

*Pacific
Journal of
Mathematics*

SCATTERED REPRESENTATIONS OF $SL(n, \mathbb{C})$

CHAO-PING DONG AND KAYUE DANIEL WONG

Volume 309 No. 2

December 2020

SCATTERED REPRESENTATIONS OF $SL(n, \mathbb{C})$

CHAO-PING DONG AND KAYUE DANIEL WONG

Let G be $SL(n, \mathbb{C})$. The unitary dual \widehat{G} was classified by Vogan in the 1980s. This paper aims to describe the Zhelobenko parameters and the spin-lowest K -types of the scattered representations of G , which lie at the heart of \widehat{G}^d —the set of all the equivalence classes of irreducible unitary representations of G with nonvanishing Dirac cohomology. As a consequence, we will verify a couple of conjectures of Dong for G .

1. Introduction

1.1. Preliminaries on complex simple Lie groups. Let G be a complex connected simple Lie group, and H be a Cartan subgroup of G . Let \mathfrak{g}_0 and \mathfrak{h}_0 be the Lie algebra of G and H respectively, and we drop the subscripts to stand for the complexified Lie algebras. We adopt a positive root system $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$, and let $\varpi_1, \dots, \varpi_{\text{rank}(\mathfrak{g}_0)}$ be the corresponding fundamental weights with $\rho = \varpi_1 + \dots + \varpi_{\text{rank}(\mathfrak{g}_0)}$ being the half sum of positive roots.

Fix a Cartan involution θ on G such that its fixed points form a maximal compact subgroup K of G . Then on the Lie algebra level, we have the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0.$$

We denote by $\langle \cdot, \cdot \rangle$ the Killing form on \mathfrak{g}_0 . This form is negative definite on \mathfrak{k}_0 and positive definite on \mathfrak{p}_0 . Moreover, \mathfrak{k}_0 and \mathfrak{p}_0 are orthogonal to each other under $\langle \cdot, \cdot \rangle$. We shall denote by $\| \cdot \|$ the norm corresponding to the Killing form.

Let $H = TA$ be the Cartan decomposition of H , with $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$. We make the following identifications:

$$(1) \quad \mathfrak{h} \cong \mathfrak{h}_0 \times \mathfrak{h}_0, \quad \mathfrak{t} = \{(x, -x) : x \in \mathfrak{h}_0\}, \quad \mathfrak{a} \cong \{(x, x) : x \in \mathfrak{h}_0\}.$$

Take an arbitrary pair $(\lambda_L, \lambda_R) \in \mathfrak{h}_0^* \times \mathfrak{h}_0^*$ such that $\mu := \lambda_L - \lambda_R$ is integral. Denote by $\{\mu\}$ the unique dominant weight to which μ is conjugate under the action of

MSC2010: primary 22E46; secondary 17B56.

Keywords: Dirac cohomology, unitary representations, scattered representations.

the Weyl group W . Write $\nu := \lambda_L + \lambda_R$. We can view μ as a weight of T and ν a character of A . Put

$$I(\lambda_L, \lambda_R) := \text{Ind}_B^G(\mathbb{C}_\mu \otimes \mathbb{C}_\nu \otimes \mathbf{1})_{K\text{-finite}},$$

where B is the Borel subgroup of G determined by $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$. It is not hard to show that $V_{\{\mu\}}$, the K -type with highest weight $\{\mu\}$, occurs exactly once in $I(\lambda_L, \lambda_R)$. Let $J(\lambda_L, \lambda_R)$ be the unique irreducible subquotient of $I(\lambda_L, \lambda_R)$ containing $V_{\{\mu\}}$. By [Zhelobenko 1974], every irreducible admissible (\mathfrak{g}, K) -module has the form $J(\lambda_L, \lambda_R)$. Indeed, $J(\lambda_L, \lambda_R)$ has infinitesimal character the $W \times W$ orbit of (λ_L, λ_R) , and lowest K -type $V_{\{\lambda_L - \lambda_R\}}$. We will refer to the pair (λ_L, λ_R) as the *Zhelobenko parameter* for the module $J(\lambda_L, \lambda_R)$.

1.2. Dirac cohomology. Fix an orthonormal basis Z_1, \dots, Z_l of \mathfrak{p}_0 with respect to the inner product on \mathfrak{p}_0 induced by $\langle \cdot, \cdot \rangle$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} , and put $C(\mathfrak{p})$ as the Clifford algebra of \mathfrak{p} . One checks that

$$(2) \quad D := \sum_{i=1}^l Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$$

is independent of the choice of the orthonormal basis Z_1, \dots, Z_l . The operator D , called the *Dirac operator*, was introduced by Parthasarathy [1972]. By construction, D^2 is a natural Laplacian on G , which gives rise to the Parthasarathy's Dirac inequality (see (6) below). The inequality is very effective for detecting nonunitarity of (\mathfrak{g}, K) -modules, but is by no means sufficient to classify all (non)unitary modules.

To sharpen the Dirac inequality, and to offer a better understanding of the unitary dual, Vogan [1997] formulated the notion of Dirac cohomology. Let $\text{Ad} : K \rightarrow SO(\mathfrak{p}_0)$ be the adjoint map, $\text{Spin } \mathfrak{p}_0$ be the spin group of \mathfrak{p}_0 , and denote by $p : \text{Spin } \mathfrak{p}_0 \rightarrow SO(\mathfrak{p}_0)$ the spin double covering map. Put

$$\tilde{K} := \{(k, s) \in K \times \text{Spin } \mathfrak{p}_0 \mid \text{Ad}(k) = p(s)\}.$$

As in the case of K -types, we will refer to an irreducible \tilde{K} -type with highest weight δ as V_δ .

Let π be any admissible (\mathfrak{g}, K) -module, and S be the spin module of $C(\mathfrak{p})$. Then $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, in particular the Dirac operator D , acts on $\pi \otimes S$. Now the *Dirac cohomology* is defined as the \tilde{K} -module

$$(3) \quad H_D(\pi) := \text{Ker } D / (\text{Ker } D \cap \text{Im } D).$$

It is evident from the definition that Dirac cohomology is an invariant for admissible (\mathfrak{g}, K) -modules. To compute this invariant, the Vogan conjecture, proved by Huang

and Pandžić [2002], says that whenever $H_D(\pi) \neq 0$, one would have

$$(4) \quad \gamma + \rho = w\Lambda,$$

where Λ is the infinitesimal character of π , γ is the highest weight of any \tilde{K} -type in $H_D(\pi)$, and w is some element of W .

It turns out that many interesting (\mathfrak{g}, K) -modules π , such as some $A_q(\lambda)$ -modules and all the highest weight modules, have nonzero Dirac cohomology (see [Huang et al. 2009; 2011]). One would therefore like to classify all representations with nonzero Dirac cohomology.

1.3. Spin-lowest K -type. From now on, we set π as an irreducible unitary (\mathfrak{g}, K) -module with infinitesimal character Λ . In order to get a clearer picture on $H_D(\pi)$, the first-named author introduced the notion of spin-lowest K -types. Given an arbitrary K -type V_δ , its spin norm is defined as

$$(5) \quad \|\delta\|_{\text{spin}} := \|\{\delta - \rho\} + \rho\|.$$

Then a K -type V_τ occurring in π is called a *spin-lowest K -type* of π if it achieves the minimum spin norm among all the K -types showing up in π .

As an application of spin-lowest K -type, note that D is self-adjoint on the unitarizable module $\pi \otimes S$. By writing out D^2 carefully, and by using the *PRV-component* [Parthasarathy et al. 1967], we can rephrase *Parthasarathy's Dirac operator inequality* [Parthasarathy 1980] as follows:

$$(6) \quad \|\delta\|_{\text{spin}} \geq \|\Lambda\|,$$

where V_δ is any K -type occurring in π . Moreover, one can deduce from [Huang and Pandžić 2006, Theorem 3.5.3] that $H_D(\pi) \neq 0$ if and only if the spin-lowest K -types V_τ attain the lower bound of (6). In such cases, $V_{\{\tau - \rho\}}$ will show up in $H_D(\pi)$. Put in a different way, the spin-lowest K -types of π are exactly the K -types contributing to $H_D(\pi)$ whenever the cohomology is nonvanishing (see [Dong 2013, Proposition 2.3] for more details).

1.4. Scattered representations. Based on the studies [Barbasch and Pandžić 2011; Ding and Dong 2020], we are interested in the irreducible unitarizable (\mathfrak{g}, K) -modules $J(\lambda, -s\lambda)$ such that

- (i) the weight 2λ is dominant integral, i.e., $2\lambda = \sum_{i=1}^{\text{rank}(\mathfrak{g}_0)} c_i \varpi_i$, where each c_i is a positive integer;
- (ii) the element $s \in W$ is an involution such that each simple reflection s_i , $1 \leq i \leq \text{rank}(\mathfrak{g}_0)$, occurs in one (thus in each) reduced expression of s ;

(iii) the module has nonzero Dirac cohomology, i.e., $H_D(J(\lambda, -s\lambda)) \neq 0$, or equivalently, there exists a K -type V_τ in $J(\lambda, -s\lambda)$ such that

$$(7) \quad \|\tau\|_{\text{spin}} = \|(\lambda, -s\lambda)\| = \|2\lambda\|$$

According to [Ding and Dong 2020], there are only finitely many such representations, which are called the *scattered representations*.

These representations lie at the heart of \widehat{G}^d — the set of all the irreducible unitary (\mathfrak{g}, K) -modules of G with nonzero Dirac cohomology up to equivalence. Namely, by [Ding and Dong 2020, Theorem A], any member of \widehat{G}^d is either a scattered representation, or it is cohomologically induced from a scattered representation tensored with a suitable unitary character of the Levi factor of a certain proper θ -stable parabolic subgroup. In the latter case, one can easily trace the spin-lowest K -types along with the Dirac cohomology of the modules before and after induction. It is therefore of interest to have a good understanding of scattered representations.

1.5. Overview. In this manuscript, we focus on Lie groups G of Type A . For convenience, we will start from the group $GL(n, \mathbb{C})$, written as $GL(n)$ for short. In this case, Vogan classified the unitary dual. The part that we need can be described as follows.

Theorem 1.1 [Vogan 1986]. *All irreducible unitary representations of $GL(n)$ with regular half-integral infinitesimal characters are parabolically induced from a unitary character, i.e., they are of the form*

$$\text{Ind}_{\left(\prod_{i=0}^m GL(a_i)\right)U}^{GL(n)} \left(\bigotimes_{i=0}^m \det^{p_i} \otimes \mathbf{1} \right)$$

for some $a_i \in \mathbb{Z}_{>0}$ and $p_i \in \mathbb{Z}$. For simplicity, we will write the parabolically induced module $\text{Ind}_{LU}^G(\pi \otimes \mathbf{1})$ as $\text{Ind}_L^G(\pi)$ for the rest of the manuscript.

Using [Barbasch and Pandžić 2011, Theorem 2.4], all such π have nonzero Dirac cohomology. Moreover, [Barbasch et al. 2020] proved [Barbasch and Pandžić 2011, Conjecture 4.1], which says

$$H_D(\pi) = 2^{\lceil \text{rank}(\mathfrak{go})/2 \rceil} V_{\{\tau-\rho\}},$$

where V_τ is the *unique* spin-lowest K -type appearing in π with multiplicity one. However, it is not clear what V_τ is like from the calculations in [Barbasch et al. 2020].

In Section 2, we will give an algorithm to compute V_τ for all such π (see Proposition 2.5). In Section 3, we will see how the calculations for $GL(n)$ in Section 2 can be translated to $SL(n)$, which gives a combinatorial description of scattered representations of $SL(n)$ (Proposition 3.1). As a result, we prove the following:

- The spin-lowest K -type of each scattered representation of $SL(n)$ is *unitarily small* in the sense of Salamanca-Riba and Vogan [1998] (Corollary 3.5); and
- the number of scattered representations of $SL(n)$ is equal to 2^{n-2} (Corollary 3.9).

This verifies [Ding and Dong 2020, Conjecture C] in the case of $SL(n)$, and proves [Dong 2019, Conjecture 5.2] respectively.

It is worth noting that for any nontrivial scattered representation, its spin-lowest K -type lives deeper than, and differs from the lowest K -type. We hope the effort here will shed some light on the real case in future.

2. An algorithm computing the spin-lowest K -types

In this section, we give an algorithm to find the spin-lowest K -types of the irreducible unitary modules of $GL(n)$ given by Theorem 1.1. We use a *chain*

$$\mathcal{C} := \{c, c - 2, \dots, c - (2k - 2), c - 2k\},$$

where $c, k \in \mathbb{Z}$ with $k > 0$, to denote the Zhelobenko parameter

$$\begin{pmatrix} \lambda \\ -w_0\lambda \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c & \frac{1}{2}c - 1 & \dots & \frac{1}{2}c - (k - 1) & \frac{1}{2}c - k \\ -\frac{1}{2}c + k & -\frac{1}{2}c + (k - 1) & \dots & -\frac{1}{2}c + 1 & -\frac{1}{2}c \end{pmatrix}.$$

Note that the entries of \mathcal{C} are precisely equal to 2λ . Also, this parameter corresponds to the one-dimensional module \det^{c-k} of $GL(k + 1)$. Consequently, Theorem 1.1 implies that the Zhelobenko parameters of all irreducible unitary modules with regular half-integral infinitesimal character can be expressed by the chains

$$(\lambda, -s\lambda) = \bigcup_{i=0}^m \mathcal{C}_i,$$

where all the entries of \mathcal{C}_i are disjoint.

In order to understand the spin-lowest K -types of these modules of $GL(n)$, we make the following:

Definition 2.1. (a) Two chains $\mathcal{C}_1 = \{A, \dots, a\}$, $\mathcal{C}_2 = \{B, \dots, b\}$ are *linked* if the entries of \mathcal{C}_1 and \mathcal{C}_2 are disjoint satisfying

$$A > B > a \quad \text{or} \quad B > A > b.$$

(b) We say a union of chains $\bigcup_{i \in I} \mathcal{C}_i$ is *interlaced* if for all $i \neq j$ in I , there exist indices $i = i_0, i_1, \dots, i_m = j$ in I such that $\mathcal{C}_{i_{l-1}}$ and \mathcal{C}_{i_l} are linked for all $1 \leq l \leq m$. (By convention, we also let the single chain \mathcal{C}_1 be interlaced).

For example, the parameter $\{9, 7, 5\} \cup \{6, 4, 2\} \cup \{3, 1\}$ is interlaced, while the parameter $\{10, 8\} \cup \{9, 7\} \cup \{6, 4\} \cup \{5, 3, 1\}$ is not interlaced.

We are now in the position to describe the spin-lowest K -types of the unitary modules in Theorem 1.1 using chains.

Algorithm 2.2. Let $J(\lambda, -s\lambda)$ be an irreducible unitary module of $GL(n)$ in Theorem 1.1 with $(\lambda, -s\lambda) = \bigcup_{i=0}^m \mathcal{C}_i$, where

$$\mathcal{C}_i := \{k_i + (d_i - 1), \dots, k_i - (d_i - 1)\} = \{\mathcal{C}_{i,1}, \dots, \mathcal{C}_{i,d_i}\}$$

is a chain with average value k_i and length d_i . Then the lowest K -type is equal to (a W -conjugate of) $(\mathcal{T}_0, \dots, \mathcal{T}_m)$, where

$$\mathcal{T}_i := \underbrace{(k_i, \dots, k_i)}_{d_i}.$$

By reindexing the chains when necessary, we may and we will assume that

$$(8) \quad \text{for any } 0 \leq i < j \leq m, \quad k_i > k_j \text{ or } d_i < d_j \text{ if } k_i = k_j.$$

Let us change the coordinates of \mathcal{T}_i and \mathcal{T}_j for all pairs of linked chains \mathcal{C}_i and \mathcal{C}_j such that $i < j$ by the following rule:

(a) If $C_{i,1} > C_{j,1} \geq C_{j,d_j} > C_{i,d_i}$, i.e.,

$$\{\mathcal{C}_{i,1}, \dots, \mathcal{C}_{i,d_i-p}, \overbrace{\mathcal{C}_{i,d_i-p+1}, \dots, \mathcal{C}_{i,d_i}}^p\},$$

$$\{\mathcal{C}_{j,1}, \dots, \mathcal{C}_{j,d_j}\},$$

with $C_{j,1} = C_{i,d_i} + 2p - 1$ and $d_j \leq p$, then we change the coordinates of \mathcal{T}_i and \mathcal{T}_j into

$$\mathcal{T}'_i : (*, \dots, *, \overbrace{k_i + p, k_i + (p - 1), \dots, k_i + (p - d_j + 1)}^p, *, \dots, *)$$

$$\mathcal{T}'_j : (k_j - p, k_j - (p - 1), \dots, k_j - (p - d_j + 1)),$$

where the entries marked by $*$ remain unchanged.

(b) If $C_{i,1} > C_{j,1} > C_{i,d_i} > C_{j,d_j}$, i.e.,

$$\{\mathcal{C}_{i,1}, \dots, \mathcal{C}_{i,d_i-p}, \overbrace{\mathcal{C}_{i,d_i-p+1}, \dots, \mathcal{C}_{i,d_i}}^p\}$$

$$\{\mathcal{C}_{j,1}, \dots, \mathcal{C}_{j,p}, \mathcal{C}_{j,p+1}, \dots, \mathcal{C}_{j,d_j}\}$$

with $C_{j,1} = C_{i,d_i} + 2p - 1$ and $d_j > p$, then we change the coordinates of \mathcal{T}_i and \mathcal{T}_j into

$$\begin{array}{l} \mathcal{T}'_i : (*, \dots, *, \overbrace{k_i + 1, \dots, k_i + p}^p) \\ \mathcal{T}'_j : (k_j - 1, \dots, k_j - p, *, \dots, *) \end{array}$$

where the entries marked by $*$ remain unchanged.

- (c) If $C_{j,1} > C_{i,1} > C_{j,d_j}$, then since $k_i \geq k_j$ one also have $C_{j,1} > C_{i,1} \geq C_{i,d_i} > C_{j,d_j}$, i.e.,

$$\underbrace{\{C_{j,1}, \dots, C_{j,q}\}}_q \quad \{C_{i,1}, \dots, C_{i,d_i}\} \quad C_{j,q+1}, \dots, C_{j,d_j}$$

with $C_{j,1} = C_{i,d_i} + 2q - 1$, then we change the coordinates of \mathcal{T}_i and \mathcal{T}_j into

$$\begin{array}{l} \mathcal{T}'_i : (k_i + (q - d_0 + 1), \dots, k_i + (q - 1), k_i + q) \\ \mathcal{T}'_j : (*, \dots, *, \underbrace{k_j - (q - d_0 + 1), \dots, k_j - (q - 1), k_j - q}_q, *, \dots, *) \end{array}$$

where the entries marked by $*$ remain unchanged.

In the above three cases, we only demonstrate the situation that C_i is in the first row and C_j is in the second row. The rule is the same when C_j is in the first row while C_i is in the second row.

After running through all pairs of linked chains, V_τ is defined as the K -type with highest weight τ given by (a W -conjugate of) $\bigcup_{i=0}^m \mathcal{T}'_i$.

Example 2.3. Consider

$$(\lambda, -s\lambda) = \begin{array}{cccc} \{10 & 8\} & \{6\} & \{4\} \\ \{9 & 7 & 5 & 3 & 1\}. \end{array}$$

Then the lowest K -type of $J(\lambda, -s\lambda)$ is

$$\begin{array}{cccc} (9 & 9) & (6) & (4) \\ (5 & 5 & 5 & 5 & 5). \end{array}$$

To compute V_τ , let us label the chains so that (8) holds:

$$\mathcal{T}_0 = (9 \ 9), \quad \mathcal{T}_1 = (6), \quad \mathcal{T}_2 = (5 \ 5 \ 5 \ 5 \ 5), \quad \mathcal{T}_3 = (4).$$

Then we apply (a) to the pair $\mathcal{T}_2, \mathcal{T}_3$, apply (b) to the pair $\mathcal{T}_0, \mathcal{T}_2$, and apply (c) to the pair $\mathcal{T}_1, \mathcal{T}_2$. This gives us

$$\begin{pmatrix} 9 & 10 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Thus $\tau = (10, 9, 8, 7, 5, 5, 4, 3, 2)$.

Theorem 2.4. *Let $J(\lambda, -s\lambda)$ be a unitary module of $GL(n)$ in Theorem 1.1, and V_τ be obtained by Algorithm 2.2. Then $[J(\lambda, -s\lambda) : V_\tau] > 0$.*

Proof. Let

$$J(\lambda, -s\lambda) = \text{Ind}_{\prod_{i=0}^m GL(a_i)}^{GL(n)} \left(\bigotimes_{i=0}^m V_{(k_i, \dots, k_i)} \right).$$

By rearranging the Levi factors, one can assume the chains $\mathcal{C}_0, \dots, \mathcal{C}_m$ satisfy Equation (8). We are interested in studying

$$\begin{aligned} & \left[\text{Ind}_{\prod_{i=0}^m GL(a_i)}^{GL(n)} \left(\bigotimes_{i=0}^m V_{(k_i, \dots, k_i)} \right) : V_\tau \right] \\ &= \left[\bigotimes_{i=0}^m V_{(k_i, \dots, k_i)} : V_\tau |_{\prod_{i=0}^m GL(a_i)} \right] \\ &= \left[\bigotimes_{i=0}^m V_{(k_i+t, \dots, k_i+t)} : V_\tau |_{\prod_{i=0}^m GL(a_i)} \otimes \bigotimes_{i=0}^m V_{(t, \dots, t)} \right] \\ &= \left[\bigotimes_{i=0}^m V_{(k_i+t, \dots, k_i+t)} : V_\tau |_{\prod_{i=0}^m GL(a_i)} \otimes V_{(t, \dots, t)} |_{\prod_{i=1}^m GL(a_i)} \right] \\ &= \left[\bigotimes_{i=0}^m V_{(k_i+t, \dots, k_i+t)} : V_{\tau+(t, \dots, t)} |_{\prod_{i=0}^m GL(a_i)} \right] \\ &= \left[\text{Ind}_{\prod_{i=0}^m GL(a_i)}^{GL(n)} \left(\bigotimes_{i=0}^m V_{(k_i+t, \dots, k_i+t)} \right) : V_{\tau+(t, \dots, t)} \right] \end{aligned}$$

So we can assume $k_i > 0$ for all i without loss of generality.

We prove the theorem by induction on the number of Levi components. The theorem obviously holds when there is only one Levi component—the irreducible module is a unitary character of $GL(n)$. Now suppose we have the hypothesis holds when there are m Levi factors, i.e.,

$$\left[\text{Ind}_{\prod_{i=0}^{m-1} GL(a_i)}^{GL(n')} \left(\bigotimes_{i=0}^{m-1} V_{(k_i, \dots, k_i)} \right) : V_{\tau_{m-1}} \right] > 0,$$

where $n' = n - a_m$, and τ_{m-1} is obtained by applying Algorithm 2.2 on $\bigcup_{i=0}^{m-1} \mathcal{C}_i$. Suppose now τ_m is obtained by applying Algorithm 2.2 on $\bigcup_{i=0}^m \mathcal{C}_i$. Then

$$\begin{aligned} & \left[\text{Ind}_{\prod_{i=0}^m GL(a_i)}^{GL(n)} \left(\bigotimes_{i=0}^m V_{(k_i, \dots, k_i)} \right) : V_{\tau_m} \right] \\ &= \left[\text{Ind}_{GL(n') \times GL(a_m)}^{GL(n)} \left(\text{Ind}_{\prod_{i=0}^{m-1} GL(a_i)}^{GL(n')} \left(\bigotimes_{i=0}^{m-1} V_{(k_i, \dots, k_i)} \right) \otimes V_{(k_m, \dots, k_m)} \right) : V_{\tau_m} \right] \\ &\geq \left[\text{Ind}_{GL(n') \times GL(a_m)}^{GL(n)} (V_{\tau_{m-1}} \otimes V_{(k_m, \dots, k_m)}) : V_{\tau_m} \right] \\ &= c_{\tau_{m-1}, (k_m, \dots, k_m)}^{\tau_m} \end{aligned}$$

Here $c_{\mu, \nu}^\lambda$ is the Littlewood–Richardson coefficient, and the last step uses [Goodman and Wallach 2009, Theorem 9.2.3].

Suppose $\tau_{m-1} = \bigcup_{i=0}^{m-1} \mathcal{T}_i''$. Here these \mathcal{T}_i'' are obtained by applying Algorithm 2.2 on $\mathcal{C}_0, \dots, \mathcal{C}_{m-1}$. Then τ_m is obtained from applying Algorithm 2.2 on \mathcal{T}_i'' and $\mathcal{T}_m = (k_m, \dots, k_m)$ for all linked \mathcal{C}_i and \mathcal{C}_m . More precisely, by applying Rules (a)–(c) in Algorithm 2.2, τ_m is obtained from τ_{m-1} by the following:

- (i) Construct a new partition $\tau_{m-1} \cup (k_m, \dots, k_m)$.
- (ii) For each linked \mathcal{C}_i and \mathcal{C}_m , add $(0, \dots, 0, A, A - 1, \dots, a + 1, a, 0, \dots, 0)$ on the rows of τ_{m-1} corresponding to \mathcal{T}_i'' , and subtract $(0, \dots, 0, A, A - 1, \dots, a + 1, a, 0, \dots, 0)$ on the corresponding rows of (k_m, \dots, k_m) .
- (iii) τ_m is obtained by going through (ii) for all \mathcal{C}_i linked with \mathcal{C}_m .

By the above construction of τ_m , it follows from the Littlewood–Richardson Rule as stated in [Goodman and Wallach 2009, p. 420] that

$$(9) \quad c_{\tau_{m-1}, (k_m, \dots, k_m)}^{\tau_m} \geq 1.$$

Indeed, it suffices to find *one L-R skew tableaux* of shape τ_m/τ_{m-1} and weight

$$\underbrace{(k_m, \dots, k_m)}_{d_m}$$

in the sense of [Goodman and Wallach 2009, Definition 9.3.17]. Recall that d_m is the number of entries of the chain \mathcal{C}_m .

To do so, we first describe the Ferrers diagram τ_m/τ_{m-1} . Suppose $\mathcal{C}_{i_1}, \dots, \mathcal{C}_{i_l}$ are linked to \mathcal{C}_m with $i_1 > \dots > i_l$. By Step (ii) of the above algorithm, we add $(A_j, A_j - 1, \dots, a_j + 1, a_j)$ to the rows in τ_{m-1} corresponding to the chains \mathcal{C}_{i_j} . Note that by our ordering of the chains, we must have

$$A_l > \dots > a_l > A_{l-1} > \dots > a_{l-1} > \dots > A_1 > \dots > a_1.$$

The rows of the Ferrers diagram τ_m/τ_{m-1} have lengths

$$(10) \quad \underbrace{A_1, \dots, a_1}_{:=\mathcal{R}_1}; \cdots; \underbrace{A_l, \dots, a_l}_{:=\mathcal{R}_l};$$

$$\underbrace{k_m, \dots, k_m; (k_m - a_1), \dots, (k_m - A_1); \dots; (k_m - a_l), \dots, (k_m - A_l)}_{:=\mathcal{R}_{l+1}}$$

with $\sum_{j=1}^{l+1} |\mathcal{R}_j| = d_m$, where $|\mathcal{R}_j|$ is the number of entries in \mathcal{R}_j .

Now we fill in the entries on each row of τ_m/τ_{m-1} as follows. Consider the standard Young tableau T whose row sizes are

$$\underbrace{(k_m, \dots, k_m)}_{d_m}$$

and the entries of the i -th row of T are all equal to i . Now let a sequence of subtableaux of T given by

$$T_1 \subset T_2 \subset \cdots \subset T_l \subset T_{l+1} := T$$

such that for each $1 \leq j \leq l$, T_j has the shape of the form

$$A_j > \cdots > a_j > \cdots > A_1 > \cdots > a_1.$$

Consider the skew tableau T_j/T_{j-1} for $1 \leq j \leq l+1$ (where we take T_0 to be the empty tableau), then the column sizes of T_j/T_{j-1} is the same as the parametrization for the tableau \mathcal{R}_j marked in (10).

For each $1 \leq j \leq l+1$, fill in the rows of the Ferrers diagram τ_m/τ_{m-1} corresponding to \mathcal{R}_j in (10) by filling the t -th row of \mathcal{R}_j with the t -th entries on each column of T_j/T_{j-1} counting from the top in ascending order. This will give us a *semistandard skew tableau* of shape τ_m/τ_{m-1} and weight

$$\underbrace{(k_m, \dots, k_m)}_{d_m}$$

(see [Goodman and Wallach 2009, Definition 9.3.16]), whose row word is a *reverse lattice word* by [Goodman and Wallach 2009, Definition 9.3.17]. To sum up, it is a desired L-R tableau and (9) follows. □

Proposition 2.5. *Let $J(\lambda, -s\lambda)$ be a unitary module of $GL(n)$ in Theorem 1.1, and V_τ be the K -type obtained by Algorithm 2.2. Then τ satisfies*

$$\{\tau - \rho\} = 2\lambda - \rho.$$

Consequently, V_τ is a spin-lowest K -type of $J(\lambda, -s\lambda)$ by (7).

Proof. We prove the proposition by induction on the number of chains in $(\lambda, -s\lambda) = \bigcup_{i=0}^m C_i$, where the chains are arranged so that (8) holds. Suppose that the proposition holds for $\bigcup_{i=0}^{m-1} C_i$. There are two possibilities when adding C_m :

- There exists C_i such that C_i and C_m is related by Rule (a) in Algorithm 2.2:

$$\left\{ \begin{array}{c} C_i \\ \hline \end{array} \right\} \left\{ \begin{array}{c} C_m \\ \hline \end{array} \right\}.$$

- There exist C_j and C_r, \dots, C_{m-1} , such that C_j and C_m are related by Rule (b), and $C_l, r \leq l \leq m-1$ and C_m are related by Rule (c) in Algorithm 2.2:

$$\left\{ \begin{array}{c} C_j \\ \hline \end{array} \right\} \left\{ \begin{array}{c} C_r \\ \hline \end{array} \right\} \dots \left\{ \begin{array}{c} C_{m-1} \\ \hline \end{array} \right\} \left\{ \begin{array}{c} C_m \\ \hline \end{array} \right\}.$$

We will only study the second case, and the proof of the first case is simpler. Suppose the chains in the second case are interlaced in the following fashion:

$$(11) \quad \left\{ \begin{array}{c} C_j \\ \hline \end{array} \right\} \left\{ \begin{array}{c} \overbrace{C_r}^{d_r} \\ \hline \end{array} \right\} \dots \left\{ \begin{array}{c} \overbrace{C_{m-1}}^{d_{m-1}} \\ \hline \end{array} \right\} \\ \left\{ \underbrace{C_{m,1}, \dots}_{p} \dots \underbrace{\dots}_{a_r} \dots \underbrace{\dots}_{d_r} \dots \underbrace{\dots}_{a_{r+1}} \dots \dots \underbrace{\dots}_{d_{m-1}} \dots, \dots, \underbrace{C_{m,d_m}}_{a_m} \right\}$$

for some $j + 1 \leq r \leq m - 1$, and the chains C_{j+1}, \dots, C_{r-1} — which have not been shown in (11) — are linked with C_j under Rule (a) of Algorithm 2.2.

To simplify the calculations below, we introduce the notation

$$(a)_d^\epsilon := \underbrace{a, a + \epsilon, \dots, a + (d - 1)\epsilon}_d.$$

Then 2λ is equal to the entries in (11). Since the values of the adjacent entries within the same chain differ by 2, and the values of the interlaced entries differ by 1, one can calculate $2\lambda - \rho$ up to a translation by a constant on all coordinates as follows:

$$(12) \quad \left\{ \dots (A_{r-1})_p^0 \right\} \left\{ (A_r)_{d_r}^0 \right\} \dots \left\{ (A_{m-1})_{d_{m-1}}^0 \right\} \\ \dots \left\{ (A_{r-1})_p^0 (A_r)_{d_r}^{-1} (A_r)_{d_r}^0 (A_r)_{d_r}^{-1} \dots (A_{m-1})_{d_{m-1}}^0 (A_{m-1})_{a_m}^{-1} \right\}$$

where $A_x := \sum_{l=x}^{m-1} a_{l+1}$ for $r - 1 \leq x \leq m - 1$ (note that the smallest entry of (12) is 1, appearing at the rightmost entry of the bottom chain).

On the other hand, the calculation in Algorithm 2.2 gives τ as follows:

$$\left(\dots (k_j)_p^0 \right) (k_r)_{d_r}^0 \dots (k_{m-1})_{d_{m-1}}^0 = \bigcup_{i=0}^m \mathcal{T}_i \rightarrow \bigcup_{i=0}^m \mathcal{T}'_i = \tau,$$

$$\dots \left((k_m)_p^0 (k_m)_{a_r}^0 (k_m)_{d_r}^0 (k_m)_{a_{r+1}}^0 \dots (k_m)_{d_{m-1}}^0 (k_m)_{a_m}^0 \right)$$

where $\bigcup_{i=0}^m \mathcal{T}'_i$ is given by

$$(13) \quad \begin{aligned} & (\dots (k_j+1)_p^1) \quad (k_r+(q_r-d_r+1))_{d_r}^1 \quad \dots \quad (k_{m-1}+(q_{m-1}-d_{m-1}+1))_{d_{m-1}}^1 \\ & \dots \quad ((k_m-1)_p^{-1} (k_m)_{a_r}^0 (k_m-(q_r-d_r+1))_{d_r}^{-1} (k_m)_{a_{r+1}}^0 \dots (k_m-(q_{m-1}-d_{m-1}+1))_{d_{m-1}}^{-1} (k_m)_{a_m}^0) \end{aligned}$$

and q_i are obtained by Rule (c) of Algorithm 2.2. For instance, $q_r = p + a_r + d_r$. Note that

$$k_j - (d_j - 1) = k_r + (d_r - 1) + 2a_r + 2.$$

Therefore,

$$k_j - d_j = k_r + d_r + 2a_r.$$

From this, one deduces easily that $k_j \geq k_r + q_r + 1$. Thus it makes sense to talk about the interval $[k_r + q_r + 1, k_j]$.

Before we proceed, we pay closer attention to the coordinates of \mathcal{T}'_j , which is the left-most chain on the top row of (13). More precisely, it consists of three parts:

- (i) As mentioned in the paragraph after (11), by applying Rule (a) of Algorithm 2.2 between \mathcal{C}_j and each of $\mathcal{C}_{j+1}, \dots, \mathcal{C}_{r-1}$, one can check that

$$\bigcup_{i=j+1}^{r-1} \mathcal{T}'_i \subset [k_r + q_r + 1, k_j].$$

Suppose there are $\delta \geq 0$ coordinates in $\bigcup_{i=j+1}^{r-1} \mathcal{T}'_i$, then there will be exactly δ coordinates in \mathcal{T}'_j having coordinates strictly greater than $k_j + p$.

- (ii) By applying Algorithm 2.2 to \mathcal{C}_j and \mathcal{C}_m , we have p coordinates $(k_j + 1)_p^1$ in \mathcal{T}'_j as in (13).
 (iii) The other coordinates of \mathcal{T}'_j are either equal to k_j , or smaller than k_j if they are linked with \mathcal{C}_t with $t < j$.

In conclusion, the coordinates of \mathcal{T}'_j are given by

$$\left(\overbrace{\# \dots \#}^{\delta}; (k_j + 1)_p^1; \overbrace{\flat \dots \flat}^{d_j - \delta - p} \right),$$

where $\# \dots \#$ has coordinates greater than $k_j + p$, and $\flat \dots \flat$ has coordinates smaller than $k_j + 1$.

We now arrange the coordinates of $\bigcup_{i=j}^m \mathcal{T}'_i$ in (13) as follows:

$$\begin{aligned} & \overbrace{\# \dots \#}^{\delta} > \overbrace{(k_j + 1)_p^1}^p > \overbrace{\flat \dots \flat}^{d_j - p - \delta} > \bigcup_{i=j+1}^{r-1} \mathcal{T}'_i > \mathcal{T}'_r > \dots > \mathcal{T}'_{m-1} > (k_m)_{a_r}^0 = \dots = (k_m)_{a_m}^0 \\ & > (k_m - 1)_p^{-1} > (k_m - (q_r - d_r + 1))_{d_r}^{-1} > \dots > (k_m - (q_{m-1} - d_{m-1} + 1))_{d_{m-1}}^{-1} \end{aligned}$$

Here elements in the blocks $\mathcal{T}'_r, \dots, \mathcal{T}'_{m-1}$ are still kept in the increasing manner. Note that if $x < y$, then $\mathcal{T}'_x > \mathcal{T}'_y$ in terms of their coordinates.

We index the coordinates of τ shown in (13) using the above ordering, with the smallest coordinate indexed by 1:

$$(14) \quad (\dots (d_m + D_r + d_j - p + 1)_p^1 \ ((d_m + D_{r+1} + 1)_{d_r}^1 \ \dots \ ((d_m + 1)_{d_{m-1}}^1) \\ \left((D_r + p)_p^{-1} (D_r + p + 1)_{a_r}^1 (D_r)_{d_r}^{-1} (D_r + p + a_r + 1)_{a_{r+1}}^1 \dots (D_{m-1})_{d_{m-1}}^{-1} \left(D_r + p + \sum_{l=r}^{m-1} a_l + 1 \right)_{a_m}^1 \right),$$

where $D_x := \sum_{l=x}^{m-1} d_l$ for $r \leq x \leq m - 1$. Note that the coordinates of the last row read as

$$(D_r + p, \dots, 2, 1) = ((D_r + p)_p^{-1}; (D_r)_{d_r}^{-1}; \dots; (D_{m-1})_{d_{m-1}}^{-1}), \\ (D_r + p + 1, \dots, d_m - 1, d_m) = \\ \left((D_r + p + 1)_{a_r}^1; \dots; \left(D_r + p + \sum_{l=r}^{x-1} a_l + 1 \right)_{a_x}^1; \dots; \left(D_r + p + \sum_{l=r}^{m-1} a_l + 1 \right)_{a_m}^1 \right).$$

Up to a translation of a constant of all coordinates, the difference between (13) and (14) gives (a W -conjugate of) $\{\tau - \rho\}$, which is of the form:

$$(15) \quad (\dots (\beta_j)_p^0) \quad (\beta_r)_{d_r}^0 \quad \dots \quad (\beta_{m-1})_{d_{m-1}}^0 \\ ((\alpha_j)_p^0 \ \dots \ (\alpha_r)_{d_r}^0 \ \dots \ (\alpha_{m-1})_{d_{m-1}}^0 \ \dots)$$

Our goal is to show (12) and (15) are equal up to a translation of a constant of all coordinates. So we need to show the following:

(i) $\alpha_j = \beta_j$: We need to show

$$k_m - 1 - (D_r + p) = k_j + 1 - (d_m + D_r + d_j - p + 1).$$

In fact, we have

$$C_{m,1} = C_{j,d_j} + 2p - 1, \\ k_m + (d_m - 1) = k_j - (d_j - 1) + 2p - 1, \\ k_m - p - 1 = k_j - d_j + p - d_m, \\ k_m - 1 - (D_r + p) = k_j + 1 - (d_m + D_r + d_j - p + 1),$$

as required.

(ii) $\alpha_x = \beta_x$ for all $r \leq x \leq m - 1$: This is the same as showing

$$k_m - (q_x - d_x + 1) - D_x = k_x + (q_x - d_x + 1) - (d_m + D_{x+1} + 1).$$

As in (i), we consider

$$\begin{aligned}
C_{m,1} &= C_{x,d_x} + 2q_x - 1, \\
k_m + (d_m - 1) &= k_x - (d_x - 1) + 2q_x - 1, \\
k_m - q_x + d_x - 1 &= k_x + q_x - d_m, \\
k_m - q_x + d_x - 1 - D_x + D_{x+1} + d_x &= k_x + (q_x + 1) - (d_m + 1), \\
k_m - q_x + d_x - 1 - D_x &= k_x + (q_x - d_1 + 1) - (d_m + D_{x+1} + 1),
\end{aligned}$$

as we wish to show.

(iii) $\alpha_j - \alpha_x = A_{r-1} - A_x$ for all $r \leq x \leq m - 1$: In other words, we need to show

$$[(k_m - 1) - (D_r + p)] - [(k_m - (q_x - d_x + 1)) - D_x] = A_{r-1} - A_x = a_r + \cdots + a_x.$$

Indeed, by looking at (11) and applying Rule (c) of Algorithm 2.2, one gets

$$\begin{aligned}
p + (a_r + \cdots + a_x) + (d_r + \cdots + d_x) &= q_x, \\
q_x - p &= (A_{r-1} - A_x) + (D_r - D_{x+1}), \\
(k_m - 1) - (k_m - 1) + q_x - p - D_r + D_{x+1} &= A_{r-1} - A_x, \\
[(k_m - 1) - (D_r + p)] - (k_m - 1) + q_x + (D_x - d_x) &= A_{r-1} - A_x, \\
[(k_m - 1) - (D_r + p)] - [(k_m - (q_x - d_x + 1)) - D_x] &= A_{r-1} - A_x,
\end{aligned}$$

so the result follows.

(iv): Collecting the *** entries of (15) consecutively from left to right gives

$$\underbrace{\alpha_j, \dots, \alpha_r + 1}_{a_r}; \dots; \underbrace{\alpha_x, \dots, \alpha_{x+1} + 1}_{a_{x+1}}; \dots; \underbrace{\alpha_{m-1}, \dots, \alpha_{m-1} - (a_m - 1)}_{a_m}.$$

In order for the above expression to make sense, one needs $\alpha_x - \alpha_{x+1} = a_x$ for all $r \leq x \leq m - 1$ for instance. This is indeed the case, since $\alpha_x - \alpha_{x+1} = A_x - A_{x+1}$ by (iii), and the latter is equal to a_{x+1} by the definition of A_x for $r - 1 \leq x \leq m - 1$. So it suffices to check $k_m - (D_r + p + \sum_{l=r}^x a_l + 1) = \alpha_x$.

To see it is the case, one can check that the leftmost entry of the second row of (15) is equal to

$$\begin{aligned}
\alpha_j &= k_m - 1 - (D_r + p), \\
\alpha_x + A_{r-1} - A_x &= k_m - (D_r + p + 1), && \text{(by (iii))} \\
\alpha_x + \sum_{l=r}^x a_l &= k_m - (D_r + p + 1), \\
\alpha_x &= k_m - \left(D_r + p + \sum_{l=r}^x a_l + 1 \right),
\end{aligned}$$

as follows.

Combining (i)–(iv), (15) can be rewritten as

$$\begin{pmatrix} \cdots (\alpha_j)_p^0 & & ((\alpha_r)_{d_r}^0) & \cdots \cdots & & ((\alpha_{m-1})_{d_{m-1}}^0) \\ ((\alpha_j)_p^0 & (\alpha_j)_{a_r}^{-1} & (\alpha_r)_{d_r}^0 & (\alpha_r)_{a_{r+1}}^{-1} & \cdots \cdots & (\alpha_{m-1})_{d_{m-1}}^0 & (\alpha_{m-1})_{a_m}^{-1} \end{pmatrix},$$

whose coordinates are in descending order from left to right. So it is equal to $\{\tau - \rho\}$ up to a translation of a constant. Moreover, by comparing it with (12), we have shown that all coordinates of $2\lambda - \rho$ and $\{\tau - \rho\}$ differ by a constant (note that the other coordinates on the left of \mathcal{C}_j are taken care of by induction hypothesis). To see they are exactly equal to each other, we calculate the *true* values of A_{m-1} and α_{m-1} in $2\lambda - \rho$ and τ respectively on the entry marked by \otimes below:

$$\begin{matrix} \{\dots, *, \dots, *\} & & \{*, \dots, *\} & \cdots & & \{*, \dots, *\} \\ \{*, \dots, *; & *, \dots, *; & *, \dots, *; & *, \dots, *; & \dots & ; *, \dots, \otimes; & \underbrace{*, \dots, *}_{a_m} \end{matrix}$$

For $2\lambda - \rho$, \otimes takes the value

$$C_{m, d_m - a_m} - \rho_{a_m + 2},$$

where $\rho = (\rho_n, \dots, \rho_2, \rho_1)$ with $\rho_i = \rho_1 + (i - 1)$. So it can be simplified as

$$\begin{aligned} C_{m, d_m - a_m} - \rho_{a_m + 2} &= k_m - (d_m - 1) + 2a_m - \rho_{a_m + 2} \\ &= k_m - d_m + 1 + 2a_m - \rho_1 - (a_m + 1) \\ &= k_m - d_m + a_m - \rho_1. \end{aligned}$$

On the other hand, for $\{\tau - \rho\}$, \otimes takes the value

$$k_m - q_{m-1} - \rho_1$$

(recall that we had $\alpha_{m-1} = k_m - q_{m-1} - 1$ for \otimes in our previous calculation).

By looking at (11) and applying Rule (c) of Algorithm 2.2 again, one has $q_{m-1} = d_m - a_m$, hence $2\lambda - \rho$ and $\{\tau - \rho\}$ takes the same value on the \otimes coordinate. Since we have seen that their coordinates differ by the same constant, one can conclude that $2\lambda - \rho = \{\tau - \rho\}$. \square

Example 2.6. For the interlaced chain in Example 2.3, the translate of $2\lambda - \rho$ in (12) is equal to

$$\begin{pmatrix} \{10 - 8 & 8 - 6\} & \{6 - 4\} & \{4 - 2\} \\ \{9 - 7 & 7 - 5 & 5 - 3 & 3 - 1 & 1 - 0\} \end{pmatrix} = \begin{pmatrix} \{2 & 2\} & \{2\} & \{2\} \\ \{2 & 2 & 2 & 2 & 1\} \end{pmatrix}.$$

Also, the translate of $\tau - \rho$ in (15) is given by

$$\begin{pmatrix} (9 - 8 & 10 - 9) & (8 - 7) & (2 - 1) \\ (4 - 3 & 3 - 2 & 5 - 4 & 7 - 6 & 5 - 5) \end{pmatrix} = \begin{pmatrix} (1 & 1) & (1) & (1) \\ (1 & 1 & 1 & 1 & 0) \end{pmatrix}$$

Hence their coordinates differ by the same constant 1. To see $2\lambda - \rho$ and $\{\tau - \rho\}$ are equal, where $\rho = (4, 3, 2, 1, 0, -1, -2, -3, -4)$, one can look at the *true* values of them for the rightmost entry of the bottom chain:

$$2\lambda - \rho : 1 - \rho_1 = 1 - (-4) = 5; \quad \tau - \rho : 5 - \rho_5 = 5 - 0 = 5.$$

Hence $2\lambda - \rho = \{\tau - \rho\} = (6, 6, 6, 6, 6, 6, 6, 6, 5)$, and the unique \tilde{K} -type in the Dirac cohomology of the corresponding unitary module is $V_{(6,6,6,6,6,6,6,6,5)}$.

3. Scattered representations of $SL(n)$

It is easy to parametrize irreducible unitary representations of $SL(n)$ using the parametrization for $GL(n)$. In such cases, we impose the condition on λ such that the sum of the coordinates is equal to 0. In other words, for each possible regular, half-integral infinitesimal character λ for $SL(n)$, one can shift the coordinates by a suitable scalar, so that it corresponds to an infinitesimal character λ' of $GL(n)$ whose smallest coordinate is equal to $\frac{1}{2}$.

Therefore, the irreducible unitary representations of $SL(n)$ are parametrized by chains with n coordinates whose smallest coordinate is equal to 1.

The following proposition characterizes which of these representations are scattered in the sense of Section 1.4:

Proposition 3.1. *Let $\pi := J(\lambda, -s\lambda)$ be an irreducible unitary representation of $SL(n)$ such that λ is dominant and half-integral. Then π is a scattered representation if and only if the translated Zhelobenko parameter $(\lambda', -s\lambda')$ can be expressed as a union of interlaced chains with smallest coordinate equal to 1.*

Proof. By the arguments in Section 1.4, one only needs to check that $s \in W$ involves all simple reflections in its reduced expression if and only if $(\lambda', -s\lambda') = \bigcup_{i=0}^m C_i$ are interlaced. Indeed, $s \in W$ can be read from $\bigcup_{i=0}^m C_i$ as follows: label the entries of $\bigcup_{i=0}^m C_i$ in descending order, e.g.,

$$\bigcup_{i=0}^m C_i = \begin{matrix} & \{p_{k+1}, \dots\} \cdots \\ \{p_1, p_2, \dots, p_k, p_{k+2}, \dots\} \cdots \end{matrix}$$

with $p_1 > p_2 > \dots > p_n$, then we “flip” the entries of each chain C_i by

$$\{C_{i,1}, \dots, C_{i,d_i}\} \rightarrow \{C_{i,d_i}, \dots, C_{i,1}\}.$$

Suppose we have

$$\begin{matrix} \{p_{s_{k+1}}, \dots\} \cdots \\ \{p_{s_1}, p_{s_2}, \dots, p_{s_k}, p_{s_{k+2}}, \dots\} \cdots \end{matrix}$$

after flipping each chain, then $s \in S_n$ is obtained by

$$s = \begin{pmatrix} 1 & 2 & \dots & n \\ s_1 & s_2 & \dots & s_n \end{pmatrix}$$

(see Example 3.2).

Define the equivalence class of interlaced chains by letting $\mathcal{C}_i \sim \mathcal{C}_j$ if and only if $i = j$, or $\mathcal{C}_i, \mathcal{C}_j$ are interlaced. So we have a partition of $\{p_1, \dots, p_n\}$ by the entries of chains in the same equivalence class. It is not hard to check that the entries on each partition have consecutive indices, i.e.,

$$\mathcal{E}_i = \{p_{a_i}, p_{a_i+1}, \dots, p_{b_i-1}, p_{b_i}\}$$

and $\bigcup_{i=0}^m \mathcal{C}_i$ are interlaced if and only if there is only one equivalence class.

We now prove the proposition. Suppose there exists more than one equivalence class, i.e., we have

$$\mathcal{E}_1 = \{p_1, \dots, p_a\}; \quad \mathcal{E}_2 = \{p_{a+1}, \dots, p_b\}$$

for some $1 \leq a < n$. Since the smallest element in any equivalence class must be the smallest element of a chain, and the largest element in a class must be the largest element of a chain, we have

$$\mathcal{C}_i = \{\dots, p_a\}\{p_{a+1}, \dots\} = \mathcal{C}_j.$$

By the above description of $s \in S_n$, it is obvious that $s \in S_a \times S_{n-a} \subset S_n$, which does not involve the simple reflection s_a .

Conversely, if there is only one equivalence class, we suppose on the contrary that there exists some $1 \leq a < n$ such that $s \in S_a \times S_{n-a}$. Since p_a, p_{a+1} are in the same equivalence class, then at least one of the following:

$$\{p_a, p_{a+1}\}, \quad \{p_a, p_{a+2}\}, \quad \{p_{a-1}, p_{a+1}\}$$

is in the same chain \mathcal{C}_i for some $0 \leq i \leq m$. By “flipping” \mathcal{C}_i in either case, there must be some $u \leq a < a + 1 \leq v$ such that

$$s = \begin{pmatrix} \dots & u & \dots & v & \dots \\ \dots & v & \dots & u & \dots \end{pmatrix}.$$

The reduced expression of such s must involve the simple reflection s_a , hence we obtain a contradiction. Therefore, s must involve all simple reflections in its reduced expression. □

Example 3.2. Consider the interlaced chain with smallest coordinate 1 given in Example 2.3:

$$\begin{array}{cccccc} \{10 & 8\} & \{6\} & \{4\} & & \\ & \{9 & 7 & 5 & 3 & 1\} \end{array}$$

Its corresponding irreducible representation in $SL(9)$ has Langlands parameter $(\lambda', -s\lambda')$, where

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 9 & 1 & 8 & 5 & 6 & 7 & 4 & 2 \end{pmatrix},$$

and $\lambda' = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1]$, where $[a_1, \dots, a_m]$ is defined by

$$[a_1, \dots, a_m] := a_1\varpi_1 + \dots + a_m\varpi_m.$$

In fact, the coordinates of λ' is simply obtained by taking the difference of the neighboring coordinates of $\lambda = \frac{1}{2}(10, 9, 8, 7, 6, 5, 4, 3, 1)$.

Example 2.3 implies that the spin-lowest K -type for $J(\lambda', -s\lambda')$ in $SL(8)$ is $V_{[1,1,1,2,0,1,1,1]}$.

Example 3.3. We explore the possibilities of chains $\bigcup_{i=0}^m \mathcal{C}_i$ whose corresponding Zhelobenko parameter $(\lambda', -s\lambda')$ gives a spherical representation.

In order for the lowest K -type to be trivial, we need the \mathcal{T}_i in Algorithm 2.2 to have the same average value k_i for all i , that is, the mid-point of all \mathcal{C}_i (if there is more than one) must be the same. This leaves the possibility of $\bigcup_{i=0}^m \mathcal{C}_i$ consisting of a single chain, which corresponds to the trivial representation, or there are two chains of lengths $a > b > 0$ whose entries are of different parity. Hence it must be of the form

$$\{2a - 1, 2a - 3, \dots, 3, 1\} \cup \{a + (b - 1), a + (b - 3), \dots, a - (b - 3), a - (b - 1)\},$$

where a, b are of different parity.

In other words, such representations can only occur for $SL(n)$ with $n = a + b$ is odd, and is equal to $\text{Ind}_{S(GL(a) \times GL(b))}^{SL(n)}(\text{triv} \otimes \text{triv})$, which are the unipotent representations corresponding to the nilpotent orbit with Jordan block $(2^b 1^{a-b})$ (see [Barbasch and Pandžić 2011, §5.3]). Its Langlands parameter $(\lambda', -s\lambda')$ has

$$2\lambda' = [\underbrace{2, \dots, 2}_{(a-b-1)/2}, \underbrace{1, \dots, 1}_{2b}, \underbrace{2, \dots, 2}_{(a-b-1)/2}]$$

and $s = w_0$ (see [Ding and Dong 2020, Conjecture 5.6]). Moreover, its spin-lowest K -type is given by [Barbasch and Pandžić 2011, Equation (5.5)], which matches with our calculations in Algorithm 2.2.

For the rest of this section, we give two applications of Proposition 3.1:

3.1. The spin-lowest K -type is unitarily small. To offer a unified conjectural description of the unitary dual, Salamanca-Riba and Vogan [1998] formulated the notion of unitarily small (*u-small* for short) K -type. Here we only quote them for a complex connected simple Lie group G — using the setting in the introduction,

a K -type V_δ is u -small if and only if $\langle \delta - 2\rho, \varpi_i \rangle \leq 0$ for $1 \leq i \leq \text{rank}(\mathfrak{g}_0)$ (see [Salamanca-Riba and Vogan 1998, Theorem 6.7]).

Lemma 3.4. *Let $\lambda = \sum_{i=1}^{\text{rank}(\mathfrak{g}_0)} \lambda_i \varpi_i \in \mathfrak{h}_0^*$ be a dominant weight such that $\lambda_i = \frac{1}{2}$ or 1 for each $1 \leq i \leq n$, and V_δ be the K -type with highest weight δ such that*

$$\{\delta - \rho\} = 2\lambda - \rho.$$

Then $\langle \delta - 2\rho, \varpi_i \rangle \leq 0$, $1 \leq i \leq \text{rank}(\mathfrak{g}_0)$. Therefore, the K -type V_δ is u -small.

Proof. By assumption, there exists $w \in W$ such that $\delta = w^{-1}(2\lambda - \rho) + \rho$. Thus

$$\begin{aligned} \langle \delta - 2\rho, \varpi_i \rangle &= \langle w^{-1}(2\lambda - \rho) - \rho, \varpi_i \rangle \\ &= \langle w^{-1}(2\lambda - \rho), \varpi_i \rangle - \langle \rho, \varpi_i \rangle \\ &= \langle 2\lambda - \rho, w(\varpi_i) \rangle - \langle \rho, \varpi_i \rangle. \end{aligned}$$

On the other hand, let $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_p}$ be a reduced decomposition of w into simple root reflections. Then by [Dong and Huang 2011, Lemma 5.5],

$$(16) \quad \varpi_i - w(\varpi_i) = \sum_{k=1}^p \langle \varpi_i, \beta_k^\vee \rangle s_{\beta_1} s_{\beta_2} \cdots s_{\beta_{k-1}}(\beta_k).$$

Note that $s_{\beta_1} s_{\beta_2} \cdots s_{\beta_{k-1}}(\beta_k)$ is a positive root for each k . Now we have that

$$\begin{aligned} \langle \delta - 2\rho, \varpi_i \rangle &= \left\langle 2\lambda - \rho, \varpi_i - \sum_{k=1}^p \langle \varpi_i, \beta_k^\vee \rangle s_{\beta_1} s_{\beta_2} \cdots s_{\beta_{k-1}}(\beta_k) \right\rangle - \langle \rho, \varpi_i \rangle \\ &= 2\langle \lambda - \rho, \varpi_i \rangle - \sum_{k=1}^p \langle \varpi_i, \beta_k^\vee \rangle \langle 2\lambda - \rho, s_{\beta_1} s_{\beta_2} \cdots s_{\beta_{k-1}}(\beta_k) \rangle \\ &\leq 2\langle \lambda - \rho, \varpi_i \rangle \\ &\leq 0. \end{aligned} \quad \square$$

Corollary 3.5. *The unique spin-lowest K -type V_τ of any scattered representation of $SL(n)$ is u -small. Consequently, [Ding and Dong 2020, Conjecture C] holds for $SL(n)$.*

Proof. Let $(\lambda, -s\lambda)$ be the Zhelobenko parameter for a scattered representation of $SL(n)$. Write $\lambda = \sum_{i=1}^{n-1} \lambda_i \varpi_i$ in terms of the fundamental weights. Then it is direct from our definition of the interlaced chains that each λ_i is either $\frac{1}{2}$ or 1 (recall Proposition 3.1 and Example 3.2). Let V_τ be the unique spin-lowest K -type of the scattered representation. Then $\{\tau - \rho\} = 2\lambda - \rho$ (see Proposition 2.5). Thus the result follows from Lemma 3.4. \square

3.2. Number of scattered representations. As another application of Proposition 3.1, we compute the number of scattered representations of $SL(n)$. By the proposition, it is equal to the number of interlaced chains with n entries with the smallest entry equal to 1. We now give an algorithm of constructing new interlaced chains with smallest coordinate equal to 1 from those with one less coordinate:

Algorithm 3.6. Let $\bigcup_{i=1}^p \{2A_i - 1, \dots, 2a_i - 1\} \cup \bigcup_{j=1}^q \{2B_j, \dots, 2b_j\}$ be a union of interlaced chains with such that

- $A_{i'} > A_i$ if $i' > i$, and $B_{j'} > B_j$ if $j' > j$; and
- $2a_p - 1 = 1$.

We construct two new interlaced chains with one extra coordinate as follows. (When $q = 0$, we adopt Case I only.)

Case I: If $2A_p - 1 > 2B_q + 1$, then the two new interlaced chains are

$$\begin{array}{ccccccc} & & & & \{2B_q & \dots & 2b_q\} & \dots \\ \{2A_p + 1 & 2A_p - 1 & \dots & 2a_p - 1\} & \dots & & & \end{array}$$

and

$$\begin{array}{ccccccc} & & & & \{2A_p - 2\} & & \{2B_q & \dots & 2b_q\} & \dots \\ \{2A_p - 1 & \dots & \dots & 2a_p - 1\} & \dots & & & & & \end{array}$$

Case II: If $2A_p - 1 = 2B_q + 1$, then the two new interlaced chains are

$$\begin{array}{ccccccc} & & & & \{2B_q & \dots & 2b_q\} & \dots \\ \{2A_p + 1 & 2A_p - 1 & \dots & 2a_p - 1\} & \dots & & & \end{array}$$

and

$$\begin{array}{ccccccc} \{2B_q + 2 & 2B_q & \dots & 2b_q\} & \dots & & & \\ \{2A_p - 1 & \dots & 2a_p - 1\} & \dots & & & & \end{array}$$

Case III: If $2A_p - 1 = 2B_q - 1$, then the two new interlaced chains are

$$\begin{array}{ccccccc} & & & & \{2B_q & \dots & 2b_q\} & \dots \\ \{2A_p + 1 & 2A_p - 1 & \dots & 2a_p - 1\} & \dots & & & \end{array}$$

and

$$\begin{array}{ccccccc} \{2B_q + 2 & 2B_q & \dots & 2b_q\} & \dots & & & \\ \{2A_p - 1 & \dots & 2a_p - 1\} & \dots & & & & \end{array}$$

Case IV: If $2A_p - 1 < 2B_q - 1$, then the two new interlaced chains are

$$\begin{array}{ccccccc} \{2B_q & \dots & \dots & 2B_q\} & \dots & & & \\ \{2B_q - 1\} & \{2A_p - 1 & \dots & 2a_p - 1\} & \dots & & & \end{array}$$

and

$$\{2B_q + 2 \quad 2B_q \quad \dots \quad 2b_p\} \quad \dots$$

$$\{2A_p - 1 \quad \dots \quad 2a_p - 1\} \quad \dots$$

Example 3.7. Suppose we begin with an interlaced chain $\{9, 7, 5, 3, 1\} \cup \{4, 2\}$. Then the new interlaced chains with one extra coordinate are

$$\{11, 9, 7, 5, 3, 1\} \cup \{4, 2\} \quad \text{and} \quad \{9, 7, 5, 3, 1\} \cup \{8\} \cup \{4, 2\}.$$

Proposition 3.8. *All interlaced chains with $n \geq 2$ entries with smallest coordinate equal to 1 can be obtained uniquely from the chain $\{3 \ 1\}$ by inductively applying the above algorithm.*

Proof. Suppose $\bigcup_{i=0}^m C_i$ be interlaced chains with largest coordinate equal to $M \in C_0$. We remove a coordinate from it by the following rule: If $C_i \neq \{M - 1\}$ for all i , remove the entry M from C_0 . Otherwise, remove the whole chain $\{M - 1\}$ from the original interlaced chains.

One can easily check from the definition of interlaced chain that the reduced chains are still interlaced, and one can recover the original chain by applying Algorithm 3.6 on the reduced chain.

Therefore, for all interlaced chains with smallest entry 1, we can use the reduction mentioned in the first paragraph repeatedly to get an interlaced chain with only 2 entries, which must be of the form $\{3 \ 1\}$, and repeated applications of Algorithm 3.6 on $\{3 \ 1\}$ will retrieve the original interlaced chains (along with other chains). In other words, all interlaced chains with smallest entry 1 can be obtained by Algorithm 3.6 inductively on $\{3 \ 1\}$.

We are left to show that all interlaced chains are uniquely constructed using the algorithm. Suppose on the contrary that there are two different interlaced chains that give rise to the same $\bigcup_{i=0}^m C_i$ after applying Algorithm 3.6. By the algorithm, these two chains must be obtained from $\bigcup_{i=0}^m C_i$ by removing its largest odd entry $M_o \in C_p$ or largest even entry $M_e \in C_q$. So they must be equal to

$$\bigcup_{i \neq p} C_i \cup (C_p \setminus \{M_o\}) \quad \text{and} \quad \bigcup_{i \neq q} C_i \cup (C_q \setminus \{M_e\}),$$

respectively.

Assume $M_o > M_e$ for now (and the proof for $M_e > M_o$ is similar). By applying Algorithm 3.6 to $\bigcup_{i \neq q} C_i \cup (C_q \setminus \{M_e\})$, we obtain two interlaced chains

$$\bigcup_{i \neq p,q} C_i \cup C'_p \cup (C_q \setminus \{M_e\}) \quad \text{and} \quad \bigcup_{i \neq q} C_i \cup (C_q \setminus \{M_e\}) \cup \{M_o - 1\},$$

where

$$C'_p := \{M_o + 2, \overbrace{M_o, \dots, m_o}^{C_p}\}.$$

Note that none of the above gives rise to the interlaced chains $\bigcup_{i=0}^m C_i$: Even in the case when $M_0 - 1 = M_e$, $(C_q \setminus \{M_e\}) \cup \{M_0 - 1\}$ and C_q are different — although they have the same coordinates, the first consists of two chains while the second consists of one chain only. So we have a contradiction, and the result follows. \square

Corollary 3.9. *The number of interlaced chains with n coordinates and the smallest coordinate equal to 1 is equal to 2^{n-2} .*

Since the scattered representations of $SL(n + 1)$ are in one to one correspondence with interlaced chains with $n + 1$ coordinates having smallest coordinate 1, this corollary implies that the number of scattered representations of Type A_n is equal to 2^{n-1} . This verifies [Dong 2019, Conjecture 5.2]. Moreover, by using `atlas`, the spin-lowest K -types for all scattered representations of $SL(n)$ with $n \leq 6$ are given in [Dong 2019, Tables 1–3]. One can easily check the results there match with our V_τ in Algorithm 2.2.

Example 3.10. Let us start from $SL(2, \mathbb{C})$ and the chain $\{3\ 1\}$. This chain corresponds to the trivial representation.

Now we consider $SL(3, \mathbb{C})$. By Algorithm 3.6, the chain $\{3\ 1\}$ for $SL(2)$ produces two chains

$$\{5\ 3\ 1\} \quad \begin{matrix} \{2\} \\ \{3\ \ 1\} \end{matrix}.$$

The first corresponds to the trivial representation, while the second gives the representation with $\lambda = [\frac{1}{2}, \frac{1}{2}]$ and

$$s = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

One computes by Algorithm 2.2 that the spin-lowest K -type $\tau = [1, 1]$.

Now let us consider $SL(4)$. By Algorithm 3.6, the chain $\{5\ 3\ 1\}$ for $SL(3)$ produces two chains

$$\{7\ 5\ 3\ 1\} \quad \begin{matrix} \{4\} \\ \{5\ \ 3\ 1\} \end{matrix}.$$

The first chain corresponds to the trivial representation, while the second one gives the representation with $\lambda = [\frac{1}{2}, \frac{1}{2}, 1]$ and

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}.$$

One computes by Algorithm 2.2 that the spin-lowest K -type $\tau = [2, 0, 1]$. The other chain of $SL(3)$ shall produce

$$\begin{matrix} \{2\} & \{4\ \ 2\} \\ \{5\ \ 3\ 1\} & \{3\ \ 1\} \end{matrix}$$

One computes that

$$\lambda = [1, \frac{1}{2}, \frac{1}{2}], \quad s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}, \quad \tau = [1, 0, 2];$$

and that

$$\lambda = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}], \quad s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \tau = [1, 1, 1],$$

respectively. These four representations (and their spin-lowest K -types) match precisely with [Dong 2019, Table 1].

Acknowledgements

We thank the referee sincerely for very careful reading and nice suggestions.

Dong was supported by the National Natural Science Foundation of China (grant 11571097, 2016–2019). Wong is supported by the National Natural Science Foundation of China (grant 11901491) and the Presidential Fund of CUHK(SZ).

References

- [Barbasch and Pandžić 2011] D. Barbasch and P. Pandžić, “Dirac cohomology and unipotent representations of complex groups”, pp. 1–22 in *Noncommutative geometry and global analysis*, edited by A. Connes et al., Contemp. Math. **546**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl
- [Barbasch et al. 2020] D. Barbasch, C.-P. Dong, and K. D. Wong, “Dirac series for complex classical Lie groups”, preprint, 2020. arXiv
- [Ding and Dong 2020] J. Ding and C.-P. Dong, “Unitary representations with Dirac cohomology: a finiteness result for complex Lie groups”, *Forum Math.* **32**:4 (2020), 941–964. MR Zbl
- [Dong 2013] C.-P. Dong, “On the Dirac cohomology of complex Lie group representations”, *Transform. Groups* **18**:1 (2013), 61–79. Erratum in **18**:2 (2013), 595–597. MR Zbl
- [Dong 2019] C.-P. Dong, “Unitary representations with non-zero Dirac cohomology for complex E_6 ”, *Forum Math.* **31**:1 (2019), 69–82. MR Zbl
- [Dong and Huang 2011] C.-P. Dong and J.-S. Huang, “Jacquet modules and Dirac cohomology”, *Adv. Math.* **226**:4 (2011), 2911–2934. MR Zbl
- [Goodman and Wallach 2009] R. Goodman and N. R. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics **255**, Springer, 2009. MR Zbl
- [Huang and Pandžić 2002] J.-S. Huang and P. Pandžić, “Dirac cohomology, unitary representations and a proof of a conjecture of Vogan”, *J. Amer. Math. Soc.* **15**:1 (2002), 185–202. MR Zbl
- [Huang and Pandžić 2006] J.-S. Huang and P. Pandžić, *Dirac operators in representation theory*, Mathematics: Theory and Applications, Birkhäuser, Boston, 2006. MR Zbl
- [Huang et al. 2009] J.-S. Huang, Y.-F. Kang, and P. Pandžić, “Dirac cohomology of some Harish-Chandra modules”, *Transform. Groups* **14**:1 (2009), 163–173. MR Zbl
- [Huang et al. 2011] J.-S. Huang, P. Pandžić, and V. Protsak, “Dirac cohomology of Wallach representations”, *Pacific J. Math.* **250**:1 (2011), 163–190. MR Zbl
- [Parthasarathy 1972] R. Parthasarathy, “Dirac operator and the discrete series”, *Ann. of Math. (2)* **96** (1972), 1–30. MR Zbl

- [Parthasarathy 1980] R. Parthasarathy, “Criteria for the unitarizability of some highest weight modules”, *Proc. Indian Acad. Sci. Sect. A Math. Sci.* **89**:1 (1980), 1–24. MR Zbl
- [Parthasarathy et al. 1967] K. R. Parthasarathy, R. Ranga Rao, and V. S. Varadarajan, “Representations of complex semi-simple Lie groups and Lie algebras”, *Ann. of Math. (2)* **85** (1967), 383–429. MR Zbl
- [Salamanca-Riba and Vogan 1998] S. A. Salamanca-Riba and D. A. Vogan, Jr., “On the classification of unitary representations of reductive Lie groups”, *Ann. of Math. (2)* **148**:3 (1998), 1067–1133. MR Zbl
- [Vogan 1986] D. A. Vogan, Jr., “The unitary dual of $GL(n)$ over an Archimedean field”, *Invent. Math.* **83**:3 (1986), 449–505. MR Zbl
- [Vogan 1997] J. Vogan, D. A., “Dirac operators and unitary representations”, 3 talks at MIT Lie groups seminar, MIT, 1997.
- [Zhelobenko 1974] D. P. Zhelobenko, *Harmonic analysis on complex semisimple Lie groups*, Izdat. “Nauka”, Moscow, 1974. MR

Received October 23, 2019. Revised September 19, 2020.

CHAO-PING DONG
SCHOOL OF MATHEMATICAL SCIENCES
SOOCHOW UNIVERSITY
SUZHOU
CHINA
chaopindong@163.com

KAYUE DANIEL WONG
SCHOOL OF SCIENCE AND ENGINEERING
THE CHINESE UNIVERSITY OF HONG KONG, SHENZHEN
SHENZHEN
CHINA
kayue.wong@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Matthias Aschenbrenner
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
matthias@math.ucla.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2020 is US \$520/year for the electronic version, and \$705/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2020 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 309 No. 2 December 2020

Thin subgroups isomorphic to Gromov–Piatetski-Shapiro lattices	257
SAMUEL A. BALLAS	
Value distribution properties for the Gauss maps of the immersed harmonic surfaces	267
XINGDI CHEN, ZHIXUE LIU and MIN RU	
Scattered representations of $SL(n, \mathbb{C})$	289
CHAO-PING DONG and KAYUE DANIEL WONG	
Number of singular fibres of surface fibrations over \mathbb{P}^1	313
CHENG GONG and YI GU	
Asymptotic behavior of solutions for some elliptic equations in exterior domains	333
ZONGMING GUO and ZHONGYUAN LIU	
Ulrich elements in normal simplicial affine semigroups	353
JÜRGEN HERZOG, RAHELEH JAFARI and DUMITRU I. STAMATE	
Notes on equivariant homology with constant coefficients	381
SOPHIE KRIZ	
Local normal forms for multiplicity free $U(n)$ actions on coadjoint orbits	401
JEREMY LANE	
Functional determinant on pseudo-Einstein 3-manifolds	421
ALI MAALAOU	
Distribution of distances in positive characteristic	437
THANG PHAM and LÊ ANH VINH	
Elliptic gradient estimates for a parabolic equation with V -Laplacian and applications	453
JIAN-HONG WANG and YU ZHENG	
Optimal L^2 extension of sections from subvarieties in weakly pseudoconvex manifolds	475
XIANGYU ZHOU and LANGFENG ZHU	



0030-8730(202012)309:2;1-H