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Let $H \subseteq \mathbb{N}^d$ be a normal affine semigroup, $R = K[H]$ its semigroup ring over the field K and ω_R its canonical module. The Ulrich elements for H are those h in H such that for the multiplication map by x^h from R into ω_R, the cokernel is an Ulrich module. We say that the ring R is almost Gorenstein if Ulrich elements exist in H. For the class of slim semigroups that we introduce, we provide an algebraic criterion for testing the Ulrich property. When $d = 2$, all normal affine semigroups are slim. Here we have a simpler combinatorial description of the Ulrich property. We improve this result for testing the elements in H which are closest to zero. In particular, we give a simple arithmetic criterion for when is $(1, 1)$ an Ulrich element in H.

Introduction

Let H be an affine semigroup in $\mathbb{N}^d$ and $K[H]$ its semigroup ring over the field K. In this paper we investigate the almost Gorenstein property for $K[H]$ taking into account the natural multigraded structure of this ring, under the assumption that H is normal and simplicial.

The almost Gorenstein property appeared in [Barucci and Fröberg 1997] in the context of 1-dimensional analytical unramified rings. It was extended to 1-dimensional local rings by Goto, Matsuoka and Thi Phuong [Goto et al. 2013], and later on to rings of higher dimension by Goto, Takahashi and Taniguchi [Goto et al. 2015]. Let R be a positively graded Cohen–Macaulay K-algebra with canonical module ω_R. We let $a = -\min\{k \in \mathbb{Z} : (\omega_R)_k \neq 0\}$, which is also known as the a-invariant of R. In [Goto et al. 2015], R is called (graded) almost Gorenstein (AG for short) if there exists an exact sequence of graded R-modules

$$0 \to R \to \omega_R(-a) \to E \to 0, \qquad (1)$$

where E is an Ulrich module, i.e., E is a Cohen–Macaulay graded module which is minimally generated by $e(E)$ elements. Here $e(E)$ denotes the multiplicity of E with respect to the graded maximal ideal in R.

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Let $H \subseteq \mathbb{N}^d$ be an affine semigroup whose associated group is $\mathrm{gr}(H) = \mathbb{Z}^d$. We denote C the cone over H. Assume H is normal, i.e., $H = C \cap \mathbb{Z}^d$, equivalently, the ring R is normal. Then R is a Cohen–Macaulay ring [Hochster 1972] and a K-basis for the canonical module ω_R is given by the monomials with exponents in the relative interior of the cone C [Danilov 1978; Stanley 1978], i.e., in the set $\omega_H = \mathbb{Z}^d \cap \operatorname{relint} C$. In the multigraded setting that we want to consider here, there does not seem to be any distinguished element in ω_H to replace the a-invariant in the short exact sequence (1). In this sense, we propose the following.

Definition 3.1. For $\boldsymbol{b} \in \omega_H$ consider the exact sequence

$$0 \to R \to \omega_R(\boldsymbol{b}) \to E \to 0, \tag{2}$$

where $1 \in R$ is mapped to $u = \boldsymbol{x}^{\boldsymbol{b}}$ and $E = \omega_R/uR$. Then $\boldsymbol{b}$ is called an *Ulrich element* in H, if E is an Ulrich R-module.

If H admits an Ulrich element $\boldsymbol{b}$, then we call the ring $R = K[H]$ *almost Gorenstein with respect to* $\boldsymbol{b}$, or simply AG if H has an Ulrich element.

The Gorenstein property has attracted a lot of interest due to its multifaceted algebraic and homological descriptions. For rings with a combinatorial structure behind, there are often nice characterizations of the Gorenstein property. Scratching only at the surface, we mention that Gorenstein toric rings were characterized by Hibi [1992], and for special subclasses the results are more precise; see [De Negri and Hibi 1997; Hibi 1987; Hibi et al. 2019; Dinu 2020].

The almost Gorenstein property was characterized for determinantal rings in [Taniguchi 2018], numerical semigroup rings in [Nari 2013] and Hibi rings in [Miyazaki 2018].

In this paper we investigate the Ulrich elements in H under the assumption that the normal affine semigroup $H \subset \mathbb{N}^d$ is also simplicial, i.e., the cone C over H has $d = \dim_{\mathbb{R}} \operatorname{aff}(H)$ extremal rays. That will be assumed for the rest of this introduction, too.

Next we outline the main results. We denote $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$ the primitive integer vectors in H situated on each extremal ray of the cone C, respectively, and we call them the extremal rays of H. They are part of the Hilbert basis of H, denoted B_H, which is the unique minimal generating set of H.

When H is normal and simplicial it is known that the monomials $\boldsymbol{x}^{\boldsymbol{a}_1}, \dots, \boldsymbol{x}^{\boldsymbol{a}_d}$ form a maximal regular sequence on R. A first result that we prove in Proposition 2.2 is that for any $\boldsymbol{b} \neq 0$ in H the sequence $\boldsymbol{x}^{\boldsymbol{b}}, \boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_2}, \dots, \boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_d}$ is regular, as well. Let $J = (\boldsymbol{x}^{\boldsymbol{a}_i} - \boldsymbol{x}^{\boldsymbol{a}_j} : 1 \leq i, j \leq d)R$. The next technical result is vital in our study of Ulrich elements. Namely, in Theorem 2.4, we show that J is a reduction ideal of $\mathfrak{m}$ modulo the ideal $\boldsymbol{x}^{\boldsymbol{b}}R$, if and only if for any $\boldsymbol{c} \in B_H \setminus \omega_H$ the sum of the coordinates of $\boldsymbol{c}$ with respect to the basis $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d$ is at least one. The normal and simplicial semigroups with that property are to be called *slim*.

In this notation, we provide the following characterization.

Theorem 3.2. *Let H be a slim semigroup and $\boldsymbol{b} \in \omega_H$. Then $\boldsymbol{b}$ is an Ulrich element in H if and only if $\mathfrak{m}\omega_R \subseteq (\boldsymbol{x}^{\boldsymbol{b}} R, J\omega_R)$.*

This result allows to produce first examples of semigroups with Ulrich elements; see Examples 3.5, 3.4.

In the next sections we focus on making more explicit the AG property in dimension two. Let H be any normal affine semigroup $H \subseteq \mathbb{N}^2$. It is automatically slim since it is simplicial and $B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\} \subseteq \omega_H$ (see Lemma 1.1). We denote by $\boldsymbol{a}_1, \boldsymbol{a}_2$ its extremal rays. In Theorem 4.3 we prove that any element $\boldsymbol{b} \in B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is an Ulrich element in H if and only if for all $\boldsymbol{c}_1, \boldsymbol{c}_2$ in B_H one has

$$\boldsymbol{c}_1 + \boldsymbol{c}_2 \in (\boldsymbol{a}_1 + H) \cup (\boldsymbol{a}_2 + H) \cup (\boldsymbol{b} + H).$$

Equivalently, if for all $\boldsymbol{c}_1, \boldsymbol{c}_2 \in B_H$ so that $\boldsymbol{c}_1, \boldsymbol{c}_2 \in P_H$ it follows that $\boldsymbol{c}_1 + \boldsymbol{c}_2 \in \boldsymbol{b} + H$. Here $P_H = \{\lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2 : 0 \leq \lambda_1, \lambda_2 < 1\}$ is the fundamental parallelogram of H.

Based on this result, in Section 4 we find examples with zero, one, or several Ulrich elements in B_H.

We prove in Lemma 5.1 that for any $H \subseteq \mathbb{N}^2$ as above, the semigroup ideal ω_H has a unique minimal element with respect to the componentwise partial order on $\mathbb{N}^2$. We call it the bottom element of H. This definition naturally extends to higher embedding dimension, but when $d > 2$ not all normal semigroups in $\mathbb{N}^d$ have a bottom element.

However, bottom elements, when available, are good candidates to check against the Ulrich property. We prove that when $H \subseteq \mathbb{N}^d$ is a slim semigroup such that

- (Proposition 3.6) the nonzero elements in H have all the entries positive and $\boldsymbol{b} = (1, 1, \ldots, 1) \in \omega_H$, or
- (Proposition 5.5) $d = 2$ and $\boldsymbol{b}$ the bottom element in H satisfies $2\boldsymbol{b} \in P_H$,

then $\boldsymbol{b}$ is the only possible Ulrich element in H.

These results motivate us to find more direct criteria for testing the Ulrich property of the bottom element. Our attempt is successful when $d = 2$.

In the following, H is a normal affine semigroup in $\mathbb{N}^2$ with the extremal rays $\boldsymbol{a}_1 = (x_1, y_1)$ and $\boldsymbol{a}_2 = (x_2, y_2)$ with $\boldsymbol{a}_1$ closer to the x-axis than $\boldsymbol{a}_2$. Considering $\boldsymbol{b} = (u, v)$ the bottom element of H, for $i = 1, 2$ we define H_i to be the normal semigroup with the extremal rays $\boldsymbol{b}$ and $\boldsymbol{a}_i$. We denote $H_i^* = \operatorname{relint} P_{H_i} \cap \mathbb{Z}^2$ for $i = 1, 2$. We show:

Lemma 5.12. *For $\boldsymbol{b}$ the bottom element in H the following are equivalent*:

(a) *$\boldsymbol{b}$ is an Ulrich element in H.*

(b) *For $i = 1, 2$, if $\boldsymbol{p}, \boldsymbol{q} \in H_i^*$ then $\boldsymbol{p} + \boldsymbol{q} \notin H_i^*$.*

We shall say that H is AG1 if point (b) above is satisfied for $i = 1$ and we call it AG2 if it holds for $i = 2$.

Thus the bottom element is an Ulrich element in H if and only if H is AG1 and AG2. This calls for a better understanding of the points in H_1^* and H_2^*. Lemma 5.16 shows that H_i^* has $|vx_i - uy_i| - 1$ elements, for $i = 1, 2$. An immediate consequence of independent interest is the following Gorenstein criterion.

Corollary 5.17. *With notation as above, the ring $K[H]$ is Gorenstein if and only if $vx_1 - uy_1 = uy_2 - vx_2 = 1$.*

The x-coordinates of points in H_1^* are distinct integers in the interval (u, x_1). Moreover, if for any integer i we consider the integers q_i, r_i so that $iy_1 = q_i x_1 + r_i$ with $0 \le r_i < x_1$ then any integer $k \in (u, x_1)$ is the x-coordinate of some $\boldsymbol{p} \in H_1^*$ (i.e., $k \in \pi_1(H_1^*)$) if and only if $q_k = v - 1 + q_{k-u}$, or equivalently, if $r_k \ge x_1 - (vx_1 - uy_1)$. In that case, $\boldsymbol{p} = (k, q_k + 1)$. These observations (detailed in Lemma 5.18) allow us to test the AG1 property as follows.

Proposition 5.20. *The semigroup H is AG1 if and only if $r_k + r_\ell < 2x_1 - (vx_1 - uy_1)$ for all integers $k, \ell \in \pi_1(H_1^*)$ with $k + \ell < x_1$.*

When the bottom element is $(1, 1)$ (i.e., $y_1 < x_1$ and $x_2 < y_2$) we can describe recursively the points in H_1^*.

Lemma 6.1. *Assume $(1, 1) \in \omega_H$ and $H_1^* \neq \varnothing$. Let $n = |H_1^*| = x_1 - y_1 - 1$. Recursively, we define nonnegative integers $\ell_1, \dots, \ell_n$ and $s_1, \dots, s_n$ by*

$$x_1 = \ell_1(x_1 - y_1) + s_1, \quad \textit{with} \quad s_1 < x_1 - y_1,$$

and

$$y_1 + s_{i-1} = \ell_i(x_1 - y_1) + s_i \quad \textit{with} \quad s_i < x_1 - y_1,$$

for $i = 2, \dots, n$. Then

$$H_1^* = \left\{ \boldsymbol{p}_t = (c_t, d_t) : c_t = t + \sum_{i=1}^{t} \ell_i,\ d_t = \sum_{i=1}^{t} \ell_i,\ t = 1, \dots, n \right\}.$$

A similar description is available for points in H_2^*. A little bit more effort is necessary to obtain the following arithmetic criterion for the Ulrich property of $(1, 1)$. The effort is compensated with the simplicity of the statement.

Theorem 6.3. *Assume $(1, 1) \in \omega_H$. Then $(1, 1)$ is an Ulrich element in H if and only if $x_i \equiv 1 \bmod (x_i - y_i)$ for $i = 1, 2$.*

Consequently, by Corollary 6.4, if $x_1y_1x_2y_2 \neq 0$ the ring $K[H]$ is AG if and only if $x_i \equiv 1 \bmod (x_i - y_i)$ for $i = 1, 2$.

In Section 7 we discuss another extension of the Gorenstein property for affine semigroup rings. According to the definition proposed in [Herzog et al. 2019], any Cohen–Macaulay ring $K[H]$ is called nearly Gorenstein if the trace ideal $\operatorname{tr}(\omega_{K[H]})$ contains the graded maximal ideal of $K[H]$. For one-dimensional rings, the almost Gorenstein property implies the nearly Gorenstein property, but for rings of larger dimension there is no implication between these two properties. We prove in Theorem 7.1 that when H is a normal semigroup in $\mathbb{N}^2$ the ring $K[H]$ is nearly Gorenstein. Example 7.2 shows that the statement is not valid in higher embedding dimensions.

1. Background on affine semigroups and their toric rings

In this paper all semigroups considered are fully embedded, i.e., when writing $H \subseteq \mathbb{N}^d$ we shall implicitly assume that the group generated by H is $\operatorname{gr}(H) = \mathbb{Z}^d$. A subset $H \subseteq \mathbb{N}^d$ is called an affine semigroup if there exist $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r \in H$ such that $H = \sum_{i=1}^{r} \mathbb{N}\boldsymbol{c}_i$. Moreover, H is called a normal semigroup if for all $\boldsymbol{h}$ in $\mathbb{N}^d$ and n positive integer, $n\boldsymbol{h} \in H$ implies that $\boldsymbol{h} \in H$.

Let K be any field and $H = \sum_{i=1}^{r} \mathbb{N}\boldsymbol{c}_i \subseteq \mathbb{N}^d$. The semigroup ring $K[H]$ is the subalgebra of the polynomial ring $K[x_1, \ldots, x_d]$ generated by the monomials with exponents in H. Then H is normal if and only if the semigroup ring $K[H]$ is integrally closed in its field of fractions [Bruns and Gubeladze 2009]. The normality for H is also equivalent to the fact that H contains all the lattice points of the rational polyhedral cone C that it generates, i.e., $H = C \cap \mathbb{Z}^d$, where

$$C = \left\{ \sum_{i=1}^{r} \lambda_i \boldsymbol{c}_i : \lambda_i \in \mathbb{R}_{\geq 0}, \text{ for } i = 1, \ldots, r \right\}.$$

The dimension (or rank) of H is defined as the dimension of $\operatorname{aff}(H)$, the affine subspace it generates. The latter is the same as $\operatorname{aff}(C)$.

Let $\langle \cdot, \cdot \rangle$ denote the usual scalar product in $\mathbb{R}^d$. Given $\boldsymbol{n} \in \mathbb{R}^d \setminus \{0\}$, the hyperplane $H_{\boldsymbol{n}} = \{\boldsymbol{z} \in \mathbb{R}^d : \langle \boldsymbol{z}, \boldsymbol{n} \rangle = 0\}$ is called a *support hyperplane* for C if $\langle \boldsymbol{z}, \boldsymbol{n} \rangle \geq 0$ for all $\boldsymbol{z} \in C$ and $H_{\boldsymbol{n}} \cap C \neq \varnothing$. In this case, the cone $H_{\boldsymbol{n}} \cap C$ is called a *face* of C and its dimension is $\dim \operatorname{aff}(H_{\boldsymbol{n}} \cap C)$. Let F be any face of the cone C. When $\dim F = 1$, the face F is called an *extremal ray*, and when $\dim F = d - 1$, it is called a *facet* of C. The normal vector to any hyperplane is determined up to multiplication by a nonzero factor; hence we may choose $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_s \in \mathbb{Z}^d$ to be the normals to the support hyperplanes that determine the facets of C and such that

$$C = \{\boldsymbol{z} \in \mathbb{R}^d : \langle \boldsymbol{z}, \boldsymbol{n}_i \rangle \geq 0, \text{ for } i = 1, \ldots, s\}.$$

The unique minimal set of generators for the semigroup H is called the *Hilbert basis* of H and we denote it as B_H.

It is known that the cone C has at least d facets and at least d extremal rays. When C has d facets (equivalently, that it has d extremal rays) the cone C and the semigroup H are called *simplicial*.

For any $d \geq 2$ we denote by $\mathcal{N}_d$ the class of normal simplicial affine semigroups which are fully embedded in $\mathbb{N}^d$.

Let $H \in \mathcal{N}_d$ and C the cone over H. On each extremal ray of C there exists a unique primitive element from H, which we call an *extremal ray* for H. Denote $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$ the extremal rays for H. These form an $\mathbb{R}$-basis in $\mathbb{R}^d$. For $\boldsymbol{z} \in \mathbb{R}^d$ such that $\boldsymbol{z} = \sum_{i=1}^d \lambda_i \boldsymbol{a}_i$ with $\lambda_i \in \mathbb{R}$, $i = 1, \dots, d$, we set $[\boldsymbol{z}]_i = \lambda_i$ for $i = 1, \dots, d$. In this notation, $\boldsymbol{z}$ is in the cone C if and only if $[\boldsymbol{z}]_i \geq 0$ for $i = 1, \dots, d$. Also, when $\boldsymbol{z} \in \mathbb{Z}^d$ one has that $\boldsymbol{z} \in H$ if and only if $[\boldsymbol{z}]_i \geq 0$ for $i = 1, \dots, d$.

The fundamental (semiopen) *parallelotope* of H is the set

$$P_H = \left\{ \boldsymbol{z} \in \mathbb{R}^d : \boldsymbol{z} = \sum_{i=1}^{d} \lambda_i \boldsymbol{a}_i \text{ with } 0 \leq \lambda_i < 1 \text{ for } i = 1, \dots, d \right\}.$$

Its closure in $\mathbb{R}^d$ is the set $\overline{P_H} = \{ \boldsymbol{z} \in \mathbb{R}^d : 0 \leq [\boldsymbol{z}]_i \leq 1 \text{ for } i = 1, \dots, d \}$. It is well known, and easy to see, that any $\boldsymbol{h}$ in H decomposes uniquely as $\boldsymbol{h} = \sum_{i=1}^d n_i \boldsymbol{a}_i + \boldsymbol{h}'$ with $\boldsymbol{h}' \in P_H \cap \mathbb{Z}^d$ and $n_1, \dots, n_d$ nonnegative integers. The extremal rays of H are in $B_H \setminus P_H$, but the rest of the elements in B_H belong to P_H.

Since H is simplicial, $\boldsymbol{x}^{\boldsymbol{a}_1}, \dots, \boldsymbol{x}^{\boldsymbol{a}_d}$ is a system of parameters in R; see [Goto et al. 1976, (1.11)]. As H is a normal semigroup, by [Hochster 1972], the ring $R = K[H]$ is Cohen–Macaulay of dimension d; hence any system of parameters in R is a regular sequence of maximal length. By [Danilov 1978; Stanley 1978], the canonical module ω_R of R is the ideal in R generated by the monomials $\boldsymbol{x}^{\boldsymbol{v}}$ whose exponent vector $\boldsymbol{v} = \log(\boldsymbol{x}^{\boldsymbol{v}})$ belongs to the relative interior of C, denoted by $\operatorname{relint} C$. Note that

$$\operatorname{relint} C = \left\{ \boldsymbol{c} \in \mathbb{R}^d : \boldsymbol{c} = \sum_{i=1}^{d} \lambda_i \boldsymbol{a}_i \text{ with } \lambda_i \in \mathbb{R}_{>0} \text{ for all } i = 1, \dots, d \right\}.$$

We set

$$\omega_H = \{ \boldsymbol{h} \in H : \boldsymbol{x}^{\boldsymbol{h}} \in \omega_R \} = \mathbb{Z}^d \cap \operatorname{relint} C,$$

which is a semigroup ideal of H, i.e., $\omega_H + H \subseteq \omega_H$. We note that $\boldsymbol{h} \in \mathbb{Z}^d$ is in ω_H if and only if $[\boldsymbol{h}]_i > 0$ for $i = 1, \dots, d$.

The ideal ω_R has a unique minimal system of monomial generators which we denote by $G(\omega_R)$. We set $G(\omega_H) = \{\log(u) : u \in G(\omega_R)\}$. Clearly, $G(\omega_H)$ is the unique minimal system of generators for ω_H. The situation when $G(\omega_H)$

is a singleton corresponds to the situation when R is a Gorenstein ring. When $B_H = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_d\}$ then R is a regular ring, there is no lattice point in the relative interior of $\overline{P_H}$, and $G(\omega_H) = \{\sum_{i=1}^{d} \boldsymbol{a}_i\}$.

The following easy lemma describes the minimal generators of ω_H when $H \in \mathcal{N}_2$. For completeness, we include a proof here.

Lemma 1.1. *Let H be in $\mathcal{N}_2$ with the extremal rays $\boldsymbol{a}_1, \boldsymbol{a}_2$. Then $B_H \cap \omega_H = B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. Moreover, if $\{\boldsymbol{a}_1, \boldsymbol{a}_2\} \subsetneq B_H$, then $G(\omega_H) = B_H \cap \omega_H$.*

Proof. Clearly, $B_H \cap \omega_H \subseteq B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. If $\boldsymbol{b} \in B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ then, since $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are the extremal rays, it follows that $\boldsymbol{b} \in \operatorname{relint} P_H$; hence $\boldsymbol{b} \in B_H \cap \omega_H$. Therefore, $B_H \cap \omega_H = B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$.

Assume $\{\boldsymbol{a}_1, \boldsymbol{a}_2\} \subsetneq B_H$. Let $\boldsymbol{b} \in G(\omega_H)$. The only lattice points on the boundary of the parallelogram $\overline{P_H}$ are $0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_1 + \boldsymbol{a}_2$. None of them is in $G(\omega_H)$, under our hypothesis. Thus $\boldsymbol{b} \in \operatorname{relint} P_H$. If, on the contrary, $\boldsymbol{b} \notin B_H$, then $\boldsymbol{b}$ is the sum of at least two elements in B_H, out of which at least one is not in ω_H, i.e., the latter is $\boldsymbol{a}_1$ or $\boldsymbol{a}_2$. This implies that $\boldsymbol{b} \notin P_H$, a contradiction. Consequently, $G(\omega_H) \subseteq B_H \cap \omega_H$. The reverse inclusion is clear. □

We refer to the monographs [Bruns and Herzog 1998; Bruns and Gubeladze 2009; Villarreal 2015; Ziegler 1995; Fulton 1993] for more background on affine semigroups, their semigroup rings, rational cones and the connections with algebraic geometry.

2. A regular sequence in $K[H]$ and slim semigroups

Throughout this section H is a semigroup in $\mathcal{N}_d$ having the extremal rays $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$, and $R = K[H]$. The main result is Theorem 2.4 where we present equivalent conditions for the existence of a convenient reduction for the graded maximal ideal of $K[H]/(\boldsymbol{x}^{\boldsymbol{b}})$, where $\boldsymbol{b}$ is any nonzero element in H. This result motivates us to introduce the class of slim semigroups.

The following lemma plays a crucial role in the proof of Proposition 2.2 and in several other arguments in this paper.

Lemma 2.1. *Let $\boldsymbol{b} = \sum_{i=1}^{d} \lambda_i \boldsymbol{a}_i$, with $\lambda_1, \dots, \lambda_d \in \mathbb{R}$.*

(a) *Let $n_i = \lfloor \lambda_i \rfloor + 1$ for $i = 1, \dots, d$. Then $\left(\sum_{i=1}^{d} n_i \boldsymbol{a}_i\right) - \boldsymbol{b} \in \omega_H$.*

(b) *If $\boldsymbol{b} \in \overline{P_H}$ then $\left(\sum_{i=1}^{d} \boldsymbol{a}_i\right) - \boldsymbol{b} \in H$, and in particular, if $\boldsymbol{b} \in P_H$, then $\left(\sum_{i=1}^{d} \boldsymbol{a}_i\right) - \boldsymbol{b} \in \omega_H$.*

Proof. For (a) we note that $\left(\sum_{i=1}^{d} n_i \boldsymbol{a}_i\right) - \boldsymbol{b} = \sum_{i=1}^{d} (1 - \{\lambda_i\}) \boldsymbol{a}_i$ and $0 < 1 - \{\lambda_i\} \le 1$ for all i; hence the sum of interest is in ω_H. Here we denoted $\{\lambda_i\} = \lambda_i - \lfloor \lambda_i \rfloor$ for all i. Part (b) follows immediately. □

Proposition 2.2. *For any $\boldsymbol{b} \neq 0$ in H, the sequence $\boldsymbol{x}^{\boldsymbol{b}}, \boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_2}, \dots, \boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_d}$ is a regular sequence on R.*

Proof. In order to simplify notation we set $u = \boldsymbol{x}^{\boldsymbol{b}}$ and $v_i = \boldsymbol{x}^{\boldsymbol{a}_i}$ for $i = 1, \dots, d$. Let $I = (u, v_1 - v_2, \dots, v_1 - v_d)$. We may write $\boldsymbol{b} = \sum_{i=1}^{d} \lambda_i \boldsymbol{a}_i$ with $\lambda_i \geq 0$ for $i = 1, \dots, d$. We denote $n_i = \lfloor \lambda_i \rfloor + 1$ for all i and we set $N = \sum_{i=1}^{d} n_i$.

We will show that $v_i^N \in I$ for $i = 1, \dots, d$. Since $v_i - v_j \in I$ for all i and j, it follows by symmetry that it is enough to show that $v_1^N \in I$.

We write

$$\begin{aligned} v_1^N &= (v_1^{n_2} - v_2^{n_2}) \cdot v_1^{N-n_2} + v_1^{N-n_2} v_2^{n_2} \\ &= (v_1^{n_2} - v_2^{n_2}) \cdot v_1^{N-n_2} + v_1^{N-n_2-n_3} v_2^{n_2} (v_1^{n_3} - v_3^{n_3}) + v_1^{N-n_2-n_3} v_2^{n_2} v_3^{n_3} \\ &= \sum_{i=2}^{d} v_1^{N - \sum_{j=2}^{i} n_j} v_2^{n_2} \cdots v_{i-1}^{n_{i-1}} (v_1^{n_i} - v_i^{n_i}) + v_1^{n_1} \cdots v_d^{n_d}. \end{aligned}$$

Hence by Lemma 2.1 it follows that $v_1^N \in I$.

Since H is a normal simplicial semigroup, $v_1, \dots, v_d$ is a regular sequence in R; hence $v_1^N, \dots, v_d^N$ is a regular sequence in R, as well. Since R is a Cohen–Macaulay ring of dimension d we get that $v_1^N, \dots, v_d^N$ is also a system of parameters for R. Thus $0 < \lambda(R/I) \leq \lambda(R/(v_1^N, \dots, v_d^N)) < \infty$, which implies that $u, v_1 - v_2, \dots, v_1 - v_d$ is a system of parameters, and consequently a regular sequence for R. Here $\lambda(M)$ denotes the length of an R-module M. □

In the sequel, our aim is to find a reduction ideal for the graded maximal ideal $\mathfrak{m}$ of R, modulo the ideal $\boldsymbol{x}^{\boldsymbol{b}} R$, for any $\boldsymbol{b} \in H \setminus \{0\}$. In this order, we need the following lemma which is interesting on its own.

Lemma 2.3. *For any $\boldsymbol{b}$ in H, there exists a positive integer k such that for all $\boldsymbol{c}_1, \dots, \boldsymbol{c}_k$ in ω_H, one has $\boldsymbol{c}_1 + \cdots + \boldsymbol{c}_k \in \boldsymbol{b} + H$.*

Proof. Assume $\boldsymbol{n}_1, \dots, \boldsymbol{n}_r \in \mathbb{Z}^d$ are normal vectors to the support hyperplanes of the facets of the cone C such that

$$C = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{n}_i \rangle \geq 0, \text{ for all } i = 1, \dots, r\}.$$

The map $\sigma : \mathbb{R}^d \to \mathbb{R}^r$ given by

$$\sigma(\boldsymbol{h}) = (\langle \boldsymbol{h}, \boldsymbol{n}_1 \rangle, \dots, \langle \boldsymbol{h}, \boldsymbol{n}_r \rangle), \text{ for all } \boldsymbol{h} \in \mathbb{R}^d$$

is clearly $\mathbb{R}$-linear and $\sigma(H) \subseteq \mathbb{N}^r$.

Let $k_0 = \max\{\langle \boldsymbol{b}, \boldsymbol{n}_j \rangle : j = 1, \dots, r\}$. For any integer $k > k_0$, any $\boldsymbol{c}_1, \dots, \boldsymbol{c}_k \in H \cap \operatorname{relint} C$, and any $1 \leq j \leq r$, the j-th component of $\sigma(\boldsymbol{c}_1 + \cdots + \boldsymbol{c}_k - \boldsymbol{b})$ equals $\left(\sum_{i=1}^{k} \langle \boldsymbol{c}_i, \boldsymbol{n}_j \rangle\right) - \langle \boldsymbol{b}, \boldsymbol{n}_j \rangle \geq k - k_0 > 0$; hence $\boldsymbol{c}_1 + \cdots + \boldsymbol{c}_k \in \boldsymbol{b} + H$. □

Theorem 2.4. *Let $R = K[H]$, $J = (\boldsymbol{x}^{\boldsymbol{a}_i} - \boldsymbol{x}^{\boldsymbol{a}_j} : i, j = 1, \dots, d)R$ and $0 \neq \boldsymbol{b} \in \omega_H$. Then the following statements are equivalent*:

(i) *There exists an integer k such that $\mathfrak{m}^{k+1} = J\mathfrak{m}^k$ modulo the ideal $\boldsymbol{x}^{\boldsymbol{b}}R$.*

(ii) $\sum_{j=1}^{d}[\boldsymbol{c}]_j \geq 1$, *for any $\boldsymbol{c} \in B_H \setminus \omega_H$.*

Proof. We denote $\{\boldsymbol{c}_1, \dots, \boldsymbol{c}_r\} = B_H \setminus (\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_d\} \cup \omega_H)$.

(i) ⇒ (ii): For any $\boldsymbol{c}$ in H we set $l(\boldsymbol{c}) = \sum_{j=1}^{d}[\boldsymbol{c}]_j$. Clearly, $l(\boldsymbol{a}_i) = 1$ for $i = 1, \dots, d$, so if $r = 0$, we are done. Assuming $r > 0$, we pick t such that $l(\boldsymbol{c}_t) = \min\{l(\boldsymbol{c}_j) : j = 1, \dots, r\}$. Let $k > 0$ so that $\mathfrak{m}^{k+1} + \boldsymbol{x}^{\boldsymbol{b}}R = J\mathfrak{m}^k + \boldsymbol{x}^{\boldsymbol{b}}R$, i.e., $\mathfrak{m}^{k+1} \subseteq J\mathfrak{m}^k + \boldsymbol{x}^{\boldsymbol{b}}R$.

As $\boldsymbol{c}_t \notin \omega_H$, there exists $1 \leq i_0 \leq d$ with $[\boldsymbol{c}_t]_{i_0} = 0$; hence $(k+1)\boldsymbol{c}_t \notin \boldsymbol{b} + H$. From $\boldsymbol{x}^{(k+1)\boldsymbol{c}_t} \in J\mathfrak{m}^k + \boldsymbol{x}^{\boldsymbol{b}}R$ we get that

$$\boldsymbol{x}^{(k+1)\boldsymbol{c}_t} = \boldsymbol{x}^{\boldsymbol{a}_j}\boldsymbol{x}^{\boldsymbol{u}_1 + \dots + \boldsymbol{u}_k},$$

for some $1 \leq j \leq d$ and $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k \in H \setminus (\omega_H \cup \{0\})$. Consequently,

$$(k+1)l(\boldsymbol{c}_t) = 1 + \sum_{i=1}^{k} l(\boldsymbol{u}_i) \geq 1 + k\min\{1, l(\boldsymbol{c}_t)\},$$

which implies $l(\boldsymbol{c}_t) \geq 1$.

(ii) ⇒ (i): Let $u = \boldsymbol{x}^{\boldsymbol{b}}$ and $v_i = \boldsymbol{x}^{\boldsymbol{a}_i}$ for $i = 1, \dots, d$. We decompose $\boldsymbol{b} = \sum_{i=1}^{d} \lambda_i \boldsymbol{a}_i$ with $\lambda_i \geq 0$ and we set $n_i = \lfloor \lambda_i \rfloor + 1$ for all $i = 1, \dots, d$ and $N = \sum_{i=1}^{d} n_i$.

We claim that for any positive integer t, any $i_1, \dots, i_t \in \{1, \dots, d\}$ and any $v \in \{v_1, \dots, v_d\}$ one has

$$v_{i_1} \cdots v_{i_t} \in J\mathfrak{m}^{t-1} + v^t R. \tag{3}$$

Indeed, this is a consequence of the following equations:

$$\begin{aligned} v_{i_1} \cdots v_{i_t} &= (v_{i_1} - v) \cdot v_{i_2} \cdots v_{i_t} + v \cdot v_{i_2} \cdots v_{i_t} \\ &= (v_{i_1} - v) \cdot v_{i_2} \cdots v_{i_t} + v(v_{i_2} - v) \cdot v_{i_3} \cdots v_{i_t} + v^2 v_{i_3} \cdots v_{i_t} \\ &= \Sigma_{j=1}^{d} v^{j-1} \cdot (v_{i_j} - v) \cdot v_{i_{j+1}} \cdots v_{i_t} + v^t. \end{aligned}$$

Now let $i_1, \dots, i_N \in \{1, \dots, d\}$. In the product $v_{i_1} \dots v_{i_N}$ we apply (3) to the first n_1 terms, then to the next n_2 terms, etc., and we obtain that

$$v_{i_1} \dots v_{i_N} \in \prod_{i=1}^{d}(J\mathfrak{m}^{n_i - 1}, v_i^{n_i}) \subseteq \left(J\mathfrak{m}^{N-1}, \prod_{i=1}^{d} v_i^{n_i}\right) \subseteq (J\mathfrak{m}^{N-1}, uR), \tag{4}$$

where for the last inclusion we used Lemma 2.1.

For $1 \le i \le r$ we may write $\boldsymbol{c}_i = \sum_{j=1}^{d}(p_{ij}/q_i)\boldsymbol{a}_i$, where q_i, p_{ij} are nonnegative integers and $q_i > 0$. Hence, $Nq_i \cdot \boldsymbol{c}_i = \sum_{j=1}^{d} Np_{ij}\boldsymbol{a}_j$, where by the hypothesis of (ii) we have $N \le Nq_i \le \sum_{j=1}^{d} Np_{ij}$. Thus, using (4) we derive

$$(\boldsymbol{x}^{\boldsymbol{c}_i})^{Nq_i} = \prod_{j=1}^{d}(\boldsymbol{x}^{\boldsymbol{a}_j})^{Np_{ij}} \in (J\mathfrak{m}^{\sum_{j=1}^{d} Np_{ij}-1}, uR) \subseteq (J\mathfrak{m}^{Nq_i-1}, uR).$$

We set $N_1 = \max\{Nq_i : 1 \le i \le r\}$ if $r > 0$, otherwise we let $N_1 = N$. Then

$$(\boldsymbol{x}^{\boldsymbol{c}})^{N_1} \subseteq (J\mathfrak{m}^{N_1-1}, uR) \quad \text{for all } \boldsymbol{c} \in B_H \setminus \omega_H. \tag{5}$$

Let k_0 be a positive integer satisfying the conclusion of Lemma 2.3 for the element $\boldsymbol{b}$. We set $k = k_0 + N_1 \cdot |B_H \setminus \omega_H| - 2$.

Let w be any product of $k+1$ monomial generators of $\mathfrak{m}$. If the exponents of at least k_0 of them are in relint C, then by the choice of k_0 we get that $w \in uR$. Otherwise, we may write $w = (\boldsymbol{x}^{\boldsymbol{c}})^{N_1} \cdot w'$ for some $\boldsymbol{c} \in B_H \setminus \omega_H$ and $w' \in \mathfrak{m}^{k+1-N_1}$. In the latter case, using (5) we infer that $w \in J\mathfrak{m}^k + uR$. This shows that $\mathfrak{m}^{k+1} + uR = J\mathfrak{m}^k + uR$, which completes the proof. □

The semigroups H satisfying the equivalent conditions of Theorem 2.4 deserve a special name.

Definition 2.5. A semigroup $H \in \mathcal{N}_d$ is called *slim* if $\sum_{i=1}^{d}[c]_i \ge 1$, for any $\boldsymbol{c} \in B_H \setminus \omega_H$.

We will denote by $\mathcal{H}_d$ the class of slim semigroups in $\mathbb{N}^d$.

Remark 2.6. Let $H \in \mathcal{N}_d$. If $B_H \setminus \omega_H = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_d\}$, then H is slim. In particular, by Lemma 1.1, any normal semigroup in $\mathbb{N}^2$ is slim, so $\mathcal{N}_2 = \mathcal{H}_2$.

Proposition 2.7. *Let $H \in \mathcal{N}_3$. Then H is slim if and only if $[\boldsymbol{c}]_1 + [\boldsymbol{c}]_2 + [\boldsymbol{c}]_3 = 1$ for any $\boldsymbol{c} \in B_H \setminus \omega_H$.*

Proof. Assume H is slim. If $B_H \setminus \omega_H = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$, there is nothing left to prove. Assume there exists $\boldsymbol{c} \in B_H \setminus (\omega_H \cup \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\})$ such that $\sum_{i=1}^{3}[\boldsymbol{c}]_i > 1$. Without loss of generality, we may assume that $0 < [\boldsymbol{c}]_i < 1$ for $i = 1, 2$ and $[\boldsymbol{c}]_3 = 0$. Arguing as in the proof of Lemma 2.1 we obtain that $0 \ne \boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{c} \in \mathbb{N}\boldsymbol{a}_1 + \mathbb{N}\boldsymbol{a}_2 \subset H$. We may write $\boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{h}$, where $\boldsymbol{b} \in B_H \setminus \omega_H$ and $\boldsymbol{h} \in H$. Then $\sum_{i=1}^{3}[\boldsymbol{b}]_i = [\boldsymbol{b}]_1 + [\boldsymbol{b}]_2 \le \sum_{i=1}^{2}[\boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{c}]_i = 2 - \sum_{i=1}^{2}[\boldsymbol{c}_i] < 1$, which is false since H is slim. □

Example 2.8. The semigroup $L \in \mathcal{N}_3$ with the extremal rays $\boldsymbol{a}_1 = (11, 13, 0)$, $\boldsymbol{a}_2 = (3, 4, 0)$ and $\boldsymbol{a}_3 = (0, 0, 1)$ is not slim. Indeed, $B_L = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, (4, 5, 0), (5, 6, 0)\}$ (compare with Example 4.6) and it is easy to check that $\sum_{i=1}^{3}[(4, 5, 0)]_i = \frac{4}{5} < 1$.

3. Ulrich elements and the almost Gorenstein property

The theory of almost Gorenstein rings has its origin in the theory of the almost symmetric numerical semigroups in [Barucci and Fröberg 1997]. If R is the semigroup ring of a numerical semigroup, then the semigroup is almost symmetric, if and only if there exists an exact sequence

$$0 \to R \to (\omega_R)(-a) \to E \to 0, \tag{6}$$

where E is annihilated by the graded maximal ideal of R; see [Herzog and Watanabe 2019]. Here ω_R denotes the canonical module of R and $-a$ the smallest degree of a generator of ω_R, i.e., $-a = \min\{k : (\omega_R)_k \neq 0\}$

In [Goto et al. 2013] the 1-dimensional positively graded rings which admit such an exact sequence are called almost Gorenstein.

Goto et al. [2015, Definition 8.1] extended the concept of the almost Gorenstein property to rings of higher dimension: let R be a positively graded Cohen–Macaulay K-algebra with a-invariant a. Then R is called *graded almost Gorenstein*, if there exists an exact sequence like in (6), where E is an Ulrich module.

Ulrich modules are defined as follows: let $(R, \mathfrak{m}, K)$ be a local (or positively graded) ring with (graded) maximal ideal $\mathfrak{m}$, and let M be a (graded) Cohen–Macaulay module over R. Then the minimal number of generators $\mu(M)$ of M is bounded by the multiplicity $e(M)$ of M. The module M is called an *Ulrich module*, if $\mu(M) = e(M)$. Ulrich [1984] asked whether any Cohen–Macaulay ring admits an Ulrich module M with $\dim M = \dim R$. At present this question is still open, and has an affirmative answer for example when R is a hypersurface ring [Backelin and Herzog 1989].

In the case of almost symmetric numerical semigroup rings, the module E in the exact sequence (6) is of Krull dimension zero. A graded module M with $\dim M = 0$ is Ulrich if and only if $\mathfrak{m}M = 0$. Thus the above definition [Goto et al. 2015, Definition 8.1] is a natural extension of 1-dimensional almost Gorenstein rings to higher dimensions.

We propose the following multigraded version of the almost Gorenstein property for normal semigroup rings.

Definition 3.1. Let H be a normal affine semigroup and $R = K[H]$. For $\boldsymbol{b} \in \omega_H$ consider the exact sequence

$$0 \to R \to \omega_R(\boldsymbol{b}) \to E \to 0, \tag{7}$$

where $1 \in R$ is mapped to $u = \boldsymbol{x}^{\boldsymbol{b}}$ and $E = \omega_R/uR$. Then $\boldsymbol{b}$ is called an *Ulrich element* in H, if E is an Ulrich R-module.

If H admits an Ulrich element $\boldsymbol{b}$, then we call the ring R *almost Gorenstein with respect to* $\boldsymbol{b}$, or simply AG if H has an Ulrich element.

Theorem 3.2. *Let $H \in \mathcal{H}_d$ with extremal rays $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$. Let $\mathfrak{m}$ be the graded maximal ideal of $R = K[H]$. Let $\boldsymbol{b} \in \omega_H$, $u = \boldsymbol{x}^{\boldsymbol{b}}$ and $J = (\boldsymbol{x}^{\boldsymbol{a}_i} - \boldsymbol{x}^{\boldsymbol{a}_j} : i, j = 1, \dots, d)R$. Then $\boldsymbol{b}$ is an Ulrich element in H if and only if*

$$\mathfrak{m}\omega_R \subseteq (uR, J\omega_R). \tag{8}$$

Proof. Since uR and ω_R are Cohen–Macaulay R-modules of dimension d, we see (keeping the notation from (7)) that depth $E \geq d-1$, and since uR and ω_R are rank 1 modules, we deduce that $\operatorname{Ann}(E) \neq 0$. Therefore, $\dim E \leq d-1$, and this implies that E is a Cohen–Macaulay R-module of dimension $d-1$.

Suppose that (8) holds. By [Goto et al. 2015, Proposition 2.2.(2)], it follows that E is an Ulrich module since (8) implies that $\mathfrak{m}E = JE$ and since J is generated by $d-1(=\dim E)$ elements, namely by the elements $f_j = \boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_j}$ with $j = 2, \dots, d$. Thus $\boldsymbol{b}$ is an Ulrich element in H.

Conversely, assume that $\boldsymbol{b}$ is an Ulrich element. Then E is an Ulrich module, and therefore $\lambda(E/\mathfrak{m}E) = e(E)$. It follows from Theorem 2.4 that J is a reduction ideal of $\mathfrak{m}$ with respect to E. Thus by [Bruns and Herzog 1998, Lemma 4.6.5], $e(E) = e(J, E)$, where $e(J, E)$ denotes the Hilbert–Samuel multiplicity of E with respect to J. Since E is Cohen–Macaulay of dimension $d-1$, and since J is generated by the $d-1$ elements $f_2, \dots, f_d$ and $\lambda(E/JE) < \infty$, we see that $f_2, \dots, f_d$ is a regular sequence on E. Thus [Bruns and Herzog 1998, Theorem 4.7.6] implies that $e(J, E) = \lambda(E/JE)$. Hence, $\lambda(E/\mathfrak{m}E) = \lambda(E/JE)$. Since $JE \subseteq \mathfrak{m}E$, it follows that $\mathfrak{m}E = JE$, and this implies (8). □

Remark 3.3. It follows from the proof of Theorem 3.2 that if $H \in \mathcal{N}_d$ and (8) holds for some ideal $J \subset \mathfrak{m}$, generated by $d-1$ elements, then $\boldsymbol{b}$ is an Ulrich element in H.

Example 3.4 (Ulrich elements in Gorenstein and regular rings).

(a) If $K[H]$ is a Gorenstein ring and $G(\omega_H) = \{\boldsymbol{b}\}$, then $\omega_R = \boldsymbol{x}^{\boldsymbol{b}}R$; hence (8) holds and $\boldsymbol{b}$ is an Ulrich element in H.

(b) Assume $K[H]$ is a regular ring with $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$ the extremal rays of H. Set $\boldsymbol{c} = \sum_{i=1}^{d} \boldsymbol{a}_i$. Then $\boldsymbol{a}_i + \boldsymbol{c}$ is an Ulrich element in H for any $i = 1, \dots, d$. Indeed, since $\mathfrak{m} = (\boldsymbol{x}^{\boldsymbol{a}_j} : 1 \leq j \leq d)$ and $\boldsymbol{x}^{\boldsymbol{a}_j + \boldsymbol{c}} = \boldsymbol{x}^{\boldsymbol{c}}(\boldsymbol{x}^{\boldsymbol{a}_j} - \boldsymbol{x}^{\boldsymbol{a}_i}) + \boldsymbol{x}^{\boldsymbol{c}+\boldsymbol{a}_i}$ for $j = 1, \dots, d$, we have that (8) is verified for $\boldsymbol{b} = \boldsymbol{c} + \boldsymbol{a}_i$.

Example 3.5. Let $H \in \mathcal{H}_2$ having the extremal rays $\boldsymbol{a}_1 = (11, 2)$ and $\boldsymbol{a}_2 = (31, 6)$. A computation with [Normaliz] shows that the Hilbert basis of H is

$$B_H = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{b} = (16, 3), \boldsymbol{c}_1 = (21, 4), \boldsymbol{c}_2 = (26, 5)\}.$$

Moreover, $\boldsymbol{b}, \boldsymbol{c}_1, \boldsymbol{c}_2$ are the only nonzero lattice points in P_H, and they all lie on the line $y = (x-1)/5$ passing through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Comparing componentwise, we have

$$\boldsymbol{a}_1 \preceq \boldsymbol{b} \preceq \boldsymbol{c}_1 \preceq \boldsymbol{c}_2 \preceq \boldsymbol{a}_2.$$

We note that $\boldsymbol{a}_1+\boldsymbol{a}_2=\boldsymbol{b}+\boldsymbol{c}_2=2\boldsymbol{c}_1$. It is also straightforward to check that in $K[H]$ we have

$$\begin{aligned}
&\boldsymbol{x}^{\boldsymbol{a}_1}\boldsymbol{x}^{\boldsymbol{c}_1}=(\boldsymbol{x}^{\boldsymbol{b}})^2, \quad \boldsymbol{x}^{\boldsymbol{a}_1}\boldsymbol{x}^{\boldsymbol{c}_2}=\boldsymbol{x}^{\boldsymbol{b}}\boldsymbol{x}^{\boldsymbol{c}_1}, \quad \boldsymbol{x}^{\boldsymbol{c}_1}\boldsymbol{x}^{\boldsymbol{c}_1}=\boldsymbol{x}^{\boldsymbol{b}}\boldsymbol{x}^{\boldsymbol{c}_2}, \quad \boldsymbol{x}^{\boldsymbol{c}_1}\boldsymbol{x}^{\boldsymbol{c}_2}=\boldsymbol{x}^{\boldsymbol{b}}\boldsymbol{x}^{\boldsymbol{a}_2},\\
&\boldsymbol{x}^{\boldsymbol{c}_2}\boldsymbol{x}^{\boldsymbol{c}_2}=\boldsymbol{x}^{\boldsymbol{a}_2}\boldsymbol{x}^{\boldsymbol{c}_1}=(\boldsymbol{x}^{\boldsymbol{a}_2}-\boldsymbol{x}^{\boldsymbol{a}_1})\boldsymbol{x}^{\boldsymbol{c}_1}+\boldsymbol{x}^{\boldsymbol{a}_1}\boldsymbol{x}^{\boldsymbol{c}_1}=(\boldsymbol{x}^{\boldsymbol{a}_2}-\boldsymbol{x}^{\boldsymbol{a}_1})\boldsymbol{x}^{\boldsymbol{c}_1}+(\boldsymbol{x}^{\boldsymbol{b}})^2,\\
&\boldsymbol{x}^{\boldsymbol{a}_2}\boldsymbol{x}^{\boldsymbol{c}_2}=(\boldsymbol{x}^{\boldsymbol{a}_2}-\boldsymbol{x}^{\boldsymbol{a}_1})\boldsymbol{x}^{\boldsymbol{c}_2}+\boldsymbol{x}^{\boldsymbol{a}_1}\boldsymbol{x}^{\boldsymbol{c}_2}=(\boldsymbol{x}^{\boldsymbol{a}_2}-\boldsymbol{x}^{\boldsymbol{a}_1})\boldsymbol{x}^{\boldsymbol{c}_2}+\boldsymbol{x}^{\boldsymbol{b}}\boldsymbol{x}^{\boldsymbol{c}_1}.
\end{aligned}$$

Using Theorem 3.2 we conclude that $\boldsymbol{b}$ is an Ulrich element in H, and hence $K[H]$ is AG.

In the following special case, the possible Ulrich elements can be identified.

Proposition 3.6. *Let H be a semigroup in $\mathcal{H}_d$ whose nonzero elements have all the entries positive, and assume that $(1,\dots,1)\in\omega_H$. If H has an Ulrich element $\boldsymbol{b}$, then $\boldsymbol{b}=(1,\dots,1)$.*

Proof. We set $\boldsymbol{b}'=(1,\dots,1)$. Assume, on the contrary, that $\boldsymbol{b}\neq\boldsymbol{b}'$. Then by the criterion in Theorem 3.2 and using the same notation, we get that $\boldsymbol{x}^{\boldsymbol{b}'}\cdot\boldsymbol{x}^{\boldsymbol{b}'}\in(\boldsymbol{x}^{\boldsymbol{b}}R, J\omega_R)$. This implies that $(2,\dots,2)=2\boldsymbol{b}'=\boldsymbol{b}+\boldsymbol{c}$ for some $\boldsymbol{c}\in H$, or that $2\boldsymbol{b}'=\boldsymbol{a}_i+\boldsymbol{h}$ for some $1\le i\le d$ and $\boldsymbol{h}\in\omega_H$, $\boldsymbol{h}\neq\boldsymbol{b}'$. Since $(1,\dots,1)$ is the smallest element of ω_H when comparing vectors componentwise, at least one component of $\boldsymbol{b}$ (respectively, of $\boldsymbol{h}$) is at least two; hence at least one component of $\boldsymbol{c}$ (respectively, of $\boldsymbol{a}_i$) is less than or equal to zero, which is false by the assumption that all the entries of nonzero elements of H are positive. □

4. The AG property for normal semigroups in dimension 2

As mentioned before, all 2-dimensional normal affine semigroups are slim. For them, in Theorem 4.3 we make more concrete the criterion for Ulrich elements given in Theorem 3.2. We first prove a couple of lemmas.

Throughout this section, unless otherwise stated, H is a semigroup in $\mathcal{N}_2=\mathcal{H}_2$ with extremal rays $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. We denote by $\mathfrak{m}$ the graded maximal ideal of $R=K[H]$.

Lemma 4.1. *Let $\boldsymbol{b}$ be an element in $B_H\setminus\{\boldsymbol{a}_1,\boldsymbol{a}_2\}$. For any $\boldsymbol{c}\in\omega_H$ such that $\boldsymbol{c}\notin\boldsymbol{b}+H$, there exists $t\in\{1,2\}$ such that $\boldsymbol{c}+\boldsymbol{a}_t\in\boldsymbol{b}+H$.*

Proof. Let C be the cone with the extremal rays $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be vectors normal to the facets of the cone C such that $\langle\boldsymbol{a}_i,\boldsymbol{n}_i\rangle=0$ for $i=1,2$ and $\boldsymbol{x}\in C$ if and only if $\langle\boldsymbol{x},\boldsymbol{n}_1\rangle\ge 0$ and $\langle\boldsymbol{x},\boldsymbol{n}_2\rangle\ge 0$.

Since $\boldsymbol{c}\notin\boldsymbol{b}+H$ it follows that $\boldsymbol{c}-\boldsymbol{b}\notin C$. We may assume that $\langle\boldsymbol{c}-\boldsymbol{b},\boldsymbol{n}_1\rangle<0$, and claim then that $\boldsymbol{c}+\boldsymbol{a}_2\in\boldsymbol{b}+H$. Indeed,

$$\langle\boldsymbol{c}+\boldsymbol{a}_2-\boldsymbol{b},\boldsymbol{n}_1\rangle=\langle\boldsymbol{c},\boldsymbol{n}_1\rangle+\langle\boldsymbol{a}_1+\boldsymbol{a}_2-\boldsymbol{b},\boldsymbol{n}_1\rangle>0,$$

since $\langle \boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{b}, \boldsymbol{n}_1 \rangle > 0$, by Lemma 2.1, and

$$\langle \boldsymbol{c} + \boldsymbol{a}_2 - \boldsymbol{b}, \boldsymbol{n}_2 \rangle = \langle \boldsymbol{c} - \boldsymbol{b}, \boldsymbol{n}_2 \rangle > 0,$$

since otherwise $\boldsymbol{c} - \boldsymbol{b} \in -C = \{-\boldsymbol{a} : \boldsymbol{a} \in C\}$, a contradiction to $\boldsymbol{b} \in B_H$. □

Lemma 4.2. *Let $\boldsymbol{b}$ belong to $B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. We set $I = (\boldsymbol{x}^{\boldsymbol{b}} R, (\boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_2})\omega_R) \subset R$. Let $\boldsymbol{c} \in \omega_H$. The following conditions are equivalent*:

(a) $\boldsymbol{x}^{\boldsymbol{c}} \in I$.

(b) $\boldsymbol{c} \in (\boldsymbol{b} + H) \cup (\boldsymbol{a}_1 + \omega_H) \cup (\boldsymbol{a}_2 + \omega_H)$.

(c) $\boldsymbol{c} \in (\boldsymbol{b} + H) \cup (\boldsymbol{a}_1 + H) \cup (\boldsymbol{a}_2 + H)$.

Proof. (a) ⇒ (b): Note that $\boldsymbol{x}^{\boldsymbol{c}} \in \boldsymbol{x}^{\boldsymbol{b}} R$ if and only if $\boldsymbol{c} \in \boldsymbol{b} + H$. If $\boldsymbol{x}^{\boldsymbol{c}} \notin \boldsymbol{x}^{\boldsymbol{b}} R$, then there exist $0 \neq f$ in ω_R and g in R such that

$$\boldsymbol{x}^{\boldsymbol{c}} = (\boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_2}) \cdot f + \boldsymbol{x}^{\boldsymbol{b}} \cdot g.$$

Therefore, there exists a monomial $\boldsymbol{x}^{\boldsymbol{a}}$ in ω_R such that $\boldsymbol{x}^{\boldsymbol{c}} = \boldsymbol{x}^{\boldsymbol{a}_1} \cdot \boldsymbol{x}^{\boldsymbol{a}}$ or $\boldsymbol{x}^{\boldsymbol{c}} = \boldsymbol{x}^{\boldsymbol{a}_2} \cdot \boldsymbol{x}^{\boldsymbol{a}}$, equivalently $\boldsymbol{c} \in (\boldsymbol{a}_1 + \omega_H) \cup (\boldsymbol{a}_2 + \omega_H)$.

(b) ⇒ (a): If $\boldsymbol{c} \in \boldsymbol{b} + H$ then clearly $\boldsymbol{x}^{\boldsymbol{c}} \in I$. Assume $\boldsymbol{c} \notin \boldsymbol{b} + H$. By symmetry, it is enough to consider the case $\boldsymbol{c} \in \boldsymbol{a}_1 + \omega_H$.

By Lemma 4.1, since $0 \neq \boldsymbol{c} - \boldsymbol{a}_1 \in \omega_H$, $\boldsymbol{c} - \boldsymbol{a}_1 \notin \boldsymbol{b} + H$ and $(\boldsymbol{c} - \boldsymbol{a}_1) + \boldsymbol{a}_1 = \boldsymbol{c} \notin \boldsymbol{b} + H$ it follows that $\boldsymbol{c} - \boldsymbol{a}_1 + \boldsymbol{a}_2 \in \boldsymbol{b} + H$.

As we may write

$$\boldsymbol{x}^{\boldsymbol{c}} = \boldsymbol{x}^{\boldsymbol{c} - \boldsymbol{a}_1} \cdot (\boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_2}) + \boldsymbol{x}^{\boldsymbol{c} - \boldsymbol{a}_1 + \boldsymbol{a}_2},$$

we conclude that $\boldsymbol{x}^{\boldsymbol{c}} \in I$.

(b) ⇒ (c) is trivial.

For (c) ⇒ (b) it is enough to consider the case when $\boldsymbol{c} \notin \boldsymbol{b} + H$. We may assume $\boldsymbol{c} \in \boldsymbol{a}_1 + H$. If $\boldsymbol{c} \notin \boldsymbol{a}_1 + \omega_H$, then there exists a positive integer n such that either $\boldsymbol{c} - \boldsymbol{a}_1 = n\boldsymbol{a}_1$, or $\boldsymbol{c} - \boldsymbol{a}_1 = n\boldsymbol{a}_2$. In the first case we get that $\boldsymbol{c} = (n+1)\boldsymbol{a}_1 \notin \omega_H$, a contradiction. In the second case we get that $\boldsymbol{c} = \boldsymbol{a}_1 + n\boldsymbol{a}_2 \in \boldsymbol{b} + H$, since $\boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{b} \in H$ by Lemma 2.1. This is again a contradiction. Thus $\boldsymbol{c} \in \boldsymbol{a}_1 + \omega_H$. □

Theorem 4.3. *An element $\boldsymbol{b} \in B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is an Ulrich element in H, if and only if*

$$\boldsymbol{c}_1 + \boldsymbol{c}_2 \in (\boldsymbol{b} + H) \cup (\boldsymbol{a}_1 + H) \cup (\boldsymbol{a}_2 + H) \quad \textit{for all } \boldsymbol{c}_1, \boldsymbol{c}_2 \in B_H.$$

Proof. Let $B_H = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_m\}$. Then $\mathfrak{m} = (\boldsymbol{x}^{\boldsymbol{a}_1}, \boldsymbol{x}^{\boldsymbol{a}_2}, \boldsymbol{x}^{\boldsymbol{c}_1}, \ldots, \boldsymbol{x}^{\boldsymbol{c}_m})$ and $\omega_R = (\boldsymbol{x}^{\boldsymbol{c}_1}, \ldots, \boldsymbol{x}^{\boldsymbol{c}_m})$.

If $\boldsymbol{b}$ is an Ulrich element, then $\mathfrak{m}\omega_R \subseteq (\boldsymbol{x}^{\boldsymbol{b}} R, (\boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_2})\omega_R)$, and therefore $\boldsymbol{x}^{\boldsymbol{c}_i} \boldsymbol{x}^{\boldsymbol{c}_j} \in (\boldsymbol{x}^{\boldsymbol{b}} R, (\boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_2})\omega_R)$ for all i, j. Thus the desired conclusion follows from Lemma 4.2.

Conversely, let $\boldsymbol{x}^{\boldsymbol{c}} \in \mathfrak{m}\omega_R$. Then $\boldsymbol{c} = \boldsymbol{c}_i + \boldsymbol{c}_j + h$, or $\boldsymbol{c} = \boldsymbol{a}_i + \boldsymbol{c}_j + h$ for some $h \in H$. In both cases our assumptions imply that $\boldsymbol{c} \in (\boldsymbol{b}+H) \cup (\boldsymbol{a}_1 + H) \cup (\boldsymbol{a}_2 + H)$. Thus $\boldsymbol{x}^{\boldsymbol{c}} \in (\boldsymbol{x}^{\boldsymbol{b}} R, (\boldsymbol{x}^{\boldsymbol{a}_1} - \boldsymbol{x}^{\boldsymbol{a}_2})\omega_R)$, by Lemma 4.2. This shows that $\boldsymbol{b}$ is an Ulrich element in H. □

Example 4.4. Let H be the semigroup in $\mathcal{H}_2$ with the extremal rays $\boldsymbol{a}_1 = (5, 2)$ and $\boldsymbol{a}_2 = (2, 5)$. Then the Hilbert basis of H is

$$B_H = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{c}_1 = (1, 1), \boldsymbol{c}_2 = (2, 1), \boldsymbol{c}_3 = (1, 2)\}.$$

Using Theorem 4.3, we may check that none of $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ or $\boldsymbol{c}_3$ is an Ulrich element in H. The same conclusion could be reached by using Proposition 3.6 together with Theorem 6.3.

Here is one immediate application of Theorem 4.3.

Proposition 4.5. *Let $H \in \mathcal{H}_2$ such that $\boldsymbol{c} + \boldsymbol{c}' \notin P_H$ for all $\boldsymbol{c}, \boldsymbol{c}' \in B_H \cap P_H$. Then any $\boldsymbol{b} \in B_H \cap P_H$ is an Ulrich element in H.*

Proof. By the hypothesis, if $\boldsymbol{c}, \boldsymbol{c}' \in B_H \cap P_H$ then $\boldsymbol{c} + \boldsymbol{c}' \in (\boldsymbol{a}_1 + H) \cup (\boldsymbol{a}_2 + H)$. Theorem 4.3 yields the conclusion. □

One may check that the semigroup H in Example 3.5 satisfies the hypothesis of Proposition 4.5; hence H admits three Ulrich elements.

In the following example there is exactly one Ulrich element in the Hilbert basis of H.

Example 4.6. For the semigroup $H \in \mathcal{H}_2$ with $\boldsymbol{a}_1 = (11, 13)$ and $\boldsymbol{a}_2 = (3, 4)$, a computation with [Normaliz] shows that $B_H = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{c}_1 = (4, 5), \boldsymbol{c}_2 = (5, 6)\}$. We note that the points $2\boldsymbol{c}_2 - \boldsymbol{c}_1 = (6, 7)$ and $2\boldsymbol{c}_2 - \boldsymbol{a}_2 = (7, 8)$ are not in ω_H since the slope of the line through the origin and each of these respective points is not in the interval $\left(\frac{13}{11}, \frac{4}{3}\right)$. Also, clearly, $2\boldsymbol{c}_2 - \boldsymbol{a}_1 = (-1, -1) \notin H$. Therefore, by Theorem 4.3 we get that $\boldsymbol{c}_1$ is not an Ulrich element in H.

On the other hand, since $2\boldsymbol{c}_1 = (8, 10) = \boldsymbol{c}_2 + \boldsymbol{a}_2$ and $B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$, by Theorem 4.3 we conclude that $\boldsymbol{c}_2$ is an Ulrich element in H.

5. Bottom elements as Ulrich elements in dimension 2

In the multigraded situation which we consider in Definition 3.1, there is in general no distinguished multidegree with $(\omega_{K[H]})_{\boldsymbol{b}} \neq 0$. Inspired by Proposition 3.6, we are prompted to test the Ulrich property for elements in ω_H with smallest entries. First we present the following lemma.

Lemma 5.1. *For any $H \in \mathcal{H}_2$, the set ω_H has a unique minimal element with respect to the componentwise partial ordering.*

Proof. Let C be the cone of H, and let $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ be points in the relative interior of C. We claim that $\boldsymbol{a} \wedge \boldsymbol{b} = (\min\{a_1, b_1\}, \min\{a_2, b_2\}) \in \operatorname{relint} C$. This will imply the existence of the unique minimal element of ω_H.

For the proof of the claim, it is enough to consider the case when $a_1 < b_1$ and $a_2 > b_2$. Since $a_2/a_1 > b_2/a_1 > b_2/b_1$, it follows that the point in the plane with coordinates $\boldsymbol{a} \wedge \boldsymbol{b} = (a_1, b_2)$ lies inside the cone with vertex the origin and passing through the points with coordinates $\boldsymbol{a}$ and $\boldsymbol{b}$. Since the latter cone is in $\operatorname{relint} C$ it follows that $\boldsymbol{a} \wedge \boldsymbol{b} \in \operatorname{relint} C$. □

We call the unique minimal element of ω_H with respect to the componentwise partial ordering, *the bottom element* of H.

Remark 5.2. For $H \in \mathcal{H}_2$, since the elements in ω_H have only nonnegative entries, it follows that the bottom element of H is also the smallest element in $G(\omega_H)$ with respect to the componentwise order. Moreover, if $K[H]$ is not a regular ring then the bottom element of H is componentwise the smallest element in $B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$; see Lemma 1.1.

In arbitrary embedding dimension we give the following definition.

Definition 5.3. For $H \in \mathcal{H}_d$, an element $\boldsymbol{b} \in \omega_H$ is called the *bottom element* of H if $\boldsymbol{c} - \boldsymbol{b} \in \mathbb{N}^d$ for all $\boldsymbol{c} \in \omega_H$.

Remark 5.4. In general, a semigroup $H \in \mathcal{H}_d$ with $d > 2$ may not have a unique minimal element in ω_H with respect to the componentwise partial ordering $\preceq$. For instance, let $d = 3$, $\boldsymbol{a}_1 = (5, 3, 1)$, $\boldsymbol{a}_2 = (1, 5, 2)$, $\boldsymbol{a}_3 = (8, 3, 5)$. Then, a calculation with [Normaliz] shows that

$$B_H = \big\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, (1, 2, 1), (2, 1, 1), (2, 2, 1), (2, 5, 2), (3, 2, 1), (3, 2, 2), (3, 5, 2), (3, 5, 3), (4, 5, 2), (5, 2, 3), (5, 5, 2), (5, 5, 4), (7, 5, 5)\big\}.$$

One can check that the vectors $\boldsymbol{n}_1 = (19, 11, -37), \boldsymbol{n}_2 = (-12, 17, 9), \boldsymbol{n}_3 = (1, -9, 22)$ are normal to the planes generated by $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, by $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$, by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ respectively. Also, that no element in $B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ lies on any of the three planes just mentioned. Consequently, there are no inner lattice points on the faces of $\overline{P_H}$ and $G(\omega_H) = B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$. It follows that $\boldsymbol{b}_1 = (1, 2, 1)$ and $\boldsymbol{b}_2 = (2, 1, 1)$ are both minimal elements in ω_H with respect to $\preceq$.

Using Theorem 4.3 we show that sometimes the bottom element may be the only Ulrich element in B_H.

Proposition 5.5. *Let $\boldsymbol{b}$ be the bottom element of $H \in \mathcal{H}_2$. If $2\boldsymbol{b} \in P_H$, then $\boldsymbol{b}$ is the only possible Ulrich element in B_H.*

Proof. Assume $\boldsymbol{b}' \in B_H$ is an Ulrich element in H. Then $2\boldsymbol{b} \in (\boldsymbol{a}_1 + H) \cup (\boldsymbol{a}_2 + H) \cup (\boldsymbol{b}' + H)$, by Theorem 4.3. Since $2\boldsymbol{b} \in P_H$, we get that $2\boldsymbol{b} \in \boldsymbol{b}' + H$; hence $2\boldsymbol{b} = \boldsymbol{b}' + \boldsymbol{h}$

for some $\boldsymbol{h} \in P_H$. Moreover, comparing componentwise, $\boldsymbol{b} \preceq \boldsymbol{b}'$ and $\boldsymbol{b} \preceq \boldsymbol{h}$ since $\boldsymbol{b}$ is the bottom element for H; thus $\boldsymbol{b}' = \boldsymbol{b}$. □

Remark 5.6. In general, as Example 4.6 shows, even when the Hilbert basis of H contains a unique Ulrich element, the latter need not be the bottom element.

In the following, we discuss when the bottom element $\boldsymbol{b}$ of $H \in \mathcal{H}_2$ is Ulrich.

Notation 5.7. To avoid repetitions, in the rest of the section $H \in \mathcal{H}_2$ has the extremal rays $\boldsymbol{a}_1 = (x_1, y_1)$ and $\boldsymbol{a}_2 = (x_2, y_2)$ such that ($y_1/x_1 < y_2/x_2$ when $x_1, x_2 > 0$) or $x_2 = 0$.

We define H_1 and H_2 to be the semigroups in $\mathcal{H}_2$ with the extremal rays $\boldsymbol{a}_1$ and $\boldsymbol{b}$, respectively $\boldsymbol{a}_2$ and $\boldsymbol{b}$. We denote $\mathbb{Z}^2 \cap \operatorname{relint} P_{H_i}$ by H_i^*, for $i = 1, 2$.

By an easy argument, the following proposition presents a class of semigroups in $\mathcal{H}_2$ with Ulrich bottom element.

Proposition 5.8. *Let $\boldsymbol{b}$ be the bottom element of H. If $(x_2, y_1) \preceq \boldsymbol{b}$, then $\boldsymbol{b}$ is an Ulrich element in H.*

Proof. If $K[H]$ is a regular ring, $\boldsymbol{b}$ is an Ulrich element in H, since $G(\omega_H) = \{\boldsymbol{b}\}$.

Assume $K[H]$ is not a regular ring. Then $\boldsymbol{b} \in G(\omega_H) = B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$, by Lemma 1.1(d). Let $\boldsymbol{c}_1, \boldsymbol{c}_2 \in B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$, and $\boldsymbol{c}_1 + \boldsymbol{c}_2 = (c, d)$, $\boldsymbol{b} = (u, v)$.

If $(c, d) \in H_1$, then $(c, d) = r_1(x_1, y_1) + r_2(u, v)$ for some $r_1, r_2 \in \mathbb{R}_{\geq 0}$. Since $d \geq 2v \geq y_1 + v$, we have $r_1 \geq 1$ or $r_2 \geq 1$. Consequently, $(c, d) \in (\boldsymbol{b} + H_1) \cup (\boldsymbol{a}_1 + H_1) \subset (\boldsymbol{b} + H) \cup (\boldsymbol{a}_1 + H)$.

A similar argument shows that if $(c, d) \in H_2$, then $(c, d) \in (\boldsymbol{b} + H) \cup (\boldsymbol{a}_2 + H)$. The conclusion follows by Theorem 4.3. □

Example 5.9. Let H be the semigroup with extremal rays $\boldsymbol{a}_1 = (a, 1)$ and $\boldsymbol{a}_2 = (1, b)$, where $a, b \geq 2$. Since $1/a < 1 < b$ we get that $\boldsymbol{b} = (1, 1)$ is in ω_H and it is the bottom element in H. Then Proposition 5.8 implies that $\boldsymbol{b}$ is an Ulrich element in H.

Clearly, $H = H_1 \cup H_2$ and $H_1 \cap H_2 = \mathbb{N}\boldsymbol{b}$. The following lemma states some nice properties regarding H_1 and H_2.

Lemma 5.10. *Let $\boldsymbol{b}$ be the bottom element of H. Then*

(a) $\boldsymbol{p} + \boldsymbol{q} \in \boldsymbol{b} + H$ *for all* $\boldsymbol{p} \in H_1 \setminus \{0\}$ *and* $\boldsymbol{q} \in H_2 \setminus \{0\}$,

(b) $(\boldsymbol{b} + H) \cup (\boldsymbol{a}_1 + H) \cup (\boldsymbol{a}_2 + H) = H \setminus (H_1^* \cup H_2^*)$.

Proof. (a) If $\boldsymbol{p} - \boldsymbol{b} \in H$ or $\boldsymbol{q} - \boldsymbol{b} \in H$, then clearly $\boldsymbol{p} + \boldsymbol{q} \in \boldsymbol{b} + H$. Let us assume that $\boldsymbol{p} - \boldsymbol{b} \notin H$ and $\boldsymbol{q} - \boldsymbol{b} \notin H$. Let C' be the cone generated by the extremal rays $\boldsymbol{p}, \boldsymbol{q}$. Since $\boldsymbol{b} \in C'$, we have $\boldsymbol{b} = r_1\boldsymbol{p} + r_2\boldsymbol{q}$ for some $r_1, r_2 \in \mathbb{R}_{>0}$. If $r_1 > 1$, then $\boldsymbol{b} - \boldsymbol{p} = (r_1 - 1)\boldsymbol{p} + r_2\boldsymbol{q}$; hence $\boldsymbol{b} - \boldsymbol{p} \in C' \cap \omega_H$, a contradiction with $\boldsymbol{b}$ being the bottom element in H. Therefore, $r_1 \leq 1$, and also $r_2 \leq 1$ by a similar argument. Now, $\boldsymbol{p} + \boldsymbol{q} - \boldsymbol{b} = (1 - r_1)\boldsymbol{p} + (1 - r_2)\boldsymbol{q} \in C' \cap \mathbb{Z}^2 \subset H$.

(b) Note that for any $\boldsymbol{p} \in H$ we have

$$\boldsymbol{p} \in H_1 \setminus H_1^* \Leftrightarrow \boldsymbol{p} \in (\boldsymbol{b} + H_1) \cup (\boldsymbol{a}_1 + H_1),$$
$$\boldsymbol{p} \in H_2 \setminus H_2^* \Leftrightarrow \boldsymbol{p} \in (\boldsymbol{b} + H_2) \cup (\boldsymbol{a}_2 + H_2).$$

Therefore, $H \setminus (H_1^* \cup H_2^*) = (H_1 \cup H_2) \setminus (H_1^* \cup H_2^*) \subseteq (\boldsymbol{b}+H) \cup (\boldsymbol{a}_1+H) \cup (\boldsymbol{a}_2+H)$.

In order to check the reverse inclusion, let $\boldsymbol{p} \in (\boldsymbol{b} + H) \cup (\boldsymbol{a}_1 + H) \cup (\boldsymbol{a}_2 + H)$.

We first consider the case $\boldsymbol{p} \in H_1$. Then clearly, $\boldsymbol{p} \notin H_2^*$. If we assume, on the contrary, that $\boldsymbol{p} \in H_1^*$, then $\boldsymbol{p} = r_1\boldsymbol{a}_1 + r_2\boldsymbol{b}$ with $r_1, r_2 \in (0, 1)$. We decompose $\boldsymbol{b} = \alpha_1\boldsymbol{a}_1 + \alpha_2\boldsymbol{a}_2$ with $\alpha_1, \alpha_2 \in (0, 1]$. This gives $\boldsymbol{p} = (r_1 + r_2\alpha_1)\boldsymbol{a}_1 + r_2\alpha_2\boldsymbol{a}_2$. Since $r_2\alpha_2 < 1$ and $r_2\alpha_2 < \alpha_2$ we infer that $\boldsymbol{p} \notin (\boldsymbol{a}_2 + H) \cup (\boldsymbol{b} + H)$. Thus $\boldsymbol{p} \in \boldsymbol{a}_1 + H$ and $r_1 \geq 1$, a contradiction. Consequently, $\boldsymbol{p} \notin H_1^* \cup H_2^*$.

A similar argument works for the case $\boldsymbol{p} \in H_2$. □

Lemma 5.11. *Let $\boldsymbol{p} = (k, r) \in H_1^*$ and $\boldsymbol{q} = (\ell, s) \in H_2^*$. If $\boldsymbol{b} = (u, v)$ is the bottom element of H, then*

(a) *$u < k < x_1$ and $v \leq r \leq y_1$,*

(b) *$u \leq \ell \leq x_2$ and $v < s < y_2$.*

Proof. We only show (a), part (b) is proved similarly. Clearly, $\boldsymbol{b} \neq \boldsymbol{p} \in \omega_H$; thus $0 < u \leq k$ and $0 < v \leq r$. If $u = k$, then since $\boldsymbol{p} \neq \boldsymbol{b}$, we have $v < r$. Then $r/k > v/u > y_1/x_1$, which gives that $\boldsymbol{p} \notin H_1$, which is false. Thus $u < k$.

On the other hand, by Lemma 2.1 applied in $H_1 \in \mathcal{H}_2$ for $\boldsymbol{p}$, the point

$$(u, v) + (x_1, y_1) - (k, r) = (u + x_1 - k, v + y_1 - r) \in H_1^*,$$

hence $u < u + x_1 - k$ and $v \leq v + y_1 - r$. That gives $k < x_1$ and $r \leq y_1$. □

The following result restricts the verification of the bottom element being Ulrich to verifying that the sum of any two points in H_i^* is not in H_i^*, for $i = 1, 2$.

Lemma 5.12. *Assume $\boldsymbol{b}$ is the bottom element of H. The following conditions are equivalent*:

(a) *$\boldsymbol{b}$ is an Ulrich element in H.*

(b) *For $i = 1, 2$, if $\boldsymbol{p}, \boldsymbol{q} \in H_i^*$ then $\boldsymbol{p} + \boldsymbol{q} \notin H_i^*$.*

Proof. We know that $\boldsymbol{b} \in G(\omega_H)$ since it is the bottom element in H. If $K[H]$ is a regular ring, then $\boldsymbol{b} = \boldsymbol{a}_1 + \boldsymbol{a}_2$. Hence statement (a) holds by Example 3.4, and (b) is true since $H_1^* = H_2^* = \varnothing$.

Assume that $K[H]$ is not a regular ring, hence $\boldsymbol{b} \in B_H \setminus \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. According to Theorem 4.3, $\boldsymbol{b}$ is an Ulrich element in H if and only if for all $\boldsymbol{p}, \boldsymbol{q} \in B_H$ one has

$$\boldsymbol{p} + \boldsymbol{q} \in (\boldsymbol{b} + H) \cup (\boldsymbol{a}_1 + H) \cup (\boldsymbol{a}_2 + H). \tag{9}$$

It is of course equivalent to check (9) for all $\boldsymbol{p}$ and $\boldsymbol{q}$ nonzero in H.

If ($\boldsymbol{p} \in H_1$ and $\boldsymbol{q} \in H_2$) or ($\boldsymbol{p} \in H_2$ and $\boldsymbol{q} \in H_1$) then $\boldsymbol{p}+\boldsymbol{q} \in \boldsymbol{b}+H$, by Lemma 5.10. Thus, for (a) it suffices to check (9) for nonzero $\boldsymbol{p}, \boldsymbol{q}$ both in H_1 or both in H_2. For $i = 1, 2$, the semigroup H_i is normal and simplicial; hence any $\boldsymbol{p} \in H_i$ is of the form $\boldsymbol{p} = n_1\boldsymbol{b} + n_2\boldsymbol{a}_i + \boldsymbol{p}'$ with $n_1, n_2 \in \mathbb{N}$ and $\boldsymbol{p}' \in H_i^* \cup \{0\}$. Consequently, $\boldsymbol{b}$ is an Ulrich element in H if and only if property (b) holds. □

Definition 5.13. We say that H *is AG1* if condition (b) in Lemma 5.12 is satisfied for $i = 1$, and we call it *AG2* if the said condition is satisfied for $i = 2$.

Thus the bottom element is an Ulrich element in H if and only if H is AG1 and AG2.

Remark 5.14. Using Lemma 5.11, property AG1 means that for any $\boldsymbol{p} = (k, r)$ and $\boldsymbol{q} = (\ell, s) \in H_1^*$ such that $k + \ell < x_1$ and $r + s \leq y_1$ one has that $\boldsymbol{p}+\boldsymbol{q} \notin H_1^*$.

Similarly, the AG2 condition means that when $\boldsymbol{p} = (k, r)$ and $\boldsymbol{q} = (\ell, s) \in H_2^*$ such that $k + \ell \leq x_2$ and $r + s < y_2$, then $\boldsymbol{p}+\boldsymbol{q} \notin H_2^*$.

Remark 5.15. Assume $\boldsymbol{b} = (u, v)$ is the bottom element in H. If $y_1 = 0$ then $v = 1$, since otherwise the inequalities $v/u > (v-1)/u > y_1/x_1 = 0$ would give that $(u, v-1) \in \omega_{H_1}$, a contradiction to the fact that (u, v) is the bottom element in H. Then, by Lemma 5.11 we get that $H_1^* = \varnothing$; hence H satisfies condition AG1.

Similarly, if $x_2 = 0$ then $u = 1$ and $H_2^* = \varnothing$; hence H is AG2.

In order to check the AG1 and AG2 conditions we need to have a better understanding of the points in H_1^* and H_2^*. We can count their elements.

Lemma 5.16. *Let* $\boldsymbol{b} = (u, v)$ *be the bottom element for* H. *Then*

(a) $|H_1^*| = vx_1 - uy_1 - 1$ *and* $|H_2^*| = uy_2 - vx_2 - 1$,

(b) $1 \leq vx_1 - uy_1 \leq x_1$ *and* $1 \leq uy_2 - vx_2 - 1 \leq y_2$,

(c) *if* $H_1^* \neq \varnothing$ *then* $vx_1 - uy_1 \leq x_1 - u$,

(d) *if* $H_2^* \neq \varnothing$ *then* $uy_2 - vx_2 \leq y_2 - v$.

Proof. We only show the first parts of (a) and (b), since the second parts are proved similarly.

(a) The area of the parallelogram spanned by $\boldsymbol{b}$ and $\boldsymbol{a}_1$ equals $\det\begin{pmatrix} x_1 & u \\ y_1 & v \end{pmatrix} = vx_1 - uy_1$. Since the boundary of that parallelogram contains precisely four lattice points, the vertices, (here we use the fact that $\gcd(u, v) = \gcd(x_1, y_1) = 1$), Pick's theorem [Beck and Robins 2015, Theorem 2.8] implies that P_{H_1} has $vx_1 - uy_1 - 1$ inner lattice points, which proves the claim.

(b) The inequality $1 \leq vx_1 - uy_1$ follows from (a). Since (u, v) is the bottom element of H, it follows that $(u, v-1)$ is not in ω_H and in relint P_{H_1}. As $(v-1)/u < v/u$, and $y_1/x_1 < v/u$ by our assumption, we get that $(v-1)/u \leq y_1/x_1$, i.e., $vx_1 - uy_1 \leq x_1$.

Parts (c) and (d) will be proved after Remark 5.19. □

One nice consequence of Lemma 5.16 is a Gorenstein criterion for $K[H]$ in terms of the coordinates of the bottom element in H.

Corollary 5.17. *If $\boldsymbol{b} = (u, v)$ is the bottom element in H, then the K-algebra $K[H]$ is Gorenstein if and only if $vx_1 - uy_1 = uy_2 - vx_2 = 1$.*

Proof. The ring $K[H]$ is Gorenstein if and only if ω_H is a principal ideal, and hence generated by $\boldsymbol{b}$, which is equivalent to saying that P_{H_1} and P_{H_2} have no inner points. By Lemma 5.16 this is the case if and only if $vx_1 - uy_1 = uy_2 - vx_2 = 1$. □

Lemma 5.18. *Let $\boldsymbol{b} = (u, v)$ be the bottom element of H. We assume that H_1^* is not the empty set. For any integer i we consider the integers q_i, r_i such that $iy_1 = q_i x_1 + r_i$ with $0 \le r_i < x_1$.*

Assume the integer k satisfies $u < k < x_1$. The following statements are equivalent:

(i) *k is the x-coordinate of some $\boldsymbol{p} \in H_1^*$.*

(ii) $\lceil ky_1/x_1 \rceil \le v + \lfloor (k-u)y_1/x_1 \rfloor$.

(iii) $\lceil ky_1/x_1 \rceil = v + \lfloor (k-u)y_1/x_1 \rfloor$.

(iv) $q_k \le v - 1 + q_{k-u}$.

(v) $q_k = v - 1 + q_{k-u}$.

(vi) $r_k \ge r_{k-u} + x_1 - (vx_1 - uy_1)$.

(vii) $r_k = r_{k-u} + x_1 - (vx_1 - uy_1)$.

(viii) $r_k \ge x_1 - (vx_1 - uy_1)$.

If any of these conditions holds, then $\boldsymbol{p} = (k, \lceil ky_1/x_1 \rceil) = (k, q_k + 1)$.

Proof. Since $H_1^* \neq \varnothing$ we have that $y_1 > 0$ and $u < x_1$ by Remark 5.15 and Lemma 5.11, respectively. We note that for any integer $u < k < x_1$, the fractions ky_1/x_1 and $v + (k-u)y_1/x_1$ are not integers. Thus $\lceil ky_1/x_1 \rceil = q_k + 1$ and $\lfloor v + (k-u)y_1/x_1 \rfloor = v + q_{k-u}$. This shows that (ii) $\Longleftrightarrow$ (iv) and (iii) $\Longleftrightarrow$ (v). Since $q_k = (ky_1 - r_k)/x_1$, simple manipulations give that (iv) $\Longleftrightarrow$ (vi) and (v) $\Longleftrightarrow$ (vii).

We also infer that the number of points in H_1^* whose x-coordinate is k equals the number of lattice points on the line $x = k$ located strictly between the lines $y = (y_1/x_1)x$ and $y = (y_1/x_1)(x-u) + v$, which is

$$\left\lfloor \frac{y_1}{x_1}(k-u) + v \right\rfloor - \left\lceil \frac{y_1}{x_1}k \right\rceil + 1 = v + q_k + \left\lfloor \frac{r_k - uy_1}{x_1} \right\rfloor - (q_k + 1) + 1 \tag{10}$$

$$= \left\lfloor \frac{vx_1 - uy_1 + r_k}{x_1} \right\rfloor \in \{0, 1\}. \tag{11}$$

The latter statement is due to the fact that $r_k < x_1$ and $vx_1 - uy_1 \le x_1$, by Lemma 5.16.

Consequently, $k \in (u, x_1)$ is the x-coordinate of some point in H_1^* if and only if the value in equation (10) is at least (and actually equal to) 1, which is equivalent to property (ii), respectively to (iii). That is, moreover, equivalent (using (11)) to

$$1 \leq \frac{vx_1 - uy_1 + r_k}{x_1},$$

which can be rewritten as $r_k \geq x_1 - (vx_1 - uy_1)$, namely statement (viii).

From (10) and (11) we obtain that if k is the x-coordinate of some point $\boldsymbol{p} \in H_1^*$, then $\boldsymbol{p} = (k, \lceil (y_1/x_1)k \rceil) = (k, q_k + 1)$. □

Remark 5.19. A similar result holds for the points in H_2^* in terms of the integers q'_i, r'_i such that $ix_2 = q'_i y_2 + r'_i$ with $0 \leq r'_i < y_2$.

Now we can finish the proof of Lemma 5.16.

Proof of Lemma 5.16, continued. (c) By Lemma 5.18, for each $u < k < x_1$ there is at most one point in H_1^* whose x-coordinate is k; therefore $|H_1^*| \leq x_1 - u - 1$. Using point (a) we obtain the inequality at (c). Part (d) is proved similarly. □

It will be convenient to denote $\pi_1(H_1^*) = \{k : \text{ there exists } (k, \ell) \in H_1^*\}$. The next result is a criterion to verify the AG1 property in terms of the remainders r_i introduced in Lemma 5.18, with $i \in \pi_1(H_1^*)$. A similar statement characterizes the AG2 property in terms of the r'_j's from Remark 5.19, with $j \in \pi_2(H_2^*)$.

Proposition 5.20. *For any integer i let $r_i \equiv iy_1 \bmod x_1$ with $0 \leq r_i < x_1$. Then H is AG1 if and only if $r_k + r_\ell < 2x_1 - (vx_1 - uy_1)$ for all integers $k, \ell \in \pi_1(H_1^*)$ with $k + \ell < x_1$.*

Proof. If $H_1^* = \varnothing$ then there is nothing to prove. Assume H_1^* is not empty. If $k, \ell \in \pi_1(H_1^*)$, then by Lemma 5.18, $\boldsymbol{p}_1 = (k, \lceil ky_1/x_1 \rceil)$ and $\boldsymbol{p}_2 = (\ell, \lceil \ell y_1/x_1 \rceil)$ are the corresponding points in H_1^*. By definition, H is AG1 if and only if $\boldsymbol{p}_1 + \boldsymbol{p}_2 \notin H_1^*$ for all $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as above. When $k + \ell \geq x_1$, Lemma 5.11 implies already that $\boldsymbol{p}_1 + \boldsymbol{p}_2 \notin H_1^*$. If $k + \ell < x_1$, then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \notin H_1^*$ if and only if

$$\left\lceil \frac{ky_1}{x_1} \right\rceil + \left\lceil \frac{\ell y_1}{x_1} \right\rceil \geq (k + \ell - u)\frac{y_1}{x_1} + v, \quad \text{equivalently} \tag{12}$$

$$\frac{ky_1 - r_k}{x_1} + 1 + \frac{\ell y_1 - r_\ell}{x_1} + 1 \geq (k + \ell - u)\frac{y_1}{x_1} + v,$$

$$ky_1 - r_k + \ell y_1 - r_\ell + 2x_1 \geq (k + \ell)y_1 - uy_1 + vx_1,$$

$$2x_1 - (vx_1 - uy_1) \geq r_k + r_\ell. \tag{13}$$

Since $u < k + \ell < x_1$, the term of the right-hand side of (12) is not an integer; hence the inequality at (12) (and equivalently, at (13)) can not become an equality. □

6. A criterion for (1, 1) to be an Ulrich element

Our aim in this section is to obtain a complete classification of when $\boldsymbol{b} = (1, 1)$ is an Ulrich element. The setup in Notation 5.7 is in use. The element $(1, 1)$ is in ω_H if and only if $y_1/x_1 < 1 < y_2/x_2$. If that is the case, it is clear that $(1, 1)$ is the bottom element in H. It suffices to verify the AG1 and AG2 conditions, by Lemma 5.12.

Set $n = x_1 - y_1 - 1$, which equals $|H_1^*|$, by Lemma 5.16. If $n = 0$, then H is clearly AG1.

We consider the case $n > 0$. The next result presents an explicit way to determine H_1^*. Recursively, we define nonnegative integers $\ell_1, \dots, \ell_n$ and $s_1, \dots, s_n$ by

$$x_1 = \ell_1(x_1 - y_1) + s_1 \quad \text{with} \quad s_1 < x_1 - y_1,$$
$$y_1 + s_{i-1} = \ell_i(x_1 - y_1) + s_i \quad \text{with} \quad s_i < x_1 - y_1,$$

for $i = 2, \dots, n$.

Lemma 6.1. *Assume that* $(1, 1)$ *belongs to* ω_H *and* $H_1^* \neq \varnothing$. *Then*

$$H_1^* = \left\{ \boldsymbol{p}_t = (c_t, d_t) : c_t = t + \sum_{i=1}^{t} \ell_i,\ d_t = \sum_{i=1}^{t} \ell_i,\ t = 1, \dots, n \right\},$$

Proof. For $k = 1, \dots, x_1 - 1$, let $ky_1 = q_k x_1 + r_k$ with integers $q_k \geq 0$ and $x_1 > r_k \geq 0$.

By Lemma 5.18, the integer $k > 1$ is the x-coordinate of an element of H_1^* if and only if $q_k = q_{k-1}$. In this case, $(k, 1 + q_k) \in H_1^*$.

Now, let $t \geq 1$. Summing up the equations $x_1 = \ell_1(x_1 - y_1) + s_1$ and $y_1 + s_{i-1} = \ell_i(x_1 - y_1) + s_i$, for $i = 2, \dots, t$, we get

$$x_1 + (t-1)y_1 + s_1 + s_2 + \cdots + s_{t-1} = \sum_{i=1}^{t} \ell_i(x_1 - y_1) + s_1 + s_2 + \cdots + s_t,$$

and consequently,

$$\left(t - 1 + \sum_{i=1}^{t} \ell_i\right) y_1 = \left(\sum_{i=1}^{t} \ell_i - 1\right) x_1 + s_t.$$

Then

$$\left(t + \sum_{i=1}^{t} \ell_i\right) y_1 = \left(\sum_{i=1}^{t} \ell_i - 1\right) x_1 + s_t + y_1,$$

with $s_t + y_1 < x_1$. Therefore, $q_k = q_{k-1} = \left(\sum_{i=1}^{t} \ell_i - 1\right)$ for $k = t + \sum_{i=1}^{t} \ell_i$.

Note that

$$\begin{aligned} n+\sum_{i=1}^{n}\ell_i &= n+\frac{x_1-s_1}{x_1-y_1}+\sum_{i=2}^{t}\frac{y_1-s_{i-1}+s_i}{x_1-y_1} \\ &= n+\frac{x_1+(n-1)y_1-s_n}{x_1-y_1} = n+1+\frac{ny_1-s_n}{x_1-y_1} \\ &< n+1+y_1 = x_1. \end{aligned}$$

Hence $\boldsymbol{p}_t = \left(t+\sum_{i=1}^{t}\ell_i, \sum_{i=1}^{t}\ell_i\right) \in H_1^*$ for $t=1,\dots,n$.

We know from Lemma 5.16, that H_1^* has exactly $n = x_1-y_1-1$ elements, so $\boldsymbol{p}_1,\dots,\boldsymbol{p}_n$ are the only elements of H_1^*. □

Examples 6.2. Let $x_1 = \ell_1(x_1-y_1)+s_1$ and $y_1+s_{i-1} = \ell_i(x_1-y_1)+s_i$, for $i=2,\dots,n=x_1-y_1-1$ as before.

(a) If $y_1 = 1$, then $H_1^* = \{(m,1) : m = 2,\dots,x_1-1\}$. In this case, H is AG1 by Lemma 5.12.

(b) If $x_1-y_1 \in \{1,2\}$ then by Lemma 5.16, H_1^* is either empty, or it consists of one element, which is different from $(0,0)$. Hence H is AG1.

(c) If $2 < 2y_1 < x_1 < 3y_1$, then $\ell_1 = \ell_2 = 1$. Therefore, $\boldsymbol{p}_1 = (2,1)$ and $\boldsymbol{p}_2 = (4,2) = 2\boldsymbol{p}_1$ belong to H_1^*. Then, by definition, H is not AG1.

Next, we give a simple arithmetic criterion to check the AG1 or AG2 property:

Theorem 6.3. *Assume that* $(1,1)$ *belongs to* ω_H. *Assuming Notation 5.7, then*

(a) *H is AG1 if and only if* $x_1 \equiv 1 \bmod (x_1-y_1)$;

(b) *H is AG2 if and only if* $y_2 \equiv 1 \bmod (y_2-x_2)$;

(c) $(1,1)$ *is an Ulrich element in H if and only if* $x_i \equiv 1 \bmod (x_i-y_i)$ *for* $i=1,2$.

Proof. (a) Let $n = x_1-y_1-1 = |H_1^*|$. If $n \in \{0,1\}$, then H is AG1 by Examples 6.2(b). On the other hand, if $n=0$ then $x_1-y_1 = 1$ and clearly, $x_1 \equiv 1 \bmod (x_1-y_1)$. When $n=1$ we have $x_1-y_1 = 2$. Since $\gcd(x_1,y_1) = 1$ we get that x_1 is odd; hence $x_1 \equiv 1 \bmod (x_1-y_1)$, too.

We further prove the stated equivalence when $n \geq 2$. Let $\ell_1,\dots,\ell_n \geq 0$ and $x_1-y_1 > s_1,\dots,s_n \geq 0$ such that

$$x_1 = \ell_1(x_1-y_1)+s_1, \quad y_1+s_{i-1} = \ell_i(x_1-y_1)+s_i$$

for $i=2,\dots,n$. Then

$$H_1^* = \left\{\boldsymbol{p}_t = (c_t,d_t) : c_t = t+\sum_{i=1}^{t}\ell_i,\ d_t = \sum_{i=1}^{t}\ell_i,\ t=1,\dots,n\right\},$$

by Lemma 6.1. We note that since $y_1 > 0$ (see Remark 5.15) we have $x_1 > x_1-y_1$; hence $\ell_1 \geq 1$.

Assume that $x_1 \equiv 1 \bmod (x_1 - y_1)$. Then it is easy to check that $s_i = i$ and $\ell_i = \ell_1 - 1$ for $i = 2, \ldots, n$. Consequently,

$$H_1^* = \{(t\ell_1 + 1, t(\ell_1 - 1) + 1) : t = 1, \ldots, n\},$$

and therefore, the sum of any two elements of H_1^* is not in H_1^*, i.e., H is AG1.

Conversely, assume that H is AG1. As $n > 0$ we get that $x_1 - y_1 > 1$ and $y_1 > 0$. In case $y_1 = 1$, then clearly, $x_1 \equiv 1 \bmod (x_1 - y_1)$.

We consider the case $y_1 \geq 2$. As

$$1 = \gcd(x_1, y_1) = \gcd(x_1, x_1 - y_1) = \gcd(s_1, x_1 - y_1)$$

and $x_1 - y_1 > 1$, we have that $s_1 > 0$. We need to prove that $s_1 = 1$.

Assume, on the contrary, that $s_1 \neq 1$. Then $s_1 \geq 2$. Since

$$\begin{aligned}(\ell_1 - 1)(x_1 - y_1) + s_1 = y_1 \leq y_1 + s_{i-1} = \ell_i(x_1 - y_1) + s_i,\\ y_1 + s_{i-1} < y_1 + (x_1 - y_1) = x_1 = \ell_1(x_1 - y_1) + s_1,\end{aligned}$$

we have $\ell_1 - 1 \leq \ell_i \leq \ell_1$, for $i = 2, \ldots, n$.

If $\ell_2 = \ell_1$, then $\boldsymbol{p}_2 = (2 + 2\ell_1, 2\ell_1) = 2\boldsymbol{p}_1$, which contradicts the AG1 property. Now, we consider the case $\ell_2 = \ell_1 - 1$. By subtracting the equations

$$x_1 = \ell_1(x_1 - y_1) + s_1 \quad \text{and} \quad y_1 + s_1 = \ell_2(x_1 - y_1) + s_2,$$

we get that $s_2 = 2s_1$; hence $s_2 > s_1$.

If $\ell_2 = \cdots = \ell_n$ then $s_1 < s_2 < \cdots < s_n$ is an increasing sequence of n positive integers less than $n + 1$, and hence $s_1 = 1$, which is false. Thus $\ell_i = \ell_1$ for some $i \geq 3$. Let i be the smallest index with this property, i.e., $\ell_2 = \cdots = \ell_{i-1} = \ell_1 - 1$ and $\ell_i = \ell_1$. Then

$$\begin{aligned}\boldsymbol{p}_i &= \big(i + (i-2)(\ell_1 - 1) + 2\ell_1,\ (i-2)(\ell_1 - 1) + 2\ell_1\big)\\ &= (1 + \ell_1, \ell_1) + \big(i - 1 + (i-2)(\ell_1 - 1) + \ell_1,\ (i-2)(\ell_1 - 1) + \ell_1\big)\\ &= \boldsymbol{p}_1 + \boldsymbol{p}_{i-1},\end{aligned}$$

which is a contradiction. This shows that when H is AG1, then $x_1 \equiv 1 \bmod (x_1 - y_1)$.

For part (b) we let H' be the semigroup in $\mathcal{H}_2$ with the extremal rays $\boldsymbol{a}_1' = (y_2, x_2)$ and $\boldsymbol{a}_2' = (y_1, x_1)$. We remark that H is AG2 if and only if H' is AG1, and we use (a). Part (c) is a consequence of (a) and (b). □

Corollary 6.4. *Let H be a semigroup in $\mathcal{H}_2$ with extremal rays $\boldsymbol{a}_i = (x_i, y_i)$ for $i = 1, 2$. Assume $(1, 1) \in \omega_H$ and $x_1x_2y_1y_2 \neq 0$. Then $K[H]$ is AG if and only if $x_i \equiv 1 \bmod (x_i - y_i)$ for $i = 1, 2$.*

Proof. By Proposition 3.6, the only possible Ulrich element in H is $(1, 1)$. The conclusion follows by Theorem 6.3. □

Remark 6.5. In the statement of Corollary 6.4, the assumption $x_1 x_2 y_1 y_2 \neq 0$ can not be dropped. For instance, let $H \in \mathcal{H}_2$ with the extremal rays $\boldsymbol{a}_1 = (1, 0)$ and $\boldsymbol{a}_2 = (2, 5)$. Its Hilbert basis is

$$B_H = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{c}_1 = (1, 1), \boldsymbol{c}_2 = (2, 3), \boldsymbol{c}_3 = (1, 2)\}.$$

The bottom element in H is $\boldsymbol{c}_1$, and by Theorem 6.3 it follows that H is not AG2.

Still, H is AG. Since $2\boldsymbol{c}_1 = (2, 2) = \boldsymbol{c}_3 + \boldsymbol{a}_1$, $2\boldsymbol{c}_2 = (4, 6) = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{c}_1$ and $\boldsymbol{c}_1 + \boldsymbol{c}_2 = (3, 4) = \boldsymbol{a}_1 + 2\boldsymbol{c}_3$, by Theorem 4.3 we get that $\boldsymbol{c}_3$ is an Ulrich element in H.

7. Nearly Gorenstein semigroup rings

In this section we prove the nearly Gorenstein property for semigroup rings $K[H]$ when $H \in \mathcal{H}_2$.

Nearly Gorenstein rings approximate Gorenstein rings in a different way as almost Gorenstein rings. In [Herzog et al. 2019], a local (or graded) Cohen–Macaulay ring which admits a canonical module ω_R is called *nearly Gorenstein* if the trace of ω_R contains the (graded) maximal ideal of R. In the case that R is a domain, the canonical module can be realized as an ideal of R and its trace in R, which we denote by $\operatorname{tr}(\omega_R)$, is the ideal $\sum_f f\omega_R$, where the sum is taken over all f in the quotient field of R for which $f\omega_R \subseteq R$; see [Herzog et al. 2019, Lemma 1.1].

A one-dimensional almost Gorenstein ring is nearly Gorenstein, but the converse does not hold in general. In higher dimension there is in general no implication valid between these two concepts; see [Herzog et al. 2019].

Theorem 7.1. *Let H be a simplicial affine semigroup in $\mathcal{H}_2$. Then $R = K[H]$ is a nearly Gorenstein ring.*

Proof. Let $\boldsymbol{a}_1 = (c, d)$ and $\boldsymbol{a}_2 = (e, f)$ be the extremal rays of H. We may assume that $d/c < f/e$ and that R is not already a Gorenstein ring.

The vector $\boldsymbol{n}_1 = (-d, c)$ is orthogonal to $\boldsymbol{a}_1$ and $\boldsymbol{n}_2 = (f, -e)$ is orthogonal to $\boldsymbol{a}_2$. Moreover, $\boldsymbol{c}$ is in C, the cone over H, if and only if $\langle \boldsymbol{n}_1, \boldsymbol{c}\rangle \geq 0$ and $\langle \boldsymbol{n}_2, \boldsymbol{c}\rangle \geq 0$.

Let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_t, \boldsymbol{c}_{t+1}, \boldsymbol{c}_{t+2}$ be the Hilbert basis of H, where $\boldsymbol{c}_{t+i} = \boldsymbol{a}_i$ for $i = 1, 2$. Then ω_R is generated by $v_i = \boldsymbol{x}^{\boldsymbol{c}_i}$ for $i = 1, \dots, t$; see Lemma 1.1.

In order to prove that R is nearly Gorenstein, it suffices to show that for each element $\boldsymbol{c}_i$ of the Hilbert basis there exist $\boldsymbol{c} \in \mathbb{Z}^2$ and an integer $k \in \{1, \dots, t\}$ such that

(i) $\boldsymbol{c} + \boldsymbol{c}_j \in C$ for $j = 1, \dots, t$, and

(ii) $\boldsymbol{c} + \boldsymbol{c}_k = \boldsymbol{c}_i$.

If $i \in \{1, \dots, t\}$, we may choose $\boldsymbol{c} = 0$ and $k = i$. It suffices to consider the cases $i = t + 1$ and $i = t + 2$. By symmetry we may assume that $i = t + 1$, and have to find $\boldsymbol{c} \in \mathbb{Z}^2$ and $k \in \{1, \dots, t\}$ such that (i) is satisfied and such that $\boldsymbol{c} + \boldsymbol{c}_k = \boldsymbol{a}_1$.

Let $k \in \{1, \dots, t\}$ be chosen such that $\langle \boldsymbol{n}_1, \boldsymbol{c}_k \rangle = \min\{\langle \boldsymbol{n}_1, \boldsymbol{c}_j \rangle : j = 1, \dots, t\}$. Set $\boldsymbol{c} = \boldsymbol{a}_1 - \boldsymbol{c}_k$. Then $\boldsymbol{c} + \boldsymbol{c}_k = \boldsymbol{a}_1$. Moreover, by the choice of k for $j = 1, \dots, t$ we have

$$\langle \boldsymbol{n}_1, \boldsymbol{c} + \boldsymbol{c}_j \rangle = \langle \boldsymbol{n}_1, \boldsymbol{a}_1 \rangle - \langle \boldsymbol{n}_1, \boldsymbol{c}_k \rangle + \langle \boldsymbol{n}_1, \boldsymbol{c}_j \rangle = 0 - \langle \boldsymbol{n}_1, \boldsymbol{c}_k \rangle + \langle \boldsymbol{n}_1, \boldsymbol{c}_j \rangle \geq 0,$$
$$\langle \boldsymbol{n}_2, \boldsymbol{c} + \boldsymbol{c}_j \rangle = \langle \boldsymbol{n}_2, \boldsymbol{a}_1 \rangle - \langle \boldsymbol{n}_2, \boldsymbol{c}_k \rangle + \langle \boldsymbol{n}_2, \boldsymbol{c}_j \rangle.$$

Since $\boldsymbol{c}_j \in H$, we have $\langle \boldsymbol{n}_2, \boldsymbol{c}_j \rangle \geq 0$. Let L be the line passing through $\boldsymbol{c}_k$ which is parallel to $L_2 = \mathbb{R}\boldsymbol{a}_2$, and L' be the line passing through $\boldsymbol{a}_1$ parallel to L_2. Since $\boldsymbol{c}_k \in P_H$, the line L has a smaller distance to L_2 than the line L'. This implies that $\langle \boldsymbol{n}_2, \boldsymbol{a}_1 \rangle > \langle \boldsymbol{n}_2, \boldsymbol{c}_k \rangle$; hence $\langle \boldsymbol{n}_2, \boldsymbol{c} + \boldsymbol{c}_j \rangle > 0$. Thus we conclude that $\boldsymbol{c} + \boldsymbol{c}_j \in C$, as desired. □

Theorem 7.1 is no longer valid when $\dim K[H] > 2$, as the following example shows.

Example 7.2. We consider again the semigroup $H \in \mathcal{H}_3$ from Remark 5.4. It turns out that $K[H]$ is not nearly Gorenstein for this semigroup H. One can see that $\boldsymbol{a}_1$ does not satisfy the two conditions (i) and (ii) in the proof of Theorem 7.1.

In fact, if we consider the set A of all $\boldsymbol{a}_1 - \boldsymbol{c}_i$ for $i = 1, \dots, 13$, then the third component of elements in A belongs to $\{0, -1, -2, -3, -4\}$. Adding the elements with negative third component to $(1, 2, 1)$, we get a vector with third component less than 1, which does not belong to C, the cone over H. Adding those elements in A with zero third component to either $(2, 1, 1)$ or $(1, 2, 1)$, we again get a vector which does not belong to C.

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JÜRGEN HERZOG
FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT DUISBURG-ESSEN
ESSEN
GERMANY
juergen.herzog@uni-essen.de

RAHELEH JAFARI
MOSAHEB INSTITUTE OF MATHEMATICS
KHARAZMI UNIVERSITY
TEHRAN
IRAN
and
SCHOOL OF MATHEMATICS
INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM)
TEHRAN
IRAN
rjafari@ipm.ir

DUMITRU I. STAMATE
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF BUCHAREST
BUCHAREST
ROMANIA
dumitru.stamate@fmi.unibuc.ro

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