{lj}u$$

$$+ \left(\frac{1}{4v^3}\sum_{i,j=1}^{n} a^{ij}\nabla_i\gamma\nabla_j\gamma - \frac{1}{2v}\sum_{i=1}^{n} F_{p_i}\nabla_i\gamma - \frac{1}{2v}\sum_{i,j=1}^{n} a^{ij}\nabla_{ij}\gamma\right)$$

$$- \sum_{i,j,k,l=1}^{n} \frac{a^{ij}}{v}\nabla_k u \nabla_l u R_{likj}$$

$$:= I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.$$

Therefore, by substituting (3-3) into term I_{11}, we get

$$I_{11} := -\sum_{k=1}^{n} \frac{\nabla_k u}{v}\left(\sum_{i,j=1}^{n} a^{ij}\nabla_{kji}u + \sum_{i=1}^{n} F_{p_i}\nabla_{ik}u\right)$$

$$= \sum_{k=1}^{n} \frac{\nabla_k u}{v}\left[-\frac{2}{v^4}\left(\nabla_1 u\nabla_{11}u\nabla_{1k}u\gamma + \sum_{\alpha=2}^{n}\nabla_{1\alpha}u\nabla_{\alpha k}u\nabla_1 u v^2\right) + \frac{\nabla_{11}u(\nabla_1 u)^2\nabla_k\gamma}{v^4}\right.$$

$$\left. - \sum_{i=1}^{n}\frac{\nabla_{ik}\gamma\nabla_i u}{2\gamma} + \frac{\nabla_k\gamma}{2\gamma^2}\langle\nabla\gamma, \nabla u\rangle - \sum_{i=1}^{n}\frac{\nabla_{ik}\gamma\nabla_i u}{2v^2} + \langle\nabla\gamma, \nabla u\rangle\frac{\nabla_k\gamma}{2v^4}\right] - \varepsilon\frac{|\nabla u|^2}{v}$$

$$= \left[-\frac{2(\nabla_1 u)^2\gamma}{v^5}(\nabla_{11}u)^2 - \frac{2(\nabla_1 u)^2}{v^3}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2 + \frac{(\nabla_1 u)^3\nabla_1\gamma}{v^5}\nabla_{11}u\right]$$

$$+ \left[\frac{\langle\nabla\gamma, \nabla u\rangle^2}{2\gamma^2 v} - \frac{\nabla^2\gamma(\nabla u, \nabla u)}{2\gamma v} - \frac{\nabla^2\gamma(\nabla u, \nabla u)}{2v^3} + \frac{(\nabla_1 u)^2(\nabla_1\gamma)^2}{2v^5}\right] - \varepsilon\frac{|\nabla u|^2}{v}$$

$$:= I_{111} + I_{112} + I_{113}.$$

While for term I_2, by using (3-3) again, we obtain

$$I_2 := \sum_{k=1}^{n} \nabla_k d\varphi \cdot \left(\sum_{i,j=1}^{n} a^{ij}\nabla_{jik}u + \sum_{i=1}^{n} F_{p_i}\nabla_{ik}u\right)$$

$$= \sum_{k=1}^{n} \nabla_k d\varphi \cdot \left(\sum_{i,j=1}^{n} a^{ij}\nabla_{kji}u + \sum_{i=1}^{n} F_{p_i}\nabla_{ik}u + \sum_{i,j,l=1}^{n} a^{ij}\nabla_l u R_{ikj}^l\right)$$

$$= \left(\frac{2\gamma \nabla_1 d \nabla_1 u}{v^4} (\nabla_{11} u)^2 \varphi - \frac{2\varphi \nabla_1 d \nabla_1 u}{v^2} \sum_{\alpha=2}^{n} (\nabla_{1\alpha} u)^2 \right)$$

$$+ \left(\frac{2\gamma \nabla_1 u}{v^4} \nabla_{11} u \sum_{\alpha=2}^{n} \nabla_\alpha d \nabla_{1\alpha} u \varphi - \frac{2\varphi \nabla_1 u}{v^2} \sum_{\alpha=2}^{n} \nabla_\alpha d \nabla_{1\alpha} u \nabla_{\alpha\alpha} u \right.$$

$$\left. - \frac{(\nabla_1 u)^2}{v^4} \nabla_{11} u \langle \nabla d, \nabla \gamma \rangle \right)$$

$$+ \left[- \sum_{k=1}^{n} \frac{\nabla_k d \nabla_{1k} \gamma \nabla_1 u}{2\gamma} \varphi + \frac{\varphi \nabla_1 \gamma}{2\gamma^2} \nabla_1 u \langle \nabla d, \nabla \gamma \rangle - \sum_{k=1}^{n} \frac{\nabla_k d \nabla_{1k} \gamma \nabla_1 u \varphi}{2v^2} \right.$$

$$\left. + \frac{\varphi \nabla_1 \gamma \nabla_1 u}{2v^4} \langle \nabla d, \nabla \gamma \rangle + \mathrm{Ric}_{\mathbb{L}} (\nabla u, \nabla d) \varphi \right] + \varepsilon \langle \nabla u, \nabla d \rangle \varphi$$

$$:= I_{21} + I_{22} + I_{23} + I_{24}.$$

We assume that $\nabla_1 u(x_0) \geq \sup_M \sqrt{\gamma}$, otherwise we have completed the proof. Due to a simple observation about the extra terms I_{113} and I_{24}, we find that

$$I_{113} + I_{24} := -\varepsilon \frac{|\nabla u|^2}{v} + \varepsilon \langle \nabla u, \nabla d \rangle \varphi$$

$$= -\varepsilon u_1 \left(\frac{u_1}{v} - d_1 \varphi \right) \leq 0,$$

where the last inequality follows from taking $\delta_0 \leq 1/\sqrt{2}$ with $\|\varphi\|_{C^0(M)} \leq \delta_0$. While the rest all terms can be handled as same as in Case 3 in the proof of Theorem 2.2.

Hence, eventually, by choosing

$$(3\text{-}4) \qquad 0 < \delta_0 \leq \min \left\{ \frac{1}{2}, \frac{\kappa}{16L_1 + 32}, \frac{k_0}{2C_1 + 1} \right\},$$

we conclude that $v(x_0) \leq C$, where C is a positive constant which is independent of ε and $\|u\|_{C^0(M)}$. So we have finished the proof of the Lemma 3.1. $\qquad \square$

Now we are going to give the proof of Theorem 1.2.

Proof of Theorem 1.2. First of all, we show the existence of solution u_ε to problem (3-1) for any fixed $\varepsilon \in (0, 1)$. Let $\phi \in C^2(\overline{M})$ be the smooth function satisfying

$$\nabla_\nu \phi \leq \varphi \sqrt{\gamma + |\nabla \phi|^2} \quad \text{on } \partial M,$$

and $\phi \in C^2(\overline{M})$. In fact, the existence of function ϕ can be constructed as follows. Define $d(x) := \mathrm{dist}_g(x, \partial M)$ for x in the near neighborhood of ∂M, afterward, smoothly extends it to $\overline{M}$, which we still denote as $d(x)$. Let α be a positive constant such that $\alpha \leq \inf_M(\varphi) \sqrt{\inf_M \gamma + \alpha^2}$. Then $\phi := \alpha d(x)$ would satisfy our requirement. Assume $\phi - u_\varepsilon$ attains its minimum value at $x_0 \in \overline{M}$. If $x_0 \in \partial M$, we get $\nabla'(\phi - u_\varepsilon)(x_0) = 0$ and $\nabla_\nu(\phi - u_\varepsilon)(x_0) > 0$, that is, $\nabla' u_\varepsilon(x_0) = \nabla' \phi(x_0) := q$ and $\nabla_\nu u_\varepsilon(x_0) < \nabla_\nu \phi(x_0)$, where we denote ∇' and ν as the tangential and normal

part of ∇ on boundary ∂M. On the other hand, from the boundary value condition in (3-1), it follows that

$$\frac{\nabla_\nu u_\varepsilon}{\sqrt{\gamma + q^2 + |\nabla_\nu u_\varepsilon|^2}} = \varphi(x_0) \geq \frac{\nabla_\nu \phi}{\sqrt{\gamma + q^2 + |\nabla_\nu \phi|^2}},$$

which is a contradiction with the fact that function $s/\sqrt{\gamma(x_0) + q^2 + s^2}$ is strictly increasing with respect to $s \in \mathbb{R}$ and $\nabla_\nu u_\varepsilon(x_0) < \nabla_\nu \phi(x_0)$. Hence $x_0 \in M$, and it follows that $\nabla(\phi - u_\varepsilon)(x_0) = 0$ and $\nabla^2(\phi - u_\varepsilon)(x_0) \geq 0$. Thus at $x = x_0$, we get

$$\begin{aligned}
C &\geq \sum_{i,j=1}^n a^{ij}(x_0, \nabla\phi)\nabla_{ij}\phi - \tfrac{1}{2}\Big(\frac{1}{\gamma} + \frac{1}{\gamma + |\nabla\phi|^2}\Big)\langle\nabla\gamma, \nabla\phi\rangle \\
&\geq \sum_{i,j=1}^n a^{ij}(x_0, \nabla u_\varepsilon)\nabla_{ij}u_\varepsilon - \tfrac{1}{2}\Big(\frac{1}{\gamma} + \frac{1}{\gamma + |\nabla u_\varepsilon|^2}\Big)\langle\nabla\gamma, \nabla u_\varepsilon\rangle \\
&= \varepsilon u_\varepsilon(x_0).
\end{aligned}$$

It yields that, for all $x \in \overline{M}$,

$$\varepsilon u_\varepsilon(x) \leq \varepsilon\phi(x) - \varepsilon\big(\phi(x_0) - u_\varepsilon(x_0)\big) \leq C.$$

Similarly, we can also get the lower bound of $\varepsilon u_\varepsilon$, that is $\varepsilon u_\varepsilon(x) \geq -C$ for all $x \in \overline{M}$. Therefore, we obtain

$$\sup_M |\varepsilon u_\varepsilon| \leq C,$$

where C is a positive constant depending only on n, γ and ϕ. Hence the existence of solutions u_ε to problem (3-1) follows from the Schauder estimates and the continuity method, see [Gilbarg and Trudinger 1977] or [de Lira and Wanderley 2014] for example.

Second of all, we show the existence of the limit solution when $\varepsilon \to 0^+$. To begin with, we introduce the normalized function

$$\hat{u}_\varepsilon := u_\varepsilon - \frac{1}{|M|}\int_M u_\varepsilon\, dV.$$

It is easy to check that $\hat{u}_\varepsilon$ satisfies

$$(3\text{-}5) \quad \begin{cases} \sum_{i,j=1}^n a^{ij}(x, \nabla\hat{u}_\varepsilon)\nabla_{ij}\hat{u}_\varepsilon - \tfrac{1}{2}\big(\tfrac{1}{\gamma} + 1/(\gamma + |\nabla\hat{u}_\varepsilon|^2)\big)\langle\nabla\gamma, \nabla\hat{u}_\varepsilon\rangle \\ \qquad\qquad\qquad\qquad = \varepsilon\hat{u}_\varepsilon + \frac{1}{|M|}\int_M \varepsilon u_\varepsilon\, dV & \text{in } M, \\ \nabla_\nu \hat{u}_\varepsilon = \varphi\sqrt{\gamma + |\nabla\hat{u}_\varepsilon|^2} & \text{on } \partial M. \end{cases}$$

Since there exists some $x_0 \in \overline{M}$ such that $\hat{u}_\varepsilon(x_0) = 0$, by combining this with the gradient estimate from Lemma 3.1,

$$|\nabla\hat{u}_\varepsilon| = |\nabla u_\varepsilon| \leq C.$$

Thus we know $\sup_M |\hat{u}_\varepsilon| \leq C$ and note that $\left|\frac{1}{|M|}\int_M \varepsilon u_\varepsilon \, dV\right| \leq C$. Using the Schauder theory, we obtain that for some $\alpha \in (0, 1)$ such that

$$\|\hat{u}_\varepsilon\|_{C^{2,\alpha}(\overline{M})} \leq C,$$

where C is a positive constant, independent of ε. By taking $\varepsilon \to 0$, we know that $\hat{u}_\varepsilon$ converges to some $\hat{u} \in C^2(\overline{M})$ and $\varepsilon\hat{u}_\varepsilon + \frac{1}{|M|}\int_M \varepsilon\hat{u}_\varepsilon \, dV \to \tau$ for some $\tau \in [-2C, 2C]$, which yields that $(\tau, \hat{u})$ solves (1-5).

Lastly, we show the uniqueness in Theorem 1.2. Assume that $(\tau_i, \hat{u}_i)$ for $i = 1, 2$ are solutions to (1-5). Without loss of generality, we assume that $\tau_2 \leq \tau_1$. Then it follows that

$$\mathcal{L}(\hat{u}_1 - \hat{u}_2) := \tau_1 - \tau_2 \geq 0 \quad \text{in } M,$$

where $\mathcal{L}$ denotes the elliptic operator as

$$\mathcal{L}h := \sum_{i,j=1}^{n} A^{ij}\nabla_{ij}h + \sum_{i=1}^{n} b^i \cdot \nabla_i h,$$

with

$$A^{ij}(\nabla\hat{u}_1, \nabla\hat{u}_2) := \int_0^1 a^{ij}\big(x, s\nabla\hat{u}_1 + (1-s)\nabla\hat{u}_2\big)\, ds,$$

and

$$
\begin{aligned}
b^i(\nabla\hat{u}_1, \nabla\hat{u}_2) :=\ & -\frac{\nabla_i\gamma}{2\gamma} - \int_0^1 \frac{\nabla_i\gamma}{2\big(\gamma + |s\nabla\hat{u}_1 + (1-s)\nabla\hat{u}_2|^2\big)}\, ds \\
& + \int_0^1 \frac{\langle\nabla\gamma, s\nabla\hat{u}_1 + (1-s)\nabla\hat{u}_2\rangle}{\big(\gamma + |s\hat{u}_1 + (1-s)\nabla\hat{u}_2|^2\big)^2} \cdot \big[(1-s)\nabla_i\hat{u}_2 + s\nabla\hat{u}_1\big]\, ds \\
& + \sum_{k,l=1}^{n} \int_0^1 a_{,p_i}^{kl}\big(x, s\nabla\hat{u}_1 + (1-s)\nabla\hat{u}_2\big) \cdot \nabla_{kl}\big(s\hat{u}_1 + (1-s)\hat{u}_2\big)\, ds.
\end{aligned}
$$

Hence, it follows that $\hat{u}_1 - \hat{u}_2$ attains the maximum value at ∂M, say x_0. we get $\nabla'(\hat{u}_1 - \hat{u}_2)(x_0) = 0$ and $\nabla_\nu(\hat{u}_1 - \hat{u}_2)(x_0) < 0$, that is, $\nabla'\hat{u}_1(x_0) = \nabla'\hat{u}_2(x_0) := q$ and $\nabla_\nu\hat{u}_1(x_0) < \nabla_\nu\hat{u}_2(x_0)$, where we denote ∇' and ν as the tangential and normal part of ∇ on boundary ∂M. On the other hand, from the boundary value condition in (1-5), it yields that

$$\frac{\nabla_\nu\hat{u}_1}{\sqrt{\gamma + q^2 + |\nabla_\nu\hat{u}_1|^2}} = \varphi(x_0) = \frac{\nabla_\nu\hat{u}_2}{\sqrt{\gamma + q^2 + |\nabla_\nu\hat{u}_2|^2}},$$

which is a contradiction with the fact that function $s/\sqrt{\gamma(x_0) + q^2 + s^2}$ is strictly increasing with respect to $s \in \mathbb{R}$ and $\nabla_\nu\hat{u}_1(x_0) < \nabla_\nu\hat{u}_2(x_0)$. Therefore, we get $\hat{u}_1 - \hat{u}_2 = \text{const}$. Combining this with the first equation in (1-5), it gives us that $\tau_1 = \tau_2$. Hence we have completed the proof. $\qquad\square$

4. Translating surfaces

In this section, we switch to the dimension $n = 2$ case, and we could release the range of the variable contact angle. To be more precise, assume that (M, σ) is a bounded domain with smooth boundary in $\mathbb{L}^2$, we denote $f := u|_{\partial M}$, $\Gamma = \gamma|_{\partial M}$, $\Phi := \varphi|_{\partial M}$, and $\chi := \nabla_\nu u|_{\partial M}$. We use the arclength s to parametrize the boundary ∂M, thus $\left\{ e_1 := \frac{\partial}{\partial s}, e_2 := \nu \right\}$ forms an orthonormal frame near the boundary ∂M. Then

$$v(s) := v|_{\partial M} = \sqrt{\Gamma(s) + f'(s)^2 + \chi^2(s)},$$

and combining with boundary value condition $\chi = \Phi v$ on ∂M, it follows that

$$(4\text{-}1) \qquad \chi^2 = \frac{\Phi^2}{1 - \Phi^2}(\Gamma + f'(s)^2),$$

$$(4\text{-}2) \qquad f'(s)^2 = (1 - \Phi^2)v^2 - \Gamma.$$

In particular, on ∂M, we have the following identities.

$$a^{11} = 1 - \frac{f'(s)^2}{v^2} = \frac{\Gamma + \chi^2(s)}{v^2},$$

$$a^{12} = -\frac{f'(s)\chi}{v^2} = -\frac{f'(s)\Phi}{v} = a^{21},$$

$$a^{22} = 1 - \frac{\chi(s)^2}{v^2} = \frac{\Gamma + f'(s)^2}{v^2}.$$

Now we are going to show the gradient estimate, which will be divided into two parts. Firstly, we get the boundary gradient estimate under the appropriate condition of geodesic curvature of ∂M, and secondly, we adopt the maximum principle to get the global gradient estimate.

Lemma 4.1. *Let M be a strictly convex domain in $\mathbb{L}^2$ with κ the geodesic curvature of ∂M, satisfying*

$$\kappa \geq \left(\frac{|\nabla^T \varphi|}{\sqrt{1 - \varphi^2}} + |\varphi| \cdot |\nabla \log \sqrt{\gamma}| \right) + \delta_1,$$

for some positive constant $\delta_1 > 0$, where $\nabla^T \varphi$ is the tangential part of $\nabla \varphi$ restricted to the boundary. Suppose $u \in C^4(\overline{M})$ such that

$$(4\text{-}3) \qquad \begin{cases} C_0 = \sum_{i,j=1}^2 a^{ij} \nabla_{ij} u - \frac{1}{2}\left(\frac{1}{\gamma} + 1/(\gamma + |\nabla u|^2)\right)\langle \nabla \gamma, \nabla u \rangle & \text{in } M, \\ \nabla_\nu u = \varphi(x)\sqrt{\gamma + |\nabla u|^2} & \text{on } \partial M. \end{cases}$$

If v attains its maximum at somewhere on the boundary, then we have

$$\sup_{\partial M} v \leq C,$$

where C is a constant only depending on φ, M, γ, n and δ_1.

Proof. Firstly, we notice that

$$\nabla^2 u\left(\frac{\partial}{\partial s}, v\right) = \langle \nabla_{\partial/\partial s} \nabla u, v \rangle = \frac{\partial}{\partial s} \langle \nabla u, v \rangle - \langle \nabla u, \nabla_{\partial/\partial s} v \rangle$$

$$= \chi'(s) - f'(s)\left\langle \frac{\partial}{\partial s}, \nabla_{\partial/\partial s} v \right\rangle$$

$$= \Phi'(s)v + \Phi v'(s) + f'(s)\kappa,$$

where the last equality follows from combining with the boundary value condition $\chi = \Phi v$. Hence by direct computation, it yields that

$$(4\text{-}4) \qquad \frac{\partial v}{\partial v} = \frac{1}{2v}\left(\frac{\partial \gamma}{\partial v} + 2\nabla^2 u(\nabla u, v)\right)$$

$$= \frac{1}{2v}\left(\frac{\partial \gamma}{\partial v} + 2\nabla^2 u\left(\frac{\partial}{\partial s}, v\right)f'(s) + 2\nabla^2 u(v, v)\chi\right)$$

$$= \frac{\chi}{v}\nabla^2 u(v, v) + \frac{1}{2v}\frac{\partial \gamma}{\partial v} + \frac{f'(s)}{v}\left(\Phi'(s)v + \Phi v'(s) + f'(s)\kappa\right).$$

Similarly, it holds that

$$(4\text{-}5) \qquad \nabla^2 u\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = \left\langle \nabla_{\partial/\partial s} \nabla u, \frac{\partial}{\partial s} \right\rangle = \frac{\partial}{\partial s}\left\langle \nabla u, \frac{\partial}{\partial s} \right\rangle - \left\langle \nabla u, \nabla_{\partial/\partial s} \frac{\partial}{\partial s} \right\rangle$$

$$= f''(s) - \chi(s)\kappa.$$

Hence, from the first equation in (4-3), which follows that on ∂M,

$$(4\text{-}6) \quad C_0 = \sum_{i,j=1}^{2} a^{ij}\nabla_{ij}u - \tfrac{1}{2}\left(\frac{1}{\Gamma} + \frac{1}{v^2}\right)\langle \nabla \gamma, \nabla u \rangle$$

$$= \frac{\Gamma + \chi^2(s)}{v^2} \cdot \left(f''(s) - \chi(s)\kappa\right) - 2\frac{f'(s)\Phi}{v}\left(\chi'(s) + f'(s)\kappa\right)$$

$$+ \frac{\Gamma + f'(s)^2}{v^2}\nabla^2 u(v, v) - \tfrac{1}{2}\left(\frac{1}{\Gamma} + \frac{1}{\Gamma + f'(s)^2 + \chi^2(s)}\right) \cdot f'(s)\Gamma'(s)$$

$$- \tfrac{1}{2}\left(\frac{1}{\Gamma} + \frac{1}{\Gamma + f'(s)^2 + \chi^2(s)}\right)\chi\frac{\partial \gamma}{\partial v}.$$

Since v attains its maximum value somewhere on the boundary, say $x_0 \in \partial M$ whose local parameter is s_0. We may assume that $|f'(s_0)| \geq 1$ in the below, otherwise we have done the gradient estimate for v, due to Equation (4-2). In the sequel, we do all the computation at $s = s_0$. Now we have

$$(4\text{-}7) \qquad\qquad v'(s_0) = 0, \quad \text{and} \quad \frac{\partial v}{\partial v}(s_0) \leq 0,$$

that is,

$$(4\text{-}8) \qquad\qquad \Gamma' + 2f'f'' + 2\chi\chi' = 0,$$

and respectively,

$$(4\text{-}9) \qquad 0 \geq \frac{\partial v}{\partial \nu} = \frac{\chi}{v} \nabla^2 u(v, v) + \frac{1}{2v} \frac{\partial \gamma}{\partial \nu} + f' \Phi' + \frac{f'(s)^2}{v} \kappa.$$

Note that from boundary value equation in (4-3), we get

$$(4\text{-}10) \qquad \chi'(s_0) = \Phi' v + \Phi v' = \Phi' v.$$

Substituting above equation into (4-8) gives us

$$(4\text{-}11) \quad f''(s_0) = -\frac{1}{2f'(s)} \left[2\Phi\Phi' v^2 - (1 - \Phi^2) 2vv' + \Gamma' \right] = -\frac{\Phi\Phi'}{f'} v^2 - \frac{\Gamma'}{2f'}.$$

We rewrite (4-9) in the following expression,

$$(4\text{-}12) \qquad \chi \cdot \nabla^2 u(v, v) \leq -f'(s)^2 \kappa - f' \Phi' v - \frac{1}{2} \frac{\partial \gamma}{\partial \nu}.$$

By multiplying χ into (4-6) first, then substituting (4-10), (4-11) and (4-12) into it, we obtain

$$(4\text{-}13) \quad \chi \cdot C_0 = \chi \frac{\Gamma + \chi^2}{v^2} \cdot \left(-\frac{\Phi\Phi'}{f'} v^2 - \frac{\Gamma'}{2f'} - \chi\kappa \right) - 2 \frac{\chi(s) f'(s) \Phi}{v} \left(\Phi' v + f'(s)\kappa \right)$$

$$+ \frac{\Gamma + f'(s)^2}{v^2} \chi \cdot \nabla^2 u(v, v) - \frac{\chi}{2} \left(\frac{1}{\Gamma} + \frac{1}{\Gamma + f'(s)^2 + \chi^2(s)} \right)$$

$$\cdot \left[f'(s) \Gamma'(s) + \chi \frac{\partial \gamma}{\partial \nu} \right]$$

$$\leq \left[-\frac{v}{f'} \Phi^2 \Phi' (\Gamma + \Phi^2 v^2) - \frac{\Phi\Gamma'(\Gamma + \Phi^2 v^2)}{2f'v} - \kappa \Phi^2 (\Gamma + \Phi^2 v^2) \right]$$

$$- 2f'(s) v \Phi^2 \Phi' - 2\kappa f'(s)^2 \Phi^2$$

$$- \frac{\Gamma + f'(s)^2}{v^2} \left(f'(s)^2 \kappa + f' \Phi' v + \frac{1}{2} \frac{\partial \gamma}{\partial \nu} \right) - \frac{\Phi\Gamma'}{2\Gamma} vf' - \frac{\Phi^2}{2\Gamma} \frac{\partial \gamma}{\partial \nu} \cdot v^2$$

$$- \frac{\Gamma' \Phi}{2} \frac{f'v}{\Gamma + f'^2 + \chi^2} - \frac{\Phi^2}{2} \frac{v^2}{\Gamma + f'(s)^2 + \chi^2(s)} \frac{\partial \gamma}{\partial \nu},$$

which is equivalent to

$$(4\text{-}14) \quad \kappa \underbrace{\left[(1 - \Phi^2) f'^2 + \Phi^4 v^2 + \Phi^2 \Gamma + 2f'^2 \Phi^2 \right]}_{:=J}$$

$$+ \left[\frac{v}{f'} \Phi^2 \Phi' (\Gamma + \Phi^2 v^2) + 2f'v\Phi^2\Phi' + \frac{\Gamma + f'(s)^2}{v} f'\Phi' + \frac{\Phi^2}{2\Gamma} \frac{\partial \gamma}{\partial \nu} \cdot v^2 + \frac{\Phi\Gamma'}{2\Gamma} vf' \right]$$

$$\leq -\Phi v C_0 - \frac{\Phi\Gamma'(\Gamma + \Phi^2 v^2)}{2f'v} - \frac{\Gamma + f'(s)^2}{2v^2} \frac{\partial \gamma}{\partial \nu} - \frac{\Gamma' \Phi}{2} \frac{f'v}{\Gamma + f'^2 + \chi^2}$$

$$- \frac{\Phi^2}{2} \frac{v^2}{\Gamma + f'(s)^2 + \chi^2(s)} \frac{\partial \gamma}{\partial \nu}.$$

Note that the right-hand side of above inequality are at most the linear terms of v or f', while the left-hand side contains all the possible quadratic terms of v or f'. Firstly, we tackle term J in the left-hand side of above inequality, which can be written as

$$J := (1 - \Phi^2)f'^2 + \Phi^4 v^2 + \Phi^2 \Gamma + 2f'^2 \Phi^2$$
$$= f'^2 + \Phi^2 v^2 = v^2 - \Gamma,$$

where we have used that on boundary, it holds that

$$f'(s) = \sqrt{(1 - \Phi^2)v^2 - \Gamma}.$$

On the other hand, the rest terms of the left-hand side of above inequality gives us

$$\left| \frac{v}{f'} \Phi^2 \Phi'(\Gamma + \Phi^2 v^2) + 2f'v\Phi^2\Phi' + \frac{\Gamma + f'(s)^2}{v}f'\Phi' + \frac{\Phi^2}{2\Gamma}\frac{\partial \gamma}{\partial v} \cdot v^2 + \frac{\Phi\Gamma'}{2\Gamma}vf' \right|$$

$$= \left| \frac{\Phi'(v^2 - \Gamma)}{f'}v + \frac{\Phi^2}{2\Gamma}\frac{\partial \gamma}{\partial v} \cdot v^2 + \frac{\Phi\Gamma'}{2\Gamma}v\sqrt{(1 - \Phi^2)v^2 - \Gamma} \right|$$

$$\leq v^2 \left[\frac{|v\Phi'|}{\sqrt{(1 - \Phi^2)v^2 - \Gamma}} + \frac{\Phi^2}{2\Gamma}\left| \frac{\partial \gamma}{\partial v} \right| + \frac{|\Phi\Gamma'|\sqrt{1 - \Phi^2}}{2\Gamma} \right] + |\Phi'\Gamma|\frac{v}{|f'|}.$$

Notice that, by using the Cauchy–Schwarz inequality, we have

$$\left(\frac{\Phi^2}{2\Gamma}\left| \frac{\partial \gamma}{\partial v} \right| + \frac{|\Phi\Gamma'|\sqrt{1 - \Phi^2}}{2\Gamma} \right) \leq \left[(\Phi^4 + \Phi^2(1 - \Phi^2)) \cdot \left(\frac{1}{4\Gamma^2}\left| \frac{\partial \gamma}{\partial v} \right|^2 + \frac{\Gamma'^2}{4\Gamma^2} \right) \right]^{1/2}$$

$$= |\Phi| \cdot |\nabla \log \sqrt{\gamma}|.$$

Substituting above inequality into (4-14) yields that

$$\left[v^2 \left(\kappa - \frac{|v\Phi'|}{\sqrt{(1 - \Phi^2)v^2 - \Gamma}} - |\Phi| \cdot |\nabla \log \sqrt{\gamma}| \right) \right]$$

$$\leq \kappa \Gamma - \Phi v u_t - \frac{\Phi\Gamma'(\Gamma + \Phi^2 v^2)}{2f'v} - \frac{\Gamma + f'(s)^2}{2v^2}\frac{\partial \gamma}{\partial v} - \frac{\Gamma'\Phi}{2}\frac{f'v}{\Gamma + f'^2 + \chi^2}$$
$$- \frac{\Phi^2}{2}\frac{v^2}{\Gamma + f'(s)^2 + \chi^2(s)}\frac{\partial \gamma}{\partial v}$$

$$\leq C_1 + C_2 v,$$

where C_1, C_2 are positive constants only depending on Γ, Φ, n, and M. Under the assumption that

$$\kappa - \left(\frac{|\nabla^T \varphi|}{\sqrt{1 - \varphi^2}} + |\varphi| \cdot |\nabla \log \sqrt{\gamma}| \right) \geq \delta_1,$$

finally note that

$$\lim_{v \to +\infty} \frac{|v\Phi'|}{\sqrt{(1 - \Phi^2)v^2 - \Gamma}} = \frac{|\Phi'|}{\sqrt{1 - \Phi^2}}.$$

Hence, we can obtain the gradient estimate for v from above, that is

$$v(s_0) \le C,$$

where C is a positive constant only depending on γ, φ, n, δ_1 and M. $\qquad\square$

Theorem 4.2. *Let M be a strictly convex domain in $\mathbb{L}^2$ with κ the geodesic curvature of ∂M, satisfying*

$$\kappa - \left(\frac{|\nabla^T \varphi|}{\sqrt{1 - \varphi^2}} + |\varphi| \cdot |\nabla \log \sqrt{\gamma}|\right) \ge \delta_1,$$

for some positive constant $\delta_1 > 0$ and $\nabla^T \varphi$ is the tangential part of $\nabla \varphi$ restricted on the boundary. If K and γ satisfies the compatible condition

$$K + \lambda_1(\nabla^2 \rho) \ge 0 \quad and \quad \Delta\gamma - \frac{|\nabla\gamma|^2}{2\gamma} \ge 0, \quad in \ M,$$

where K is the Gauss curvature of M and λ_1 is the minimum eigenvalue of the Hessian $\nabla^2 \rho$ with $\rho := \log \sqrt{\gamma}$. Suppose $u \in C^4(\overline{M})$ such that

$$(4\text{-}15) \quad \begin{cases} u_t = \sum_{i,j=1}^2 a^{ij} \nabla_{ij} u \\ \qquad - \frac{1}{2}\left(\frac{1}{\gamma} + 1/(\gamma + |\nabla u|^2)\right)\langle \nabla\gamma, \nabla u \rangle & in \ M \times [0, T), \\ \nabla_\nu u = \varphi(x)\sqrt{\gamma + |\nabla u|^2} & on \ \partial M \times [0, T), \\ u(x, 0) = u_0(x) & on \ M. \end{cases}$$

Then we have the gradient estimate,

$$\sup_M |\nabla u| \le C,$$

where C is a constant only depending on φ, M, γ, n, u_0 and δ_1.

Proof. From the computation in the previous section, we recall that

$$(4\text{-}16) \qquad \mathcal{L}v := \left(\partial_t - \sum_{i,j=1}^2 a^{ij} \nabla_{ij} - \sum_{i=1}^2 F_{p_i} \nabla_i\right)v$$

(4-16 cont.)
$$= \sum_{k=1}^{2} \frac{\nabla_k u}{v} \left(\nabla_k u_t - \sum_{i,j=1}^{2} a^{ij} \nabla_{kji} u - \sum_{i=1}^{2} F_{p_i} \nabla_{ik} u \right)$$

$$+ \frac{1}{v} \sum_{i,j,k=1}^{2} \left(-a^{ij} \nabla_{ki} u \nabla_{kj} u + \frac{1}{v^2} \sum_{l=1}^{2} a^{ij} \nabla_k u \nabla_l u \nabla_{ki} u \nabla_{lj} u \right)$$

$$+ \left(\frac{1}{v^3} \sum_{i,j,l=1}^{2} a^{ij} \nabla_i \gamma \nabla_l u \nabla_{lj} u - \frac{1}{2v} \sum_{i=1}^{2} F_{p_i} \nabla_i \gamma - \frac{1}{2v} \sum_{i,j=1}^{2} a^{ij} \nabla_{ij} \gamma \right.$$

$$\left. + \frac{1}{4v^3} \sum_{i,j=1}^{2} a^{ij} \nabla_i \gamma \nabla_j \gamma - \sum_{i,j,k,l=1}^{2} \frac{a^{ij} \nabla_k u \nabla_l u R_{likj}}{v} \right)$$

$$= P_1 + P_2 + P_3.$$

Notice that

$$2v \nabla_i v = \nabla_i \gamma + 2 \sum_{l=1}^{2} \nabla_l u \nabla_{li} u,$$

by using (2-16), it follows that

$$P_1 = \sum_{k=1}^{2} \frac{\nabla_k u}{v} \left(\nabla_k u_t - \sum_{i,j=1}^{2} a^{ij} \nabla_{kji} u - \sum_{i=1}^{2} F_{p_i} \nabla_{ik} u \right)$$

$$= \sum_{k=1}^{2} \frac{\nabla_k u}{v} \left[- \sum_{i,j=1}^{2} \frac{2 \nabla_{ij} u \nabla_{ki} u \nabla_j u}{v^2} + \sum_{i,j=1}^{2} \frac{\nabla_{ij} u \nabla_i u \nabla_j u}{v^4} \left(\nabla_k \gamma + 2 \sum_{l=1}^{2} \nabla_l u \nabla_{kl} u \right) \right.$$

$$\left. + \frac{1}{2} \left(\frac{\nabla_k \gamma}{\gamma^2} + \frac{\nabla_k \gamma}{v^4} \right) \langle \nabla u, \nabla \gamma \rangle - \frac{1}{2} \left(\frac{1}{\gamma} + \frac{1}{v^2} \right) \sum_{i=1}^{2} \nabla_{ik} \gamma \nabla_i u \right]$$

$$= - \frac{1}{2v^3} \sum_{i=1}^{2} (2v \nabla_i v - \nabla_i \gamma)^2 + \frac{2 \nabla^2 u (\nabla u, \nabla u)}{v^4} \langle \nabla u, \nabla v \rangle + \frac{\langle \nabla u, \nabla \gamma \rangle^2}{2v} \left(\frac{1}{\gamma^2} + \frac{1}{v^4} \right)$$

$$- \frac{\nabla^2 \gamma (\nabla u, \nabla u)}{2v} \left(\frac{1}{\gamma} + \frac{1}{v^2} \right)$$

$$= - \frac{|\nabla \gamma|^2}{2v^3} + \frac{\langle \nabla u, \nabla \gamma \rangle^2}{2v} \left(\frac{1}{\gamma^2} + \frac{1}{v^4} \right) - \frac{\nabla^2 \gamma (\nabla u, \nabla u)}{2v} \left(\frac{1}{\gamma} + \frac{1}{v^2} \right) \quad \text{mod } \nabla v.$$

Since $[a^{ij}]$ is positive definite, $A^{kl} := \sum_{i,j=1}^{n} a^{ij} \nabla_{ki} u \nabla_{lj} u$ is also positive definite. Using the Cauchy–Schwarz inequality yields that

$$P_2 := \frac{1}{v^3} \sum_{i,j,k,l=1}^{2} a^{ij} \nabla_k u \nabla_l u \nabla_{ki} u \nabla_{lj} u - \frac{1}{v} \sum_{i,j,k=1}^{2} a^{ij} \nabla_{ki} u \nabla_{kj} u$$

$$= \frac{1}{v^3} \left(\sum_{k,l=1}^{2} A^{kl} \nabla_k u \nabla_l u - v^2 \sum_{k=1}^{2} A^{kk} \right) \le 0.$$

Besides, we have

$$P_3 := \frac{1}{v^3} \sum_{i,j,l=1}^{2} a^{ij} \nabla_i \gamma \nabla_l u \nabla_{lj} u + \frac{1}{2v} \sum_{i,j=1}^{2} \left(\frac{1}{2v^2} a^{ij} \nabla_i \gamma \nabla_j \gamma - a^{ij} \nabla_{ij} \gamma \right)$$

$$- \frac{1}{2v} \sum_{i=1}^{2} F_{p_i} \nabla_i \gamma - \sum_{i,j,k,l=1}^{2} \frac{a^{ij} \nabla_k u \nabla_l u R_{likj}}{v}$$

$$= \frac{|\nabla \gamma|^2}{4v\gamma} - \frac{\Delta \gamma}{2v} + \frac{\nabla^2 \gamma (\nabla u, \nabla u)}{2v^3} - \frac{\langle \nabla \gamma, \nabla u \rangle^2}{4v^5} - \frac{|\nabla u|^2}{v} K \quad \text{mod } \nabla v.$$

Thus we add the above three terms P_1, P_2 and P_3 together, and take advantage of the assumption in Theorem 4.2, to conclude that

$$(4\text{-}17) \quad \mathcal{L}v := \left(\partial_t - \sum_{i,j=1}^{2} a^{ij} \nabla_{ij} - \sum_{i=1}^{2} F_{p_i} \nabla_i \right) v := P_1 + P_2 + P_3$$

$$\leq \frac{1}{v} \sum_{i,j=1}^{2} \left(\frac{\nabla_i \gamma \nabla_j \gamma}{2\gamma^2} - \frac{\nabla_{ij} \gamma}{2\gamma} - K\sigma_{ij} \right) \cdot \nabla_i u \nabla_j u + \frac{1}{2v} \left(\frac{|\nabla \gamma|^2}{2\gamma} - \Delta \gamma \right)$$

$$- \frac{|\nabla \gamma|^2}{2v^3} + \frac{\langle \nabla \gamma, \nabla u \rangle^2}{4v^5}$$

$$\leq 0 \quad \text{mod } \nabla v.$$

Hence the maximum principle implies that v attains its maximum value at (x_0, t_0) for either $x_0 \in \partial M$ or $t_0 = 0$. If $x_0 \in \partial M$, since we have the estimate for $|u_t| \leq C_0$ from Lemma 2.1, then combining with Lemma 4.1 we have $v \leq C$. If $t_0 = 0$, then we get $v \leq \sup_M \sqrt{\gamma + |\nabla u_0|^2}$.

Therefore, we have

$$v \leq C,$$

where C is a constant only depending on φ, M, γ, n, u_0 and δ_1.   □

5. Stationary equation and asymptotic behavior

In this section, we use the approach and argument in [Altschuler and Wu 1994] to prove Theorem 1.1. Firstly, we use an equivalent way to rewrite the first equation in (1-5), which has a nice weighted divergence structure. Recall that $\rho := \log \sqrt{\gamma}$, thus first equation in (1-5) turns out to be

$$(5\text{-}1) \qquad \text{div}_\rho \left(\frac{\nabla u}{\sqrt{\gamma + |\nabla u|^2}} \right) = \frac{\tau}{\sqrt{\gamma + |\nabla u|^2}} \quad \text{in } M,$$

where the weighted divergence operator div_ρ is defined as

$$\mathrm{div}_\rho(X) = e^\rho \, \mathrm{div}(e^{-\rho} X),$$

for any vector field X in TM. From this, one can see that the constant $\tau \in \mathbb{R}$ in (5-1) is a uniquely determined constant. In fact, by using integration by parts over (5-1), yields that

$$(5\text{-}2) \qquad \tau = \frac{\int_{\partial M} \varphi / \sqrt{\gamma}\, d\sigma}{\int_M \sqrt{\gamma(\gamma + |\nabla u|^2)}\, dV}.$$

Moreover, we see that if $\hat{u} := \hat{u}(x)$ solves (5-1), then $\tilde{u}(x,t) := \hat{u}(x) + \tau t$ satisfies

$$(5\text{-}3) \qquad \begin{cases} \tilde{u}_t = \sum_{i,j=1}^n a^{ij}(x, \nabla\tilde{u})\nabla_{ij}\tilde{u} \\ \qquad -\frac{1}{2}\left(\frac{1}{\gamma} + 1/(\gamma + |\nabla\tilde{u}|^2)\right)\langle\nabla\gamma, \nabla\tilde{u}\rangle & \text{in } M \times [0, +\infty), \\ \nabla_\nu\tilde{u} = \varphi(x)\sqrt{\gamma + |\nabla\tilde{u}|^2} & \text{on } \partial M \times [0, +\infty), \\ \tilde{u}(x,0) = \hat{u}(x) & \text{on } \overline{M} \times \{0\}. \end{cases}$$

Therefore, based on the maximum principle and boundary value condition similar to the one used in Theorem 1.2, we can obtain the following oscillation bound on the solutions to the parabolic problem (1-3) as in [Altschuler and Wu 1994] (see Corollary 2.7 there).

Corollary 5.1. *For a solution $u(x,t)$ to (1-3), there exists a positive constant C independent of t such that*

$$(5\text{-}4) \qquad |u(x,t) - \tau t| \leq C.$$

In particular, one has

$$\frac{u(x,t)}{t} \to \tau \quad \text{uniformly as } t \to \infty.$$

Proof. Define the new function $U(x,t) = u(x,t) - \tilde{u}(x,t)$, by direct calculation, we find that U satisfies

$$\partial_t U = \sum_{i,j=1}^n A^{ij}(\nabla u, \nabla\tilde{u}) \cdot \nabla_{ij}U + \sum_{i=1}^n b^i(\nabla u, \nabla\tilde{u}) \cdot \nabla_i U,$$

where A^{ij}, b^i are defined as previously in the proof of Theorem 1.2. From the maximum principle, we know $U(x,t)$ attains its maximum value at the point (x_0, t_0) with either $x_0 \in \partial M$ or $t_0 = 0$. If $x_0 \in \partial M$, using a similar argument to that in the proof of Theorem 1.2, we reach a contradiction. So $t_0 = 0$, hence we have

$$|u(x,t) - \tau t| \leq \hat{u}(x) + \sup_M |u_0(x) - \hat{u}(x)| \leq 2\|\hat{u}\|_{C^0(M)} + \|u_0\|_{C^0(M)},$$

for all $(x,t) \in M \times [0, +\infty)$. Thus we have completed the proof. $\square$

Eventually, based on the above preparation, combining Lemma 2.1, Theorem 2.2 and Corollary 5.1 together, one can follow an argument of Schnürer used in [Schnürer 2002, Section 6] to show the asymptotic behavior of solutions to (1-3), that is, u converges to the translating solution as $t \to +\infty$. For completeness, we give the proof of Theorem 1.1 with a few minor modifications of [Schnürer 2002] to our situation here.

Proof of Theorem 1.1. Recall that

$$U(x, t) := u(x, t) - \tilde{u}(x, t),$$

from the proof of Corollary 5.1, and

$$\partial_t U = \sum_{i,j=1}^n A^{ij}(\nabla u, \nabla \tilde{u}) \nabla_{ij} U + \sum_{i=1}^n b^i(\nabla u, \nabla \tilde{u}) \nabla_i U \quad \text{in } M.$$

Firstly, we let

$$h(p) := \frac{\langle p, \nu \rangle}{\sqrt{\gamma + |p|^2}} - \varphi \quad \text{for } p \in \mathbb{R}^n.$$

By direct computation, we find that

$$h_{p_i} = \frac{\nu^i}{\sqrt{\gamma + |p|^2}} - \frac{\langle p, \nu \rangle}{(\gamma + |p|^2)^{\frac{3}{2}}} p_i,$$

and

$$h(p) - h(q) = \sum_{i=1}^n \int_0^1 h_{p_i}(sp + (1-s)q)ds \cdot (p - q)_i := \langle \beta(p, q), p - q \rangle.$$

Combining this with the boundary value condition in (1-3), we have

$$\langle \nabla U, \beta(\nabla u(x, t), \nabla \tilde{u}(x, t)) \rangle = 0 \quad \text{on } \partial M.$$

Note that this is a uniformly strictly oblique boundary condition, since

$$\langle \beta, \nu \rangle = \int_0^1 \frac{1}{\sqrt{\gamma + |s\nabla u(x, t) + (1-s)\nabla \tilde{u}(x, t)|^2}} ds$$
$$- \int_0^1 \frac{\langle s\nabla u(x, t) + (1-s)\nabla \tilde{u}(x, t), \nu \rangle^2}{(\gamma + |s\nabla u(x, t) + (1-s)\nabla \tilde{u}(x, t)|^2)^{3/2}} ds$$
$$\geq \int_0^1 \frac{\gamma}{(\gamma + |s\nabla u(x, t) + (1-s)\nabla \tilde{u}(x, t)|^2)^{3/2}} ds$$
$$> 0,$$

where the last inequality in above follows from the uniform gradient estimate.

Next we define the oscillation of function U as

$$\mathrm{osc}(U)(t) := \sup_M U - \inf_M U.$$

Then by the strong maximum principle and the Hopf lemma, we know that $\mathrm{osc}(U)(t)$ is either a strictly decreasing function in t or a constant function.

Claim.
$$\mathrm{osc}(U)(t) \to 0 \quad \text{as } t \to \infty.$$

In fact, we can verify this by contradiction. If $\mathrm{osc}(U)(t) \to \alpha_0$ as $t \to \infty$ for some $\alpha_0 > 0$, we can choose a sequence $t_k \to +\infty$ as $k \to \infty$. Consider $(x, t) \in \overline{M} \times [-t_k, \infty)$ and for fixed $x_0 \in M$, we consider the function

(5-5) $\qquad \tilde{u}^{k,1}(x, t) := u(x, t + t_k) - \tau t_k, \quad \tilde{u}^{k,2}(x, t) := \tilde{u}(x, t + t_k) - \tau t_k,$

both of which satisfies (1-3) in $M \times [-t_k, \infty)$.

And for any $t_k > T$ and $(x, t) \in \overline{M} \times [-T, T]$, using Corollary 5.1 and Lemma 2.1, we obtain that

$$|\tilde{u}^{k,1}(x, t)| \le |u(x, t + t_k) - u(x, t_k)| + |u(x, t_k) - \tau t_k|$$
$$\le T \cdot \sup_M |\dot{u}| + C,$$

and

$$|\tilde{u}^{k,2}(x, t)| = |\hat{u}(x) + \tau t| \le C.$$

Combining Lemma 2.1, Theorem 2.2, Corollary 5.1 and Schauder theory together, we get the locally uniform bounds for any C^l-norm ($l \ge 0$) for both sequences $\{\tilde{u}^{k,1}\}_{k\ge1}$ and $\{\tilde{u}^{k,2}\}_{k\ge1}$. By applying the Arzelà–Ascoli theorem to both sequences in (5-5) we can extract a subsequence of t_k (still denoted as t_k) such that the limits of both subsequences are $\tilde{u}^{\infty,1}$ and $\tilde{u}^{\infty,2}$, and satisfy the Equations (1-3) in $\overline{M} \times [-T, T]$.

Now we define the new function $\tilde{u} := \tilde{u}^{\infty,1} - \tilde{u}^{\infty,2}$, due to the uniform convergence of above two sequences, it follows that, for any fixed $t \in \mathbb{R}$,

$$\mathrm{osc}(\tilde{u})(t) = \mathrm{osc}\Big[\lim_{k\to\infty} \big(\tilde{u}^{k,1}(x, t) - \tilde{u}^{k,2}(x, t) \big) \Big]$$
$$= \mathrm{osc}\Big[\lim_{k\to\infty} \big(u(x, t + t_k) - \tilde{u}(x, t + t_k) \big) \Big]$$
$$= \lim_{k\to\infty} \mathrm{osc}\Big[\big(u(x, t + t_k) - \tilde{u}(x, t + t_k) \big) \Big]$$
$$= \lim_{s\to\infty} \mathrm{osc}(U)(s) = \alpha_0 > 0.$$

However, this is a contradiction with the fact that $\tilde{u} \equiv \text{const}$, which follows from the strong maximum principle applied to the function $\tilde{u}$, as $\tilde{u}$ satisfies

$$\begin{cases} \partial_t \tilde{u} = \sum_{i,j=1}^n A^{ij}(\nabla \tilde{u}^{\infty,1}, \nabla \tilde{u}^{\infty,2}) \nabla_{ij} \tilde{u} \\ \qquad\qquad\qquad + \sum_{i=1}^n b^i(\nabla \tilde{u}^{\infty,1}, \nabla \tilde{u}^{\infty,2}) \nabla_i \tilde{u} & \text{in } M \times \mathbb{R}, \\ \langle \nabla \tilde{u}, \beta(\nabla \tilde{u}^{\infty,1}, \nabla \tilde{u}^{\infty,2}) \rangle = 0 & \text{on } \partial M \times \mathbb{R}. \end{cases}$$

Hence we have proved the Claim.

The Claim yields that

$$\lim_{t\to\infty} \sup_M U = \lim_{t\to\infty} \inf_M U = c,$$

for some constant c. Finally, combining with Theorem 1.2, up to an additive constant,

$$\tilde{u}(x,t) := \hat{u}(x) + \tau t$$

is the only translating solution of (1-5), we finish the proof that any solution of flow (1-3) tends smoothly to a translating solution. $\qquad\square$

Similarly, we can also prove Theorem 1.3 for $n = 2$. Since we have established the gradient estimate in Theorem 4.2, and under the assumption of Theorem 1.3, one can get the existence of the translating solution firstly, then adopt the same above argument and approach to show Theorem 1.3. For conciseness, we omit it here.

Acknowledgements

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LIANGJUN WENG
SCHOOL OF MATHEMATICAL SCIENCES
SHANGHAI JIAO TONG UNIVERSITY
SHANGHAI
CHINA
ljweng08@sjtu.edu.cn

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