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CLOSURES OF 3-BRAIDS AND DETECTION

FRASER BINNS

We give some new link detection results for link Floer homology, Khovanov homology and annular Khovanov homology. The links we detect arise via different closure operations on 3-braids. Examples of our results include that link Floer homology detects the Mazur link, that annular Khovanov homology detects the Mazur pattern, and that Khovanov homology detects L6a2 and L9n15. The Mazur pattern detection result depends on a new bound on the rank of the annular Khovanov homology of certain links.

Braids are of wide mathematical interest; see the survey article [14]. In this paper we will consider the four types of links obtained from braids, as shown in Figure 1.

Let α be a braid. The first two types of link we obtain from α have been widely studied. We have the *braid-closure* of α, $b(\alpha)$, which for the purposes of this paper is the link in the thickened annulus obtained by attaching n parallel strands as in Figure 1(a). Secondly, we have the *augmented braid-closure* of α, $\mathring{b}(\alpha)$, which is the link obtained by adding the annular axis to $b(\alpha)$, as shown in Figure 1(b).

For the remaining two types of link we move beyond the usual setting of braid-closures. The *clasp-closure* of α, $c(\alpha)$, can be thought of as the annular link formed by $b(\alpha)$ and replacing two parallel strands in a ball with a *clasp*, as shown in Figure 1(c). Note that the clasp is between the two rightmost strands of α, though this is simply a matter of convention and plays no significant role. The *augmented clasp-closure* of α, $\mathring{c}(\alpha)$, is defined analogously to the augmented braid-closure of α; see Figure 1(d).

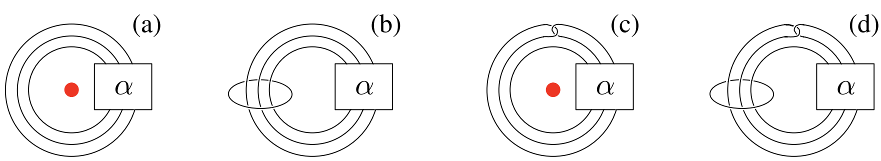

Figure 1. The four types of links we study: (a) a braid-closure; (b) an augmented braid-closure; (c) a clasp-closure; (d) an augmented clasp-closure. The red dots indicate the axes.

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As motivation for studying clasp-closures, we mention a result of Martin [32, Proposition 1] stating that the links with the simplest link Floer homology — in an appropriate sense — are augmented braid-closures. A result of the author and Dey [11, Theorem 5.1] says that augmented clasp-closures are examples of links with second simplest link Floer homology, in the same sense. Thus one might reasonably expect that understanding the behavior of categorified link invariants of braid- and clasp-closures of braids might be easier than understanding other types of closure.

(Augmented) braid-closures of 1- and 2-braids are readily classified up to isotopy. Braid-closures of 3-braids were classified completely by Murasugi [33]. In particular, he showed that there are three braid-closures of 3-braids representing the unknot, namely $\sigma_1\sigma_2$, $\sigma_1^{-1}\sigma_2^{-1}$ and $\sigma_1\sigma_2^{-1}$. Here we use the standard Artin generators for the braid group. The augmented braid-closures of these three links are $T(2,6)$, $T(2,-6)$ and L6a2 respectively. More generally, Birman and Menasco showed that for $|n| \neq 1$ there are two 3-braid representatives of the torus links $T(2,n)$, namely $\sigma_1^{\pm 1}\sigma_2^{n}$ [15]. Note that $\mathring{b}(\sigma_1\sigma_2^3)$ is L9n15 while $\mathring{b}(\sigma_1^{-1}\sigma_2^3)$ is L9n16.

One cannot take the clasp-closure of a 1-braid. Clasp-closures of 2-braids are the twisted Whitehead patterns. The case of (augmented) clasp-closures of 3-braids is more complicated. Baldwin and Sivek classified 3-braids with clasp-closures representing the unknot, up to isotopy of the clasp-closure; see the proof of [4, Theorem 6.1]. Up to mirroring and reversal these braids are as follows:

(1) σ_1^{-1}. The augmentation of this link is L7a6, i.e., the mirror of the Mazur link.
(2) $\sigma_1^3\sigma_2^{-1}\sigma_1^2\sigma_2$.
(3) $\sigma_1^n\sigma_2^{-1}\sigma_1\sigma_2$. For $n = 1$ the augmentation of this link is L7a5.

Our goal in this paper is to exploit the Baldwin–Sivek, Murasugi and Birman–Menasco classification results to obtain detection results for various categorified link invariants. There are three different invariants we will study; link Floer homology and two versions of Khovanov homology.

Link Floer homology is an invariant of oriented links defined by Ozsváth and Szabó using symplectic topology [36]. For two-component links it takes value in the category of triply graded vector spaces. Our first results are the following:

Theorem 1.1. *Link Floer homology detects* L6a2.

Theorem 1.3. *Link Floer homology with rational coefficients detects* L9n15.

L6a2 is the augmented braid-closure of $\sigma_1\sigma_2^{-1}$. The author and Martin showed in [13] that link Floer homology detects the augmented braid-closures of the other two braids that represent the unknot; i.e., it was shown that link Floer homology detects $T(2,\pm 6)$ endowed with any orientation. The author and Dey showed in [10] that link Floer homology detects all of the augmented braid-closures of 2-braids.

The augmented braid-closure of the 1-braid is also detected by link Floer homology since it is simply a Hopf link.

The proof strategies for Theorem 1.1 and Theorem 1.3 are that used by the author and Martin in [13]. That is, we use the fact that the link Floer homology of a link L contains various pieces of topological information about L. In particular we appeal to Martin's result that link Floer homology detects braid axes [32, Proposition 1].

We now address 3-braids with unknotted clasp-closures. For the first type we have detection.

Theorem 1.5. *Link Floer homology detects the Mazur link.*

The author and Dey showed that link Floer homology detects the augmented clasp-closures of all but two 2-braids and that the remaining two augmented clasp-closures are the unique links of their link Floer homology type [11, Theorems 6.1 and 6.2].

For the final type of 3-braid with unknotted clasp-closure we do not get detection. Nevertheless we can give the following classification result:

Theorem 1.6. *Let L be a link.* $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(\mathring{c}(\sigma_2^{-1}\sigma_1\sigma_2))$ *if and only if L is of the form* $\mathring{c}(\sigma_1^n\sigma_2^{-1}\sigma_1\sigma_2)$ *for some* $n \in \mathbb{Z}$.

The proof strategies for these two theorems are similar to that used in the proof of Theorem 1.1. The chief difference is that we appeal to the classification of links with link Floer homology of next to minimal rank in certain gradings [11, Theorem 5.1], as opposed to Martin's braid axis detection result, which was a classification of links with link Floer homology of minimal rank in certain gradings [32, Proposition 1].

We now turn to *Khovanov homology*. This is a combinatorial link invariant due to Khovanov that takes values in the category of bigraded vector spaces [26]. We have the following two results:

Theorem 2.1. *Khovanov homology with integer coefficients detects* L6a2.

Theorem 2.2. *Khovanov homology with integer coefficients detects* L9n15.

For context recall that Khovanov homology detects the Hopf link [7]. It also detects the augmented link associated to all 2-braid representatives of the unknot, namely, $T(2, \pm 4)$. This was first proven by using instanton Floer homology [44], but see also [13] for a proof that is more in line with that of Theorem 2.1. Martin showed that Khovanov homology detects $T(2, 6)$, one of the braid-closures of a 3-braid representing the unknot.

The main tool we use to prove these results is Dowlin's spectral sequence [17] from Khovanov homology to knot Floer homology — a version of link Floer homology introduced independently in [35] by Ozsváth and Szabó and in [39] by J. Rasmussen. This allows us to reduce the question of detection for Khovanov homology to problems in link Floer homology.

Finally we study *annular Khovanov homology*, a version of Khovanov homology for links in the thickened annulus due to Asaeda, Przytycki and Sikora [1]. We have the following family of results:

Theorem 3.11. *Annular Khovanov homology with integer coefficients detects* $b(\sigma_1\sigma_2^n)$ *for* $-2 \leq n \leq 5$.

For context, recall that annular Khovanov homology detects the braid-closure of the identity braids [2], and all braid-closures of 2-braids by a combination of work of Grigsby and Ni [19] and Grigsby, Licata and Wehrli [19]. The author and Martin also showed the $n = 1$ case of Theorem 3.11 in [13]. For the proof of our result we use the Birman–Menasco classification of 3-braids with fixed closures [15].

We can also prove the following:

Theorem 3.13. *Annular Khovanov homology with integer coefficients detects the Mazur pattern.*

Note that annular Khovanov homology detects the clasp-closures of all 2-braids, amongst annular knots [11, Theorem 8.1]. For the proofs of the two preceding theorems we use a version of the following rank bound:

Theorem 3.1. *Let* β *be an* n*-braid with* $n \geq 2$*. Then*:

(1) $\text{rank}(\text{AKh}(b(\beta); \mathbb{C})) \geq 2n$.[1]
(2) $\text{rank}(\text{AKh}(c(\beta); \mathbb{C})) \geq 4n$.

This result is inspired by the proof of a structurally similar rank bound in knot Floer homology due to Baldwin, Vela and Vick [5]. The proof relies on the left orderability of the braid group. See Lemma 3.2 for the more technical version of the result that we apply to prove Theorem 3.13 and Theorem 3.11. A number of other consequences of Theorem 3.1 are noted in Section 3C. The clasp-closure statement version of Theorem 3.1 is perhaps more interesting because there is currently no analogous result in the link Floer homology context.

Remark. Link Floer homology, Khovanov homology and annular Khovanov homology are invariant under overall orientation reversal. All of the detection and classification results in this paper are thus up to overall orientation reversal, if any relevant link and its reverse are distinct.

We end the introduction with two questions:

Question 1. Is there a complete classification of clasp-closures of 3-braids in the style of the Birman–Menasco classification of braid-closures of 3-braids?

Such a classification might allow one to obtain more classification results for links with categorified link invariants taking certain values.

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[1] This bound appears in an unpublished note of the author written while he was a graduate student.

Question 2. Does annular Khovanov homology detect all clasp-closures of 3-braids representing the unknot? Does Khovanov homology detect all of their augmentations? Does link Floer homology detect the links $\mathring{b}(\sigma_1\sigma_2^n)$?

Outline. In Section 1 we prove our results for link Floer homology. In Section 2 we prove our Khovanov homology detection results. In Section 3 we prove our annular Khovanov homology detection results as well as our two rank bounds.

1. Link Floer homology

In this section we collect our detection results for link Floer homology. In Section 1B we show that link Floer homology detects L6a2 and L9n15. In Section 1C we give a partial classification of links with the link Floer homology types of augmentations of clasp-closures of index 3-braids that represent the unknot.

1A. ***A review.*** Link Floer homology is an invariant introduced by Ozsváth and Szabó [36]. It assigns to each oriented n-component link a finitely generated vector space equipped with $n+1$ gradings. The first n of these gradings are called *Alexander gradings*, and the last is called the *Maslov grading*. The Alexander gradings takes value in $\frac{1}{2}\mathbb{Z}$, while the Maslov grading takes value in $\mathbb{Z}$. For each component K of a link L, there is a spectral sequence from $\widehat{\mathrm{HFL}}(L)$ to

$$\widehat{\mathrm{HFL}}(L\setminus K)\otimes V\left[\frac{\ell\mathrm{k}(K,L\setminus L_i)}{2}\right]$$

corresponding to allowing pseudoholomorphic disks to "cross basepoints" in Heegaard diagrams [36, Proposition 7.1]. Throughout this paper we consider link Floer homology with $\mathbb{Z}/2$ coefficients unless explicitly stated otherwise.

Link Floer homology detects the Thurston norm, under mild hypotheses, by a result of Ozsváth and Szabó [37]. It also detects braid closures, by a result of Martin [32, Proposition 1].

1B. ***Braid-closures.***

Theorem 1.1. *Link Floer homology detects* L6a2.

For the reader's convenience we recall that the link Floer homology of L6a2 is as follows:

(1)

$A_1 \rightarrow$ / $A_2 \downarrow$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
$\frac{3}{2}$		$\mathbb{F}_{-1}$	$\mathbb{F}_0$	
$\frac{1}{2}$	$\mathbb{F}_{-3}$	$\mathbb{F}^3_{-2}$	$\mathbb{F}^3_{-1}$	$\mathbb{F}_0$
$-\frac{1}{2}$	$\mathbb{F}_{-4}$	$\mathbb{F}^3_{-3}$	$\mathbb{F}^3_{-2}$	$\mathbb{F}_{-1}$
$-\frac{3}{2}$		$\mathbb{F}_{-4}$	$\mathbb{F}_{-3}$	

This can be deduced from, say, the fact that L6a2 is alternating, the multivariable Alexander polynomial of L6a2, the signature of L6a2, and an application of [36, Theorem 1.3].

Proof of Theorem 1.1. Our strategy is to argue that if a link has the link Floer homology type as L6a2 then it is the augmentation of a braid-closure of a 3-braid by applying Martin's braid axes detection result [32, Proposition 1]. We then appeal to Murasugi's classification of 3-braids whose braid-closures are unknotted and note that link Floer homology distinguishes the corresponding links.

Suppose L is a link with the link Floer homology of L6a2. Then L cannot be split, since its link Floer homology is not of the correct form. More specifically, if L were split then it could be written as $K_1\#(U \sqcup K_2)$ for K_i knots and U the unknot, where the connect sum is between U and K_1. Consequently, the Künneth formula for link Floer homology [36, Theorem 1.4] would imply that each Alexander bigrading would have rank at least two, which we can observe is not the case from Table (1). Since the rank of $\widehat{\mathrm{HFL}}(L)$ in the maximal nontrivial A_1 grading is 2, it follows from [32, Proposition 1] that the first component of L, L_1, is a braid axis. Observe that the Conway polynomial of two-component links — and hence knot Floer homology and link Floer homology — detects the linking number of two-component links [21]. It follows that L is the augmentation of the braid-closure of a 3-braid, L_2.

For the remainder of this section, let V be a rank-two vector space supported in Alexander grading 0 and Maslov gradings 0 and -1, and let $[a]$ indicate a shift in Alexander grading by a. There are spectral sequences from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(L_i) \otimes V[\ell\mathrm{k}(L)/2]$ for each i corresponding to allowing pseudoholomorphic disks to "cross basepoints" in Heegaard diagrams [36, Proposition 7.1]. Hence $\widehat{\mathrm{HFL}}(L_i)$ can be supported only in Alexander grading zero, so that each L_i is the unknot. Thus, L is the augmentation of a braid-closure of a 3-braid representing the unknot. By Murasugi's classification of 3-braids up to conjugacy there are exactly three 3-braid representatives of the unknot; namely $(\sigma_1\sigma_2)^{\pm 1}$, and $\sigma_1\sigma_2^{-1}$ [33]. Taking augmentations of the braid-closures of either of the first two braids yields $T(2, \pm 6)$, which have distinct link Floer homology from L. The result follows. □

Remark 1.2. Of course, a two-component unoriented link can, a priori, be endowed with four distinct orientations. However, $\overline{\text{L6a2}}$ is isotopic to the link obtained from L6a2 by reversing the orientation of either component. Likewise, the reverse of L6a2 is isotopic to L6a2. Thus, Theorem 1.1 holds for oriented links.

We proceed to our next detection result.

Theorem 1.3. *Link Floer homology with rational coefficients detects* L9n15.

We will not compute the link Floer homology of $\mathring{b}(\sigma_1^3\sigma_2)$. Instead, we will rely on formal properties of link Floer homology. The reason we take rational coefficients is that we will use the Khovanov homology of L9n15 to obtain information about link Floer homology via Dowlin's spectral sequence [17], which is defined over the rational numbers.

Proof. We first study the link Floer homology of $\widehat{\mathrm{HFL}}(\mathring{b}(\sigma_1^3\sigma_2); \mathbb{Q})$. Observe that, perhaps after relabeling components, $\widehat{\mathrm{HFL}}(\mathring{b}(\sigma_1^3\sigma_2); \mathbb{Q})$ has maximal A_2 grading $\frac{3}{2}$, since we may take the second component to be the braid axis for the braid-closure $\mathring{b}(\sigma_1^2\sigma_2)$. Now, $\mathring{b}(\sigma_1^2\sigma_2)$ bounds a 3-punctured torus, so the maximal A_1-grading is at most $\frac{5}{2}$. In fact, the maximal A_1 grading must be at least $\frac{5}{2}$ since $\widehat{\mathrm{HFL}}(L; \mathbb{Q})$ admits a spectral sequence to

$$\widehat{\mathrm{HFL}}(T(2, -3); \mathbb{Q}) \otimes V\left[-\tfrac{3}{2}\right],$$

so that $\widehat{\mathrm{HFL}}(L; \mathbb{Q})$ must have generators of A_1 grading $\pm\frac{5}{2}$. From Knot atlas [28] we have that $\mathrm{rank}(\mathrm{Kh}(\mathrm{L9n15}; \mathbb{Z}/2)) = 12$ — see also Table (5) — so that $6 = \mathrm{rank}(\mathrm{Khr}(\mathrm{L9n15}; \mathbb{Z}/2)) \geq \mathrm{rank}(\mathrm{Khr}(\mathrm{L9n15}; \mathbb{Q}))$ by the universal coefficient theorem and [40, Corollary 3.2.C]. It follows that $\mathrm{rank}(\widehat{\mathrm{HFL}}(\mathrm{L9n15}; \mathbb{Q})) \leq 12$ by an application of the rank bound from Dowlin's spectral sequence [17] together with some properties of pointed Khovanov homology [6, Lemma 2.11].

Suppose L is a link with the same link Floer homology with rational coefficients as $\mathring{b}(\sigma_1^3\sigma_2)$. Since $\widehat{\mathrm{HFL}}(L; \mathbb{Q})$ determines the Conway polynomial of L and the Conway polynomial of L determines the linking number of two-component links [21], it follows that L has linking number -3. In particular L is nonsplit. Now, the link Floer polytope of L agrees with that of L9n15. Since the link Floer polytope detects the Thurston polytope [37] for nonsplit links, it follows that the Thurston polytopes of L and L9n15 agree. Consequently L_2 bounds a surface in the exterior of L of Euler characteristic -2, just as does the corresponding component in L9n15. Since such a surface necessarily has at least four boundary components, it follows that it is, in fact, a 4-punctured disk, so that L_2 is an unknot. Since the rank in the maximum nontrivial Alexander grading is 2, it follows that L_2 is a braid axis by [32, Proposition 1].

We now study the first component of L. From the link Floer polytope of L we can see that L_1 bounds a surface in the exterior of L of Euler characteristic -4. Since the linking number of L is three, it follows that L_1 has Seifert-genus at most one. If L_1 has a Seifert surface of genus zero, then it is an unknot. Consequently L is an augmented braid closure of a 3-braid representing the unknot. As previously noted, these were classified by Murasugi [33] and are each detected by link Floer homology by [13] and Theorem 1.1. Thus L_1 bounds a genus one surface. In particular, $\widehat{\mathrm{HFL}}(L_1; \mathbb{Q})$ is nontrivial in Alexander gradings ± 1 and trivial in Alexander gradings i with $|i| > 1$. Since $\widehat{\mathrm{HFL}}(L_1; \mathbb{Q})$ has the same rank in Alexander gradings

± 1 and the total rank of $\widehat{\mathrm{HFL}}(L;\mathbb{Q})$ must be odd, it follows that $\widehat{\mathrm{HFL}}(L_1;\mathbb{Q})$ must be nontrivial in Alexander grading zero.

We now prove that L_1 is fibered. By [36, Theorem 1.4], there is a spectral sequence from $\widehat{\mathrm{HFL}}(L;\mathbb{Q})$ to $\widehat{\mathrm{HFL}}(L_1)\otimes V\big[-\frac{3}{2}\big]$. This latter space must be of rank at least two for $k=-\frac{1}{2},-\frac{3}{2},-\frac{5}{2}$, and so the same is true of $\widehat{\mathrm{HFL}}(L;A_1=k;\mathbb{Q})$. In fact, by symmetry properties of link Floer homology, $\widehat{\mathrm{HFL}}(L,A_1=k;\mathbb{Q})$ must also be of rank at least two for $k=\frac{1}{2},\frac{3}{2},\frac{5}{2}$. It follows that $\mathrm{rank}(\widehat{\mathrm{HFL}}(L;\mathbb{Q}))=12$ and indeed that $\widehat{\mathrm{HFL}}(L;\mathbb{Q})$ is of rank two in the maximal nontrivial A_1 grading. Martin's braid axis detection result implies that L_1 is a braid axis for L_2 and so, in particular, fibered [32, Proposition 1]. Indeed, since the maximal nontrivial A_1 grading is $1+\frac{3}{2}$, L_1 must be a genus one fibered knot. It follows that L_1 is a trefoil or a figure eight knot. To see that L_1 is a left handed trefoil, observe that $\widehat{\mathrm{HFL}}\big(\mathrm{L9n15},A_2=-\frac{5}{2};\mathbb{Q}\big)$ must be supported in Maslov gradings 0 and -1 since it has a left-handed trefoil component and the linking number is -3. This in turn implies that L_1 must have a left handed trefoil component, since in the right handed trefoil case and Figure eight case $\widehat{\mathrm{HFL}}\big(L,A_1=-\frac{5}{2};\mathbb{Q}\big)$ would have to be supported in Maslov gradings -2 and -3 or -1 and -2 respectively.

Now by Birman and Menasco's classification theorem for 3-braids [15], there are exactly two 3-braids with braid-closures representing $T(2,-3)$, namely $\sigma_1^{-1}\sigma_2^{-3}$ and $\sigma_1\sigma_2^{-3}$, which is L9n16. These two links are distinguished by their Alexander polynomials, so the result follows. □

Remark 1.4. Once again there are — apriori — four possible orientations with which L9n15 can be endowed. One pair of these have linking number -3 while the other has linking number 3. It can be checked that each pair with the same linking number are, in fact, isotopic as links. That is, Theorem 1.3 holds as a statement for oriented links.

1C. ***Clasp-closures.*** In this section we study the links with the link Floer homology type of augmentations of clasp-closures of 3-braids representing the unknot. We prove two of the theorems advertised in the introduction:

Theorem 1.5. *Link Floer homology detects the Mazur link.*

Theorem 1.6. *Let L be a link. $\widehat{\mathrm{HFL}}(L)\cong\widehat{\mathrm{HFL}}(\mathring{c}(\sigma_2^{-1}\sigma_1\sigma_2))$ if and only if L is of the form $\mathring{c}(\sigma_1^n\sigma_2^{-1}\sigma_1\sigma_2)$ for some $n\in\mathbb{Z}$.*

We begin by discussing some structural properties of the link Floer homology of links that are augmentations of clasp-closures of 3-braids representing the unknot. Let L be such a link, with the first component of L, L_1, being the clasp-closure of the 3-braid and the second component, L_2, its axis. The maximal A_2 grading in which $\widehat{\mathrm{HFL}}(L,A_2)$ has nontrivial support is $\frac{3}{2}$. This follows from [37, Theorem 1.1].

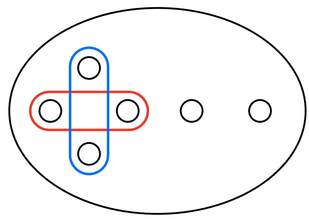

Figure 2. A sutured Heegaard diagram for the sutured manifold obtained by decomposing an augmented clasp-closure along a longitudinal surface for its axis. The outer boundary component of the surface corresponds to a longitude of L_2, while the inner boundary components correspond to meridians of L_1.

Lemma 1.7. *Suppose L is the augmentation of a clasp-closure, L_1, of a braid representing a knot. Then the component of $\widehat{\mathrm{HFL}}(L)$ with maximal nontrivial A_2 grading is given by $\mathbb{F}_{-1}[-1] \oplus \mathbb{F}_0^2[0] \oplus \mathbb{F}_1[1]$ up to overall shifts in the Maslov and A_1 gradings.*

A version of this result without the Maslov grading is given in [11, Lemma 5.9]. The proof of this lemma requires techniques from sutured Floer homology. The reader is directed to Juhász' papers [23; 24; 25] for necessary background.

Proof. Suppose L is as in the statement of the lemma. A sutured Heegaard diagram for the sutured manifold (Y, γ) obtained by decomposing the exterior of L along an appropriate maximal Euler characteristic longitudinal surface for L_2 is shown in Figure 2.

A priori, $\mathrm{SFH}(Y, \gamma)$ only comes with a relative Maslov grading in each spin^c structure. However, in the case at hand these Maslov gradings can be upgraded to a relative Maslov grading that applies across all spin^c structures. To see this, observe that capping off the sutures corresponding to meridians of L_1 results in another sutured manifold, $(\widehat{Y}, \hat{\gamma})$ in which all four of the generators of $\mathrm{CF}(\widehat{Y}, \hat{\gamma})$ are supported in a single spin^c-structure. The claim follows. It remains to check that the maps $\hat{p}$ from [25, Proposition 5.4] respect this relative Maslov grading which applies across all spin^c structures. However, this follows by repeating Juhász' proof of [25, Proposition 5.4]. Specifically, there is a Heegaard diagram for (Y, γ) that can be obtained by doubling a Heegaard diagram for the exterior of L along a certain subsurface [24, Proposition 5.2], and pseudoholomorphic disks from the doubled Heegaard diagram correspond to disks in the Heegaard diagram for L [24, Proposition 7.6]. This correspondence still holds if we fill in boundary components, yielding the desired result. □

We can now prove that link Floer homology detects augmented clasp-closures of 3-braids. Our proof depends on the much more general classification of links with link Floer homology of next to minimal rank in the maximal nontrivial Alexander grading of a given component due to the author and Dey [11, Theorem 5.1].

Lemma 1.8. *Suppose that a link L has the link Floer homology type of an augmented clasp-closure of a 3-braid representing the unknot. Then L is an augmented clasp-closure of a 3-braid.*

Proof. Suppose L is as in the statement of the lemma. We first claim that L has linking number ± 1. Note that augmented clasp-closures of 3-braids have linking number ± 1. Now recall that the Conway polynomial — and hence link Floer homology — detects the linking number of two-component links [21]. The claim follows.

After relabeling the components of L if necessary, we may assume that the component of $\widehat{\mathrm{HFL}}(L)$ with maximal nontrivial Alexander grading of rank four is L_2 and that the maximal nontrivial A_2 grading is $\frac{3}{2}$ and that $\widehat{\mathrm{HFL}}\big(L, A_2 = \frac{3}{2}\big)$ is given by $\mathbb{F}_{-1}[-1] \oplus \mathbb{F}_0^2[0] \oplus \mathbb{F}_1[1]$, up to shifts in the A_1 and Maslov gradings by Lemma 1.7. We now bound the genus of the component L_2. Recall that there is a spectral sequence from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(L_2) \otimes V\big[\pm\frac{1}{2}\big]$. It follows that the maximum nontrivial Alexander grading in which L_1 can have nontrivial support is at most one.

By [11, Theorem 5.1] we have four cases to treat:

1. L_2 is a genus one fibered knot and L_1 is a clasp-braid with axis L_2.
2. L_2 is a genus one nearly fibered knot and L_1 is a braid-closure with axis L_2.
3. L_2 is a fibered knot and L_1 can be isotoped to a simple closed curve in a minimal genus Seifert surface for L_2.
4. L_1 is a clasp-closure with L_2 its unknotted axis.

For definitions of "nearly fibered" see [4]. For a definition of what it is to be braided with respect to a nearly fibered knot see [11, Section 3]. We rule out the first three of the four possibilities.

Case 1: The maximal Euler characteristic of a longitudinal surface for L_2 would be -3, so that the maximal A_2 grading in which $\widehat{\mathrm{HFL}}(L)$ would be nontrivial support would be $\frac{5}{2}$ by [37, Theorem 1.1], a contradiction.

Case 2: Recall that there is a spectral sequence from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFK}}(L_2) \otimes V\big[\pm\frac{1}{2}\big]$. Since in the maximal nontrivial A_2 grading $\widehat{\mathrm{HFL}}(L)$ is of rank four, as is the rank of the maximal nontrivial Alexander grading of $\widehat{\mathrm{HFK}}(L_2) \otimes V\big[\pm\frac{1}{2}\big]$, it follows that this spectral sequence collapses immediately. In particular, it follows that the component of $\widehat{\mathrm{HFK}}(L_2) \otimes V$ in maximal nontrivial Alexander grading is given up to an overall shift in Maslov grading by $\mathbb{F}_{-1} \oplus \mathbb{F}_0^2 \oplus \mathbb{F}_1$. Now, Baldwin and Sivek classified all genus one nearly fibered knots [4]. All such knots have the property

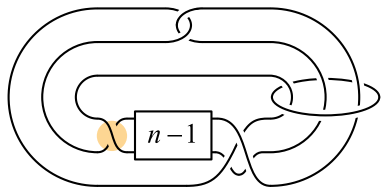

Figure 3. The link $\mathring{c}(\sigma_1^n\sigma_2^{-1}\sigma_1\sigma_2)$. We consider the unoriented resolution of the crossing highlighted in orange.

that their knot Floer homology in Alexander grading one is supported in exactly one Maslov grading — see [4, Table 1] — a contradiction.

Case 3: This is immediately excluded by the fact that the linking number of L is nonzero. □

Lemma 1.9. *Suppose L is a link with the link Floer homology of either the Mazur link or $\mathring{c}(\sigma_2^{-1}\sigma_1\sigma_2)$. Then L has unknotted components each of which is an unknotted clasp closure with respect to the other.*

Proof. Suppose L is as in the statement of the lemma. First observe that the for both the Mazur link and $\mathring{c}(\sigma_2^{-1}\sigma_1\sigma_2)$, each component is a clasp-closure of a 3-braid with respect to the other. The claim now follows from the same argument as given in the previous lemma, but now applied to both components of L. □

By the preceding lemmas, to complete the proofs of Theorems 1.5 and 1.6 it suffices to show that link Floer homologies distinguishes between appropriate augmentations of clasp-closures of 3-braids representing the unknot.

We first address the links corresponding to the infinite family of braids $\sigma^n\sigma_2^{-1}\sigma_1\sigma_2$. It can be checked that the unoriented resolution of this link at the crossing shown in Figure 3 is the split sum of a Hopf link and an unknot. J. Wang has shown that if L_b is a band sum of the split union of two links $L_1 \sqcup L_2$ then the link Floer homology of L_b does not change after adding twists to the band [42, Remark 1.18]. Thus the links $\mathring{c}(\sigma^n\sigma_2^{-1}\sigma_1\sigma_2)$ all have the same link Floer homology.

We can now conclude the proofs of two of the results promised in the introduction.

Proof of Theorems 1.5 and 1.6. Suppose that L is a link as in one the two theorem statements. By Lemma 1.8 and Lemma 1.9, both components of L are clasp-closures of index 3-braids with respect to the other component. In particular, the maximum A_i gradings in which $\widehat{\mathrm{HFL}}(L)$ are nontrivial are $\frac{3}{2}$. On the other hand, $\mathring{c}(\sigma_1^3\sigma_2^{-1}\sigma_1^2\sigma_2)$ does not have this property; one of the components — say L_1 — is the clasp-closure of an index 5 braid. Consequently, the maximum A_1 grading in which $\widehat{\mathrm{HFL}}(\mathring{c}(\sigma_1^3\sigma_2^{-1}\sigma_1^2\sigma_2))$ is nontrivial is $\frac{5}{2}$. It follows that L cannot be $\mathring{c}(\sigma_1^3\sigma_2^{-1}\sigma_1^2\sigma_2)$. By [42, Remark 1.18], $\widehat{\mathrm{HFL}}(\mathring{c}(\sigma_1^n\sigma_2^{-1}\sigma_1\sigma_2)) \cong \widehat{\mathrm{HFL}}(\mathring{c}(\sigma_2^{-1}\sigma_1\sigma_2))$

for all n. It thus suffices to show that no two of the Mazur link, $\mathring{c}(\sigma_2^{-1}\sigma_1\sigma_2)$, and their mirrors have the same link Floer homology.

Given that link Floer homology detects the linking number of two-component links, the transformation property of link Floer homology under changing the orientation of a link component implies that for a two-component link, L, $\widehat{\mathrm{HFL}}(L)$ determines the Alexander polynomial of L endowed with an arbitrary orientation. For the two links at hand, these are given as follows:

1. $\Delta(\mathring{c}(\sigma_1^{-1})) = 2 - 5t + 5t^2 - 2t^3$.
2. $\Delta(\mathring{c}'(\sigma_1^{-1})) = 1 - 3t + 3t^2 - 3t^3 + 3t^4 - t^5$.
3. $\Delta(\mathring{c}(\sigma_2^{-1}\sigma_1\sigma_2)) = 2 - 7t + 7t^2 - 2t^3$,
4. $\Delta(\mathring{c}'(\sigma_2^{-1}\sigma_1\sigma_2)) = 1 - 3t + 5t^2 - 5t^3 + 3t^4 - t^5$.

Here the primes indicate that the orientation of the axis has been reversed. The author used [27] for these computations. Note that the Alexander polynomial of a link is unchanged under mirroring. Thus link Floer homology distinguishes the Mazur link and its mirror, from $\mathring{c}(\sigma_2^{-1}\sigma_1\sigma_2)$ and its mirror. Since the Mazur link and its mirror have linking numbers ± 1, and likewise for $\mathring{c}(\sigma_2^{-1}\sigma_1\sigma_2)$, the Alexander polynomial distinguishes each pair of links. Consequently link Floer homology distinguishes between the four links, concluding the proof. □

Remark 1.10. The Mazur link and its reverse are isotopic, so for this link we have oriented link detection on the nose.

2. Khovanov homology

The goal of this section is to prove the following two results:

Theorem 2.1. *Khovanov homology with integer coefficients detects* L6a2.

Theorem 2.2. *Khovanov homology with integer coefficients detects* L9n15.

From Martin's result that Khovanov homology detects $T(2,6)$ oriented as a 2-braid closure [32] we know that Khovanov homology detects $T(2,\pm 6)$ with both orientations. Thus Khovanov homology detects all augmentations of braid-closures of 3-braids representing the unknot.

2A. ***A review.*** *Khovanov homology* is an invariant of oriented links introduced in [26]. It assigns to each link a $\mathbb{Z}$-module equipped with two gradings: the *quantum grading* and the *homological grading*. The Khovanov homology of an n-component link is supported in quantum gradings whose parity agrees with that of n. Taking coefficients in a field, the Khovanov homology of L admits a link splitting spectral sequence to the Khovanov homology of the disjoint union of the underlying components of L. This is due to Batson and Seed [9]. Taking coefficients in $\mathbb{Q}$, Khovanov homology admits a spectral sequence to an invariant called *Lee homology*, a result from [30]. Lee homology is a finitely generated $\mathbb{Q}$-vector space equipped with a homology grading compatible with that of Khovanov homology under the Lee spectral sequence. For a two component link L, the Lee homology of L is of rank four, with rank two in homological grading zero and rank four in homological grading the linking number of L.

There is a version of Khovanov homology due to Baldwin, Levine and Sarkar [6], called *pointed Khovanov homology*. Taking coefficients in $\mathbb{Q}$, pointed Khovanov homology admits a spectral sequence to knot Floer homology, a result of Dowlin [17]. Pointed Khovanov homology is a generalization of an invariant called *reduced Khovanov homology.* Taking coefficients in $\mathbb{Z}/2$, the rank of reduced Khovanov homology is exactly half that of Khovanov homology, a result of Shumakovitch [40, Corollary 3.2.C].

2B. *Detection results.* For convenience we record the Khovanov homology of L6a2 (see [28]):

$h \rightarrow$ $q \downarrow$	-6	-5	-4	-3	-2	-1	0
-2							$\mathbb{Z}$
-4						$\mathbb{Z}$	$\mathbb{Z}$
-6					$\mathbb{Z}\oplus\mathbb{Z}/2$		
-8				$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}$		
-10			$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}$			
-12		$\mathbb{Z}/2$	$\mathbb{Z}$				
-14	$\mathbb{Z}$	$\mathbb{Z}$					
-16	$\mathbb{Z}$						

Lemma 2.3. *Suppose L is a link with the Khovanov homology of the* L6a2*. Then L is a two-component link with each component an unknot. Moreover $\ell k(L) = -3$.*

Proof. Suppose L is as in the statement of the theorem. Observe that L has an even number of components since the quantum grading is supported in even gradings. The Batson–Seed link splitting spectral sequence [9], together with an application of the universal coefficient theorem, implies that

$$\text{rank}(\text{Kh}(L;\mathbb{Z}/2)) = 20 \geq \prod \text{rank}(\text{Kh}(L_i;\mathbb{Z}/2)). \tag{2}$$

Here, the product is taken over components L_i of L. Since $\text{rank}(\text{Kh}(L_i;\mathbb{Z}/2))$ is of the form $2+4k_i$ for some $k_i \geq 0$, we must have that L has at most four components. If L has exactly four components, then each component is an unknot by [29]. To check that this is impossible, we use the refined version of the Batson–Seed link splitting spectral sequence. Equip $\text{Kh}(L;\mathbb{Q})$ with the $l := h-q$ grading. Then [9, Corollary 4.4] implies that for some constant t

$$\operatorname{rank}^l(\operatorname{Kh}(L;\mathbb{Z}/2)) \geq \operatorname{rank}^{l+t}(W^{\otimes 4}). \tag{3}$$

Here, and for the remainder of this proof, W is the rank-two vector space supported in l gradings 1 and -1. In particular, there is some l grading in which $\operatorname{rank}(\operatorname{Kh}^l(L;\mathbb{Z}/2)) \geq 6$. This is false by inspection. Thus, L has exactly two components. Observe that equation (2) implies that at least one component of L is an unknot, since the unknot is the unique knot K with $\operatorname{rank}(\operatorname{Kh}(K;\mathbb{Z}/2)) = 2$ [29]. The remaining component K_2 of L has $\operatorname{rank}(\operatorname{Kh}(K_2;\mathbb{Z}/2)) \leq 6$, so that it is either an unknot or a trefoil by [29] and [8].

To see that $\ell k(L) = -3$, recall that $\operatorname{Kh}(L;\mathbb{Q})$ — which can be obtained from $\operatorname{Kh}(L;\mathbb{Z})$ by an application of the universal coefficient theorem — admits a spectral sequence to Lee homology [30]. Lee homology carries a homological grading and the spectral sequence respects this grading. Moreover, the Lee homology of a two-component link L is supported in homological gradings 0 and $2\ell k(L)$, and that each such grading contributes a $\mathbb{Q}^2$ summand. By inspection of $\operatorname{Kh}(L;\mathbb{Q})$ we see that L must have linking number -3.

To check that the remaining component of L is also unknotted, we use the refined version of the Batson–Seed link splitting spectral sequence again. Equip $\operatorname{Kh}(L;\mathbb{Q})$ with the $l := h - q$ grading; then [9, Corollary 4.4] implies that

$$\operatorname{rank}^l(\operatorname{Kh}(L;\mathbb{Q})) \geq \operatorname{rank}^{l-3}(\operatorname{Kh}(K_2;\mathbb{Q}) \otimes W), \tag{4}$$

where W is the rank-two vector space supported in l gradings 1 and -1. Since $\operatorname{Kh}(T(2,3);\mathbb{Q})$ has a generator in l grading -6, $\operatorname{Kh}(K_2;\mathbb{Q}) \otimes W$ has a generator of l grading -7, violating the rank bound. Likewise since $\operatorname{Kh}(T(2,-3);\mathbb{Q})$ has two generators in l grading 3, $\operatorname{Kh}(T(2,-3);\mathbb{Q}) \otimes W$ has two generators in l grading 2 violating the rank bound. □

The remainder of the proof of Theorem 2.1 amounts to showing that there is a component of L that is a braid axis for the other. To do so, we use Dowlin's spectral sequence from an appropriate version of Khovanov homology to $\widehat{\operatorname{HFK}}(L;\mathbb{Q})$ to reduce this question to a question about link Floer homology [17].

Proof of Theorem 2.1. Suppose L is as in the statement of the Theorem. By the previous lemma, L has two components. Since L has δ-thin Khovanov homology, $\widehat{\operatorname{HFL}}(L;\mathbb{Q})$ is δ-thin. A result of the author and Dey [12, Proposition 6.1] implies in turn that $\widehat{\operatorname{HFL}}(L;\mathbb{Q})$ decomposes as a direct sum of vector spaces of the form

$$W_a[b,c] :=$$
$$\mathbb{Q}_{a-1}\big[b-\tfrac{1}{2}, c-\tfrac{1}{2}\big] \oplus \mathbb{Q}_a\big[b+\tfrac{1}{2}, c-\tfrac{1}{2}\big] \oplus \mathbb{Q}_a\big[b-\tfrac{1}{2}, c+\tfrac{1}{2}\big] \oplus \mathbb{Q}_{a+1}\big[b+\tfrac{1}{2}, c+\tfrac{1}{2}\big].$$

Here $\mathbb{Q}_a[b, c]$ is a $\mathbb{Q}$ summand in (A_1, A_2) grading (b, c) of Maslov grading a. There are at most five of these summands since

$$20 = \operatorname{rank}(\operatorname{Kh}(L; \mathbb{Z}/2)) \geq \operatorname{rank}(\widehat{\operatorname{HFK}}(L; \mathbb{Q})),$$

as follows form Dowlin's spectral sequence [17] together with the same steps applied in the corresponding stage of the proof of Theorem 1.3.

Observe that if the span of an A_i grading is $\left[\frac{-1}{2}, \frac{1}{2}\right]$ then L_i is a meridian of the other component, so that L is a Hopf link, since each component is unknotted by Lemma 2.3. Neither Hopf link has the correct Khovanov homology, so the span of each Alexander gradings must be strictly larger than $\left[\frac{-1}{2}, \frac{1}{2}\right]$.

If $\widehat{\operatorname{HFL}}(L; \mathbb{Q})$ contains an odd number of $W_a[b, c]$ summands, the symmetry of link Floer homology implies that $\widehat{\operatorname{HFL}}(L; \mathbb{Q})$ contains a $W_b[0, 0]$ summand, for some b. Thus we are in one of the following cases:

1. $\widehat{\operatorname{HFL}}(L; \mathbb{Q})$ has a $W_a[m, n]^{\oplus 2} \oplus W_{a-2m-2n}[-m, -n]^{\oplus 2}$ summand, where $m, n \in \frac{1}{2}\mathbb{Z}$, $m, n \geq \frac{1}{2}$ and $a = b + n + m$, if there is a $W_b[0, 0]$ summand.
2. $\widehat{\operatorname{HFL}}(L; \mathbb{Q})$ has a

$$W_a[m, n] \oplus W_{a-2m}[-m, n] \oplus W_{-a-2m-2n}[-m, -n] \oplus W_{-a-2n}[m, -n]$$

summand, where $m, n \in \frac{1}{2}\mathbb{Z}$, $m, n \geq \frac{1}{2}$ and $a = b + n + m$, if there is a $W_b[0, 0]$ summand.
3. $\widehat{\operatorname{HFL}}(L)$ is of rank two, in a maximal nontrivial A_i grading,

Suppose that we are in one of the first two cases. L is nonsplit because both components are unknotted and the Khovanov homology of the two-component unlink is of rank four. Thus, we can apply [11, Theorem 5.1]. Since L_1 is unknotted, we deduce that L_2 is a clasp-closure with respect to L_1. However, Lemma 1.7 then implies that the maximal nontrivial A_1 grading is given, up to affine isomorphism, by $\mathbb{Q}[-1] \oplus \mathbb{Q}^2[0] \oplus \mathbb{Q}[1]$. This is a direct contradiction in case 1. In case 2 $\widehat{\operatorname{HFL}}(L; \mathbb{Q})$ would contain a $W_a\left[n, \frac{1}{2}\right] \oplus W_{a-1}\left[n, -\frac{1}{2}\right] \oplus W_{a-1-2n}\left[-n, -\frac{1}{2}\right] \oplus W_{a-2n}\left[-n, \frac{1}{2}\right]$ summand, a contradiction since L has odd linking number so that $\widehat{\operatorname{HFL}}(L; \mathbb{Q})$ must be supported in $\mathbb{Z} + \frac{1}{2}$ valued Alexander gradings.

Thus $\widehat{\operatorname{HFL}}(L; \mathbb{Q})$ is of rank two in one of the maximal nontrivial Alexander gradings. By [32, Proposition 1] one component — say L_1 — is a braid axis for the other — say L_2. Since the linking number of the two links is -3, it follows that L_2 is the braid-closure of a 3-braid. Since L_2 represents the unknot, the desired result follows from Murasugi's classification of 3-braids with unknotted braid-closures up to conjugacy [33] and the fact that $T(2, \pm 6)$ has distinct Khovanov homology from L6a2. □

Remark 2.4. Using the same argument given in Remark 1.2, it can be shown that

Khovanov homology detects L6a2 regardless of the orientation.

We now proceed to our next detection result, for L9n15, whose Khovanov homology we recall [28]:

(5)

$h \to$ / $q \downarrow$	-6	-5	-4	-3	-2	-1	0
-6							$\mathbb{Z}$
-8					$\mathbb{Z}/2$		$\mathbb{Z}$
-10					$\mathbb{Z}$		
-12			$\mathbb{Z}$				
-14			$\mathbb{Z}$	$\mathbb{Z}$			
-16	$\mathbb{Z}$	$\mathbb{Z}$					
-18	$\mathbb{Z}$	$\mathbb{Z}$					

Proof of Theorem 2.2. Suppose L is a link with $\mathrm{Kh}(L;\mathbb{Z}) \cong \mathrm{Kh}(\mathrm{L9n15};\mathbb{Z})$. We first determine the components of L. There is an even number of them, since $\mathrm{Kh}(L;\mathbb{Z})$ is supported in even quantum gradings. An application of the universal coefficient theorem gives $\mathrm{Kh}(L;\mathbb{Z}/2) \cong \mathrm{Kh}(\mathrm{L9n15};\mathbb{Z}/2)$, so $\mathrm{Kh}(L;\mathbb{Z}/2)$ has rank 12. Consider the Batson–Seed link splitting spectral sequence [9]. Every knot has Khovanov homology with $\mathbb{Z}/2$ coefficients of rank $2+4m$ for some m, so that L has at most two components. One of them, say L_1, has $2\,\mathrm{rank}(\mathrm{Khr}(L_1;\mathbb{Q}) \leq \mathrm{rank}(\mathrm{Kh}(L_1;\mathbb{Z}/2)) = 2$, thanks to [40, Corollary 3.2.C]; therefore L_1 is unknotted by [29]. The remaining component of L, L_2, has $\mathrm{rank}(\mathrm{Kh}(L_2;\mathbb{Z}/2)) \leq 6$ and so, in turn, $\mathrm{rank}(\mathrm{Khr}(L_2;\mathbb{Z}/2)) \leq 3$ by [40, Corollary 3.2.C]. It follows from [8] and [29] that L_2 is an unknot or a trefoil.

An application of the universal coefficient theorem shows that $\mathrm{Kh}(L;\mathbb{Q}) \cong \mathrm{Kh}(\mathrm{L9n15};\mathbb{Q})$. Consider the spectral sequence from Khovanov homology to Lee homology. Since this spectral sequence respects the homological grading and the Lee homology of a two-component link consists of two $\mathbb{Q}\oplus\mathbb{Q}$ summands supported in homological gradings 0 and $2\ell\mathrm{k}(L)$, we can see by inspection that $\ell\mathrm{k}(L) = -3$. We can then apply equation (4) again; or rather to the same equation but where we take $\mathbb{Z}/2$-coefficients rather than rational coefficients. Let U denote the unknot. Note that $\mathrm{Kh}(U;\mathbb{Z}/2)\otimes W$ has support in $l := h-q$ grading 2 so that L_1 cannot be the unknot. Likewise $\mathrm{Kh}(T(2,3);\mathbb{Z}/2)\otimes W$ has support in $l := h-q$ grading -7 so that in fact L_2 is $T(2,-3)$.

By an application of [40, Corollary 3.2.C] and the universal coefficient theorem we have $\mathrm{rank}(\mathrm{Khr}(L;\mathbb{Q})) \leq 6$, so that $\mathrm{rank}(\widehat{\mathrm{HFK}}(L;\mathbb{Q})) \leq 12$ by the rank bound coming from Dowlin's spectral sequence [17]. Recall that there is a spectral sequence from $\widehat{\mathrm{HFL}}(L;\mathbb{Q})$ to $\widehat{\mathrm{HFL}}(L_2;\mathbb{Q})\otimes V\left[-\frac{3}{2}\right]$. Since L has linking number -3 and the second component is a copy of $T(2,-3)$ it follows that in A_2 grading $-\frac{5}{2}$ there is a $\mathbb{Q}_0\oplus\mathbb{Q}_{-1}$-summand, in A_2 grading $-\frac{3}{2}$ there is a $\mathbb{Q}_1\oplus\mathbb{Q}_0$-summand

and that in A_2 grading $-\frac{1}{2}$ there is a $\mathbb{Q}_1 \oplus \mathbb{Q}_2$-summand. This completely determines the A_2-graded version of the link Floer homology of L by symmetry properties and the fact that the rank is at most 12. Now, since L_1 is unknotted and the linking number of L is -3, $\widehat{\mathrm{HFL}}(L; \mathbb{Q})$ must have support in Alexander gradings $\pm\frac{3}{2}$. Since homogeneous summands with A_2-grading at least 0 must all die under the spectral sequence to $\widehat{\mathrm{HFL}}(T(2,-3); \mathbb{Q}) \otimes V\left[-\frac{3}{2}\right]$, we have that the pairs of generators in each A_2 grading must be of distinct A_1 gradings.

The span of δ-graded $\mathrm{Kh}(L; \mathbb{Z}/2)$ is 4, so the span of δ-graded $\mathrm{Khr}(L; \mathbb{Z}/2)$ is 2. Thus, the span of δ-graded pointed Khovanov homology, $\widetilde{\mathrm{Kh}}(L, \boldsymbol{p}; \mathbb{Z}/2)$ where $\boldsymbol{p}$ consists of a point on each component of L, is at most 2 by [6, Lemma 2.11] and so finally the span of δ-graded $\widehat{\mathrm{HFL}}(L; \mathbb{Q})$ is at most 2. It follows that at most three of the homogeneous $\mathbb{Q}$ summands with A_2-grading at most $-\frac{1}{2}$ occur in extremal A_1 gradings. It follows in turn that $\widehat{\mathrm{HFL}}(L); \mathbb{Q}$ is of rank two in the maximal nontrivial A_1 grading, so that U is a braid axis for $T(2,-3)$. Since the linking number is -3, the corresponding braid is a 3-braid. Now, by Birman and Menasco's classification of 3-braids with braid-closures representing the unknot, the only two such augmented braid-closures are L9n15 and L9n16. These are distinguished by their Khovanov homology, see [28]. The result holds for oriented links by Remark 1.4. □

3. Annular Khovanov homology

In this section we study annular Khovanov homology. In Section 3A we review structural properties of the invariant we will use in the rest of the section. In Section 3B we prove rank bounds for the annular Khovanov homology of clasp-closures and braid-closures and give some applications of the rank bounds to the study of braid-closures. In Section 3C we give two braid-closure detection results. In Section 3E we apply a rank bound from Section 3B to prove that annular Khovanov homology detects the Mazur pattern.

3A. ***A review.*** We begin with a brief review of annular Khovanov homology, which was introduced by Asaeda, Przytycki and Sikora in [1]. We work with coefficients in R, where R is either $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{Q}$ or $\mathbb{Z}/2$. Annular Khovanov homology is an R-module-valued invariant of links in the thickened annulus. The underlying chain complex for the annular Khovanov homology of an annular link L is freely generated by complete resolutions of a fixed diagram for L where each circle is decorated with a 1 or an X. The resulting homology groups carry three gradings. The first of these gradings is called the *homological grading* which we shall denote by i, the second is the *quantum grading* which we shall denote by j. These two gradings are defined just as in the Khovanov homology case. The third grading is called the *annular grading*, which we shall denote by k. This is defined as the difference between the number of resolutions encircling the annular axis marked with a 1 and those marked with an X. The differential on annular Khovanov homology is then simply defined as the components of the differential on the Khovanov complex of the underlying link that preserve the annular grading.

We will use two exact triangles for annular Khovanov homology. Recall — say from [11, Lemma 8.2] — that annular Khovanov admits the following skein exact triangle corresponding to resolving a negative crossing:

$$\begin{array}{ccc} \mathrm{AKh}(L) & \longrightarrow & \mathrm{AKh}(L_0)[n_-^0 - n_-]\{3n_-^0 - 3n_- + 1\} \\ & \nwarrow \quad \swarrow_{\delta} & \\ & \mathrm{AKh}(L_1)\{-1\} & \end{array} \tag{6}$$

Here n_- is the number of negative crossings in the diagram for L, n_-^0 is the number of negative crossings in the diagram for L_0, $\{a\}$ is a shift in the quantum grading by a and $[b]$ is a shift in the homological grading by b. The map δ increases the homological grading by one. Corresponding to resolving a positive crossing we have the following exact triangle:

$$\begin{array}{ccc} \mathrm{AKh}(L) & \longrightarrow & \mathrm{AKh}(L_0)\{1\} \\ & \nwarrow \quad \swarrow_{\delta} & \\ & \mathrm{AKh}(L_1)[n_-^1 - n_- + 1]\{3n_-^1 - 3n_- + 2\} & \end{array} \tag{7}$$

Here the map δ again increases the homological grading by 1. Grigsby, Licata and Wehrli showed that for annular Khovanov homology with complex coefficients carries the structure of an $\mathfrak{sl}_2(\mathbb{C})$ representation [20]. This entails that, up to an overall grading shift in the quantum grading, $\mathrm{AKh}(L;\mathbb{C})$ decomposes as a direct sum of vector spaces V_n^i, where V_n^i is the rank $n+1$ vector space supported in homological grading i and quantum and annular gradings $(-n+2p, -n+2p)$ for all $0 \leq p \leq n$. It is not hard to see that annular Khovanov homology of an annular link L is supported in annular gradings j with $|j| \leq w(L)$, where $w(L)$ is the *wrapping number* of L; i.e., the minimum *geometric* intersection number of L with a meridional disk for the thickened annulus. Consequently, $\mathrm{AKh}(L;\mathbb{C})$ can only contain V_n summands with $n \leq w(L)$.

Annular Khovanov homology admits a spectral sequence to the Khovanov homology of the underlying link. The differential on annular Khovanov homology inducing this spectral sequence increases the homological grading by one, preserves the quantum grading and decreases the annular grading. Moreover, the differential forms part of an action of the $\mathfrak{sl}_2(\wedge)$ current algebra on $\mathrm{AKh}(L;\mathbb{C})$ — a stronger structural property than being an $\mathfrak{sl}_2(\mathbb{C})$ representation. See [20, Section 6] for details.

It is not hard to see that $\mathrm{AKh}(b(\beta), k = -n;\, R)$ consists of a single copy of R. This summand is generated by Plamenevskaya's *transverse invariant* [38]. This class consists of n concentric circles about the braid axis, each decorated with X. The quantum grading of this generator is the self-linking number of β — which we denote $sl(\beta)$ — and the homological grading is zero.

3B. *From orderability to rank bounds.* In this section we prove Theorem 3.1 and related results. The most concise version of our result is that stated in the introduction:

Theorem 3.1. *Let β be an n-braid with $n \geq 2$. Then*:

(1) $\mathrm{rank}(\mathrm{AKh}(b(\beta); \mathbb{C})) \geq 2n$.

(2) $\mathrm{rank}(\mathrm{AKh}(c(\beta); \mathbb{C})) \geq 4n$.

Of course, if $n = 1$, then $c(\beta)$ is undefined, while $\mathrm{rank}(\mathrm{AKh}(b(\beta); \mathbb{C})) = 2$. Theorem 3.1 is a direct consequence of the next — stronger — result. To state it recall that the braid group is left orderable. There are many different interpretations of the ordering on the braid group (see [16]), two of which we will use in this section. The first is the following: we write $\beta < 1$ if there is a word for β in the letters given by the standard Artin generators and their inverses which is *σ-negative*; i.e., if among the letters that occur in that word, the letter of the lowest index occurs only with negative powers. See [16, Chapter II, Section 1.2] for details.

Lemma 3.2. *Suppose β is a σ-negative n-braid. Then* $\mathrm{AKh}(c(\beta); \mathbb{C})$ *contains a*

$$V_n^{a_1}\{a_2\} \oplus V_n^{b_1}\{b_2\} \oplus V_{n-2}^{a_1-1}\{a_2 - 2\} \oplus V_{n-2}^{b_1-1}\{b_2 - 2\}$$

summand. Moreover, the $\mathfrak{sl}_2(\wedge)$ action sends the lowest annular grading generator in $V_{n-2}^{a_1-1}\{a_2 - 2\}$ to the lowest annular grading generator in $V_n^{a_1}\{a_2\}$ and the lowest annular grading generator in $V_{n-2}^{b_1-1}\{b_2 - 2\}$ to the lowest annular grading generator in $V_n^{b_1}\{b_2\}$.

Similarly, $\mathrm{AKh}(b(\beta); \mathbb{C})$ *contains a $V_n^0\{n + sl(\beta)\} \oplus V_{n-2}^{-1}\{n - 2 + sl(\beta)\}$ summand. Moreover, the $\mathfrak{sl}_2(\wedge)$ action sends the lowest annular grading generator in $V_{n-2}^{-1}\{n + sl(\beta) - 2\}$ to the lowest annular grading generator in $V_n^0\{n + sl(\beta)\}$.*

Remark 3.3. As we shall see, one can write down the values of the quantum and homological gradings of $\mathrm{AKh}(c(\beta), k = -n;\, R)$ in terms of diagrammatic data for β. Comparing this to the braid closure case — where there is a single generator whose homological grading is zero and quantum grading is the self-linking number — it is natural to ask what topological or contact geometry-theoretic information these numbers contain. We do not pursue this question here.

Our strategy for the proof of Lemma 3.2 is to use properties of σ-negative words to control the annular Khovanov homology of closures of braids in the next to minimal annular grading.

Given an annular link L view $\mathrm{CKh}(L;\, R)$ as a chain complex filtered with respect to the annular filtration. The differential comes in two pieces, $\partial_0 + \partial_{-2}$, where ∂_0 preserves the annular grading on $\mathrm{CKh}(L;\, R)$ and ∂_{-2} decreases it by 2. $\mathrm{AKh}(L;\, R)$ can be viewed as $(\mathrm{CKh}(L;\, R), \partial_0)$, the first page of the corresponding spectral sequence.

Lemma 3.4. *Let β be a σ-negative braid which is not of index* 1. *Then there are chain maps*

$$f_c : \mathrm{CKh}(c(\beta);\, i,\, j,\, k \leq -n;\, R) \to \mathrm{CAKh}(c(\beta);\, i-1,\, j,\, 2-n;\, R),$$

$$f_b : \mathrm{CKh}(b(\beta);\, i,\, j,\, k \leq -n;\, R) \to \mathrm{CAKh}(b(\beta);\, i-1,\, j,\, 2-n;\, R).$$

Moreover, ∂^*_{-2} *is a left inverse to f_c or f_b on the E_1 page of the spectral sequence from* $\mathrm{CAKh}(c(\beta);\, R)$ *or* $\mathrm{CAKh}(b(\beta);\, R)$ *to* $\mathrm{Kh}(b(\beta);\, R)$.

Here by $\mathrm{CKh}(c(\beta);\, i,\, j,\, k \leq -n;\, R)$ we mean the k-filtered part of filtration level n. It follows that $\mathrm{AKh}(c(\beta), k = 2-n;\, R)$ has an $\mathrm{AKh}(c(\beta), k = -n;\, R)[-1]$ summand while $\mathrm{AKh}(b(\beta), k = 2-n;\, R)$ has a $\mathrm{AKh}(b(\beta), k = -n;\, R)[-1]$ summand. Here, $[-1]$ indicates a shift in the homological grading by -1.

Proof. We treat the case of clasp-closures. The proof in the braid-closure case is the same in essence and strictly easier in practice.

Since β is σ-negative β is isotopic to a braid β' that contains the inverse of an Artin generator σ_i^{-1} but not the corresponding Artin generator, σ_i. Consider the diagram D for $c(\beta')$ as in Figure 1(c). There are three complete resolutions D_1, D_2 and D_3 corresponding to $\mathrm{CAKh}(c(\beta), k = -n;\, R)$; these are shown in Figure 4. There are four generators of $\mathrm{CAKh}(c(\beta), k = -n;\, R)$. They can be described as follows. For each i we have a generator $\boldsymbol{X}_i$ where every circle in the resolution D_i is decorated with an X. We have a final generator $\mathbf{1}$ which corresponds to decorating the homologically essential circles in diagram D_1 with Xs and the homologically inessential circle with a 1. The nontrivial components of the differential are given by $\partial_0(\boldsymbol{X}_2) = \partial_0(\boldsymbol{X}_3) = \boldsymbol{X}_1$ for an appropriate sign assignment.

Pick one of the crossings corresponding in D to a letter σ_i^{-1} and label it y. Consider the resolutions D_i' of D that are identical to the resolutions D_i aside from at y. Given a generator $\boldsymbol{x} \in \mathrm{CAKh}(c(\beta), R;\, k = -n)$ define $f_c(\boldsymbol{x}) \in \mathrm{CAKh}(c(\beta), R;\, k = 2-n)$ to be the generator which agrees with $\boldsymbol{X}_i$ on every circle in the resolution that does not involve y, and is labeled with an X on the remaining circle. Observe that if $\boldsymbol{x}$ is of (i, j, k) grading $(a, b, -n)$ then $f_c(\boldsymbol{x})$ is of grading $(a-1, b, 2-n)$.

Since β is σ-negative, f_c is a chain map viewed as a map $(\mathrm{CKh}(c(\beta);\, R), \partial_0) \to (\mathrm{CKh}(c(\beta);\, R), \partial_0)$. To verify this notice that the maps corresponding to changing the resolutions in the β' part of the diagram correspond to merging circles decorated with X's. Thus the only contributions to the differential on $\mathrm{CAKh}(c(\beta), k = 2-n;\, R)$ involve the crossings contained in the part of the diagram for the clasp.

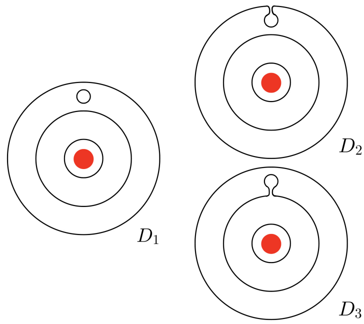

Figure 4. The resolutions for the canonical diagram, as in Figure 1(c), for a clasp-closure yielding generators of $\mathrm{CAKh}(c(\beta), k = -n; R)$. The solid red dot is the annular axis.

One can check that ∂^*_{-2} is a left-inverse to f_c similarly; the only contributions to the differential which lower the annular filtration level correspond to changing the resolution at y. This corresponds to splitting a circle labeled with an X, resulting in two circles both labeled with X's. □

Proof of Lemma 3.2. We treat only the braid-closure case, since the clasp-closure case is essentially the same. Observe first that by the previous lemma, f_b induces an injection

$$f_b^* : \mathrm{AKh}(b(\beta), k = -n; \mathbb{C}) \cong \mathbb{C}[0, sl(\beta), -n] \hookrightarrow \mathrm{AKh}(b(\beta), (-1, sl(\beta), 2-n); \mathbb{C}).$$

Here $\mathbb{C}[0, sl(\beta), -n]$ indicates a $\mathbb{C}$ summand supported in (i, j, k) grading $(0, sl(\beta), -n)$. Now, $\mathrm{AKh}(b(\beta); \mathbb{C})$ carries the structure of an $\mathfrak{sl}_2(\mathbb{C})$-representation, where each summand is supported in a single homological grading. The summand $\mathbb{C}[0, sl(\beta), -n]$ and its image under f_b^* are supported in different homological gradings and hence cannot be part of the same irreducible $\mathfrak{sl}_2(\mathbb{C})$ representation. The image $f_b^*(\mathbb{C}[0, sl(\beta), -n])$ is a minimal annular grading $\mathbb{C}$-summand of its $\mathfrak{sl}_2(\mathbb{C})$ representation, as else we would have $\mathrm{rank}(\mathrm{AKh}(b(\beta), k = -n; \mathbb{C})) \geq 2$, which is a contradiction since $b(\beta)$ is a braid-closure. It follows that the $\mathrm{AKh}(b(\beta); \mathbb{C})$ contains the two desired representations as summands.

The structure of these summands as an $\mathfrak{sl}_2(\wedge)$ representation follow from the fact that ∂^*_{-2} is part of the $\mathfrak{sl}_2(\wedge)$ action. □

We can now extract a rank bound for annular Khovanov homology with $\mathbb{Z}/2$ coefficients from Lemma 3.2 and the proof of Lemma 3.4.

Lemma 3.5. *Let β be a nonidentity $n > 1$-braid. Then*

$$\mathrm{rank}(\mathrm{Kh}(b(\beta)); \mathbb{Z}/2) \leq \mathrm{rank}(\mathrm{AKh}(b(\beta); \mathbb{Z}/2)) - 2(n-1),$$

$$\mathrm{rank}(\mathrm{Kh}(c(\beta); \mathbb{Z}/2))) \leq \mathrm{rank}(\mathrm{AKh}(c(\beta); \mathbb{Z}/2)) - 4(n-1).$$

Proof. Suppose β is as in the statement of the lemma. Since any nonidentity braid β is either σ-negative or σ-positive, we have four cases to consider. We prove the result in the σ-negative braid-closure case. The other three cases are similar.

We show that the rank of the map $\partial^*_{-2} : \mathrm{AKh}(b(\beta); \mathbb{Z}/2) \to \mathrm{AKh}(b(\beta); \mathbb{Z}/2)$ is at least $n-1$; this implies the result. The universal coefficient theorem for homology is functorial, so considering the map ∂^*_{-2} we obtain the following commutative diagram, where μ is the map defined on elementary tensors by $\mu(\boldsymbol{x} \otimes a) \mapsto a\boldsymbol{x}$, and where we have suppressed $b(\beta)$ from the notation in each nontrivial entry to make room.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{AKh}_i(\mathbb{Z}) \otimes \mathbb{F} & \xrightarrow{\mu} & \mathrm{AKh}_i(\mathbb{F}) & \longrightarrow & \mathrm{Tor}^1_{\mathbb{Z}}(\mathrm{AKh}_{i+1}(\mathbb{Z}), \mathbb{F}) & \xrightarrow{h} & 0 \\ & & \downarrow{\scriptstyle \partial^*_{-2} \otimes \mathbf{1}} & & \downarrow{\scriptstyle \partial^*_{-2}} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{AKh}_{i+1}(\mathbb{Z}) \otimes \mathbb{F} & \xrightarrow{\mu} & \mathrm{AKh}_{i+1}(\mathbb{F}) & \longrightarrow & \mathrm{Tor}^1_{\mathbb{Z}}(\mathrm{AKh}_{i+2}(\mathbb{Z}), \mathbb{F}) & \xrightarrow{h} & 0 \end{array} \tag{8}$$

Taking $\mathbb{F} = \mathbb{C}$, so that $\mathrm{Tor}^1_{\mathbb{Z}}(\mathrm{AKh}_j(b(\beta); \mathbb{Z}), \mathbb{F})$ vanishes for all j, and setting $i = -1$ we obtain

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{AKh}_{-1}(b(\beta); \mathbb{Z}) \otimes \mathbb{C} & \xrightarrow{\mu} & \mathrm{AKh}_{-1}(b(\beta); \mathbb{C}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial^*_{-2} \otimes \mathbf{1}} & & \downarrow{\scriptstyle \partial^*_{-2}} & & \\ 0 & \longrightarrow & \mathrm{AKh}_0(b(\beta); \mathbb{Z}) \otimes \mathbb{C} & \xrightarrow{\mu} & \mathrm{AKh}_0(b(\beta); \mathbb{C}) & \longrightarrow & 0 \end{array} \tag{9}$$

Now, from the proof of Lemma 3.4, $\partial^*_{-2} : \mathrm{AKh}_{-1}(b(\beta); \mathbb{C}) \to \mathrm{AKh}_0(b(\beta); \mathbb{C})$ has a component given by the identity map $\mathbb{C}^{\oplus(n-1)} \to \mathbb{C}^{\oplus(n-1)}$ with respect to the canonical basis for $\mathrm{AKh}(b(\beta); \mathbb{C})$. More specifically, since ∂^*_{-2} is nontrivial on a bottom generator of $V^{-1}_{n-2}\{n-2+sl(\beta)\}$ and ∂^*_{-2} is part of the structure of $\mathrm{AKh}(b(\beta); \mathbb{C})$ as an $\mathfrak{sl}_2(\wedge)$-representation, we have that ∂^*_{-2} is nontrivial on the entire $V^{-1}_{n-2}\{n-2+sl(\beta)\}$ summand. It follows from diagram (9) that $\mathrm{AKh}_{-1}(b(\beta); \mathbb{Z})$ and $\mathrm{AKh}_0(b(\beta); \mathbb{Z})$ contain $\mathbb{Z}^{\oplus(n-1)}$ summands and that ∂^*_{-2} acts as the identity map between these two summands.

Now take $\mathbb{F} = \mathbb{Z}/2$ in diagram (8). By the commutativity of the diagram we deduce that $\partial^*_{-2} : \mathrm{AKh}_0(b(\beta); \mathbb{Z}/2) \to \mathrm{AKh}_0(b(\beta); \mathbb{Z}/2)$ has a component given by the identity map $(\mathbb{Z}/2)^{\oplus(n-1)} \to (\mathbb{Z}/2)^{\oplus(n-1)}$. The result follows. ☐

Proof of Theorem 3.1. Suppose β is a braid. Then β is either σ-positive, σ-negative or the identity. If β is the identity then $\mathrm{rank}(\mathrm{AKh}(b(\beta); \mathbb{C})) = 2^n \geq 2n$.

If β is σ-negative the result follows immediately from Lemma 3.2. If β is σ-positive, the result follows from applying Lemma 3.2 to the mirror of β, which is σ-negative, and applying symmetry properties of annular Khovanov homology. ☐

We can also prove an annular Khovanov homology analogue of a result of Ni from knot Floer homology [34, Theorem A.1]. To do so, we exploit a geometric interpretation of the ordering of the braid group in terms of curve diagrams; see [16, Chapter 10]. Recall that n-braids can be viewed as mapping classes of n-punctured disks. Recall too that a braid is *right* (*left*) *veering* if it sends every *admissible* arc to the right (left). See [2, Section 3.1] for a definition of admissible. If a braid is non-right (left) veering then it is conjugate to a σ-negative (positive) braid; see the proof of [2, Proposition 3.1].

Proposition 3.6. *Suppose β is a non-right-veering and non-left-veering n-braid, with $n \geq 4$. Then* $\operatorname{rank}(\mathrm{AKh}(b(\beta);\mathbb{C})) \geq 4n-4$.

Proof. Suppose β is as in the statement of the proposition. Since V is an n-braid there is a V_n^0 summand. Since β is non-right-veering, it is conjugate to a braid β' that is σ-negative. Since $b(\beta')=b(\beta)$, $\mathrm{AKh}(b(\beta);\mathbb{C})$ has a V_{n-2}^{-1}-summand by Lemma 3.2. Similarly, since β is non-left-veering, $\mathrm{AKh}(b(\beta);\mathbb{C})$ contains a V_{n-2}^1 summand by Lemma 3.2.

Assume towards a contradiction that there is a unique generator in (i,k)-grading $(0,4-n)$. Consider the generator $\boldsymbol{x}$ of (i,k)-grading $(-1,6-n)$ and form $\partial_{-2}^*:\mathrm{AKh}(b(\beta);\mathbb{C})\to\mathrm{AKh}(b(\beta);\mathbb{C})$. Since $b(\beta)$ is isotopic to the braid-closures of σ-positive and σ-negative words, Lemma 3.2 and the symmetry properties of the spectral sequence from annular Khovanov homology to Khovanov homology under mirroring imply that $(\partial_{-2}^*)^2(\boldsymbol{x})\neq 0$, a contradiction. The result now follows from noting that $\mathrm{AKh}(b(\beta);\mathbb{C})$ carries the structure of an $\mathfrak{sl}_2(\mathbb{C})$-representation; specifically there must be at least one more generator in (i,k)-grading $(0,4-n)$, and so in turn another V_{n-4}-summand. □

In the case that $n=3$ we have:

Proposition 3.7. *Suppose β is a 3-braid that is non-right-veering and non-left-veering. Then* $\operatorname{rank}(\mathrm{AKh}(b(\beta);\mathbb{C}) \geq 10$.

Proof. Since V is a 3-braid there is a V_3^0 summand. Since β is non-right-veering there is a V_1^{-1} summand by Lemma 3.2. Since β is non-left-veering there is a V_1^1 summand by Lemma 3.2. We claim that $\operatorname{rank}(\mathrm{AKh}(b(\beta);\mathbb{C}))>8$. Otherwise, with $\operatorname{rank}(\mathrm{AKh}(b(\beta);\mathbb{C}))\leq 8$, we would in fact have $\mathrm{AKh}(b(\beta);\mathbb{C})\cong V_1^{-1}\oplus V_3^0\oplus V_1^1$. Then Lemma 3.2 would imply that all generators die under the spectral sequence from $\mathrm{AKh}(b(\beta);\mathbb{C})$ to $\mathrm{Kh}(b(\beta);\mathbb{C})$; this is not possible, proving the claim. The result now follows from the fact that $\operatorname{rank}(\mathrm{AKh}(b(\beta);\mathbb{C}))$ is even for 3-braids — which have wrapping number 3 — since it splits as a direct sum of V_3 and V_1 summands. □

It is unclear to the author if similar results could be obtained for clasp-closures, since clasp-closures of conjugate braids are not necessarily isotopic, so the proof strategy above breaks down.

3C. ***Applications of the rank bound.*** Let β_n denote the n-braid $\sigma_1\sigma_2\dots\sigma_{n-1}$, and $\mathbb{1}_n$ denote the identity n-braid.

Proposition 3.8. *Suppose α is an n-braid, with $n > 2$. If* $\operatorname{rank}(\operatorname{AKh}(b(\alpha);\mathbb{C})) = 2n$ *then* $\operatorname{AKh}(b(\alpha);\mathbb{C}) \cong \operatorname{AKh}(b(\beta_n^{\pm 1});\mathbb{C})$.

The $n = 1$ case is uninteresting, as is the $n = 2$ case since the annular Khovanov homology of all 2-braids is known [20]. Indeed, in the 2-braid case, the proposition is false; $\operatorname{rank}(\operatorname{AKh}(b(\mathbb{1}_2);\mathbb{Z}) = \operatorname{rank}(\operatorname{AKh}(b(\beta_2^{\pm 1});\mathbb{Z})) = 4$ [20].

Proof. Suppose α is neither σ-positive nor σ-negative. Then α is the identity braid, and one can readily check that $\operatorname{rank}(\operatorname{AKh}(b(\alpha);\mathbb{C})) = 2^n$. It follows that $n = 1$ or $n = 2$, contradicting our assumption.

Suppose now that α is σ-negative. Then $\operatorname{AKh}(b(\alpha);\mathbb{C})$ contains a summand of the form $V_n^0\{n + sl(\beta)\} \oplus V_{n-2}^{-1}\{sl(\beta) - 2\}$ by Lemma 3.2, so must in fact be $V_n^0\{sl(\beta)\} \oplus V_{n-2}^{-1}\{sl(\beta) - 2\}$ The σ-positive case follows by a similar argument. □

Since annular Khovanov homology detects $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_8, \beta_{10}$ by [12; 13; 18], it follows from Proposition 3.8 that $\operatorname{rank}(\operatorname{AKh}(-;\mathbb{Z}/2))$ detects each of these braids amongst braids of the correct index. That is, we have:

Corollary 3.9. *Suppose α is an n-braid with $n \in \{3, 4, 5, 6, 8, 10\}$. If the rank of* $\operatorname{AKh}(b(\alpha);\mathbb{Z}/2))$ *equals $2n$ then $\alpha = \beta_n^{\pm 1}$.*

This allows one to cut some casework from Baldwin, Hu and Sivek's proof that Khovanov homology detects $T(2, 5)$ with $\mathbb{Z}/2$ coefficients [8, Sections 4 and 5]. In fact, we can generalize part of their argument as follows:

Proposition 3.10. *Suppose K is an m-periodic link with axis of symmetry A,* $\operatorname{rank}(\operatorname{Kh}(K;\mathbb{Z}/m)) \leq 2n$ *and $\ell k(A, K) \geq n$. Let J denote the quotient of K viewed as an annular link about A. Then* $\operatorname{AKh}(J;\mathbb{C}) \cong \operatorname{AKh}(b(\beta_n^{\pm 1});\mathbb{C})$. *Moreover, if $n \in \{3, 4, 5, 6, 8, 10\}$, then $J = b(\beta_n^{\pm 1})$.*

Proof. We follow closely the argument in [8]. Let J denote the quotient of K, viewed as an annular link about A. A result of Stoffregen and Zhang [41] implies that

$$\operatorname{rank}(\operatorname{AKh}(J;\mathbb{Z}/m)) \leq \operatorname{rank}(\operatorname{Kh}(K;\mathbb{Z}/m)).$$

Thus $\operatorname{rank}(\operatorname{AKh}(J;\mathbb{C})) \leq \operatorname{rank}(\operatorname{AKh}(J;\mathbb{Z}/m)) \leq 2n$, with the first inequality coming from the universal coefficient theorem. Xie defined a spectral sequence from annular Khovanov homology to an invariant called *annular instanton Floer homology* which respects the annular grading [43]. Xie and Zhang showed in [45, Theorem 1.6] that the maximum nontrivial annular grading of the annular instanton Floer homology of J is given by

$$\min\{2g(S) + |S \cap J| : S \text{ is a meridional surface}\}.$$

Here, a meridional surface in the thickened annulus is any surface which meets the boundary of the thickened annulus in a meridian, i.e., a curve that bounds a disk. Note that $\min\{2g(S)+|S\cap J|\} \geq \ell\mathrm{k}(A, J) \geq n$. It follows that

$$\mathrm{rank}(\mathrm{AKh}(J; \mathbb{C}, k=n)) > 0.$$

Therefore, J is a braid by [19, Theorem 1.1], since $\mathrm{AKh}(J; \mathbb{C})$ contains a copy of V_{n+1} and so cannot have rank greater than one in the maximum annular grading. The result then follows directly from Proposition 3.8 in general and Corollary 3.9 in the special cases. □

3D. ***Braid-closures.*** In this section we prove the following result:

Theorem 3.11. *Annular Khovanov homology with integer coefficients detects* $b(\sigma_1\sigma_2^n)$ *for* $-2 \leq n \leq 5$.

We remind the reader that the $n=1$ case was already proven in [13], so we do not discuss it here. We begin with some computations.

We compute the annular Khovanov homology of the annular links $b(\sigma_1\sigma_2^n)$. It is readily checked that

$$\mathrm{AKh}(b(\sigma_1); \mathbb{C}) \cong V_3^0\{1\} \oplus V_1^0\{1\} \oplus V_1^1\{3\} \tag{10}$$

and that $\mathrm{AKh}(b(\sigma_1); \mathbb{Z}/2)$ can be obtained by replacing each homogeneous $\mathbb{C}$ summand in $\mathrm{AKh}(c(\sigma_1); \mathbb{C})$ with a $\mathbb{Z}/2$ summand. We compute the annular Khovanov homology of the remaining links.

Lemma 3.12. *For* $n \geq 1$, $\mathrm{AKh}(b(\sigma_1\sigma_2^n); \mathbb{C})$ *is given by*

$$V_3^0\{n+1\} \oplus V_1^1\{n+3\} \oplus \bigoplus_{1\leq i\leq n-1} V_1^{1+i}\{n+1+2i\}.$$

For $n \leq -1$, $\mathrm{AKh}(b(\sigma_1\sigma_2^n); \mathbb{C})$ *is given by*

$$V_3^0\{1+n\} \oplus V_1^1\{3+n\} \oplus V_1^0\{1+n\} \oplus \bigoplus_{-1\geq i\geq n} V_1^i\{n+1+2i\}.$$

In each case $\mathrm{AKh}(b(\sigma_1\sigma_2^n); \mathbb{Z}/2)$ *is given by replacing each homogeneous* $\mathbb{C}$*-summand with a* $\mathbb{Z}/2$*-summand.*

Proof. Consider the standard diagram for $b(\sigma_1\sigma_2^n)$, as in Figure 1(a). We consider two cases; that in which $n \geq 1$ and that in which $n \leq -1$. We proceed by induction in both instances.

In the $n \geq 1$ case, note that $\mathrm{AKh}(b(\sigma_1\sigma_2); \mathbb{C}) \cong V_3^0\{2\} \oplus V_1^1\{4\}$.

For the inductive step we resolve $b(\sigma_1\sigma_2^n)$ at one of the crossings corresponding to a σ_2. Observe that the 1-resolution is the braid-closure of the identity 1-braid, while the 0-resolution is $b(\sigma_1\sigma_2^{n-1})$. Applying the exact triangle (7) and the fact that $n_- = 0$ and $n_-^1 = n-1$, we obtain

$$\begin{array}{ccc} \mathrm{AKh}(b(\sigma_1\sigma_2^n);\mathbb{C}) & \longrightarrow & \mathrm{AKh}(b(\sigma_1\sigma_2^{n-1});\mathbb{C})\{1\} \\ & \nwarrow \quad \swarrow_{\delta} & \\ & \mathrm{AKh}(b(\mathbf{1}_1);\mathbb{C})[n]\{3n-1\} & \end{array}$$

Now,

$$\mathrm{AKh}(\mathbf{1}_1;\mathbb{C}) \cong V_1^0. \tag{11}$$

For $n > 2$ this map splits by the inductive hypothesis, since there are no generators in $\mathrm{AKh}(b(\sigma_1\sigma_2^{n-1});\mathbb{C})$ of the correct gradings to map nontrivially to $V_1^0[n]\{3n-1\}$. For $n = 2$ the result can be computed by hand or one can note that the connecting map δ, which increases the i grading by 1, must vanish as $\mathrm{Kh}(b(\sigma_1\sigma_2^2);\mathbb{C}) \cong \mathrm{Kh}(T(2,2));\mathbb{C})$ has two generators with i grading 2.

We now proceed to the $n \leq -1$ case. Note that

$$\mathrm{AKh}(b(\sigma_1\sigma_2^{-1});\mathbb{C}) \cong V_3^0 \oplus V_1^1\{2\} \oplus V_1^0 \oplus V_1^{-1}\{-2\}.$$

For the inductive step we resolve $b(\sigma_1\sigma_2^n)$ at one of the crossings corresponding to a σ_2^{-1}. Applying the exact triangle (6) and noting that $n_- = -n$ and $n_-^0 = 0$ we obtain

$$\begin{array}{ccc} \mathrm{AKh}(b(\sigma_1\sigma_2^n);\mathbb{C}) & \longrightarrow & \mathrm{AKh}(b(\mathbf{1}_1);\mathbb{C})[n]\{1+3n\} \\ & \nwarrow \quad \swarrow & \\ & \mathrm{AKh}(b(\sigma_1\sigma_2^{n+1});\mathbb{C})\{-1\} & \end{array}$$

Given equation (11) and the inductive hypothesis, the grading data implies that the exact triangle must split. The result follows.

Finally, to see that $\mathrm{AKh}(b(\sigma_1\sigma_2^n);\mathbb{Z}/2))$ is as claimed, observe that the proofs above from the case of complex coefficients carry through to the case of $\mathbb{Z}/2$ coefficients verbatim. □

For $\mathbb{F} \in \{\mathbb{Z}/2, \mathbb{C}\}$, $\mathrm{AKh}(b(\sigma_1^{-1}\sigma_2^n);\mathbb{F})$ can be determined from Lemma 3.12 using symmetry properties of annular Khovanov homology. Since, by the Birman–Menasco classification, the 3-braid representatives of the link $T(2,n)$ with $n \neq 0$ are exactly links of the form $b(\sigma_1^{-1}\sigma_2^n)$ and $b(\sigma_1\sigma_2^n)$, this means we have computed the annular Khovanov homology of all 3-braid representatives of $T(2,n)$.

Proof of Theorem 3.11. The strategy is to use the spectral sequence from the annular Khovanov homology of an annular link to Khovanov homology of the underlying link to determine the underlying link type then to exploit the Birman–Menasco classification [15].

Suppose L is an annular link with $\mathrm{AKh}(L;\mathbb{Z}) \cong \mathrm{AKh}(b(\sigma_1\sigma_2^n);\mathbb{Z})$ for some n. Note that $\mathrm{AKh}(L;R) \cong \mathrm{AKh}(b(\sigma_1\sigma_2^n);R)$ for $R \in \{\mathbb{Q},\mathbb{C},\mathbb{Z}/2\}$ by the universal coefficient theorem. Since L has rank one in the maximum nontrivial k grading it follows that L is isotopic to the closure of a braid β [19]. Since the maximum nontrivial k grading is 3 it follows that β has index 3. We now split our analysis into three cases; $n = -1$, $n = -2$ and $n \geq 0$.

<u>Case $n = -1$:</u> We claim that $b(\beta)$ is an unknot. First, since $\mathrm{AKh}(b(\beta);\mathbb{C})$ has support in odd quantum gradings, $b(\beta)$ has an odd number of components. Consider the spectral sequence from $\mathrm{AKh}(b(\beta);\mathbb{Z}/2)$ to $\mathrm{Kh}(b(\beta);\mathbb{Z}/2)$. Suppose L is neither σ-positive nor σ-negative. Then L is the identity 3-braid. This is a contradiction, since the identity 3-braid has annular Khovanov homology of rank 8. It follows that L is either σ-positive or σ-negative. Thus $\mathrm{rank}(\mathrm{Kh}(b(\beta);\mathbb{Z}/2)) \leq 6$ by Lemma 3.5. It follows that L has at most two components, since $\mathrm{rank}(\mathrm{Kh}(b(\beta);\mathbb{Z}/2)) \geq 2^m$ where m is the number of components of L. Thus, $\mathrm{rank}(\mathrm{Khr}(b(\beta);\mathbb{Z}/2)) \leq 3$ by [40, Corollary 3.2.C]. Thus L is either a trefoil or an unknot [3; 29]. But L cannot be a trefoil, since $\mathrm{Kh}(T(2,\pm 3);\mathbb{Q})$ has support in quantum gradings ± 9. It follows that L is an unknot. Since there are only three 3-braids representing the unknot up to conjugation by Murasugi's classification [33], it suffices to show that β is not $\sigma_1\sigma_2$ or $\sigma_1^{-1}\sigma_2^{-1}$. But these two braids have braid-closures with annular Khovanov homology of rank 6 over $\mathbb{C}$, rather than 10, completing the proof in this case.

<u>Case $n = -2$:</u> First, since $\mathrm{AKh}(b(\beta);\mathbb{C})$ has support in even quantum gradings $b(\beta)$ has an even number of components. Since β is a 3-braid it follows that β has two components. Observe that $\mathrm{rank}(\mathrm{Kh}(b(\beta);\mathbb{Z}/2)) \leq 8$ by Lemma 3.5, so by [44, Corollary1.4] $b(\beta)$ represents a two-component unlink, $T(2,\pm 2)$ or $T(2,\pm 4)$. By Birman–Menasco, $c(\beta)$ must be of the form $\sigma_1^{\pm 1}\sigma_2^n$ for some even $|n| \leq 4$. Annular Khovanov homology distinguishes each of these links, concluding the proof in this case.

<u>Case $n \neq 0$:</u> Observe that β cannot be the identity braid, since its annular Khovanov homology is not of the correct form. Moreover, β cannot be σ-negative as there are no generators of $\mathrm{AKh}(L;\mathbb{C})$ in homological grading -1. It follows that β is σ-positive. An application of Lemmas 3.5 and 3.12 implies that for $n \geq 1$

$$\mathrm{rank}(\mathrm{Kh}(b(\beta);\mathbb{Z}/2)) \leq \mathrm{rank}(\mathrm{AKh}(b(\beta);\mathbb{Z}/2)) - 4 = 2n. \tag{12}$$

On the other hand, for $n = 0$, equation (10) and Lemma 3.5 imply that

$$\mathrm{rank}(\mathrm{Kh}(b(\beta);\mathbb{Z}/2)) \leq \mathrm{rank}(\mathrm{AKh}(b(\beta);\mathbb{Z}/2)) - 4 = 4.$$

Thus $\mathrm{rank}(\mathrm{Khr}(b(\beta);\mathbb{Z}/2)) \leq \max\{n, 2\}$ by [40, Corollary 3.2.C]. We now treat three subcases:

<u>$n = 3$</u>. In this case, $\text{rank}(\text{Khr}(b(\beta); \mathbb{Z}/2)) \leq 3$. Since $\text{AKh}(b(\beta); \mathbb{Q})$ is supported in odd quantum gradings it has an odd number of components. Note that $b(\beta)$ can have no more than two components, since $\text{rank}(\text{Kh}(b(\beta); \mathbb{Q})) \geq 2^m$, where m is the number of components of $c(\beta)$. It follows that $b(\beta)$ is a knot. Now, $\text{rank}(\text{Khr}(b(\beta); \mathbb{Z}/2)))$ is odd, so that $\text{rank}(\text{Khr}(b(\beta); \mathbb{Z}/2))) = 1$ or $\text{rank}(\text{Khr}(b(\beta); \mathbb{Z}/2))) = 3$. If $\text{rank}(\text{Khr}(b(\beta); \mathbb{Z}/2))) = 1$ then $b(\beta)$ represents the unknot by [29]. But the three braid-closures of 3-braids representing the unknot have different annular Khovanov homology from $\text{AKh}(b(\beta); \mathbb{Z})$, so $n \neq 1$. It follows that $n = 3$ and $b(\beta)$ represents a trefoil by [3]. There are four 3-braids representing trefoils by [15]. They each have distinct annular Khovanov homology by Lemma 3.12, so the result follows.

<u>$n = 5$</u>. Since $\text{AKh}(b(\beta); \mathbb{C})$ is supported in odd quantum gradings, $b(\beta)$ has an odd number of components. Since β is a 3-braid, $b(\beta)$ has either one or three components. If β has three components, then since $\text{rank}(\text{Kh}(b(\beta); \mathbb{Z}/2)) \leq 10$ by Table (12), the Batson–Seed link splitting spectral sequence implies that each component of $b(\beta)$ is unknotted. More specifically, for any component K of $b(\beta)$, we have $2\,\text{rank}(\text{Khr}(K; \mathbb{Z}/2)) \leq \text{rank}(\text{Kh}(K; \mathbb{Z}/2)) \leq 2$ by the universal coefficient theorem and [40, Corollary 3.2.C]. Consequently, $\text{rank}(\text{Khr}(K; \mathbb{Z}/2)) = 1$ and K is the unknot by [29]. Birman and Menasco's classification theorem [15] implies that the only 3-braid representative of the three-component unlink is the identity 3-braid. However, the identity 3-braid has distinct annular Khovanov homology from $\text{AKh}(b(\beta); \mathbb{C})$, so that L in fact does not have 3 components. It follows that $b(\beta)$ is a knot. Now, if $\text{rank}(\text{Khr}(b(\beta); \mathbb{Z}/2)) \leq 3$ we can proceed as in the $n = 3$ case and deduce that $b(\beta)$ represents a trefoil or the unknot. This is a contradiction, since the annular Khovanov homology of the corresponding braid-closures are not of the correct form. It follows that $\text{rank}(\text{Khr}(b(\beta); \mathbb{Z}/2)) = 5$. In turn it follows from [40, Corollary 3.2.C] that $\text{rank}(\text{Kh}(b(\beta); \mathbb{Z}/2)) = 10$. Consider the spectral sequence from $\text{AKh}(b(\beta); \mathbb{Z}/2))$ to $\text{Kh}(b(\beta); \mathbb{Z}/2))$. The proof of Lemma 3.5, together with the symmetry properties of the spectral sequences under mirroring, implies that the spectral sequence kills the generators in (i, j, k) gradings given by

$$(0, 9, 3), \ (1, 9, 1), \ (0, 7, 1), \ (1, 7, -1).$$

Since the E_∞-page $\text{Kh}(b(\beta); \mathbb{Z}/2))$ must have rank 10, the remaining ten generators must survive. By examining their homological and quantum gradings we find that $\text{Kh}(b(\beta); \mathbb{Z}/2)) \cong \text{Kh}(T(2, 5); \mathbb{Z}/2))$. Thus $b(\beta)$ is $T(2, 5)$ by [8, Theorem 1.1]. Birman and Menasco's classification implies that $\beta = \sigma^{\pm 1}\sigma_2^5$ up to conjugation. But $b(\sigma^{-1}\sigma_2^5)$ has the wrong annular Khovanov homology, so the result follows.

<u>$n \in \{0, 2, 4\}$</u>. In this case $b(\beta)$ has an even number of components, because $\text{AKh}(b(\beta); \mathbb{C})$ is supported in even quantum gradings. Since $b(\beta)$ is a 3-braid it has

exactly two components. Now, $\operatorname{rank}(\mathrm{Kh}(b(\beta); \mathbb{Z}/2)) \leq 8$, so by [44, Corollary1.4] $b(\beta)$ is either a two-component unlink, $T(2, \pm 2)$ or $T(2, \pm 4)$. By Birman and Menasco's classification result [15], $c(\beta)$ must be of the form $\sigma_1^{\pm 1}\sigma_2^n$ for some even $|n| \leq 4$. Annular Khovanov homology distinguishes these links. □

3E. *Clasp-closures.* We now study the annular Khovanov homology of clasp-closures of 3-braids. The results are dependent on the rank bound from Section 3B. We first state our main result, followed by a necessary lemma.

Theorem 3.13. *Annular Khovanov homology with integer coefficients detects the Mazur pattern.*

Lemma 3.14. *Suppose K is a clasp-closure of a 3-braid. If K' is an annular knot with $\mathrm{AKh}(K; \mathbb{C}) \cong \mathrm{AKh}(K'; \mathbb{C})$ then K' is also a clasp-closure of a 3-braid.*

Proof. Suppose K' is as in the statement of the lemma. Consider Xie's spectral sequence from $\mathrm{AKh}(K'; \mathbb{C})$ to $\mathrm{AHI}(K'; \mathbb{C})$ [43]. The maximum nontrivial annular grading of $\mathrm{AHI}(K'; \mathbb{C})$ is either 3 or 1. By [45, Theorem 1.6], if it is 1 then there is a meridional surface of Euler characteristic zero, i.e., K' is an annular link of wrapping number 1. Such links have annular Khovanov homology with maximal nontrivial annular grading one — which can be seen by viewing K' as a connect sum of the 1-braid and a wrapping number zero link and applying the Künneth formula for annular Khovanov homology — a contradiction.

It follows that $\mathrm{AHI}(K'; \mathbb{C})$ is of rank 2 in annular grading 3, the maximum annular grading in which $\mathrm{AHI}(K'; \mathbb{C})$ is nontrivial. It follows from [11, Proposition 8.6] that K' is a clasp-braid-closure of index 3. □

To prove Theorem 3.13 it remains to show that annular Khovanov homology distinguishes the Mazur pattern from the other clasp-closures of 3-braids representing unknots. To that end, we give a partial computation for the annular Khovanov homology of the three types of clasp-closures representing unknots.

First, we consider the mirror of the Mazur pattern, $c(\sigma_1^{-1})$.

Lemma 3.15. $\mathrm{AKh}(c(\sigma^{-1}); \mathbb{C})$ *is given by*

$k \rightarrow$ / $i \downarrow$	-3	-1	1	3
0	$\mathbb{C}_{-5}$	$\mathbb{C}^2_{-3}$	$\mathbb{C}^2_{-1}$	$\mathbb{C}_{1}$
-1	$\mathbb{C}_{-7}$	$\mathbb{C}^2_{-5}$	$\mathbb{C}^2_{-3}$	$\mathbb{C}_{-1}$
-2		$\mathbb{C}_{-7}$	$\mathbb{C}_{-5}$	

$\mathrm{AKh}(c(\sigma^{-1}); \mathbb{Z}/2)$ can be obtained by replacing every homogeneous $\mathbb{C}$-summand and replacing it with a $\mathbb{Z}/2$-summand.

Proof. Consider the 0- and 1-resolutions of the crossing at the top of the diagram shown in Figure 1(c). The 0-resolution yields the $b(\sigma_1^{-1}\sigma_2^{-1})$ while the 1-resolution is $b(\sigma_1^{-1})$. Recall that $\mathrm{AKh}(b(\sigma_1^{-1}\sigma_2^{-1});\mathbb{C})$ is given by

(13)

$k \rightarrow$ / $i \downarrow$	-3	-1	1	3
0	$\mathbb{C}_{-5}$	$\mathbb{C}_{-3}$	$\mathbb{C}_{-1}$	$\mathbb{C}_{1}$
-1		$\mathbb{C}_{-5}$	$\mathbb{C}_{-3}$	

This can be computed by hand. On the other hand, $\mathrm{AKh}(b(\sigma_1^{-1});\mathbb{C})$ is given by

(14)

$k \rightarrow$ / $i \downarrow$	-3	-1	1	3
0	$\mathbb{C}_{-4}$	$\mathbb{C}^2_{-2}$	$\mathbb{C}^2_{0}$	$\mathbb{C}_{2}$
-1		$\mathbb{C}_{-4}$	$\mathbb{C}_{-2}$	

Now observe that $n^- = 3$, $n_0^- = 2$, and the exact triangle (6) reduces to

$$\begin{array}{ccc} \mathrm{AKh}(c(\sigma^{-1});\mathbb{C}) & \longrightarrow & \mathrm{AKh}(b(\sigma_1^{-1}\sigma_2^{-1});\mathbb{C})[-1]\{-2\} \\ \nwarrow & & \swarrow \scriptstyle{\delta} \\ & \mathrm{AKh}(b(\sigma_1^{-1});\mathbb{C})\{-1\} & \end{array}$$

Comparing the gradings listed in (13) and (14) and noting that the connecting homomorphism δ preserves the quantum grading, we see that the lower right-hand map in this triangle vanishes, yielding the desired result for $\mathrm{AKh}(c(\sigma_1^{-1});\mathbb{C})$. The computation for $\mathrm{AKh}(c(\sigma_1^{-1});\mathbb{Z}/2)$ is identical. $\square$

We now give a partial computation for $\mathrm{AKh}(b(\sigma_1^{-3}\sigma_2\sigma_1^{-2});\mathbb{C})$.

Lemma 3.16. *$b(\sigma_1^{-3}\sigma_2\sigma_1^{-2})$ has annular Jones polynomial given by*

$$t^{-3}(-q+q^{-1}) + t^{-1}(-q^3+2q) + t(-q^5+2q^3) + t^3(-q^7+q^5).$$

Moreover, in annular grading 3 *the annular Khovanov homology has rank two and is supported in* (i, j) *gradings* $(3,7)$ *and* $(2,5)$.

Proof. Consider the 0 and 1-resolutions of the crossing at the top of the diagram shown in Figure 1(c) taking $\alpha = \sigma_1^{-3}\sigma_2\sigma_1^{-2}$. The 0-resolution yields the braid $b'(\sigma_1^{-3}\sigma_2\sigma_1^{-2}\sigma_2^{-1})$ but with the orientation of the component which is not a braid-closure of the 1-braid endowed with the opposite orientation — which we have indicated with the $'$. The 1-resolution is $b(\sigma_1^{-3}\sigma_2\sigma_1^{-2})$.

The annular Khovanov homology of the two braids can be computed using Hunt, Keese, Licata and Morrison's program [22]. In particular we find that

$\mathrm{AKh}(b(\sigma_1^{-3}\sigma_2\sigma_1^{-2}\sigma_2^{-1});\mathbb{C})$ is given by

$k \rightarrow$ / $i \downarrow$	-3	-1	1	3
0	$\mathbb{C}_{-8}$	$\mathbb{C}_{-6}$	$\mathbb{C}_{-4}$	$\mathbb{C}_{-2}$
-1		$\mathbb{C}_{-8}$	$\mathbb{C}_{-6}$	
-2		$\mathbb{C}^2_{-8}$	$\mathbb{C}^2_{-6}$	
-3		$\mathbb{C}_{-10}$	$\mathbb{C}_{-8}$	
-4		$\mathbb{C}_{-12}$	$\mathbb{C}_{-10}$	
-5		$\mathbb{C}_{-14}$	$\mathbb{C}_{-12}$	

To correct for the fact that one of the components is given the nonbraid orientation we have to shift the homological grading by [2] and the quantum grading by $\{6\}$. This holds because (annular) Khovanov homology is defined by applying a cube of resolution procedure to an unoriented (annular) link diagram, and then shifting the (diagram dependent) homological and quantum gradings by quantities determined by the orientation — namely the quantum grading by $\{n_+ - 2n_-\}$ and the homological grading by $-n_-$, where $n_\pm$ represents the number of positive and negative intersections — to obtain the diagram independent homological and quantum gradings. It is then straightforward to check that for the diagram at hand, reversing the orientation of the relevant component induces the shifts in homological and quantum grading as claimed above.

On the other hand, $\mathrm{AKh}(b(\sigma_1^{-3}\sigma_2\sigma_1^{-2});\mathbb{C})$ is given by

$k \rightarrow$ / $i \downarrow$	-3	-1	1	3
1		$\mathbb{C}_{-5}$	$\mathbb{C}_{-3}$	
0	$\mathbb{C}_{-7}$	$\mathbb{C}^2_{-5}$	$\mathbb{C}^2_{-3}$	$\mathbb{C}_{-1}$
-1		$\mathbb{C}_{-7}$	$\mathbb{C}_{-5}$	
-2		$\mathbb{C}_{-9}$	$\mathbb{C}_{-7}$	
-3		$\mathbb{C}_{-11}$	$\mathbb{C}_{-9}$	
-4		$\mathbb{C}_{-13}$	$\mathbb{C}_{-11}$	
-5		$\mathbb{C}_{-15}$	$\mathbb{C}_{-13}$	

Since $n_- = 4$ and $n_-^1 = 6$, we have the exact triangle

$$\begin{array}{ccc} \mathrm{AKh}(L;\mathbb{C}) & \longrightarrow & \mathrm{AKh}(b'(\sigma_1^{-3}\sigma_2\sigma_1^{-2}\sigma_2^{-1});\mathbb{C})\{1\} \\ & \nwarrow \qquad \swarrow & \\ & \mathrm{AKh}(b(\sigma_1^{-3}\sigma_2\sigma_1^{-2});\mathbb{C})[3]\{8\} & \end{array}$$

This isn't enough information to show that the exact triangle splits. However, it does split in annular gradings ± 3, and the decategorification of the exact triangle determines the annular Jones polynomial, as desired. □

Let S be the annular link given by the split sum of $b(\mathbf{1}_1)$ and an unknot. Observe that the annular Jones polynomial of S is given by $J(S) = t^{-1}(1+q^{-2}) + t(1+q^2)$.

Lemma 3.17. *The annular Jones polynomial of* $c(\sigma_1^n \sigma_2^{-1}\sigma_1\sigma_2)$ *is given by*

$$J(c(\sigma_1^n \sigma_2^{-1}\sigma_1\sigma_2)) = \frac{q+(-1)^{n+1}q^{1-2n}}{1+q^2} J(S) + (-1)^n q^{-2n} J(c(\sigma_2^{-1}\sigma_1\sigma_2)).$$

Moreover, in annular grading 3 *the annular Khovanov homology has rank two and is supported in* (i, j) *gradings* $(-n, 3-2n)$ *and* $(-1-n, 1-2n)$.

Proof. We first compute the annular Khovanov homology of $c(\sigma_1)$, which is isotopic to $c(\sigma_2^{-1}\sigma_1\sigma_2)$.

Consider the 0 and 1-resolutions of the crossing at the top of the diagram shown in Figure 1(c) taking $\alpha = \sigma_1$. The 0-resolution is $b(\sigma_1\sigma_2^{-1})$. The 1-resolution is $b(\sigma_1)$. Since $n_-^0 = 1$ and $n_- = 2$, we have

$$\mathrm{AKh}(c(\sigma_1);\mathbb{C}) \longrightarrow \mathrm{AKh}(b(\sigma_1\sigma_2^{-1});\mathbb{C})[-1]\{-2\}$$
$$\mathrm{AKh}(b(\sigma_1\sigma_2^{-1});\mathbb{C})[-1]\{-2\} \xrightarrow{\delta} \mathrm{AKh}(b(\sigma_1);\mathbb{C})\{-1\}$$
$$\mathrm{AKh}(b(\sigma_1);\mathbb{C})\{-1\} \longrightarrow \mathrm{AKh}(c(\sigma_1);\mathbb{C})$$

$\mathrm{AKh}(b(\sigma_1\sigma_2^{-1});\mathbb{C})[-1]\{-2\}$ and $\mathrm{AKh}(b(\sigma_1);\mathbb{C})\{-1\}$ are given respectively by

$k \rightarrow$ / $i \downarrow$	-3	-1	1	3
0		$\mathbb{C}_{-1}$	$\mathbb{C}_1$	
-1	$\mathbb{C}_{-5}$	$\mathbb{C}^2_{-3}$	$\mathbb{C}^2_{-1}$	$\mathbb{C}_1$
-2		$\mathbb{C}_{-5}$	$\mathbb{C}_{-3}$	

and

$k \rightarrow$ / $i \downarrow$	-3	-1	1	3
1		$\mathbb{C}_1$	$\mathbb{C}_3$	
0	$\mathbb{C}_{-3}$	$\mathbb{C}^2_{-1}$	$\mathbb{C}^2_1$	$\mathbb{C}_3$

Thus, since δ preserves the quantum grading and increases the homological grading by one, the exact triangle splits and $\mathrm{AKh}(c(\sigma_1);\mathbb{C})$ is given by

$k \rightarrow$ / $i \downarrow$	-3	-1	1	3
1		$\mathbb{C}_1$	$\mathbb{C}_3$	
0	$\mathbb{C}_{-3}$	$\mathbb{C}^3_{-1}$	$\mathbb{C}^3_1$	$\mathbb{C}_3$
-1	$\mathbb{C}_{-5}$	$\mathbb{C}^2_{-3}$	$\mathbb{C}^2_{-1}$	$\mathbb{C}_1$
-2		$\mathbb{C}_{-5}$	$\mathbb{C}_{-3}$	

We now proceed to the general case. Remove the axis from the diagram shown in Figure 3 to obtain a diagram for the link. Observe that the 1-resolution of the highlighted crossing is the link S, which has annular Khovanov homology given by

$k \rightarrow$ $i \downarrow$	-1	1
0	$\mathbb{C}_{-2} \oplus \mathbb{C}_0$	$\mathbb{C}_0 \oplus \mathbb{C}_2$

Now, since we can take $n_-^0 - n_- = -1$, the exact triangle (6) reduces to

$$\mathrm{AKh}(\sigma_1^n \sigma_2^{-1} \sigma_1 \sigma_2; \mathbb{C}) \longrightarrow \mathrm{AKh}(\sigma_1^{n-1} \sigma_2^{-1} \sigma_1 \sigma_2; \mathbb{C})[-1]\{-2\} \longrightarrow \mathrm{AKh}(S; \mathbb{C})\{-1\} \longrightarrow \mathrm{AKh}(\sigma_1^n \sigma_2^{-1} \sigma_1 \sigma_2; \mathbb{C})$$

Since $\mathrm{AKh}(S; \mathbb{C})$ is trivial in annular grading 3 this proves the second part of the result. For the first part, observe that decategorifying either of the above exact triangles we obtain

$$J(c(\sigma_1^n \sigma_2^{-1} \sigma_1 \sigma_2)) = q^{-1} J(S) - q^{-2} J(c(\sigma_1^{n-1} \sigma_2^{-1} \sigma_1 \sigma_2)).$$

The desired result follows by induction. □

Remark 3.18. One could perhaps give a complete computation of the annular Khovanov homology of the infinite family of clasp-closures using techniques of J. Wang [42]. The annular Jones polynomial was enough for our purposes, however, so we do not pursue this.

Let $r(\beta)$ denote the reverse of the braid word β written in terms of the standard Artin generators.

Lemma 3.19. *Suppose β_1 and β_2 are 3-braids such that $c(\beta_1)$ and $c(\beta_2)$ represent unknots. If $\mathrm{AKh}(c(\beta_1); \mathbb{C}) \cong \mathrm{AKh}(c(\beta_2); \mathbb{C})$ then $c(\beta_1) = c(\beta_2)$ or $c(\beta_1) = c(r(\beta_2))$.*

Proof. Lemmas 3.15, 3.16, and 3.17 determine the annular Khovanov homology of all of the clasp-closures up to mirroring. The annular Khovanov homology of their mirrors can be determined using formal properties of annular Khovanov homology. We can then see that no two clasp-closures of 3-braids representing unknots have the same annular Khovanov homology in annular grading 3 so the result follows. □

Proof of Theorem 3.13. Suppose that K is an annular link with $\mathrm{AKh}(K; \mathbb{Z}) \cong \mathrm{AKh}(c(\sigma^{-1}); \mathbb{Z})$. Since $\mathrm{AKh}(K; \mathbb{Z})$ is supported in odd quantum gradings it follows that K has an odd number of components. K is a clasp-closure of a 3-braid by Lemma 3.14. Consider the Batson–Seed link splitting sequence for $\mathrm{Kh}(K; \mathbb{C})$. Observe that $\mathrm{rank}(\mathrm{Kh}(K; \mathbb{C})) \geq 2^m$, where m is the number of components of K. Now observe that Lemma 3.2 implies that the rank of ∂_{-2}^* is at least 4 since the two V_{n-2} summands in the statement of the lemma are mapped nontrivially under ∂_{-2}^*, which we recall is part of the $\mathfrak{sl}_2(\mathbb{C})$ action on annular Khovanov homology. In turn it follows that $\operatorname{rank}(\mathrm{Kh}(K;\mathbb{C})) \leq 6$. Thus K has a single component.

We now show that K represents the unknot. An application of the universal coefficient theorem shows that $\mathrm{AKh}(K;\mathbb{Z}/2) \cong \mathrm{AKh}(c(\sigma^{-1});\mathbb{Z}/2)$. Consider the spectral sequence from $\mathrm{AKh}(K;\mathbb{Z}/2)$ to $\mathrm{Kh}(K;\mathbb{Z}/2)$. Lemma 3.5 implies that $\operatorname{rank}(\mathrm{Kh}(K;\mathbb{Z}/2)) \leq 6$. It follows that $\operatorname{rank}(\mathrm{Khr}(K;\mathbb{Q})) \leq \operatorname{rank}(\mathrm{Khr}(K;\mathbb{Z}/2)) \leq 3$, so that L is either a trefoil or an unknot by [29] and [3]. However, K cannot be a trefoil because $\mathrm{AKh}(K;\mathbb{C})$, and hence $\mathrm{Kh}(K;\mathbb{C})$, does not contain a summand in quantum grading ± 9.

It follows that K is a clasp-closure of one of Baldwin and Sivek's 3-braid types. By Lemma 3.19, if two such annular links have the same annular Khovanov homology then they differ only up to reversal. But of course, $c(\sigma_1^{-1})$ and $c(r(\sigma_1^{-1}))$ are isotopic, so the result follows. □

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References

[1] M. M. Asaeda, J. H. Przytycki, and A. S. Sikora, "Categorification of the Kauffman bracket skein module of I-bundles over surfaces", *Algebr. Geom. Topol.* **4** (2004), 1177–1210. MR

[2] J. A. Baldwin and J. E. Grigsby, "Categorified invariants and the braid group", *Proc. Amer. Math. Soc.* **143**:7 (2015), 2801–2814. MR

[3] J. A. Baldwin and S. Sivek, "Khovanov homology detects the trefoils", *Duke Math. J.* **171**:4 (2022), 885–956. MR

[4] J. A. Baldwin and S. Sivek, "Floer homology and non-fibered knot detection", *Forum Math. Pi* **13** (2025), art. id. e1, 65 pp. MR

[5] J. Baldwin and D. S. Vela-Vick, "A note on the knot Floer homology of fibered knots", *Algebr. Geom. Topol.* **18**:6 (2018), 3669–3690. MR

[6] J. A. Baldwin, A. S. Levine, and S. Sarkar, "Khovanov homology and knot Floer homology for pointed links", *J. Knot Theory Ramifications* **26**:2 (2017), art. id. 1740004, 49 pp. MR

[7] J. A. Baldwin, S. Sivek, and Y. Xie, "Khovanov homology detects the Hopf links", *Math. Res. Lett.* **26**:5 (2019), 1281–1290. MR

[8] J. A. Baldwin, Y. Hu, and S. Sivek, "Khovanov homology and the cinquefoil", *J. Eur. Math. Soc. (JEMS)* **27**:6 (2025), 2443–2465. MR

[9] J. Batson and C. Seed, "A link-splitting spectral sequence in Khovanov homology", *Duke Math. J.* **164**:5 (2015), 801–841. MR

[10] F. Binns and S. Dey, "Cable links, annuli and sutured Floer homology", *Math. Res. Lett.* **31**:5 (2024), 1315–1337. MR

[11] F. Binns and S. Dey, "Floer homology, clasp-braids and detection results", preprint, 2024. arXiv 2405.11224

[12] F. Binns and S. Dey, "Rank bounds in link Floer homology and detection results", *Quantum Topol.* **16**:4 (2025), 743–794. MR

[13] F. Binns and G. Martin, "Knot Floer homology, link Floer homology and link detection", *Algebr. Geom. Topol.* **24**:1 (2024), 159–181. MR

[14] J. S. Birman and T. E. Brendle, "Braids: a survey", pp. 19–103 in *Handbook of knot theory*, Elsevier, Amsterdam, 2005. MR

[15] J. S. Birman and W. W. Menasco, "Studying links via closed braids. III. Classifying links which are closed 3-braids", *Pacific J. Math.* **161**:1 (1993), 25–113. MR

[16] P. Dehornoy, I. Dynnikov, D. Rolfsen, and B. Wiest, *Ordering braids*, Mathematical Surveys and Monographs **148**, American Mathematical Society, Providence, RI, 2008. MR

[17] N. Dowlin, "A spectral sequence from Khovanov homology to knot Floer homology", *J. Amer. Math. Soc.* **37**:4 (2024), 951–1010. MR

[18] E. Farber, B. Reinoso, and L. Wang, "Fixed-point-free pseudo-Anosov homeomorphisms, knot Floer homology and the cinquefoil", *Geom. Topol.* **28**:9 (2024), 4337–4381. MR

[19] J. E. Grigsby and Y. Ni, "Sutured Khovanov homology distinguishes braids from other tangles", *Math. Res. Lett.* **21**:6 (2014), 1263–1275. MR

[20] J. E. Grigsby, A. M. Licata, and S. M. Wehrli, "Annular Khovanov homology and knotted Schur–Weyl representations", *Compos. Math.* **154**:3 (2018), 459–502. MR

[21] J. Hoste, "The first coefficient of the Conway polynomial", *Proc. Amer. Math. Soc.* **95**:2 (1985), 299–302. MR

[22] H. Hunt, H. Keese, A. Licata, and S. Morrison, "Computing annular Khovanov homology", 2015. arXiv 1505.04484

[23] A. Juhász, "Holomorphic discs and sutured manifolds", *Algebr. Geom. Topol.* **6** (2006), 1429–1457. MR

[24] A. Juhász, "Floer homology and surface decompositions", *Geom. Topol.* **12**:1 (2008), 299–350. MR

[25] A. Juhász, "The sutured Floer homology polytope", *Geom. Topol.* **14**:3 (2010), 1303–1354. MR

[26] M. Khovanov, "A categorification of the Jones polynomial", *Duke Math. J.* **101**:3 (2000), 359–426. MR

[27] "KLO: Knot-Like Objects", software, version 0.979, available at https://community.middlebury.edu/~mathanimations/klo.

[28] "The Knot Atlas", online resource, available at https://katlas.org/wiki/Main_Page.

[29] P. B. Kronheimer and T. S. Mrowka, "Khovanov homology is an unknot-detector", *Publ. Math. Inst. Hautes Études Sci.* 113 (2011), 97–208. MR

[30] E. S. Lee, "An endomorphism of the Khovanov invariant", *Adv. Math.* **197**:2 (2005), 554–586. MR

[31] C. Livingston and A. H. Moore, "LinkInfo: table of link invariants", 2024, available at http://linkinfo.math.indiana.edu.

[32] G. Martin, "Khovanov homology detects $T(2, 6)$", *Math. Res. Lett.* **29**:3 (2022), 835–849. MR

[33] K. Murasugi, *On closed 3-braids*, Memoirs of the American Mathematical Society **151**, American Mathematical Society, Providence, RI, 1974. MR

[34] Y. Ni, "Exceptional surgeries on hyperbolic fibered knots", preprint, 2020. arXiv 2007.11774

[35] P. Ozsváth and Z. Szabó, "Holomorphic disks and knot invariants", *Adv. Math.* **186**:1 (2004), 58–116. MR

[36] P. Ozsváth and Z. Szabó, "Holomorphic disks, link invariants and the multi-variable Alexander polynomial", *Algebr. Geom. Topol.* **8**:2 (2008), 615–692. MR

[37] P. Ozsváth and Z. Szabó, "Link Floer homology and the Thurston norm", *J. Amer. Math. Soc.* **21**:3 (2008), 671–709. MR

[38] O. Plamenevskaya, "Transverse knots and Khovanov homology", *Math. Res. Lett.* **13**:4 (2006), 571–586. MR

[39] J. Rasmussen, "Floer homology and knot complements", preprint, 2003. arXiv math/0306378

[40] A. N. Shumakovitch, "Torsion of Khovanov homology", *Fund. Math.* **225**:1 (2014), 343–364. MR

[41] M. Stoffregen and M. Zhang, "Localization in Khovanov homology", *Geom. Topol.* **28**:4 (2024), 1501–1585. MR

[42] J. Wang, "The cosmetic crossing conjecture for split links", *Geom. Topol.* **26**:7 (2022), 2941–3053. MR

[43] Y. Xie, "Instantons and annular Khovanov homology", *Adv. Math.* **388** (2021), art. id. 107864, 51 pp. MR

[44] Y. Xie and B. Zhang, "On links with Khovanov homology of small ranks", *Math. Res. Lett.* **29**:4 (2022), 1261–1277. MR

[45] Y. Xie and B. Zhang, "Instanton Floer homology for sutured manifolds with tangles", *J. Differential Geom.* **130**:3 (2025), 701–769. MR

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FLAT BRAID GROUPS, RIGHT-ANGLED ARTIN GROUPS, AND COMMENSURABILITY

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For every $n \geq 1$, the flat braid group FB_n is an analogue of the braid group B_n that can be described as the fundamental group of the configuration space $\{\{x_1, \dots, x_n\} \in \mathbb{R}^n/\mathrm{Sym}(n) \mid$ there exist at most two indices i, j such that $x_i = x_j\}$. Alternatively, FB_n can be described as the right-angled Coxeter group $C(P_{n-2}^{\mathrm{opp}})$, where P_{n-2}^{opp} denotes the opposite graph of the path P_{n-2} of length $n-2$. We prove that, for every $n = 7$ or ≥ 11, PFB_n is not virtually a right-angled Artin group, disproving a conjecture of Naik, Nanda, and Singh. In the opposite direction, we observe that FB_7 turns out to be commensurable to the right-angled Artin group $A(P_4)$.

1. Introduction

Recall that, given a graph Γ, the corresponding *right-angled Artin group* is

$$A(\Gamma) := \langle u \text{ vertex of } \Gamma \mid [u, v] = 1 \text{ if } u \text{ and } v \text{ are adjacent in } \Gamma \rangle,$$

and that the corresponding *right-angled Coxeter group* is

$$C(\Gamma) := \langle u \text{ vertex of } \Gamma \mid u^2 = 1 \text{ for every } u,\ [u, v] = 1 \text{ if } u \text{ and } v \text{ are adjacent in } \Gamma \rangle.$$

It is well-known that right-angled Artin and Coxeter groups are tightly connected. Most notably, every right-angled Artin group is isomorphic to a finite-index subgroup of some right-angled Coxeter group [Davis and Januszkiewicz 2000]. Conversely, despite the fact that every right-angled Coxeter group can be virtually described

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as a subgroup of some right-angled Artin group, right-angled Coxeter groups may not be commensurable to right-angled Artin groups. (Recall that two groups are (*abstractly*) *commensurable* whenever they contain isomorphic finite-index subgroups.) A simple example is given by $C(C_n)$, where C_n is a cycle of length $n \geq 5$. Indeed, $C(C_n)$ can be described as the reflection group associated to a right-angled n-gon in the hyperbolic plane $\mathbb{H}^2$, so $C(C_n)$ is virtually the fundamental group of a closed surface of genus ≥ 2. However, a right-angled Artin group that does not admit $\mathbb{Z}^2$ as a subgroup is automatically free, so no right-angled Artin group can be commensurable to $C(C_n)$. Consequently, one can think of the family of right-angled Coxeter groups as being strictly larger than the family of right-angled Artin groups. A natural problem, then, is to understand when a group from the bigger family belongs to the smaller family. More precisely:

Question 1.1. Given a graph Γ, when is the right-angled Coxeter group $C(\Gamma)$ commensurable to a right-angled Artin group?

Recall that two groups are (*abstractly*) *commensurable* whenever they contain two isomorphic finite-index subgroups.

Question 1.1, and its analogue where "commensurable" is replaced with "quasi-isometric", is well-known in geometric group theory. Nevertheless, it is poorly understood. Some invariants are available in order to distinguish some right-angled Artin and Coxeter groups, such as divergence and thickness [Behrstock and Charney 2012; Dani and Thomas 2015; Levcovitz 2022] or Morse boundaries [Charney and Sultan 2015; Cordes and Hume 2017; Behrstock 2019] and Morse subgroups [Genevois 2022b]. In the other directions, a few constructions are known in order to produce (finite-index) subgroups in right-angled Coxeter groups that are right-angled Artin groups. See for instance [Januszkiewicz and Świątkowski 2001], and most notably [Dani and Levcovitz 2024] (based on [LaForge 2017] and further studied in [Cashen and Edletzberger 2024]). As a concrete but very specific application, it is determined in [Dani and Levcovitz 2024] precisely when some two-dimensional one-ended right-angled Coxeter groups defined by planar graphs are commensurable to right-angled Artin groups. (These examples are not representative of the general case because they are based on the large-scale geometry of graph manifolds [Behrstock and Neumann 2008; Nguyen and Tran 2019].) Despite all these results available in the literature, no global picture seems to emerge and an answer to Question 1.1 seems currently to be out of reach in full generality.

In this article, we focus on a specific family of right-angled Coxeter groups, known as *flat braid groups*. For every $n \geq 1$, the flat braid group FB_n on n strands is the fundamental group of the configuration space

$$\big\{\{x_1, \dots, x_n\} \in \mathbb{R}^n/\mathrm{Sym}(n) \mid \text{there exist at most two indices } i,\, j \text{ such that } x_i = x_j\big\}.$$

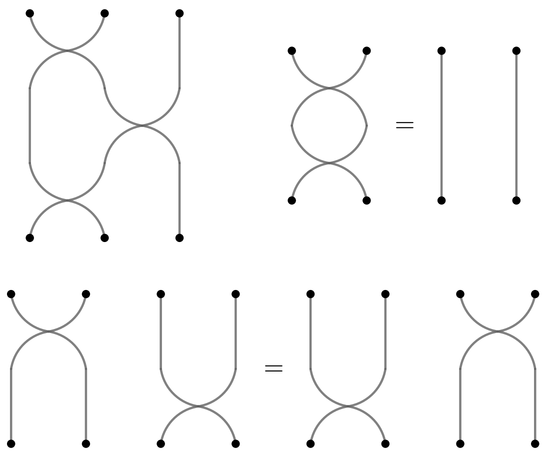


Figure 1. A flat braid from FB_3 (namely, $\sigma_1\sigma_2\sigma_1$), and the two typical relations between flat braids (namely, $\sigma_1^2 = 1$ and $\sigma_1\sigma_3 = \sigma_3\sigma_1$).

One can think of an element of FB_n as a configuration of n arcs in the infinite strip $\mathbb{R} \times [0, 1]$ connecting n marked points on each of the parallel lines $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{0\}$ such that each arc is monotonic and no three arcs intersect at a common point. Two such configurations are considered as equivalent if one can be deformed into the other by a homotopy of such configurations in $\mathbb{R} \times [0, 1]$ keeping the end points of the arcs fixed. See Figure 1. From this description, one can define a natural morphism $\mathrm{FB}_n \to \mathrm{Sym}(n)$, encoding how a flat braid permutes the n strands. The kernel of this morphism is the *pure flat braid group* PFB_n.

If σ_i denotes the element of FB_n that twists the i-th and $(i+1)$-st strands, then it is clear that FB_n is generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$. Moreover, FB_n admits

$$\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i^2 = 1 \text{ for every } 1 \leq i \leq n-1,\ [\sigma_i, \sigma_j] = 1 \text{ whenever } |i-j| \geq 2 \rangle$$

as a presentation. Thus, FB_n coincides with the right-angled Coxeter group $C(P_{n-2}^{\mathrm{opp}})$ where P_{n-2}^{opp} denotes the opposite graph of the path P_{n-2} of length $n-2$.

Flat braid groups have been introduced in the literature under various names, often independently. For instance, one can meet *Grothendieck cartographic groups* in [Shabat and Voevodsky 1990; Voevodsky 1990]; *traid groups* in [Harshman and Knapp 2020] for applications to physics; *twin groups* in [Khovanov 1996; 1997] in connection with *doodles*; *pseudobraids groups* in [Genevois 2020] when studying *diagram groups* (see also [Farley 2021]); *planar braid groups* in [Mostovoy and Roque-Márquez 2020; Mostovoy 2020]. For notational convenience, we call our groups *flat braid groups* in reference to flat braids from [Merkov 1999].

Several articles in the literature investigate the algebraic structure of (pure) flat braid groups. See for instance [Dey and Gongopadhyay 2019] about presentations and ranks of the commutator subgroups of flat braid groups; [Bardakov et al. 2019] about similar results for pure flat braid groups; [Mostovoy and Roque-Márquez 2020] about the algebraic structures of pure flat braid groups on ≤ 6 strands; [Naik et al. 2020a] about the conjugacy problem and the structure of automorphism groups of flat braid groups; [Naik et al. 2020b] about Property R_∞ of flat braid groups; and [Bellingeri et al. 2024] about a connection with the so-called *cactus groups* (see also [Genevois 2022a]).

Motivated by the fact that pure virtual twin groups are right-angled Artin groups [Naik et al. 2023] and by the structure of pure flat braid groups on ≤ 6 strands [Mostovoy and Roque-Márquez 2020], it has been conjectured that:

Conjecture 1.2 [Naik et al. 2024]. For every $n \geq 3$, PFB_n is a right-angled Artin group.

In this article, we disprove this conjecture by showing that most pure flat braid groups are not right-angled Artin groups, even up to a finite index.

Theorem 1.3. *For every $n = 7$ or ≥ 11, the pure flat braid group PFB_n is not virtually a right-angled Artin group.*

Despite the fact that Conjecture 1.2 is false, we stress that the following (vague) question is still interesting: to which extent does a (pure) flat braid group look like a right-angled Artin group? For instance, Theorem 1.3 shows that FB_n does not contain a right-angled Artin group of finite index in PFB_n, but is there such a subgroup elsewhere in FB_n? More precisely:

Question 1.4. Is FB_n virtually a right-angled Artin group?

For instance, a natural candidate to check would be the commutator subgroup of FB_n, which is torsion-free. In case of a negative answer to Question 1.4, it would be natural to ask whether FB_n contains a finite-index subgroup that, despite not being a right-angled Artin group, turns out to be isomorphic to a finite-index subgroup of some right-angled Artin groups. In other words:

Question 1.5. Is FB_n commensurable (or at least quasi-isometric) to a right-angled Artin group?

Since, as said, flat braid groups are right-angled Coxeter groups, Question 1.5 is a particular case of Question 1.1. Perhaps surprisingly, in view on Theorem 1.3, we answer positively Question 1.5 for $n = 7$.

Theorem 1.6. *The flat braid group FB_7 is commensurable to the right-angled Artin group $A(P_4)$.*

In order to prove Theorem 1.6, we exploit the observation that FB_7 and $A(P_4)$ are both (virtually) fundamental groups of compact flip manifolds. It is known by [Behrstock and Neumann 2008] that right-angled Artin groups defined by finite trees can be described as fundamental groups of such 3-manifolds. For FB_7, we start by proving that FB_7 contains an index-8 normal subgroup isomorphic to an index-2 subgroup of $\langle a, t \mid [a, tat^{-1}] = 1\rangle$ (see Section 6.2). This group can be easily described as the fundamental group of a compact flip manifold M_2. Given another such 3-manifold M_1 whose fundamental group is isomorphic to $A(P_4)$, we construct a common finite-sheeted cover $M_0 \to M_1, M_2$ (see Section 6.3), which allows us to deduce Theorem 1.6.

About the proof of Theorem 1.3. The first step is to prove the theorem for the flat braid group on seven strands, namely:

Theorem 1.7. *The flat braid group* PFB_7 *is not virtually a right-angled Artin group.*

This is the smallest number of strands for which the statement holds (see [Naik et al. 2020a]). Theorem 1.7 is proved as follows. First, we observe that, since FB_7 does not contain a subgroup isomorphic to $\mathbb{Z}^3$, to $\mathbb{F}_2 \times \mathbb{F}_2$, or to the fundamental group of a closed surface of genus ≥ 2 (Lemma 4.2), every right-angled Artin group that appears as a subgroup of FB_7 is defined by a forest (Corollary 4.4). As a consequence, it suffices to show that PFB_7 does not contain as a finite-index subgroup a right-angled Artin group $A(T)$ defined by a finite tree T. For this, we define the *thick subgroup* $\mathrm{Thick}(G)$ of a group G as the subgroup generated by the centralisers of all the elements whose centralisers are not virtually abelian. Then, by showing that $\mathrm{Thick}(A(T)) = A(T)$ (Lemma 4.6) but that $\mathrm{Thick}(\mathrm{PFB}_7)$ has infinite index in PFB_7 (Lemma 4.7), we conclude that $A(T)$ cannot be a finite-index subgroup of PFB_7.

Once Theorem 1.7 is proved, it is not so difficult to deduce that PFB_n is not a right-angled Artin group for $n \geq 11$. The trick is that PFB_n contains a natural copy of $\mathrm{PFB}_7 \times \mathrm{PFB}_{n-7}$, which turns out to be a maximal product subgroup. But, in a right-angled Artin group, maximal product subgroups are well-understood; their factors, in particular, are also right-angled Artin groups. Since this product decomposition is unique (Corollary 3.6), it follows that, if PFB_n were a right-angled Artin group, then PBF_7 would be a right-angled Artin group as well, contradicting Theorem 1.7.

To prove that PFB_n is not virtually a right-angled Artin group, the strategy is basically the same, but some technicality is required. In Section 3, we introduce the notion of *IMC generating sets*, which is of independent interest, and we prove that, under the assumption that our groups admit IMC generating sets, factors of finite-index subgroups in products must be contained in factors of the whole product (Lemma 3.5).

2. Preliminaries on graph products

Let Γ be a graph and $\mathcal{G} = \{G_u \mid u \in \Gamma\}$ a collection of groups indexed by the vertices of Γ. The *graph product* $\Gamma\mathcal{G}$ is

$$\langle G_u \ (u \in \Gamma) \mid [G_u, G_v] = 1 \ (\{u, v\} \in E(\Gamma)) \rangle$$

where $E(\Gamma)$ denotes the edge-set of Γ and where $[G_u, G_v] = 1$ is a shorthand for $[g, h] = 1$ for all $g \in G_u$, $h \in G_v$. The groups of $\mathcal{G}$ are referred to as the *vertex groups*.

Graph products of groups will allow us to state and prove results simultaneously about right-angled Artin groups (i.e., when vertex groups are infinite cyclic) and about right-angled Coxeter groups (i.e., when vertex groups are cyclic of order two).

Convention. We always assume that the groups in $\mathcal{G}$ are nontrivial. Notice that it is not a restrictive assumption, since a graph product with some trivial factors can be described as a graph product over a smaller graph all of whose factors are nontrivial.

A *word* in $\Gamma\mathcal{G}$ is a product $g_1 \cdots g_n$ where $n \geq 0$ and where, for every $1 \leq i \leq n$, $g_i \in G$ for some $G \in \mathcal{G}$; the g_i's are the *syllables* of the word, and n is the *length* of the word. Clearly, each of the following operations on a word does not modify the element of $\Gamma\mathcal{G}$ it represents:

Cancellation: delete the syllable g_i if $g_i = 1$.

Amalgamation: if $g_i, g_{i+1} \in G$ for some $G \in \mathcal{G}$, replace the two syllables g_i and g_{i+1} by the single syllable $g_i g_{i+1} \in G$.

Shuffling: if g_i and g_{i+1} belong to two adjacent vertex groups, switch them.

A word is *graphically reduced* if its length cannot be shortened by applying these elementary moves. Every element of $\Gamma\mathcal{G}$ can be represented by a graphically reduced word, and this word is unique up to the shuffling operation. This allows us to define the *support* of an element as the subgraph of Γ induced by the vertices labelling the syllables of a graphically reduced word representing our element. Similarly, a word is *graphically cyclically reduced* if all its cyclic permutations are graphically reduced. An element of $\Gamma\mathcal{G}$ that can be represented by a graphically cyclically reduced word is *graphically cyclically reduced*. Every element is conjugate to a graphically cyclically reduced word. One can define the *essential support* of an element as the support of a graphically cyclically reduced element that is conjugate to it. For more information on graphically reduced words, we refer to [Green 1990] (see also [Hsu and Wise 1999; Genevois 2019]).

Parabolic subgroups. Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. Given a subgraph $\Lambda \subset \Gamma$, we denote by $\langle \Lambda \rangle$ the subgroup of $\Gamma\mathcal{G}$ generated by the vertex groups labelling the vertices of Λ. A subgroup of $\Gamma\mathcal{G}$ of the form $g\langle \Phi \rangle g^{-1}$ for some element $g \in \Gamma\mathcal{G}$ and subgraph $\Phi \subset \Gamma$ is a *parabolic subgroup*. Here, we record a few basic results about parabolic subgroups. They will be useful in order to prove some preliminary results later in this section (but they will not be used in the next sections).

Lemma 2.1. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. The graph product $\Gamma\mathcal{G}$ is virtually cyclic if and only if one of the following conditions holds*:

- Γ *is a complete graph all of whose vertices are labelled by finite groups.*
- $\Gamma = \Xi * \{u\}$ *where Ξ is a complete graph all of whose vertices are labelled by finite groups and where u is a vertex labelled by a virtually-$\mathbb{Z}$ group.*
- $\Gamma = \Xi * \{u, v\}$ *where Ξ is a complete graph all of whose vertices are labelled by finite groups and where u, v are two nonadjacent vertices both labelled by $\mathbb{Z}_2$.*

Recall that, given two graphs Φ and Ψ, their *join* $\Phi * \Psi$ is the graph obtained from $\Phi \sqcup \Psi$ by connecting with an edge every vertex of Φ with every vertex of Ψ.

Proof of Lemma 2.1. If Γ is complete, then $\Gamma\mathcal{G}$ is the product of its vertex groups. In this case, $\Gamma\mathcal{G}$ is virtually cyclic if and only if either all its vertex groups are finite or one vertex group is virtually-$\mathbb{Z}$ and all its other vertex groups are finite. This corresponds to the first and second items of our lemma.

Now, assume that Γ is not complete. Fix two nonadjacent vertices $u, v \in \Gamma$. Since the subgroup $\langle u, v \rangle$ of $\Gamma\mathcal{G}$ decomposes as the free product $\langle u \rangle * \langle v \rangle$, the only possibility for $\Gamma\mathcal{G}$ to be virtually cyclic is that $\langle u \rangle$ and $\langle v \rangle$ are both cyclic of order two. Next, if Γ contains a vertex w that is not adjacent to both u and v, then $\langle u, v, w \rangle$ decomposes as one of the following free products: $\langle u \rangle * \langle v \rangle * \langle w \rangle$, $(\langle u \rangle \times \langle w \rangle) * \langle v \rangle$, or $\langle u \rangle * (\langle v \rangle \times \langle w \rangle)$. In any case, $\Gamma\mathcal{G}$ cannot be virtually cyclic. Therefore, Γ decomposes as a join $\Xi * \{u, v\}$. Algebraically, this implies that $\Gamma\mathcal{G}$ decomposes as the product $\langle \Xi \rangle \times \langle u, v \rangle \simeq \langle \Xi \rangle * \mathbb{D}_\infty$. Then, $\Gamma\mathcal{G}$ is virtually cyclic if and only if $\langle \Xi \rangle$ is finite, which amounts to saying that Ξ is a complete graph all of whose vertices are labelled by finite groups. This corresponds to the third item of our lemma. □

Lemma 2.2. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. For every subgraph $\Lambda \subset \Gamma$, $\langle \Lambda \rangle$ has finite index in $\Gamma\mathcal{G}$ if and only if $\Gamma = \Lambda * \Xi$, where Ξ is a complete graph all of whose vertices are labelled by finite groups.*

Proof. First, assume that there exist two nonadjacent vertices $u \in \Gamma \setminus \Lambda$ and $v \in \Lambda$. Fix two nontrivial elements $a \in \langle u \rangle$ and $b \in \langle v \rangle$. If there exist distinct integers $p, q \geq 1$ such that $(ba)^p$ and $(ba)^q$ belong to the same $\langle \Lambda \rangle$-coset, then we find an integer $\ell \geq 1$ such that $(ba)^\ell \in \langle \Lambda \rangle$. But this is impossible since, as a graphically reduced word, $(ba)^\ell$ cannot represent an element of $\langle \Lambda \rangle$. Consequently, $\langle \Lambda \rangle$ must have infinite index in $\Gamma\mathcal{G}$.

From now on, assume that every vertex in $\Gamma \setminus \Lambda$ is adjacent to every vertex in Λ; i.e., Γ decomposes as a join $\Lambda * \Xi$. Algebraically, $\Gamma\mathcal{G}$ decomposes as the product $\langle \Lambda \rangle * \langle \Xi \rangle$. Then, $\langle \Xi \rangle$ has finite index in $\Gamma\mathcal{G}$ if and only if $\langle \Xi \rangle$ is finite, which amounts to saying that Ξ is a complete graph all of whose vertices are labelled by finite groups. □

Lemma 2.3. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. For all subgraphs $\Phi, \Psi \subset \Gamma$ and element $a \in \Gamma\mathcal{G}$, the inclusion $a\langle \Phi \rangle a^{-1} \leq \langle \Psi \rangle$ holds if and only if $\Phi \subset \Psi$ and $a \in \langle \Psi \rangle \cdot \langle \mathrm{star}(\Phi) \rangle$.*

Recall that, given a subgraph Λ, its *link* $\mathrm{link}(\Lambda)$ refers to the subgraph induced by the vertices that are adjacent to all the vertices in Λ, and its *star* $\mathrm{star}(\Lambda)$ refers to the subgraph induced by $\mathrm{link}(\Lambda) \cup \Lambda$.

A proof of this lemma can be found in [Genevois and Martin 2019, Lemma 3.17]. Here is an immediate consequence:

Corollary 2.4. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. For all subgraph $\Phi \subset \Gamma$ and element $a \in \Gamma\mathcal{G}$, if $a\langle \Phi \rangle a^{-1} \leq \langle \Phi \rangle$ then $a\langle \Phi \rangle a^{-1} = \langle \Phi \rangle$.*

Join subgroups. Parabolic subgroups given by join subgraphs play a central role in the study of graph products of groups. For us, they will be fundamental in the proof of Theorem 5.1, allowing us to reduced a problem about flat braid groups on arbitrarily many strands to a problem about flat braid groups on seven strands. As explained by our next lemma, Join subgroups can be used in order to characterise maximal product subgroups in graph products. Here, we refer to a *maximal product subgroup* of a given group as a maximal member of the collection of the subgroups that decompose as products of two nontrivial groups.

Lemma 2.5. *Let Γ be a finite graph and $\mathcal{G}$ a collection of groups indexed by Γ. A subgroup of $\Gamma\mathcal{G}$ is a maximal product subgroup if and only if either it is conjugate to $\langle \Lambda \rangle$ for some maximal join $\Lambda \subset \Gamma$ or it is a maximal product subgroup in a conjugate of a vertex group given by an isolated vertex of Γ.*

A proof of this lemma can be found in [Genevois 2024, Proposition 2.8]. It is worth noticing that the maximality given by the previous lemma behaves nicely with respect to commensurability:

Lemma 2.6. *Let Γ be a finite graph and $\mathcal{G}$ a collection of groups indexed by Γ. A subgroup of $\Gamma\mathcal{G}$ commensurable to a product of two infinite groups is contained in a conjugate of a vertex group or in a maximal product subgroup.*

Proof. Let $H \leq \Gamma\mathcal{G}$ be a subgroup that contains a finite-index subgroup $\dot{H}$ in common with a product $A \times B$ of two infinite groups. Given an $h \in H$, there exists some $p \geq 1$ such that $h^p \in \dot{H}$. Thinking of h^p as an element of $A \times B$, it can be written as (a, b) for some $a \in A$ and $b \in B$. Fix a $q \geq 1$ such that a^q and b^q either are trivial or have infinite order. Then the centraliser of $h^{pq} = (a^q, b^q)$ in $A \times B$ is $A \times B$ if $a^q = b^q = 1$; contains $A \times \langle b \rangle \simeq A \times \mathbb{Z}$ if $a^q = 1$ and if b has infinite order; contains $\langle a \rangle \times B \simeq \mathbb{Z} \times B$ if a has infinite order and if $b^q = 1$; contains $\langle a \rangle \times \langle b \rangle \simeq \mathbb{Z}^2$ if a and b both have infinite order. In any case, the centraliser of h^{pq} in $A \times B$, and a fortiori in H is not virtually cyclic. Thus, we have proved that every element of H has a nontrivial power whose centraliser is not virtually cyclic.

Now, assume that H is not contained in a conjugate of a vertex group nor in a maximal product subgroup. We claim that H must contain an element all of whose nontrivial powers have a virtually cyclic centraliser. As a consequence of the previous observation, this will conclude the proof of our lemma. Let $g\langle\Lambda\rangle g^{-1}$ denote the smallest parabolic subgroup containing H. (Such a subgroup exists according to [Antolín and Minasyan 2015].) According to [Minasyan and Osin 2015, Corollary 6.20], there exists a tree on which $g\langle\Lambda\rangle g^{-1}$ acts such that H contains a WPD element $h \in H$. Every nontrivial power of such an element must have a virtually cyclic centralisers in $g\langle\Lambda\rangle g^{-1}$. But, as a consequence of Proposition 2.8 below, the centraliser of an element of H in $\Gamma\mathcal{G}$ decomposes as the product of the centraliser of the element in $g\langle\Lambda\rangle g^{-1}$ with $g\langle\text{link}(\Lambda)\rangle g^{-1}$. Since H cannot be contained in a join subgroup, link$(\Lambda)\rangle$ must be empty. Consequently, the centraliser of a nontrivial power of h in $\Gamma\mathcal{G}$, and a fortiori in H, is virtually cyclic. □

Finally, let us observe no two maximal product subgroups in graph products can be commensurable:

Lemma 2.7. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. Let $\Phi, \Psi \subset \Gamma$ be two maximal joins and let $g, h \in \Gamma\mathcal{G}$. If $g\langle\Phi\rangle g^{-1}$ has a finite-index subgroup contained in $h\langle\Psi\rangle h^{-1}$, then $g\langle\Phi\rangle g^{-1} = h\langle\Psi\rangle h^{-1}$.*

Proof. According to [Antolín and Minasyan 2015] (see also [Genevois 2022c]), an intersection of two parabolic subgroups is again a parabolic subgroup, so there exist $k \in \Gamma\mathcal{G}$ and $\Lambda \subset \Gamma$ such that

$$g\langle\Phi\rangle g^{-1} \cap h\langle\Psi\rangle h^{-1} = k\langle\Lambda\rangle k^{-1}.$$

Since $g\langle\Phi\rangle g^{-1}$ contains a finite-index subgroup that is also contained in $h\langle\Psi\rangle h^{-1}$, necessarily $k\langle\Lambda\rangle k^{-1}$ has finite index in $g\langle\Phi\rangle g^{-1}$. Notice that, as a consequence of Lemma 2.3, $\Lambda \subset \Phi$ and $g^{-1}k$ belongs to $\langle\text{star}(\Phi)\rangle$. This implies that $\langle\Lambda\rangle$ is conjugate in $\langle\Phi\rangle$ (by an element of $\langle\Phi\rangle$) to a finite-index subgroup. Since $\langle\Lambda\rangle$ then must have finite index in $\langle\Phi\rangle$, we deduce from Lemma 2.2 that Φ decomposes as a join with Λ as a factor. But Φ is by assumption a maximal join in Γ, so we must have $\Phi = \Lambda$. So far, we have proved that

$$g\langle\Phi\rangle g^{-1} \cap h\langle\Psi\rangle h^{-1} = k\langle\Phi\rangle k^{-1}.$$

From the inclusion $k\langle\Phi\rangle k^{-1} \leq g\langle\Phi\rangle g^{-1}$, we deduce from Corollary 2.4 that $k\langle\Phi\rangle k^{-1} = g\langle\Phi\rangle g^{-1}$. Then, the centred equality above amounts to saying that $g\langle\Phi\rangle g^{-1} \leq h\langle\Psi\rangle h^{-1}$. We know from Lemma 2.3 that $\Phi \subset \Psi$. But Ψ is a maximal join in Γ, so we must have $\Phi = \Psi$. Then, we conclude from Corollary 2.4 that $g\langle\Phi\rangle g^{-1} = h\langle\Psi\rangle h^{-1}$, as desired. □

Centralisers in graph products. Below, we record the structure of centralisers in graph products of groups, from which we will extract some information about stable centralisers (Definition 2.9) and virtual centres (Definition 2.12), two central ingredients in the rigidity later obtained with Lemma 3.5.

Proposition 2.8 [Barkauskas 2007]. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. Fix a graphically cyclically reduced element $g \in \Gamma\mathcal{G}$ and decompose its support as a join*

$$\mathrm{supp}(g) = \Phi_1 * \cdots * \Phi_r * \Psi_1 * \cdots * \Psi_s$$

such that Φ_i is reduced to a single vertex for every $1 \leq i \leq r$ and such that Ψ_i is an irreducible subgraph with at least two vertices for every $1 \leq i \leq s$. Write g as a graphically reduced word $a_1 \cdots a_r \cdot b_1 \cdots b_s$ such that $\mathrm{supp}(a_i) = \Phi_i$ for every $1 \leq i \leq r$ and $\mathrm{supp}(b_i) = \Psi_i$ for every $1 \leq i \leq s$. Then the centraliser of g in $\Gamma\mathcal{G}$ is

$$C_{\Gamma\mathcal{G}}(g) = C_{\langle\Phi_1\rangle}(a_1) \times \cdots \times C_{\langle\Phi_r\rangle}(a_r) \times \langle h_1\rangle \times \cdots \times \langle h_s\rangle \times \big\langle \mathrm{link}(\mathrm{supp}(g))\big\rangle,$$

where each h_i is a primitive element of $\langle\Psi_i\rangle$ such that $b_i \in \langle h_i\rangle$.

As a first application of this description of centralisers, we observe that, in many graph products, the centraliser of a nontrivial power of an element is pretty much the same as the centraliser of the element itself. More formally:

Definition 2.9. A group G has *stable centralisers* if, for all $g \in G$ and $k \geq 1$, $C(g)$ equals $C(g^k)$. The group has *almost stable centralisers* if, for all $g \in G$ and $k \geq 1$, $C(g)$ has finite index in $C(g^k)$.

As an easy application of Proposition 2.8:

Lemma 2.10. *Let Γ be a graph and $\mathcal{G}$ a collection of torsion-free groups indexed by Γ. If the groups in $\mathcal{G}$ have (almost) stable centralisers, then $\Gamma\mathcal{G}$ has (almost) stable centralisers.*

Proof. Let $g \in \Gamma\mathcal{G}$ be an element. Up to conjugating g, we assume for convenience that it is graphically cyclically reduced. Decompose supp(g) as a join $\Phi_1 * \cdots * \Phi_r * \Psi_1 * \cdots * \Psi_s$ and write g as a product $a_1 \cdots a_r \cdot b_1 \cdots b_s$ as in Proposition 2.8. So

$$C_{\Gamma\mathcal{G}}(g) = C_{\langle \Phi_1 \rangle}(a_1) \times \cdots \times C_{\langle \Phi_r \rangle}(a_r) \times \langle h_1 \rangle \times \cdots \times \langle h_s \rangle \times \big\langle \text{link}(\text{supp}(g)) \big\rangle.$$

Given a $k \geq 1$, because vertex groups are torsion-free, Proposition 2.8 applies to the decomposition $g^k = a_1^k \cdots a_r^k \cdot b_1^k \cdots b_s^k$ and shows that

$$C_{\Gamma\mathcal{G}}(g^k) = C_{\langle \Phi_1 \rangle}(a_1^k) \times \cdots \times C_{\langle \Phi_r \rangle}(a_r^k) \times \langle h_1 \rangle \times \cdots \times \langle h_s \rangle \times \big\langle \text{link}(\text{supp}(g)) \big\rangle.$$

If vertex groups have stable centralisers (resp. almost stable centralisers), then each $C_{\langle \Phi_i \rangle}(a_i)$ agrees with (resp. has finite index in) $C_{\langle \Phi_i \rangle}(a_i^k)$. This implies that the centraliser of g^k in $\Gamma\mathcal{G}$ agrees with (resp. has finite index in) the centraliser of g in $\Gamma\mathcal{G}$. □

Corollary 2.11. *Right-angled Artin groups, as well as their subgroups, have stable centralisers.*

There exist also many graph products that do not have almost stable centralisers. This is the case, for instance, of the product $\mathbb{D}_\infty \times \mathbb{D}_\infty$ of two infinite dihedral groups. Indeed, given an infinite-order element $a \in \mathbb{D}_\infty$ and an element of order two $b \in \mathbb{D}_\infty$, the centraliser of (a, b) is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$ but the centraliser of its square $(a^2, 1)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Virtual centres. Motivating by the observation that the centres of a group and of its finite-index subgroups may be quite different, we introduce the notion of *virtual centre*. It will be a central ingredient in the rigidity later obtained with Lemma 3.5.

Definition 2.12. The *virtual centre* VZ(G) of a group G is the set of the elements that centralise some finite-index subgroups of G.

Lemma 2.13. *The virtual centre of a group is a normal subgroup.*

Proof. Let G be a group and $a, b \in G$ two elements. We make two observations:

- If a (resp. b) commutes with all the elements of a finite-index subgroup $H \leq G$ (resp. $K \leq G$), then ab^{-1} commutes with all the elements of the finite-index subgroup $H \cap K$. Hence $ab^{-1} \in \text{VZ}(G)$.
- If a commutes with all the elements of a finite-index subgroup $H \leq G$, then bab^{-1} commutes with all the elements of the finite-index subgroup bHb^{-1}. Hence $bab^{-1} \in \text{VZ}(G)$.

We conclude that VZ(G) is indeed a normal subgroup of G. □

For future use, let us describe virtual centres of graph products of groups:

Lemma 2.14. *Let* Γ *be a finite graph and* $\mathcal{G}$ *a collection of groups indexed by* Γ. *Decompose* Γ *as a join*

$$\Gamma = \{u_1\} * \cdots * \{u_r\} * \{a_1, b_1\} * \cdots * \{a_s, b_s\} * \Lambda_1 * \cdots * \Lambda_n$$

where $u_1, \ldots, u_r \in \Gamma$ *are single vertices, where each* $a_i, b_i \in \Gamma$ *are two nonadjacent vertices labelled by* $\mathbb{Z}_2$*, and where each* Λ_i *is an irreducible subgraph containing at least two vertices not both labelled by* $\mathbb{Z}_2$*. Then*

$$\mathrm{VZ}(\Gamma\mathcal{G}) = \mathrm{VZ}(\langle u_1 \rangle) \times \cdots \times \mathrm{VZ}(\langle u_r \rangle) \times \langle a_1 b_1 \rangle \times \cdots \times \langle a_s b_s \rangle.$$

Proof. Let $g \in \Gamma\mathcal{G}$ be an element that belongs to the virtual centre of $\Gamma\mathcal{G}$. This amounts to saying that the centraliser of g has finite index in $\Gamma\mathcal{G}$. Our goal is to prove that g belongs to the subgroup

$$V := \mathrm{VZ}(\langle u_1 \rangle) \times \cdots \times \mathrm{VZ}(\langle u_r \rangle) \times \langle a_1 b_1 \rangle \times \cdots \times \langle a_s b_s \rangle.$$

Because V clearly lies in the virtual centre of $\Gamma\mathcal{G}$, this will be sufficient to conclude the proof of our lemma. Notice also that V is a normal subgroup, so, up to conjugating g, we assume without loss of generality that g is graphically cyclically reduced. Following Proposition 2.8, decompose the support of g as a join

$$\mathrm{supp}(g) = \Phi_1 * \cdots * \Phi_p * \Psi_1 * \cdots * \Psi_q$$

such that Φ_i is reduced to a single vertex for every $1 \leq i \leq p$ and such that Ψ_i is an irreducible subgraph with at least two vertices for every $1 \leq i \leq q$. Write g as a graphically reduced word $a_1 \cdots a_p \cdot b_1 \cdots b_q$ such that $\mathrm{supp}(a_i) = \Phi_i$ for every $1 \leq i \leq p$ and $\mathrm{supp}(b_i) = \Psi_i$ for every $1 \leq i \leq q$. According to Proposition 2.8, the centraliser of g in $\Gamma\mathcal{G}$ is

$$C_{\Gamma\mathcal{G}}(g) = C_{\langle \Phi_1 \rangle}(a_1) \times \cdots \times C_{\langle \Phi_r \rangle}(a_p) \times \langle h_1 \rangle \times \cdots \times \langle h_q \rangle \times \big\langle \mathrm{link}(\mathrm{supp}(g)) \big\rangle$$

where each h_i is a primitive element of $\langle \Psi_i \rangle$ such that $b_i \in \langle h_i \rangle$. Since this centraliser has finite index in $\Gamma\mathcal{G}$, the following assertions hold:

- for every $1 \leq i \leq p$, $C_{\langle \Phi_i \rangle}(a_i)$ has finite-index in $\langle \Phi_i \rangle$, i.e., $a_i \in \mathrm{VZ}(\langle \Phi_i \rangle)$;
- $\langle h_i \rangle$ has finite index in $\langle \Psi_i \rangle$ for every $1 \leq i \leq q$;
- $\langle \Phi_1 \cup \cdots \cup \Phi_p \cup \Psi_1 \cup \cdots \cup \Psi_q \cup \mathrm{link}(\mathrm{supp}(g)) \rangle$ has finite index in $\Gamma\mathcal{G}$.

The second item implies that each $\langle \Psi_i \rangle$ is virtually cyclic. According to Lemma 2.1, the only possibility is that Ψ_i is given by two nonadjacent vertices both labelled by $\mathbb{Z}_2$. And the third item implies, according to Lemma 2.2, that

$$\Gamma = \Phi_1 * \cdots * \Phi_r * \Psi_1 * \cdots * \Psi_s * \mathrm{link}(\mathrm{supp}(g)).$$

It follows that $p = r$ and $q = s$; that $\mathrm{link}(\mathrm{supp}(g)) = \Lambda_1 * \cdots * \Lambda_n$; and that, up to reordering our subgraphs, $\Phi_i = \{u_i\}$ for every $1 \leq i \leq r$ and $\Psi_i = \{a_i, b_i\}$ for every

$1 \leq i \leq s$. Notice that, for every $1 \leq i \leq s$, b_i is an infinite-order element of the infinite dihedral group $\langle \Psi_i \rangle = \langle a_i, b_i \rangle$, so we can take $h_i = a_i b_i$. We conclude that g indeed belongs to V, as desired. □

Corollary 2.15. *For every $k \geq 4$, the* FB_k *and* PFB_k *have trivial virtual centres.*

Proof. Thinking of the flat braid group FB_k as the right-angled Coxeter group $C(P_{k-2}^{\mathrm{opp}})$, it follows immediately from Lemma 2.14 that FB_k has a trivial virtual centre as soon as $k \geq 4$. Then, notice that an element of the virtual centre of PFB_k also belongs to the virtual centre of FB_k, so it must be trivial. □

Acylindrical hyperbolicity. In order to prove Corollaries 2.17 and 3.3, we will use a few tools coming from the theory of *acylindrically hyperbolic groups*. We refer the reader to [Osin 2018] for more information on the subject. Recall that an acylindrically hyperbolic group G has a unique maximal finite normal subgroup, referred to as its *finite radical* [Dahmani et al. 2017, Theorem 2.24].

Proposition 2.16. *Let Γ be a finite graph and $\mathcal{G}$ a collection of groups indexed by Γ. If Γ is not a join and has at least two vertices, then $\Gamma\mathcal{G}$ is acylindrically hyperbolic and its finite radical is trivial.*

Proof. The acylindrical hyperbolicity of $\Gamma\mathcal{G}$ is given by [Minasyan and Osin 2015, Corollary 2.13]. It remains to verify that the finite radical R of $\Gamma\mathcal{G}$ is trivial. Let $g \in \Gamma\mathcal{G}$ be an element and $\Theta \subset \Gamma$ a subgraph such that $g\langle\Theta\rangle g^{-1}$ is the unique smallest parabolic subgroup containing R (see [Antolín and Minasyan 2015, Proposition 3.10]). As it is well-known that finite subgroups in graph products are contained in clique subgroups (see for instance [Genevois 2017, Theorem 2.115 and Corollary 8.7] for a geometric proof), we deduce from Lemma 2.3 that Θ is complete. Moreover, since $\Gamma\mathcal{G}$ normalises R, necessarily $g\langle\Theta\rangle g^{-1}$ is normalised by $\Gamma\mathcal{G}$. But, according to [Antolín and Minasyan 2015, Proposition 3.13], the normaliser of $g\langle\Theta\rangle g^{-1}$ is $g\langle\mathrm{star}(\Theta)\rangle g^{-1}$. It follows from Corollary 2.4 that $\Gamma = \mathrm{star}(\Theta)$. Since Γ is not a join and contains at least two vertices, this implies that $\Theta = \varnothing$. In other words, R must be trivial, as desired. □

Corollary 2.17. *For every $k \geq 4$,* PFB_k *is not virtually a product of two infinite groups.*

Proof. For $k \geq 4$, it follows from Proposition 2.16 that FB_k, thought of as the right-angled Coxeter group $C(P_{k-2}^{\mathrm{opp}})$, is acylindrically hyperbolic. As a finite-index subgroup, PFB_k must be acylindrically hyperbolic as well. This prevents PFB_k from being virtually a product of two infinite groups, for instance as a consequence of [Minasyan and Osin 2015, Lemma 6.24]. □

3. Morphisms to products

In this section, our goal is to show that a morphism between two products of groups satisfying mild assumptions has to send a factor to a factor. The main result in this direction is Lemma 3.5, using the notion of *IMC generating sets* which we now define and study.

3.1. *IMC generating sets.*

Definition 3.1. Given a group G, $S \subset G$ is an IMC generating set if it satisfies the following conditions:

Independence: for all distinct $s_1, s_2 \in S$ and all integers $p, q \geq 1$, $[s_1^p, s_2^q] \neq 1$;

Maximal centralisers: for all $s \in S$ and $g \in G$, if $C(s) \subsetneq C(g)$ then $g = 1$.

Our goal now is to show that most acylindrically hyperbolic groups admit IMC generating sets.

Proposition 3.2. *Let G be an acylindrically hyperbolic group. If the finite radical of G is trivial, then G admits an IMC generating set.*

Proof. Recall that every generalised loxodromic element $g \in G$ belongs to a unique maximal virtually cyclic subgroup of G, which we denote by $E(g)$ [Dahmani et al. 2017, Lemma 6.5]. According to [ibid., Corollary 6.6], $E(g)$ coincides with $\{h \in G \mid \exists n \geq 1, hg^nh^{-1} = g^{\pm n}\}$. As a consequence, $E(g^k) = E(g)$ for every $k \geq 1$.

Fix a nonelementary acylindrical action of G on some hyperbolic space. Following [Antolín et al. 2016], we refer to an element $g \in G$ as *special* if g is loxodromic and $E(g) = \langle g \rangle$. Let S_0 denote the set of all the special elements of G. Fix a set of representatives $S \subset S_0$ with respect to the equivalence relation: for all $r, s \in S_0$, $r \sim s$ whenever $r = s^{\pm 1}$. Let us verify that S is an IMC generating set of G.

According to [Antolín et al. 2016, Proposition 5.14], S_0 generates G, so S is a generating set of G.

Now, let $s \in S$ and $g \in G$ be two elements satisfying $C(s) \subset C(g)$. Since s and g commute, we have

$$g \in C(s) \subset E(s) = \langle s \rangle.$$

Either $g = 1$, in which case there is nothing to prove; or g is a nontrivial power of s, which implies that

$$C(s) \subset C(g) \subset E(g) = E(s) = \langle s \rangle \subset C(s),$$

hence $C(g) = C(s)$.

Next, let $r, s \in S$ and $p, q \geq 1$ be such that $[r^p, s^q] = 1$. We have

$$r^p \in C(s^q) \subset E(s^q) = E(s) = \langle s \rangle,$$

from which we deduce that

$$\langle r \rangle = E(r) = E(r^p) = E(s) = \langle s \rangle.$$

Therefore, we must have $r = s^{\pm 1}$, hence $r = s$ by definition of S. □

Corollary 3.3. *Let* Γ *be a finite graph and* $\mathcal{G}$ *a collection of groups indexed by* Γ. *If* Γ *contains at least two vertices and is not a join, then* $\Gamma\mathcal{G}$ *has an IMC generating set.*

Proof. According to Proposition 2.16, Proposition 3.2 applies and yields the desired conclusion. □

Lemma 3.4. *Let* Γ *be a finite graph and* $\mathcal{G}$ *a collection of groups indexed by* Γ. *Let* Q *be a maximal product subgroup of* $\Gamma\mathcal{G}$ *that is not contained in a conjugate of vertex group given by an isolated vertex of* Γ. *Then* Q *decomposes as* $Q_1 \times \cdots \times Q_s$ *where each* Q_i *either is conjugate to a vertex group or admits an IMC generating set.*

Proof. According to Lemma 2.5, our maximal product subgroup $Q \leq \Gamma\mathcal{G}$ can written as $g\langle \Xi \rangle g^{-1}$ for some $g \in \Gamma\mathcal{G}$ and some maximal join $\Xi \subset \Gamma$. Decompose Ξ as $\Xi_1 * \cdots * \Xi_s$ where no Ξ_i is a join. Accordingly, Q decomposes as a product $Q_1 \times \cdots \times Q_s$ where $Q_i := g\langle \Xi_i \rangle g^{-1}$ for every $1 \leq i \leq s$. For every $1 \leq i \leq s$, either Ξ_i is reduced to a single vertex, in which case Q_i is conjugate to a vertex group; or Ξ_i contains at least two vertices, in which case Q_i admits an IMC generating set according to Corollary 3.3. □

3.2. ***Some rigidity.*** Our main motivation for the introduction of IMC generating sets is the following statement, which is inspired by [Genevois 2024, Lemma 3.4] and which will be fundamental in order to deduce Theorem 5.1 from Theorem 4.1.

Lemma 3.5. *Let* $H, K, A_1, \ldots, A_n$ *be groups such that* $H \times K$ *is a finite-index subgroup of* $A_1 \times \cdots \times A_n$. *If* $A_1, \ldots, A_n$ *have almost stable centralisers and if* H *is a noncyclic group admitting an IMC generating set, then*

$$H \leq \mathrm{VZ}(A_1) \times \cdots \times \mathrm{VZ}(A_{i-1}) \times A_i \times \mathrm{VZ}(A_{i+1}) \times \cdots \times \mathrm{VZ}(A_n)$$

for some index $1 \leq i \leq n$.

Proof. Fix an IMC generating set $S \subset H$. Since H is not cyclic, S contains at least two elements.

Fix an $s \in S$. In $A := A_1 \times \cdots \times A_n$, we can write $s = (s_1, \ldots, s_n)$. Let $k \geq 1$ be a sufficiently large integer so that $s_i^k \in H \times K$ for every $1 \leq i \leq n$. If $s_i^k \in Z(A_i)$ for every $1 \leq i \leq n$, then s^k belongs to the centre of A, and a fortiori of H. Then, $[r, s^k] = 1$ for every $r \in S \backslash \{s\}$, contradicting the fact that S is IMC. Thus, there exists some $1 \leq i \leq n$ such that $s_i^k \notin Z(A_i)$. It follows that

$$\begin{aligned} C_{H\times K}(s) &\subset C_{H\times K}(s^k) = (H\times K)\cap C_A(s^k) \\ &= (H\times K)\cap \left(C_{A_1}(s_1^k)\times\cdots\times C_{A_n}(s_n^k)\right) \\ &\subsetneq (H\times K)\cap \left(C_{A_1}(s_1^k)\times\cdots\times C_{A_{i-1}}(s_{i-1}^k)\times A_i\times C_{A_{i+1}}(s_{i+1}^k)\times\cdots\times C_{A_n}(s_n^k)\right) \\ &\subsetneq C_{H\times K}(s_1^k\cdots s_{i-1}^k s_{i+1}^k\cdots s_n^k) \end{aligned}$$

Since S is IMC, it follows that $s_1^k\cdots s_{i-1}^k s_{i+1}^k\cdots s_n^k = 1$, which amounts to saying that $s_j^k = 1$ for every $j \neq i$, or equivalently that $s^k \in A_i$.

Notice that, for every $j \neq i$, $C_{A_j}(s_j)$ has finite index in $C_{A_j}(s_j^k) = C_{A_j}(1) = A_j$ since A_j has almost stable centralisers, which amounts to saying that s_j belongs to $\mathrm{VZ}(A_j)$.

So far, we have proved that s belongs to

$$\mathrm{VZ}(A_1)\times\cdots\times\mathrm{VZ}(A_{i-1})\times A_i\times\mathrm{VZ}(A_{i+1})\times\cdots\times\mathrm{VZ}(A_n)$$

and that $s^k \in A_i$. Given an $r \in S\backslash\{s\}$, we know similarly that there exist $\ell \geq 1$ and $1 \leq j \leq n$ such that r belongs to

$$\mathrm{VZ}(A_1)\times\cdots\times\mathrm{VZ}(A_{j-1})\times A_j\times\mathrm{VZ}(A_{j+1})\times\cdots\times\mathrm{VZ}(A_n)$$

and such that $r^\ell \in A_j$. If $i \neq j$, then clearly $[r^\ell, s^k] = 1$, which is impossible as S is IMC. Hence $i = j$. We conclude that S, and a fortiori H, is contained in

$$\mathrm{VZ}(A_1)\times\cdots\times\mathrm{VZ}(A_{i-1})\times A_i\times\mathrm{VZ}(A_{i+1})\times\cdots\times\mathrm{VZ}(A_n),$$

as desired. □

We mention a first consequence of Lemma 3.5:

Corollary 3.6. *Let Ψ_1, Ψ_2 be two finite graphs that contain at least two vertices and are not joins. If $A(\Psi_1)\times A(\Psi_2) = H_1\times H_2$ for some nontrivial subgroups $H_1, H_2 \leq A(\Psi_1)\times A(\Psi_2)$, then $H_1 = A(\Psi_1)$ and $H_2 = A(\Psi_2)$ up to switching H_1 and H_2.*

Proof. We know from Corollary 3.3 that $A(\Psi_1)$ and $A(\Psi_2)$ are not cyclic and admit an IMC generating set. Moreover, as a consequence of Corollary 2.11, H_1 and H_2 have (almost) stable centralisers. Therefore, Lemma 3.5 applies to the inclusion map $A(\Psi_1)\times A(\Psi_2) \hookrightarrow H_1\times H_2$, proving that $A(\Psi_1)$ and $A(\Psi_2)$ are contained in H_1 or H_2. Since H_1 and H_2 are nontrivial, clearly $A(\Psi_1)$ and $A(\Psi_2)$ cannot be both contained in either H_1 or H_2. Up to switching H_1 and H_2, say that $A(\Psi_1) \leq H_1$ and $A(\Psi_2) \leq H_2$. From the equality $A(\Psi_1)\times A(\Psi_2) = H_1\times H_2$, we conclude that $H_1 = A(\Psi_1)$ and $H_2 = A(\Psi_2)$. □

4. Flat braids on seven strands

This section is dedicated to the proof of the following statement, which will be the key in order to deduce that most pure flat braid groups are not virtually right-angled Artin groups.

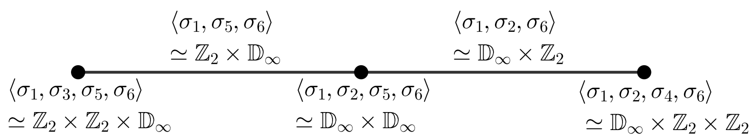
Theorem 4.1. *The group* PFB_7 *is not virtually a right-angled Artin group.*

We denote by $\sigma_1, \ldots, \sigma_6$ the usual generators of FB_7, i.e., each σ_i is an elementary twist of the i-th and $(i+1)$-st strands. Equivalently, when thinking of FB_7 as the right-angled Coxeter group $C(P_5^{\mathrm{opp}})$, $\sigma_1, \ldots, \sigma_6$ correspond to the generators given by the successive vertices along the path P_5.

From the presentation

$$\langle \sigma_1, \ldots, \sigma_6 \mid \sigma_i^2 = 1\ (1 \le i \le 6),\ [\sigma_i, \sigma_j] = 1\ (|i-j| \ge 2) \rangle$$

of FB_7, one easily verifies that FB_7 decomposes as the following graph of groups:



We refer to this decomposition of FB_7 as its *tubular decomposition*.[1] See [Serre 1980] and [Scott and Wall 1979] for more information on graphs of groups. The key point is that vertex groups are virtually $\mathbb{Z}$ or $\mathbb{Z}^2$ and that edge groups are virtually $\mathbb{Z}$.

We use this tubular decomposition in order to find restrictions on the possible subgroups of FB_7. This will allow us to show that the only right-angled Artin groups that are subgroups of PFB_7 are of the form $A(\Gamma)$ where Γ is a forest (see Corollary 4.4).

Lemma 4.2. *The group* FB_7 *does not contain a subgroup isomorphic to* $\mathbb{Z}^3$, *to* $\mathbb{F}_2 \times \mathbb{F}_2$, *or to the fundamental group of a closed surface of genus* ≥ 2.

Proof. As a consequence of its tubular decomposition, FB_7 acts on a tree T with virtually $\mathbb{Z}$ or $\mathbb{Z}^2$ vertex-stabilisers and with virtually $\mathbb{Z}$ edge-stabilisers. If $g \in \mathrm{FB}_7$ induces a loxodromic isometry on T, then its centraliser $C(g)$ in FB_7, which must stabilise the axis of γ of g in T, is necessarily (virtually cyclic)-by-$\mathbb{Z}$. (Indeed, the action of $C(g)$ on γ by translation induces an epimorphism to $\mathbb{Z}$ whose kernel fixes γ pointwise, and consequently must be virtually cyclic since edge groups are virtually $\mathbb{Z}$.)

[1] In reference to *tubular groups*, i.e., fundamental groups of graphs of groups whose vertex groups are $\mathbb{Z}^2$ and whose edge groups are $\mathbb{Z}$.

Since every element of $\mathbb{Z}^3$ has centraliser $\mathbb{Z}^3$, it follows that, if $\mathbb{Z}^3$ is a subgroup of FB_7, then it cannot contain a loxodromic isometry. Thus, it must be elliptic in T, which is impossible since vertex-stabilisers are virtually $\mathbb{Z}$ or $\mathbb{Z}^2$. Therefore, $\mathbb{Z}^3$ cannot be a subgroup of FB_7.

If $\mathbb{F}_2 \times \mathbb{F}_2$ is a subgroup of FB_7, then an $\mathbb{F}_2$-factor cannot be elliptic in T, since vertex-stabilisers are virtually $\mathbb{Z}$ or $\mathbb{Z}^2$, so it must contain a loxodromic isometry. But the centraliser of an element of an $\mathbb{F}_2$-factor always contains a nonabelian subgroup, contracting the previous observation. Therefore, $\mathbb{F}_2 \times \mathbb{F}_2$ cannot be a subgroup of FB_7.

Finally, it remains to verify that FB_7 does not contain a subgroup isomorphic to the fundamental group of a closed surface of genus ≥ 2. More generally:

Claim 4.3. *Let G be a one-ended hyperbolic group. Then G is not isomorphic to a subgroup of* FB_7.

What we need to know about hyperbolic groups is that, for every infinite-order element $g \in G$, there exists a unique maximal virtually cyclic subgroup $E(g)$ containing $\langle g \rangle$, which we will call the *elementary closure*. (Geometrically, given a quasi-axis γ of g in G, $E(g)$ corresponds to the subgroup given by the elements $h \in G$ such that the Hausdorff distance between γ and $h\gamma$ is finite. Or equivalently, to the stabiliser of the pair of points at infinity of γ.) Assume for contradiction that G is isomorphic to a subgroup of FB_7. Fix a minimal G-invariant subtree in the Bass-Serre tree associated to the tubular decomposition of FB_7. Because G is one-ended, edge groups must be virtually $\mathbb{Z}$; and, because G does not contain $\mathbb{Z}^2$, vertex subgroups must be virtually $\mathbb{Z}$ as well. Thus, vertex groups are pairwise commensurable in G. This implies that they all have the same elementary closure E. Then, E yields a normal virtually $\mathbb{Z}$ subgroup of G. The only possibility is that G is virtually $\mathbb{Z}$ itself, which is impossible as G is supposed to be one-ended. □

Corollary 4.4. *If a right-angled Artin group $A(\Gamma)$ embeds into* FB_7, *then Γ is a forest. If Γ is disconnected, then $A(\Gamma)$ has infinite index in* FB_7.

Proof. If Γ contains an induced cycle of length three (resp. four, at least five), then $A(\Gamma)$ contains a subgroup isomorphic to $\mathbb{Z}^3$ (resp. $\mathbb{F}_2 \times \mathbb{F}_2$, the fundamental group of a closed surface of genus ≥ 2 (see for instance [Servatius et al. 1989])), which prevents FB_7 from containing $A(\Gamma)$ according to Lemma 4.2. Therefore, Γ must be a forest. If Γ is disconnected, then $A(\Gamma)$ splits as a free product of two infinite groups, which prevents FB_7 from containing $A(\Gamma)$ as a finite-index subgroup since FB_7 is one-ended (which amounts to saying, when thinking of FB_7 as the right-angled Coxeter group $C(P_5^{\mathrm{opp}})$, that no complete subgraph separates P_5^{opp}). □

In order to prove Theorem 4.1, it remains to distinguish PFB_7 from right-angled Artin groups defined by trees. For this purpose, we introduce a specific subgroup.

Definition 4.5. Let G be a group. An element $g \in G$ is *thick* if its centraliser is not virtually abelian. The *thick subgroup* $\mathrm{Thick}(G)$ is the subgroup of G generated by the centralisers of all its thick elements.

It is worth noticing that, since conjugates of thick elements are thick themselves, thick subgroups are always normal.

As a consequence of Proposition 2.8, thick elements in right-angled Artin groups defined by trees coincide with nontrivial powers of generators given by vertices that are not leaves. Thus, the tree defining our right-angled Artin group can be essentially recovered from the thick elements and how they commute with each other. In other words, the structure of the group is entirely contained in its thick elements. So it makes sense to use such elements in order to determine whether or not a group can be described as a right-angled Artin group over a tree. For us, the key observation is that thick subgroups are not proper for right-angled Artin groups defined by trees while the thick subgroup of PFB_7 is rather small (in particular, it has infinite index). This is the content of the next two lemmas.

Lemma 4.6. *For every tree Γ with at least three vertices*, $\mathrm{Thick}(A(\Gamma)) = A(\Gamma)$.

Proof. For every vertex $u \in \Gamma$ of degree ≥ 2, the centraliser of the corresponding generator of $A(\Gamma)$ is $\langle \mathrm{link}(u) \rangle$, which is a free group of rank ≥ 2. Therefore, such a generator is a thick element. Since a leaf of Γ must be adjacent to some vertex of degree ≥ 2, a generator of $A(\Gamma)$ that is not thick must belong to the centraliser of a thick centraliser. Therefore, every generator of $A(\Gamma)$ belongs to the thick subgroup, hence the desired equality. □

Lemma 4.7. *The quotient* $\mathrm{FB}_7/\mathrm{Thick}(\mathrm{PFB}_7)$ *is isomorphic to the Coxeter group $C(\Gamma)$ where Γ is the labelled graph defined as follows*:

- *the underlying graph of Γ is a complete graph with six vertices $x_1, \dots, x_6$*;
- *the edge connecting x_i and x_j is labelled by 2 whenever $|i - j| \geq 2$*;
- *the edge connecting x_i and x_{i+1} is labelled by 3 if $i \neq 3$ and ∞ otherwise.*

In particular, *it is infinite.*

Proof. Let $\pi : \mathrm{FB}_7 \to C(\Gamma)$ denote the morphism that sends σ_i to x_i for every $1 \leq i \leq 6$. Our goal is to prove that $\ker(\pi) = \mathrm{Thick}(\mathrm{PFB}_7)$, which will conclude the proof of our lemma. First, let us verify that $\mathrm{Thick}(\mathrm{PFB}_7)$ is contained in $\ker(\pi)$.

Claim 4.8. *An element of* FB_7 *is thick if and only if it is conjugate to a nontrivial power of $\sigma_1\sigma_2$ or $\sigma_5\sigma_6$.*

We think of FB_7 as the right-angled Coxeter group $C(P_5^{\mathrm{opp}})$. Let $g \in \mathrm{FB}_7$ be a thick element. Up to conjugating g, we can assume that g is graphically cyclically reduced. Because g has infinite order, necessarily $\mathrm{supp}(g)$ is not complete. And, because the centraliser of g is not virtually abelian, it follows from

Proposition 2.8 that link(supp(g)) contains at least two nonadjacent vertices but is not just a pair of two nonadjacent vertices. In P_5^{opp}, there are only two possibilities: either supp(g) $= \{\sigma_1, \sigma_2\}$ and link(supp(g)) $= \{\sigma_4, \sigma_5, \sigma_6\}$, or supp($g$) $= \{\sigma_5, \sigma_6\}$ and link(supp(g)) $= \{\sigma_1, \sigma_2, \sigma_3\}$. In the first case, g is a nontrivial power of $\sigma_1\sigma_2$; and, in the second case, g is a nontrivial power of $\sigma_5\sigma_6$. Conversely, it follows from Proposition 2.8 that the centraliser of a nontrivial power $(\sigma_1\sigma_2)^k$ (resp. $(\sigma_5\sigma_6)^k$) is $\langle \sigma_1\sigma_2 \rangle \times \langle \sigma_4, \sigma_5, \sigma_6 \rangle$ (resp. $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \times \langle \sigma_5\sigma_6 \rangle$), which is isomorphic to $\mathbb{Z} \times (\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2))$. Thus, the centraliser of $\sigma_1\sigma_2$ (resp. $\sigma_5\sigma_6$) is indeed not virtually abelian. This concludes the proof of Claim 4.8.

Since an element of PFB_7 is thick in PFB_7 if and only if it is thick in FB_7, it follows from Claim 4.8 that an element of PFB_7 is thick if and only if it is conjugate to a nontrivial power of $(\sigma_1\sigma_2)^3$ or $(\sigma_5\sigma_6)^3$. We deduce from Proposition 2.8 that the centraliser of $(\sigma_1\sigma_2)^3$ in FB_7 is $\langle \sigma_1\sigma_2 \rangle \times \langle \sigma_4, \sigma_5, \sigma_6 \rangle$, so the centraliser of $(\sigma_1\sigma_2)^3$ in PFB_7 is

$$\langle (\sigma_1\sigma_2)^3 \rangle \times (\mathrm{PFB}_7 \cap \langle \sigma_4, \sigma_5, \sigma_6 \rangle) .$$

By noticing that π restricts on $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \simeq \mathrm{FB}_4$ (resp. $\langle \sigma_4, \sigma_5, \sigma_6 \rangle \simeq \mathrm{FB}_4$) to the canonical map to the permutation group $\langle x_1, x_2, x_3 \rangle \simeq \mathrm{Sym}(4)$ (resp. $\langle x_4, x_5, x_6 \rangle \simeq \mathrm{Sym}(4)$), it follows that the centraliser above is contained in the kernel of π. Symmetrically, we show that the centraliser of $(\sigma_5\sigma_6)^3$ in PFB_7 is contained in $\ker(\pi)$. Thus, we have proved that $\ker(\pi)$ contains the centraliser in PFB_7 of every thick element of PFB_7. In other words, $\mathrm{Thick}(\mathrm{PFB}_7) \leq \ker(\pi)$, as desired.

By comparing the Coxeter presentations of FB_7 and $C(\Gamma)$, it is clear that $\ker(\pi)$ coincides with the normal closure in FB_7 of $\{(\sigma_1\sigma_2)^3, (\sigma_2\sigma_3)^3, (\sigma_4\sigma_5)^3, (\sigma_5\sigma_6)^3\}$. Because $\mathrm{Thick}(\mathrm{PFB}_7)$ is a normal subgroup of FB_7, as a consequence of Fact 4.9 below, it suffices to notice that $(\sigma_1\sigma_2)^3$, $(\sigma_2\sigma_3)^3$, $(\sigma_4\sigma_5)^3$, $(\sigma_5\sigma_6)^3$ all belong to $\mathrm{Thick}(\mathrm{PFB}_7)$ in order to conclude that $\ker(\pi)$ is contained in $\mathrm{Thick}(\mathrm{PFB}_7)$. But we already know that $(\sigma_1\sigma_2)^3$ and $(\sigma_5\sigma_6)^3$ are thick elements of PFB_7, and $(\sigma_2\sigma_3)^3$ (resp. $(\sigma_4\sigma_5)^3$) belongs to the centraliser of $(\sigma_5\sigma_6)^3$ (resp. of $(\sigma_1\sigma_2)^3$).

Fact 4.9. *Let G be a group. For every normal subgroup $H \lhd G$,* Thick(H) *is a normal subgroup of G.*

The action of G on H by conjugation permutes the thick elements of H and sends centralisers to centralisers. Therefore, G permutes the centralisers of the thick elements of H, proving that Thick(H) is stabilised by conjugation, or equivalently that Thick(H) is a normal subgroup of G. □

Proof of Theorem 4.1. Assume for contradiction that PFB_7 contains a right-angled Artin group $A(\Gamma)$ as a finite-index subgroup. It follows from Corollary 4.4 that Γ must be a tree (with at least three vertices since FB_7 is not virtually abelian), hence

$\mathrm{Thick}(A(\Gamma)) = A(\Gamma)$ according to Lemma 4.6. But we clearly have $\mathrm{Thick}(A(\Gamma)) \leq \mathrm{Thick}(\mathrm{PFB}_7)$, hence

$$|\mathrm{PFB}_7/\mathrm{Thick}(\mathrm{PFB}_7)| \leq |\mathrm{PFB}_7/A(\Gamma)| < \infty.$$

This contradicts Lemma 4.7, which implies that $\mathrm{PFB}_7/\mathrm{Thick}(\mathrm{PFB}_7)$ is infinite. □

5. Pure flat braid groups are not right-angled Artin groups

In this section, we prove the main result of this article:

Theorem 5.1. *For every $n = 7$ or ≥ 11, PFB_n is not virtually a right-angled Artin group.*

We start by stating and proving a general criterion that allows us to show that, under some assumptions, if a group G_1 can be realised as a finite-index subgroup in a group G_2, then every factor of a maximal product subgroup of G_2 contains as a finite-index subgroup a factor of a maximal product subgroup of G_1.

Proposition 5.2. *Let G_1 and G_2 be two torsion-free groups. Assume that:*

- (i) *In both G_1 and G_2, a subgroup commensurable to a product of two infinite groups is contained in a maximal product subgroup.*
- (ii) *In G_1, every maximal product subgroup Q decomposes as $Q_1 \times \cdots \times Q_s$ where each Q_i either is infinite cyclic or admits an IMC generating set.*
- (iii) *In G_2, if two maximal product subgroups P_1 and P_2 are such that $P_1 \cap P_2$ has finite index in P_1, then $P_1 = P_2$.*
- (iv) *G_2 has almost stable centralisers.*

If G_1 is a finite-index subgroup of G_2 and if $P := P_1 \times \cdots \times P_s$ is a maximal product subgroup of G_2 such that $\mathrm{VZ}(P) = \{1\}$ and such that no P_i is virtually a product of two infinite groups, then there exists a maximal product subgroup $R := R_1 \times \cdots \times R_s$ of G_1 such that each R_i is a finite-index subgroup of P_i.

Proof. Let $P = P_1 \times \cdots \times P_s$ be a maximal product subgroup of G_2 such that $\mathrm{VZ}(P) = \{1\}$ and such that no P_i is virtually a product of two infinite groups. It follows from (i) that $P \cap G_1$ is contained in some maximal product subgroup $R \leq G_1$. Similarly, R must be contained in some maximal product subgroup P^+ of G_2. Since $P \cap G_1$ has finite index in P, it follows from (iii) that P^+ actually coincides with P. Let $R = R_1 \times \cdots \times R_m$ denote the decomposition of R given by (ii). Notice that, since $\mathrm{VZ}(P)$ is trivial, necessarily $\mathrm{VZ}(R)$ must be trivial as well, which implies that no R_i is (virtually) cyclic.

Notice that $P \cap G_1 \leq R \leq P$ and that $P \cap G_1$ has finite index in P, so R must have finite index in P. Applying Lemma 3.5, which is possible thanks to (ii) and (iv), we deduce that, for every $1 \leq i \leq m$, there exists $1 \leq \sigma(i) \leq s$ such that $R_i \leq P_{\sigma(i)}$. Notice that

$$P/R \equiv \prod_{i=1}^{r} \left(P_i / \prod_{j \in \sigma^{-1}(i)} R_j \right),$$

which must be finite. Consequently, $\prod_{j \in \sigma^{-1}(i)} R_j$ must have finite index in P_i for every $1 \leq i \leq r$. But we know by assumption that no P_i is virtually a product, so σ must be bijective. Thus, up to reordering the factors of P, we have proved that $m = s$ and that R_i has finite index in P_i for every $1 \leq i \leq s$, as desired. □

Thanks to Lemmas 2.6, 3.4, 2.7, and 2.10, we have a good understanding of when a graph product satisfies the assumptions of Proposition 5.2. This applies in particular to flat braid groups, from which it is not difficult to deduce similar statements for their pure subgroups:

Lemma 5.3. *For every $n \geq 2$, the following assertions hold*:

(i) *A subgroup of* PFB_n *commensurable to a product of two infinite groups is contained in a maximal product subgroup.*

(ii) *In* PFB_n, *if two maximal product subgroups P_1 and P_2 are such that $P_1 \cap P_2$ has finite index in P_1, then $P_1 = P_2$.*

(iii) PFB_n *has almost stable centralisers.*

(iv) *The maximal product subgroups of* PFB_n *are the conjugates of*

$$\mathrm{PFB}_i \times \mathrm{PFB}_{n-i} := (\mathrm{PFB}_n \cap \langle \sigma_1, \ldots, \sigma_{i-1} \rangle) \times (\mathrm{PFB}_n \cap \langle \sigma_{i+1}, \ldots, \sigma_{n-1})$$

for $3 \leq i \leq n-3$.

Proof. Let P be a product in PFB_n. According to Lemma 2.6, P is contained in a maximal product subgroup Q of FB_n. Thinking of FB_n as a right-angled Coxeter group, we deduce from Lemma 2.5 that Q is conjugate to $\langle \sigma_1, \ldots, \sigma_{i-1} \rangle \times \langle \sigma_{i+1}, \ldots, \sigma_{n-1} \rangle$ for some $3 \leq i \leq n-3$. Thus, P is contained in a conjugate of $\mathrm{PFB}_i \times \mathrm{PFB}_{n-i}$. This proves (iv). Moreover, our argument shows the following assertion, which we will use in order to prove the rest of the lemma:

Fact 5.4. *The maximal product subgroups of* PFB_n *are the intersection with* PFB_n *of the maximal product subgroups of* FB_n.

If $H \leq \mathrm{PFB}_n$ is commensurable to a product of two infinite groups, then we know from Lemma 2.6 that H is contained in a maximal product subgroup Q of FB_n. Thus, H is contained in $Q \cap \mathrm{PFB}_n$, which is a maximal product subgroup of PFB_n according to Fact 5.4. This proves (i).

Let P_1 and P_2 be two maximal product subgroups of PFB_n such that $P_1 \cap P_2$ has finite index in P_1. According to Lemma 2.6, P_1 (resp. P_2) is contained in a maximal product subgroup P_1^+ (resp. P_2^+) of FB_n. Notice that, as a consequence of Fact 5.4, $P_1^+ \cap \mathrm{PFB}_n$ (resp. $P_2^+ \cap \mathrm{PFB}_n$) is a maximal product subgroup of PFB_n. Since it contains P_1 (resp. P_2), necessarily $P_1 = P_1^+ \cap \mathrm{PFB}_n$ (resp. $P_2 = P_2^+ \cap \mathrm{PFB}_n$). Since $P_1^+ \cap P_2^+$ contains $P_1 \cap P_2$, and since the latter has finite index in P_1, and a fortiori in P_1^+, we deduce from Lemma 2.7 that $P_1^+ = P_2^+$. Hence

$$P_1 = P_1^+ \cap \mathrm{PFB}_n = P_2^+ \cap \mathrm{PFB}_n = P_2,$$

proving assertion (ii).

Finally, assertion (iii) follows from Corollary 2.11, since pure flat braid groups embed into right-angled Artin groups (as a consequence of [Genevois 2020, Example 5.40] (see also [Farley 2021]) and [Genevois 2018, Theorem 1.2]). □

Proof of Theorem 5.1. For $n = 7$, the desired conclusion is given by Theorem 4.1. From now on, assume that $n \geq 9$. Notice that, according to Lemmas 2.6, 3.4, and 5.3, assumptions (i)–(iv) of Proposition 5.2 are satisfied for G_1 a right-angled Artin group and G_2 a pure flat braid group.

Assume for contradiction that PFB_n contains a finite-index subgroup that is a right-angled Artin group $A(\Gamma)$. According to Lemma 5.3, $P := \mathrm{PFB}_7 \times \mathrm{PFB}_{n-7}$ is a maximal product subgroup of PFB_n. We know from Corollaries 2.15 and 2.17 that PFB_k is not virtually a product of two infinite groups and has a trivial virtual centre whenever $k \geq 4$. Consequently, Proposition 5.2 applies and shows that $A(\Gamma)$ contains a maximal product subgroup $R := R_1 \times R_2$ such that R_1 (resp. R_2) has finite index in PFB_7 (resp. PFB_{n-7}). Since R_1 must be a right-angled Artin group according to Corollary 3.6, we deduce that PFB_7 is virtually a right-angled Artin group, contradicting Theorem 4.1. □

6. An instance of commensurability

In contrast with Theorem 5.1, in this section we prove the following observation:

Theorem 6.1. *The flat braid group* FB_7 *is commensurable to the right-angled Artin group* $A(P_4)$.

After preparing the ground in Section 6.1, we show in Section 6.2 that FB_7 is commensurable to a *lampraag* $A(\mathbb{L}) \rtimes \mathbb{Z}$. A key observation is that both $A(\mathbb{L}) \rtimes \mathbb{Z}$ and $A(P_4)$ can be described as fundamental groups of compact 3-manifolds. Therefore, in order to deduce that these two groups are commensurable, it suffices to construct a common finite-sheeted cover for the two corresponding manifolds. This is what we do in Section 6.3.

6.1. ***Some quasi-median geometry.*** In the next section, we will use some quasi-median geometry of graph products, as introduced in [Genevois 2017]. In this section, we recall the few definitions and results that we need.

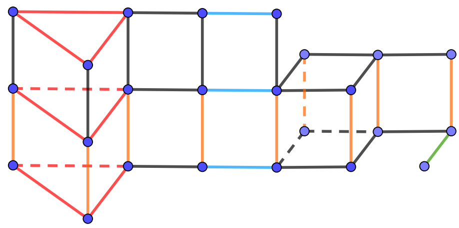


Figure 2. Some hyperplanes in a quasi-median graph.

Theorem 6.2 [Genevois 2017]. *For every graph* Γ *and every collection of groups* $\mathcal{G}$ *indexed by* Γ, *the Cayley graph*

$$\mathrm{QM}(\Gamma, \mathcal{G}) := \mathrm{Cayl}\left(\Gamma\mathcal{G}, \bigcup_{G \in \mathcal{G}} G \backslash \{1\} \right)$$

is a quasi-median graph.

Quasi-median graphs can be defined as retracts of Hamming graphs (i.e., products of complete graphs). There are many alternative characterisations of quasi-median graphs (see for instance [Bandelt et al. 1994]), but this definition is rather simple and it highlights the connection with median graphs (also known as one-skeletons of CAT(0) cube complexes; see [Genevois 2023]), which can be defined as retracts of hypercubes. It is also worth mentioning that median graphs coincide with triangle-free quasi-median graphs. As a consequence, Theorem 6.2 implies this well-known observation:

Corollary 6.3. *For every graph* Γ, *the Cayley graph* $\mathrm{Cayl}(C(\Gamma), \Gamma)$ *of the right-angled Coxeter group* $C(\Gamma)$ *is a median graph.*

Hyperplanes are the key objects in order to understand the geometry of median and quasi-median graphs. They are equivalence classes of edges with respect to the reflexive-transitive closure of the relation that identifies two edges whenever they belong to a common 3-cycle or whenever they are opposite edges in a 4-cycle. See Figure 2. Hyperplanes in quasi-median graphs of graph products are described by [Genevois 2017, Lemma 8.9 and Corollary 8.10]:

Lemma 6.4. *Let* Γ *be a graph and* $\mathcal{G}$ *a collection of groups indexed by* Γ. *Fix a vertex* $u \in \Gamma$ *and let* J_u *denote the hyperplane of* $\mathrm{QM}(\Gamma, \mathcal{G})$ *containing the edges of the clique* $\langle u \rangle$. *An edge* e *belongs to* J_u *if and only if* $e = \{g, g\ell\}$ *for some* $g \in \langle \mathrm{link}(u) \rangle$ *and* $\ell \in \langle u \rangle \backslash \{1\}$. *As a consequence, the stabiliser of* J_u *in* $\Gamma\mathcal{G}$ *is* $\langle \mathrm{star}(u) \rangle$.

As a consequence of our lemma, all the edges of a given hyperplane of $\mathrm{QM}(\Gamma, \mathcal{G})$ are naturally labelled by the same vertex of Γ. This labelling may quite useful in practice. For instance:

Lemma 6.5. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. Two transverse hyperplanes in $\mathrm{QM}(\Gamma, \mathcal{G})$ must be labelled by adjacent vertices of Γ.*

In order to state our last preliminary lemma, we need to introduce a couple of preliminary definitions.

Definition 6.6. Let G be a group acting on a quasi-median graph X. The *rotative-stabiliser* $\mathrm{stab}_{\circlearrowleft}(J)$ of a hyperplane J is the subgroup of $\mathrm{stab}(J)$ that stabilises each maximal complete subgraph of J.

In the sequel, given a group G acting on a median graph X and a collection of hyperplanes $\mathcal{J}$, we denote by $\mathrm{Rot}(\mathcal{J})$ the subgroup $\langle \mathrm{stab}_{\circlearrowleft}(J) \mid J \in \mathcal{J} \rangle$ of G.

Definition 6.7. Let X be a quasi-median graph and $\mathcal{J}$ a collection of hyperplanes. The *crossing graph* of $\mathcal{J}$ is the graph whose vertices are the hyperplanes of $\mathcal{J}$ and whose edges connect two hyperplanes whenever they are transverse.

Recall that two hyperplanes are *transverse* whenever they cover some 4-cycle. We also say that a hyperplane J is *tangent* to a subgraph Y whenever J contains an edge not in Y but with a vertex in Y.

Proposition 6.8. *Let G be a group acting on a quasi-median graph X. Fix a gated subgraph $Y \subset X$ and let $\mathcal{J}$ denote the collection of the hyperplanes of X tangent to Y. Assume that*

- *for every $J \in \mathcal{J}$, $\mathrm{stab}_{\circlearrowleft}(J)$ acts vertex-freely on X, and*
- *for all transverse $J, H \in \mathcal{J}$, every element of $\mathrm{stab}_{\circlearrowleft}(J)$ commutes with every element of $\mathrm{stab}_{\circlearrowleft}(H)$.*

Let Δ denote the crossing graph of $\mathcal{J}$ and let $\mathcal{G} = \{\mathrm{stab}_{\circlearrowleft}(J) \mid J \in \mathcal{J}\}$. Then

$$\langle \mathrm{Rot}(\mathcal{J}), \mathrm{stab}(Y) \rangle = \mathrm{Rot}(\mathcal{J}) \rtimes \mathrm{stab}(Y)$$

and the map $\Delta\mathcal{G} \to \mathrm{Rot}(\mathcal{J})$ that restricts on each vertex group $\mathrm{stab}_{\circlearrowleft}(J)$ to the identity is an isomorphism.

The proposition is a rather straightforward consequence of the ping-pong lemma [Genevois 2017, Proposition 8.44]. It is also a consequence of [Genevois 2017, Theorem 10.54]. For brevity, we refer the reader to [Genevois 2017] for details and we only mention that Proposition 6.8 applies to subgroups $G \leq \Gamma\mathcal{G}$ of graph products $\Gamma\mathcal{G}$, to quasi-median graphs $X = \mathrm{QM}(\Gamma, \mathcal{G})$, and to subgraphs of the form $\langle \Lambda \rangle$, where $\Lambda \subset \Gamma$ as justified by [Genevois 2017, Lemma 8.46] and [Genevois 2022c, Corollary 6.6].

6.2. ***Lampraag over*** $\mathbb{Z}$**.** We now introduce a family of groups we call *lampraags*, in analogy with lamplighter groups. They are particular examples of graph-wreath products [Kropholler and Martino 2016], and more generally of halo products [Genevois and Tessera 2024]. The lampraag over $\mathbb{Z}$ will provide a convenient model for FB_7 up to commensurability.

Definition 6.9. Let G be a group and $S \subset G$ a subset. The *lampraag over* (G, S) is the semidirect product

$$A(\mathrm{Cayl}(G, S)) \rtimes G,$$

where G permutes the generators of $A(\mathrm{Cayl}(G, S))$ according to its action by left-multiplication on $\mathrm{Cayl}(G, S)$.

In the definition, we do not require S to be a generating set. For instance, taking $S = \varnothing$ is allowed, in which case $A(\mathrm{Cayl}(G, \varnothing)) \rtimes G \simeq \mathbb{Z} * G$. In practice, however, we will be mainly interested in the case where $\mathrm{Cayl}(G, S)$ is connected, i.e., when S is a generating set. In fact, in the sequel, we will focus on the lampraag over $\mathbb{Z}$, where $\mathbb{Z}$ is endowed with its canonical generating set $\{1\}$. For convenience, we will denote by $\mathbb{L}$ the Cayley graph of $\mathbb{Z}$ with respect to $\{1\}$, i.e., the bi-infinite line; and by $A(\mathbb{L}) \rtimes \mathbb{Z}$ the corresponding lampraag.

Fact 6.10. *The lampraag* $A(\mathbb{L}) \rtimes \mathbb{Z}$ *admits* $\langle a, t \mid [a, tat^{-1}] = 1\rangle$ *as a presentation.*

Proof. It is clear that

$$\langle \dots, a_{-1}, a_0, a_1, \dots, t \mid ta_it^{-1} = a_{i+1} \text{ and } [a_i, a_{i+1}] = 1 \text{ for every } i \in \mathbb{Z}\rangle$$

is a presentation of our lampraag. Since $a_i = t^i a_0 t^{-i}$ for every $i \in \mathbb{Z}$, this infinite presentation can be simplified into the finite presentation given above. □

Interestingly, the lampraag over $\mathbb{Z}$ turns out to be connected to many other families of groups. For instance:

- As a consequence of Fact 6.10, this is a one-relator group. One-relator groups have been extensively studied in combinatorial group theory.
- As already mentioned, this is an example of a graph-wreath product [Kropholler and Martino 2016], and more generally of a halo product [Genevois and Tessera 2024].
- As we will see in Section 6.3, this is the fundamental group of a compact flip 3-manifold with boundary.
- Its finite-index subgroup $A(\mathbb{L}) \rtimes 2\mathbb{Z}$, which admits $\langle a, b, t \mid [a, b] = [a, tbt^{-1}] = 1\rangle$ as a presentation, is a *diagram group* [Genevois 2020, Example 5.43]. Interestingly, it is also proved in [Genevois 2020] that $A(\mathbb{L}) \rtimes 2\mathbb{Z}$ is not a right-angled Artin group.

• The lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$ is the fundamental group of a *Cartesian graph of groups* [Genevois 2017, Example 11.38].

• It is not difficult to deduce from the description of centralisers in right-angled Artin groups (see Proposition 2.8) that the lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$ cannot be a subgroup of a right-angled Artin group. However, its finite-index subgroup $A(\mathbb{L}) \rtimes 2\mathbb{Z}$ is a simple subgroup of $A(P_3)$. Indeed, if we denote by p, q, r, s the four vertices successively met along P_3, then the subgroup $\langle q, r, sp\rangle$ is isomorphic to $A(\mathbb{L}) \rtimes 2\mathbb{Z}$.

The rest of the section is dedicated to the proof of the following statement, which shows that the lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$ can be thought of as a good model of FB_7 up to commensurability.

Proposition 6.11. *The group* FB_7 *is commensurable to the lampraag* $A(\mathbb{L}) \rtimes \mathbb{Z}$.

Our argument allows us to be more explicit. To be precise, the proof of Proposition 6.11 shows that the map

$$\sigma_1\sigma_2 \mapsto a, \quad \sigma_3\sigma_4 \mapsto t^2, \quad \sigma_5\sigma_6 \mapsto tat^{-1}$$

induces an isomorphism from the index-8 normal subgroup $\langle \sigma_1\sigma_2, \sigma_3\sigma_4, \sigma_5\sigma_6\rangle$ of FB_7 to the index-2 subgroup $\langle a, tat^{-1}, t^2\rangle = A(\mathbb{L}) \rtimes 2\mathbb{Z}$ of $A(\mathbb{L}) \rtimes \mathbb{Z}$, when presented as $\langle a, t \mid [a, tat^{-1}] = 1\rangle$ (see Fact 6.10).

Proof of Proposition 6.11. As a first step, let us consider the subgroup

$$H := \langle \sigma_1\sigma_2, \sigma_3, \sigma_4, \sigma_5\sigma_6\rangle$$

of FB_7. One easily checks that this is a normal subgroup. The quotient FB_7/H is the product of two cyclic groups of order 2 generated by the images of σ_1 and σ_5. Therefore, H is a normal subgroup of index 4 in FB_7 with $\{1, \sigma_1, \sigma_5, \sigma_1\sigma_5\}$ as a set of representatives.

Now, let us investigate the structure of H. We know from Corollary 6.3 that the graph $\mathrm{M} := \mathrm{Cayl}(\mathrm{FB}_7, \{\sigma_1, \dots, \sigma_6\})$ is a median graph. Moreover, it follows from Lemma 6.5 that two hyperplanes of M labelled by σ_3 or σ_4 are never transverse; so the set $\mathcal{J}$ of all the hyperplanes of M labelled by σ_3 or σ_4 yields an arboreal structure on M, i.e., the graph $T(\mathcal{J})$ whose vertices are the connected components of the graph $\mathrm{M}\backslash\backslash\mathcal{J}$ (obtained from M by removing the hyperplanes from $\mathcal{J}$) and whose edges connect two components whenever they are separated by a single hyperplane is a tree. Make H act on $T(\mathcal{J})$.

Notice that there is a single H-orbit of vertices. Indeed, a vertex of $T(\mathcal{J})$ corresponds to a maximal subgraph of M all of whose edges are labelled by generators distinct from σ_3 and σ_4. In other words, the vertices of $T(\mathcal{J})$ correspond to the cosets of $K := \langle \sigma_1, \sigma_2, \sigma_5, \sigma_6\rangle$ in FB_7. Since we saw that $1, \sigma_1, \sigma_5, \sigma_1\sigma_5$ are representatives modulo H, it suffices to verify that K, $\sigma_1 K$, $\sigma_5 K$, and $\sigma_1\sigma_5 K$ all lie in the same H-orbit. But this is clear since

- $\sigma_1 K = \sigma_1\sigma_2 K$ with $\sigma_1\sigma_2 \in H$;
- $\sigma_5 K = \sigma_5\sigma_6 K$ with $\sigma_5\sigma_6 \in H$;
- $\sigma_1\sigma_5 K = \sigma_1\sigma_2\sigma_5\sigma_6 K$ with $\sigma_1\sigma_2, \sigma_5\sigma_6 \in H$.

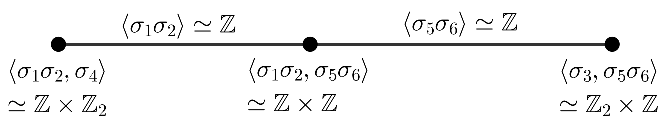
The H-stabiliser of the vertex of $T(\mathcal{J})$ given by K is $H \cap K = \langle \sigma_1\sigma_2, \sigma_5\sigma_6\rangle$. Next, notice that there are two H-orbits of edges in $T(\mathcal{J})$. Indeed, the edges of $T(\mathcal{J})$ correspond to the hyperplanes of M labelled by σ_3 or σ_4; or equivalently, according to Lemma 6.4, to the cosets of

$$E_1 := \langle \sigma_1, \sigma_3, \sigma_5, \sigma_6\rangle \quad \text{and} \quad E_2 := \langle \sigma_1, \sigma_2, \sigma_4, \sigma_6\rangle$$

in FB_7. As previously, we deduce our claim from the following easy observations:

- for every $\eta \in \{\sigma_1, \sigma_5, \sigma_1\sigma_5\}$, $\eta E_1 = E_1$ since $\eta \in E_1$;
- $\sigma_1 E_2 = E_2$ since $\sigma_1 \in E_2$;
- $\sigma_5 E_2 = \sigma_5\sigma_6 E_2$ with $\sigma_5\sigma_6 \in H$;
- $\sigma_1\sigma_5 E_2 = \sigma_5 E_2 = \sigma_5\sigma_6 E_2$ with $\sigma_5\sigma_6 \in H$.

The H-stabiliser of the edge of $T(\mathcal{J})$ given by E_1 (resp. E_2) is $H \cap E_1 = \langle \sigma_3, \sigma_5\sigma_6\rangle$ (resp. $H \cap E_2 = \langle \sigma_4, \sigma_1\sigma_2\rangle$). We conclude from these properties satisfied by the action of H on $T(\mathcal{J})$ that H decomposes as the following graph of groups:

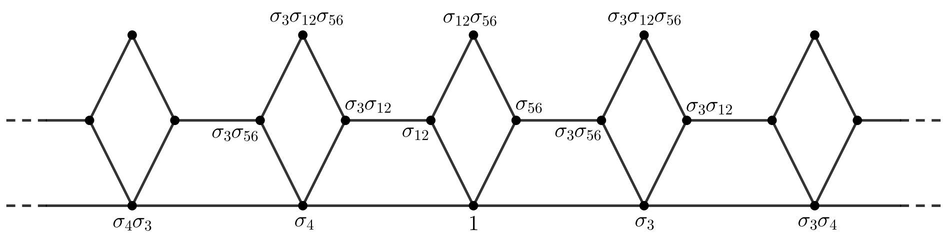


Alternatively, this amounts to describing H as the following graph product:

For convenience, we will write $\sigma_{12} := \sigma_1\sigma_2$ and $\sigma_{56} := \sigma_5\sigma_6$. Consider the action of H on the quasi-median graph QM given by this decomposition, as described in Section 6.1. Let $\mathcal{R}$ be the collection of all the hyperplanes of QM labelled by σ_{12} or σ_{56} and let Λ denote the subgraph $\langle \sigma_3, \sigma_4\rangle \subset \mathrm{QM}$. Notice that Λ is a connected component of $\mathrm{QM}\backslash\backslash\mathcal{R}$. Of course, since $\langle \sigma_3, \sigma_4\rangle$ is an infinite dihedral group, Λ is just a bi-infinite line; namely

$$\ldots,\ \sigma_4,\ \sigma_3\sigma_4,\ \sigma_4\sigma_3,\ \sigma_4,\ 1,\ \sigma_3,\ \sigma_3\sigma_4,\ \sigma_3\sigma_4\sigma_3,\ \ldots$$

The hyperplanes of $\mathcal{R}$ tangent to Λ are the $\langle \sigma_3, \sigma_4 \rangle$-translates of $J_{\sigma_{12}}$ and $J_{\sigma_{56}}$. The configuration in QM is the following:



Thus, the crossing graph of the hyperplanes of $\mathcal{R}$ tangent to Λ is also a bi-infinite line, and $\langle \sigma_3, \sigma_4 \rangle$ acts on it through two reflections fixing two adjacent vertices. Applying Proposition 6.8, we obtain the decomposition:

$$H = \langle \text{conjugates of } \sigma_1\sigma_2 \text{ and } \sigma_5\sigma_6 \rangle \rtimes \langle \sigma_3, \sigma_4 \rangle \simeq A(\mathbb{L}) \rtimes \mathbb{D}_\infty$$

where $\mathbb{D}_\infty$ acts on $\mathbb{Z}$ via two reflections (corresponding to σ_3 and σ_4) fixing 0 and 1 respectively. We conclude that the index-2 subgroup $\langle \sigma_1\sigma_2, \sigma_3\sigma_4, \sigma_5\sigma_6 \rangle$ of H, which is therefore an index-8 subgroup of FB_7, is isomorphic to the index-2 subgroup $A(\mathbb{L}) \rtimes 2\mathbb{Z}$ of the lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$. □

6.3. ***Some flip manifolds.*** In this section, our goal is to describe $A(\mathbb{L}) \rtimes \mathbb{Z}$ and $A(P_4)$ as fundamental groups of two flip manifolds M_1 and M_2, and then to construct a third flip manifold M_0 that is a common finite-sheeted cover of M_1 and M_2. Since $A(\mathbb{L}) \rtimes \mathbb{Z}$ is commensurable to PFB_7 according to Proposition 6.11, this will allow us to deduce that PFB_7 is commensurable to $A(P_4)$. Here, by a flip manifold, we mean a 3-manifold (possibly with boundary) obtained by gluing copies of $\mathbb{S}^1 \times \mathbb{S}(b)$, where $\mathbb{S}(b)$ denotes a punctured sphere with $b \geq 1$ boundary components. Two copies of $\mathbb{S}^1 \times \mathbb{S}(b)$ will be always glued along a torus boundary component in such a way that meridians and longitudinals are switched.

Let us begin by describing $A(\mathbb{L}) \rtimes \mathbb{Z}$ as the fundamental group of the flip manifold M_1 given by Figure 3. The fundamental group of $\mathbb{S}^1 \times \mathbb{S}(3)$ can be identified with $\mathbb{Z} \times \mathbb{F}_2 = \langle a, b, c \mid [a, c] = [b, c] = 1 \rangle$ where a (resp. b) corresponds to the red (resp. blue) boundary component of $\mathbb{S}(3)$ and where c is given by the $\mathbb{S}^1$-factor. The gluing identifies c with b and a with c, so the fundamental group of M_1 admits

$$\langle a, b, c, t \mid [a, c] = [b, c] = 1,\ tct^{-1} = b,\ tat^{-1} = c \rangle$$

as a presentation, which can be simplified as $\langle a, t \mid [a, tat^{-1}] = 1 \rangle$. We conclude from Fact 6.10 that $A(\mathbb{L}) \rtimes \mathbb{Z}$ is indeed isomorphic to the fundamental group of M_1.

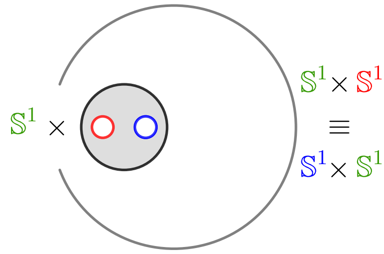


Figure 3. The flip manifold M_1 whose fundamental group is $A(\mathbb{L}) \rtimes \mathbb{Z}$.

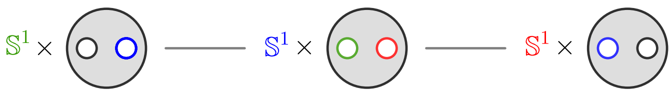


Figure 4. The flip manifold M_2 whose fundamental group is $A(P_4)$.

Next, let us describe $A(P_4)$ as the fundamental group of the flip manifold M_2 given by Figure 4. The three copies of $\mathbb{S}^1 \times \mathbb{S}(3)$ have fundamental groups isomorphic to $\mathbb{Z} \times \mathbb{F}_2$, say respectively $\langle a_i, b_i, c_i \mid [a_i, c_i] = [b_i, c_i] = 1 \rangle$ for $i = 1, 2, 3$ where a_i corresponds to the left (resp. right) inner boundary component of $\mathbb{S}(3)$ and where c_i is given by the $\mathbb{S}^1$-factor. The first gluing identifies c_1 with a_2 and b_1 with c_2, and the second gluing identifies c_2 with a_3 and b_2 with c_3. Therefore, the fundamental group of M_2 admits

$$\left\langle \begin{array}{l} a_1, a_2, a_3, \\ b_1, b_2, b_3, \\ c_1, c_2, c_3 \end{array} \middle| \begin{array}{l} [a_1, c_1] = [b_1, c_1] = [a_2, c_2] = [b_2, c_2] = [a_3, c_3] = [b_3, c_3] = 1, \\ c_1 = a_2, \;\; b_1 = c_2, \;\; c_2 = a_3, \;\; b_2 = c_3 \end{array} \right\rangle$$

as a presentation, which can be simplified as

$$\langle a_1, a_2, b_2, c_2, b_3 \mid [a_1, a_2] = [a_2, c_2] = [c_2, b_2] = [b_2, b_3] = 1 \rangle.$$

This is clearly a presentation of $A(P_4)$, concluding the proof that $A(P_4)$ is indeed isomorphic to the fundamental group of M_2.

Finally, let us construct a common finite cover of M_1 and M_2. Our flip manifold M_0 is described by Figure 5. In order to describe the covering maps $M_0 \to M_1, M_2$, we need two specific covering maps $\alpha, \beta : \mathbb{S}(6) \to \mathbb{S}(3)$. They are respectively described by Figures 6 and 7. It is worth noticing that the restriction of α to each boundary component is a 2-sheeted cover; and that the restriction of β to a red (resp. blue) boundary component is a 4-sheeted (resp. 1-sheeted) cover.

The covering map $\mu : M_0 \to M_1$ is defined as follows. The two pieces $\mathbb{S}^1 \times \mathbb{S}(6)$ of M_0 are sent to the piece $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_1 through $(1, \alpha)$. And the four pieces $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_0 are sent to the piece $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_1 through $(2, \mathrm{id})$. By

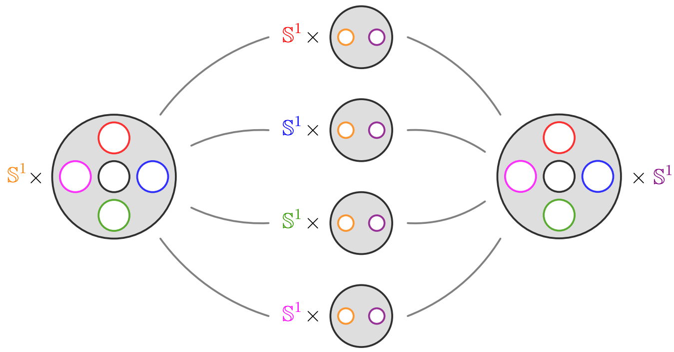


Figure 5. The flip manifold M_0, a common cover of M_1 and M_2.

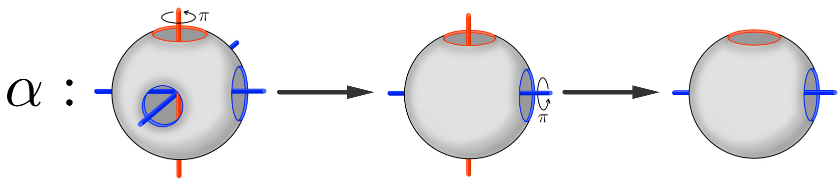


Figure 6. The 4-sheeted covering map $\alpha : \mathbb{S}(6) \to \mathbb{S}(3)$.

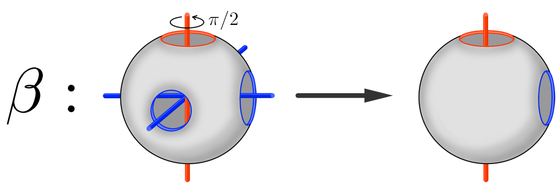


Figure 7. The 4-sheeted covering map $\beta : \mathbb{S}(6) \to \mathbb{S}(3)$.

construction, these isolated maps are compatible with the gluings defining M_0, and we get a 4-sheeted covering map $\mu : M_0 \to M_1$.

The covering map $\nu : M_0 \to M_2$ is defined as follows. The two pieces $\mathbb{S}^1 \times \mathbb{S}(6)$ of M_0 are sent to the left and right pieces $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_2 through $(1, \beta)$. And the four pieces $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_0 are sent to the middle piece $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_2 through $(4, \beta)$. By construction, these isolated maps are compatible with the gluings defining M_0, and we get a 4-sheeted covering map $\nu : M_0 \to M_2$.

Since we have constructed 4-sheeted covering maps from M_0 to M_1 and M_2, the fundamental groups of M_1 and M_2 share isomorphic index-4 subgroups. Hence:

Proposition 6.12. *The lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$ and the right-angled Artin group $A(P_4)$ share an isomorphic index-4 subgroup.*

Combined with Proposition 6.11, we conclude that Theorem 6.1 holds. □

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References

[Antolín and Minasyan 2015] Y. Antolín and A. Minasyan, "Tits alternatives for graph products", *J. Reine Angew. Math.* **704** (2015), 55–83. MR

[Antolín et al. 2016] Y. Antolín, A. Minasyan, and A. Sisto, "Commensurating endomorphisms of acylindrically hyperbolic groups and applications", *Groups Geom. Dyn.* **10**:4 (2016), 1149–1210. MR

[Bandelt et al. 1994] H.-J. Bandelt, H. Mulder, and E. Wilkeit, "Quasi-median graphs and algebras", *J. Graph Theory* **18**:7 (1994), 681–703. MR

[Bardakov et al. 2019] V. Bardakov, M. Singh, and A. Vesnin, "Structural aspects of twin and pure twin groups", *Geom. Dedicata* **203** (2019), 135–154. MR

[Barkauskas 2007] D. Barkauskas, "Centralizers in graph products of groups", *J. Algebra* **312**:1 (2007), 9–32. MR

[Behrstock 2019] J. Behrstock, "A counterexample to questions about boundaries, stabilities, and commensurability", in *Beyond hyperbolicity*, London Math. Soc. Lecture Note Ser. **454**, Cambridge Univ. Press, 2019.

[Behrstock and Charney 2012] J. Behrstock and R. Charney, "Divergence and quasimorphisms of right-angled Artin groups", *Math. Ann.* **352**:2 (2012), 339–356. MR

[Behrstock and Neumann 2008] J. Behrstock and W. Neumann, "Quasi-isometric classification of graph manifold groups", *Duke Math. J.* **141**:2 (2008), 217–240. MR

[Bellingeri et al. 2024] P. Bellingeri, H. Chemin, and V. Lebed, "Cactus groups, twin groups, and right-angled Artin groups", *J. Algebraic Combin.* **59**:1 (2024), 153–178. MR

[Cashen and Edletzberger 2024] C. Cashen and A. Edletzberger, "Visual right-angled Artin subgroups of two-dimensional right-angled Coxeter groups", preprint, 2024. To appear in *J. Group Theory*. arXiv 2405.04817

[Charney and Sultan 2015] R. Charney and H. Sultan, "Contracting boundaries of CAT(0) spaces", *J. Topol.* **8**:1 (2015), 93–117. MR

[Cordes and Hume 2017] M. Cordes and D. Hume, "Stability and the Morse boundary", *J. Lond. Math. Soc.* (2) **95**:3 (2017), 963–988. MR

[Dahmani et al. 2017] F. Dahmani, V. Guirardel, and D. Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Mem. Amer. Math. Soc. **1156**, 2017. MR

[Dani and Levcovitz 2024] P. Dani and I. Levcovitz, "Right-angled Artin subgroups of right-angled Coxeter and Artin groups", *Algebr. Geom. Topol.* **24**:2 (2024), 755–802. MR

[Dani and Thomas 2015] P. Dani and A. Thomas, "Divergence in right-angled Coxeter groups", *Trans. Amer. Math. Soc.* **367**:5 (2015), 3549–3577. MR

[Davis and Januszkiewicz 2000] M. Davis and T. Januszkiewicz, "Right-angled Artin groups are commensurable with right-angled Coxeter groups", *J. Pure Appl. Algebra* **153**:3 (2000), 229–235. MR

[Dey and Gongopadhyay 2019] S. Dey and K. Gongopadhyay, "Commutator subgroups of twin groups and Grothendieck's cartographical groups", *J. Algebra* **530** (2019), 215–234. MR

[Farley 2021] D. Farley, "The planar pure braid group is a diagram group", preprint, 2021. arXiv 2109.02815

[Genevois 2017] A. Genevois, *Cubical-like geometry of quasi-median graphs and applications to geometric group theory*, Ph.D. thesis, Aix-Marseille Université, 2017. arXiv 1712.01618

[Genevois 2018] A. Genevois, "Hyperplanes of Squier's cube complexes", *Algebr. Geom. Topol.* **18**:6 (2018), 3205–3256. MR

[Genevois 2019] A. Genevois, "On the geometry of van Kampen diagrams of graph products of groups", preprint, 2019. arXiv 1901.04538

[Genevois 2020] A. Genevois, "Contracting isometries of CAT(0) cube complexes and acylindrical hyperbolicity of diagram groups", *Algebr. Geom. Topol.* **20**:1 (2020), 49–134. MR

[Genevois 2022a] A. Genevois, "Cactus groups from the viewpoint of geometric group theory", preprint, 2022. arXiv 2212.03494

[Genevois 2022b] A. Genevois, "Quasi-isometrically rigid subgroups in right-angled Coxeter groups", *Algebr. Geom. Topol.* **22**:2 (2022), 657–708. MR

[Genevois 2022c] A. Genevois, "Rotation groups, mediangle graphs, and periagroups: a unified point of view on Coxeter groups and graph products of groups", preprint, 2022. arXiv 2212.06421

[Genevois 2023] A. Genevois, "Why CAT(0) cube complexes should be replaced with median graphs", preprint, 2023. arXiv 2309.02070

[Genevois 2024] A. Genevois, *Automorphisms of graph products of groups and acylindrical hyperbolicity*, Mem. Amer. Math. Soc. **1509**, 2024. MR

[Genevois and Martin 2019] A. Genevois and A. Martin, "Automorphisms of graph products of groups from a geometric perspective", *Proc. Lond. Math. Soc.* (3) **119**:6 (2019), 1745–1779. MR

[Genevois and Tessera 2024] A. Genevois and R. Tessera, "Lamplighter-like geometry of groups", preprint, 2024. arXiv 2401.13520

[Green 1990] E. R. Green, *Graph products of groups*, Ph.D. thesis, University of Leeds, 1990, available at http://etheses.whiterose.ac.uk/236.

[Harshman and Knapp 2020] N. Harshman and A. Knapp, "Anyons from three-body hard-core interactions in one dimension", *Ann. Physics* **412** (2020), 168003, 18. MR

[Hsu and Wise 1999] T. Hsu and D. Wise, "On linear and residual properties of graph products", *Michigan Math. J.* **46** (1999), 251–259.

[Januszkiewicz and Świątkowski 2001] T. Januszkiewicz and J. Świątkowski, "Commensurability of graph products", *Algebr. Geom. Topol.* **1** (2001), 587–603. MR

[Khovanov 1996] M. Khovanov, "Real $K(\pi, 1)$ arrangements from finite root systems", *Math. Res. Lett.* **3**:2 (1996), 261–274. MR

[Khovanov 1997] M. Khovanov, "Doodle groups", *Trans. Amer. Math. Soc.* **349**:6 (1997), 2297–2315. MR

[Kropholler and Martino 2016] P. Kropholler and A. Martino, "Graph-wreath products and finiteness conditions", *J. Pure Appl. Algebra* **220**:1 (2016), 422–434. MR

[LaForge 2017] G. LaForge, *Visible Artin subgroups of right-angled Coxeter groups*, Ph.D. thesis, 2017, available at https://www.proquest.com/docview/1986002761. MR

[Levcovitz 2022] I. Levcovitz, "Characterizing divergence and thickness in right-angled Coxeter groups", *J. Topol.* **15**:4 (2022), 2143–2173. MR

[Merkov 1999] A. Merkov, "Vassiliev invariants classify flat braids", pp. 83–102 in *Differential and symplectic topology of knots and curves*, Amer. Math. Soc. Transl. Ser. 2 **190**, Amer. Math. Soc., Providence, RI, 1999. MR

[Minasyan and Osin 2015] A. Minasyan and D. Osin, "Acylindrical hyperbolicity of groups acting on trees", *Math. Ann.* **362**:3-4 (2015), 1055–1105. MR

[Mostovoy 2020] J. Mostovoy, "A presentation for the planar pure braid group", preprint, 2020. arXiv 2006.08007

[Mostovoy and Roque-Márquez 2020] J. Mostovoy and C. Roque-Márquez, "Planar pure braids on six strands", *J. Knot Theory Ramifications* **29**:1 (2020), 1950097, 11. MR

[Naik et al. 2020a] T. Naik, N. Nanda, and M. Singh, "Conjugacy classes and automorphisms of twin groups", *Forum Math.* **32**:5 (2020), 1095–1108. MR

[Naik et al. 2020b] T. Naik, N. Nanda, and M. Singh, "Some remarks on twin groups", *J. Knot Theory Ramifications* **29**:10 (2020), 2042006, 14. MR

[Naik et al. 2023] T. Naik, N. Nanda, and M. Singh, "Structure and automorphisms of pure virtual twin groups", *Monatsh. Math.* **202**:3 (2023), 555–582. MR

[Naik et al. 2024] T. Naik, N. Nanda, and M. Singh, "Virtual planar braid groups and permutations", *J. Group Theory* **27**:3 (2024), 443–483. MR

[Nguyen and Tran 2019] H. T. Nguyen and H. C. Tran, "On the coarse geometry of certain right-angled Coxeter groups", *Algebr. Geom. Topol.* **19**:6 (2019), 3075–3118. MR

[Osin 2018] D. Osin, "Groups acting acylindrically on hyperbolic spaces", pp. 919–939 in *Proceedings of the International Congress of Mathematicians*, *vol. II*: *Invited lectures* (Rio de Janeiro, 2018), edited by B. Sirakov et al., World Sci. Publ., Hackensack, NJ, 2018. MR

[Scott and Wall 1979] P. Scott and T. Wall, "Topological methods in group theory", pp. 137–203 in *Homological group theory* (Durham, 1977), London Math. Soc. Lecture Note Ser. **36**, Cambridge Univ. Press, 1979. MR

[Serre 1980] J.-P. Serre, *Trees*, Springer, 1980. MR

[Servatius et al. 1989] H. Servatius, C. Droms, and B. Servatius, "Surface subgroups of graph groups", *Proc. Amer. Math. Soc.* **106**:3 (1989), 573–578. MR

[Shabat and Voevodsky 1990] G. Shabat and V. Voevodsky, "Drawing curves over number fields", pp. 199–227 in *The Grothendieck Festschrift*, vol. III, Progr. Math. **88**, Birkhäuser, Boston, 1990. MR

[Voevodsky 1990] V. Voevodsky, "Flags and Grothendieck cartographical group in higher dimensions", CSTARCI Math. Preprints, 1990.

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CLASSIFICATION OF UNIMODAL ISOLATED COMPLETE INTERSECTION SINGULARITIES IN POSITIVE CHARACTERISTIC

HONGRUI MA, STEPHEN S.-T. YAU AND HUAIQING ZUO

We classify unimodal isolated complete intersection singularities in arbitrary characteristic under contact equivalence. The classification over $\mathbb{C}$ has been done by A. Dimca and C. G. Gibson. We continue and generalize their work. To complete the classification, we generalized the complete transversal method into positive characteristic field, which is also useful in many other classification problem.

1. Introduction

Classification is one of the oldest topics in singularity theory. The modality of singularities for real and complex hypersurfaces was first introduced by V. I. Arnold in [2]. He also finished the classification of hypersurface singularities with small modality over $\mathbb{C}$ in [1]. G. M. Greuel and H. D. Nguyen generalized the notion of modality to the algebraic setting in [8], so that one can define modality over any algebraically closed field of arbitrary characteristic. They also classified simple (modality 0) hypersurface singularities in positive characteristic field under right equivalence.

The classification of isolated complete intersection singularities (ICIS) under contact equivalence was studied in the 1980s. Assume $(X, 0) = (f^{-1}(0), 0)$ to be a complete intersection germ with an isolated singularity defined by $f : F^n \to F^p$. We call such germs $I_{n,p}$. M. Giusti has shown that only $I_{1,1}$, $I_{2,2}$ and $I_{3,2}$ can be simple in [7]. He then classified all simple ICIS over $F = \mathbb{C}$. The classification of unimodal (i.e., modality 1) germs from plane to plane ($I_{2,2}$ case) was completed by A. Dimca and C. G. Gibson. C. T. C. Wall then classified unimodal germs of $I_{n,p}$ with $n > p$. Recently, T. H. Pham, G. Pfister, and G. M. Greuel generalized the modality of hypersurface singularities to ICIS, and classified zero-dimensional ICIS ($I_{2,2}$ case) over any algebraically closed field of arbitrary characteristic. In this

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paper, we continue their work. We use a different method to classify unimodal zero-dimensional ICIS of $I_{2,2}$ over any algebraically closed field of arbitrary characteristic. In Corollary 2.24, we show that for all zero-dimensional ICIS, only $I_{2,2}$ and $I_{3,3}$ can be unimodal. Unfortunately, the classification of $I_{3,3}$ seems very complicated and needs more new tools.

In Dimca and Gibson's previous work [5], the finite determinacy theorem works well in zero characteristic fields. T. H. Pham and G. M. Greuel generalized it to positive characteristic in [10]. Although the given boundary is sharp, it is still insufficient to deal with complicated (higher order) problems. For example, let $h = (x^3 + xy^3 + y^5, x^2y + y^4 + y^5)$ be the 5-jet of an ICIS f over an algebraically closed field F. If $\operatorname{char} F = 0$, then h is 5-determined. Therefore f is contact equivalent to $(x^3 + xy^3 + y^5, x^2y + y^4 + y^5)$, and then to $(x^3 + xy^3 + y^5, x^2y + y^4)$ after a transformation. Otherwise, we can only know that h is 7-determined in positive characteristic, which gives few idea about the form of f.

To solve the problem, we develop new tools. We generalize the complete transversal method introduced by Bruce, Kirk and du Plessis in [4] to the case of a positive characteristic field; this is useful in many classification problem (see Corollary 2.17). It can be used for semi-quasihomogeneous singularities over fields of arbitrary characteristic. For the above example, f is a semi-quasihomogeneous singularity with initial term $(x^3 + xy^3, x^2y + y^4)$. Using our method we can show that f is contact equivalent to $(x^3 + xy^3 + y^5, x^2y + y^4)$.

The main result is Theorem 8.4. Surprisingly, the classification result in positive characteristic turns out to be similar as the zero characteristic case except for some special characteristics.

In the following, we let F be an algebraically closed field with arbitrary characteristic, and write $R = F[\![x_1, \dots, x_n]\!]$, $\mathfrak{m} = \langle x_1, \dots, x_n \rangle \subset R$ and $p = \operatorname{char}(F)$.

2. Basic settings

We first talk about some basic concepts of ICIS. Then we introduce the notion of finite determinacy in positive characteristic from [10]. After that, we generalize the complete transversal method into positive characteristic. The estimation of modality is also given in this section.

2.1. *Basic concepts.*

Definition 2.1. (1) An ideal $I \subset R$ defines a complete intersection if I can be generated by $f_1, \dots, f_m$ with $f_i \in \mathfrak{m}$ for all i such that f_i is a non-zerodivisor of $R/\langle f_1, \dots, f_{i-1} \rangle$ for $i = 1, \dots, m$. Then $\dim R/I = n - m$.

(2) We call $f = (f_1, \dots, f_m)$ an isolated complete intersection singularity (ICIS) if $I = \langle f_1, \dots, f_m \rangle$ defines a complete intersection and there exists $k \in \mathbb{N}$ such that

$\mathfrak{m}^k \subset I + I_m(J(f))$, where $J(f) = (\partial f_i/\partial x_j)_{ij}$ is the $m \times n$ Jacobian matrix and $I_m(J(f))$ is the ideal generated by all $m \times m$ minors of $J(f)$. Define

$$I_{m,n} = \{f = (f_1, \ldots, f_m) \in R^m \mid f \text{ is an ICIS with codimension } n - m\}.$$

In this article, we focus on $I_{2,2}$, which denotes the zero-dimensional isolated complete intersection singularity in the plane.

Remark 2.2. A complete intersection f is isolated if and only if the corresponding Tjurina number

$$\tau(f) = \dim_F R^m \Big/ \Big(\langle f_1, \ldots, f_m\rangle \cdot R^m + \Big\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big\rangle\Big)$$

is finite.

Definition 2.3. The contact group $\mathcal{K}$ is defined as

$$\mathcal{K} = \mathrm{GL}(m, R) \rtimes \mathrm{Aut}(R),$$

and the action of $\mathcal{K}$ on R^m is defined as

$$(U, \phi, f) \mapsto U \cdot \phi(f),$$

with $U \in \mathrm{GL}(m, R)$, $\phi \in \mathrm{Aut}(R)$, $f = (f_1, \ldots, f_m) \in R^m$ and

$$\phi(f) = (f_1(\phi(\boldsymbol{x})), \ldots, f_m(\phi(\boldsymbol{x})),$$

where $\phi(\boldsymbol{x}) = (\phi(x_1), \ldots, \phi(x_n))$.

Let f and g define two isolated complete intersections of the same codimension $n - m$. f is called contact equivalent to g, denoted by $f \sim g$, if $g \in \mathcal{K}f$, that is, if there exists $U \in \mathrm{GL}(m, R)$ and $\phi \in \mathrm{Aut}(R)$ such that $g = U \cdot \phi(f)$.

2.2. ***Tangent image and finite determinacy.*** To classify the ICIS under contact equivalence, we need to work on jet spaces.

Definition 2.4. (1) The k-jet space of R^m is defined as $J_k = R^m/\mathfrak{m}^{k+1}R^m$. For $f \in R^m$, the k-jet of f is the image in J_k, denoted by $j_k(f)$. Let $\pi : J_l \to J_k$ be the natural projection and denote the kernel as $P_{k,l}$. If $f \in J_k$ is a k-jet, we denote the submanifold $J_l(f) = f + P_{k,l}$.

(2) We say that f is k-determined if for any $g \in R^m$ with $j_k(g) = j_k(f)$, we always have $g \sim f$.

Let $J_k = R^m/\mathfrak{m}^{k+1}R^m$ denote the k-jet space of R^m. Let

$$\mathcal{K}_k = \{(j_k(U), j_k(\phi)) \mid U \in \mathrm{GL}(m, R), \phi \in \mathrm{Aut}(R)\}$$

be the k-jet algebraic group, with the algebraic action of $\mathcal{K}_k$ on the affine space J_k defined by

$$(j_k(U),\, j_k(\phi),\, j_k(f)) \mapsto j_k(U\cdot\phi(f)).$$

The tangent space $T_e(\mathcal{K}_k)$ of the algebraic group $\mathcal{K}_k$ has a natural Lie algebra structure (see [6, Chapter 4]). The orbit map $\pi : \mathcal{K}_k \to \mathcal{K}_k \cdot f$ induces the tangent map $d\pi : \mathrm{Lie}(\mathcal{K}_k) \to T_f(\mathcal{K}_k f)$. We denote the image of $d\pi$ by $\widetilde{T}_f(\mathcal{K}_k f)$, which coincides with $\mathrm{Lie}(\mathcal{K}_k)\cdot f$. When char $F = 0$, $d\pi$ is surjective and

$$\widetilde{T}_f(\mathcal{K}_k f) = T_f(\mathcal{K}_k f) \tag{2-1}$$

(and therefore $\widetilde{T}_f(\mathcal{K} f) = T_f(\mathcal{K} f)$). But when char $F > 0$, (2-1) may not hold. For details one can see [10, Section 2].

The tangent image is computed in [10] as follows.

Proposition 2.5. *The tangent image is identified with the submodule*

$$\widetilde{T}_f(\mathcal{K}_k f) = \Big(\langle f_1, \dots, f_m\rangle \cdot R^m + \mathfrak{m}\cdot\Big\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big\rangle + \mathfrak{m}^{k+1}R^m\Big)\Big/\mathfrak{m}^{k+1}R^m.$$

The tangent image at f to the orbit $\mathcal{K} f$ is the submodule

$$\widetilde{T}_f(\mathcal{K} f) = \langle f_1, \dots, f_m\rangle \cdot R^m + \mathfrak{m}\cdot\Big\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big\rangle,$$

where $\langle f_1, \dots, f_m\rangle$ is regarded as an ideal of R and $\Big\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big\rangle$ is regarded as an ideal of R^m.

Finite determinacy is strongly related to the tangent image.

Theorem 2.6 (cf. [10, Theorem 3.2]). *Let $f = (f_1, \dots, f_m) \in R^m$. If there exists $k \in \mathbb{N}$ such that*

$$\mathfrak{m}^{k+2}\cdot R^m \subset \mathfrak{m}\cdot \widetilde{T}_f(\mathcal{K} f),$$

then f is $(2k-\mathrm{ord}(f)+2)$-determined, where $\mathrm{ord}(f) = \min\{\mathrm{ord}(f_i) \mid i = 1, \dots, m\}$. That is, any $g \in R^m$ such that $j_{2k-\mathrm{ord}(f)+2}(g) = j_{2k-\mathrm{ord}(f)+2}(f)$ satisfies $g \sim f$.

Remark 2.7. If f is an isolated complete intersection singularity, then f is $(2\tau(f) - \mathrm{ord}(f)+2)$-determined.

2.3. *Complete transversals.* We next introduce the complete transversal method.

Let $a = (a_1, \dots, a_n)$ be a given sequence of positive integers and $d = (d_1, \dots, d_m)$ a given sequence of nonnegative integers. Then $f = (f_1, \dots, f_m)$ is said to be weighted homogeneous of degree r (with respect to $(a; d)$) if

$$f_i(t^{a_1}x_1, \dots, t^{a_n}x_n) = t^{r+d_i} f_i(x_1, \dots, x_n)$$

for any $t \in F$ and $i = 1, 2, \dots, m$.

Let f be a k-jet in J_k, weighted homogeneous of degree 0 with respect to $(a; d)$.

Assume that

$$\max(d_i) < (k+1)\min(a_j) \quad \text{or} \quad \min(d_i) > (k+1)\max(a_j). \tag{2-2}$$

For $l > k$, let $P_{k,l}$, $J_l(f)$ be the subsets of J_l from Definition 2.4(1). We have the following useful theorem from [5].

Theorem 2.8. *For f as above, let $C \subset P_{k,l}$ be a linear subspace of $P_{k,l}$ satisfying*

$$P_{k,l} \subset C + \widetilde{T}_f(\mathcal{K}_l f) \cap P_{k,l};$$

we call C a complete transversal. Then C has the following property: every $g \in J_l(f)$ is in the same $\mathcal{K}_l$-orbit as an l-jet of the form $f + c$, for some $c \in C$.

Proof. For the case of $F = \mathbb{C}$, this is Proposition 1.3 in [5]. The original proof there carries over to positive characteristic without changes. □

Remark 2.9. Condition (2-2) is necessary. In its absence, one finds a counterexample in [3, Remark 1]: Let $f = y^2 + xy^3 \in F[[x, y]]$ with char $F = 2$. One can check that f does not satisfy (2-2). Computation shows $\widetilde{T}_f(\mathcal{K}_l f) = \mathfrak{m}^4/\mathfrak{m}^l$, which implies that the complete transversal C equals 0 for any $l > 4$ and then for any $g \in f + \mathfrak{m}^5$, $g \sim f$. But in fact $f + x^5$ is not contact equivalent to f.

2.4. ***Complete transversals and homogeneous filtrations.*** In this subsection we introduce the generalization of the complete transversal method following [4]. Then we generalize the work of [4] to any positive characteristic field.

Let $F^r_{a,d}R^m$ denote the submodule of R^m generated by the monomials of degree equal or greater than r with respect to $(a; d)$. The sequence $\{F^r_{a,d}R^m\}_{r\geq 0}$ defines a filtration of the module $F^0_{a,d}R^m$.

We introduce a filtration of the contact group $\mathcal{K}$ compatible with the weighted filtration. For details one may see [4, Section 2.3].

Definition 2.10. (i) For $r \geq 0$, define

$$F^r\mathcal{R} = (I_n + F^r_{a,a}R^n) \cap \mathcal{R}.$$

(ii) For $r \geq 0$, define

$$F^r\mathcal{C} = (I_{n+m} + F^r_{a\cup d,a\cup d}\widetilde{R}^{n+m}) \cap \mathcal{C},$$

where

$$\widetilde{R} = F[[x_1, \ldots, x_n, y_1, \ldots, y_m]]$$

and $a \cup d$ denotes the $(n+m)$-tuple $(a_1, \ldots, a_n, d_1, \ldots, d_m)$.

(iii) Since the contact group $\mathcal{K}$ equals $\mathcal{R} \rtimes \mathcal{C}$, we define

$$F^r\mathcal{K} = F^r\mathcal{R} \rtimes F^r\mathcal{C}.$$

Remark 2.11. For a survey of the standard Mather groups $\mathcal{K}$, $\mathcal{R}$, $\mathcal{C}$, one can refer to [12].

Proposition 2.12. (i) *$F^r\mathcal{K}$ respects the filtration $\{F^r_{a,d}R^m\}$; that is, for every $r, s \geq 0$, $(U, \phi) \in F^r\mathcal{K}$, $f \in F^sR^m$, $U \cdot \phi(f) \in F^sR^m$, where $F^sR^m = F^s_{a,d}R^m$ is the submodule of R^m generated by the monomials of degree equal or greater than s.*

(ii) *For $r, s, l \leq 0$, the action of $F^r\mathcal{K}$ induces an action on $F^sR^m/F^{s+l}R^m$.*

(iii) *The Lie algebra action satisfies the following: for any $f - g \in F^t_{a,d}R^m$ with $f, g \in F^0_{a,d}R^m$ and $l \in \mathrm{Lie}(F^r_{a,d}R^m)$, we have $l \cdot f - l \cdot g \in F^{r+t}_{a,d}R^m$.*

After a computation of tangent spaces (similar to the one in [10, Proposition 2.5]), the tangent image of $F^r\mathcal{K}f$ can be regarded as

$$\widetilde{T}_f(F^r_{a,d}\mathcal{K} \cdot f) = F^r_{a\cup d,d}(\langle f_1, \dots, f_m\rangle \cdot R^m) + \sum_j F^r_{a,a_j}(\mathfrak{m}) \cdot \frac{\partial f}{\partial x_j}, \tag{2-3}$$

and clearly

$$\widetilde{T}_f(F^r_{a,d}\mathcal{K} \cdot f) \subset T_f(F^r_{a,d}\mathcal{K} \cdot f)$$

also holds. We denote $\langle f_1, \dots, f_m\rangle \cdot R^m + \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right\rangle$ by $\widetilde{T}^e_f(\mathcal{K}f)$, using the same notation found in [10].

We have a similar complete transversal result for F^rR^m and $F^r\mathcal{K}$.

Proposition 2.13. *Let $f \in F^0_{a,d}R^m$. If T is a subspace of $F^r_{a,d}R^m$ satisfying*

$$F^{k+1}_{a,d}R^m \subset T + \mathrm{Lie}(F^1_{a,d}\mathcal{K}) \cdot f + F^{k+2}_{a,d}R^m,$$

then any g with $g - f \in F^{k+1}_{a,d}R^m$ is contact equivalent to $f + t + \overline{f}$ for some $t \in T$ and $\overline{f} \in F^{k+2}_{a,d}R^m$.

To prove the proposition, we first need a lemma, which can be seen as Taylor series in positive characteristic. We omit the proof.

Lemma 2.14. *Let $f(x) = \sum_\alpha a_\alpha x^\alpha \in F[\![x_1, \dots, x_n]\!]$ and $\xi = (\xi_1, \dots, \xi_n) \in F^n$, then*

$$f(x+\xi) = f(x) + \sum_{i=1}^n \xi_i \frac{\partial f}{\partial x_i} + \textit{terms of } f(x) \textit{ with more than two } x_i \textit{ replaced by } \xi_i. \tag{2-4}$$

Now we can prove Proposition 2.13.

Proof. Take the subset $W = F^{k+1}_{a,d}R^m \setminus F^{k+2}_{a,d}R^m$ of R^m, which is a finite-dimensional F-vector space. We claim that for any $f \in F^0_{a,d}R^m$ and $w \in W$, we have

$$\text{(i)} \qquad \mathrm{Lie}(F^1\mathcal{K}) \cdot (f + w) - \mathrm{Lie}(F^1\mathcal{K}) \cdot f \in F^{k+2}_{a,d}R^m$$

and

$$\text{(ii)} \qquad f + \{\mathrm{Lie}(F^1\mathcal{K}) \cdot f \cap W\} + F^{k+2}_{a,d}R^m \subset F^1\mathcal{K} \cdot f \cap \{f + W\} + F^{k+2}_{a,d}R^m.$$

Inclusion (i) follows from Proposition 2.12(iii). For (ii), let $w = (w_1, \dots, w_m)$ be in $\{\mathrm{Lie}(F^1\mathcal{K}) \cdot f \cap W\}$. Then

$$w_j = \sum_{i=1}^{n} \xi_i \frac{\partial f_j}{\partial x_i} + \sum_{i=1}^{m} \lambda_{ji} f_i$$

is weighted homogeneous of degree $k+1$ with respect to $(a; d_j)$. For $r \in R$, write $r = \sum_\alpha r_\alpha x^\alpha$. Let $v_a(r) = \min_{r_\alpha \neq 0}\{v_a(x^\alpha)\}$, where $v_a(x^\alpha)$ denotes the weighted degree of the monomial x^α with respect to $(a; 0)$. Then $v_a(\xi_i) = a_i + k + 1$, $v_a(\lambda_{ji} f_i) = d_j + k + 1$. Hence $v_a(\xi_i) \geq v_a(x_i) + k + 1$ and $v_a(\lambda_{ji}) \geq d_j - d_i + k + 1$.

Let $\phi \in \mathrm{Aut}(R)$ be such that $\phi(x_1, \dots, x_n) = (x_1 - \xi_1, \dots, x_n - \xi_n)$. Set $U = (\lambda_{ji}) \in M_{m\times m}(R)$. We have

$$\phi\Big(f + \sum_{i=1}^{n} \xi_i \frac{\partial f_j}{\partial x_i} - Uw\Big) = f - \sum_{i=1}^{n} \xi_i \frac{\partial f_j}{\partial x_i} + h_1 + \sum_{i=1}^{n} \xi_i \frac{\partial f_j}{\partial x_i} + h_2 - Uw + h_3$$

by Lemma 2.14, where the h_i are the higher-order terms defined in (2-4). We have $h_i \in F^{k+2}_{a,d} R^m$ for $i = 1, 2, 3$ since the ξ_i appear twice in each term and $v_a(\xi_i) \geq v_a(x_i) + k + 1$. Moreover $Uw \in F^{k+2}_{a,d} R^m$ since for each j we have $v_a(\lambda_{ji} w_i) = v_a(\lambda_{ji}) + v_a(w_i) \geq d_j - d_i + k + 1 + d_i + k + 1 \geq d_j + k + 2$.

Set $\widetilde{w} = h_1 + h_2 + h_3 - Uw$. By the preceding discussion, $\widetilde{w} \in F^{k+2}_{a,d} R^m$. Without loss of generality, we can assume $v_a(f_1) \leq \cdots \leq v_a(f_m)$. Then $U(0)$ is an upper triangular matrix all of whose principal diagonal elements are zero. Hence $\mathrm{Id} - U$ must be invertible. We have $f + w \sim (\mathrm{Id} - U) \cdot \phi(f + w) = f + \widetilde{w}$. Denote $g = (\mathrm{Id} - U, \phi) \in \mathcal{K}$, we have $f + w = g^{-1}(f) + g^{-1}(\widetilde{w})$.

To finish (ii), it remains to show $g \in F^1\mathcal{K}$. Since $\xi = (\xi_1, \dots, \xi_n) \in F^1_{a,a} R^n$ we have $\phi \in I_n + F^1_{a,a} R^n = F^1\mathcal{R}$ by Definition 2.10. Similarly $\mathrm{Id} - U \in I_m + F^1_{a\cup d, a\cup d} R^m \subset F^1\mathcal{C}$. Now we get $g = (\mathrm{Id} - U, \phi) \in F^1\mathcal{K}$. The claim is proved.

From inclusions (i) and (ii) we have

$$\begin{aligned}
\bigcup_{t\in T} F^1\mathcal{K} \cdot (f + t + F^{k+2}_{a,d} R^m) &\supset \bigcup_{t\in T} \{f + t + \mathrm{Lie}(F^1\mathcal{K}) \cdot (f+t) \cap W + F^{k+2}_{a,d} R^m\} \\
&= \bigcup_{t\in T} \{f + t + \mathrm{Lie}(F^1\mathcal{K}) \cdot f \cap W\} + F^{k+2}_{a,d} R^m \\
&= f + T + \mathrm{Lie}(F^1\mathcal{K}) \cdot f \cap W + F^{k+2}_{a,d} R^m \\
&= f + (T + \mathrm{Lie}(F^1\mathcal{K}) \cdot f) \cap W + F^{k+2}_{a,d} R^m \\
&= f + W + F^{k+2}_{a,d} R^m.
\end{aligned}$$

That is, for any $g = f + w + \bar{g}$ with $w \in W$ and $\bar{g} \in F^{k+2}_{a,d} R^m$, g is contact equivalent to $f + t + \bar{f}$ with $t \in T$ and $\bar{f} \in F^{k+2}_{a,d} R^m$. □

Remark 2.15. In [4], Mather's lemma is used to show that (i) ⇒ (ii). However, the proof of Mather's lemma relies on analysis in the complex field. We can complete the proof in this special case without Mather's lemma, using M. Giusti's proof of [7, Proposition 1].

Using induction, we can show:

Proposition 2.16. *Suppose that f is weighted homogeneous of weight r with respect to $(a; d)$. Take $s > r$. If T is a subspace of $F^{r+1}_{a,d}R^m$ such that*

$$F^{r+1}_{a,d}R^m \subset T + \widetilde{T}_f(F^1_{a,d}\mathcal{K}\cdot f) + F^{s+1}_{a,d}R^m, \tag{2-5}$$

then any g with $g - f \in F^{r+1}_{a,d}R^m$ is $F^1\mathcal{K}$-equivalent to $f + t + \phi$ where $t \in T$ and $\phi \in F^{s+1}_{a,d}R^m$.

Proof. See Theorem 2.28 in [4] and note that $\mathrm{Lie}(F^1_{a,d}\mathcal{K})\cdot f = \widetilde{T}_f(F^1_{a,d}\mathcal{K}\cdot f)$ is shown above. □

For an isolated complete intersection singularity, f is always finite determined. Choosing s sufficiently large in the preceding proof, we get:

Corollary 2.17. *Suppose that f is an ICIS of weight r with respect to (a, d). Let T be a subspace of $F^{r+1}_{a,d}R^m$ such that*

$$F^{r+1}_{a,d}R^m \subset T + \widetilde{T}_f(F^1_{a,d}\mathcal{K}\cdot f).$$

Then any g with $g - f \in F^{r+1}_{a,d}R^m$ is contact equivalent to $f + t$ for some $t \in T$.

2.5. *Modality.* V. I. Arnold introduced the notion of modality in his famous [2] as follows: The modality of a point $x \in X$ under the action of a Lie group G on a manifold X is the smallest m such that a sufficiently small neighborhood of x may be covered by a finite number of m-parameter orbit families. G. M. Greuel and H. D. Nguyen generalized the notion and gave a detailed discussion in [8; 13]. For the definition of the modality of an ICIS, we refer to [11, Remark 1.13(3)].

Definition 2.18. An ICIS is called unimodal if $\mathcal{K}$-mod(f), the $\mathcal{K}$-modality of f, is equal to 1.

In this section, we give some methods to estimate a lower bound for the modality and give a criterion for non-unimodality.

Our first lower bound, from [5], uses complete transversals. It will be useful in next section. Let C be a complete transversal of f in J_l $(l > k)$. For $a \in C$, we define

$$\mathrm{cod}(f + a) = \text{codimension of } \widetilde{T}_f(\mathcal{K}_l f) \cap P_{k,l} \text{ in } P_{k,l}, \tag{2-6}$$

$$\mathrm{cod}_0(f) = \inf_{a\in C}\{\mathrm{cod}(f + a)\}. \tag{2-7}$$

Note that there exists a Zariski open subset $U \subset C$ such that $\mathrm{cod}(f + a) = \mathrm{cod}_0(f)$ if and only if $a \in U$.

Proposition 2.19. *Let $f \in J_k$ be a k-jet of weighted homogeneous type and degree* 0 *with respect to* $(a_1, \dots, a_n; d_1, \dots, d_m)$ *and satisfies condition* (2-2). *For any $a \in U$, $f + a$ has modality* $\operatorname{cod}_0(f)$ *in $J_l(f)$ under the action of the subgroup $\mathcal{K}_l(f)$ of $\mathcal{K}_l$ that stabilizes f. Thus, any jet h in $J_l(f)$ has $\mathcal{K}_l(f)$-$\operatorname{mod}(h) \geq \operatorname{cod}_0(f)$ in J_l.*

Proof. The main idea comes from [5, Proposition 1.4]. We rewrite the proof using tangent images instead of tangent spaces for the sake of fields with positive characteristic.

Find a subspace $\langle e_1, \dots, e_c\rangle \in P_{k,l}$ with $\langle e_1, \dots, e_c\rangle \oplus \widetilde{T}_f(\mathcal{K}_l(f+a)) \cap P_{k,l} = P_{k,l}$. Then $\operatorname{cod}(f+a) = c$. Since $\widetilde{T}_f(\mathcal{K}_l(f+b))$ varies continuously for $b \in U$, we have $\langle e_1, \dots, e_c\rangle \cap \widetilde{T}_f(\mathcal{K}_l(f+b)) = \{0\}$ for b in a Zariski open neighborhood V of a.

Consider

$$\phi : (F^c, 0) \to (J_l(f), f+a), \quad (t_1, \dots, t_c) \mapsto f + a + \sum_{i=1}^{c} t_i e_i. \tag{2-8}$$

We claim that, for any $\mathcal{K}_l(f)$-orbit X in $J_l(f)$, $\phi^{-1}(X)$ consists of finitely many points in a neighborhood of 0; hence ϕ is a minimal deformation of $f + a$ and $\mathcal{K}_l(f)$-$\operatorname{mod}(f+a) = c = \operatorname{cod}_0(f)$.

For any $g \in J_l(f)$, write $g = f + \tilde{g} = (f_1 + \tilde{g}_1, \dots, f_m + \tilde{g}_m)$, where $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_m)$ with weighted degree of $\tilde{g}_i > d_i$. We have

$$\begin{aligned} g(x) \sim g_t(x) &= (t^{-d_1} g_1(t_1^{a_1} x_1, \dots, t_n^{a_n} x_n), \dots, t^{-d_n} g_m(t_1^{a_1} x_1, \dots, t_n^{a_n} x_n)) \\ &= (f_1 + t^{-d_1} \tilde{g}_1(t_1^{a_1} x_1, \dots, t_n^{a_n} x_n), \dots, f_m + t^{-d_n} \tilde{g}_m(t_1^{a_1} x_1, \dots, t_n^{a_n} x_n)). \end{aligned}$$

Then condition (2-2) ensures any neighborhood of f intersects $\mathcal{K}_l(f) \cdot g$; that is, any neighborhood of f intersects all $\mathcal{K}_l(f)$-orbits in $J_l(f)$. Hence for any $\mathcal{K}_l(f)$-orbit X in $J_l(f)$, there exists $b \in V$, $X = \mathcal{K}_l(f) \cdot (f+b)$. But we have

$$\widetilde{T}_{f+b}(\mathcal{K}_l(f) \cdot (f+b)) = \widetilde{T}_{f+b}(\mathcal{K}_l \cdot (f+b)) \cap P_{k,l}; \tag{2-9}$$

hence $\widetilde{T}_{f+b}(\mathcal{K}_l(f) \cdot (f+b)) \cap \langle e_1, \dots, e_c\rangle = \{0\}$, meaning that $\phi^{-1}(X)$ has only finitely many points in a neighborhood of 0. This finishes the proof. □

Remark 2.20. In fact, we can show $\mathcal{K}$-$\operatorname{mod}(f)$ is semicontinuous. That is, let $F(\boldsymbol{x}, \boldsymbol{t}) \in F[\boldsymbol{t}][[\boldsymbol{x}]]$ be such that $F_{\boldsymbol{t}_0} = F(\boldsymbol{x}, \boldsymbol{t}_0)$ is an ICIS for a $\boldsymbol{t}_0 \in F^k$. Then there is a Zariski open subset $U \in F^k$ such that $F(\boldsymbol{x}, \boldsymbol{t})$ is an ICIS for any $\boldsymbol{t} \in U$. The sets $U_i = \{\boldsymbol{t} \in U \mid \mathcal{K}\text{-}\operatorname{mod}(F(\boldsymbol{x}, \boldsymbol{t})) \leq i\}$ are open for all $i \in \mathbb{N}$. Now let $\operatorname{mod}_{\min} = \min\{\mathcal{K}\text{-}\operatorname{mod}(F(\boldsymbol{x}, \boldsymbol{t})) \mid \boldsymbol{t} \in F^k\}$; then $U_{\min} = \{\boldsymbol{t} \in U \mid \mathcal{K}\text{-}\operatorname{mod}(F(\boldsymbol{x}, \boldsymbol{t})) = \operatorname{mod}_{\min}\}$ is open and dense.

We use the following facts from [13] to give a criterion for non-unimodality.

Proposition 2.21. *Let the algebraic group G act on a variety X.*

(1) *If the subvariety $X' \subset X$ is invariant under G and $x \in X'$, then*

$$G\text{-}\operatorname{mod}(x) \text{ in } X \geq G\text{-}\operatorname{mod}(x) \text{ in } X'.$$

(2) *Let additionally the algebraic group G' act on a variety X' and let $p : X \to X'$ be a morphism of varieties, with p open and*

$$G \cdot x \subset p^{-1}(G' \cdot p(x)),\ \forall x \in X.$$

Then

$$G\text{-mod}(x) \geq G'\text{-mod}(p(x)),\ \forall x \in X.$$

(3) *If X is irreducible, for $x \in X$, we have*

$$G\text{-mod}(x) \geq \dim X - \dim G.$$

Proposition 2.22. *Let $f \in I_{m,n}$. Then $\mathcal{K}$-mod$(f) = \mathcal{K}_k$-mod$(j_k(f))$ for k sufficiently large.*

Proof. See [13], Chapter 3. □

The following proposition is the main result of this section.

Proposition 2.23. *Let $f \in I_{n,n}$ with* $\operatorname{ord}(f) = l$ *and f unimodal. Then one of the following holds*:

(1) $n = 2$, $l \leq 3$.

(2) $n = 3$, $l = 2$.

Proof. Choose k large enough and let $X = \mathfrak{m}^l/\mathfrak{m}^{k+1}$. It follows from Propositions 2.22 and 2.21(1) that

$$1 = \mathcal{K}\text{-mod}(f) = \mathcal{K}_k\text{-mod}(f) \text{ in } J_k \geq \mathcal{K}_k\text{-mod}(f) \text{ in } X.$$

Let $X' = \mathfrak{m}^l/\mathfrak{m}^{l+1}$. The action of $\mathcal{K}_k$ on X induces the action of the algebraic group $\mathcal{K}' = \mathrm{GL}(m, F) \times \mathrm{GL}(m, F)$ on X', and it is easy to check that $p : X \to X'$ is open and $\mathcal{K}_k \cdot f \subset p^{-1}(\mathcal{K}' \cdot p(f))$. Then by Proposition 2.21(2) we have

$$\mathcal{K}_k\text{-mod}(f) \text{ in } X \geq \mathcal{K}'\text{-mod}(p(f)) \text{ in } X'.$$

It is easy to see that

$$\dim X' = n\binom{n-1+l}{l},$$

while for any $g \in X'$, $\dim(\mathcal{K}' \cdot g) \leq \dim \mathcal{K}' - 1$, since $\left\{\left(a^l E_n, \frac{1}{a} E_n\right) \mid a \in F^\times\right\} \subset \mathcal{K}'$ stabilizes g, where $F^\times = F \setminus \{0\}$ denotes the units in F.

After a small change of the proof of Proposition 2.21(3) in [13], we have

$$1 \geq \mathcal{K}'\text{-mod}(p(f)) \text{ in } X' \geq \dim X' - (\dim \mathcal{K}' - 1),$$

which is

$$1 \geq n\binom{n-1+l}{l} - (2n^2 - 1).$$

The only solutions are $n = 2$, $l \leq 3$ and $n = 3$, $l = 2$. □

Corollary 2.24. *If f is a unimodal zero-dimensional ICIS, then $f \in I_{2,2}$ or $I_{3,3}$.*

In the following we discuss the case $n = 2$. The $n = 3$ case will be presented in a later article. From now on we assume $R = F[\![x, y]\!]$, $\mathfrak{m} = \langle x, y \rangle$.

3. The classification of order 2

Some classification of ICIS of order 2 has already been discussed in [11], namely, ICIS of modality 0. Here we continue their work and finish the classification of order 2 ICIS with modality 1. In this section $f = (f_1, f_2) \in R^2$.

First, we assume char $F \neq 2$. The following two propositions are from [11].

Proposition 3.1. (i) *If some $j_2(f_i)$ is nondegenerate, then $f \sim (xy, x^n + y^m)$ for some $m, n \geq 2$, which is of modality* 0.

(ii) *If $j_2(f)$ is degenerate, then*

$$f \sim \Big(x^2 + \alpha y^s, \sum_{i \geq t} a_i y^i + x \sum_{j \geq q} b_j y^j\Big), \tag{3-1}$$

where $s \geq 3$, $\alpha \in \{0, 1\}$, $t \geq 2$, $q \geq 1$ and $a_i, b_j \in F$.

Proposition 3.2. *Let $f = (f_1, f_2) = \big(x^2 + \alpha y^s, \sum_{i \geq t} a_i y^i + x \sum_{j \geq q} b_j y^j\big)$ be an ICIS such that $s \geq 3$, $t \geq 2$, $q \geq 1$ and $\alpha \in \{0, 1\}$.*

(i) *If $a_i = 0$ for all i and $b_q \neq 0$, then $\alpha = 1$ and $f \sim (x^2 + y^s, xy^q)$.*

(ii) *If $b_j = 0$ for all j and $a_t \neq 0$, then $f \sim (x^2 + \alpha y^s, y^t)$. If $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$.*

Now assume that $a_t b_q \neq 0$.

(iii) *If $t \leq q$, then $f \sim (x^2 + \alpha y^s, y^t)$. If additionally $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$.*

(iv) *If $t > q$ and $\alpha = 0$, then $f \sim (x^2, y^t + xy^q)$.*

(v) *Let $t > q$ and $\alpha = 1$. Then $f \sim (x^2 + y^s, y^t + exy^q)$ for a suitable unit $e \in F[\![y]\!]$. If $2t - 2q - s \neq 0$ and $p = 0$ (or $p \nmid (2t - 2q - s)$), then $f \sim (x^2 + y^s, y^t + xy^q)$.*

(vi) *If $t = q + 1$ and $p \nmid t$, then $f \sim (x^2 + \alpha y^s, y^t)$.*

Next we give a criterion for modality 1:

Proposition 3.3. *Assume f is of the form* (3-1)*. Then if $s \geq 5$, $t \geq 6$ and $q \geq 3$, then f is of modality at least* 2.

Proof. Let $g = j_3(f) = (x^2, 0)$. Then an open subset in $J_5'(g)$ is formed by jets equivalent to

$$h \sim (x^2 + y^5, axy^3),$$

where$J_5'(g)$ is formed by jets in $J_5(g)$ with $s \geq 5$, $t \geq 6$ and $q \geq 3$.

For all $a, b \in F$, computation shows $(y^4, 0)$, $(0, y^4)$, $(0, y^5) \notin \widetilde{T}_h(\mathcal{K}_5 h)$. Hence the codimension of $\widetilde{T}_h(\mathcal{K}_5 h) \cap P_{3,5}$ in $P_{3,5}$ is at least 2. By Proposition 2.19, $\mathcal{K}$-mod$(f) \geq \mathcal{K}_5$-mod$(f) \geq 2$. □

Remark 3.4. By (3-1), $h \sim (x^2 + \alpha y^5,\ y^6 + axy^3 + bxy^4 + cxy^5)$. One can show, for example, by applying $\phi(y) = y - \frac{1}{6}cx$, that $h \sim (x^2 + \alpha y^5,\ y^6 + axy^3 + bxy^4)$. This simplifies the computation of the codimension, but does not affect the result.

The following proposition is also from [11].

Proposition 3.5. *Let $p = \operatorname{char} F$. The following ICIS are the only candidates for being simple* (*i.e.*, *of modality* 0):

(**0**) $j_2(f_i)$ *is nondegenerate*; *then* $f \sim (xy, x^s + y^m)$, $s, m \geq 2$.
(**1**) $a_i = 0$ *for all* i; *then* $f \sim (x^2 + y^3, xy^q)$, $q \geq 3$ *or* $f \sim (x^2 + y^s, xy^2)$, $s \geq 3$.
(**2**) $b_i = 0$ *for all* i, *and*
 (**2.*a***) $\alpha = 0$ *or* $t \leq s$; *then* $f \sim (x^2, y^t)$, $t = 2, 3, 4$.
 (**2.*b***) $\alpha = 1$ *and* $t > s$; *then* $f \sim (x^2 + y^3, y^t)$, $t \geq 4$.
(**3**) $a_t b_q \neq 0$ *and*
 (**3.*a***) $t \leq q$ *and*
  (**3.*a.i***) $\alpha = 0$ *or* $t \leq s$; *then* $f \sim (x^2, y^t)$, $t = 2, 3, 4$.
  (**3.*a.ii***) $\alpha = 1$ *and* $t > s$; *then* $f \sim (x^2 + y^3, y^t)$, $t \geq 4$.
 (**3.*b***) $t > q$ *and*
  (**3.*b.i***) $t = q + 1$ *and*
   (**3.*b.i.*1**) $p \nmid t$ *and*
    (**3.*b.i.*1.1**) $\alpha = 0$ *or* $t \leq s$; *then* $f \sim (x^2, y^t)$, $t = 2, 3, 4$.
    (**3.*b.i.*1.2**) $\alpha = 1$ *and* $t > s$; *then* $f \sim (x^2 + y^3, y^t)$, $t \geq 4$.
   (**3.*b.i.*2**) $p \mid t$ *and*
    (**3.*b.i.*2.1**) $\alpha = 0$; *then* $f \sim (x^2, xy^2 + y^3)$.
    (**3.*b.i.*2.2**) $\alpha = 1$ *and*
     (**3.*b.i.*2.2.1**) $s = 3$; *then* $f \sim (x^2, xy^2 + y^3)$ *for* $p = t = 3$, *or* $f \sim (x^2 + y^3, y^t + xy^{t-1})$ *for* $t \geq 4$.
     (**3.*b.i.*2.2.2**) $s > 3$, $t = 3$, $p = 3$; *then* $f \sim (x^2, y^3 + xy^2)$.
  (**3.*b.ii***) $t > q + 1$ *and*
   (**3.*b.ii.*1**) $\alpha = 0$ *and*
    (**3.*b.ii.*1.1**) $q = 1$; *then* $f \sim (xy, x^2 + y^{2t-2})$, $t \geq 2$.
    (**3.*b.ii.*1.2**) $q = 2$; *then* $f \sim (x^2 + y^{2t-4}, xy^2)$, $t \geq 4$.
   (**3.*b.ii.*2**) $\alpha = 1$ *and*

(**3.*b*.*ii*.2.1**) *$s \geq 3$, $q \leq 2$; then $f \sim (xy, x^2 + y^m)$ for some m and $q = 1$, and $f \sim (x^2 + y^m, xy^2)$ for some m and $q = 2$.*

(**3.*b*.*ii*.2.2**) *$s = 3$, $t \geq q + 3$; then $f \sim (x^2 + y^3, xy^q)$.*

(**3.*b*.*ii*.2.3**) *$s = 3$, $t = q + 2$; then $f \sim (x^2 + y^3, xy^q + y^{q+2})$.*

Next we classify ICIS of modality 1 based on Proposition 3.5.

Proposition 3.6. *The following ICIS are the only candidates of modality* 1:

symbol	*form*	*condition*
h_q	$(x^2 + y^4, xy^q)$	$q \geq 3$
i	(x^2, y^5)	
$\tilde{i}$	$(x^2, y^5 + xy^3)$	
i^5	$(x^2, y^5 + xy^4)$	$p = 5$
j_t	$(x^2 + y^4, y^t)$	$t \geq 5$
$\tilde{j}_t$	$(x^2 + y^4, y^t + xy^{t-1})$	$t \geq 5,\ p \mid t$
k_q	$(x^2 + y^4, y^{q+3} + xy^q)$	$q \geq 3$
$l_{q,\lambda}$	$(x^2 + y^4, y^{q+2} + \lambda xy^q)$	$q \geq 3, \lambda^2 \notin \{0, -1\}$
$\tilde{l}_{q,t,t'}$	$(x^2 + y^4, y^{q+2} + \lambda xy^q + uxy^t + xy^{t'})$, *where* $u = u_0 + u_1 y^p + u_2 y^{2p} + \cdots$	$\lambda^2 = -1, q \geq 3,$ $t \geq q + 1, t' \geq t + 1,$ $p \mid t - q, p \nmid t' - q$

Table 1. Candidate ICIS of modality 1.

Proof. (**0**) If $j_2(f_i)$ is nondegenerate, then $f \sim (xy, x^s + y^m)$, $s, m \geq 2$, which is simple.

(**1**) $\forall a_i = 0$, then $f \sim (x^2 + y^4, xy^q)$, $q \geq 3$.

(**2**) $\forall b_i = 0$,

(**2.*a***) $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$, $t = 5$.

(**2.*b***) $\alpha = 1$ and $t > s$, then $f \sim (x^2 + y^4, y^t)$, $t \geq 5$.

(**3**) $a_t b_q \neq 0$,

(**3.*a***) If $t \leq q$, then $f \sim (x^2 + \alpha y^s, y^t)$.

(**3.*a*.*i***) $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$, $t = 5$.

(**3.*a*.*ii***) $\alpha = 1$ and $t > s$, then $f \sim (x^2 + y^4, y^t)$, $t \geq 5$.

(**3.*b***) If $t > q$,

(**3.*b*.*i***) when $t = q + 1$,

(**3.*b*.*i*.1**) If $p \nmid t$, then $f \sim (x^2 + \alpha y^s, y^t)$ by Proposition 3.2(vi).

(**3.*b*.*i*.1.1**) $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$, $t = 5$.

(**3.*b.i.*1.2**) $\alpha = 1$ and $t > s$, then $f \sim (x^2 + y^3, y^t)$, $t \geq 4$, which is simple.

(**3.*b.i.*2**) If $p \mid t$,

(**3.*b.i.*2.1**) $\alpha = 0$, then $f \sim (x^2, y^3 + xy^2)$, which is of modality 0 by Proposition 3.5.

(**3.*b.i.*2.2**) $\alpha = 1$, then $f \sim (x^2 + y^s, y^t + exy^q)$ by Proposition 3.2(v). And we have $s = 3, 4$ or $p = t = 3, 5$ by Proposition 3.3.

(**3.*b.i.*2.2.1**) If $s = 3$, then $f \sim (x^2, xy^2 + y^3)$ for $p = t = 3$, or $f \sim (x^2 + y^3, y^t + xy^{t-1})$ for $t \geq 4$ by Proposition 3.2(v), which is of modality 0.

(**3.*b.i.*2.2.2**) If $s = 4$, then $f \sim (x^2 + y^4, y^t + xy^{t-1})$ for $t \geq 4$ by Proposition 3.2(v). If $p = t = 3$, then $f \sim (x^2 + y^4 - y(y^3 + xy^2), y^3 + xy^2) \sim (x^2, y^3 + xy^2)$ additionally, which is simple as shown in Proposition 3.5. Hence $f \sim (x^2 + y^4, y^t + xy^{t-1})$ for $p \mid t$, $t \geq 5$.

(**3.*b.i.*2.2.3**) If $p = t = 3$, $s > 4$, same as the process in [11, Proposition 2.5], $f \sim (x^2, y^3 + xy^2)$, which is simple.

(**3.*b.i.*2.2.4**) If $p = t = 5$, $s > 4$, then

$$
\begin{aligned}
f &\sim (x^2 + y^s, y^5 + e(y)xy^4), \text{ where } e(y) \in F[\![x, y]\!] \text{ is a unit} \\
&\sim (x^2 + e_0 y^s, y^5 + xy^4), \ e_0 \in F \\
&\sim (x^2 + e_0 y^s - e_0 y^{s-5}(y^5 + xy^4), y^5 + xy^4) \\
&\sim (x^2 - e_0 xy^{s-1}, y^5 + xy^4) \\
&\sim \big((x - \tfrac{1}{2}e_0 y^{s-1})^2 - \tfrac{1}{4}e_0^2 y^{2s-2}, y^5 + xy^4\big).
\end{aligned} \tag{3-2}
$$

Using the automorphism $\phi(x) = x - \frac{1}{2}e_0 y^{s-1}$, $\phi(y) = y$, then

$$
\begin{aligned}
f &\sim \big(x^2 - \tfrac{1}{4}e_0^2 y^{2s-1}, y^5 + xy^4 + \tfrac{1}{2}e_0 y^{s+3}\big) \\
&\sim \big(x^2 - \tfrac{1}{4}e_0^2 y^{2s-1}, y^5 + \tilde{e}xy^4\big),
\end{aligned} \tag{3-3}
$$

where $\tilde{e} = 1/\big(1 + \frac{1}{2}e_0 y^{s-2}\big)$. Applying $\phi(x) = (1/\tilde{e})x$ and $\phi(y) = y$, we have $f \sim (x^2 + e_1 y^{2s-1}, y^5 + xy^4)$, where $e_1 = -\frac{1}{4}\tilde{e}^2 e_0^2$. Repeating the process, we get $f \sim (x^2, y^5 + xy^4)$ with $p = \operatorname{char} F = 5$.

(**3.*b.ii***) Now assume $t > q + 1$.

(**3.*b.ii.*1**) If $\alpha = 0$, by Proposition 3.3, we have $1 \leq q \leq 2$ or $q = 3$, $t = 5$.

(**3.*b.ii.*1.1**) If $q = 1$, then $j_2(f_2)$ is nondegenerate; hence f is simple.

(**3.*b.ii.*1.2**) If $q = 2$, then $f \sim (x^2 + y^{2t-4}, xy^2)$, $t \geq 4$, same as Proposition 3.5, which is also simple.

(**3.*b.ii.*1.3**) If $q = 3$, $t = 5$, then $f \sim (x^2, y^5 + xy^3)$.

(**3.*b.ii.*2**) If $\alpha = 1$, we have $f \sim (x^2 + y^s, y^t + exy^q)$ by Proposition 3.2(v). By Proposition 3.3, we have $s = 3, 4$ or $t = 2, 3, 4, 5$ or $q = 1, 2$.

(**3.*b.ii.*2.1**) If $q = 1, 2$ holds, then f is simple as shown in Proposition 3.5.

(**3.*b*.*ii*.2.2**) If $s = 3$, then $f \sim (x^2 + y^3, xy^q)$ or $(x^2 + y^3, y^{q+2} + xy^q)$ for $q \geq 3$, which is also simple.

(**3.*b*.*ii*.2.3**) If $s = 4$, $q > 2$,

(**3.*b*.*ii*.2.3.1**) If $t \geq q+4$, $f \sim (x^2+y^4, y^t+e(y)xy^q - y^{t-4}(x^2+y^4)) = (x^2+y^4, (e(y)-xy^{t-q-4})xy^q) \sim (x^2+y^4, xy^q)$, where $e(y) = b_q + b_{q+1}y + \cdots$ is a unit in $F[\![x, y]\!]$ and b_q, b_{q+1} are defined in (3-1).

(**3.*b*.*ii*.2.3.2**) If $t = q+3$, then $f \sim (x^2+y^4, y^{q+3}+xy^q)$ by Proposition 3.2(v), since in this case $2t - 2q - s = 2$.

(**3.*b*.*ii*.2.3.3**) If $t = q + 2$. Let $g = (x^2 + y^4, y^{q+2} + \lambda xy^q)$, where $\lambda = e(0) \in F^\times$, $e(0)$ is the constant term of $e(y)$ in (**3.*b*.*ii*.2.3.1**). Then g is weighted homogeneous of degree 0 with respect to $(a; d)$, where $a = (2, 1)$, $d = (4, q+2)$.

(**3.*b*.*ii*.2.3.3.1**) If $\lambda^2 \neq -1$, we claim that $F^1_{a,d}R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$ (the proof will be presented later). Hence by Proposition 2.16, $f \sim (x^2 + y^4, y^{q+2} + \lambda xy^q)$, $\lambda \in F^\times$ and $\lambda^2 + 1 \neq 0$.

(**3.*b*.*ii*.2.3.3.2**) If $\lambda^2+1=0$, from the proof of (**3.*b*.*ii*.2.3.3.1**), we can see that $F^1_{a,d}R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$, with $T = \operatorname{span}\langle (0, xy^t) \mid t > q\rangle$. Then by Proposition 2.16,

$$f \sim (x^2 + y^4, y^{q+2} + \lambda xy^q + e(y)xy^t)$$

for $\lambda^2 + 1 = 0$, $t > q$ and $e(y)$ a unit in $F[\![y]\!]$. In fact, we can show that for some $e(y)$ and l, $f \sim (x^2 + y^4, y^{q+2} + \lambda xy^q + xy^l)$, while for others, we get a family of unimodal ICIS with the form

$$(x^2 + y^4, y^{q+2} + \lambda xy^q + u(y)xy^t + xy^{t'}), \tag{3-4}$$

where $u(y) = u_0 + u_1y^p + u_2y^{2p} + \cdots$ is a unit, $t' > t \geq q+1$, $p \mid t - q$, $p \nmid t' - q$. The details will be shown later.

(**3.*b*.*ii*.2.4**) The last remaining case is $t = 5$, $q = 3$, $s > 4$. Then $f \sim (x^2 + y^s, y^5 + exy^3)$. Using the same method as (**3.*b*.*i*.2.2.4**), one can show $f \sim (x^2, y^5 + xy^3)$. □

Proof of the claim in (**3.*b*.*ii*.2.3.3.1**). We need to show $F^1_{a,d}R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$ for $g = (x^2 + y^4, y^{q+2} + \lambda xy^q)$, where $a = (2, 1)$, $d = (4, q+2)$ and $\lambda^2 \neq -1$.

Denote $e_1 = (x^2 + y^4, 0)$, $e_2 = (0, x^2 + y^4)$, $e_3 = (y^{q+2} + \lambda xy^q, 0)$, $e_4 = (0, y^{q+2} + \lambda xy^q)$, $e_5 = (2x, \lambda y^q)$, $e_6 = (4y^3, (q+2)y^{q+1} + \lambda qxy^{q-1})$.

Set the weights $\mathrm{wt}(x) = 2$, $\mathrm{wt}(y) = 1$, $\mathrm{wt}(x^2+y^4) = 4$, $\mathrm{wt}(y^{q+2}+\lambda xy^q) = q+2$. Then by (2-3), $\widetilde{T}_g(F^1\mathcal{K}g)$ has the elements $x^iy^je_k$ with:

(a) $k = 1, 4$, $\mathrm{wt}(x^iy^je_k) \geq (5, q+3)$;

(b) $k = 5$, $\mathrm{wt}(x^iy^j) \geq 3$;

(c) $k = 6$, $\mathrm{wt}(x^iy^j) \geq 2$.

We have

$$xe_6 = (4xy^3,\ (q+2)xy^{q+1} + \lambda q x^2 y^{q-1}) \in \widetilde{T}_g(F^1\mathcal{K}g)$$

and

$$ye_4 - y^{q-1}e_2 = (0,\ -x^2y^{q-1} + \lambda xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g).$$

Therefore

$$xe_6 + \lambda q(ye_4 - y^{q-1}e_2) = (4xy^3,\ (\lambda^2 q + q + 2)xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g).$$

We also have

$$y^{q-1}e_2 = (0,\ x^2y^{q-1} + y^{q+3}) \in \widetilde{T}_g(F^1\mathcal{K}g);$$

hence,

$$xe_6 - \lambda q y^{q-1}e_2 = (4xy^3,\ (q+2)xu^{q+1} - \lambda q y^{q+3}) \in \widetilde{T}_g(F^1\mathcal{K}g).$$

Then

$$\begin{aligned}(q+2)(xe_6 - \lambda q y^{q-1}e_2) + \lambda q y^2 e_6 \\ = (4(q+2)xy^3 + 4\lambda q y^5,\ ((q+2)^2 + \lambda^2 q^2)xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g).\end{aligned}$$

Denote this element by e_7.

Note that

$$xye_5 - 2ye_1 = (-2y^5,\ \lambda xy^{q+1}),\quad y^3e_5 - \lambda ye_4 = (2xy^3,\ -\lambda^2 xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g),$$

and thus

$$\begin{aligned}e_7 + 2\lambda q(xye_5 - 2ye_1) - 2(q+2)(y^3e_5 - \lambda ye_4) \\ = (0,\ ((q+2)^2 + \lambda^2 q^2 + 2\lambda^2 q + 2\lambda^2(q+2))xy^{q+1}) \\ = (0,\ (q+2)^2(\lambda^2+1)xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g).\end{aligned}$$

Since $\lambda^2+1 \neq 0$, we have $(0, xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g)$. The other elements in $F^1_{a,d}R^2$ follow easily. □

Proof of the claim in (**3.*b.ii.*2.3.3.2**). We have $f \sim (x^2+y^4,\ y^{q+2}+\lambda xy^q+e(y)xy^t)$. If $p \nmid t-q$, we can use an "α, β-trick" based on the implicit function theorem (see Lemma 5.10) to show that $f \sim (x^2+y^4,\ y^{q+2}+\lambda xy^q+xy^t)$ for $t \geq q+1$. See Remarks 5.11. For the case $p \mid (t-q)$, we write

$$e(y) = \sum_{i\geq 0} e_i y^{q+ip} + \sum_{\substack{j\geq 0 \\ p\nmid(t'+j)}} e'_j y^{t'+j}, \tag{3-5}$$

then $f \sim (x^2+y^4,\ y^{q+2}+\lambda xy^q(1+e_1y^p+e_2y^{2p}+\cdots)+xy^{t'}(e'_0+e'_1y+e'_2y^2+\cdots))$ with $p \nmid t'-q$. Therefore we can use α, β-trick again to reduce $e'_0+e'_1y+\cdots$. Then

$f \sim (x^2+y^4,\ y^{q+2}+\lambda xy^q+u(y)xy^t+xy^{t'})$, where $u(y)=u_0+u_1y^p+u_2y^{2p}+\cdots$ as we want. □

4. The classification of order 2 in char $F = 2$

In this section, we will show:

Proposition 4.1. *A unimodal ICIS of order* 2 *in any field with characteristic equal to* 2 *must have the form in Table 2.*

symbol	*form*	*condition*
h^2_λ	$(x^2+\lambda xy^2,\ y^3)$	$\lambda \in \{0, 1\}$
i^2_k	$(x^2+y^k,\ xy^2)$	$k \geq 3$, *k is odd*
$i^2_{k,\lambda}$	$(x^2+y^k+\lambda y^{k+1},\ xy^2)$	$k \geq 3$, *k is even,* $\lambda \in \{0, 1\}$
j^2_λ	$(x^2+y^3,\ y^4+\lambda xy^3)$	$\lambda \in \{0, 1\}$
$k^2_{\lambda,\mu}$	$(x^2+\lambda xy^3,\ y^4+\mu xy^3)$	$\lambda, \mu \in \{0, 1\}$
l^2	$(x^2+xy^2,\ y^4)$	
m^2_s	$(x^2+y^s,\ xy^3)$	$s \geq 3$, *s is odd*
$m^2_{s,\lambda}$	$(x^2+\lambda x^2y+y^s,\ xy^3)$	$\lambda \in F$, $s \geq 4$, *s is even*
n^2_s	$(x^2+xy^2+y^s,\ xy^3)$	$s \geq 3$, *s is odd*
$\tilde{n}^2_{s,\lambda}$	$(x^2+xy^2+\lambda x^2y+y^s,\ xy^3)$	$\lambda \in F$, $s \geq 4$, *s is even*

Table 2. Possible ICIS of order 2 when char $F = 2$.

The following result is from [11]:

Proposition 4.2. *Let* char $F = 2$, $f \in I_{2,2}$ *and* $\operatorname{ord}(f) = 2$. *Then one of the following cases occurs*:

(a) $f \sim (xy,\ g)$ *for some* $g \in \mathfrak{m}^2$. *In this case,* $f \sim (xy,\ x^m+y^n)$ *for some* $m, n \geq 2$, *which is simple.*

(b) $f \sim (x^2+h,\ g)$ *for* $h \in \mathfrak{m}^3$ *and* $g \in \mathfrak{m}^2$. *Moreover, if* $g \notin \mathfrak{m}^3$, *then* f *is simple.*

Similarly to Proposition 3.3, we can show:

Proposition 4.3. *If* $g \in \mathfrak{m}^5$, *then the modality of* $f \sim (x^2+h,\ g)$ *is at least* 2.

Proof. For $g \in \mathfrak{m}^5$, then $l = j_4(f)$ is of the form

$$l \sim (x^2+axy^2+by^3+cxy^3+dy^4,\ 0) \sim (x^2+axy^2+by^3+dy^4,\ 0).$$

It is easy to compute that the codimension of $\widetilde{T}_l(\mathcal{K}_4 l)$ in $P_{2,4}$ is at least 2 (note that $(0, xy^3)$, $(0, y^4) \notin P_{2,4}$). Therefore $\mathcal{K}$-mod$(f) \geq \mathcal{K}_4$-mod$(l) \geq 2$ by Proposition 2.19. $\square$

Therefore, we need to work on the case $f \sim (x^2 + h, g)$ with $g \in \mathfrak{m}^4 \setminus \mathfrak{m}^5$.

Proposition 4.4. *If $g \in \mathfrak{m}^3 \setminus \mathfrak{m}^4$, then $f \sim (x^2 + \lambda xy^2, y^3)$, $\lambda \in \{0, 1\}$ or $f \sim (x^2 + y^k, xy^2)$, k is odd or $f \sim (x^2 + y^k + \mu y^{k+1}, xy^2)$, k is even, $\mu \in \{0, 1\}$.*

Proof. Set $j_3(g) = ax^3 + bx^2y + cxy^2 + dy^3$.

If $d \neq 0$, let $f_0 = (x^2, dy^3) \sim (x^2, y^3)$ be the weighted 0-jet of f with respect to (a, d), where $a = (4, 3)$, $d = (8, 9)$. Then

$$F^1_{a,d} R^2 \subset \operatorname{span}\langle (xy^2, 0)\rangle + \widetilde{T}_{f_0}(F^1 \mathcal{K} f_0).$$

By Proposition 2.16, $f \sim (x^2 + cxy^2, y^3) \sim (x^2 + \lambda xy^2, y^3)$, $\lambda \in \{0, 1\}$.

If $d = 0$ and $c \neq 0$, we still let $f_0 = (x^2, cxy^2) \sim (x^2, xy^2)$ be the weighted 0-jet of f with respect to (a, d), where $a = (4, 3)$, $d = (8, 10)$. In this case

$$F^1_{a,d} R^2 \subset \operatorname{span}\langle (y^k, 0),\ k \geq 4\rangle + \widetilde{T}_{f_0}(F^1 \mathcal{K} f_0).$$

Then $f \sim (x^2 + e(y)y^k, xy^2) \sim (e(y)^{-1}x^2 + y^k, xy^2)$, $e(y) \in F[\![y]\!]$ is a unit.

If k is odd, then there exists $\tilde{e}(y)^k = e(y)$. Apply $\phi(x) = \tilde{e}(y)x$, $\phi(y) = y$, then $f \sim (x^2 + y^k, xy^2)$.

If k is even, write $e(y)^{-1} = e_0 + e_1 y + \cdots$, then

$$\begin{aligned} f &\sim (e_0x^2 + e_1x^2y + e_2x^2y^2 + \cdots + y^k, xy^2) \\ &\sim (e_0x^2 + e_1x^2y + y^k, xy^2) \\ &\sim ((e_0x^2 + e_1x^2y + y^k)(1 - (e_1/e_0)y), xy^2) \\ &\sim (e_0x^2 + y^k - (e_1/e_0)y^{k+1}, xy^2) \\ &\sim (x^2 + y^k + \lambda y^{k+1}, xy^2), \end{aligned} \tag{4-1}$$

where $\lambda \in \{0, 1\}$.

If $c = d = 0$, then $j_3(f) \sim (x^2 + h, 0)$. Then $g \in \mathfrak{m}^4$, a contradiction. $\square$

Proposition 4.5. *If $g \in \mathfrak{m}^4$, then f is equivalent to other forms in Table 2, that is, j^2_λ, $k^2_{\lambda,\mu}$, l^2, m^2_s, m^2_{s,e_1}, n^2_s or $\tilde{n}^2_{s,e_1}$.*

Proof. Computing the complete transversal of $j_2(f) = (x^2, 0)$, we have $f \sim (x^2 + a(y)xy^r + b(y)y^s, c(y)xy^u + d(y)y^v)$ with $a(y), b(y), c(y), d(y)$ either units or 0. If $a(y)$ (resp. $b(y)$, $c(y)$, $d(y)$)$= 0$, we regard r (resp. s, u, v) as ∞. By Proposition 4.3, we have either $u = 3$ or $v = 4$.

(i) $v = 4$, $s = 3$. Then $f \sim (x^2 + a(y)xy^r + b(y)y^3, c(y)xy^u + d(y)y^4)$. Take $l = (x^2 + b_0y^3, d_0y^4) \sim (x^2 + y^3, y^4)$ to be the weighted 0-jet with respect to $a = (3, 2)$, $d = (6, 8)$. We have $F^1_{a,d} \subset \operatorname{span}\langle (0, xy^3)\rangle + \widetilde{T}_l(F^1 \mathcal{K} l)$. Therefore $f \sim (x^2 + y^3, y^4 + \lambda xy^3)$ for $\lambda \in \{0, 1\}$, which is j^2_λ in Table 2.

(ii) $v = 4$, $s > 3$. If $r \geq 3$, then we choose $l = (x^2, y^4)$ be the weighted 0-jet of f with respect to $a = (2, 1)$, $d = (4, 4)$. We have $F^1_{a,d} \subset \operatorname{span}\langle (xy^3, 0), (0, xy^3)\rangle + \widetilde{T}_l(F^1\mathcal{K}l)$. Thus $f \sim (x^2 + \lambda xy^3, y^4 + \mu xy^3)$, $\lambda, \mu \in \{0, 1\}$ after a scaling. That is, $f \sim k^2_{\lambda,\mu}$ in Table 2. If $r = 2$, then the weighted 0-jet of f with respect to $a = (2, 1)$, $d = (4, 4)$ becomes $l = (x^2 + a_0xy^2, y^4) \sim (x^2 + xy^2, y^4)$. Computation shows that $F^1_{a,d} \subset \widetilde{T}_l(F^1\mathcal{K}l)$, then we have $f \sim (x^2 + xy^2, y^4) \sim l^2$, which is in Table 2.

(iii) $v > 4$. Then we have $s = 3$ and then $f \sim (x^2 + a(y)xy^r + b(y)y^s, xy^3 + d(y)y^v)$. Choose $l = (x^2, xy^2)$ as the weighted 0-jet of f with respect to $a = (1, 1)$, $d = (2, 4)$. We have $F^1_{a,d} \subset \operatorname{span}\langle (xy^2, 0), (y^k, 0) \mid k \geq 3\rangle + \widetilde{T}_l(F^1\mathcal{K}l)$. Then $f \sim (x^2 + \mu xy^2 + e(y)y^s, xy^3)$, $\mu \in \{0, 1\}$, $s \geq 3$. After a scaling we can assume $e(0) = 1$.

If $\mu = 0$ and s is odd, using the α, β-trick in Remarks 5.11, we have $f \sim (x^2 + y^s, xy^3) \sim m^2_s$. If $\mu = 0$ and s is even, we have

$$f \sim (e(y)^{-1}x^2 + y^s, xy^3) \sim ((1 + e_1y + e_2y^2 + \cdots)x^2 + y^s, xy^3) \sim (x^2 + e_1x^2y + e_2x^2y^2 + y^s, xy^3), \tag{4-2}$$

where $e(y)^{-1} = e_0 + e_1y + e_2y^2 + \cdots$, and the e_3-term vanishes since $e_3x^2y^3$ is killed by the xy^3 term in the second component. We apply

$$\phi(x) = \frac{x}{1 + e_2^{1/2}y}, \phi(y) = y;$$

then $f \sim (x^2 + e_1(1 + e_2y^2)^{-1}x^2y + y^s, xy^3) \sim (x^2 + e_1x^2y + y^s, xy^3)$, $e_1 \in F$. That is, $f \sim m^2_{s,e_1}$.

If $\mu = 1$ and s is odd, we have $f \sim (x^2 + xy^2 + y^s, xy^3) \sim n^2_s$ by α, β-trick. If $\mu = 1$ and s is even, as above, we have

$$f \sim (e(y)^{-1}(x^2 + xy^2) + y^s, xy^3) \sim ((1 + e_1y + e_2y^2 + \cdots)x^2 + y^s, xy^3) \sim (x^2 + xy^2 + e_1x^2y + e_2x^2y^2 + y^s, xy^3). \tag{4-3}$$

Then applying

$$\phi(x) = \frac{x}{1 + e_2^{1/2}y}, \quad \phi(y) = y,$$

we have

$$f \sim (x^2 + (1 + e_2^{1/2}y)^{-1}xy^2 + e_1(1 + e_2y^2)^{-1}x^2y + y^s, xy^3) \sim (x^2 + xy^2 + e_1x^2y + y^s, xy^3) \tag{4-4}$$

with $e_1 \in F$. That is, $f \sim \tilde{n}^2_{s,e_1}$. □

5. The classification of order 3

In this section we assume $f = (f_1, f_2) \in F[x, y]^2$ with $\operatorname{ord}(f_1) = 3$, $\operatorname{ord}(f_2) \geq 3$. We also assume char $F = p > 3$ in this part. We begin by classifying 3-jets.

5.1. ***The classification of 3-jets.*** Choose a suitable coordinate system such that $j_3(f_1) = ax^3 + bx^2y + cxy^2 + dy^3$ with $a, b, c, d \in F$ and $a \neq 0$. Then $j_3(f_1) \sim x^3 + \frac{b}{a}x^2y + \frac{c}{a}xy^2 + \frac{d}{a}y^3 \sim (x - e_1y)(x - e_2y)(x - e_3y) \sim l_1l_2l_3$ since F is algebraically closed, where l_i, $i = 1, 2, 3$, are linear forms in R.

I. $l_1 = l_2 = l_3$. Let $\phi(x) = l_1$, $\phi(y) = y$, then $\phi(j_3(f_1)) = x^3$, i.e., $j_3(f_1) \sim x^3$. We have $j_3(f_1, f_2) \sim (x^3, ay^3 + bxy^2 + cx^2y)$.

I.1 If $a = b = c = 0$, $j_3(f) \sim (x^3, 0)$.

I.2 If $a = b = 0$, $c \neq 0$, $j_3(f) \sim (x^3, x^2y)$.

I.3 If $a = c = 0$, $b \neq 0$, $j_3(f) \sim (x^3, xy^2)$.

I.4 If $a = 0$, $b, c \neq 0$, $j_3(f) \sim (x^3, x^2y + b'xy^2)$, where $b' = \frac{b}{c}$. Let $\tilde{x} = 2b'x$, $\tilde{y} = y$, then $j_3(f) \sim (\tilde{x}^3, 2\tilde{x}^2\tilde{y} + \tilde{x}\tilde{y}^2) \sim (x^3, x^3 + 2x^2y + xy^2) \sim (x^3, x(x + y)^2)$. Let $\phi(x) = x$, $\phi(y) = y - x$, then $j_3(f) \sim (x^3, xy^2)$.

I.5 If $a \neq 0$, $b^2 \neq 3ac$, we have

$$
\begin{aligned}
j_3(f) &\sim (x^3, y^3 + b'xy^2 + c'x^2y) \\
&\sim \left(x^3, \left(y + \tfrac{1}{3}b'x\right)^3 + c'x^2y - \tfrac{1}{3}b'^2x^2y - \tfrac{1}{27}b'^3x^3\right) \\
&\sim \left(x^3, \left(y + \tfrac{1}{3}b'x\right)^3 + \left(c' - \tfrac{1}{3}b'^2\right)x^2\left(y + \tfrac{1}{3}b'x\right)\right),
\end{aligned}
\tag{5-1}
$$

where $b' = b/a$, $c' = c/a$. Using the automorphism $\phi(x) = x$, $\phi(y) = y - \frac{1}{3}b'x$ we have

$$j_3(f) \sim \left(x^3, y^3 + \left(c' - \tfrac{1}{3}b'^2\right)x^2y\right).$$

Using the automorphism $\phi(x) = \left(c' - \frac{1}{3}b'^2\right)^{-1/2}x$, $\phi(y) = y$ since $c' - \frac{1}{3}b'^2 \neq 0$, we have $j_3(f) \sim (x^3, y^3 + x^2y)$.

I.6 If $a \neq 0$, $b^2 = 3ac$, as above, we have $j_3(f) \sim \left(x^3, y^3 + \left(c' - \frac{1}{3}b'^2\right)x^2y\right) \sim (x^3, y^3)$.

II. $l_1 = l_2 \neq l_3$. Let $\phi(x) = l_1$, $\phi(y) = l_2$, then $j_3(f_1) \sim x^2y$. We have $j_3(f) \sim (x^2y, ax^3 + bxy^2 + cy^3)$.

II.1 If $a = b = c = 0$, $j_3(f) \sim (x^2y, 0)$.

II.2 If $a = b = 0$, $c \neq 0$, $j_3(f) \sim (x^2y, y^3)$. Let $\phi(x) = y$, $\phi(y) = x$, we get $j_3(f) \sim (x^3, xy^2)$.

II.3 If $b = c = 0$, $a \neq 0$, $j_3(f) \sim (x^2y, x^3) \sim (x^3, x^2y)$.

II.4 If $a = c = 0$, $b \neq 0$, $j_3(f) \sim (x^2y, xy^2)$.

II.5 If $c=0,\ a,b\neq 0,\ j_3(f)\sim(x^2y,\,ax^3+bxy^2)$. Let $\phi(x)=x,\ \phi(y)=\sqrt{a/b}\,y$, we have $j_3(f)\sim(x^2y,\,x^3+xy^2)$.

II.6 If $c\neq 0,\ a\neq 0$, we have

(5-2) $$j_3(f)\sim(x^2y,\,y^3+a'xy^2+b'x^3)\sim\big(x^2y,\,\big(y+\tfrac{1}{3}a'x\big)^3+\big(b'-\tfrac{1}{27}a'^3\big)x^3\big),$$

where $a'=a/c,\ b'=b/c$. Let $\phi(x)=x,\ \phi(y)=y+\frac{1}{3}a'x$, we have

(5-3) $$\begin{aligned} j_3(f)&\sim\big(x^2\big(y-\tfrac{1}{3}a'x\big),\,y^3+\big(b'-\tfrac{1}{27}a'^3\big)x^3\big)\\ &\sim\big(x^2y-\tfrac{1}{3}a'x^3,\,y^3+\tfrac{1}{a'}3\big(b'-\tfrac{1}{27}a'^3\big)x^2y\big).\end{aligned}$$

Let $\phi(x)=x,\ \phi(y)=-\frac{1}{3}a'y$, we have $j_3(f)\sim(x^3+x^2y,\ y^3+\lambda x^2y)$, where $\lambda=(27b'-a'^3)/a'^3=(27bc^2-a^3)/a^3\in F$. If $\lambda=0$, we are back in case I. Hence we assume $\lambda\neq 0$ here.

II.7 If $c\neq 0,\ a=0$, as above, $j_3(f)\sim\big(x^2\big(y-\frac{1}{3}a'x\big),\,y^3+\big(b'-\frac{1}{27}a'^3\big)x^3\big)\sim(x^2y,\,y^3+b'x^3)\sim(x^2y,\,x^3+y^3)$.

III. $l_1\neq l_2\neq l_3$. Multiplying by a unit, one can assume $l_3=\frac{1}{2}l_1+\frac{1}{2}l_2$. Suppose $l_1=ux+vy,\ l_2=rx+sy$. Let

$$\phi(x)=\frac{u+r}{2}x+\frac{v+s}{2}y,\quad \phi(y)=\frac{u-r}{2i}x+\frac{v-s}{2i}y,$$

where $i^2=-1$. Then $\phi(l_1)=x+iy,\ \phi(l_2)=x-iy,\ \phi(l_3)=x$ and $j_3(f_1)\sim\phi(l_1l_2l_3)=x^3+xy^2$. Hence $j_3(f)\sim(x^3+xy^2,\,axy^2+bx^2y+cy^3)$.

III.1 If $a=b=c=0,\ j_3(f)\sim(x^3+xy^2,0)$.

III.2 If $b=c=0,\ a\neq 0,\ j_3(f)\sim(x^3,xy^2)$.

III.3 If $c=0,\ b\neq 0,\ j_3(f)\sim(x^3+xy^2,\,axy^2+bx^2y)\sim(x^3+xy^2,\,-ax^3+bx^2y)\sim(x^3+xy^2,\,x^2(y-(a/b)x))$. Let $\phi(x)=x,\ \phi(y)=y+(a/b)x$; we have $j_3(f)\sim(x^2y,\,x^3+x(y+(a/b)x)^2)$, which goes back to II.

III.4 If $c\neq 0,\ a=b=0,\ j_3(f)\sim(x^3+xy^2,\,y^3)$, which goes back to I.

III.5 If $c\neq 0$, one of $a,b\neq 0$, then write $j_3(f)\sim(x^3+xy^2,\,y^3+uxy^2+vx^2y)$, where $u,v\in F$ and one of $u,v\neq 0$. We have $j_3(f)\sim(x^3+xy^2,\,y^3+uxy^2+vx^2y+\alpha(x^3+xy^2))$. Choose $\alpha,r,s\in F$ such that

(5-4) $$s\neq 0,\ u+\alpha=r+2s,\ v=2rs+s^2,\ \alpha=rs^2.$$

These equations then reduce to

(5-5) $$s^4+(3-v)s^2-2us-v=0,\ 2r=\frac{v}{s}-s,\ \alpha=rs^2;$$

hence such α,r,s exist. Then $j_3(f)\sim(x^3+xy^2,\,(y+rx)(y+sx)^2)$. Using the automorphism $\phi(x)=x+y/s,\ \phi(y)=y+rx$, we have reduced it to case I or II.

Hence:

Proposition 5.1. *Let $f \in F[x, y]^2$ with $\operatorname{ord}(f) = 3$ be a unimodal complete intersection singularity, then $j_3(f)$ is equivalent to one of the following*:

$$(x^3, 0),\ (x^3, x^2y),\ (x^3, xy^2),\ (x^3, y^3 + x^2y),\ (x^3, y^3),$$
$$(x^2y, 0),\ (x^2y, xy^2),\ (x^2y, x^3 + xy^2),\ (x^3 + x^2y, y^3 + \lambda x^2y)(\lambda \neq 0), \tag{5-6}$$
$$(x^2y, x^3 + y^3),\ (x^3 + xy^2, 0).$$

5.2. *The classification of unimodals.* We have the following classification of unimodal ICIS of order 3:

Proposition 5.2. *A unimodal ICIS of order* 3 *in any field with characteristic not equal to* 2, 3 *must have one of the forms in Table 3.*

symbol	*form*	*condition*
H	$(x^3 + x^2y, y^3 + \lambda x^2y)$	$\lambda \neq 0$
I	$(x^3, y^3 + x^2y)$	
J	(x^3, y^3)	
K_r	$(x^3 + xy^2, x^2y + y^r)$	$r \geq 4$
$L_{r,s}$	$(x^2y + y^r, xy^2 + x^s)$	$r, s \geq 4$
M_r	$(x^3 + y^r, xy^2)$	$r \geq 4$
N_λ	$(x^3 + \lambda xy^3, x^2y + y^4)$	$\lambda \notin \{1, 12\}$
$P_{r,\infty}$	$(x^3 + xy^3 + xy^r, x^2y + y^4)$	$r \geq 4$
$P_{\infty,s}$	$(x^3 + xy^3 + y^s, x^2y + y^4)$	$s \geq 5$
$P_{r,s,\lambda}$	$(x^3 + xy^3 + xy^r + \lambda y^s, x^2y + y^4)$	$r \geq 4,\ s \geq 5,$ $\lambda \in F$
$\widetilde{P}_{r,s}$	$(x^3 + xy^3 + uxy^r + vy^s, x^2y + y^4)$, *where* $u = u_0 + u_1y + \cdots,\ v = v_0 + v_1y + \cdots,$	$r \geq 4,\ s \geq 5,$ $p \mid 2r - 2s + 3$
R_t	$(x^3 + xy^3, x^2y + y^t)$	$t \geq 5$
X_λ	$(x^3 + y^4, x^2y + \lambda y^4)$	$\lambda \in \{0, 1\}$
Y_λ	$(x^3 + y^5, x^2y + \lambda y^5)$	$\lambda \in \{0, 1\}$
Z_λ	$(x^3 + 12xy^3 + \lambda y^5, x^2y + y^4)$	$\lambda \in \{0, 1\}$

Table 3. Possible ICIS of order 3 when char $F \neq 2, 3$.

We will prove Proposition 5.2 step by step:

Proposition 5.3. *If $j_3(f)$ is contact equivalent to one of the following forms*:

$$(5\text{-}7)\quad (x^3,\, y^3+x^2y),\ (x^3,\, y^3),\ (x^2y,\, x^3+y^3),\ (x^3+x^2y,\, y^3+\lambda x^2y)(\lambda\neq 0),$$

then $j_3(f)$ is 3*-determined. In particular*, *f is contact equivalent to $j_3(f)$ of the above forms.*

Proof. After some computation, one can show $\mathfrak{m}^4 \subset \mathfrak{m}\cdot\widetilde{T}_f(\mathcal{K}f)$ when $j_3(f)$ has forms in 5.2. Hence by Theorem 2.6, $j_3(f)$ is 3-determined. Here we compute the case when $j_3(f)\sim(x^3,\, y^3+x^2y)$ as an example.

Let

$$e_1=(x^3,0),\ e_2=(0,x^3),\ e_3=(y^3+x^2y,0),$$

$$e_4=(0,\, y^3+x^2y),\ e_5=(3x^2,2xy),\ e_6=(0,3y^2+x^2).$$

Then $\widetilde{T}_f(\mathcal{K}f)$ is generated by $e_1,\ e_2,\ e_3,\ e_4,\ xe_5,\ ye_5,\ xe_6,\ ye_6$, and we have

$$\begin{aligned}
(0,x^4)&=xe_2 && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
(0,x^3y)&=ye_2 && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
(0,x^2y^2)&=\tfrac{1}{3}(x^2e_6-xe_2) && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
(0,xy^3)&=\tfrac{1}{3}(xye_6-(0,x^3y)) && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
(0,y^4)&=ye_4-(0,x^2y^2) && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
(x^4,0)&=xe_1 && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
(x^3y,0)&=ye_1 && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
(x^2y^2,0)&=\tfrac{1}{3}(y^2e_5-2(0,xy^3)) && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
(xy^3,0)&=xe_3-(x^3y,0) && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
(y^4,0)&=ye_3-(x^2y^2,0) && \in\mathfrak{m}\widetilde{T}_f(\mathcal{K}f),
\end{aligned}$$

This implies $\mathfrak{m}^4\cdot R^2\subset\mathfrak{m}\cdot\widetilde{T}_f(\mathcal{K}f)$, as claimed. □

Proposition 5.4. *If $j_3(f)\sim$ $(x^3,0)$, $(x^2y,0)$, $(x^3+xy^2,0)$, then $\mathcal{K}$*-mod(f) *is at least* 2.

Proof. We just prove the case when $j_3(f)\sim(x^2y+xy^2,0)$. The others are similar.

In this case a complete transversal in J_4 is spanned by $(0,x^4)$, $(0,x^2y^2)$, $(0,y^4)$; hence $j_4(f)\sim(x^3+xy^2,\, ax^4+bx^2y^2+cy^4)$ by Theorem 2.8. Let $g=(x^3+xy^2,\, ax^4+bx^2y^2+cy^4)$. After computation, we have $(0,xy^3)$, $(0,y^4)\notin P_{3,4}$ for almost all a,b,c. Hence, the codimension of $\widetilde{T}_g(\mathcal{K}_4g)$ in $P_{3,4}$ is 2, which is, the modality of $f\geq 2$ by Proposition 2.19. □

Proposition 5.5. *If $j_3(f)\sim(x^3,xy^2)$, then $f\sim(x^3+y^r,xy^2)$, $r\geq 4$.*

Proof. We can compute the complete transversal as follows: By Proposition 2.5, $\widetilde{T}_f(\mathcal{K}f)$ is generated by

$$e_1 = (x^3, 0),\ e_2 = (0, x^3),\ e_3 = (xy^2, 0),\ e_4 = (0, xy^2),\ xe_5,\ ye_5,\ xe_6,\ ye_6,$$

where $e_5 = (3x^2, y^2)$, $e_6 = (0, 2xy)$. A similar computation shows that

$$(x^4, 0),\ (x^3y, 0),\ (x^2y^2, 0),\ (xy^3, 0),\ (0, x^4),\ (0, x^3y),\ (0, x^2y^2),\ (0, xy^3),\ (0, y^4)$$

lie in $\widetilde{T}_f(\mathcal{K}f)$ while $(y^4, 0) \notin \widetilde{T}_f(\mathcal{K}f)$. In fact, one can easily show that $(y^l, 0) \notin \widetilde{T}_f(\mathcal{K}f)$ for any $l > 3$. Hence, a complete transversal is spanned by $\{(y^l, 0) \mid l \geq 4\}$.

By Theorem 2.8, we have

$$\begin{aligned} f &\sim \Big(x^3 + \sum_{l\geq 4} b_l y^l, xy^2\Big) \sim (x^3 + e(y)y^r, xy^2) \quad (r \geq 4,\ e(y) \text{ a unit}) \\ &\sim (x^3 + y^r, xy^2). \end{aligned} \tag{5-8}$$

In the last line, we take the automorphism $\phi(x) = e(y)^{1/3}x$ and $\phi(y) = y$. □

Proposition 5.6. *If $j_3(f) \sim (x^2y, xy^2)$, then $f \sim (x^2y + y^r, xy^2 + x^s)$, $r, s \geq 4$.*

Proof. Using a computation like Proposition 5.5, a complete transversal is given by $\{(y^l, 0), (0, x^l) \mid l \geq 4\}$. Hence, by Theorem 2.8,

$$\begin{aligned} f &\sim \Big(x^2y + \sum_{l\geq 4} a_l y^l, xy^2 + \sum_{l\geq 4} b_l x^l\Big) \\ &\sim (x^2y + a(y)y^r, xy^2 + b(x)x^s), \end{aligned} \tag{5-9}$$

where $a(y), b(x)$ are units in $F[\![x, y]\!]$ and $r, s \geq 4$.

Using the automorphism $\phi(x) = a(y)^{1/2}x$ and $\phi(y) = y$, we have

$$\begin{aligned} f &\sim (x^2y + y^r, a(y)^{1/2}xy^2 + a(y)^{s/2}b(a(y)^{1/2}x)x^s) \\ &\sim (x^2y + y^r, xy^2 + e(x, y)x^s), \end{aligned} \tag{5-10}$$

where $e(x, y)$ is a unit. We write $e(x, y) = \sum_{i\geq 0} e_i(x)y^i$; then

$$\begin{aligned} f &\sim \Big(x^2y + y^r, xy^2 + \Big(\sum_{i\geq 0} e_i(x)y^i\Big)x^s\Big) \\ &\sim \Big(x^2y + y^r, xy^2 + \Big(\sum_{i\geq 0} e_i(x)y^i\Big)x^s - x^{s-2}(x^2y + y^r)\Big(\sum_{k\geq 1} e_k(x)y^{k-1}\Big)\Big) \\ &\sim \Big(x^2y + y^r, xy^2 + e_0(x)x^s - \sum_{k\geq 1} x^{s-2}e_k(x)y^{r+k-1}\Big). \end{aligned}$$

Note that the order of x changes from s to $s-2$, and the order of y changes from 0 to r. Repeating the operation, we get $f \sim (x^2y + y^r, xy^2 + e_0(x)x^s + d(x, y))$, where $d(x, y) = y^{rs/2}e'(x, y)$ or $xy^{r(s-1)/2}e'(x, y)$ depending on whether s is even or odd, and $e'(x, y) = \sum_{k\geq 1} e_k(x)y^{k-1}$, which is a unit in $F[\![x, y]\!]$. The order of d is $\geq \frac{1}{2}r(s-1) + 1$.

Using the automorphism $\phi(x) = x$ and $\phi(y) = e_0(x)^{1/2}y$. Then

$$\begin{aligned}(5\text{-}11)\qquad f &\sim (e_0(x)^{1/2}x^2y + e_0(x)^{r/2}y^r,\, xy^2 + x^s + d_1(x, y))\\ &\sim (x^2y + e_0(x)^{(r-1)/2}y^r,\, xy^2 + x^s + d_1(x, y)),\end{aligned}$$

where $d_1(x, y) = d(x, e_0(x)^{1/2}y)$. Write $e_0(x)^{(r-1)/2} = \sum_{i\ge 0} u_i x^i$; then

$$\begin{aligned}f &\sim \Big(x^2y+\Big(\sum_{i\ge 0} u_i x^i\Big)y^r,\, xy^2+x^s+d_1(x,y)\Big)\\ &\sim \Big(x^2y+\Big(\sum_{i\ge 0} u_i x^i\Big)y^r - y^{r-2}(xy^2+x^s+d_1(x,y))\Big(\sum_{k\ge 1} u_k x^{k-1}\Big),\, xy^2+x^s+d_1(x,y)\Big)\\ &\sim \Big(x^2y+u_0y^r - \sum_{k\ge 1} u_k y^{r-2}x^{s+k-1} - \sum_{k\ge 1} u_k y^{r-2} d_1(x,y),\, xy^2+x^s+d_1(x,y)\Big).\end{aligned}$$

Repeating the operation, we get $f \sim (x^2y + u_0y^r + d_2(x, y), xy^2 + x^s + d_1(x, y))$, where the order of $d_2(x, y)$ is $\ge \frac{1}{2}s(r-1)+1$.

Taking the automorphism $\phi(x) = \alpha x$, $\phi(y) = \beta y$, where $\alpha, \beta \in F$ satisfy $\alpha^2\beta = u_0\beta^r$, $\alpha\beta^2 = \beta^s$, we have

$$(5\text{-}12)\qquad f \sim (x^2y + y^r + \tilde{d}_2(x, y),\, xy^2 + x^s + \tilde{d}_1(x, y)).$$

Specifically, $d_1(\tilde{x, y})$ has order $\frac{1}{2}rs$ if s is even and $\frac{1}{2}r(s-1)+1$ if s is odd, while $\tilde{d}_2(x, y)$ has order $\frac{1}{2}rs$ if r is even and $\frac{1}{2}(r-1)s+1$ if r is odd.

Now we exchange the position of x, y so that $r \ge s$. Let

$$(5\text{-}13)\qquad g = j_r(f) = (x^2y + y^r,\, xy^2 + x^s).$$

Using a similar computation as in the proof of (**3.*b.ii.*2.3.3.1**) in Proposition 3.6, we can show $\mathfrak{m}^{r+1}R^2 \subset \mathfrak{m}\widetilde{T}_g(\mathcal{K}g)$. This means g is $(2r-3)$-determined by Theorem 2.6. Since $s \ge 4$, $\min\{\operatorname{ord}\tilde{d}_1(x, y), \operatorname{ord}\tilde{d}_2(x, y)\} > 2r-3$. Therefore $f \sim j_{2r-3}(f) = (x^2y + y^r, xy^2 + x^s)$. □

Proposition 5.7. *If $j_3(f) \sim (x^2y, x^3 + xy^2)$, then $f \sim (x^2y + y^r, x^3 + xy^2)$, $r \ge 4$.*

Proof. A complete transversal is given by $\{(y^l, 0) \mid l \ge 4\}$; hence

$$f \sim \Big(x^2y + \sum_{l\ge 4} a_l y^l,\, x^3 + xy^2\Big) \sim (x^2y + e(y)y^r,\, x^3 + xy^2),$$

where $r \ge 4$.

Using the automorphism $\phi(x) = e(y)^{1/2}x$ and $\phi(y) = y$, we have

$$f \sim (x^2y + y^r,\, e(y)x^3 + xy^2).$$

Write $e(y)=\sum_{i\geq 0} e_i y^i$. Then

$$f \sim \Big(x^2y+y^r,\, \Big(\sum_{i\geq 0} e_i y^i\Big)x^3+xy^2-x(x^2y+y^r)\Big(\sum_{k\geq 1} e_k y^{k-1}\Big)\Big) \tag{5-14}$$
$$\sim \Big(x^2y+y^r,\, e_0x^3+xy^2-\sum_{k\geq 1} e_k xy^{r+k-1}\Big)$$
$$\sim \Big(x^2y+y^r,\, e_0x^3+xy^2\Big(1-\sum_{k\geq 1} e_k xy^{r+k-3}\Big)\Big).$$

Set $u_0(y)=e(y)$, $v_1(y)=1-\sum_{k\geq 1} e_k xy^{r+k-3}$. Then $v_1(y)\in 1+\mathfrak{m}^{r-2}$ and

$$f \sim (x^2y+y^r,\, e_0x^3+u_1(y)xy^2) \sim (x^2y+y^r,\, e_0v_1(y)^{-1}x^3+xy^2). \tag{5-15}$$

Since $v_1(y)\in 1+\mathfrak{m}^{r-2}$, we have $v_1(y)^{-1}\in 1+\mathfrak{m}^{r-2}$. Set $u_1(y)=e_0v_1(y)^{-1}$; then $u_1(y)\in e_0+\mathfrak{m}^{r-2}$ and $f\sim (x^2y+y^r,\, u_1(y)x^3+xy^2)$. Writing

$$u_1(y)=e_0+\sum_{k\geq r-2} v_k y^k, \tag{5-16}$$

we have

$$f \sim \Big(x^2y+y^r,\, \Big(e_0+\sum_{k\geq r-2} v_k y^k\Big)x^3+xy^2\Big)$$
$$\sim \Big(x^2y+y^r,\, \Big(e_0+\sum_{k\geq r-2} v_k y^k\Big)x^3+xy^2-x(x^2y+y^r)\Big(\sum_{k\geq r-2} v_k y^{k-1}\Big)\Big)$$
$$\sim \Big(x^2y+y^r,\, e_0x^3+xy^2-\sum_{k\geq r-2} v_k xy^{r+k-1}\Big)$$
$$\sim \Big(x^2y+y^r,\, e_0x^3+xy^2\Big(1-\sum_{k\geq r-2} v_k y^{r+k-3}\Big)\Big).$$

Set $v_2(y)=1-\sum_{k\geq r-2} v_k y^{r+k-3}$; then $v_2(y)\in 1+\mathfrak{m}^{2r-5}$. Setting $u_2(y)=e_0v_2(y)^{-1}\in e_0+\mathfrak{m}^{2r-5}$, we have $f\sim (x^2y+y^r,\, u_2(y)x^3+xy^2)$.

Repeating the operation, we can get a sequence of units $u_1(y), u_2(y), \ldots, u_n(y), \ldots$ with $u_n(y)\in e_0+\mathfrak{m}^{n(r-3)+1}$ and $f\sim (x^2y+y^r,\, u_n(y)x^3+xy^2)$.

Since $r\geq 4$, the orders of $u_n(y)-e_0$ are strictly increasing. Noting that $(x^2y+y^r,\, e_0x^3+xy^2)$ has finite Tjurina number (and hence is finite determined), we have $f\sim (x^2y+y^r,\, e_0x^3+xy^2)\sim (x^2y+y^r,\, x^3+xy^2)$ by using Remark 2.7 and applying the automorphism $\phi(x)=\alpha x$, $\phi(y)=\beta y$, where $\alpha^2\beta=\beta^r$ and $e_0\alpha^3=\alpha\beta^2$. □

The situation becomes complicated when $j_3(f)\sim (x^3, x^2y)$. A complete transversal is given by $\{(xy^{l-1}, 0), (y^l, 0), (0, y^l) \mid l\geq 4\}$. Then

$$f \sim \Big(x^3+\sum_{i\geq 3} a_i xy^i+\sum_{j\geq 4} b_j y^j,\, x^2y+\sum_{k\geq 4} c_k y^k\Big) \tag{5-17}$$
$$\sim (x^3+a(y)xy^r+b(y)y^s,\, x^2y+c(y)y^t),$$

where $r\geq 3$, $s\geq 4$, $t\geq 4$, and $a(y), b(y), c(y)$ are units or 0.

First we have the following criterion:

Proposition 5.8. *The modality of* $f \sim (x^3 + a(y)xy^r + b(y)y^s, x^2y + c(y)y^t)$ *is at least* 2 *if* $r \geq 4$, $s \geq 6$ *and* $t \geq 5$.

Proof. If $r \geq 4$, $s \geq 6$ and $t \geq 5$, set $g = j_4(f) = (x^3, x^2y)$, then all jets in $J_5'(g)$ are equivalent to $g_{ac} := (x^3 + axy^4, x^2y + cy^5)$, where $J_5'(g)$ is formed by jets in $J_5(g)$ with $r \geq 4$, $s \geq 6$ and $t \geq 5$.

Analogously to Proposition 3.3, we can show $(y^5, 0), (0, y^5) \notin \widetilde{T}_{g_{ac}}(\mathcal{K}_5 g_{ac})$. Hence $\operatorname{cod}(g_{ac}) \geq 2$. By Propositions 2.19 and 2.21, $\mathcal{K}$-mod$(f) \geq \mathcal{K}_5$-mod$(f) \geq 2$. □

Remark 5.9. If $a(y)$ (resp. $b(y), c(y)$) $= 0$, we regard r (resp. s, t) $= \infty$ here.

Now we assume $r \leq 4$ or $s \leq 6$ or $t \leq 5$ and $a(y), b(y), c(y)$ are units. We can simplify (5-17) by the trick using in [11].

First we recall the implicit function theorem:

Lemma 5.10 (cf. [9, Theorem 6.2.17]). *Let* $\mathcal{K}$ *be a field and* $F \in \mathcal{K}[[x_1, \dots, x_n, y]]$ *be such that*

$$F(x_1, \dots, x_n, 0) \in \langle x_1, \dots, x_n \rangle, \qquad \frac{\partial F}{\partial y}(x_1, \dots, x_n, 0) \notin \langle x_1, \dots, x_n \rangle. \tag{5-18}$$

Then there exists a unique $y(x_1, \dots, x_n) \in \langle x_1, \dots, x_n \rangle \mathcal{K}[[x_1, \dots, x_n]]$ *such that*

$$F(x_1, \dots, x_n, y(x_1, \dots, x_n)) = 0.$$

Back to f in (5-17): if

$$3r - 2s \neq 0 \quad \text{and} \quad p \nmid 3r - 2s, \tag{5-19}$$

applying $\phi(x) = a(y)^{1/2}x$ and $\phi(y) = y$, we get

$$f \sim (x^3 + xy^r + y^s \tilde{b}(y), x^2y + y^t c_1(y)),$$

where $\tilde{b}(y) = b(y)/a(y)$ and $c_1(y) = c(y)/a(y)$.

Write $\tilde{b}(y) = \sum_{i \geq 0} b_i y^i$. Consider the function

$$F(z) = z^{2s-3r} \sum_{i \geq 0} b_i y^i z^{2i} - b_0.$$

We have $F(1) \in \langle y \rangle F[[y]]$, and

$$F'(1) = (2s - 3r) \sum_{i \geq 0} b_i y^i - 2 \sum_{i \geq 1} i b_i y^i$$

is a unit by (5-19). Applying Lemma 5.10 to the function $G(z) = F(z+1)$, we find $\tilde{z}(y)$ with $G(\tilde{z}(y)) = 0$. Let $z(y) = \tilde{z}(y) + 1$; then $z(y)$ is a unit and $F(z(y)) = 0$.

Using the automorphism $\phi(x) = z(y)^r x$ and $\phi(y) = z(y)^2 x$, we have $f \sim (x^3 + xy^r + b_0 y^s, x^2y + y^t c_2(y))$. Then apply $\xi(x) = \alpha x$, $\xi(y) = \beta(y)$ with

$\alpha,\ \beta \in F$ satisfying $\alpha^3 = \alpha\beta^r$, $\alpha\beta^r = b_0\beta^s$ (such $\alpha,\ \beta$ exist since $3r - 2s \neq 0$). We have

$$f \sim (x^3 + xy^r + y^s,\ x^2y + y^t\tilde{c}(y)), \tag{5-20}$$

where $\tilde{c}(y)$ is the image of $c_2(y)$ under the automorphism ξ.

Similarly, if

$$2s - 3t + 3 \neq 0 \quad \text{and} \quad p \nmid 2s - 3t + 3, \tag{5-21}$$

we have

$$f \sim (x^3 + \tilde{a}(y)xy^r + y^s,\ x^2y + y^t), \tag{5-22}$$

where $\tilde{a}(y)$ is the image of $a(y)$ under the similar automorphism.

If

$$r + 1 - t \neq 0 \quad \text{and} \quad p \nmid r + 1 - t, \tag{5-23}$$

we have

$$f \sim (x^3 + xy^r + \tilde{b}(y)y^s,\ x^2y + y^t). \tag{5-24}$$

If (5-19), (5-21), (5-23) all fail, then r, s, t satisfy

$$3r - 2s = ap, \quad 2s - 3t + 3 = bp, \quad r + 1 - t = cp. \tag{5-25}$$

But the minimal solution of (5-25) with $r \geq 3$, $s \geq 4$, $t \geq 4$ and $p \geq 5$ is exactly $r = 4$, $s = 6$, $t = 5$, in which case f is not unimodal.

Remarks 5.11. (i) We call the technique we use here the α, β-trick, since we can easily apply $\phi(x) = \alpha x$, $\phi(y) = \beta y$ in (5-17) and get the result.

For example, when (5-19) holds, choose α, β as the simultaneous solution of

$$\alpha^3 = \alpha\beta^r a(\beta y), \quad \alpha\beta^r a(\beta y) = \beta^s b(\beta y); \tag{5-26}$$

then apply ϕ on (5-17), we can get (5-20). The trick was shown in [5]. The implicit function theorem provides a complete proof with the same result. But it is useless in some special characteristics, e.g., if $p \mid 3r - 2s$ here.

(ii) We call the technique used in Propositions 5.6 and 5.7 the r, s-trick. It has no restriction on characteristic, although the process is a little tedious. Later we will use the r, s-trick again but omit the process.

Proposition 5.12. *If $s = 4$, then $f \sim (x^3 + y^4,\ x^2y + \lambda y^4) \sim X_\lambda$ in Table 3, where* $\lambda = 0$ or 1.

Proof. Let $h = (x^3 + y^4, x^2y)$ be the weighted 0-jet of f with respect to (a, d), where $a = (4, 3)$, $d = (12, 11)$.

We choose $T = \mathrm{span}\langle(0, y^4)\rangle$ as a complete transversal. Then we have $F^1_{a,d} \subset T + \widetilde{T}_h(F^1_{a,d}\mathcal{K}h)$. By Proposition 2.16, $f \sim (x^3 + y^4, x^2y + ay^4)$ for some $a \in F$. After an obvious scaling, $f \sim (x^3 + y^4, x^2y + y^4)$ or $(x^3 + y^4, x^2y)$. □

Proposition 5.13. *If $s = 5$ and $r = 3$, then $f \sim N_\lambda$, R_t, $P_{r,s}$, Z_λ in Table 3.*

Proof. In this case $p \nmid 3r - 2s$. By (5-20), we have $j_4(f) \sim (x^3 + xy^3, x^2y + \lambda_0 y^4)$, which is weighted homogeneous of degree 0 with respect to $(a; d)$, where $a = (3, 2)$, $d = (9, 8)$. Write $g = j_4(f)$. Next we find $T \subset F^1_{a,d}R^2 \setminus F^r_{a,d}R^2$ such that $F^1_{a,d}R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$. Then by Proposition 2.16, $f \sim g + t$ with $t \in T$.

After some computation (easily done by hand), we know that for $\lambda_0 \neq 0, 1, \frac{1}{12}$, $F^1_{a,d}R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$ (note that char $F \neq 2, 3$, hence $\frac{1}{12}$ is well-defined). In that case $f \sim g = (x^3 + xy^3, x^2y + \lambda_0 y^4) \sim (x^3 + \lambda xy^3, x^2y + y^4)$, where $\lambda = 1/\lambda_0$.

If $\lambda_0 = 0$, then $F^1_{a,d}R^2 \subset T + \widetilde{T}_g(\mathcal{K}\cdot g)$ where T is spanned by $\{(0, y^5), (0, y^6), \dots\}$. Hence $f \sim (x^3 + xy^3, x^2y + ey^t)$ with $t \geq 5$ or $f \sim (x^3 + xy^3, x^2y)$ (which is not an ICIS). Using the r, s-trick we can show $f \sim (x^3 + xy^3, x^2y + y^t)$.

If $\lambda_0 = 1$ or $\lambda_0 = \frac{1}{12}$, we write $g \sim (x^3 + \lambda xy^3, x^2y + y^4)$ with $\lambda = 1, 12$. The process will be shown later in Proposition 5.15. □

Proposition 5.14. *If $s = 5$ and $r \geq 4$, then $f \sim N_0$ or Y_λ in Table 3, where $\lambda \in \{0, 1\}$.*

Proof. We have $f \sim (x^3 + a(y)xy^r + b(y)y^5, x^2y + c(y)y^t)$.

If $t = 4$, let $g = j_4(f) = (x^3, x^2y + cy^4) \sim (x^3, x^2y + y^4)$ be the 4-jet of f. Then g is weighted homogeneous of degree 0 with respect to $(a; d)$, where $a = (3, 2)$, $d = (9, 8)$ as in Proposition 5.13. After the same computation, we have $F^1_{a,d}R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$. That is, $f \sim (x^3, x^2y + y^4)$.

If $t \geq 5$, let $h = (x^3 + by^5, x^2y) \sim (x^3 + y^5, x^2y)$ be the weighted 0-jet with respect to (a, d), where $a = (5, 3)$, $d = (15, 13)$. Computation shows that $F^1_{a,d}R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$ for $T = \mathrm{span}\langle(0, y^5)\rangle$ (whether $p = 5$ or not). Hence $f \sim (x^3 + y^5, x^2y + ay^5)$. An obvious scaling shows $f \sim (x^3 + y^5, x^2y + \lambda y^5)$ for $\lambda \in \{0, 1\}$. □

Proposition 5.15. *If $s \geq 6$ and $t = 4$, then $f \sim N_\lambda$, $P_{r,\infty}$, $P_{\infty,s}$, $P_{r,s,\lambda}$, $\widetilde{P}_{r,s}$ or Z_λ in Table 3.*

Proof. Consider the 4-jet $j_4(f) = (x^3 + axy^3, x^2y + cy^4) \sim (x^3 + \lambda xy^3, x^2y + y^4)$, where $\lambda = a/c$. If $\lambda \neq 1, 12$, a computation similar to that of Proposition 5.13 shows $j_4(f) \sim (x^3 + \lambda xy^3, x^2y + y^4)$ and furthermore $f \sim (x^3 + \lambda xy^3, x^2y + y^4)$. We have

$$F^1_{a,d}R^2 \subset \mathrm{span}\langle(y^5, 0)\rangle + \widetilde{T}_g(F^1\mathcal{K}g).$$

Then, by Proposition 2.16, $f \sim (x^3 + 12xy^3, x^2y + y^4)$ or $(x^3 + 12xy^3 + y^5, x^2y + y^4)$.

If $\lambda_0 = 1$, we write $g \sim (x^3 + xy^3, x^2y + y^4)$. Then $F^1_{a,d}R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$ with $T = \operatorname{span}\langle (y^l, 0), (xy^j, 0) \mid l \geq 5,\ j \geq 4\rangle$. Hence

$$f \sim (x^3 + xy^3 + u(y)xy^r + v(y)y^s, x^2y + y^4)$$

with $r \geq 5,\ s \geq 4$.

If $v(y) = 0$, write $u(y) = u_0 + u_1y + \cdots$. Through the process

$$\begin{aligned}(5\text{-}27)\quad f &\sim (x^3+xy^3+u(y)xy^r, x^2y+y^4)\\ &\sim \Big((x^3+xy^3+u(y)xy^r)\Big(1-\frac{u_1}{u_0}y\Big), x^2y+y^4\Big)\\ &\sim \Big(x^3+xy^3+\Big(u_0+\Big(u_2-\frac{u_1^2}{u_0}\Big)y^2+\cdots\Big)xy^r - \frac{u_1}{u_0}x(x^2y+y^4), x^2y+y^4\Big)\\ &\sim \Big(x^3+xy^3+\Big(u_0+\Big(u_2-\frac{u_1^2}{u_0}\Big)y^2+\cdots\Big)xy^r, x^2y+y^4\Big),\end{aligned}$$

we reduce u_1 to 0. Repeating the process, we can reduce $u(y)$ to $u_0 \in F$ and finally to 1. Then $f \sim (x^3 + xy^3 + xy^r, x^2y + y^4) \sim P_{r,\infty}$.

If $u(y) = 0$, similarly we can get $f \sim (x^3 + xy^3 + y^s, x^2y + y^r) \sim P_{\infty,s}$.

If $u(y), v(y) \neq 0$, apply $\phi(x) = \alpha(y)^{-3}, \phi(y) = \alpha(y)^{-2}$. This leads to $f \sim (x^3+xy^3+u(y)\alpha(y)^{2r-6}xy^r+v(y)\alpha(y)^{2s-9}y^s, x^2y+y^s)$. Using the α, β-trick, if $p \nmid 2r-2s-3$, there exists $\alpha(y)$ such that $\alpha(y)2r-6u(y) = \alpha(y)^{2s-9}v(y)$. By the same process as for (5-27), we have $f \sim (x^3 + xy^3 + u_0xy^r + u_0y^s, x^2y + y^4) \sim (x^3 + xy^3 + xy^r + \lambda y^s, x^2y + y^4) \sim P_{r,s,\lambda}$.

If $p \mid 2r-2s-3$, we get a family $\widetilde{P}_{r,s}$. □

Proposition 5.16. *If $r = 3$, $s \geq 6$, $t \geq 5$, then $f \sim R_t$ in Table 3.*

Proof. In this case, let $g = j_4(f) = (x^3 + xy^3, x^2y)$. An ordinary complete transversal is given by $T = \operatorname{span}\langle (0, y^5), (0, y^6), \ldots\rangle$. Hence $f \sim (x^3 + xy^3, x^2y + e(y)y^t)$. Using the r, s-trick we get $f \sim (x^3 + xy^3, x^2y + y^t)$, $t \geq 5$. □

The above propositions finish the proof of Proposition 5.2.

6. The classification of order 3 when char $F = 2$

The process of classification in the field of characteristic 2 is quite similar to that of other characteristics. We first classify 3-jets and then classify all germs.

6.1. ***The classification of 3-jets.*** Same as in Section 5, for $f = (f_1, f_2)$ with $\operatorname{ord}(f_1) = 3$, we have $j_3(f_1) \sim (x - e_1y)(x - e_2y)(x - e_3y) \sim l_1l_2l_3$ since F is algebraically closed, where e_1, e_2, e_3 are the roots of $j_3(f_1(x, 1))$, and l_i, $i = 1, 2, 3$, are linear forms in R. Here we repeat the discussion at the beginning of Section 5.

I. $l_1 = l_2 = l_3$. Let $\phi(x) = l_1$, $\phi(y) = y$, then $\phi(j_3(f_1)) = x^3$, i.e., $j_3(f_1) \sim x^3$. We have $j_3(f_1, f_2) \sim (x^3, ay^3 + bxy^2 + cx^2y)$.

I.1 If $a=b=c=0$, $j_3(f)\sim(x^3, 0)$.

I.2 If $a=b=0$, $c\neq 0$, $j_3(f)\sim(x^3, x^2y)$.

I.3 If $a=c=0$, $b\neq 0$, $j_3(f)\sim(x^3, xy^2)$.

I.4 If $a=0$, $b, c\neq 0$, $j_3(f)\sim(x^3, bxy^2+cx^2y)\sim(x^3, xy^2+x^2y)$.

I.5 If $a\neq 0$ and $b^2\neq ac$, then we have $j_3(f)\sim(x^3, y^3+b'xy^2+c'x^2y)\sim(x^3, (y+b'x)^3+(c'-b'^2)x^2y)$, where $b'=b/a$, $c'=c/a$. Using the automorphism $\phi(x)=x$, $\phi(y)=y-b'x$, we have $j_3(f)\sim(x^3, y^3+(c'-b'^2)x^2(y-b'x))\sim(x^3, y^3+x^2y)$ since $b^2\neq ac$.

I.6 If $a\neq 0$, $b^2=3ac$, as above, we have $j_3(f)\sim(x^3, y^3+(c'-b'^2)x^2(y-b'x))\sim(x^3, y^3)$.

II. $l_1=l_2\neq l_3$. Let $\phi(x)=l_1$, $\phi(y)=l_2$, then $j_3(f_1)\sim x^2y$. We have $j_3(f)\sim(x^2y, ax^3+bxy^2+cy^3)$

II.1 If $a=b=c=0$, $j_3(f)\sim(x^2y, 0)$.

II.2 If $a=b=0$, $c\neq 0$, $j_3(f)\sim(x^2y, y^3)$. Let $\phi(x)=y$, $\phi(y)=x$, we get $j_3(f)\sim(x^3, xy^2)$.

II.3 If $b=c=0$, $a\neq 0$, $j_3(f)\sim(x^2y, x^3)\sim(x^3, x^2y)$.

II.4 If $a=c=0$, $b\neq 0$, $j_3(f)\sim(x^2y, xy^2)$.

II.5 If $c=0$, $a, b\neq 0$, $j_3(f)\sim(x^2y, ax^3+bxy^2)$. Let $\phi(x)=x$, $\phi(y)=\sqrt{a/b}y$, we have $j_3(f)\sim(x^2y, x^3+xy^2)$.

II.6 If $c\neq 0$, $b\neq 0$, then $j_3(f)\sim(x^2y, y^3+b'xy^2+a'x^3)\sim(x^2y, (y+b'x)^3+(a'-b'^3)x^3)$, where $a'=a/c$, $b'=b/c$. Letting $\phi(x)=x$, $\phi(y)=y-b'x$, we have $j_3(f)\sim(x^2(y-b'x), y^3+(a'-b'^3)x^3)\sim(x^2y-b'x^3, y^3+((a'-b'^3)/b')x^2y)$. Letting $\phi(x)=x$, $\phi(y)=-b'y$, we have $j_3(f)\sim(x^3+x^2y, y^3+\lambda x^2y)$, where $\lambda=(a'-b'^3)/b'^3=(ac^2-b^3)/b^3\in F$. We still assume $\lambda\neq 0$ here.

II.7 If $c\neq 0$, $b=0$, as above, $j_3(f)\sim(x^2(y-b'x), y^3+(a'-b'^3)x^3)\sim(x^2y, y^3+a'x^3)\sim(x^2y, x^3+y^3)$.

III. $l_1\neq l_2\neq l_3$. Multiplying by a unit, one can assume $l_3=l_1+l_2$. Suppose $l_1=x$, $l_2=y$; then $j_3(f_1)\sim x^2y+xy^2$. Hence $j_3(f)\sim(x^2y+xy^2, ax^3+bx^2y+cy^3)$.

III.1 If $a=b=c=0$, then $j_3(f)\sim(x^2y+xy^2, 0)$.

III.2 If $a=c=0$, $b\neq 0$, then $j_3(f)\sim(x^2y+xy^2, x^2y)\sim(x^2y, xy^2)$.

III.3 If exactly one of a, c is equal to 0, assume $a\neq 0$ and $c=0$. Then $j_3(f)\sim(x^2y+xy^2, x^3+b'x^2y)\sim(x^2y+xy^2, x^2(x+b'y))$, where $b'=b/a$. Using the automorphism $\phi(x)=x$, $\phi(y)=(y-x)/b'$, we reduce the situation to I or II depending on whether $b'=0$.

III.4 If $a, c \neq 0$ and $b = 0$, then $j_3(f) \sim (x^2y + xy^2, x^3 + c'y^3) \sim (x^2y + xy^2, x^3 + c'^{1/2}x^2y + c'^{1/2}xy^2 + c'y^3) \sim (x^2y + xy^2, (x + cy)(x + c'^{1/2}y)^2)$, where $c' = c/a$. It's easily to see that this reduces to I or II depending on whether $c' = 1$.

III.5 If $a, b, c \neq 0$, $j_3(f) \sim (x^2y + xy^2, x^3 + b'x^2y + c'xy^2)$, where $b' = b/a$, $c' = c/a$. Let α be the root of $\alpha^2 + b'\alpha = c'$. Then

$$\begin{aligned} f &\sim (x^2y + xy^2, x^3 + (\alpha + b')x^2y + \alpha xy^2 + c'y^3 \\ &\sim (x^2y + xy^2, x^2(x + \lambda y) + \alpha y^2(x + \lambda y)) \\ &\sim (x^2y + xy^2, (x + \lambda y)(x + \alpha^{1/2}y)^2), \end{aligned}$$

where $\lambda = \alpha + b = c/\alpha$. This reduces it to case I or II.

Hence we get the result:

Proposition 6.1. *Let $f \in F[[x, y]]^2$ be a unimodal complete intersection singularity with* $\operatorname{ord}(f) = 3$ *in a field F with characteristic* 2, *then $j_3(f)$ is contact equivalent to one of the following*:

$$\begin{aligned} &(x^3, 0),\ (x^3, x^2y),\ (x^3, xy^2),\ (x^3, x^2y + xy^2),\ (x^3, y^3 + x^2y),\ (x^3, y^3), \\ &(x^2y, 0),\ (x^2y, xy^2),\ (x^2y, x^3 + xy^2),\ (x^3 + x^2y, y^3 + \lambda x^2y)\ (\lambda \neq 0), \\ &(x^2y, x^3 + y^3),\ (x^2y + xy^2, 0). \end{aligned} \tag{6-1}$$

6.2. *The classification of unimodals.*

Proposition 6.2. *A unimodal ICIS of order* 3 *in any field with characteristic* 2 *must have one of the forms in Table 5* (on this page and the next).

symbol	*form*	*condition*
H_λ	$(x^3 + x^2y, y^3 + \lambda x^2y)$	$\lambda \neq 0$
I	$(x^3, y^3 + x^2y)$	
J	(x^3, y^3)	
K_r	$(x^3 + xy^2, x^2y + y^r)$	$r \geq 4$
$L_{r,s}$	$(x^2y + y^r, xy^2 + x^s)$	$r, s \geq 4$
M^2	$(x^3 + y^3, x^2y)$	
M_r	$(x^3 + y^r, xy^2)$	$r \geq 4$
M_r^2	$(x^3 + y^r, x^2y + xy^2)$	$r \geq 4$, *r is even*
$\widetilde{M}_r^2$	$(x^3 + y^r + ey^l, x^2y + xy^2)$, *where* $e = e_0 + e_1y^2 + e_2y^4 + \cdots$	$r \geq 4$, *r is odd,* $l > r$, *l is even*
N_λ	$(x^3 + \lambda xy^3, x^2y + y^4)$	$\lambda \neq 1$

symbol	*form*	*condition*
N^2_λ	$(x^3, x^2y+y^4+\lambda xy^3)$	$\lambda \in F$
$P_{r,\infty}$	$(x^3+xy^3+xy^r, x^2y+y^4)$	$r \geq 4$
$P_{\infty,s}$	$(x^3+xy^3+y^s, x^2y+y^4)$	$s \geq 5$
$P_{r,s,\lambda}$	$(x^3+xy^3+xy^r+\lambda y^s, x^2y+y^4)$	$r\geq 4,\ s\geq 5,\ \lambda\in F$
R_t	(x^3+xy^3, x^2y+y^t)	$t \geq 5$
X^2_μ	$(x^3+y^4, x^2y+\mu xy^3)$	$\mu \in \{0, 1\}$
$\widetilde{X}^2_\lambda$	$(x^3+y^4+xy^3, x^2y+\lambda xy^3)$	$\lambda \in F$
Y_λ	$(x^3+y^5, x^2y+\lambda y^5)$	$\lambda \in \{0, 1\}$

Table 5. Unimodal ICIS of order 3 when char $F = 2$ (starts on previous page).

The remaining propositions in this section finish the proof.

Proposition 6.3. *If $j_3(f)$ is contact equivalent to one of the following forms*:

(6-2) (x^3, y^3+x^2y), (x^3, y^3), $(x^3+x^2y, y^3+\lambda x^2y)$ $(\lambda \neq 1)$, (x^2y, x^3+y^3),

then $j_3(f)$ is 3*-determined*, *and so f is contact equivalent to the $j_3(f)$ in* (6-2).

Proof. If g is one of the first three germs, we can show $\mathfrak{m}^4 \subset \mathfrak{m}\cdot\widetilde{T}_g(\mathcal{K}g)$ as before; hence g is 3-determined.

For $g = j_3(f) = (x^2y, x^3+y^3)$, $\widetilde{T}_g(\mathcal{K}g)$ is spanned by $\{(x^2y, 0), (0, x^2y), (x^3+y^3, 0), (0, x^3+y^3), (0, x^2), (x^2, y^2)\}$, and a complete transversal is spanned by $\{(x^r, 0) \mid r \geq 4\}$. Hence $f \sim (x^2y+e(x)x^r, x^3+y^3)$ by Theorem 2.8 where $e(x) \in F[\![x]\!]$ is a unit and $r \geq 4$. Applying the automorphism $\phi(x) = x, \phi(y) = y-e(x)x^{r-2}$, $f \sim (x^2y, x^3+(y+e(x)x^{r-2})^3) \sim (x^2y, x^3+y^3+e(x)^3x^{3r-6}) \sim (x^2y, u(x)x^3+y^3)$, where $u(x) = 1+e(x)^3x^{3r-9}$ is a unit in $F[\![x]\!]$. Applying $\phi(x) = x, \phi(y) = u(x)^{1/3}y$, then $f \sim (x^2y, x^3+y^3)$. □

Proposition 6.4. *If $j_3(f) \sim (x^3, xy^2)$, then $f \sim (x^3+y^r, xy^2) \sim M_r$, $r \geq 4$ in Table 5.*

Proof. The complete transversal is still given by $\{(y^r, 0) \mid r \geq 4\}$. Then the process is same as Proposition 5.5. □

Proposition 6.5. *If $j_3(f) \sim (x^3, x^2y+xy^2)$, then $f \sim M^2_r$ or $\widetilde{M}^2_r$ in Table 5.*

Proof. A complete transversal is $T = \operatorname{span}\langle(y^r, 0),\ r \geq 4\rangle$; then $f \sim (x^3+u(y)y^r, x^2y+xy^2)$.

If r is even, using the α, β-trick we have $f \sim (x^3+y^r, x^2y+xy^2)$ since $2\nmid r-3$. If r is odd, write

$$\begin{aligned} f &\sim (x^3+e_0y^{2k+1}+e_1y^{2k+2}+\cdots, x^2y+xy^2) \\ &= (x^3+(e_0+e_2y^2+e_4y^4+\cdots)y^{2k+1}+(e_1+e_3y^2+\cdots)y^{2k+2}, x^2y+xy^2). \end{aligned}$$

There exists $e(x)^2 = e_0+e_2y^2+\cdots$ that allows us to use the α, β-trick again. Resetting the symbols, we get a family $f \sim (x^3+y^r+(e_0+e_1y^2+\cdots)y^l, x^2y+xy^2)$. □

Proposition 6.6. *If $j_3(f) \sim (x^2y, xy^2)$, then $f \sim (x^2y+y^r, xy^2+y^s) \sim L_{r,s}$, $r, s \geq 4$ in Table 5.*

Proof. The complete transversal is the same as Proposition 5.6, and the later process also follows from Proposition 5.6. □

Proposition 6.7. *If $j_3(f) \sim (x^2y, x^3+xy^2)$, then $f \sim (x^2y+y^r, x^3+xy^2) \sim K_r$ in Table 5, $r \geq 4$.*

Proof. Same as Proposition 5.7. □

If $j_3(f) \sim (x^3, x^2y)$, a complete transversal is given by

$$\{(xy^r, 0), (y^s, 0), (0, xy^u), (0, y^v) \mid r, u \geq 3,\ s, v \geq 4\}.$$

Then

$$f \sim (x^3+a(y)xy^r+b(y)y^s, x^2y+c(y)xy^u+d(y)y^v), \tag{6-3}$$

with $r, u \geq 3$, $s, v \geq 4$

Proposition 6.8. *If f is of the form* (6-3) *and $r \geq 4$, $s \geq 6$, $v \geq 5$, then f has modality at least* 2.

Proof. For $r \geq 4$, $s \geq 6$, $v \geq 5$, we write $g = j_4(f) = (x^3, x^2y)$, and any 5-jet in an open dense subset of $J'_5(g)$ is of the form $g_{ac} = (x^3+axy^4, x^2y+cxy^3+dy^5)$ with $a, c, d \in F$, where $J'_5(g)$ is formed by jets in $J_5(g)$ with $r \geq 4$, $s \geq 6$, $v \geq 5$. Computation, perhaps in a program such as Singular, shows $(y^5, 0), (0, y^5) \notin P_{4,5}$ in each case; hence $\mathrm{cod}(g_{ac}) = 2$ for all $a, c \in F$. By Propositions 2.19 and 2.21, $\mathcal{K}\text{-mod}(f) \geq \mathcal{K}_5\text{-mod}(f) \geq \inf\{\mathrm{cod}(g_{ac})\} \geq 2$. □

Proposition 6.9. *If f is of the form* (6-3) *with $s = 4$, then f is contact equivalent to the form $\sim (x^3+y^4, x^2y+\mu xy^3)$ with $\mu = 0, 1$ or $(x^3+y^4+xy^3, x^2y+\lambda xy^3)$ with $\lambda \in F$. That is, $f \sim X^2_\mu$ or $\widetilde{X}^2_\lambda$ in Table 5.*

Proof. In this case $f \sim (x^3+y^4+a(y)xy^r, x^2y+c(y)xy^u+d(y)y^v)$ after a scalar transform. Let $g = (x^3+y^4, x^2y)$, then g is weighted homogeneous of degree 0 with respect to (a, d), where $a = (4, 3)$, $d = (12, 11)$. Let $T = \mathrm{span}\langle (xy^3, 0), (0, xy^3)\rangle$, then we check that $F^1_{a,d}R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$, which means $f \sim (x^3+y^4+axy^3,$

$x^2y+bxy^3)$ for $a, b \in F$. Take a scalar transform; we have $f \sim (x^3+y^4, x^2y)$, (x^3+y^4, x^2y+xy^3) or $(x^3+y^4+xy^3, x^2y+\lambda xy^3)$ with $\lambda \in F$. □

Proposition 6.10. *If f is of the form* (6-3) *with $v=4$, $s>4$, then $f \sim N_\lambda$, N_λ^2 or $P_{r,\infty}$, $P_{\infty,s}$, $P_{r,s,\lambda}$ in Table 5.*

Proof. If $r \geq 4$, let $g=(x^3, x^2y+y^4)$ be the weighted 0-jet of f with respect to (a, d), where $a=(3, 2)$, $d=(9, 8)$. Let $T=\operatorname{span}\langle(0, xy^3)\rangle$; then $F^1_{a,d}R^2 \subset T+\widetilde{T}_g(F^1\mathcal{K}g)$, which means $f \sim (x^3, x^2y+y^4+\lambda xy^3)$ for $\lambda \in F$.

If $r=3$, let $g=(x^3+a(0)xy^3, x^2y+y^4)$, $a(0) \in F^\times$ be the weighted 0-jet of f with respect to (a, d), where $a=(3, 2)$, $d=(9, 8)$. If $a(0) \neq 1$, then $F^1_{a,d}R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$ automatically holds; hence $f \sim (x^3+\lambda xy^3, x^2y+y^4)$, $\lambda \in F^\times$.

In the case $a(0)=1$, T is spanned by $\{(xy^r, 0), (y^s, 0) \mid r \geq 4, s \geq 5\}$, and $f \sim (x^3+xy^3+u(y)xy^r+v(y)y^s, x^2y+y^4)$. Similar to Proposition 5.15, for $v(y)=0$ (resp. $u(y)=0$), we have $f \sim P_{r,\infty}$ (resp. $P_{\infty,s}$). Otherwise, since $p=2$, we have $p \nmid 2r-2s-3$ for any r, s. Then $f \sim (x^3+xy^3+e(y)xy^r+e(y)y^s, x^2y+y^4) \sim (x^3+xy^3+e_0xy^r+e_0y^s, x^2y+y^4) \sim (x^3+xy^3+xy^r+\lambda y^s, x^2y+y^4) \sim P_{r,s,\lambda}$. □

Proposition 6.11. *If f is of the form* (6-3) *with $s=5$, then $f \sim Y_\lambda$ or R_t in Table 5.*

Proof. If $r \geq 4$, let $g=(x^3+by^5, x^2y)$ with $b \in F^\times$ be the weighted 0-jet of f with respect to (a, d), where $a=(5, 3)$, $d=(15, 13)$. As in Proposition 5.14, $f \sim (x^3+y^5, x^2y+\lambda y^5)$, $\lambda \in \{0, 1\}$.

If $r=3$, let $g=(x^3+axy^3, x^2y)$ with $a \in F^\times$ be the weighted 0-jet of f with respect to (a, d), where $a=(3, 2)$, $d=(9, 8)$. As in Proposition 5.16, we have $f \sim (x^3+xy^3, x^2y+y^t)$, $t \geq 5$. □

Proposition 6.12. *If f is of the form* (6-3) *with $s \geq 6$, $v \geq 5$, then $f \sim R_t$ in Table 5.*

Proof. In this case we have $r=3$ by Proposition 6.8. Under the assumption $s, v>4$, we can choose $g=(x^3+axy^3, x^2y)$ to be the weighted 0-jet of f with respect to (a, d), where $a=(3, 2)$, $d=(9, 8)$. Let $T=\operatorname{span}\langle(0, y^l) \mid l \geq 5\rangle$; then $F^1_{a,d}R^2 \subset T+\widetilde{T}_g(F^1\mathcal{K}g)$. Therefore $f \sim (x^3+xy^3, x^2y+e(y)y^t)$. Using the r, s-trick as in Proposition 5.16, we have $f \sim (x^3+xy^3, x^2y+y^t)$, $t \geq 5$. □

7. The classification of order 3 when char $F=3$

Next we repeat the discussion in the case char $F=3$.

7.1. ***The classification of 3-jets.*** As in Section 5, for $f=(f_1, f_2)$ with $\operatorname{ord}(f_1)=3$, we have $j_3(f_1) \sim x^3, x^2y$ or x^3+xy^2.

I. $j_3(f_1) \sim x^3$. We have $j_3(f_1, f_2) \sim (x^3, ay^3+bxy^2+cx^2y)$.

I.1 If $c=0$, then $j_3(f)\sim(x^3,x^2y)$ or (x^3,xy^2) or (x^3,y^3+xy^2) or $(x^3,0)$, depending on whether $a,b=0$.

I.2 If $c\neq 0$, $b=0$, then $j_3(f)\sim(x^3,y^3+x^2y)$.

I.3 If $b,c\neq 0$, we can write $j_3(f)\sim\bigl(x^3,y\bigl(x+\frac{1}{2}by\bigr)^2+\bigl(a-\frac{1}{4}b^2\bigr)y^3\bigr)\sim\bigl(x^3,\frac{1}{4}b^2y\bigl(y+\frac{1}{b}2x\bigr)^2+\bigl(a-\frac{1}{4}b^2\bigr)(y+(2/b)x)^3\bigr)$. Using $\phi(x)=x$, $\phi(y)=y+(2/b)x$, we then get $j_3(f)\sim\bigl(x^3,\frac{1}{4}b^2(y-(2/b)x)y^2+\bigl(a-\frac{1}{4}b^2\bigr)y^3\bigr)\sim\bigl(x^3,ay^3-\frac{1}{2}bxy^2\bigr)\sim(x^3,xy^2)$ or (x^3,y^3+xy^2) depending on whether $a=0$.

II. $j_3(f_1)\sim(x^2y,0)$. We have $j_3(f)\sim(x^2y,ax^3+bxy^2+cy^3)$.

II.1 If $c=0$, then $j_3(f)\sim(x^2y,ax^3+bxy^2)\sim(x^2y,x^3)$ or (x^2y,xy^2) or (x^2y,x^3+xy^2) or $(x^2y,0)$ depending on whether a,b equal 0.

II.2 If $c\neq 0$, $b=0$, then $j_3(f)\sim(x^2y,ax^3+cy^3)\sim(x^2,(a^{1/3}x+c^{1/3}y)^3)$, which is back to case I.

II.3 If $b,c\neq 0$, $a=0$, then $j_3(f)\sim(x^2y,y^3+xy^2)$.

II.4 If $a,b,c\neq 0$, we change notation, obtaining $j_3(f)\sim(x^2y,y^3+ax^3+bxy^2)$. Applying the automorphism $\phi(x)=x$, $\phi(y)=y+\alpha x$, where α is a nonzero root of $\alpha^3-b\alpha^2+a=0$, we have

$$\begin{aligned}j_3(f)&\sim(x^2y+\alpha x^3,\,y^3+(\alpha^3+b\alpha^2+a)x^3+bxy^2+2b\alpha x^2y)\\&\sim(x^2y+\alpha x^3,\,y^3+bxy^2+2b\alpha(x^2y+\alpha x^3))\\&\sim(x^2y+\alpha x^3,\,y^3+bxy^2)\sim(x^3+x^2y,\,y^3+\lambda xy^2),\end{aligned}$$

with $\lambda\neq 1$; here we used that $\alpha\neq b$ (since $a\neq 0$).

III. $j_3(f_1)\sim x^3+xy^2$. We have $j_3(f)\sim(x^3+xy^2,axy^2+bx^2y+cy^3)$. The argument is the same as in Section 5, except for the characteristic being 3, with no other changes.

In conclusion:

Proposition 7.1. *Let $f\in F[[x,y]]^2$ be a unimodal complete intersection singularity with* $\operatorname{ord}(f)=3$ *in a field F with characteristic* 3, *then $j_3(f)$ is contact equivalent to one of the following, where $\lambda\neq 1$:*

$(x^3,0)$, (x^3,x^2y), (x^3,xy^2), (x^3,y^3+xy^2), (x^3,y^3+x^2y), (x^3,y^3),
$(x^2y,0)$, (x^2y,xy^2), (x^2y,x^3+xy^2), (x^2y,y^3+xy^2), $(x^3+x^2y,y^3+\lambda xy^2)$,
$(x^3+xy^2,0)$.

7.2. *The classification of unimodals.*

Proposition 7.2. *A unimodal ICIS of order* 3 *in any field with characteristic* 3 *must have one of the forms in Table 6.*

symbol	*form*	*condition*
H	$(x^3+x^2y,\ y^3+\lambda x^2y)$	$\lambda \neq 0$
I^3_λ	$(x^3+\lambda y^4,\ y^3+x^2y)$	$\lambda \in \{0, 1\}$
$\tilde{I}^3_\lambda$	$(x^3+\lambda y^4,\ y^3+xy^2)$	$\lambda \in \{0, 1\}$
$J^3_{\lambda,\mu}$	$(x^3+\lambda x^2y^2,\ y^3+\mu x^2y^2)$	$\lambda, \mu \in \{0, 1\}$
K_r	$(x^3+xy^2,\ x^2y+y^r)$	$r \geq 4$
K^3_r	$(y^3+xy^2+x^r,\ x^2y)$	$r \geq 4$
$\widetilde{K}^3_{r,\lambda}$	$(y^3+xy^2+\lambda xy^3+x^r,\ x^2y)$	$r \geq 4, \lambda \in F, 3 \mid r$
$L_{r,s}$	$(x^2y+y^r,\ xy^2+x^s)$	$r, s \geq 4$
M^3_r	$(x^3+y^r,\ xy^2)$	$r \geq 4$
$\widetilde{M}^3_{r,\lambda}$	$(x^3+\lambda x^3y+y^r,\ xy^2)$	$r \geq 4, \lambda \in F, 3 \mid r$
N^3_λ	$(x^3+\lambda xy^3,\ x^2y+y^4)$	$\lambda \neq 1$
$\widetilde{N}^3_s$	$(x^3+y^s,\ x^2y+y^4)$	$s \in \{5, 6\}$
$P_{r,\infty}$	$(x^3+xy^3+xy^r,\ x^2y+y^4)$	$r \geq 4$
$P_{\infty,s}$	$(x^3+xy^3+y^s,\ x^2y+y^4)$	$s \geq 5$
$P_{r,s,\lambda}$	$(x^3+xy^3+xy^r+\lambda y^s,\ x^2y+y^4)$	$r \geq 4,\ s \geq 5,\ \lambda \in F$
$\widetilde{P}_{r,s}$	$(x^3+xy^3+uxy^r+vy^s,\ x^2y+y^4)$, *where* $u=u_0+u_1y+\cdots,\ v=v_0+v_1y+\cdots,$	$r \geq 4,\ s \geq 5,$ $3 \mid 2r-2s$
R_t	$(x^3+xy^3,\ x^2y+y^t)$	$t \geq 5$
X_λ	$(x^3+y^4,\ x^2y+\lambda y^4)$	$\lambda \in \{0, 1\}$
Y_λ	$(x^3+y^5,\ x^2y+\lambda y^5)$	$\lambda \in \{0, 1\}$

Table 6. Unimodal ICIS of order 3 when char $F = 3$.

As for the assignment of cases, most of the discussion is the same as before, so we just make a table to present the results. See Table 7 on the next page.

When $j_3(f) \sim (x^3, xy^2)$, (x^2y, y^3+xy^2), (x^3, x^2y), we need further discussion.

Proposition 7.3. *If* $j_3(f) \sim (x^3, xy^2)$, *then* $f \sim M^3_r$ *or* $\widetilde{M}^3_{r,\lambda}$.

Proof. We have $f \sim (x^3+e(y)y^r, xy^2)$ with $r \geq 4$, $e(y) \in F[\![y]\!]$ is a unit.

If $3 \nmid r$, using α, β-trick we have $f \sim (x^3+y^r, xy^2)$.

If $3 \mid r$, reset notations, write $f \sim (e(y)x^3+y^r, xy^2) \sim (x^3+e_1x^3y+y^r, xy^2)$, where $e_1 \in F$. Then $f \sim \widetilde{M}^3_{r,e_1}$. □

$j_3(f)$	*complete transversal*	f
(x^3, xy^2)	$(y^r, 0)$, $r \geq 4$	$(x^3 + e(y)y^r, xy^2)$, $r \geq 4$
$(x^3, y^3 + xy^2)$	$(y^4, 0)$	$(x^3 + \lambda y^4, y^3 + xy^2)$, $\lambda \in \{0, 1\}$
$(x^3, y^3 + x^2y)$	$(y^4, 0)$	$(x^3 + \lambda y^4, y^3 + x^2y)$, $\lambda \in \{0, 1\}$
(x^3, y^3)	$(x^2y^2, 0)$, $(0, x^2y^2)$	$(x^3 + \lambda x^2y^2, y^3 + \mu x^2y^2)$, $\lambda, \mu \in \{0, 1\}$
(x^2y, xy^2)	$(y^r, 0)$, $(0, x^s)$, $r, s \geq 4$	$(x^2y + y^r, xy^2 + x^s)$, $r, s \geq 4$
$(x^2y, x^3 + xy^2)$	$(y^r, 0)$, $r \geq 4$	$(x^2y + y^r, x^3 + xy^2)$, $r \geq 4$
$(x^2y, y^3 + xy^2)$	$(0, x^r)$, $r \geq 4$	$(x^2y, y^3 + xy^2 + e(x)x^r)$, $r \geq 4$
$(x^3{+}x^2y, y^3{+}\lambda xy^2)$	3-*determined*	$(x^3 + x^2y, y^3 + \lambda xy^2)$
(x^3, x^2y)	$(y^r, 0)$, $(xy^s, 0)$, $(0, y^t)$	$(x^3 + a(y)y^r + b(y)y^s, xy^2 + c(y)y^t)$

Table 7. Determination of cases for unimodal ICIS of order 3 when char $F = 3$.

Proposition 7.4. *If $j_3(f) \sim (x^2y, y^3 + xy^2)$, then $f \sim K^3_r$ or $\widetilde{K}^3_{r,\lambda}$.*

Proof. We have $f \sim (x^2y, y^3 + xy^2 + e(x)x^r)$.

If $3 \nmid r$, using α, β-trick we have $f \sim (x^2y, y^3 + xy^2 + x^r) \sim K^3_r$.

If $3 \mid r$, similar as above, $f \sim (y^3 + xy^2 + \lambda xy^3 + x^r, x^2y) \sim \widetilde{K}^3_{r,\lambda}$ with $\lambda \in F$. □

When $j_3(f) \sim (x^3, x^2y)$, the following result is the same as the case char $F > 3$.

Proposition 7.5. *If $j_3(f) \sim (x^3, x^2y)$, then $f \sim (x^3 + a(y)y^r + b(y)y^s, xy^2 + c(y)y^t)$ with $r, t \geq 4$, $s \geq 3$, $a(y), b(y), c(y)$ are units. And when $r \geq 4$, $s \geq 6$, $t \geq 5$, f is not unimodal.*

We omit the discussion, which is similar to that of Proposition 5.16, and give the result directly in Table 8. This finishes the proof of Proposition 7.2.

8. Checking the modality

Let $T^{1,\sec}(f) = R^m \big/ \big(\langle f_1, \ldots, f_m\rangle \cdot R^m + \mathfrak{m}\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle\big)$. Choose an F-basis $g_1, \ldots, g_d$ of $T^{1,\sec}(f)$. T. H. Pham, G. Pfister, and G. M. Greuel have shown in [11] that $F_t(x) = F(x, t_1, \ldots, t_d) = f + t_1g_1 + \cdots + t_dg_d$ represents a formally semiuniversal deformation of f, where $\boldsymbol{t} = (t_1, \ldots, t_d)$. If $F(x, t)$ is equivalent to a family of ICIS of at most one parameter for $t \in F^d$, then f is unimodal. Here we check $f \sim l_{q,\lambda} \sim (x^2 + y^4, y^{q+2} + \lambda xy^q)$, $q \geq 3$, $\lambda^2 \notin \{0, -1\}$ in Table 1 as an example.

	weighted jet	*weight*	*complete transversal*	*form*
$s=4$	(x^3+y^4, x^2y)	$(4, 3; 12, 11)$	$(0, y^4)$	$(x^3+y^4, x^2y+\mu y^4)$, $\mu\in\{0, 1\}$
$s=5$, $r=3$	$(x^3+xy^3, x^2y+\lambda y^4)$ $\lambda\notin\{0, 1\}$	$(3, 2; 9, 8)$	0	$(x^3+xy^3, x^2y+\lambda y^4)$, $\lambda\notin\{0, 1\}$
$s=5$, $r=3$	$(x^3+xy^3, x^2y+\lambda y^4)$ $\lambda=0$	$(3, 2; 9, 8)$	$(0, y^t)$, $t\geq 5$	(x^3+xy^3, x^2y+y^t), $t\geq 5$
$s=5$, $r=3$	$(x^3+xy^3, x^2y+\lambda y^4)$ $\lambda=1$	$(3, 2; 9, 8)$	$(xy^r, 0)$, $(y^s, 0)$	$(x^3+xy^3+u(y)xy^r$ $+v(y)y^s, x^2y+y^4)$ *where* $u(y)$, $v(y)$ *are units in* $F[\![x, y]\!]$, $r\geq 4$, $s\geq 5$
$s=5$, $r\geq 4$, $t=4$	(x^3, x^2y+y^4)	$(3, 2; 9, 8)$	$(y^5, 0)$, $(y^6, 0)$	$(x^3+\lambda y^s, x^2y+y^4)$, $s=5, 6, \lambda=0, 1$
$s=5$, $r\geq 4$, $t\geq 5$	(x^3+y^5, x^2y)	$(5, 3; 15, 13)$	$(0, y^5)$	$(x^3+y^5, x^2y+\mu y^5)$, $\mu\in\{0, 1\}$
$s\geq 6$, $t=4$	$(x^3+\lambda xy^3, x^2y+y^4)$ $\lambda\notin\{0, 1\}$	$(3, 2; 9, 8)$	0	$(x^3+\lambda xy^3, x^2y+y^4)$, $\lambda\notin\{0, 1\}$
$s\geq 6$, $t=4$	$(x^3+\lambda xy^3, x^2y+y^4)$ $\lambda=0$	$(3, 2; 9, 8)$	$(y^5, 0)$, $(y^6, 0)$	$(x^3+\lambda y^s, x^2y+y^4)$, $s=5, 6, \lambda=0, 1$
$s\geq 6$, $t=4$	$(x^3+\lambda xy^3, x^2y+y^4)$	$(3, 2; 9, 8)$	$(xy^r, 0)$, $(y^s, 0)$	$(x^3+xy^3+u(y)xy^r$ $+v(y)y^s, x^2y+y^4)$ *where* $u(y)$, $v(y)$ *are units in* $F[\![x, y]\!]$, $r\geq 4$, $s\geq 5$
$r=3$, $s\geq 6$, $t\geq 5$	(x^3+xy^3, x^2y)	$(3, 2; 9, 8)$	$(0, y^t)$, $t\geq 5$	(x^3+xy^3, x^2y+y^t), $t\geq 5$

Table 8. Summary of omitted arguments completing the proof of Proposition 7.2.

First we choose generators

$$(y, 0),\ (y^2, 0),\ (y^3, 0),\ (0, y),\ (0, y^2),\ \dots,\ (0, y^{q+2}) \in T^{1,\sec}(f).$$

Note that $T^{1,\sec}(f) = (\mathfrak{m}R^2)/\widetilde{T}_f(\mathcal{K}f)$. In the proof of (**3.*b.ii.*2.3.3.1**) in Proposition 3.6, we have shown that $(0, xy^{q+1}) \in \widetilde{T}_f(\mathcal{K}f)$, so $(0, y^{q+3}) \in \widetilde{T}_f(\mathcal{K}f)$. Also $(0, y^{q+2})$ generates $(0, xy^q)$, while $(y^3, 0)$ and $(0, y^{q+1})$ generate $(0, xy^{q-1})$. Then we add $(0, x), (0, xy), \dots, (0, xy^{q-2})$ as generators. These generators form a basis of $T^{1,\mathrm{sec}}(f)$.

Let

$$g_1 = (y^2, 0),\ g_2 = (y^3, 0),\ g_3 = (0, y^2),\ \dots,\ g_{q+3} = (0, y^{q+2}),$$
$$g_{q+4} = (0, xy),\ \dots,\ g_{2q+1} = (0, xy^{q-2}).$$

We consider $F(x, \boldsymbol{t}) = f + t_1 g_1 + \dots + t_{2q+1} g_{2q+1}$, where $\boldsymbol{t} = (t_1, \dots, t_{2q+1})$. We write $F(x, \boldsymbol{t}) = (G_1, G_2) \in R^2$.

If $t_1 \neq 0$ or $t_3 \neq 0$ or $t_{q+4} \neq 0$, then $j_2(G_1)$ is nondegenerate, which means G is simple by Proposition 3.5. From now we assume $t_1 = t_3 = t_{q+4} = 0$.

If $t_2 \neq 0$ or $t_4 \neq 0$, then $j_2(G_1) \sim (x^2 + y^3)$. By Proposition 3.5, G is simple. From now we assume $t_2 = t_4 = 0$.

Now G is of the form

$$\begin{aligned} G &\sim (x^2 + y^4, y^{q+2} + \lambda y^q + t_5 y^4 + \dots + t_{q+3} y^{q+2} + t_{q+5} x y^2 + \dots + t_{2q+1} x y^{q-2}) \\ &\sim \Big(x^2 + y^4, \sum_{i \ge u} t_{i+1} y^i + x \sum_{j \ge v} t_{j+q+3} y^j\Big), \end{aligned}$$

where $q \ge 3$, $\lambda^2 \notin \{0, 1\}$, $u \ge 4$, $v \ge 2$. Comparing with (3-1) and Proposition 3.6, this corresponds to the case $\alpha = 1$, $s = 4$, $u = t \ge 4$, $v = q \ge 2$, where α, s, t, q are taken in the sense of (3-1).

By Proposition 3.6 (**3.*a.i***), in most cases, G is at most unimodal. The only unsure case is that there exists $q' < t < t'$ such that

$$G \sim (x^2 + y^4, y^{q'+2} + \lambda' x y^{q'} + u_0 x y^t + x y^{t'} + u_1 x y^{t+p})$$

with $q' \ge 3$, $t + p \le q - 2$, $\lambda'^2 = -1$ and $p \mid t - q'$, $p \nmid t' - q'$, which is of the form $\tilde{l}_{q',t,t'}$ in Table 1 containing two parameters u_0 and u_1. Then we must have

$$p + 3 \le p + q' \le t \le q - p - 2,$$

that is, $q \ge 2p + 5$. If we set $q \le 2p + 4$, then this will not happen. That is, $l_{q,\lambda}$ is unimodal for $q \le 2p + 4$.

Using the same method, we can give tables to show when the modality of the above class is 1.

Remark 8.1. If it's false that $(x^2 + y^4, y^5 + \lambda x y^3 + a x y^{p+3} + x y^{4+p} + b x y^{2p+3})$ is contact equivalent to $(x^2 + y^4, y^5 + \lambda x y^3 + c x y^{p+3} + x y^{4+p} + d x y^{2p+3})$ for general $a, b, c, d \in F$, then we can ensure that the singularities given in Table 9 are the only unimodal ICIS. Conversely, if all singularities of the form $\tilde{l}_{q,t,t'}$ are equivalent (or at least can be presented as a one-parameter family), then all the singularities given in Table 1 are unimodal. Unfortunately, we cannot judge this equivalence yet. We post it as a conjecture.

symbol	*form*	*condition*	*when is it unimodal*
h_q	$(x^2+y^4,\, xy^q)$	$q \geq 3$	$q \leq 2p+3$
i	$(x^2,\, y^5)$		
$\tilde{i}$	$(x^2,\, y^5+xy^3)$		
i^5	$(x^2,\, y^5+xy^4)$	$p=5$	
j_t	$(x^2+y^4,\, y^t)$	$t \geq 5$	$t \leq 2p+4$
$\tilde{j}_t$	$(x^2+y^4,\, y^t+xy^{t-1})$	$t \geq 5,\ p \mid t$	$t \leq 2p+4$
k_q	$(x^2+y^4,\, y^{q+3}+xy^q)$	$q \geq 3$	$q \leq 2p+3$
$l_{q,\lambda}$	$(x^2+y^4,\, y^{q+2}+\lambda xy^q)$	$q \geq 3, \lambda^2 \notin \{0,-1\}$	$q \leq 2p+4$
$\tilde{l}_{q,t,t'}$	$(x^2+y^4,\, y^{q+2}+\lambda xy^q+uxy^t+xy^{t'})$, *where* $u=u_0+u_1y^p+u_2y^{2p}+\cdots$	$\lambda^2=-1, q\geq 3$, $t \geq q+1, t' \geq t+1$, $p \mid t-q,\ p \nmid t'-q$	$q \leq 2p+3$

Table 9. Unimodularity criteria (see Remark 8.1).

Conjecture 8.2. *Let F be an algebraically closed field with characteristic p. Then the isolated complete intersection singularity*

$$(x^2+y^4,\, y^5+\lambda xy^3+axy^{p+3}+xy^{4+p}+bxy^{2p+3})$$

is not contact equivalent to

$$(x^2+y^4,\, y^5+\lambda xy^3+cxy^{p+3}+xy^{4+p}+dxy^{2p+3})$$

for general $a, b, c, d \in F$.

The modality of singularities in Table 2 (i.e., of order 2 in characteristic 2 field) does not need to be checked, since every germ of the form $(x^2+h,\, g)$ with $g \in \mathfrak{m}^4 \setminus \mathfrak{m}^5$ is equivalent to a form in Table 2, which has at most one parameter.

Using the same method to check Table 3, we find the results in Table 10. Thus every singularity in Table 3 is unimodal. This is because those of modality 2 must have a deformation to $\widetilde{P}_{r,s}$. But $(xy^k, 0)$, $(y^l, 0)$ $(k \geq 4,\ l \geq 5)$ do not exist in a basis of $T^{1,\mathrm{sec}}(f)$ at the same time for every f. Then such a deformation does not exist.

For the case $\operatorname{ord}(f)=3$, char $F=2$, see Table 11.

When an ICIS in Table 11 can deform to $\widetilde{M}^2_r$, it may have two parameters. This leads to another conjecture:

symbol	*form*	*condition*	*when is it unimodal*
H	$(x^3+x^2y,\, y^3+\lambda x^2y)$	$\lambda \neq 0$	
I	$(x^3,\, y^3+x^2y)$		
J	$(x^3,\, y^3)$		
K_r	$(x^3+xy^2,\, x^2y+y^r)$	$r \geq 4$	
$L_{r,s}$	$(x^2y+y^r,\, xy^2+x^s)$	$r, s \geq 4$	
M_r	$(x^3+y^r,\, xy^2)$	$r \geq 3$	
N_λ	$(x^3+\lambda xy^3,\, x^2y+y^4)$	$\lambda \notin \{1, 12\}$	
$P_{r,\infty}$	$(x^3+xy^3+xy^r,\, x^2y+y^4)$	$r \geq 4$	
$P_{\infty,s}$	$(x^3+xy^3+y^s,\, x^2y+y^4)$	$s \geq 5$	
$P_{r,s,\lambda}$	$(x^3+xy^3+xy^r+\lambda y^s,\, x^2y+y^4)$	$r \geq 4,\ s \geq 5,$ $\lambda \in F$	
$\widetilde{P}_{r,s}$	$(x^3+xy^3+uxy^r+vy^s,\, x^2y+y^4)$, *where* $u=u_0+u_1y+\cdots,\ v=v_0+v_1y+\cdots,$	$r \geq 4,\ s \geq 5$ $p \mid 2r-2s+3$	
R_t	$(x^3+xy^3,\, x^2y+y^t)$	$t \geq 5$	
X_λ	$(x^3+y^4,\, x^2y+\lambda y^4)$	$\lambda \in \{0, 1\}$	
Y_λ	$(x^3+y^5,\, x^2y+\lambda y^5)$	$\lambda \in \{0, 1\}$	
Z_λ	$(x^3+12xy^3+\lambda y^5,\, x^2y+y^4)$	$\lambda \in \{0, 1\}$	

Table 10. Unimodularity criteria when $\operatorname{ord}(f)=3$, char $F \neq 2, 3$.

Conjecture 8.3. *Let F be an algebraically closed field with characteristic 2. Then the isolated complete intersection singularity $(x^3+y^5+ay^6+by^8,\, x^2y+xy^2)$ is not contact equivalent to $(x^3+y^5+cy^6+dy^8,\, x^2y+xy^2)$ for general $a, b, c, d \in F$.*

If the conjecture holds, then all unimodal ICIS of order 3 in a characteristic 2 field are presented in Table 11.

For the case of $\operatorname{ord}(f)=3$, char $F=3$, we can check that every singularity in Table 6 is unimodal.

In conclusion, we get the following classification theorem:

Theorem 8.4. *Let F be an algebraically closed field with arbitrary characteristic. Then every unimodal isolated complete intersection singularity (ICIS) in $F[\![x, y]\!]$ has the form in Tables 1, 2, 3, 5, 6. Besides Tables 1, 5, every form in the other tables is unimodal. If additionally Conjecture 8.2 (resp. Conjecture 8.3) holds, then all the unimodal ICIS in Table 1 (resp. Table 5) are presented in Table 9 (resp. Table 11).*

symbol	*form*	*condition*	*when is it unimodal*
H	$(x^3+x^2y,\, y^3+\lambda x^2y)$	$\lambda \neq 0$	
I	$(x^3,\, y^3+x^2y)$		
J	$(x^3,\, y^3)$		
K_r	$(x^3+xy^2,\, x^2y+y^r)$	$r \geq 4$	
$L_{r,s}$	$(x^2y+y^r,\, xy^2+x^s)$	$r, s \geq 4$	
M^2	$(x^3+y^3,\, x^2y)$		
M_r	$(x^3+y^r,\, xy^2)$	$r \geq 4$	$r \leq 8$
M_r^2	$(x^3+y^r,\, x^2y+xy^2)$	$r \geq 4$, *r is even*	$r \leq 8$
$\widetilde{M}_r^2$	$(x^3+y^r+ey^l,\, x^2y+xy^2)$, *where* $e = e_0+e_1y^2+e_2y^4+\cdots$	$r \geq 4$, *r is odd*, $l > r$, *l is even*	$l \leq 7$
N_λ	$(x^3+\lambda xy^3,\, x^2y+y^4)$	$\lambda \neq 1$	
N_λ^2	$(x^3,\, x^2y+y^4+\lambda xy^3)$	$\lambda \in F$	
$P_{r,\infty}$	$(x^3+xy^3+xy^r,\, x^2y+y^4)$	$r \geq 4$	$r \leq 7$
$P_{\infty,s}$	$(x^3+xy^3+y^s,\, x^2y+y^4)$	$s \geq 5$	$s \leq 8$
$P_{r,s,\lambda}$	$(x^3+xy^3+xy^r+\lambda y^s,\, x^2y+y^4)$	$r \geq 4$, $s \geq 5$, $\lambda \in F$	$r \leq s+1$, $r \leq 7$ *or* $s \leq r$, $s \leq 8$
R_t	$(x^3+xy^3,\, x^2y+y^t)$	$t \geq 5$	
X_μ^2	$(x^3+y^4,\, x^2y+\mu xy^3)$	$\mu \in \{0, 1\}$	
$\widetilde{X}_\lambda^2$	$(x^3+y^4+xy^3,\, x^2y+\lambda xy^3)$	$\lambda \in F$	
Y_λ	$(x^3+y^5,\, x^2y+\lambda y^5)$	$\lambda \in \{0, 1\}$	

Table 11. Unimodularity criteria when $\mathrm{ord}(f) = 3$, char $F = 2$.

References

[1] V. I. Arnold, "Local normal forms of functions", *Invent. Math.* **35** (1976), 87–109. MR

[2] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps*, *I*: *Classification of critical points*, Birkhäuser, 1985. Reprinted 2012 in series Modern Birkhäuser Classics. MR

[3] Y. Boubakri, G.-M. Greuel, and T. Markwig, "Invariants of hypersurface singularities in positive characteristic", *Rev. Mat. Complut.* **25**:1 (2012), 61–85. MR

[4] J. W. Bruce, N. P. Kirk, and A. A. du Plessis, "Complete transversals and the classification of singularities", *Nonlinearity* **10**:1 (1997), 253–275. MR

[5] A. Dimca and C. G. Gibson, “Contact unimodular germs from the plane to the plane”, *Quart. J. Math. Oxford Ser.* (2) **34**:135 (1983), 281–295. MR

[6] W. Ferrer Santos and A. Rittatore, *Actions and invariants of algebraic groups*, Pure and Applied Mathematics **269**, Chapman & Hall/CRC, Boca Raton, FL, 2005. MR

[7] M. Giusti, “Classification des singularités isolées simples d’intersections complètes”, pp. 457–494 in *Singularities* (Arcata, CA, 1981), Proc. Sympos. Pure Math. **40**, Amer. Math. Soc., Providence, RI, 1983. MR

[8] G.-M. Greuel and H. D. Nguyen, “Right simple singularities in positive characteristic”, *J. Reine Angew. Math.* **712** (2016), 81–106. MR

[9] G.-M. Greuel and G. Pfister, *A singular introduction to commutative algebra*, 2nd ed., Springer, 2008. MR

[10] G.-M. Greuel and T. H. Pham, “On finite determinacy for matrices of power series”, *Math. Z.* **290**:3-4 (2018), 759–774. MR

[11] T. Huong Pham, G. Pfister, and G.-M. Greuel, “Classification of simple 0-dimensional isolated complete intersection singularities”, *Int. Math. Res. Not.* **2025**:14 (2025), art. id. rnaf191, 22 pp. MR

[12] J. N. Mather, “Stability of C^∞ mappings, III: Finitely determined map-germs”, *Inst. Hautes Études Sci. Publ. Math.* **35** (1968), 127–256. MR

[13] H. D. Nguyen, *Classification of singularities in positive characteristic*, PhD thesis, Technische Universität Kaiserslautern, 2013, available at https://tinyurl.com/HDNguyen-PhD.

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COUNTING MATRICES OVER FINITE RANK MULTIPLICATIVE GROUPS

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Motivated by recent works on statistics of matrices over sets of number theoretic interest, we study matrices with entries from arbitrary finite subsets $\mathcal{A}$ of finite rank multiplicative groups in fields of characteristic zero. We obtain upper bounds, in terms of the size of $\mathcal{A}$, on the number of such matrices of a given rank, with a given determinant and with a prescribed characteristic polynomial. In particular, in the case of ranks, our results can be viewed as a statistical version of work by Alon and Solymosi (2023).

1. Introduction

1.1. ***Motivation and set-up.*** For a finite subset $\mathcal{A}$ of a field $\mathbb{K}$, we define $\mathcal{M}_{m,n}(\mathcal{A})$ to be the set of $m \times n$ matrices with entries from $\mathcal{A}$. It is also convenient to omit one of the subscripts when $m = n$, writing $\mathcal{M}_n(\mathcal{A}) = \mathcal{M}_{n,n}(\mathcal{A})$. Various counting questions regarding matrices in $\mathcal{M}_{m,n}(\mathcal{A})$ where $\mathcal{A}$ is a set of arithmetic significance have been studied in a number of works. Here we are interested in the case when m and n are fixed and the size of $\mathcal{A}$ grows, that is, when

$$A = \#\mathcal{A} \to \infty.$$

Thus this is dual to the set-up when $\mathcal{A}$ is fixed, typically $\mathcal{A} = \{0, 1\}$ or $\mathcal{A} = \{-1, 1\}$,

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and $m, n \to \infty$, which has also received a lot of attention; see [11; 17; 19; 25; 26; 32; 36; 37; 41; 42; 51] as well as the survey [52].

The direction originates from the case of $\mathcal{A} = \mathbb{K} = \mathbb{F}_q$, where $\mathbb{F}_q$ is the finite field of q elements [13; 24; 27; 28; 29; 44; 45; 54].

In characteristic zero, the most studied case is the case of integer entries, bounded (in, say, $\mathcal{L}^2$ or $\mathcal{L}^\infty$ norms) by some parameter $H \to \infty$. This direction originates from works of Duke, Rudnick and Sarnak [18], Eskin, Mozes and Shah [22] and Katznelson [33; 34]; see also [10; 21; 30; 47; 48; 53] for further developments of these techniques, based on geometry of numbers and homogeneous dynamics. More recently, several different approaches to problems of arithmetic statistics for matrices have emerged [1; 2; 3; 4; 5; 9; 14; 16; 15; 20; 31; 38; 39; 43; 50]. These works are based on a variety of other techniques, including some inputs from Diophantine geometry and analytic number theory. In particular, these new ideas have given the means to approach various counting question for matrices with rational entries whose numerators and denominators are bounded by a given height H, see [4], and for matrices with entries which are polynomial values of integers from $[-H, H]$, see [9; 39], in the same regime of fixed m and n and $H \to \infty$.

There is also an emerging direction of studying matrices with entries from a completely general set, where, surprisingly some nontrivial bounds are possible [8; 35; 40; 49].

More precisely, most of the above works study the following three subsets of $\mathcal{M}_{m,n}(\mathcal{A})$:

- matrices of given determinant $d \in \mathbb{K}$,

$$\mathcal{D}_n(\mathcal{A}; d) = \{\boldsymbol{X} \in \mathcal{M}_n(\mathcal{A}) : \det \boldsymbol{X} = d\}, \tag{1-1}$$

- matrices with a given characteristic polynomial $f \in \mathbb{K}[T]$,

$$\mathcal{P}_n(\mathcal{A}; f) = \{\boldsymbol{X} \in \mathcal{M}_n(\mathcal{A}) : \det(T I_n - \boldsymbol{X}) = f\}, \tag{1-2}$$

- matrices of given rank $r \in \mathbb{N}$,

$$\mathcal{R}_{m,n}(\mathcal{A}; r) = \{\boldsymbol{X} \in \mathcal{M}_{m,n}(\mathcal{A}) : \operatorname{rank} \boldsymbol{X} = r\}. \tag{1-3}$$

As with $\mathcal{M}_n(\mathcal{A})$, we also adopt the notation $\mathcal{R}_n(\mathcal{A}; r) = \mathcal{R}_{n,n}(\mathcal{A}; r)$.

Here we consider the above questions in a new setting, when the set $\mathcal{A}$ is an arbitrary finite subset of a multiplicative subgroup Γ of finite rank in a field $\mathbb{K}$ of characteristic zero. Besides the aforementioned works, our motivation also comes from a result of Alon and Solymosi [6, Theorem 1], which shows that $n \times n$ matrices with entries from finitely generated subgroups Γ of $\mathbb{C}^*$ have a rank growing with their dimension n. In our notation, the result of [6, Section 4] can be formulated as $\mathcal{R}_n(\mathcal{A}; r) = \varnothing$, provided $r < (c_1 \log n)^{c_2}$ for some positive constants c_1 and c_2, depending only on n and the rank of Γ.

It is also interesting to note that both the approach of Alon and Solymosi [6] and our approach are based on the celebrated *Subspace Theorem* of Schmidt [46]. More precisely we use its implication for the number of nondegenerate solutions to linear equations solved over Γ, given in the currently strongest form by Amoroso and Viada [7, Theorem 6.2]. This in turn has been used in [12, Corollary 16] to estimate the total number of solutions; see Section 3 for details.

We emphasise that our bounds on the above quantities $\mathcal{D}_n(\mathcal{A}; d)$ and $\mathcal{P}_n(\mathcal{A}; f)$ are uniform with respect to d and f, and the implied constants, while the bounds on $\mathcal{R}_{m,n}(\mathcal{A}; r)$ may only depend on the dimensions m and n of a matrix, and the rank ϱ of Γ. In Lemmas 3.2, 3.3, and 3.4 on the number of solutions to linear equations, the implied constant may only depend on the number of summands n and the rank ϱ of Γ.

1.2. ***Notation.*** We recall that the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leqslant cV$ for some positive constant c, which, as above, may depend only on m, n and ϱ throughout this work.

We also write $U \asymp V$ as a shorthand for when both $U \ll V$ and $V \ll U$ hold.

When S is a finite set, we use $\#S$ to denote its cardinality.

Throughout this work we also use

$$A = \#\mathcal{A}$$

to denote the cardinality of $\mathcal{A}$.

Finally, $\mathbb{F}_q$ denotes the finite field of q elements and I_n denotes the $n \times n$ identity matrix.

1.3. ***Trivial upper bounds.*** Before we formulate our results, we record the following trivial bounds, which we use as benchmarks to illustrate the strength of our results.

Clearly, for any $n \geq 1$ and $\mathcal{A} \subseteq \mathbb{K}$ of cardinality A,

$$\#\mathcal{D}_n(\mathcal{A}; d) \ll A^{n^2-1}. \tag{1-4}$$

In fact, for $\mathcal{A} = \mathbb{K} = \mathbb{F}_q$ the bound (1-4) is tight. However, recent work by Shkredov and Shparlinski [49] shows that a better bound is possible for real matrices when $n \geq 3$, without any further restrictions on the entries.

Also, for $f = T^n + c_{n-1}T^{n-1} + \cdots + c_0 \in \mathbb{K}[T]$,

$$\#\mathcal{P}_n(\mathcal{A}; f) \ll A^{n^2-2}. \tag{1-5}$$

Indeed, writing $\boldsymbol{X} = (x_{i,j})_{i,j=1}^n$ and using $f = \det(T I_n - \boldsymbol{X})$ we see that $\boldsymbol{X}$ has a fixed trace $\operatorname{tr} \boldsymbol{X} = -c_{n-1}$, and hence we can express $x_{n,n}$ via other diagonal elements. After this the equation $\det \boldsymbol{X} = (-1)^n c_0$ becomes an algebraic equation in $n^2 - 1$

variables. This equation is nontrivial as one can see by specialising all nondiagonal elements of $\boldsymbol{X}$ to 0.

Furthermore, for any $n \ge m \ge r \ge 1$ and $\mathcal{A} \subseteq \mathbb{K}$ of finite cardinality A,

$$\#\mathcal{R}_{m,n}(\mathcal{A}; r) \ll A^{nr+mr-r^2}. \tag{1-6}$$

Indeed, without loss of generality, we can assume that the top left $r \times r$ submatrix of $\boldsymbol{X} \in \mathcal{M}_{m,n}(\mathcal{A})$ is nonsingular. Then we see that after fixing nr elements in the top r rows of $\boldsymbol{X}$ and, the $(m-r)r$ remaining elements in the first r columns of $\boldsymbol{X}$, the remaining elements are uniquely defined.

2. Main results

2.1. ***Matrices of given rank.*** Recall the definition of $\mathcal{R}_{m,n}(\mathcal{A}; r)$ given in (1-3) as the set of $m \times n$ matrices over $\mathcal{A}$ of rank r.

Theorem 2.1. *Let $\mathbb{K}$ be a field of characteristic zero, and Γ a rank ϱ subgroup of $\mathbb{K}^*$. If $n, m \ge 2$ with $n \ge m \ge r > 0$, for any finite subset $\mathcal{A}$ of Γ with cardinality A, we have*

$$\#\mathcal{R}_{m,n}(\mathcal{A}; r) \ll \begin{cases} A^{nr+m-r} & \text{if } 2m \le n+r, \\ A^{nr+m-r+\lfloor (r-1)/2 \rfloor (2m-n-r)} & \text{otherwise.} \end{cases}$$

When $2m > n+r$, Theorem 2.1 gives us a saving against the trivial bound (1-6), of

$$\frac{A^{nr+mr-r^2}}{A^{nr+m-r+\lfloor (r-1)/2 \rfloor (2m-n-r)}} = \begin{cases} A^{(n-r)(r-1)/2} & \text{for } r \text{ odd}, \\ A^{r(n-r)/2+(m-n)} & \text{for } r \text{ even}. \end{cases}$$

When $2m \le n+r$, the bound of Theorem 2.1 is tight. For instance, take $\mathbb{K} = \mathbb{Q}$, $\Gamma = \langle 2 \rangle$ and $\mathcal{A}_k = \{2^s : 1 \le s \le 2k\}$, defining $A_k = \#\mathcal{A}_k = 2k$. In this case we have $k^{nr} \asymp A_k^{nr}$ ways of fixing the first r rows with elements of the form 2^s for $1 \le s \le k$. We then have $k^{m-r} \asymp A_k^{m-r}$ ways of choosing all the other rows to be 2^s multiplied by the first row for each $1 \le s \le k$ (to guarantee elements stay within $\mathcal{A}_k$). Thus the number of matrices of rank at most r satisfies

$$\sum_{j=1}^{r} \#\mathcal{R}_{m,n}(\mathcal{A}_k; j) \asymp A_k^{nr+m-r}. \tag{2-1}$$

Therefore

$$\begin{aligned} \#\mathcal{R}_{m,n}(\mathcal{A}_k; r) &= \sum_{j=1}^{r} \#\mathcal{R}_{m,n}(\mathcal{A}_k; j) - \sum_{j=1}^{r-1} \#\mathcal{R}_{m,n}(\mathcal{A}_k; j) \\ &\gg \sum_{j=1}^{r} \#\mathcal{R}_{m,n}(\mathcal{A}_k; j) + O(A_k^{n(r-1)+m-(r-1)}) \gg A_k^{nr+m-r}. \end{aligned}$$

2.2. ***Matrices of given determinant.*** We recall the definition of $\mathcal{D}_n(\mathcal{A}; d)$ given in (1-1) as the set of $n \times n$ matrices over $\mathcal{A}$ with determinant d.

Theorem 2.2. *Let $\mathbb{K}$ be a field of characteristic zero, and Γ a rank ϱ subgroup of $\mathbb{K}^*$. For any $d \in \mathbb{K}$ and finite subset $\mathcal{A}$ of Γ with cardinality A,*

$$\#\mathcal{D}_n(\mathcal{A}; d) \ll \begin{cases} A^{n^2-\lceil n/2 \rceil} & \text{if } d = 0, \\ A^{n^2-\lceil (n+1)/2 \rceil} & \text{if } d \neq 0. \end{cases}$$

Clearly Theorem 2.2 always improves the bound (1-4) (and also the stronger bound from [49]), except when $n = 2$ and $d = 0$, in which case the bound is tight. Indeed, specialising the lower bound (2-1) to the case when $m = n$ and $r = n - 1$, we immediately see that

$$\#\mathcal{D}_n(\mathcal{A}; 0) \gg A^{n^2-n+1}.$$

Hence for $d = 0$, Theorem 2.2 is tight when $n = 2$ and $n = 3$.

2.3. ***Matrices of given characteristic polynomial.*** Recall the definition of $\mathcal{P}_n(\mathcal{A}; f)$ given in (1-2) as the set of $n \times n$ matrices over $\mathcal{A}$ with characteristic polynomial f.

We first consider the case $n = 2$ separately, due to the compatibility of the formulae for the trace and determinant in this case, which allows for a tighter bound to be acquired than in the general case.

Theorem 2.3. *Let $\mathbb{K}$ be a field of characteristic zero, and Γ a rank ϱ subgroup of $\mathbb{K}^*$. For any $d, t \in \mathbb{K}$ not both zero, and finite subset $\mathcal{A}$ of Γ with cardinality A,*

$$\#\mathcal{P}_2(\mathcal{A}; T^2 - tT + d) \ll A.$$

We do not specify a bound in Theorem 2.3 when $d = t = 0$ because in this case, the trivial bound of $O(A^2)$ is tight. This can be seen with the following construction. Let $\mathcal{A}_k = \{\pm 2^s : 0 \le s < k\}$, with $A_k = \#\mathcal{A}_k = 2k$. A matrix $\boldsymbol{X} = (x_{i,j})_{i,j=1}^2$ is in $\mathcal{P}_2(\mathcal{A}_k; T^2)$ if and only if

$$\det \boldsymbol{X} = x_{1,1}x_{2,2} - x_{1,2}x_{2,1} = 0 \quad \text{and} \quad \operatorname{tr} \boldsymbol{X} = x_{1,1} + x_{2,2} = 0,$$

or equivalently,

$$x_{1,1}^2 = -x_{1,2}x_{2,1} \quad \text{and} \quad x_{1,1} = -x_{2,2}.$$

By writing $x_{1,1} = 2^{a+b}$, $x_{1,2} = 2^{2a}$ and $x_{2,1} = -2^{2b}$ with nonnegative integers $a, b < k/2$, we see that

$$\#\mathcal{P}_2(\mathcal{A}_k; T^2) \gg A_k^2.$$

We now turn our attention to the case $n \ge 3$. Our bound is based on fixing only the coefficients of T^{n-1} and T^{n-2} in the characteristic polynomial $f \in \mathbb{K}[T]$, as motivated by the techniques of [4]. While we do not present a matching lower bound, the strength of [4, Theorem 2.3] which uses this approach indicates its unexpected power.

Before stating Theorem 2.4 for fixed characteristic polynomial, we introduce the function

$$\alpha(n) = \frac{n(n-1)}{2} + \max\left\{\left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n(n-1)}{4} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n(n-1)}{4} - \frac{1}{2} \right\rfloor\right\}. \tag{2-2}$$

Theorem 2.4. *Let $\mathbb{K}$ be a field of characteristic zero, and Γ a rank ϱ subgroup of $\mathbb{K}^*$. For any monic polynomial f of degree $n \ge 3$, and any $\mathcal{A} \subseteq \Gamma$ of finite cardinality A, we have*

$$\#\mathcal{P}_n(\mathcal{A}; f) \ll A^{\alpha(n)}$$

where $\alpha(n)$ is given by (2-2).

We note that

$$\lim_{n\to\infty} \alpha(n)/n^2 = \tfrac{3}{4}.$$

Direct calculations show that Theorem 2.4 improves (1-5) and also the bound

$$\#\mathcal{P}_n(\mathcal{A}; f) \ll \#\mathcal{D}_n(\mathcal{A}; (-1)^n c_0) \ll \begin{cases} A^{n^2-\lceil n/2\rceil} & \text{if } f(0) = 0, \\ A^{n^2-\lceil (n+1)/2\rceil} & \text{if } f(0) \ne 0, \end{cases}$$

which follows from Theorem 2.2.

Remark 2.5. In the proof of Theorem 2.4 we derive more precise bounds which depend on some properties of the coefficients of X^{n-1} and X^{n-2} from f. See Appendix B, where these bounds are presented.

Finally, we observe that Theorems 2.3 and 2.4 imply upper bounds on the number of *cyclotomic* matrices $\boldsymbol{X} \in \mathcal{M}_n(\mathcal{A})$, that is, matrices with $\boldsymbol{X}^k = I_n$ for some positive integer k.

3. Linear equations in finite rank multiplicative groups

3.1. *Counting nondegenerate solutions.* We start with the best known bound in the case of arbitrarily many summands in an arbitrary field of characteristic zero due to Amoroso and Viada [7, Theorem 6.2], however as in [12] the previous bound of Evertse, Schlickewei and Schmidt [23] is also suitable for our purpose (as well as other bounds of this kind).

Let $\mathbb{K}$ be a field of characteristic zero, and let Π be a subgroup of $(\mathbb{K}^*)^n$. We say that a solution to the equation

$$a_1x_1 + \cdots + a_nx_n = 1, \quad (x_1, \ldots, x_n) \in \Pi, \tag{3-1}$$

is nondegenerate if

$$\sum_{i\in\mathcal{I}} a_ix_i \ne 0$$

for all $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$.

Lemma 3.1. *Let $\mathbb{K}$ be a field of characteristic zero, and Π a rank ϱ subgroup of $(\mathbb{K}^*)^n$. For any $a_1, \dots, a_n \in \mathbb{K}^*$, the number of nondegenerate solutions to* (3-1) *is at most $(8n)^{4n^4(n+\varrho+1)}$.*

3.2. *Counting arbitrary solutions.* Since the entries of our matrices are drawn from a subgroup Γ of $\mathbb{K}^*$, we specialise Lemma 3.1 to the case $\Pi = \Gamma^n$.

The following result is essentially [12, Corollary 16]. Although it is presented in [12] for $\mathbb{K} = \mathbb{C}$ with integer coefficients, it extends to arbitrary fields of characteristic zero in the natural way.

Lemma 3.2. *Let $\mathbb{K}$ be a field of characteristic zero, and Γ a rank ϱ subgroup of $\mathbb{K}^*$. Suppose that $\mathcal{A} \subseteq \Gamma$ is a finite set of cardinality A. For any $a_1, \dots, a_n \in \mathbb{K}^*$, the number of solutions to*

$$a_1 x_1 + \cdots + a_n x_n = 0, \quad x_1, \dots, x_n \in \mathcal{A},$$

is $O(A^{\lfloor n/2 \rfloor})$.

It is easy to see that the bound of Lemma 3.2 is tight, since for any choice of Γ and $\mathcal{A}$, if $n = 2k$ then we can choose

$$a_1 = \cdots = a_k = 1 \quad \text{and} \quad a_{k+1} = \cdots = a_{2k} = -1,$$

allowing us to construct $A^k = A^{\lfloor n/2 \rfloor}$ solutions by setting $x_i = x_{k+i}$ for all $i \in \{1, \dots, k\}$. If $n = 2k+1$, then we may similarly consider

$$a_1 = \cdots = a_{k-1} = 1, \quad a_k = \cdots = a_{2k} = -1, \quad \text{and} \quad a_{2k+1} = 2,$$

which allows us to once again construct $A^k = A^{\lfloor n/2 \rfloor}$ solutions by setting $x_i = x_{k+i-1}$ for all $i \in \{1, \dots, k-1\}$ and $x_{2k-1} = x_{2k} = x_{2k+1}$.

For problems such as counting matrices of a given nonzero determinant, we also require a non-homogeneous (and a slightly stronger) version of Lemma 3.2 where the right-hand side of the corresponding equation is an arbitrary $a_0 \in \mathbb{K}^*$. We derive it as an application of Lemma 3.2, which we use to handle the vanishing subsums present in degenerate solutions.

Lemma 3.3. *Let $\mathbb{K}$ be a field of characteristic zero, and Γ a rank ϱ subgroup of $\mathbb{K}^*$. Suppose that $\mathcal{A} \subseteq \Gamma$ is a finite set of cardinality A. For any $a_0, a_1, \dots, a_n \in \mathbb{K}^*$, the number of solutions to*

$$a_1 x_1 + \cdots + a_n x_n = a_0, \quad x_1, \dots, x_n \in \mathcal{A},$$

is $O(A^{\lfloor (n-1)/2 \rfloor})$.

Proof. Dividing all coefficients of the above equation by a_0 we see that it is sufficient to consider the equation

$$a_1 x_1 + \cdots + a_n x_n = 1, \quad x_1, \dots, x_n \in \mathcal{A}. \tag{3-2}$$

Let $\mathfrak{A}$ denote the number of solutions to (3-2), and for each such solution $\boldsymbol{x} = (x_1, \dots, x_n)$, associate a subset $\mathcal{I}(\boldsymbol{x}) \subseteq \{1, \dots, n\}$ with the largest cardinality such that

$$\sum_{i \in \mathcal{I}(\boldsymbol{x})} a_i x_i = 0.$$

For each $\mathcal{I} \subseteq \{1, \dots, n\}$, let $\mathfrak{A}_{\mathcal{I}}$ denote the number of solutions of (3-2) such that $\mathcal{I}(\boldsymbol{x}) = \mathcal{I}$. As such, there is a particular set $\mathcal{J}$ which maximises the number of corresponding solutions such that

$$\mathfrak{A} = \sum_{\mathcal{I} \subsetneq \{1,\dots,n\}} \mathfrak{A}_{\mathcal{I}} \ll \mathfrak{A}_{\mathcal{J}}.$$

Considering now just solutions $\boldsymbol{y}$ for which $\mathcal{I}(\boldsymbol{y}) = \mathcal{J}$ in the interest of bounding $\mathfrak{A}_{\mathcal{J}}$, we may split (3-2) into the maximal degenerate part

$$\sum_{i \in \mathcal{J}} a_i x_i = 0 \tag{3-3}$$

and nondegenerate part

$$\sum_{i \in \{1,\dots,n\} \setminus \mathcal{J}} a_i x_i = 1. \tag{3-4}$$

Because solutions to (3-4) are nondegenerate by construction, the number of solutions is $\mathfrak{B} \ll 1$ by Lemma 3.1 with $\Pi = \Gamma^{n - \#\mathcal{J}}$.

By construction, $\#\mathcal{J} \le n-1$ and hence the number of solutions $\mathfrak{C}$ to (3-3) satisfies

$$\mathfrak{C} \ll A^{\lfloor (n-1)/2 \rfloor}$$

by Lemma 3.2 (except at $n = 1$, in which case the theorem we presently prove is trivial). This leads to the overall bound

$$\mathfrak{A} \ll \mathfrak{A}_{\mathcal{J}} \ll \mathfrak{B}\mathfrak{C} \ll A^{\lfloor (n-1)/2 \rfloor},$$

concluding the proof. □

As we saw when illustrating the tightness of Lemma 3.2, for the appropriate choice of $a_1, \dots, a_{n-1}$ we have $A^{\lfloor (n-1)/2 \rfloor}$ solutions to

$$a_1 x_1 + \cdots + a_{n-1} x_{n-1} = 0.$$

Choosing now $a_n = 1$ and $a_0 = x_n$ for some fixed $x_n \in \mathcal{A}$, we see that Lemma 3.3 is also tight.

We also require a bound on the number of solutions to a rather special system of two equations with elements of Γ.

Lemma 3.4. *Let $\mathbb{K}$ be a field of characteristic zero, and Γ a rank ϱ subgroup of $\mathbb{K}^*$. Suppose that $\mathcal{A} \subseteq \Gamma$ is a finite set of cardinality A. The number of solutions to the system of equations*

$$x_1 + \cdots + x_n = x_1^2 + \cdots + x_n^2 = 0, \quad x_1, \ldots, x_n \in \mathcal{A}, \tag{3-5}$$

is $O(A^{\lfloor 2n/5 \rfloor})$.

Proof. For each partition

$$\{1, \ldots, n\} = \bigsqcup_{i=1}^{h} \mathcal{I}_i$$

into $h \geq 1$ disjoint sets $\mathcal{I}_j$, with $\#\mathcal{I}_j \geq 2$, $j = 1, \ldots, h$, we count solutions to $x_1 + \cdots + x_n = 0$, which form nondegenerate solutions to each of the equations

$$\sum_{i \in \mathcal{I}_j} x_i = 0, \quad j = 1, \ldots, h.$$

Fixing one term of each equation and counting solutions in the remainder using Lemma 3.1, there are $O(A^h)$ such solutions.

Let k be the number of sets $\mathcal{I}_j$ with $\#\mathcal{I}_j = 2$, where, without loss of generality, we can assume that

$$\mathcal{I}_j = \{2j-1, 2j\}, \quad j = 1, \ldots, k.$$

Hence $h \leq k + \lfloor (n-2k)/3 \rfloor$ and thus there are at most

$$T_1 \ll A^{k + \lfloor (n-2k)/3 \rfloor} = A^{\lfloor (n+k)/3 \rfloor} \tag{3-6}$$

such solutions.

On the other hand, since we now have $x_{2j} = -x_{2j-1}$ for $j \leq k$, the equation $x_1^2 + \cdots + x_n^2 = 0$ becomes

$$2 \sum_{j=1}^{k} x_{2j-1}^2 + \sum_{j=2k+1}^{n} x_j^2 = 0,$$

which by Lemma 3.2 has at most

$$T_2 \ll A^{\lfloor (n-k)/2 \rfloor} \tag{3-7}$$

solutions, after which the remaining variables x_{2j}, $j = 1, \ldots, k$, are uniquely defined.

Choosing, for each $k \in \{0, \ldots, \lfloor n/2 \rfloor\}$, one of the bounds (3-6) or (3-7), whichever is smaller, we deduce that the number of solutions to (3-5) is $O(A^{\kappa_n})$, where

$$\kappa_n = \max_{k \in \{0, \ldots, \lfloor n/2 \rfloor\}} \min\big\{ \lfloor (n+k)/3 \rfloor, \lfloor (n-k)/2 \rfloor \big\}.$$

By noticing that if $k \le n/5$ then $(n+k)/3 \le 2n/5$, and similarly that if $k \ge n/5$ then $(n-k)/2 \le 2n/5$, it follows that

$$\kappa_n \le \frac{2n}{5},$$

and by the integrality of κ_n the result follows. □

Remark 3.5. It is not difficult to further show that $\kappa_n = \lfloor 2n/5 \rfloor$ in the proof of Lemma 3.4, with the maximum attained at $k = \lfloor n/5 \rfloor$.

4. Proofs of main results

4.1. ***Proof of Theorem 2.1.*** Our proof employs several ideas introduced in [39, Theorem 2.1], with the appropriate alterations made to use Lemmas 3.2 and 3.3, which are the new tools available in our setting.

As in the derivation of (1-6), we simplify by counting the size of the set $\mathcal{R}^*_{m,n}(\mathcal{A}; r)$ of matrices in $\mathcal{R}_{m,n}(\mathcal{A}; r)$ in which the top left $r \times r$ submatrix is nonsingular.

For arbitrary $\boldsymbol{X} = (x_{i,j})_{i,j=1}^n \in \mathcal{R}^*_{m,n}(\mathcal{A}; r)$, we may write $\boldsymbol{X}$ as the block matrix

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 \\ \boldsymbol{X}_3 & \boldsymbol{X}_4 \end{bmatrix},$$

where

$$\boldsymbol{X}_1 = (x_{i,j})_{i,j=1}^r$$

is the $r \times r$ nonsingular submatrix which exists by assumption.

There are at most

$$\mathfrak{A} \ll A^{r^2}$$

possible values for the entries of $\boldsymbol{X}_1$.

Observe that for each integer $k \in \{r+1, \dots, m\}$, the k-th row of $\boldsymbol{X}$ is a unique linear combination of the first r rows, given by coefficients $\rho_1(k), \dots, \rho_r(k) \in \mathbb{K}$. We say that $\boldsymbol{X}_3$, the matrix immediately below $\boldsymbol{X}_1$, is of type $t \in \{1, \dots, r\}$ if t is the largest number of nonzero values among the coefficients $\rho_1(k), \dots, \rho_r(k)$ taken over each $k \in \{r+1, \dots, m\}$. Suppose that, in particular the h-th row is such that t of the coefficients are nonzero, that is, h corresponds with the row which maximises the value of t. Without loss of generality we assume that it is the first t coefficients $\rho_1(h), \dots, \rho_t(h)$ which are nonzero. It is therefore possible to choose a nonsingular $t \times t$ submatrix of

$$(x_{i,j})_{1 \le i \le t, 1 \le j \le r},$$

which we assume, without loss of generality, to be

$$(x_{i,j})_{i,j=1}^{t}.$$

This means that each of the A^t choices for $(x_{h,1}, \dots, x_{h,t})$ defines fully the coefficients $(\rho_1(h), \dots, \rho_t(h))$ and subsequently $(\rho_1(h), \dots, \rho_r(h))$ by including the zero values. Thus the values of $x_{h,j}$ for $j \in \{t+1, \dots, r\}$ are also fixed, that is, the rest of the corresponding row of $\boldsymbol{X}_3$. Since h has been chosen to maximise the value of t, we can apply the same bound to each row of $\boldsymbol{X}_3$ to deduce that for each $\boldsymbol{X}_1$ there are

$$\mathfrak{B}_t = \prod_{j=r+1}^{m} A^t \ll A^{t(m-r)}$$

corresponding possible matrices $\boldsymbol{X}_3$ of type t.

Given such an h as described above, for each column indexed by $j \in \{r+1, \dots, n\}$ we have an equation

$$\rho_1(h)x_{1,j} + \dots + \rho_r(h)x_{r,j} = x_{h,j} \tag{4-1}$$

determining the value of $x_{h,j}$ in terms of the value in the j-th column of the first r rows.

Solving this equation in $(x_{1,j}, \dots, x_{r,j}, x_{h,j})$ for each j as described above fixes the upper right $r \times (n-r)$ submatrix $\boldsymbol{X}_2$, along with the remainder of the h-th row. This means that for each $i \in \{r+1, \dots, m\} \setminus \{h\}$ the analogous equation

$$\rho_1(i)x_{1,j} + \dots + \rho_r(i)x_{r,j} = x_{i,j},$$

with potentially fewer nonzero coefficients, has a fixed left-hand side, and thus $x_{i,j}$ on the right-hand side is uniquely determined.

Let $\mathfrak{C}_t$ be the maximum number of solutions to (4-1) where there are exactly t nonzero coefficients amongst $\rho_1(h), \dots, \rho_r(h)$ in variables $(x_{1,j}, \dots, x_{r,j}, x_{h,j})$ for each $j \in \{r+1, \dots, n\}$. Given that we require $n-r$ such equations for each j to count all remaining values of $\boldsymbol{X}$, summing over all possibles types t, we have an overall bound of

$$\#\mathcal{R}_{m,n}(\mathcal{A}; r) \ll \#\mathcal{R}^*_{m,n}(\mathcal{A}; r) \ll \mathfrak{A} \sum_{t=1}^{r} \mathfrak{B}_t \mathfrak{C}_t^{n-r}. \tag{4-2}$$

Subtracting $x_{h,j}$ from both sides of (4-1), we have an equation of the same form as in Lemma 3.2 with $t+1$ nonzero coefficients, and so we have

$$\mathfrak{C}_t \ll A^{\lfloor (t+1)/2 \rfloor + r - t},$$

where the factor of $A^{r-t} = A^{(r+1)-(t+1)}$ counts the number of solutions in the "free" variables corresponding to the zero coefficients.

Now, computing the bound in (4-2) we have

$$\begin{aligned}\#\mathcal{R}_{m,n}(\mathcal{A};r) &\ll \mathfrak{A}\sum_{t=1}^{r}\mathfrak{B}_t\mathfrak{C}_t^{n-r} \ll A^{r^2}\sum_{t=1}^{r}A^{t(m-r)}(A^{\lfloor (t+1)/2\rfloor+r-t})^{n-r}\\ &\ll \sum_{t=1}^{r}A^{r^2+t(m-r)+\lfloor (t+1)/2\rfloor(n-r)+(r-t)(n-r)}\\ &\ll \max_{t\in\{1,\dots,r\}}A^{r^2+t(m-r)+\lfloor (t+1)/2\rfloor(n-r)+(r-t)(n-r)}.\end{aligned}$$

By defining

$$\delta(n,m,r,t)=r^2+t(m-r)+\left\lfloor\frac{t+1}{2}\right\rfloor(n-r)+(r-t)(n-r),$$

we may write

$$\#\mathcal{R}_{m,n}(\mathcal{A};r)\ll \max_{t\in\{1,\dots,r\}}A^{\delta(n,m,r,t)}. \tag{4-3}$$

Simplifying, we find that

$$\begin{aligned}\delta(n,m,r,t)&=mt+\left\lfloor\frac{t+1}{2}\right\rfloor(n-r)-nt+nr\\ &=nr+t\left(m-\frac{n+r}{2}\right)+\begin{cases}\frac{n-r}{2} & \text{for } t \text{ odd},\\ 0 & \text{for } t \text{ even}.\end{cases}\end{aligned}$$

If $2m\le n+r$, then the maximum value of δ over t corresponds to

$$t=1. \tag{4-4}$$

If $2m>n+r$, then $\delta(n,m,r,t)$ is strictly monotonically increasing over integers t of the same parity. Thus it suffices to check the two possibilities $t\in\{r,r-1\}$. As such, we consider that

$$\begin{aligned}&\delta(n,m,r,r)-\delta(n,m,r,r-1)\\ &\qquad=r\left(m-\frac{n+r}{2}\right)-(r-1)\left(m-\frac{n+r}{2}\right)+(-1)^{r+1}\frac{n-r}{2}\\ &\qquad=\left(m-\frac{n+r}{2}\right)+(-1)^{r+1}\frac{n-r}{2}\\ &\qquad=\begin{cases}m-r & \text{for } r \text{ odd},\\ m-n & \text{for } r \text{ even}.\end{cases}\end{aligned}$$

In particular, for odd r, we have $\delta(n,m,r,r)\ge\delta(n,m,r,r-1)$, while for even r, we have $\delta(n,m,r,r)\le\delta(n,m,r,r-1)$. Therefore the choice of t which maximises δ is given by

$$t=\begin{cases}r & \text{if } r \text{ is odd},\\ r-1 & \text{if } r \text{ is even},\end{cases}$$

or equivalently

$$t = 2\left\lfloor \frac{r-1}{2} \right\rfloor + 1. \tag{4-5}$$

Therefore, in the case when $2m \le n+r$, with the choice of t in (4-4),

$$\delta(n, m, r, t) = nr + \left(m - \frac{n+r}{2}\right) + \frac{n-r}{2} = nr + m - r,$$

while for $2m > n+r$, with the choice of t in (4-5), we have

$$\begin{aligned}\delta(n, m, r, t) &= nr + \left(2\left\lfloor \frac{r-1}{2} \right\rfloor + 1\right)\left(m - \frac{n+r}{2}\right) + \frac{n-r}{2} \\ &= nr + m - r + \left\lfloor \frac{r-1}{2} \right\rfloor (2m - n - r).\end{aligned}$$

Substituting these into (4-3), we conclude the proof.

4.2. ***Proof of Theorem 2.2.*** For the case when $d = 0$, we may write $\mathcal{D}_n(\mathcal{A}; 0)$ as the set of matrices which have rank strictly less than n. Therefore,

$$\#\mathcal{D}_n(\mathcal{A}; 0) = \sum_{r=1}^{n-1} \#\mathcal{R}_n(\mathcal{A}; r).$$

Applying now Theorem 2.1 (when $2m > n+r$, which in our case $m = n$ is equivalent to $r < n$), we deduce

$$\begin{aligned}\#\mathcal{D}_n(\mathcal{A}; 0) &\ll \sum_{r=1}^{n-1} A^{nr+n-r+\lfloor (r-1)/2 \rfloor (n-r)} \\ &\ll \max_{r \in \{1,\dots,n-1\}} A^{nr+n-r+\lfloor (r-1)/2 \rfloor (n-r)}.\end{aligned} \tag{4-6}$$

Defining

$$\delta(n, r) = nr + n - r + \left\lfloor \frac{r-1}{2} \right\rfloor (n - r)$$

for the exponent in the above expression, we have

$$\begin{aligned}\delta(n, r) &= nr + n - r + \frac{r}{2}(n-r) + (r-n) \cdot \begin{cases} \frac{1}{2} & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even,} \end{cases} \\ &= r\left(\frac{3}{2}n - \frac{r}{2} - 1\right) + n + (r-n) \cdot \begin{cases} \frac{1}{2} & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even.} \end{cases}\end{aligned}$$

Straightforward computations similar to those in the proof of Theorem 2.1 show that $\delta(n, r)$ is increasing over integers r of the same parity, and that as a function of $r \in \{1, \dots, n-1\}$, it is maximised at $r = n-1$. Substituting this back in (4-6) we deduce

$$\#\mathcal{D}_n(\mathcal{A}; 0) \ll A^{n^2 - \lceil n/2 \rceil}, \tag{4-7}$$

proving the case $d = 0$.

Suppose now that $d \neq 0$. Take a matrix $\boldsymbol{X} \in \mathcal{D}_n(\mathcal{A}; d)$ and consider the Laplace expansion for the determinant across the first row given by

$$\det \boldsymbol{X} = d = \sum_{j=1}^{n} (-1)^{j+1} x_{1,j} \det \boldsymbol{X}_{1,j}, \tag{4-8}$$

where $\boldsymbol{X}_{1,j}$ is the submatrix of $\boldsymbol{X}$ obtained by removing the first row and the j-th column.

Suppose firstly that none of the minors $\det \boldsymbol{X}_{1,j}$ in (4-8) are zero. In this case, we have at most

$$\mathfrak{A} \ll A^{n^2-n}$$

possibilities for the bottom $n-1$ rows of $\boldsymbol{X}$. Given that none of the minors are zero, we solve (4-8) in the variables $x_{1,j}$ in

$$\mathfrak{B} \ll A^{\lfloor (n-1)/2 \rfloor}$$

ways by Lemma 3.3, leading to an overall bound of

$$\mathfrak{A}\mathfrak{B} \ll A^{n^2-n+\lfloor (n-1)/2 \rfloor} = A^{n^2-\lceil (n+1)/2 \rceil}. \tag{4-9}$$

Now, suppose that at least one of the minors $\det \boldsymbol{X}_{1,j}$ is zero, which now excludes the possibility $n = 2$. We can assume, without loss of generality, that in particular, $\det \boldsymbol{X}_{1,1} = 0$. Thus, there are at most

$$\mathfrak{C} = \#\mathcal{D}_{n-1}(\mathcal{A}; 0) \ll A^{(n-1)^2-\lceil (n-1)/2 \rceil} = A^{n^2-2n+1-\lceil (n-1)/2 \rceil}$$

possibilities for $\boldsymbol{X}_{1,1}$ by (4-7).

We can then fix the elements $x_{1,1}, x_{2,1}, \ldots, x_{n,1}$ in

$$\mathfrak{D} = A^n$$

ways, leaving only the first row less the top left entry unfixed. Under these assumptions, (4-8) becomes

$$d = \sum_{j=2}^{n} (-1)^{j+1} x_{1,j} \det \boldsymbol{X}_{1,j}. \tag{4-10}$$

Let $\mathfrak{E}_t$ be the number of solutions to (4-10) under the assumption that exactly t of the matrix minors are nonzero. We assume, without loss of generality, that it is the first t coefficients of the variables $x_{1,2}, \ldots, x_{1,(t+1)}$ which are nonzero.

If $t = 1$ then

$$d = -x_{1,2} \det \boldsymbol{X}_{1,2},$$

where $\det \boldsymbol{X}_{1,2} \neq 0$ is already fixed. This defines $x_{1,2}$ uniquely, while the remaining variables can be fixed in A^{n-2} ways leading to a bound of

$$\mathfrak{E}_1 \ll A^{n-2}.$$

If $t = 2$, then we have an equation

$$d = -x_{1,2} \det \boldsymbol{X}_{1,2} + x_{1,3} \det \boldsymbol{X}_{1,3}$$

with $O(1)$ solutions by Lemma 3.3, while the remaining elements can be fixed in A^{n-3} ways leading to a bound of

$$\mathfrak{E}_2 \ll A^{n-3}.$$

If $3 \le t \le n-1$, which may only happen when $n \ge 4$, we can solve for the nonzero coefficients in $A^{\lfloor (t-1)/2 \rfloor}$ by Lemma 3.3 ways and the remaining coefficients in A^{n-1-t} ways leading to

$$\mathfrak{E}_t \ll A^{n-1-t+\lfloor (t-1)/2 \rfloor}.$$

We observe that for $t = 1$ this also formally coincides with the above bound on $\mathfrak{E}_1$.

Combining these, we have a total bound on the number of matrices in $\mathcal{D}_n(\mathcal{A}; d)$ which have a singular submatrix in the Laplace expansion as

$$\mathfrak{C}\mathfrak{D}\mathfrak{E}_t \ll A^{n^2-2n+1-\lceil (n-1)/2 \rceil} \cdot A^n \cdot \begin{cases} A^{n-3} & \text{if } t = 2, \\ A^{n-1-t+\lfloor (t-1)/2 \rfloor} & \text{if } t = 1 \text{ or } 3 \le t \le n-1, \end{cases}$$

$$= \begin{cases} A^{n^2-\lceil (n+3)/2 \rceil} & \text{if } t = 2, \\ A^{n^2-\lceil (n-1)/2 \rceil - t + \lfloor (t-1)/2 \rfloor} & \text{if } t = 1 \text{ or } 3 \le t \le n-1. \end{cases}$$

One can easily check that the expression is maximised at $t = 1$. Therefore, for $1 \le t \le n-1$ we have

$$\mathfrak{C}\mathfrak{D}\mathfrak{E}_t \ll A^{n^2-\lceil (n+1)/2 \rceil}. \tag{4-11}$$

Hence, combining (4-9) and (4-11), we have overall

$$\#\mathcal{D}_n(\mathcal{A}; d) = \mathfrak{A}\mathfrak{B} + \sum_{t=2}^{n-1} \mathfrak{C}\mathfrak{D}\mathfrak{E}_t \ll A^{n^2-\lceil (n+1)/2 \rceil},$$

concluding the proof.

Remark 4.1. We note that while the bound of Theorem 2.2 for $d \neq 0$ is dominated by (4-9), we can eliminate the other bottleneck coming from (4-11), which corresponds to the case $t = 1$, by showing that this case is impossible. Since this general argument can be useful for other similar questions, we present it in Appendix A; see Proposition A.1.

4.3. ***Proofs of Theorems 2.3 and 2.4.***

4.3.1. *The case $n = 2$ (Theorem 2.3).* For each matrix

$$\boldsymbol{X} = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} \in \mathcal{P}_2(\mathcal{A};\, T^2 - tT + d),$$

the entries are related to the coefficients of the characteristic polynomial by the equations

$$x_{1,1}x_{2,2} - x_{1,2}x_{2,1} = d = \det \boldsymbol{X} \tag{4-12}$$

and

$$x_{1,1} + x_{2,2} = t = \operatorname{tr} \boldsymbol{X}. \tag{4-13}$$

Suppose firstly that $t = 0$ and $d \neq 0$. Hence by (4-13) we have $x_{1,1} = -x_{2,2}$, which, when substituted into (4-12) yields

$$-(x_{1,1})^2 - x_{1,2}x_{2,1} = d.$$

For each possible value of $x_{1,2}$, we have an equation in $(x_{1,1})^2$ and $x_{2,1}$ with $O(1)$ solutions by Lemma 3.3, solving over the set $\mathcal{A} \cup \{x^2 : x \in \mathcal{A}\}$, which contains no more than twice the number of elements in $\mathcal{A}$. This induces at most two values for $x_{1,1}$ and then for each of these, a unique value of $x_{2,2}$ by (4-13). Hence up to a constant, the value of $x_{1,2}$ determines the rest of the matrix, so there are only $O(A)$ such matrices.

Now, suppose that $t \neq 0$. The equation (4-13) has $O(1)$ solutions in $x_{1,1}$ and $x_{2,2}$ by Lemma 3.3. Assuming these values are now fixed, we trivially have $O(A)$ solutions to (4-12) in $x_{1,2}$ and $x_{2,1}$ because either value uniquely determines the other. Thus there exist $O(A)$ such matrices.

4.3.2. *The case $n \geq 3$ (Theorem 2.4).* We construct an upper bound on $\#\mathcal{P}_n(\mathcal{A};\, f)$ by acquiring an upper bound on the larger set of matrices $\boldsymbol{X} = (x_{i,j})_{i,j=1}^n \in \mathcal{M}_n(\mathcal{A})$ for which only the coefficients c_{n-1} and c_{n-2} of the characteristic polynomial

$$f = \det(T I_n - \boldsymbol{X}) = \sum_{k=0}^{n} c_k T^k$$

are fixed.

Given that c_{n-1} and c_{n-2} are given by

$$c_{n-1} = -\operatorname{tr} \boldsymbol{X} \quad \text{and} \quad c_{n-2} = \tfrac{1}{2}((\operatorname{tr} \boldsymbol{X})^2 - \operatorname{tr} \boldsymbol{X}^2), \tag{4-14}$$

we may instead equivalently fix $t_1 = \operatorname{tr} \boldsymbol{X}$ and $t_2 = \operatorname{tr} \boldsymbol{X}^2$.

Fixing t_1 leads to an equation

$$t_1 = \sum_{i=1}^{n} x_{i,i}, \tag{4-15}$$

while fixing t_2 we have

$$t_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} x_{j,i} = \sum_{i=1}^{n} x_{i,i}^2 + 2 \sum_{1 \le i < j \le n} x_{i,j} x_{j,i},$$

which leads to the equation

$$\frac{1}{2}\left(t_2 - \sum_{i=1}^{n} x_{i,i}^2\right) = \sum_{1 \le i < j \le n} x_{i,j} x_{j,i}. \tag{4-16}$$

We first note that there are at most

$$\mathfrak{A} \ll A^{n(n-1)/2}$$

possibilities for the elements $x_{i,j}$ for $1 \le i < j \le n$.

We begin by first counting the set of matrices for which

$$t_2 = \sum_{i=1}^{n} x_{i,i}^2, \tag{4-17}$$

and within this consider two cases,

$$(t_1, t_2) = (0, 0) \quad \text{and} \quad (t_1, t_2) \ne (0, 0).$$

If $(t_1, t_2) = (0, 0)$, then the number of possible values for the main diagonal is $A^{\lfloor 2n/5 \rfloor}$ by Lemma 3.4. If either $t_1 \ne 0$ or $t_2 \ne 0$, then we may solve (4-15) or (4-17) respectively for all the $x_{i,i}$ in $A^{\lfloor (n-1)/2 \rfloor}$ ways by Lemma 3.3, where in the second case we count the solutions over $\{x^2 : x \in \mathcal{A}\} \subseteq \Gamma$ which is no larger than $\mathcal{A}$, thus fixing each $x_{i,i}$ up to a constant. This means that overall the number of possibilities for the main diagonal is given by

$$\mathfrak{B} \ll \begin{cases} A^{\lfloor 2n/5 \rfloor} & \text{if } (t_1, t_2) = (0, 0), \\ A^{\lfloor (n-1)/2 \rfloor} & \text{otherwise.} \end{cases}$$

From (4-17), the left-hand side of (4-16) is zero and hence the number of possibilities for $x_{i,j}$ with $1 \le j < i \le n$ is

$$\mathfrak{C} \ll A^{\lfloor n(n-1)/4 \rfloor}$$

by Lemma 3.2.

This means that the set of matrices with a fixed trace and trace squared which satisfy (4-17) is given by

$$\mathfrak{A}\mathfrak{B}\mathfrak{C} \ll A^{n(n-1)/2 + \lfloor n(n-1)/4 \rfloor} \cdot \begin{cases} A^{\lfloor 2n/5 \rfloor} & \text{if } (t_1, t_2) = (0, 0), \\ A^{\lfloor (n-1)/2 \rfloor} & \text{otherwise.} \end{cases} \tag{4-18}$$

Now we consider the complementary case to (4-17) of matrices for which

$$t_2 \neq \sum_{i=1}^{n} x_{i,i}^2. \tag{4-19}$$

By considering solutions to (4-15), we can see that the number of possibilities for $x_{1,1}, \dots, x_{n,n}$ is

$$\mathfrak{D} \ll \begin{cases} A^{\lfloor n/2 \rfloor} & \text{if } t_1 = 0, \\ A^{\lfloor (n-1)/2 \rfloor} & \text{if } t_1 \neq 0, \end{cases}$$

by Lemma 3.2 and Lemma 3.3 respectively.

Now, in this case, from (4-19), the left-hand side of (4-16) is nonzero and hence the number of possibilities for $x_{i,j}$ with $1 \le j < i \le n$ is

$$\mathfrak{E} \ll A^{\lfloor n(n-1)/4 - 1/2 \rfloor}$$

by Lemma 3.3.

Thus the number of matrices of a fixed trace and trace squared satisfying (4-19) is

$$\mathfrak{A}\mathfrak{D}\mathfrak{E} \ll A^{n(n-1)/2 + \lfloor n(n-1)/4 - 1/2 \rfloor} \cdot \begin{cases} A^{\lfloor n/2 \rfloor} & \text{if } t_1 = 0, \\ A^{\lfloor (n-1)/2 \rfloor} & \text{if } t_1 \neq 0. \end{cases} \tag{4-20}$$

Clearly, the second bound in (4-20) is always dominated by the first bound in (4-20). One also checks that for $n \ge 3$ we have $\lfloor 2n/5 \rfloor \le \lfloor (n-1)/2 \rfloor$. Hence, the first bound in (4-18) is always dominated by the second one.

Therefore, the bounds (4-18) and (4-20) imply

$$\mathcal{P}_n(\mathcal{A}; f) \ll A^{\alpha(n)},$$

where $\alpha(n)$ is given by (2-2).

5. Further questions

There are various possible generalisations and extensions of the problems we consider here. Firstly, it is quite natural to consider special types of matrices such as those with symmetry constraints including symmetric, skew symmetric, or Hermitian matrices, as considered in [18; 21; 27]. We expect these questions require new ideas. For instance, the method of fixing the trace and trace of the square as in Theorem 2.4 lends itself particularly poorly to counting symmetric matrices.

Motivated by recent work on counting commuting pairs of matrices [14; 40], one can also ask about an upper bound on the number of commuting pairs $\boldsymbol{XY} = \boldsymbol{YX}$ with $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{M}_n(\mathcal{A})$. Once can also ask about multiplicative dependencies in s-tuples of matrices from $\mathcal{M}_n(\mathcal{A})$, similarly to questions studied in [16; 31]. For example, Theorem 2.2 combined with Theorems 2.3 and 2.4 enables us to apply some ideas from [31] for such questions (at least for sets $\mathcal{A} \subseteq \mathbb{Z}$).

Since the work of Blomer and Li [9] is a part of our motivation, it is natural to investigate the same type of applications as in [9] and thus study the statistics of gaps between values of linear forms in elements of finite rank multiplicative subgroups of $\mathbb{R}^*$. One can also generalise the results here to matrices with polynomials entries, evaluated on elements of $\mathcal{A} \subseteq \Gamma$, similarly to [9; 39].

Finally, one can ask about similar questions over fields of positive characteristic, for example for subsets of finitely generated multiplicative group in the field of rational functions over a finite field.

Appendix A. Vanishing minors in Laplace expansion

Proposition A.1. *Let $\mathbb{K}$ be a field, and suppose $n \geq 2$. For any nonsingular matrix $\boldsymbol{X} \in \mathcal{M}_n(\mathbb{K})$ with nonzero entries, the Laplace expansion about any row or column has at most $n-2$ zero minors.*

We also remark that the assumption that $\boldsymbol{X}$ has nonzero entries is stronger than necessary. It is sufficient to require that the span of the rows or columns not including the one being expanded about contains a vector with no zero entries.

Proof. Without loss of generality, we can consider expansion about the top row. Let $\boldsymbol{X} = (x_{i,j})_{i,j=1}^n \in \mathcal{M}_n(\mathbb{K})$, and similarly define

$$\widetilde{\boldsymbol{X}} = \begin{bmatrix} x_{n,1} & x_{n,2} & \cdots & x_{n,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} \end{bmatrix} \in \mathcal{M}_n(\mathbb{K})$$

as the matrix obtained by replacing the first row of $\boldsymbol{X}$ with the bottom row. Clearly $\widetilde{\boldsymbol{X}}$ is singular, and hence the Laplace expansion about the first row yields

$$0 = \det \widetilde{\boldsymbol{X}} = \sum_{j=1}^{n} (-1)^{j+1} x_{n,j} \det \boldsymbol{X}_{1,j}, \tag{A-1}$$

where, as before, $\boldsymbol{X}_{1,j}$ is the submatrix of $\boldsymbol{X}$ obtained by removing the first row and the j-th column of $\boldsymbol{X}$, or equivalently $\widetilde{\boldsymbol{X}}$.

It is impossible for $\det \boldsymbol{X}_{1,j}$ to be zero for all $j \in \{1, \ldots, n\}$, otherwise $\det \boldsymbol{X} = 0$, contradicting the assumed nonsingularity of $\boldsymbol{X}$. It is likewise impossible for exactly $n-1$ of the cofactors $\det \boldsymbol{X}_{1,j}$ to be zero, otherwise, because in this case, since all the $x_{n,j}$ are nonzero, the right-hand side of (A-1) is also nonzero.

Therefore, at most $n-2$ of the cofactors may be zero. ☐

Appendix B. Tighter bounds from the proof of Theorem 2.4

By careful consideration of the bounds in (4-18) and (4-20) over distinct cases based on the value of t_1, t_2, and n, determining the particular maximum in each instance, one can prove a tighter bound on $\#\mathcal{P}_n(\mathcal{A}; f)$ than that of Theorem 2.4. We present this bound below, without proof.

It is convenient to define

$$\beta(n) = \tfrac{3}{4}n^2 - \tfrac{1}{4}n. \tag{B-1}$$

Theorem B.1. *Let $\mathbb{K}$ be a field of characteristic zero, and Γ a rank ϱ subgroup of $\mathbb{K}^*$. For any monic polynomial*

$$f = \sum_{k=0}^{n} c_k T^k \in \mathbb{K}[T]$$

of degree $n \geq 3$, and $\mathcal{A} \subseteq \Gamma$ of finite cardinality A, we have

$$\#\mathcal{P}_n(\mathcal{A}; f) \ll A^{\beta(n)} \cdot \begin{cases} A^{-\lambda(n)} & \text{if } c_{n-1} = c_{n-2} = 0, \\ A^{-\mu(n)} & \text{if } c_{n-1} = 0 \text{ and } c_{n-2} \neq 0, \\ A^{-\nu(n)} & \text{if } c_{n-1} \neq 0, \end{cases}$$

where $\beta(n)$ is given by (B-1) *and furthermore*

$$\lambda(n) = \begin{cases} 1 & \text{if } n \equiv 0, 3 \pmod 4, \\ \frac{3}{2} & \text{if } n \equiv 1 \pmod 4 \text{ and } n \neq 5, \\ \frac{1}{2} & \text{if } n \equiv 2 \pmod 4 \text{ or } n = 5, \end{cases}$$

$$\mu(n) = \begin{cases} 1 & \text{if } n \equiv 0, 3 \pmod 4, \\ \frac{1}{2} & \text{if } n \equiv 1, 2 \pmod 4, \end{cases}$$

$$\nu(n) = \begin{cases} 1 & \text{if } n \equiv 0, 3 \pmod 4, \\ \frac{1}{2} & \text{if } n \equiv 1 \pmod 4, \\ \frac{3}{2} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

In the case when $\mathbb{K} = \mathbb{R}$ we can further improve the bound in a few particular cases by noticing that when $t_2 = 0$, the left-hand side of (4-16) must be nonzero, allowing the use of Lemma 3.3 rather than possibly Lemma 3.2. This leads to the following result.

Theorem B.2. *Suppose Γ is a rank ϱ subgroup of $\mathbb{R}^*$ and $n \geq 3$ with $n \equiv 0, 1 \pmod 4$. For any monic polynomial*

$$f = \sum_{k=0}^{n} c_k T^k \in \mathbb{R}[T]$$

of degree n and $\mathcal{A} \subseteq \Gamma$ *of finite cardinality* A, *we have*:

- *If* $c_{n-1} \neq 0$ *and* $2c_{n-2} = c_{n-1}^2$,

$$\#\mathcal{P}_n(\mathcal{A};\, f) \ll A^{\beta(n)} \cdot \begin{cases} A^{-2} & \text{if } n \equiv 0 \pmod 4, \\ A^{-3/2} & \text{if } n \equiv 1 \pmod 4, \end{cases}$$

  where $\beta(n)$ *is given by* (B-1).

- *If* $c_{n-1} = c_{n-2} = 0$ *and* $n = 5$,

$$\#\mathcal{P}_5(\mathcal{A};\, f) \ll A^{\beta(5)-3/2}.$$

Note that for $n \equiv 2, 3 \pmod 4$, the fact that $\Gamma \subseteq \mathbb{R}^*$ does not offer any advantage.

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References

[1] A. Abrams, Z. Landau, J. Pommersheim, and N. Srivastava, “On eigenvalue gaps of integer matrices”, *Math. Comp.* **94**:352 (2025), 853–862. MR

[2] M. Afifurrahman, “Some counting questions for matrix products”, *Bull. Aust. Math. Soc.* **110**:1 (2024), 32–43. MR

[3] M. Afifurrahman, “A uniform formula on the number of integer matrices with given determinant and height”, preprint, 2024. arXiv 2407.08191

[4] M. Afifurrahman, V. Kuperberg, A. Ostafe, and I. E. Shparlinski, “Statistics of ranks, determinants and characteristic polynomials of rational matrices”, *Forum Math.* **37**:4 (2025), 1259–1289. MR

[5] O. Ahmadi and I. E. Shparlinski, “Distribution of matrices with restricted entries over finite fields”, *Indag. Math.* (*N.S.*) **18**:3 (2007), 327–337. MR

[6] N. Alon and J. Solymosi, “Rank of matrices with entries from a multiplicative group”, *Int. Math. Res. Not.* **2023**:14 (2023), 12383–12399. MR

[7] F. Amoroso and E. Viada, “Small points on subvarieties of a torus”, *Duke Math. J.* **150**:3 (2009), 407–442. MR

[8] L. M. Arutyunyan, “On the growth of the number of determinants with restricted entries”, *Mat. Zametki* **109**:6 (2021), 803–809. In Russian; translated in *Mathem. Notes*, **109** (2021), 843–848. MR

[9] V. Blomer and J. Li, “Correlations of values of random diagonal forms”, *Int. Math. Res. Not.* **2023**:23 (2023), 20296–20336. MR

[10] V. Blomer and C. Lutsko, "Hyperbolic lattice point counting in unbounded rank", *J. Reine Angew. Math.* **812** (2024), 257–274. MR

[11] J. Bourgain, V. H. Vu, and P. M. Wood, "On the singularity probability of discrete random matrices", *J. Funct. Anal.* **258**:2 (2010), 559–603. MR

[12] J. Bourgain, M. Z. Garaev, S. V. Konyagin, and I. E. Shparlinski, "Multiplicative congruences with variables from short intervals", *J. Anal. Math.* **124** (2014), 117–147. MR

[13] R. P. Brent and B. D. McKay, "Determinants and ranks of random matrices over Z_m", *Discrete Math.* **66**:1-2 (1987), 35–49. MR

[14] T. Browning, W. Sawin, and V. Y. Wang, "Pairs of commuting integer matrices", preprint, 2024. arXiv 2409.01920

[15] K. Bulinski and I. E. Shparlinski, "Counting elements of the congruence subgroup", *Canad. Math. Bull.* **67**:4 (2024), 955–969. MR

[16] K. Bulinski, A. Ostafe, and I. E. Shparlinski, "Counting embeddings of free groups into $\mathrm{SL}_2(\mathbb{Z})$ and its subgroups", To appear in *Ann. Scuola Normale Pisa*.

[17] M. Campos, M. Jenssen, M. Michelen, and J. Sahasrabudhe, "The singularity probability of a random symmetric matrix is exponentially small", *J. Amer. Math. Soc.* **38**:1 (2025), 179–224. MR

[18] W. Duke, Z. Rudnick, and P. Sarnak, "Density of integer points on affine homogeneous varieties", *Duke Math. J.* **71**:1 (1993), 143–179. MR

[19] S. Eberhard, "The characteristic polynomial of a random matrix", *Combinatorica* **42**:4 (2022), 491–527. MR

[20] D. El-Baz, M. Lee, and A. Strömbergsson, "Effective equidistribution of primitive rational points on expanding horospheres", preprint, 2022. arXiv 2212.07408

[21] A. Eskin and Y. R. Katznelson, "Singular symmetric matrices", *Duke Math. J.* **79**:2 (1995), 515–547. MR

[22] A. Eskin, S. Mozes, and N. Shah, "Unipotent flows and counting lattice points on homogeneous varieties", *Ann. of Math.* (2) **143**:2 (1996), 253–299. MR

[23] J.-H. Evertse, H. P. Schlickewei, and W. M. Schmidt, "Linear equations in variables which lie in a multiplicative group", *Ann. of Math.* (2) **155**:3 (2002), 807–836. MR

[24] W. Feit and N. J. Fine, "Pairs of commuting matrices over a finite field", *Duke Math. J.* **27** (1960), 91–94. MR

[25] A. Ferber, K. Luh, and G. McKinley, "Resilience of the rank of random matrices", *Combin. Probab. Comput.* **30**:2 (2021), 163–174. MR

[26] A. Ferber, V. Jain, A. Sah, and M. Sawhney, "Random symmetric matrices: rank distribution and irreducibility of the characteristic polynomial", *Math. Proc. Cambridge Philos. Soc.* **174**:2 (2023), 233–246. MR

[27] J. Fulman, "Random matrix theory over finite fields", *Bull. Amer. Math. Soc.* (*N.S.*) **39**:1 (2002), 51–85. MR

[28] J. Fulman and L. Goldstein, "Stein's method and the rank distribution of random matrices over finite fields", *Ann. Probab.* **43**:3 (2015), 1274–1314. MR

[29] M. Gerstenhaber, "On the number of nilpotent matrices with coefficients in a finite field", *Illinois J. Math.* **5** (1961), 330–333. MR

[30] A. Gorodnik, A. Nevo, and G. Yehoshua, "Counting lattice points in norm balls on higher rank simple Lie groups", *Math. Res. Lett.* **24**:5 (2017), 1285–1306. MR

[31] P. Habegger, A. Ostafe, and I. E. Shparlinski, "Integer matrices with a given characteristic polynomial and multiplicative dependence of matrices", preprint, 2022. arXiv 2203.03880

[32] V. Jain, A. Sah, and M. Sawhney, "Singularity of discrete random matrices", *Geom. Funct. Anal.* **31**:5 (2021), 1160–1218. MR

[33] Y. R. Katznelson, "Singular matrices and a uniform bound for congruence groups of $\mathrm{SL}_n(\mathbf{Z})$", *Duke Math. J.* **69**:1 (1993), 121–136. MR

[34] Y. R. Katznelson, "Integral matrices of fixed rank", *Proc. Amer. Math. Soc.* **120**:3 (1994), 667–675. MR

[35] D. Koh, T. Pham, C.-Y. Shen, and L. A. Vinh, "On the determinants and permanents of matrices with restricted entries over prime fields", *Pacific J. Math.* **300**:2 (2019), 405–417. MR

[36] S. Koplewitz, "The corank of a rectangular random integer matrix", *Linear Algebra Appl.* **591** (2020), 160–168. MR

[37] M. Kwan and L. Sauermann, "On the permanent of a random symmetric matrix", *Selecta Math.* (*N.S.*) **28**:1 (2022), art. id. 15, 29 pp. MR

[38] L. Mérai and I. E. Shparlinski, "Number of characteristic polynomials of matrices with bounded height", *Linear Algebra Appl.* **712** (2025), 1–8. MR

[39] A. Mohammadi, A. Ostafe, and I. E. Shparlinski, "On some matrix counting problems", *J. Lond. Math. Soc.* (2) **110**:6 (2024), art. id. e70044, 28 pp. MR

[40] A. Mudgal, "On commuting pairs in arbitrary sets of 2×2 matrices", preprint, 2024. arXiv 2411.10404

[41] H. H. Nguyen and E. Paquette, "Surjectivity of near-square random matrices", *Combin. Probab. Comput.* **29**:2 (2020), 267–292. MR

[42] H. H. Nguyen and M. M. Wood, "Random integral matrices: universality of surjectivity and the cokernel", *Invent. Math.* **228**:1 (2022), 1–76. MR

[43] A. Ostafe and I. E. Shparlinski, "On the sparsity of non-diagonalisable integer matrices and matrices with a given discriminant", preprint, 2023. arXiv 2312.12626

[44] I. Reiner, "On the number of matrices with given characteristic polynomial", *Illinois J. Math.* **5** (1961), 324–329. MR

[45] C. Sanna, "On the distribution of the entries of a fixed-rank random matrix over a finite field", *Finite Fields Appl.* **93** (2024), art. id. 102333, 15 pp. MR

[46] W. M. Schmidt, "Norm form equations", *Ann. of Math.* (2) **96** (1972), 526–551. MR

[47] N. A. Shah, "Counting integral matrices with a given characteristic polynomial", *Sankhyā Ser. A* **62**:3 (2000), 386–412. MR

[48] U. Shapira and C. Zheng, "Limiting distributions of translates of divergent diagonal orbits", *Compos. Math.* **155**:9 (2019), 1747–1793. MR

[49] I. D. Shkredov and I. E. Shparlinski, "On the determinants of matrices with elements from arbitrary sets", *Mathematika* **71**:2 (2025), art. id. e70018, 18 pp. MR

[50] I. E. Shparlinski, "Some counting questions for matrices with restricted entries", *Linear Algebra Appl.* **432**:1 (2010), 155–160. MR

[51] T. Tao and V. H. Vu, "Inverse Littlewood–Offord theorems and the condition number of random discrete matrices", *Ann. of Math.* (2) **169**:2 (2009), 595–632. MR

[52] V. H. Vu, "Recent progress in combinatorial random matrix theory", *Probab. Surv.* **18** (2021), 179–200. MR

[53] D. Wei and F. Xu, "Counting integral points in certain homogeneous spaces", *J. Algebra* **448** (2016), 350–398. MR

[54] M. M. Wood, "Random integral matrices and the Cohen–Lenstra heuristics", *Amer. J. Math.* **141**:2 (2019), 383–398. MR

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SHALIKA NEWFORMS FOR GL(n)

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Let (π, V) be a generic irreducible representation of a general linear group over a p-adic field. Jacquet, Piatetski-Shapiro, and Shalika gave an open compact subgroup K, so that the subspace V^K consisting of $v \in V$ fixed by K is one-dimensional. If π has a Shalika model Λ, then we call vectors in $\Lambda(V)$ the Shalika forms of π, and those in $\Lambda(V^K)$ the Shalika newforms. In this article, in the case where π is supercuspidal, we show the nonvanishing of Shalika newforms at a minimal point in a sense. This point is not the identity, and the Shalika newform vanishes at the identity if the character defining the Shalika model is ramified. In view of this result, in this case, we give another Shalika form with nice properties.

1. Main results

Let $n = 2m$ be an even integer. The so-called Shalika period for an irreducible cuspidal automorphic representation of GL_n over a number field was introduced by Jacquet and Shalika [8] to characterize its pole at $1 \in \mathbb{C}$ of the partial exterior square L-function. The local analogue of the Shalika period, which we call the Shalika form, is defined as follows.

Let F be a p-adic field, and S the Shalika subgroup of $G_n = \mathrm{GL}_n(F)$ consisting of matrices

$$s = \begin{bmatrix} a & b \\ & a \end{bmatrix}.$$

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Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial continuous additive character. Let $\chi : F^\times \to \mathbb{C}^\times$ be a continuous character. Let π be an irreducible smooth representation of G_n. If π is realized in a subspace $\mathbb{S}_\pi(\chi)$ of the space consisting of continuous functions $J : G_n \to \mathbb{C}$ such that

$$J(sg) = \chi \circ \det(a)\psi(\mathrm{tr}(a^{-1}b))J(g),$$

then we call $J \in \mathbb{S}_\pi(\chi)$ a Shalika form of π relevant to χ. If a globally generic cuspidal automorphic representation $\Pi = \bigotimes_v \Pi_v$ of GL_n over a number field has a Shalika period relevant to a global character $\prod_v \xi_v$, then each Π_v has a realization in $\mathbb{S}_{\Pi_v}(\xi_v)$, i.e., Shalika model relevant to ξ_v, where v indicate the places of the number field, and Π_v, ξ_v indicate the v-components. Shalika forms and models for archimedean places are defined similarly. In the archimedean case, the uniqueness of Shalika models relevant to the trivial character was showed by Aizenbud, Gourevitch, Jacquet [1]. In the nonarchimedean case, that relevant to the trivial one was showed by Jacquet and Rallis [7], and those relevant to nontrivial ones by Chen and Sun [3]. Further, in this case, for generic square-integrable π, the Shalika model relevant to the trivial character characterizes the pole at $0 \in \mathbb{C}$ of the local exterior square L-function defined in [8], which is a local analogue of the original work.

Now let us dive into the heart of the matter. For various arithmetic applications, it is important to discover good Shalika forms and study their properties. In the case where π is unramified, the $\mathrm{GL}_n(\mathfrak{o})$-invariant Shalika form, unique up to a scalars, will play the expected roles, and its values are computed by Sakellaridis [15]. Our concern is the ramified case. At first, we consider the specific Shalika form — we call the Shalika newform, determined by the following newform theory due to Jacquet, Piatetski-Shapiro, and Shalika [10]. Suppose that π is generic with representation space V, conductor $\mathfrak{c}_\pi (\geq 0)$ and central character ω_π. Then, there exists a unique $v^{\mathrm{new}} \in V$ up to a scalar such that

$$\pi(k)v^{\mathrm{new}} = \omega_\pi(k_{n,n})v^{\mathrm{new}}$$

for k lying in the open compact subgroup

$$\begin{aligned}\Gamma_n(\mathfrak{c}_\pi) = \Gamma(\mathfrak{c}_\pi) &:= \mathrm{GL}_n(\mathfrak{o}) \cap \begin{bmatrix} 1_{n-1} & \\ & \varpi^{\mathfrak{c}_\pi}\end{bmatrix} \mathrm{GL}_n(\mathfrak{o}) \begin{bmatrix} 1_{n-1} & \\ & \varpi^{\mathfrak{c}_\pi}\end{bmatrix}^{-1} \\ &= \{k \in \mathrm{GL}_n(\mathfrak{o}) \mid k_{n,1}, \ldots, k_{n,n-1} \in \mathfrak{p}^{\mathfrak{c}_\pi}\}\end{aligned}$$

where $\mathfrak{o}$ indicates the ring of integers of F, $\mathfrak{p} = \varpi\mathfrak{o}$ the prime ideal, and 1_{n-1} the unit matrix of G_{n-1}. With setting V as $\mathbb{S}_\pi(\chi)$, we call v^{new} the Shalika newform of π relevant to a fixed χ, and denote by J^{new}.

To describe our main result, we make the following assumption on generic irreducible (π, V) with $V = \mathbb{S}_\pi(\chi)$, since our preparation is not yet sufficient to deal with the general case. Let $P_n \subset G_n$ denote the mirabolic subgroup. It is known that, taking Bernstein and Zelevinsky's derivatives of V repeatedly, a smooth irreducible P_n-submodule $V_l \subset V$ (the nondegenerate part of V) is obtained finally. The assumption is:

$$\text{there exists a } J \in V_l \text{ such that } J(1_n) \neq 0. \tag{1-1}$$

If π is supercuspidal (and admits a Shalika model relevant to χ), then $V = V_l$, and this condition is empty. Let e denote the conductor of χ. Our main result is this:

Theorem 1.1. *Suppose that $\psi(\mathfrak{o}) = \{1\} \neq \psi(\mathfrak{p}^{-1})$, and keep assumption* (1-1).

(i) *If $e = 0$, J^{new} does not vanish at the identity.*

(ii) *If $e > 0$, J^{new} does not vanish at*

$$g_n := \left[\begin{array}{ccccc|c} \varpi^e & & & & 1 & \\ & \varpi^{3e} & & & \varpi^e & \\ & & \ddots & & \vdots & \\ & & & \varpi^{(n-3)e} & \varpi^{(m-2)e} & \\ & & & & \varpi^{(m-1)e} & \\ \hline & & & & & 1_m \end{array}\right].$$

Remark 1.2. Although we do not give a proof in this article, it holds that

$$\operatorname{supp}(J^{\text{new}}|_{P_n}) \begin{cases} \supsetneq (S \cap P_n)P_n(\mathfrak{o}) & \text{if } e = 0, \\ = (S \cap P_n)g_n P_n(\mathfrak{o}) & \text{if } e > 0, \end{cases}$$

when the standard L-function of π equals 1, where $P_n(\mathfrak{o}) = P_n \cap \mathrm{GL}_n(\mathfrak{o})$, and supp indicates the support.

In the case of $e > 0$, an elementary argument shows that J^{new} vanishes at 1_n. In the case of $e = 0$ (resp. $e > 0$), 1_n (resp. g_n) is minimal in a certain order among the points at which $P_n(\mathfrak{o})$-invariant Shalika forms do not necessarily vanish (cf. Lemmas 4.7 and 4.9).

Apart from our work, Grobner and Matringe [5] showed the nonvanishing at 1_n, when π is unitary and $e = 0$ without the assumption (1-1). However, our concern is beyond the unitary case, and since there exist many non-unitary representations satisfying (1-1), the nonvanishing at the point is expected in general.

But here a problem arises. From our aesthetic and various application perspectives, we think it is not desirable that J^{new} vanishes at 1_n in the case of $e > 0$, contrary to the nonvanishing of the Whittaker newform at 1_n. Further, from some results on representations (τ, V) of connected reductive groups over the algebraic closure of F, in the case where an L-function $L(s, \tau)$ is defined as a generator of a fractional ideal of the principal ideal domain $\mathbb{C}[q^s, q^{-s}]$, $s \in \mathbb{C}$ (q is the cardinality of $\mathfrak{o}/\mathfrak{p}$) spanned by some zeta integrals, and a functional equation between $L(s, \tau)$

and $L(1-s, \tau^\vee)$ for the contragredient $\tau^\vee$ holds, we infer that the desirable newform theory should satisfy the following:

(i) For a suitable open compact subgroup K, and a homomorphism $\Omega : K \to \mathbb{C}^\times$, the subspace $V_\Omega := \{v \in V \mid \tau(k)v = \Omega(k)v\}$ is spanned by a vector v_τ.

(ii) The open compact subgroup K depends on the choice of the realization V, and that of L-function (there are several L-functions and zeta integrals corresponding to the representations of the L-group.).

(iii) If V consists of functionals on G of a certain type, then v_τ does not vanish at the identity.

(iv) The zeta integral of v_τ, or its "suitable" arrangement, coincides with $L(s, \tau)$, and its suitable conjugate coincides with the product of $L(s, \tau^\vee)$ and the root number appearing in the functional equation.

For these reasons, in the case of $e > 0$, we construct a Shalika form J_π from J^{new} via some translations and integrations assuming that

- J^{new} does not vanish at g_n in the case of $e > 0$, and
- $\mathfrak{c}_\pi \geq me$.

In the case of $e = 0$, assuming $J^{\mathrm{new}}(1_n) \neq 0$, we set $J_\pi = J^{\mathrm{new}}$. We will describe some properties of J_π below. We set

$$l = \mathfrak{c}_\pi - (m-1)e$$

and let $\mathcal{O}_r$ be the ring of $r \times r$ matrices with entries in $\mathfrak{o}$. For $t \in \mathbb{Z}$, we have

$$\mathcal{O}_r(t) = \begin{bmatrix} 1_{r-1} & \\ & \varpi^t \end{bmatrix} \mathcal{O}_r \begin{bmatrix} 1_{r-1} & \\ & \varpi^t \end{bmatrix}^{-1}, \quad \mathcal{R}_r(t) = \mathcal{O}_r \cap \mathcal{O}_r(t).$$

We define the ring

$$R_{\mathfrak{c}_\pi} = \mathcal{R}_n(l) \cap \begin{bmatrix} 1_m & \\ & \varpi^e 1_m \end{bmatrix} \mathcal{O}_n \begin{bmatrix} 1_m & \\ & \varpi^e 1_m \end{bmatrix}^{-1},$$

and denote by $\mathbb{K}(\mathfrak{c}_\pi)$ the group of units of $R_{\mathfrak{c}_\pi}$, which is an open compact subgroup of G_n and equals $\Gamma(\mathfrak{c}_\pi)$ in the case of $e = 0$. Explicitly, $R_{\mathfrak{c}_\pi}$ consists of matrices of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \text{with } D \in \mathcal{R}_m(l) \text{ and } A, B, \begin{bmatrix} \varpi^e 1_{m-1} & \\ & \varpi^l \end{bmatrix}^{-1} C \in \mathcal{O}_m. \tag{1-2}$$

In the case of $n = 4$,

$$R_{\mathfrak{c}_\pi} = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^e & \mathfrak{p}^e & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^l & \mathfrak{p}^l & \mathfrak{p}^l & \mathfrak{o} \end{bmatrix}.$$

The first property of J_π is that

$$\pi(k)J_\pi = \chi \circ \det(d_k)J_\pi, \quad k \in \mathbb{K}(\mathfrak{c}_\pi),$$

where d_k indicates the $m \times m$ block matrix of k in the lower right corner. For $i \in \mathbb{Z}$, let

$$B_{m,i} = \{b \in B_m \cap \mathcal{O}_m \mid \det(b) \in \varpi^i \mathfrak{o}^\times\},$$

where B_m indicates the Borel subgroup of G_m. The second property of J_π is

$$L(s,\pi) = \sum_{i=0}^{\infty} c_i q^{i(-s+1/2)}, \quad c_i = \sum_{b \in B_{m,i}/B_{m,0}} J_\pi\left(\begin{bmatrix} b & \\ & 1_m \end{bmatrix}\right).$$

Here c_i has another expression:

$$c_i = \int_{\{g \in \mathrm{GL}_m(F) \mid \det(g) = \varpi^i \mathfrak{o}^\times\}} J_\pi\left(\begin{bmatrix} g & \\ & 1_m \end{bmatrix}\right) \mathrm{d}g \tag{1-3}$$

(cf. Proposition 5.3), where $\mathrm{d}g$ indicates the Haar measure on $\mathrm{GL}_m(F)$ normalized so that $\mathrm{vol}(\mathrm{GL}_m(\mathfrak{o})) = 1$. Observe that $J_\pi(1_n) = 1$. Set

$$\upsilon_{\mathfrak{c}_\pi} = \begin{bmatrix} \varpi^l & & \\ & \varpi^e 1_{m-1} & \\ & & w_m \end{bmatrix} \in G_n,$$

where w_m indicates the standard antidiagonal Weyl element in G_m. Define $J_\pi^* \in \mathbb{S}_{\pi^\vee}(\chi^{-1})$ by

$$J_\pi^*(g) = J_\pi(w_n {}^t g^{-1} \upsilon_{\mathfrak{c}_\pi}). \tag{1-4}$$

Let $\mathbb{K}(\mathfrak{c}_\pi)^* = \upsilon_{\mathfrak{c}_\pi}^{-1 t}\mathbb{K}(\mathfrak{c}_\pi)\upsilon_{\mathfrak{c}_\pi}$, which is the units group of the ring $R^*_{\mathfrak{c}_\pi}$ consisting of matrices of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \text{with}$$
$$D \in w_m {}^t\mathcal{R}_m(l) w_m, \quad C \in \mathcal{O}_m \begin{bmatrix} \varpi^l & \\ & \varpi^e 1_{m-1} \end{bmatrix}, \quad A, B \in w_m {}^t\mathcal{O}_m(l') w_m, \tag{1-5}$$

where $l' := l - e$. In the case of $n = 4$,

$$R^*_{\mathfrak{c}_\pi} = \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-l'} & \mathfrak{o} & \mathfrak{p}^{-l'} \\ \mathfrak{p}^{l'} & \mathfrak{o} & \mathfrak{p}^{l'} & \mathfrak{o} \\ \mathfrak{p}^l & \mathfrak{p}^e & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^l & \mathfrak{p}^e & \mathfrak{p}^l & \mathfrak{o} \end{bmatrix}.$$

From the first property of J_π, it follows that

$$\pi(k)J_\pi^* = \chi \circ \det(d_k)^{-1} J_\pi^*, \quad k \in \mathbb{K}(\mathfrak{c}_\pi)^*.$$

For $i \in \mathbb{Z}$, let

$$B^e_{m,i} = \{b \in B_m \mid \det(b) \in \varpi^i \mathfrak{o}^\times, b_{11} \in \mathfrak{o}, b_{1j} \in \mathfrak{p}^{-l'}; j > 1, b_{hj} \in \mathfrak{o}; h > 1\}.$$

Then,

$$\varepsilon_\pi L(s, \pi^\vee) = \sum_{i=0}^{\infty} c_i^* q^{i(-s+1/2)}, \ c_i^* = \sum_{b \in B^e_{m,i}/B^e_{m,0}} J_\pi^*\left(\begin{bmatrix} b & \\ & 1_m \end{bmatrix}\right),$$

where c_i^* also has other expressions:

$$c_i^* = \int_{\{g \in \mathrm{GL}_m(F) \mid \det(g) = \varpi^i \mathfrak{o}^\times\}} J_\pi^*\left(\begin{bmatrix} g & \\ & 1_m \end{bmatrix}\right) \mathrm{d}g = \int_{\{g \in \mathrm{GL}_m(F) \mid \det(g) = \varpi^{i-\mathfrak{c}_\pi} \mathfrak{o}^\times\}} J_\pi\left(\begin{bmatrix} & 1_m \\ g^{-1} & \end{bmatrix}\right) \mathrm{d}g. \tag{1-6}$$

Observe that $J_\pi^*(1_n)$ equals ε_π, the root number of π. So, our J_π and $\mathbb{K}(\mathfrak{c}_\pi)$ satisfy the above conditions except for (i) in the case of $e > 0$, the one-dimensionality. This problem will be discussed in a future work.

We remark that for a cohomological cuspidal automorphic representation Π of GL_n over a number field, if one chooses local Shalika forms J_{Π_v} and $K(\mathfrak{c}_{\Pi_v})$ at bad places v, then the corresponding period in Theorem 6.7.1 of [6] becomes just $L(1/2, \Pi)/\omega^{\varepsilon_0}(\Pi_f)\omega(\Pi_\infty)$.

Another motivation of the above construction is a theta lift from G_4 to $\mathrm{GSp}_4(F)$. It is known that, in most cases, an irreducible, admissible representations τ of $\mathrm{GSp}_4(F)$ is a theta lift from a generic, irreducible smooth representation π with a Shalika model relevant to the central character of τ. In a forthcoming paper, we will construct a Whittaker "newform" for a generic, irreducible, admissible representation τ of $\mathrm{GSp}_4(F)$ using the above J_π, and show that the inequality $\mathfrak{c}_\pi \geq 2e$ holds for any generic π with a Shalika model. We think that this supports the validity of our Shalika form J_π and open compact subgroup $\mathbb{K}(\mathfrak{c}_\pi)$.

Outline. In Section 2, we introduce fundamental terminologies used throughout. In Section 3, we show the uniqueness of pre-Shalika models of V_l. In Section 4, we define an equivalence relation and ordering for the set consisting of elements in P_n at which $P_n(\mathfrak{o})$-invariant Shalika forms do not vanish necessarily (this set changes depending on e). According to this order, 1_n (resp. g_n) is minimal in the case of $e = 0$ (resp. $e > 0$). Using Hecke operators, we show that all $P_n(\mathfrak{o})$-invariant Shalika forms vanish on P_n, assuming that J^{new} vanishes at the minimal point. Theorem 1.1 follows from this result and the uniqueness of pre-Shalika models. The above J_π is constructed in Section 5.

Notation. Throughout, F denotes a p-adic field with ring of integers $\mathfrak{o}$. Let $\mathfrak{p} = \varpi\mathfrak{o}$ denote the prime ideal of $\mathfrak{o}$ and q the cardinality of the residue field. We fix an additive character $\psi : F \to \mathbb{C}^\times$ such that

$$\psi(\mathfrak{o}) = \{1\} \neq \psi(\mathfrak{p}^{-1}),$$

where $\mathbb{C}^\times$ indicates the multiplicative group of the complex number field $\mathbb{C}$. Let $\chi : F^\times \to \mathbb{C}^\times$ denote a continuous homomorphism, and e its (order of) conductor. Let $o(x)$ denote the p-adic order of $x \in F$, and $\nu(x)$ the p-adic norm: $\nu(x) = q^{-o(x)}$.

If G is a group, h, g are elements in G and H is a subgroup of G, then we write h^g and H^g for ghg^{-1} and $\{h^g \mid h \in H\}$, respectively. We use this notation for the groups:

$$\begin{aligned}
G_r &= \mathrm{GL}_r(F),\\
B_r &= \text{the standard Borel subgroup of } G_r,\\
D_r &= \text{the diagonal matrices in } G_r,\\
N_r &= \text{the unipotent matrices in } B_r,\\
P_r &= \{(p_{ij}) \in G_r \mid p_{rj} = 0 \text{ for } j < r,\ p_{rr} = 1\},\\
U_r &= \{(u_{ij}) \in N_r \mid u_{ij} = \delta_{ij} \text{ for } j < r\},\\
K_r &= \mathrm{GL}_r(\mathfrak{o}),\\
\mathfrak{S}_r &= \text{the symmetric group of degree } r.
\end{aligned}$$

For a permutation w of the set $\{1, \dots, r\}$, we also denote by w the permutation matrix in G_r defined by

$$w[g_{i,j}]w^{-1} = [g_{w(i),w(j)}],$$

and identify $\mathfrak{S}_r$ with a subgroup of G_r naturally. When a positive integer l is clear from the context, let $\acute{*} : G_r \to G_{r+l}$ denote the embedding

$$g \mapsto \acute{g} = \begin{bmatrix} g & \\ & 1_l \end{bmatrix},$$

and for a function f on a subset of G_{r+l} containing $\acute{G}_r$, let $\acute{f}$ denote the function on G_r by the pullback, where $1_l \in G_l$ indicates the identity. For a set X and a subset $A \subset X$, let $\mathrm{Ch}(x; A)$, $x \in X$, denote the characteristic (indicator) function of A.

2. Preparation

Let r be an integer larger than 1. Denote also by ψ the homomorphism

$$N = N_r \ni n \mapsto \prod_{1 \le i \le r-1} \psi(n_{i,i+1}) \in \mathbb{C}^\times. \tag{2-1}$$

Let π be an irreducible representation of $G = G_r$. It is known that $\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi)$, the $\mathbb{C}$-space of Whittaker models, is at most one-dimensional, where Ind indicates the induction functor. If π has a Whittaker model, then π is called *generic*. In this case, let $\mathbb{W}_\pi$ denote the image of π under it, and call $W \in \mathbb{W}_\pi$ *Whittaker forms of* π. For a positive integer m, let $\Gamma_r(m) = \Gamma(m) \subset G_r$ be the open compact subgroup

$$\{k \in K_r \mid k_{r,1}, \dots, k_{r,r-1} \in \mathfrak{p}^m\}. \tag{2-2}$$

By the work of Jacquet, Piatetski-Shapiro, and Shalika [10], there exists a unique $W^{\text{new}} \in \mathbb{W}_\pi$ such that

$$W^{\text{new}}(1_r) = 1, \qquad \pi(k)W = \omega_\pi(k_{rr})W, \quad k \in \Gamma(\mathfrak{c}_\pi),$$

where $\mathfrak{c}_\pi$ and ω_π indicate the conductor and central character of π, respectively. We call W^{new} the *canonical Whittaker newform* of π.

If r is even, then *the Shalika subgroup* $S_r = S \subset G_r$ is defined to be the subgroup consisting of the matrices

$$s = \begin{bmatrix} a & \\ & a \end{bmatrix} \begin{bmatrix} 1_{r/2} & b \\ & 1_{r/2} \end{bmatrix}.$$

For χ, let $\chi_\psi : S \to \mathbb{C}^\times$ denote the homomorphism

$$s \mapsto \chi \circ \det(a)\psi(\operatorname{tr}(b)). \tag{2-3}$$

F. Chen and B. Sun [3] showed that $\operatorname{Hom}_G(\pi, \operatorname{Ind}_S^G(\chi_\psi))$, the $\mathbb{C}$-space of Shalika models, is at most one-dimensional. If π has a Shalika model, we denote by $\mathbb{S}_\pi(\chi)$ the image of π under it, and call $J \in \mathbb{S}_\pi(\chi)$ *Shalika forms of* π *relevant to* χ. By definition,

$$J(sg) = \chi_\psi(s)J(g). \tag{2-4}$$

By the above Whittaker newform theory, there exists a unique $J \in \mathbb{S}_\pi(\chi)$ up to a scalar such that

$$\pi(k)J = \omega_\pi(k_{rr})J, \quad k \in \Gamma(\mathfrak{c}_\pi),$$

which are called *the Shalika newforms of* π (*relevant to* χ).

Let

$$S_r^\circ = S_r \cap P_r.$$

For an irreducible smooth representation τ of P_r, we call the $\mathbb{C}$-space

$$\operatorname{Hom}_{P_r}(\tau, \operatorname{Ind}_{S_r^\circ}^{P_r}(\chi_\psi)) \simeq \operatorname{Hom}_{S_r^\circ}(\tau, \chi_\psi)$$

the pre-Shalika models of τ *relevant to* χ, and will show that it is at most one-dimensional in the next section. If τ has a pre-Shalika model, we denote by $\mathbb{I}_\tau(\chi)$ its image of τ, and call vectors in $\mathbb{I}_\tau(\chi)$ *pre-Shalika forms of* τ *relevant to* χ.

3. Pre-Shalika model

In this section, firstly we show that, for an arbitrary χ, the irreducible smooth representation

$$\psi_n := \text{c-Ind}_{N_n}^{P_n}(\psi) \tag{3-1}$$

has a unique pre-Shalika model up to a scalar, where c-Ind indicates the compact induction functor. By this uniqueness, the restriction to P_n of a Shalika form of a supercuspidal representation coincides with a pre-Shalika form of ψ_n, and vice versa (cf. Propositions 3.10 and 3.11). Secondly, we compute the support of a $P_n(\mathfrak{o})$-invariant pre-Shalika form, playing the essentially important role in this article, of ψ_n constructed by the pre-Shalika model, where

$$P_n(\mathfrak{o}) := \{p \in P_n \mid p_{ij} \in \mathfrak{o}\}.$$

For an l-group G (cf. [2]), we denote by $\mathscr{A}(G)$ (resp. $\mathscr{I}(G)$) the category of smooth (resp. smooth irreducible) $\mathbb{C}[G]$-modules. For $\tau \in \mathscr{A}(P_r) \cup \mathscr{A}(G_r)$, we call *the lift of τ to P_l* with $l > r$ the representation

$$\tau_l := \begin{cases} \Psi^{l-r}(\tau) & \text{if } \tau \in \mathscr{A}(P_r) \\ \Psi^{l-r-1} \circ \Upsilon(\tau) & \text{if } \tau \in \mathscr{A}(G_r) \end{cases}$$

where $\Psi : \mathscr{A}(P_s) \to \mathscr{A}(P_{s+1})$ and $\Upsilon : \mathscr{A}(G_s) \to \mathscr{A}(P_{s+1})$ are the functors defined by

$$\Psi : \tau \mapsto \text{c-Ind}_{\acute{P}_s U_{s+1}}^{P_{s+1}}(\tau \ltimes \psi), \quad \Upsilon : \tau \mapsto \tau \ltimes \mathbf{1}_{U_{s+1}}.$$

Here $\tau \ltimes \psi$ (resp. $\tau \ltimes \mathbf{1}_{U_{s+1}}$) indicates the representation sending $\acute{p}u \in \acute{P}_s U_{s+1}$ to $\psi(u)\tau(p)$ (resp. $\tau(p)$). Abbreviate $\text{Ind}_{S_n^\circ}^{P_n}(\chi_\psi)$ as $\mathbb{I}^n(\chi)$.

Proposition 3.1. *Let $\tau \in \mathscr{I}(G_r)$. For even $n > r$, we have*:

(i) *If r is odd, then τ_n has no pre-Shalika model.*

(ii) *If r is even, then*

$$\dim \text{Hom}_{P_n}(\tau_n, \mathbb{I}^n(\chi)) = \dim \text{Hom}_{G_r}\big(\tau, \text{Ind}_{S_r}^{G_r}(\nu^{(n-r)/2}\chi_\psi)\big).$$

This follows from induction on n and the following three lemmas.

Lemma 3.2. *If r is odd, then any lift τ_{r+1} of $\tau \in \mathscr{A}(G_r)$ does not have a pre-Shalika model relevant to any χ.*

Proof. Let $\lambda \in \text{Hom}_{P_{r+1}}(\tau_{r+1}, \mathbb{I}^{r+1}(\chi))$, and $f \in \tau_{r+1}$. By the definition of Υ and that of the pre-Shalika space,

$$\lambda(f) = \lambda(\tau_{r+1}(u)f) = \psi(u_{(r+1)/2,r+1})\lambda(f), \quad u \in U_{r+1}.$$

Hence $\lambda(f)$ vanishes, and the assertion follows. □

Lemma 3.3. *For $\tau \in \mathscr{A}(G_n)$ with n even, and an arbitrary χ,*

$$\dim \mathrm{Hom}_{P_{n+2}}(\tau_{n+2}, \mathbb{I}^{n+2}(\chi)) = \dim \mathrm{Hom}_{G_n}(\tau, \mathrm{Ind}_{S_n}^{G_n}(\nu\chi_\psi)).$$

Lemma 3.4. *For $\tau \in \mathscr{A}(P_n)$ with n even, and an arbitrary χ,*

$$\dim \mathrm{Hom}_{P_{n+2}}(\tau_{n+2}, \mathbb{I}^{n+2}(\chi)) = \dim \mathrm{Hom}_{P_n}(\tau, \mathbb{I}^{n}(\nu\chi)).$$

To show the last two lemmas, we use the distributional technique of [loc. cit.] for l-spaces X. For a $\mathbb{C}$-vector space V, let $\mathscr{S}(X, V)$ denote the space of all Schwartz functions on X with values in V. Linear functionals T on $\mathscr{S}(X, V)$ are called V-distributions on X. Additionally, if X is an l-group, and T is right (resp. left) invariant under an open compact subgroup, we say T is *locally constant on the right* (resp. *left*).

Proposition 3.5. *Let G be an l-group, and V a vector space over $\mathbb{C}$. Let T be a V-distribution on G.*

(i) *If T is right* (*resp. left*) *invariant, then T is the product of a right* (*resp. left*) *Haar measure on G and a linear functional on V.*

(ii) *If T is locally constant on the right* (*resp. left*), *then T is the product of a right* (*resp. left*) *Haar measure on G, and a V^*-valued continuous function on G, where V^* indicates the full-dual of V.*

Proof. Suppose that T is right invariant. Let $\mathrm{d}_r g$ denote a right Haar measure on G. Let $\{N_\alpha \mid \alpha \in \mathfrak{A}\}$ be a fundamental system of neighborhoods of the identity consisting of open compact subgroups of G. For $v \in V$, define $\varphi_v^\alpha \in \mathscr{S}(G, V)$ by

$$\varphi_v^\alpha(g) = \mathrm{Ch}(g; N_\alpha)v.$$

For each $\alpha \in \mathfrak{A}$, define $v_\alpha^* \in V^*$ by

$$\langle v_\alpha^*, v\rangle = \mathrm{vol}(N_\alpha)^{-1} T(\varphi_v^\alpha), \quad v \in V.$$

Therefore $v_\alpha^* = v_\beta^*$ if N_α contains N_β, since $T(\varphi_v^\alpha)$ equals $[N_\alpha : N_\beta]T(\varphi_v^\beta)$ by the right invariance property of T. This implies that v_α^* is independent of the choice of α. Since each $\varphi \in \mathscr{S}(G, V)$ is right invariant under some N_α, we may express φ as a finite sum of right translations of $\varphi_{v_i}^\alpha$ with some v_i's. By the right invariance property of T again, we have

$$\begin{aligned} T(\varphi) &= T\Big(\sum_i \varphi_{v_i}^\alpha\Big) = \sum_i \mathrm{vol}(N_\alpha)\langle v_\alpha^*, v_i\rangle \\ &= \sum_i \int_G \langle v_\alpha^*, \varphi_{v_i}^\alpha(g)\rangle \,\mathrm{d}_r g = \int_G \langle v_\alpha^*, \varphi(g)\rangle \,\mathrm{d}_r g. \end{aligned}$$

This is the first assertion. The second then follows from the proof of Proposition 1.28 of [loc. cit.]. □

Let G be an l-group. Let Q_0 and U be its closed subgroups such that $Q_0 \cap U = \{1\}$, and Q_0 normalizes U. Set

$$Q = Q_0 U,$$

which is a closed subgroup of G. Let $\xi : U \to \mathbb{C}^\times$ be a continuous homomorphism stabilized by Q_0. Let $H \subset G$ be a closed subgroup, and $\rho : H \to \mathbb{C}^\times$ a continuous homomorphism. Assume that

$$QH = \left\{ g \in G \mid \xi(h^g) = \rho(h) \text{ for all } h \in H \cap U^{g^{-1}} \right\}. \tag{3-2}$$

Observe the last set is left Q-, right H-invariant, closed, and an l-space in the induced topology. Additionally, assume that

$$Q \cap H = (Q_0 \cap H)(U \cap H). \tag{3-3}$$

Proposition 3.6. *With the above assumptions, for an arbitrary $\pi \in \mathscr{A}(Q_0)$, we have*

$$\dim \mathrm{Hom}_H(\text{c-Ind}_Q^G(\pi \ltimes \xi), \rho) = \dim \mathrm{Hom}_{Q_0 \cap H}(\pi, \Delta_{Q\cap H} \Delta_H^{-1} \rho),$$

where $\Delta_{Q\cap H}$, Δ_H indicate the modular characters on the groups.

Proof. Our proof is a modification of that of Proposition 1 of [12]. Abbreviate $\pi \ltimes \xi$ and $\Delta_{Q\cap H}\Delta_H^{-1}$ as π_ξ and Δ, respectively. We will construct a linear map from the $\mathbb{C}$-space $\mathrm{Hom}_H(\text{c-Ind}_Q^G(\pi_\xi), \rho)$ to $\mathrm{Hom}_{Q_0\cap H}(\pi, \Delta\rho)$. Denote the representation space of π by V_π, on which Q also acts by π_ξ. For $\phi \in \mathscr{S}(G, V_\pi)$, define $f_\phi \in \text{c-Ind}_Q^G(\pi_\xi)$ by

$$f_\phi(g) = \int_Q \pi_\xi(q^{-1})\phi(qg)\, \mathrm{d}_r q.$$

The linear map $\phi \mapsto f_\phi$ is bijective. For $\mu \in \mathrm{Hom}_H(\text{c-Ind}_Q^G(\pi_\xi), \rho)$, define a V_π-distribution T_μ on G by $T_\mu(\phi) = \mu(f_\phi)$. Observe that $T = T_\mu$ satisfies

$$\begin{aligned} T \circ R(h) &= \rho(h) T && (h \in H), \\ T \circ L(q) &= \Delta_Q(q) T \circ \pi_\xi(q^{-1}) && (q \in Q). \end{aligned} \tag{3-4}$$

Consider the double coset space $Q\backslash G/H$. By (3-2), if g does not lie in QH, then $\rho(h) \neq \xi(h^g)$ for some $h \in H$, and $T(g)$ is zero by (3-4). Therefore, the support of T is contained in QH, and we may regard T as a V_π-distribution on the closed subset QH (an l-space by (3-2)).

Let $\varphi \in \mathscr{S}(Q \times H, V_\pi)$. Define $\overline{\varphi} \in \mathscr{S}(QH, V_\pi)$ by

$$\overline{\varphi}(q^{-1}h) = \int_{Q\cap H} \Delta_Q(aq)\pi_\xi((aq)^{-1})\rho(ah)\varphi(aq, ah)\, \mathrm{d}_r a.$$

The linear map $\varphi \mapsto \overline{\varphi}$ is bijective. Therefore any V_π-distribution T' on $Q \times H$ is derived from a V_π-distribution T on QH satisfying (3-4) by setting $T'(\varphi) = T(\overline{\varphi})$.

For $q' \in Q, h' \in H$, we compute

$$\begin{aligned}
&\overline{R(q',h')\varphi}(q^{-1}h) \\
&\qquad = \int_{Q\cap H} \Delta_Q(aq)\pi_\xi((aq)^{-1})\rho(ah)\varphi(aqq', ahh')\,\mathrm{d}_r a \\
&\qquad = \frac{\pi_\xi(q')}{\Delta_Q(q')\rho(h')} \int_{Q\cap H} \Delta_Q(aqq')\pi_\xi((aqq')^{-1})\rho(ahh')\varphi(aqq', ahh')\,\mathrm{d}_r a \\
&\qquad = \frac{\pi_\xi(q')}{\Delta_Q(q')\rho(h')} L(q')R(h')\overline{\varphi}(q^{-1}h).
\end{aligned}$$

Therefore T' is right invariant by (3-4). By Proposition 3.5, there exists a linear functional μ' on V_π such that

$$T'(\varphi) = \int_H \int_Q \langle \mu', \varphi(q,h)\rangle\,\mathrm{d}_r q\,\mathrm{d}_r h. \tag{3-5}$$

For $b \in Q \cap H$, we compute

$$\begin{aligned}
\overline{L(b,b)\varphi}(q^{-1}h) &= \int_{Q\cap H} \Delta_Q(aq)\pi_\xi((aq)^{-1})\rho(ah)\varphi(b^{-1}aq, b^{-1}ah)\,\mathrm{d}_r a \\
&= \Delta_{Q\cap H}(b) \int_{Q\cap H} \Delta_Q(baq)\pi_\xi((aq)^{-1}b^{-1})\rho(bah)\varphi(aq, ah)\,\mathrm{d}_r a \\
&= \Delta_{Q\cap H}(b)\Delta_Q(b)\rho(b)\overline{\pi_\xi(b^{-1})\varphi}(q^{-1}h).
\end{aligned}$$

Thus

$$T' \circ L(b,b) = \Delta_{Q\cap H}(b)\Delta_Q(b)\rho(b)T' \circ \pi_\xi(b^{-1}).$$

Now from (3-5),

$$\Delta_H(b)\langle \mu', \varphi(q,h)\rangle = \Delta_{Q\cap H}(b)\rho(b)\langle \mu', \pi_\xi(b^{-1})\varphi(q,h)\rangle.$$

This means that μ' lies in $\mathrm{Hom}_{Q\cap H}(\pi_\xi, \Delta\rho)$. By restricting to Q_0, we obtain a $\mu'' \in \mathrm{Hom}_{Q_0\cap H}(\pi, \Delta\rho)$. The linear map $\mu' \mapsto \mu''$ is bijective by (3-3). We have constructed the desired map $\mu \mapsto \mu''$. One can reverse the above steps, and easily find that this map is bijective. □

Remark 3.7. The spaces $\mathrm{Hom}_H(\text{c-Ind}_Q^G(\pi_\xi), \rho)$ and $\mathrm{Hom}_{Q\cap H}(\pi_\xi, \Delta\rho)$ are both zero if $\Delta|_{U\cap H} \not\equiv \mathbf{1}$.

Let $n = 2m$ be an even integer. By Frobenius duality,

$$\mathrm{Hom}_{P_{n+2}}(\tau_{n+2}, \mathbb{I}^{n+2}(\chi)) \simeq \mathrm{Hom}_{S_{n+2}^\circ}(\tau_{n+2}, \chi_\psi),$$
$$\mathrm{Hom}_{P_n}(\tau, \mathbb{I}^n(\nu\chi)) \simeq \mathrm{Hom}_{S_n^\circ}(\tau, \nu\chi_\psi).$$

We will prove Lemma 3.4 by showing the spaces in the right hands are same-dimensional. By the previous proposition, it suffices to check (3-2), (3-3) in the situation considered. Set $G = P_{n+2}$. Let $h_{n+1} \in \mathfrak{S}_{n+1}$ be the permutation

$$\begin{pmatrix} 1 & \cdots & m & m+1 & m+2 & m+3 & \cdots & n+1 \\ 1 & \cdots & m & 2m+1 & m+1 & m+2 & \cdots & n \end{pmatrix}. \tag{3-6}$$

Abbreviate

$$\eta = \acute{h}_{n+1} \in P_{n+2}. \tag{3-7}$$

Set $H = \eta S^{\circ}_{n+2} \eta^{-1}$. Let $\rho : H \to \mathbb{C}^{\times}$ be the homomorphism

$$h \mapsto \chi_{\psi}(\eta^{-1} h \eta).$$

If we write a typical $s \in S^{\circ}_{n+2}$ as

$$\begin{bmatrix} A & {}^t\alpha & X & {}^t\beta \\ & 1 & \gamma & y \\ & & A & {}^t\alpha \\ & & & 1 \end{bmatrix},$$

where A, X are $m \times m$ matrices, α, β, γ are m-dimensional row vectors, and y is an element of F, then

$$s^{\eta} = \begin{bmatrix} A & X & {}^t\alpha & {}^t\beta \\ & A & & {}^t\alpha \\ & \gamma & 1 & y \\ & & & 1 \end{bmatrix}, \quad \rho(s^{\eta}) = \chi \circ \det(A) \psi(y + \operatorname{tr}(A^{-1}X - A^{-1\,t}\alpha\gamma)).$$

Set $Q_0 = \acute{P}_n \subset G_{n+2}$, and $U = U_{n+1} U_{n+2}$. Set $\xi = \psi|_U$. Observe that

(a) $\tau_{n+2} = \text{c-Ind}_Q^G(\tau \ltimes \xi)$,

(b) $\rho(\acute{t}) = \chi_{\psi}(t)$ for $t \in S_n$,

(c) $\acute{S}^{\circ}_n = Q_0 \cap H$,

(d) $\Delta_{Q \cap H} \Delta_H^{-1}(\acute{t}) = |\det(A)|$ for $t = \begin{bmatrix} A & * \\ & A \end{bmatrix} \in S^{\circ}_n$.

It is easy to check (3-3). For (3-2), it suffices to see that any matrix p in the RHS of (3-2) satisfies:

$$p_{n,1} = \cdots = p_{n,m} = p_{n+1,1} = \cdots = p_{n+1,m} = 0.$$

Let $E_{i,j}$ denote the i-th row and j-th column matrix unit (having all entries 0 apart from 1 in position (i, j)). If $p_{n+1,i} \neq 0$ for $i \in \{1, \ldots, m\}$, then for some $h = 1_{n+2} + x E_{i,n+2} \in H$,

$$\rho(h) = 1 \neq \psi(x p_{n+1,i}) = \xi(h^p).$$

Hence, $p_{n+1,1} = \cdots = p_{n+1,m} = 0$. Now we may assume that p lies in $\acute{P}_{n+1} U_{n+2}$, since the RHS of (3-2) is right H-invariant. If $p_{n,i} \neq 0$ for $i \in \{1, \ldots, m\}$, then for some $h' = 1_{n+2} + y E_{i,n+1} + y E_{i+m,n+2} \in H$, we have $\rho(h') = 1 \neq \psi(y p_{n,i}) = \xi(h'^p)$. Hence, $p_{n,1} = \cdots = p_{n,m} = 0$. Now (3-2) is checked, and the proof of the lemma is completed.

For Lemma 3.3, reset $Q_0 = \acute{G}_n$. Then (a) and (b) above hold. If one replaces S_n° with S_n at (c) and (d), then they hold. It is also easy to check (3-3), and the same argument for (3-2) works. This means that $\mathrm{Hom}_{S_{n+2}^\circ}(\tau_{n+2}, \chi_\psi)$ and $\mathrm{Hom}_{S_n}(\tau, \nu\chi_\psi)$ have the same dimension for $\tau \in \mathscr{A}(G_n)$.

Similar to Proposition 3.1, the following holds.

Proposition 3.8. *For an arbitrary χ, ψ_n has a unique pre-Shalika model relevant to χ, up to a scalar.*

Proof. Since ψ_n equals $\Psi^{n-2}(\psi_2)$, the assertion is reduced to the evaluation of the dimension of $\mathrm{Hom}_{P_2}(\psi_2, \mathbb{I}^2(\chi))$ by Lemma 3.4. Consider the space of corresponding $\mathbb{C}$-distributions on P_2, and the corresponding double coset space $N_2 \backslash P_2 / N_2$. As a complete system of its representatives, using $\{\acute{t} \in P_2 \mid t \in F^\times\}$, we find that all supports of the distributions are contained in N_2. □

Since any irreducible smooth representation of P_n is equivalent to ψ_n, or a lift from $\mathscr{I}(G_r)$, $r < n$ (cf. [2]), we obtain:

Theorem 3.9. *An irreducible smooth representation of P_n has no or a unique pre-Shalika model relevant to χ, up to a scalar.*

From now, let $(\pi, V) \in \mathscr{I}(G_n)$ be generic. There exists a Jordan–Hölder sequence of smooth P_n-modules $V_l \subset \cdots \subset V_0 = V$ with the following properties (cf. [2]):

- V_l is equivalent to ψ_n.
- Each V_i / V_{i+1} is equivalent to some lift from $\mathscr{I}(G_r)$, $r < n$.
- π is supercuspidal if and only if $V = V_l$.

Proposition 3.10. *If* (1-1) *is assumed*, *then*, *for any pre-Shalika form of ψ_n relevant to χ, there exists a Shalika form in V_l whose restriction to P_n coincides with it.*

Proof. By the assumption, for a Shalika form in V_l, its restriction to P_n is nontrivial, and can be regarded as a pre-Shalika form of ψ_n. The assertion follows from the irreducibility of $V_l \simeq \psi_n \simeq \mathbb{I}_{\psi_n}(\chi)$. □

Proposition 3.11. *Assume that each V_i / V_{i+1} has no pre-Shalika model relevant to χ. Then*, (1-1) *is satisfied. Further*, *for a Shalika form J*, *the following statements are equivalent.*

(i) *J does not vanish on P_n.*

(ii) *J lies in V_l.*

In this case, *$J|_{P_n}$ coincides with a pre-Shalika form of ψ_n.*

Proof. Let V_i^0 be the P_n-submodule

$$\{J \in V_i \mid J|_{P_n} \equiv 0\} \subset V_i.$$

Take the minimal V_r containing a Shalika form that does not vanish at the identity. By definition, if $r \neq l$, then we have $V_{r+1} = V_{r+1}^0 \subset V_r^0$ and V_r / V_{r+1} has a pre-Shalika model, conflicting with the assumption. Hence the first assertion follows. This argument also implies the equivalence of (i) and (ii). The last assertion follows from the proof of Proposition 3.10. □

The following is an easy sufficient condition for the lack of Shalika models.

Lemma 3.12. *Let $n = 2m$ be an even integer, and $\pi \in \mathscr{I}(G_n)$ be unramified. If χ is ramified, then π has no Shalika model relevant to χ.*

Proof. Assume that π has a Shalika model relevant to a ramified χ. Then there exists a nontrivial K_n-invariant $J \in \mathbb{S}_\pi(\chi)$. Let $\acute{J}$ denote the restriction to G_m. By the Iwasawa decomposition of G_n, $J|_{B_n} \not\equiv 0$. By (2-4), $\acute{J}|_{B_m} \not\equiv 0$. By (2-4) and the Cartan decomposition of G_m, there exists a $d \in D_m$ such that $\acute{J}(d) \neq 0$. By (2-4) and the K_n-invariance property of J,

$$\chi(t)\acute{J}(d) = \acute{J}(\mathrm{diag}(t, \overbrace{1, \dots, 1}^{m-1})\, d) = \acute{J}(d\, \mathrm{diag}(t, \overbrace{1, \dots, 1}^{m-1})) = \acute{J}(d), \quad t \in \mathfrak{o}^\times.$$

Since χ is ramified, it follows that $\acute{J}(d) = 0$, a contradiction. □

We are also interested in the question whether the lift of $\tau \in \mathscr{I}(G_r)$ to P_n, $n > r$ has a $P_n(\mathfrak{o})$-invariant vector, or not.

Lemma 3.13. *If $\tau \in \mathscr{I}(G_r)$ is ramified, then the lift τ_n, $n > r$ to P_n has no nontrivial $P_n(\mathfrak{o})$-invariant vector.*

Proof. Let $f \in \tau_n$ be $P_n(\mathfrak{o})$-invariant. Define the subgroups T, $N_n^r \subset P_n$ by

$$T = \{\mathrm{diag}(t_1, \dots, t_{n-1}, 1) \mid t_1 = \dots = t_r = 1\},$$
$$N_n^r = \{u \in N_n \mid u_{ij} = 0, i < j \leq r\}.$$

By the inclusion $P_n \subset N_n^r \acute{G}_r T P_n(\mathfrak{o})$ and the definition of τ_n, it suffices to show that f vanishes on $\acute{G}_r T$. The restriction to G_r of the right translation of f by $t \in T$ is K_r-invariant. However, it is identically zero since τ is ramified. □

Combining these lemmas and Proposition 3.1, we have:

Proposition 3.14. *Any lift of $\tau \in \mathscr{I}(G_r)$ has no $P_n(\mathfrak{o})$-invariant pre-Shalika form relevant to χ if one of τ, χ is ramified and another is unramified.*

4. A vanishing theorem for Shalika forms

Let $n = 2m$ be an even integer. Let $\pi \in \mathscr{I}(G_n)$. For a nonnegative integer M, let $\Gamma(M) \subset G_n$ be the open compact subgroup defined in Section 1, and define

$$\Gamma^\circ = \Gamma^\circ(M) = \{k \in \Gamma(M) \mid k_{n,n} - 1 \in \mathfrak{p}^M\}.$$

For $f = (f_1, \dots, f_{n-1}) \in \mathbb{Z}^{n-1}$, let

$$\varpi^f = \operatorname{diag}(\varpi^{f_1}, \dots, \varpi^{f_{n-1}}),$$

and let T_f denote the Hecke operator that acts on right Γ°-invariant vectors ξ in π by

$$T_f \xi = \int_{G_n} \operatorname{Ch}(k; \Gamma^\circ \acute{\varpi}^f \Gamma^\circ) \pi(k) \xi \, \mathrm{d}k,$$

where $\mathrm{d}k$ indicates the Haar measure on G_n normalized so that $\operatorname{vol}(\Gamma^\circ) = 1$. In this section, we show:

Theorem 4.1. *Let χ be a character of F with conductor e. Let $\pi \in \mathscr{I}(G_n)$ be generic with a Shalika model relevant to χ. Let $J \in \mathbb{S}_\pi(\chi)$ be $\Gamma^\circ(M)$-invariant for an M. If J is a (simultaneous) eigenvector for all T_f, $f \in \mathbb{Z}^{n-1}$, and vanishes at 1_n (resp. g_n) in the case of $e = 0$ (resp. $e > 0$), then J vanishes on $S_n P_n \Gamma(M)$.*

To show the vanishing of J on $S_n P_n \Gamma(M)$, it suffices to show it on $\acute{G}_m$. We use the notation

$$u(x) = u_x = \begin{bmatrix} 1_r & {}^t x \\ & 1 \end{bmatrix} \in U_{r+1}, \quad x \in F^r \tag{4-1}$$

and introduce reduced elements of G_m. Let $\beta \in F^{m-1}$ and $d = \operatorname{diag}(d_1, \dots, d_m) \in D_m$. Set

$$c = o(d_m), \qquad a_i = o(d_i),\ b_i = -o(\beta_i), \quad i \in \{1, \dots, m-1\}.$$

If $a_1 \le \cdots \le a_{m-1}$, then we say d is *aligned*, and set $j_1, \dots, j_r$ by

$$a_{j_1} = a_1 = \cdots = a_{j_2-1} < a_{j_2} = \cdots = a_{j_r-1} < a_{j_r} = \cdots = a_{m-1}. \tag{4-2}$$

Understand $j_{r-1} = 0$ if $r = 1$. If $j_s \le i < j_{s+1}$, $s < r$ (resp. $j_r \le i \le m-1$), then let $s(i)$ denote s (resp. r). Let $S(b) = \{s \mid \beta_{j_s} \ne 0\}$. We say du_β is *reduced* if, in addition, the following conditions are satisfied:

(a) $\beta_i = 0$ for all $i \notin \{j_1, \dots, j_r\}$.

(b) $\beta_{j_s} = 0$ for all $s \in \{1, \dots, r\}$ such that $\beta_{j_s} \in \mathfrak{o}$.

(c) For $s, t \in S(b)$, if $s < t$, then $b_{j_s} < b_{j_t}$.

(d) For $s, t \in S(b)$, if $s < t$, then $a_{j_s} - b_{j_s} < a_{j_t} - b_{j_t}$.

For $g_1, g_2 \in G_m$, if $\acute{K}^1_{m-1} g_1 K_m = \acute{K}^1_{m-1} g_2 K_m$ (resp. $\acute{K}_{m-1} g_1 K_m = \acute{K}_{m-1} g_2 K_m$), where $K^1_{m-1} = \mathrm{SL}_{m-1}(\mathfrak{o})$, then we say g_1 is equivalent (resp. quasi-equivalent) to g_2, and denote

$$g_1 \approx g_2 \ \text{ (resp. } g_1 \sim g_2\text{)}.$$

A reason why we define the above relations for elements in G_m is the following.

Lemma 4.2. *Any $(S_n^\circ, P_n(\mathfrak{o}))$-double coset in P_n contains the embedding of an element of the form $du_\beta \in G_m$.*

Proof. Since P_n is isomorphic to $G_{n-1} \ltimes U_n$, any left coset of $P_n(\mathfrak{o})$ intersects with B_n by the Iwasawa decomposition of G_{n-1}. Therefore, any $(S_n^\circ, P_n(\mathfrak{o}))$-double coset intersects with $\acute{B}_m$. The assertion follows from the Cartan decomposition of G_m. □

For a while, in preparation for the proof of the theorem, we will discuss the above concepts. In case of $m > 2$, define $w'_{m-1} \in K^1_{m-1}$ by

$$w'_{m-1} = \begin{cases} w_{m-1} & \text{if } m \in 1+4\mathbb{Z}, \\ \mathrm{diag}(-1, \overbrace{1, \dots, 1}^{m-2}) w_{m-1} & \text{otherwise,} \end{cases}$$

where w_{m-1} indicates the standard antidiagonal Weyl element in G_{m-1}, and for $g \in G_m$, denote

$$g^\iota = g^{\acute{w}'_{m-1}}.$$

Observe that $g \sim g^\iota$, and that, if $g = du_\beta$, then

$$g^\iota = \mathrm{diag}(d_{m-1}, d_{m-2}, \dots, d_1, d_m) \times \begin{cases} u(\beta_{m-1}, \beta_{m-2}, \dots, \beta_1) & \text{if } m \in 1+4\mathbb{Z}, \\ u(-\beta_{m-1}, \beta_{m-2}, \dots, \beta_1) & \text{otherwise.} \end{cases}$$

Proposition 4.3. *We keep the notation from above.*

(i) *Let $d \in D_m$ be aligned. For an arbitrary $\beta \in F^{m-1}$, there exist $d' \in D_m$, $\gamma \in F^{m-1}$ with the following properties.*
 - *$o(d'_i) = o(d_i)$ for all i.*
 - *$d'u_\gamma$ is equivalent to du_β and reduced.*

(ii) *Any equivalence class of G_m contains a reduced element.*

Proof. (i) There exists a Weyl element $w \in K^1_{m-1}$ such that

$$o((\acute{w}d\acute{w}^{-1})_i) = o(d_i), \quad o((\beta^t w)_{j_{s(i)}}) \le o((\beta^t w)_i), \quad i \in \{1, \dots, m-1\}.$$

Set $d = \acute{w}d\acute{w}^{-1}$ and $\beta' = \beta^t w$. Then,

$$du_\beta \approx \acute{w}du_\beta\acute{w}^{-1} = \acute{w}d\acute{w}^{-1}\acute{w}u_\beta\acute{w}^{-1} = d'u_{\beta'}.$$

Preserving the equivalence class, we will translate β' step by step to satisfy conditions (a)–(d), and attain γ. Assume that there exists an $i \notin \{j_1, \dots, j_r\}$ such that $\beta'_i \ne 0$. Since $o(\beta'_{j_{s(i)}}) \le o(\beta'_i)$, we can choose a $v \in {}^tN_{m-1}(\mathfrak{o})$ such that

$$(\beta'^t v)_k = \begin{cases} 0 & \text{if } k = i, \\ \beta'_k & \text{otherwise.} \end{cases}$$

By the lemma below, $d'u(\beta''^t v) \approx d'u_{\beta'}$. Iterating such translations, we attain a β'' satisfying $d'u_{\beta''} \approx d'u_{\beta'}$ and condition (a). Similarly, if there exist $s, t \in S(b'')$ (b'' is defined by $b''_i = -o(\beta''_i)$) such that $s < t$ and $b''_{j_s} \geq b''_{j_t}$, then we can choose a $v' \in {}^t N_{m-1}(\mathfrak{o})$ such that

$$(\beta''^t v')_k = \begin{cases} 0 & \text{if } k = j_t, \\ \beta'_k & \text{otherwise.} \end{cases}$$

Iterating such translations, we attain β''' satisfying $d'u_{\beta'''} \approx d'u_{\beta''}$ and also condition (c). Set $\beta'''' \in F^{m-1}$ by

$$\beta''''_i = \begin{cases} 0 & \text{if } \beta'''_i \in \mathfrak{o}, \\ \beta''_i & \text{otherwise.} \end{cases}$$

Then, $d'u_{\beta''''} = d'u_{\beta'''}u(\beta'''' - \beta''') \approx d'u_{\beta'''}$, and β'''' satisfies also condition (b). Assume that some β''''_{j_s} does not lie in $\bigcup_{k=s+1}^{r}(d'_{j_k}\beta''''_{j_k}/d'_{j_s})\mathfrak{o}$. Let $d'^{\circ} = \operatorname{diag}(d'_1, \ldots, d'_{m-1})$. Then, we can choose a $v'' \in (d'^{\circ})^{-1}N_{m-1}(\mathfrak{o})d'^{\circ} (\subset N_{m-1(\mathfrak{o})})$ such that

$$(\beta''''^t v'')_k = \begin{cases} 0 & \text{if } k = j_s, \\ \beta''''_k & \text{otherwise.} \end{cases}$$

Observe that

$$d'u(\beta''''^t v'') = d'\acute{v}''u(\beta'''')\acute{v}''^{-1} = d'\acute{v}''d'^{-1}d'u(\beta'''')\acute{v}''^{-1} \approx d'u(\beta'''').$$

Iterating such translations, we attain the desired γ.

(ii) By the proof of Lemma 4.2, any equivalence class contains an element of the form du_β. We may choose a Weyl element w in K^1_{m-1} so that $\acute{w}d\acute{w}^{-1}$ is aligned. Now the assertion follows from (i). □

Lemma 4.4. *Let $d \in D_m$ be aligned. For $\beta \in F^{m-1}$, $v \in N_{m-1}(\mathfrak{o})$, it holds that*

$$du(\beta v) \approx du(\beta), \quad d^\iota u(\beta^t v) \approx d^\iota u(\beta).$$

Proof. Since d is aligned, $({}^t\acute{v})^d \in \acute{K}^1_{m-1}$, and $du(\beta v) = d^t\acute{v}u_\beta{}^t\acute{v}^{-1} = ({}^t\acute{v})^d du_\beta \acute{v}^{-1} \approx du_\beta$. Another equivalence is proved similarly. □

Let

$$\widetilde{P}_m = \{g \in G_m \mid g_{mi} = 0, i \in \{1, \ldots, m-1\}\}.$$

For

$$p = \begin{bmatrix} h & * \\ & * \end{bmatrix} \in \widetilde{P}_m, \quad h \in G_{m-1},$$

we call $o(\det(h))$ the *weight* of p, and denote it by $\mathrm{wt}(p)$. Observe that

$$\mathrm{wt}(p_1p_2) = \mathrm{wt}(p_1) + \mathrm{wt}(p_2), \quad p_1, p_2 \in \widetilde{P}_m. \tag{4-3}$$

Lemma 4.5. *Let $p, p' \in \widetilde{P}_m$. If $p \sim p'$, then $\mathrm{wt}(p) = \mathrm{wt}(p')$.*

Proof. Assume $p' = \acute{h}pk$ for some $h \in K^1_{m-1}$, $k \in K_m$. Write

$$p = \begin{bmatrix} g & {}^t\beta \\ & t \end{bmatrix}, \quad p' = \begin{bmatrix} g' & {}^t\beta' \\ & t' \end{bmatrix}, \quad k = \begin{bmatrix} x & {}^t y \\ z & w \end{bmatrix} \in K_m$$

by $\beta, \beta', y, z \in \mathfrak{o}^{m-1}$, $w, t, t' \in \mathfrak{o}$ and $m-1$ by $m-1$ matrices g, g', x with entries in $\mathfrak{o}$. Then we have

$$\begin{bmatrix} g' & {}^t\beta' \\ & t' \end{bmatrix} = \begin{bmatrix} h(gx + {}^t\beta z) & h(g^t y + {}^t\beta w) \\ tz & tw \end{bmatrix}.$$

It follows that $z = 0$, and $x \in K_{m-1}$, and

$$\mathrm{wt}(p') = o(\det(g')) = o(\det(h(gx + {}^t\beta z))) = o(\det(gx)) = \mathrm{wt}(p).$$

This completes the proof. □

For $(a, b, c) \in \mathbb{Z}^{m-1} \times \mathbb{Z}^{m-1} \times \mathbb{Z}$, let

$$p(a, b, c) = \varpi^{(a,c)} u(\varpi^{-b_1}, \dots, \varpi^{-b_{m-1}}) \in \widetilde{P}_m.$$

Proposition 4.6. *We keep the notation from above.*

(i) *Any quasi-equivalence class of G_m contains a $p(a, b, c)$.*

(ii) *If both $p(a, b, c)$ and $p(a', b', c')$ are reduced, then*

$$p(a, b, c) \sim p(a', b', c') \ (\Longleftrightarrow) \ (a, b, c) = (a', b', c').$$

Proof. (i) By Proposition 4.3, any equivalence class contains a reduced du_β. Set $c = o(d_m)$, and $a_i = o(d_i)$, $b_i = -o(\beta_i)$ for $i < m$. Then, $p(a, b, c) \sim du_\beta$.

(ii) We only show that $(a, b, c) = (a', b', c')$ if there exist $h \in K_{m-1}$ and $k \in K_m$ such that $\acute{h}p(a, b, c)k = p(a', b', c')$. From Lemma 4.5, $c = c'$ is derived. Set $\beta = (\varpi^{-b_1}, \dots, \varpi^{-b_m})$, $\beta' = (\varpi^{-b'_1}, \dots \varpi^{-b'_m})$. Write

$$k = \begin{bmatrix} x & {}^t y \\ z & w \end{bmatrix}$$

by $y, z \in \mathfrak{o}^{m-1}$, $w \in \mathfrak{o}$ and $m-1$ by $m-1$ matrix x with entries in $\mathfrak{o}$. Then,

$$\begin{bmatrix} \varpi^{a'} & \varpi^{a'\,t}\beta' \\ & \varpi^c \end{bmatrix} = \begin{bmatrix} h\varpi^a(x + {}^t\beta z) & h\varpi^a({}^t y + {}^t\beta w) \\ \varpi^c z & \varpi^c w \end{bmatrix}.$$

Therefore, $w = 1$ and $z = 0$. Since k lies in K_m, x lies in K_{m-1}. Therefore, $\varpi^{a'} = h\varpi^a x$, and $a = a'$ follows from the Cartan decomposition of G_{m-1}. Now, set $j_1, \dots, j_r$ by (4-2). Since both $p(a, b, c)$ and $p(a, b', c)$ are reduced,

$$\beta_i = \beta'_i = 0, \quad i \notin \{j_1, \dots, j_r\}. \tag{4-4}$$

The remained task is to show $\beta_{j_s} = \beta'_{j_s}$, $s \in \{1, \dots, r\}$. By the symmetry argument, it suffices to show $\beta_{j_s} = \beta'_{j_s}$ for each $s \in S(b')$. Since $x = \varpi^{-a}(h^{-1})\varpi^a$ lies in $K_{m-1} \cap \varpi^{-a} K_{m-1} \varpi^a$, we have

$$x_{ik} \in \begin{cases} \mathfrak{p}^{a_{j_{s(k)}} - a_{j_{s(i)}}} & \text{if } i < k, \\ \mathfrak{o} & \text{otherwise.} \end{cases} \tag{4-5}$$

Therefore, we have

$$(x_{ik})_{j_r \le i,k < m} \in K_{m-j_r}, \quad (x_{ik})_{j_s \le i,k < j_{s+1}} \in K_{j_{s+1}-j_s} \quad (s < r). \tag{4-6}$$

From the identity $\varpi^{a\,t}\beta' = h\varpi^a({}^t y + {}^t\beta)$, we obtain $\beta - \beta'^t x \in \mathfrak{o}^{m-1}$, that is,

$$\beta_i \in \mathfrak{o} + \sum_{k=1}^{m-1} x_{ik}\beta'_k = \mathfrak{o} + \sum_{t \in S(b')} x_{ij_t}\beta'_{j_t}.$$

Assume that $x_{j_s j_s} \notin \mathfrak{o}^\times$. Then, by (4-6), there exists an $l \in \{j_s + 1, \dots, j_{s+1} - 1\}$ such that $x_{lj_s} \in \mathfrak{o}^\times$. By (4-5), β_l lies in

$$\begin{aligned} \mathfrak{o} + \sum_{t \in S(b')} x_{lj_t}\beta'_{j_t} &= \mathfrak{o} + x_{lj_s}\beta'_{lj_s} + \sum_{s > t \in S(b')} x_{lj_t}\beta'_{j_t} + \sum_{s < t \in S(b')} x_{lj_t}\beta'_{j_t} \\ &\in \mathfrak{o} + \varpi^{-b'_{js}}\mathfrak{o}^\times + \mathfrak{p}^{1-b'_{js}} + \sum_{s<t \in S(b')} \mathfrak{p}^{a_{j_t} - a_{j_s} - b'_{j_t}} \\ &= \varpi^{-b'_{js}}\mathfrak{o}^\times, \end{aligned}$$

conflicting with (4-4). Hence, $x_{j_s j_s} \in \mathfrak{o}^\times$, and

$$\beta_{j_s} \in \mathfrak{o} + \sum_{t \in S(b')} x_{j_s j_t}\beta'_{j_t} = \varpi^{-b'_{js}}\mathfrak{o}^\times.$$

This completes the proof. □

Now, let J be a $P_n(\mathfrak{o})$-invariant Shalika form relevant to a character χ. In order to know the value $\acute{J}(g)$, $g \in G_m$, it suffices to know that at $p(a, b, c)$ quasi-equivalent to g by the identity

$$\acute{J}(\acute{h}gk) = \chi(\det(h))\acute{J}(g), \quad h \in K_{m-1}, k \in K_m. \tag{4-7}$$

A necessary condition for $\acute{J}(p(a, b, c)) \neq 0$ is as follows.

Lemma 4.7. *With notation as above, assume that* $\acute{J}(p(a, b, c)) \neq 0$.

(i) $c, a_i, a_i - b_i \ge 0$ *for all* i.

(ii) *If* $p(a, b, c)$ *is reduced, then for* $S(b) = \{s_1 < \dots < s_l\}$,

$$0 \le a_{j_{s_1}} - b_{j_{s_1}} \le \dots \le a_{j_{s_l}} - b_{j_{s_l}}.$$

Proof. This follows from the next (obvious) lemma and the identity

$$J\left(\begin{bmatrix} p & \\ & 1_m \end{bmatrix}\begin{bmatrix} 1_m & X \\ & 1_m \end{bmatrix}\right) = \psi\left(\varpi^c x_{mm} + \sum_{i=1}^{m-1} \varpi^{a_i}(x_{ii} + \varpi^{-b_i} x_{mi})\right) \acute{J}(p) \tag{4-8}$$

where $p = p(a, b, c)$. □

Lemma 4.8. *Let Ω be a field. Let G be a group, and H, K be its subgroups. Let ξ and ω be homomorphisms into $\Omega^\times$ of H and K, respectively. Let J be a Ω-valued function on G such that*

$$J(hgk) = \xi(h)\omega(k)J(g), \quad h \in H,\ g \in G,\ k \in K.$$

Then $J(g_0)$ vanishes at $g_0 \in G$ if there exists an $h \in H$ such that $h^{g_0} \in K$ and $\xi(h) \neq \omega(g_0^{-1}hg_0)$. □

In the case where χ is ramified, the following stronger statement can be made. Let $e > 0$ be the conductor of χ. Let $p_e(a, b, c) = p(a^+, b^+, c^+)$ with

$$\begin{aligned} a^+ &= (a_1 + e, a_2 + 3e, \ldots, a_{m-1} + (2m-3)e), \\ b^+ &= (b_1 + e, \ldots, b_{m-1} + (m-1)e), \\ c^+ &= c + (m-1)e. \end{aligned}$$

Lemma 4.9. *With notation as above, assume that $\acute{J}(p_e(a, b, c)) \neq 0$. Then:*

(i) $0 \le b_1 \le \cdots \le b_{m-1}$.

(ii) $0 \le a_1 - b_1 \le \cdots \le a_{m-1} - b_{m-1} \le c$ (*therefore*, $0 \le a_1 \le \cdots \le a_{m-1}$).

In particular, $p_e(a, b, c) = p(a^+, b^+, c^+)$ is reduced and $j_1 = 1, \ldots, j_{m-1} = m-1$, where $j_1, \ldots, j_{m-1}$ are defined for a^+.

Proof. Let $d = \varpi^{(a,c)}$, and $\beta = (\varpi^{-b_1^+}, \ldots, \varpi^{-b_{m-1}^+})$. For $s = \mathrm{diag}(s_1, \ldots, s_{m-1}, 1)$ with $s_j \in \mathfrak{o}^\times$, it holds that

$$\acute{J}(du(s_1\beta_1, \ldots, s_{m-1}\beta_{m-1})) = \acute{J}(dsu_\beta s^{-1}) = \chi(\det(s))\acute{J}(du_\beta).$$

By Lemma 4.8, $b_i^+ \ge e$ for all i. For $x \in \mathfrak{o}$ and $i \le m-2$, there exists a $v \in {}^tN_{m-1}(\mathfrak{o})$ such that

$$(\beta^t v)_j = \begin{cases} \beta_{i+1}(1 + x\beta_i/\beta_{i+1}) & \text{if } j = i+1, \\ \beta_j & \text{otherwise.} \end{cases}$$

By (4-7) and Lemma 4.4, $\acute{J}(du(\beta)) = \acute{J}(du(\beta^t v))$. Therefore, if $b_i > b_{i+1}$ for some $i \le m-2$, then $o(\beta_i/\beta_{i+1})$ is less than e, and $\acute{J}(du_\beta)$ vanishes by Lemma 4.8. Hence (i) follows. If $a_1 < b_1$, then $\acute{J}(du_\beta) = 0$ by (4-8) and Lemma 4.8. Hence, $a_1 \ge b_1$. Set $\gamma = (d_1\beta_1/d_m, \ldots, d_{m-1}\beta_{m-1}/d_m)$. Similar to (i), noting that

$$\acute{J}(du_\beta) = \chi(\det(d))J\left(\begin{bmatrix} 1_m & \\ & d^{-1}u_\gamma \end{bmatrix}\right),$$

we can show that $\acute{J}(du_\beta) = 0$ assuming $o(\gamma_{m-1}) > -e$ (which is equivalent to $a_{m-1} - b_{m-1} - c > 0$) or $o(\gamma_{i+1}/\gamma_i) < e$ (which is equivalent to $a_{i+1} - b_{i+1} < a_i - b_i$), for $i \le m-2$. Hence, (ii) follows. □

Now we begin the proof of the theorem dividing the cases whether e, the conductor of χ, is zero or not. Let J be a Shalika form as in the theorem. In the case $e = 0$, by (4-7) and Proposition 4.6, it suffices to show $\acute{J}(p(a, b, c)) = 0$ for all reduced $p(a, b, c)$ satisfying the conditions in Lemma 4.7. Equipping the set of these elements with a suitable linear order so that the set is well-ordered, and $p(0, 0, 0)$ is minimal, we prove $\acute{J}(p(a, b, c)) = 0$ by transfinite induction, assuming $\acute{J}(p(0, 0, 0)) = 0$.

Our proof for the case of $e > 0$ is similar, but the set consists of $p_e(a, b, c)$ satisfying the conditions in Lemma 4.9, and is equipped with another order so that $p_e(0, 0, 0)$ is minimal. To this end, we use the Hecke operators T_f, $f \in \mathbb{Z}^{n-1}$. Choose a p' less than p among the set. Since p' is less than p and J is a Hecke eigenvector, $\acute{J}(p') = 0$ by the induction hypothesis and $T_f \acute{J}(p') = 0$. An elementary computation shows that

$$\begin{aligned} T_f \acute{J}(p') &= \sum_{\sigma \in \mathfrak{S}_{n-1}/\mathfrak{S}(f)} \sum_{v \in \mathcal{N}(f^\sigma)} J\left(\begin{bmatrix} p' & \\ & 1_m \end{bmatrix} v \acute{\varpi}^{f^\sigma} \right) \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1}/\mathfrak{S}(f)} \sum_{v \in \mathcal{N}(f^\sigma)} \psi(\operatorname{tr}(p' X_v)) J\left(\begin{bmatrix} p' & \\ & 1_m \end{bmatrix} \begin{bmatrix} v_+ \varpi^{f_+^\sigma} & \\ & v_- \acute{\varpi}^{f_-^\sigma} \end{bmatrix} \right), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{S}(f) &= \{\sigma \in \mathfrak{S}_{n-1} \mid f^\sigma = f\}, \\ f_i^\sigma &= f_{\sigma(i)}, \\ f_+^\sigma &= (f_1^\sigma, \dots, f_m^\sigma), \\ f_-^\sigma &= (f_{(m+1)}^\sigma, \dots, f_{(n-1)}^\sigma), \\ \mathcal{N}(f^\sigma) &= \left\{ v \in N_n \mid v_{in} \in \mathfrak{o}/\mathfrak{p}^{\max\{f_i^\sigma, 0\}},\ v_{ij} \in \mathfrak{o}/\mathfrak{p}^{\max\{f_i^\sigma - f_j^\sigma, 0\}}\ (j \le n-1) \right\}, \\ v &= \begin{bmatrix} 1_m & X_v \\ & 1_m \end{bmatrix} \begin{bmatrix} v_+ & \\ & v_- \end{bmatrix}, \quad v_+, v_- \in N_m(\mathfrak{o}). \end{aligned}$$

Here, X_v is an m by m matrix with entries in $\mathfrak{o}$. Since p' satisfies the conditions in Lemma 4.7, $\psi(\operatorname{tr}(p' X_v)) = 1$, and thus each term of the above expansion is of the form

$$\chi(\det(\varpi^{f_-^\sigma})) \acute{J}(\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}). \tag{4-9}$$

We prove $\acute{J}(p) = 0$ by showing that this term equals $c\acute{J}(p)$ for a nonzero constant c independent of the choice of σ, or the element $\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}$ is quasi-equivalent to a p'' which is less than p or does not satisfy the conditions in

Lemma 4.7, 4.9. We will use the identity

$$\mathrm{wt}(\varpi^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}) = \mathrm{wt}(p') + \sum_{i=1}^{m-1} f_i^\sigma - \sum_{i=m+1}^{n-1} f_i^\sigma (\le \mathrm{wt}(p)), \tag{4-10}$$

derived from (4-3), repeatedly. For positive integers $s < t$, let

$$e_s^t = (\overbrace{0, \dots, 0}^{s-1}, 1, 0, \dots, 0) \in \mathbb{Z}^t.$$

Case of $e = 0$: In this case, the quasi-equivalence preserves the value of J by (4-7). The order for the set of reduced $p(a, b, c)$ satisfying the conditions in Lemma 4.7 is as follows. Let

$$A = \{a \in \mathbb{Z}_{\ge 0}^{m-1} \mid a_1 \le \cdots \le a_{m-1}\}. \tag{4-11}$$

We equip A with the following linear order. Set $a < a'$ if one of the following conditions holds:

(i) $\sum_{i=1}^{m-1} a_i < \sum_{i=1}^{m-1} a_i'$.

(ii) $\sum_{i=1}^{m-1} a_i = \sum_{i=1}^{m-1} a_i'$ and $a_j < a_j'$ for the last j such that $a_j \ne a_j'$.

Let

$$B = \left\{b \in \mathbb{Z}_{\ge 0}^{m-1} \mid b_i < b_j \text{ if } i < j \text{ and } b_i b_j \ne 0\right\}.$$

We equip B with a linear order by the following rule. For $b, b' \in B$, let $S(b) = \{i_1 < \cdots < i_t\}$ and $S(b') = \{j_1 < \cdots < j_u\}$ be the total sets of indices at which the entries of b and b' are nonzero, respectively. Let k be the maximal number such that

$$i_{t-k+1} = j_{u-k+1}, \dots, i_t = j_u, \quad b_{i_{t-k+1}} = b'_{j_{u-k+1}}, \dots, b_{i_t} = b'_{j_u}.$$

Understand $k = 0$ if $(i_t, b_{i_t}) \ne (j_u, b'_{j_u})$, or, at least one of $S(b)$, $S(b')$ is empty. By definition, $(i_{t-k}, b_{i_{t-k}}) \ne (j_{u-k}, b'_{j_{u-k}})$. Set $b < b'$ if one of the following conditions holds:

(i) $b_{i_{t-k}} < b'_{j_{u-k}}$.

(ii) $b_{i_{t-k}} = b'_{j_{u-k}}$, and $j_{u-k} < i_{t-k}$.

We equip $A \times B \times \mathbb{Z}_{\ge 0}$ with a linear order by setting $(a, b, c) < (a', b', c')$ if $a < a'$, or $a = a'$ and $b < b'$, or $a = a'$, $b = b'$ and $c < c'$. Obviously, $(A, <)$, $(B, <)$ are well-ordered sets, and so is $(A \times B \times \mathbb{Z}_{\ge 0}, <)$. A coordinate (a, b, c) satisfying the conditions in the lemma is identified with the quasi-equivalence class of the reduced $p(a, b, c)$ by Proposition 4.6, and such coordinates consist a subset of $A \times B \times \mathbb{Z}_{\ge 0}$, which is also well-ordered.

To begin with, we will show

$$\acute{J}(p(0,0,c)) = 0, \quad c \in \mathbb{Z}. \tag{4-12}$$

By Lemma 4.7, it suffices to show this for $c > 0$. Set

$$f = (\varpi^c, \dots, \varpi^c), \quad p' = p(0,0,0).$$

Then, $\mathfrak{S}(f) = \mathfrak{S}_{n-1}$, and (4-9) is of the form

$$\chi(\varpi)^{c(m-1)} \acute{J}(u_x \varpi^{ce_m^m})$$

where $x \in \mathfrak{o}^{m-1}$, and equals $\chi(\varpi)^{c(m-1)} \acute{J}(p(0,0,c))$. Thus (4-12) follows.

Now, we start the induction. By (4-12), we may assume that $a_{j_r} > 0$. By Lemma 4.7, $a_{j_r} \geq b_{j_r}$.

First, suppose that $b_{j_r} = 0$. Set

$$f = (0, \dots, 0, \overbrace{-1, \dots, -1}^{m-j_r}),$$
$$p' = du_\beta = p(a', b, c), \quad a' = a' = a - (0, \dots, 0, \overbrace{1, \dots, 1}^{m-j_r}).$$

Observe that p' is reduced, and satisfies the conditions in Lemma 4.7. By the induction hypothesis and (4-10) we may assume that

$$f_+^\sigma = 0, \quad v_+ = 1_m$$

since otherwise $\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}$ has the weight less than that of p, and is less than p by the definition of the order. Write

$$v_-^{-1} = \begin{bmatrix} v' & \\ & 1 \end{bmatrix}, \quad v' \in N_{m-1}(\mathfrak{o}).$$

Let $d^\circ = \mathrm{diag}(d_1, \dots, d_{m-1})$. Since d is aligned, we have

$$v'' := d^{\circ -1} v' d^\circ \in N_{m-1}(\mathfrak{o}), \quad d^{-1} v_-^{-1} d = (\acute{v''}) \in N_m(\mathfrak{o}).$$

Then,

$$\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' = \acute{\varpi}^{-f_-^\sigma} d d^{-1} v_-^{-1} d u_\beta = \acute{\varpi}^{-f_-^\sigma} d(\acute{v''}) u_\beta \approx \acute{\varpi}^{-f_-^\sigma} d u(\beta^t v'').$$

Put $\gamma = \beta^t v''$. Let $E_\sigma = \{i \in \{1, \dots, m-1\} \mid f_{m+i}^\sigma = -1\}$. By the induction hypothesis again, we may assume that

$$f_i^\sigma = 0, \quad i \leq m + j_{r-1}, \tag{4-13}$$

since otherwise, $\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' (\approx \acute{\varpi}^{-f_-^\sigma} d u_\gamma)$ is quasi-equivalent to an element less than p by Proposition 4.3, 4.6 (compare the A-part). Therefore:

- $E_\sigma = \{j_r, \dots, m-1\}$ if $a_{j_r} - a_{j_{r-1}} > 1$.
- $E_\sigma \subset \{j_{r-1}, \dots, m-1\}$ if $a_{j_r} - a_{j_{r-1}} = 1$.

If $i < k$, then we have

$$v'_{ik} \in \begin{cases} \mathfrak{o}/\mathfrak{p} & \text{if } i \notin E_\sigma \ni k, \\ \{0\} & \text{otherwise,} \end{cases}$$
$$v''_{ik} \in \begin{cases} \varpi^{a'_k - a'_i}(\mathfrak{o}/\mathfrak{p}) & \text{if } i \notin E_\sigma \ni k, \\ \{0\} & \text{otherwise.} \end{cases}$$

Since $E_\sigma \subset \{j_{r-1}, \dots, m-1\}$, and $\beta_k = 0$ for $k > j_{r-1}$ (recall that p is reduced, and $b_{j_r} = 0$ is assumed), we have

$$\gamma_i = \beta_i + \begin{cases} \sum_{k \in E_\sigma} v''_{ik}\beta_k & \text{if } i \notin E_\sigma, \\ 0 & \text{otherwise,} \end{cases} \tag{4-14}$$
$$= \beta_i + \begin{cases} v''_{i, j_{r-1}}\beta_{j_{r-1}} & \text{if } i < j_{r-1} \in E_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\gamma_{j_{r-1}} = \beta_{j_{r-1}}, \gamma_{j_{r-1}+1} = \cdots = \gamma_{m-1} = 0.$$

If $i < j_{r-1} \in E_\sigma$ and $b_{j_{s(i)}} > 0$, then by (4-14),

$$\gamma_i = \beta_i + v''_{i, j_{r-1}}\beta_{j_{r-1}} \in \mathfrak{p}^{-b_{j_{s(i)}}} + \mathfrak{p}^{a_{j_{r-1}} - a_{j_{s(i)}} - b_{j_{r-1}}} \subset \mathfrak{p}^{-b_{j_{s(i)}}},$$
$$\gamma_{j_{s(i)}} = \beta_{j_{s(i)}} + v''_{j_{s(i)}, j_{r-1}}\beta_{j_{r-1}} \in \varpi^{-b_{j_{s(i)}}}\mathfrak{o}^\times + \mathfrak{p}^{a_{j_{r-1}} - a_{j_{s(i)}} - b_{j_{r-1}}}$$
$$\subset \varpi^{-b_{j_{s(i)}}}\mathfrak{o}^\times + \mathfrak{p}^{1 - b_{j_{s(i)}}} = \varpi^{-b_{j_{s(i)}}}\mathfrak{o}^\times$$

(use Lemma 4.7, and the fact $a'_k = a_k$ for $k < j_r$). If $i < j_{r-1} \in E_\sigma$ and $b_{s(i)} = 0$ (i.e., $\beta_{s(i)} = 0$), then $\beta_i = 0$ and

$$\gamma_i = \begin{cases} v''_{i, j_{r-1}}\beta_{j_{r-1}} & \text{if } \beta_{j_{r-1}} \neq 0 \text{ and } v'_{i, j_{r-1}} \in \mathfrak{o}^\times, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\acute{\varpi}^{-f^\sigma_-} du_\gamma$ is quasi-equivalent to p if $\beta_{j_{r-1}} = 0$. From now, assume that $\beta_{j_{r-1}} \neq 0$, i.e., $b_{j_{r-1}} > 0$. From the above argument, we conclude that $\acute{\varpi}^{-f^\sigma_-} du_\gamma$ is quasi-equivalent to $p(a, b', c)$ with

$$b'_i = 0, \quad i \notin \{j_1, \dots, j_r\},$$
$$b'_{j_{s(<r-1)}} = \begin{cases} b_{j_{r-1}} + a_{j_s} - a_{j_{r-1}} \text{ or } 0 & \text{if } b_{j_s} = 0 \text{ and } j_{r-1} \in E_\sigma, \\ b_{j_s} & \text{otherwise,} \end{cases}$$
$$(b'_{j_{r-1}}, b'_{j_r}) = \begin{cases} (0, b_{j_{r-1}}) & \text{if } j_{r-1} \in E_\sigma, \\ (b_{j_{r-1}}, 0) & \text{otherwise.} \end{cases}$$

Therefore, if $j_{r-1} \in E_\sigma$, then b' is of the form

$$(b'_1, \dots, b'_{j_{r-1}-1}, \overbrace{0, \dots, 0}^{j_r - j_{r-1}}, \overbrace{b_{j_{r-1}}, \dots, 0}^{m-j_r}), \quad b'_{j_{s(<r-1)}} \le b_{j_s} (< b_{j_{r-1}})$$

and, by the proof of Proposition 4.3(i), $p(a, b', c)$ is quasi-equivalent to the reduced

$$p(a, (\dots, \overbrace{0, \dots, 0}^{j_r - j_{r-1}}, \overbrace{b_{j_{r-1}}, \dots, 0}^{m-j_r}), c),$$

which is less than p. Otherwise, $p(a, b', c) = p$. This settles the case of $b_{j_r} = 0$.

Next, suppose that $b_{j_r} > 0$. Set

$$f = (\overbrace{1, \dots, 1}^{m-j_r}, 0, \dots, 0), \qquad p' = du_\beta = p(a', b', c)^\iota,$$
$$a' = a - (\overbrace{0, \dots, 0}^{j_r - 1}, 1, \dots, 1), \quad b' = b - (\overbrace{0, \dots, 0}^{j_r - 1}, 1, 0, \dots, 0).$$

Observe that d^ι is aligned, and that $p(a', b', c)$ is reduced element satisfying the conditions in Lemma 4.7. Write

$$v_+ = \acute{v}_0 u_x, \quad v_0 \in N_{m-1}(\mathfrak{o}).$$

Let $E_\sigma = \{i \mid f_i^\sigma = 1\}$. By (4-10) and the induction hypothesis, we may assume that $E_\sigma \subset \{1, \dots, m-1\}$. Therefore, $v_- = 1_m$, and

$$\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma} = du_\beta \acute{v}_0 u_x \varpi^{f_+^\sigma} = d\acute{v}_0 u(\beta^t (v_0)^{-1}) u_x \varpi^{f_+^\sigma}.$$

Set $\beta' = \beta^t (v_0)^{-1}$. Let $y_{ik} (\in \mathfrak{o})$ denote the i, k entry of v_0^{-1}. Then,

$$\beta'_i = \beta_i + \begin{cases} \sum_{k=i+1}^{m-1} y_{ik} \beta_k & \text{if } i \in E_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Since d^ι is aligned, $d\acute{v}_0 d^{-1} \in K_{m-1}^1$, and

$$d\acute{v}_0 u_{\beta'} u_x \varpi^{f_+^\sigma} \approx du(\beta' + x)\varpi^{f_+^\sigma} = d\varpi^{f_+^\sigma} u((\beta' + x)\varpi^{-f_+^\sigma}).$$

Set $\gamma = (\beta' + x)\varpi^{-f_+^\sigma}$. Now by the induction hypothesis, we may assume

$$E_\sigma \subset \{1, \dots, m - j_{r-1}\},$$

and have

$$\gamma_i = \begin{cases} \varpi^{-1}\big(x_i + \beta_i + \sum_{k=i+1}^{m-1} y_{ik} \beta_k\big) & \text{if } i \in E_\sigma, \\ \beta_i & \text{otherwise.} \end{cases} \tag{4-15}$$

There are three cases:

(i) $b_{j_r} = 1$; (ii) $a_{j_r} - a_{j_{r-1}} > 1$ and $b_{j_r} \ge 2$; (iii) $a_{j_r} - a_{j_{r-1}} = 1$ and $b_{j_r} \ge 2$.

In case (i), all the β_i are zero, and

$$d\varpi^{f_+^\sigma} u_\gamma \sim \begin{cases} p & \text{if } x_i \in \mathfrak{o}^\times \text{ for some } i \in E_\sigma, \\ p(a,0,c)^\iota (\sim p(a,0,c)) & \text{otherwise.} \end{cases}$$

Of course, $p(a,0,c)$ is less than p.

In case (ii), since $a_{j_r} - a_{j_{r-1}} > 1$, $E_\sigma = \{1, \dots, m - j_r\}$ by the induction hypothesis. From (4-15), it follows that

$$\begin{aligned} \gamma_i &= \beta_i && (i > m - j_r), \\ o(\gamma_i) &= o\Big(x_i + \beta_i + \sum_{k=i+1}^{m-1} y_{ik}\beta_k\Big) - 1 \geq -b_{j_r} && (i \leq m - j_r), \\ o(\gamma_{m-j_r}) &= -b_{j_r}. \end{aligned}$$

Therefore, $d\varpi^{f_+^\sigma} u_\gamma$ is quasi-equivalent to p.

In case (iii), we may assume that $E_\sigma \subset \{1, \dots, m - j_{r-1}\}$ by the induction hypothesis. From (4-15),

$$\begin{aligned} \gamma_i &= \beta_i && (i > m - j_{r-1}), \\ o(\gamma_i) &\geq -b_{j_r} && (i \in E_\sigma), \\ o(\gamma_i) &\in \{0, -b_{j_{r-1}}, 1 - b_{j_r}\} && (i \in \{1, \dots, m - j_{r-1}\} \setminus E_\sigma). \end{aligned}$$

By the proof of Proposition 4.3, $d\varpi^{f_+^\sigma} u_\gamma$ is quasi-equivalent to $p(a, b', c)^\iota$ with b' of the form

$$(b_1, \dots, b_{j_{r-1}-1}, \overbrace{b'_{j_{r-1}}, 0, \dots, 0}^{j_r - j_{r-1}}, \overbrace{b'_{j_r}, 0, \dots, 0}^{m - j_r}), \quad b'_{j_{r-1}} \in \{0, b_{j_{r-1}}, b_{j_r} - 1\}, \quad b'_{j_r} \leq b_{j_r}.$$

If $b'_{j_{r-1}} = b_{j_r} - 1 (\geq 1)$, then

$$b'_{j_r} \leq b'_{j_{r-1}} \quad \text{or} \quad b'_{j_r} - b'_{j_{r-1}} = 1 (= a_{j_r} - a_{j_{r-1}}),$$

and, by the proof of Proposition 4.3 again, $p(a, b', c)^\iota$ is equivalent to

$$p(a, (b_1, \dots, b_{j_{r-1}-1}, \overbrace{b_{j_r} - 1, 0, \dots, 0}^{m - j_{r-1}}), c)^\iota \quad (< p),$$

or

$$p(a, (b_1, \dots, b_{j_{r-1}-1}, \overbrace{0, \dots, 0}^{j_r - j_{r-1}}, \overbrace{b'_{j_r}, 0, \dots, 0}^{m - j_r}), c)^\iota \quad (\leq p).$$

Otherwise, obviously, $p^\iota(a, b', c) \leq p$. We get the desired conclusion in the case of $e = 0$.

Case of $e > 0$: The order for the set of reduced $p_e(a, b, c)$ satisfying the conditions in Lemma 4.9 is as follows. Let

$$A^+ = B^+ = \{a \in \mathbb{Z}_{\geq 0}^{m-1} \mid a_1 \leq \cdots \leq a_{m-1}\}. \tag{4-16}$$

equipped with the linear order defined by setting $a < a'$ if one of the following conditions holds:

(i) $\sum_{i=1}^{m-1} a_i < \sum_{i=1}^{m-1} a'_i$.

(ii) $\sum_{i=1}^{m-1} a_i = \sum_{i=1}^{m-1} a'_i$, and $a_j > a'_j$(!) for the last j such that $a_j \neq a'_j$.

We equip $A^+ \times B^+ \times \mathbb{Z}_{\geq 0}$ with a linear order by setting $(a, b, c) < (a', b', c')$ if $a < a'$, or $a = a'$ and $b < b'$, or $a = a'$, $b = b'$ and $c < c'$. Obviously, $(A^+ \times B^+ \times \mathbb{Z}_{\geq 0}, <)$ is also an well-ordered set. A coordinate (a, b, c) satisfying the conditions in the lemma is identified with the quasi-equivalence class of the reduced $p_e(a, b, c)$, and such coordinates consist a subset of $A^+ \times B^+ \times \mathbb{Z}_{\geq 0}$.

To begin with, we will show

$$\acute{J}(p_e(0, 0, c)) = 0, \quad c \in \mathbb{Z}. \tag{4-17}$$

By Lemma 4.9, it suffices to show this for $c > 0$. Set

$$f = (\varpi^c, \ldots, \varpi^c), \quad p' = du_\beta = p_e(0, 0, 0).$$

Then, $\mathfrak{S}(f) = \mathfrak{S}_{n-1}$, and it is easy to see that (4-9) is of the form

$$\chi(\varpi)^{c(m-1)} \acute{J}(\varpi^{ce_m^m} u_x du_\beta),$$

where $x \in \mathfrak{o}^{m-1}$. Define x' by $u_{x'} = d^{-1} u_x d$. From (4-7) it follows that

$$\acute{J}(\varpi^{ce_m^m} u_x du_\beta) = \acute{J}(\varpi^{ce_m} du(x' + \beta)) = \acute{J}(t\varpi^{ce_m} du(\beta)t^{-1}) = \acute{J}(\varpi^{ce_m} du_\beta) = \acute{J}(p)$$

for

$$t := \operatorname{diag}(1 + x'_1/\beta_1, \ldots, 1 + x'_{m-1}/\beta_{m-1}, 1),$$

since $o(\beta_i) = -ie$ and $o(x'_i) \geq (m - 2i)e$ for $i \in \{1, \ldots, m - 1\}$. Therefore, (4-17) follows.

Now, we start the induction. For positive integers $r < s$, and $x \in F^r$, let

$$v_r^s(x) = \begin{bmatrix} 1_{s-r-1} & & \\ & 1 & x \\ & & 1_r \end{bmatrix}.$$

By (4-17), we may assume that $a_{m-1} > 0$. By Lemma 4.9, $a_l \geq b_l$. Let l be the first number such that $a_l > 0$.

First, suppose that $a_l = b_l$. Set

$$h = a_l = b_l, \quad f = he_1^n, \quad i = \sigma(1),$$
$$p' = du_\beta = p\big((\overbrace{0,\dots,0}^{l}, a_{l+1},\dots,a_{m-1}),\ (\overbrace{0,\dots,0}^{l}, b_{l+1},\dots,b_{m-1}), c\big)^\iota.$$

Observe that d^ι is aligned, and p' is reduced. The expansion of $T_f \acute{J}(p')$ is

$$\sum_{1\le i\le n-1}\ \sum_{x\in(\mathfrak{o}/\mathfrak{p}^h)^{n-i}} J\Big(\begin{bmatrix} p' & \\ & 1_m\end{bmatrix} v_{n-i}^n(x)\varpi^{he_i^n}\Big).$$

In case of $i \ge m$, the weight of $\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}$ is less than $\mathrm{wt}(p)$. In case of $i \le m-1$, we have $v_- = 1_m$, and (4-9) equals

$$\acute{J}(du_\beta v_{m-i}^m(x')\varpi^{he_i^m}),$$

where $x' = (x_1,\dots,x_{m-i})$. We compute

$$u_\beta v_{m-i}^m(x') = \acute{v}_{m-i-1}^{m-1}(x_1,\dots,x_{m-i-1})u_{\beta'},$$
$$\beta'_j = \begin{cases} \beta_i + (x_{m-i} - x_1\beta_{i+1} - \cdots - x_{m-i-1}\beta_{m-1}) & \text{if } j = i,\\ \beta_j & \text{otherwise.}\end{cases}$$

Here since β_k/β_j lies in $\mathfrak{p}^e$ if $k > j$ by Lemma 4.9, and x_i lies in $\mathfrak{o}$, we have

$$\frac{\beta'_j}{\beta_j} \in 1 + \mathfrak{p}^e, \quad j \in \{1,\dots,m-1\}. \tag{4-18}$$

Since d^ι is aligned, $d\acute{v}_{m-i-1}^{m-1}(x_1,\dots,x_{m-i-1})d^{-1}$ lies in $N_m(\mathfrak{o})$, and therefore,

$$\begin{aligned} du_\beta v_{m-i}^m(x')\varpi^{he_i^m} &= d\acute{v}_{m-i-1}^{m-1}(x_1,\dots,x_{m-i-1})u_{\beta'}\varpi^{he_i^m}\\ &\approx du_{\beta'}\varpi^{he_i^m} = d\varpi^{he_i^m}u(\beta''),\end{aligned}$$

where

$$\beta'' = \beta'\varpi^{-he_i^m}, \qquad \beta''_j = \begin{cases} \varpi^{-h}\beta'_i & \text{if } j = i,\\ \beta'_j & \text{otherwise.}\end{cases}$$

If $i < l$, then $d\varpi^{he_i^m}u(\beta'')$ is quasi-equivalent to an element $p_e(a', b', c)$ with $a' < a$ by the proof of Proposition 4.3. If $i > l$, then $d\varpi^{he_i^m}u(\beta'')$ is quasi-equivalent to an element which does not satisfy Lemma 4.9(ii). If $i = l$, then

$$\acute{J}(d\varpi^{he_l^m}u(\beta'')) = \acute{J}(d\varpi^{he_l^m}u(\beta'\varpi^{he_l^m})) = \acute{J}(d\varpi^{he_l^m}u(\beta\varpi^{he_l^m})) = \acute{J}(p)$$

by (4-18). This settles the case of $a_l = b_l$.

Next, suppose that $a_l > b_l$. Set

$$h = a_l - b_l, \quad f = -he_1^n, \quad i = \sigma(1),$$
$$p' = du_\beta = p\big((0, \dots, 0, b_l, a_{l+1}, \dots, a_{m-1}),\ (0, \dots, 0, b_l, b_{l+1}, \dots, b_{m-1}), c\big).$$

The expansion of $T\acute{J}(p')$ is

$$\sum_{1 \le i \le n-1} \sum_{x \in (\mathfrak{o}/\mathfrak{p}^h)^{i-1}} J\Big(\begin{bmatrix} p' & \\ & 1_m \end{bmatrix} \acute{u}(x) \varpi^{-he_i^n}\Big).$$

In case of $i \le m$, the weight of $\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}$ is less than $\mathrm{wt}(p)$. In case of $i > m$, $f_+^\sigma = 0$, $v_+ = 1_m$, and (4-9) equals

$$\chi(\varpi)^{-h} \acute{J}(\varpi^{he_{i-m}^m} \acute{u}(-x')p'),$$

where $x' = (x_{m+1}, \dots, x_{i-1})$. Since d is aligned, $d^{-1}\acute{u}(-x')d$ lies in $N_m(\mathfrak{o})$, and

$$\acute{u}(-x')p' = \acute{u}(-x')du_\beta = d(d^{-1}\acute{u}(-x')d)u_\beta = du(\beta')d^{-1}\acute{u}(-x')d \approx du(\beta'),$$

with

$$\beta'_j = \begin{cases} \beta_j(1 + x_{m+j}\varpi^{(i-m-j)e+(a_{i-m}-b_{i-m})-(a_j-b_j)}) & \text{if } j < i-m, \\ \beta_j & \text{if } j \ge i-m. \end{cases}$$

We have $\acute{J}(\varpi^{he_{i-m}^m} u(-x')p') = \acute{J}(\varpi^{he_{i-m}^m} du(\beta'))$. By the induction hypothesis and Lemma 4.9(ii) again, we may assume that $i = m + l$, and

$$\beta'_j = \beta_j(1 + x_{m+j}\varpi^{(l-j)e+(a_l-b_l)-(a_j-b_j)}) \in \beta_j(1 + \mathfrak{p}^e)$$

for $j < i - m = l$. Therefore,

$$\acute{J}(\varpi^{he_l^m} du(\beta')) = \acute{J}(\varpi^{he_l^m} du(\beta)) = \acute{J}(p).$$

This completes the proof of the theorem. □

Now, according to Reeder's oldform theory [14], any $P_n(\mathfrak{o})$-invariant vector in a generic π is a linear combination of $T'_f v^{\mathrm{new}}$, $f \in \mathbb{Z}_{\ge 0}^{n-1}$, where v^{new} is the newvector and T'_f is the Hecke operator defined by

$$T'_f \xi = \int_{G_{n-1}} \mathrm{Ch}(k;\, \acute{K}_{n-1} \acute{\varpi}^f \acute{K}_{n-1}) \pi(k)\xi \,\mathrm{d}k$$

for $P_n(\mathfrak{o})$-invariant $\xi \in \pi$, where $\mathrm{d}k$ indicates the Haar measure on G_{n-1} normalized so that $\mathrm{vol}(\acute{K}_{n-1}) = 1$. An elementary computation shows that

$$\pi(g)T'_f\xi = \sum_{\sigma \in \mathfrak{S}_{n-1}/\mathfrak{S}(f)} \sum_{v \in \mathcal{N}'(f^\sigma)} \pi(gv\varpi^{f^\sigma})\xi,$$

where

$$\mathcal{N}'(f^\sigma) = \{v \in N_n \mid v_{ij} \in \mathfrak{o}/\mathfrak{p}^{\max\{f_i^\sigma - f_j^\sigma, 0\}},\ j \le n-1,\ v_{in} = 0\}.$$

Therefore, if J is the Shalika newform, then $T'_f \acute{J}(p)$, $p \in \widetilde{P}_m$ is a linear combination of $\acute{J}(g)$, $g \in G_m$. Therefore, from Theorem 4.1, the following conclusion is deduced:

If we assume that the Shalika newform in $\mathbb{S}_\pi(\chi)$ vanishes at the minimal point, then

$$J|_{S_n P_n \Gamma(\mathfrak{c}_\pi)} \equiv 0$$

for any $P_n(\mathfrak{o})$-invariant Shalika form $J \in \mathbb{S}_\pi(\chi)$, where $\mathfrak{c}_\pi$ indicates the conductor of the generic π.

4.1. *Proof for Theorem 1.1.* There exists a nontrivial $P_n(\mathfrak{o})$-invariant vector in ψ_n. For example, we can define the $P_n(\mathfrak{o})$-invariant $\xi_n \in \psi_n$ by

$$\xi_n(\acute{\varpi}^f) = \begin{cases} 1 & \text{if } f = 0 \in \mathbb{Z}^{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 3.8, ψ_n has a nontrivial $P_n(\mathfrak{o})$-invariant pre-Shalika form. By the assumption (1-1) and Proposition 3.10, there exists a $P_n(\mathfrak{o})$-invariant Shalika form whose restriction to P_n is nontrivial. Now the assertion follows from the above conclusion.

5. Zeta integrals

Let $n = 2m$ be an even integer, and $\pi \in \mathscr{I}(G_n)$ be generic with a Shalika model relevant to χ. Let J^{new} be a Shalika newform (recall this is unique up to a scalar). Assume that J^{new} does not vanish at g_n of Theorem 1.1. This condition is empty if π is supercuspidal. Assume that

$$\mathfrak{c}_\pi \ge me.$$

Put

$$l = \mathfrak{c}_\pi - (m-1)e \quad (\ge e).$$

Let $\mathbb{K}(\mathfrak{c}_\pi)$ be as in Section 1. First of all, we construct a $J_\pi \in \mathbb{S}_\pi(\chi)$ such that

$$J_\pi(1_n) = 1, \qquad \pi(k) J_\pi = \chi \circ \det(d_k) J_\pi, \quad k \in \mathbb{K}(\mathfrak{c}_\pi),$$

where d_k indicates the $m \times m$ block matrix of k in the lower right corner. In the case where χ is unramified, there is nothing to do. Suppose that χ is ramified.

Lemma 5.1. *Let Ω be a field. Let f be a Ω-valued function on a group G such that $f(gk) = \xi(k) f(g)$ for a subgroup K and a homomorphism $\xi : K \to \Omega^\times$.*

(i) *For the right translation f^h by $h \in G$,*

$$f^h(gk) = \xi(h^{-1}kh) f^h(g), \quad k \in K^h.$$

(ii) *Let K' be a subgroup containing K. Assume that ξ is extended to $\xi' : K' \to \Omega^\times$. Define an Ω-valued function on G by*

$$f'(g) = \int_{K'/K} \xi'(k')^{-1} f(gk')\,\mathrm{d}k'.$$

Then,

$$f'(gk) = \xi'(k') f(g), \quad k' \in K'.$$

Proof. Obvious. □

Let r be a positive integer. For a set A, let $(A)^r$ denote the r-tuple product of A. Let M_r and $\mathcal{O}_r$ denote the ring of $r \times r$ matrices with entries in F and $\mathfrak{o}$, respectively. If a subgroup $K \subset G_n$ consists of matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a \in \mathfrak{A},\ b \in \mathfrak{B},\ c \in \mathfrak{C},\ d \in \mathfrak{D}$$

for subsets $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \subset M_m$, we call K the subgroup relevant to $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$. Let

$$\delta_m = \mathrm{diag}(\varpi^e, \varpi^{3e}, \ldots, \varpi^{(2m-3)e}, \varpi^{(m-1)e}) \in G_m,$$

and, for $t \in (\mathfrak{o}^\times)^{m-1}$, let

$$v_m(t) = u((\varpi^{-(m-1)e} t_1, \varpi^{-(m-2)e} t_2, \ldots, \varpi^{-e} t_{m-1})), \quad \mathfrak{J}_t = \pi\left(\begin{bmatrix} 1_m & \\ & v_m(t) \end{bmatrix}\right) J^{\mathrm{new}},$$

where $u(x)$, $x \in F^{m-1}$, indicates the element in (4-1). From (2-4) and the assumption $J^{\mathrm{new}}(g_n) \neq 0$, it follows that $\mathfrak{J}_1(\acute{\delta}_m) \neq 0$. By (2-4),

$$\mathfrak{J}_t(\acute{\delta}_m) = \chi\Big(\prod_{i=1}^{m-1} t_i\Big) \mathfrak{J}_1(\acute{\delta}_m).$$

Firstly, we set

$$J_1 = \int_{(\mathfrak{o}^\times)^{m-1}} \chi(t)^{-1} \mathfrak{J}_t\,\mathrm{d}t,$$

which is not zero at $\acute{\delta}_m$. For $u = \mathrm{diag}(u_1, \ldots, u_m)$ with $u_i \in \mathfrak{o}^\times$, it holds that

$$\begin{aligned} \pi\left(\begin{bmatrix} 1_m & \\ & u \end{bmatrix}\right) J_1 &= \int_{(\mathfrak{o}^\times)^{m-1}} \chi(t)^{-1} \pi\left(\begin{bmatrix} 1_m & \\ & u \end{bmatrix}\begin{bmatrix} 1_m & \\ & v_m(t) \end{bmatrix}\right) J\,\mathrm{d}t \\ &= \int_{(\mathfrak{o}^\times)^{m-1}} \chi(t)^{-1} \pi\left(\begin{bmatrix} 1_m & \\ & v_m(t') \end{bmatrix}\begin{bmatrix} 1_m & \\ & u \end{bmatrix}\right) J\,\mathrm{d}t = \chi\Big(\prod_{i=1}^{m} u_i\Big) J_1, \end{aligned}$$

where $t' = (t_1 u_1/u_m, \ldots, t_{m-1} u_{m-1}/u_m)$. By (i) of the lemma, $\mathfrak{J}_t$ is invariant under the subgroup relevant to

$$\mathcal{O}_m, \quad \{b \mid b_{ij} \in \mathfrak{p}^{m-1}\}, \quad \{c \mid c_{ij} \in \mathfrak{o};\ i < m,\ c_{mj} \in \mathfrak{p}^c\},$$
$$\{d \mid d_{ii} \in 1 + \mathfrak{p}^e,\ d_{ij} \in \mathfrak{o};\ i < j,\ d_{mj} \in \mathfrak{p}^c;\ j < m,\ d_{ij} \in \mathfrak{p}^{me};\ j < i < m\}.$$

Therefore,

$$\pi(k)J_1 = \chi\Big(\prod_{i=1}^{m} k_{m+i,m+i}\Big)J_1$$

for k lying in the subgroup relevant to

$$\mathcal{O}_m, \quad \{b \mid b_{ij} \in \mathfrak{p}^{m-1}\}, \quad \{c \mid c_{mj} \in \mathfrak{p}^c;\, c_{ij} \in \mathfrak{o},\, i < m\},$$
$$\{d \mid d_{ij} \in \mathfrak{o};\, i \le j,\, d_{mj} \in \mathfrak{p}^c;\, j < m,\, d_{ij} \in \mathfrak{p}^{me};\, j < i < m\}.$$

By (2-4),

$$J_1(\acute{\delta}_m) = J_1\Big(\begin{bmatrix} 1_m & \\ & \delta_m^{-1} \end{bmatrix}\Big) \neq 0$$

Secondly, we set

$$J_2 = \pi\Big(\begin{bmatrix} 1_m & \\ & \delta_m^{-1} \end{bmatrix}\Big)J_1.$$

Then $J_2(1_n) \neq 0$, and, by (i) of the lemma,

$$\pi(k)J_2 = \chi\Big(\prod_{i=1}^{m} k_{m+i,m+i}\Big)J_2$$

for k lying in the subgroup relevant to

$$\mathcal{O}_m, \quad \{b \mid b_{ij} \in \mathfrak{p}^{n-2}\}, \quad \{c \mid c_{mj} \in \mathfrak{p}^l,\, c_{ij} \in \mathfrak{o};\, i < m\},$$
$$\Big\{\begin{bmatrix} u & {}^t(\mathfrak{p}^{(m-2)e})^{m-1} \\ (\mathfrak{p}^{c_\pi})^{m-1} & * \end{bmatrix} \in K_m \mid u_{ij} \in 1 + \mathfrak{p}^{(n-4)e};\, i \neq j\Big\}.$$

Thirdly, we set

$$J_3 = \iiint \pi\left(\begin{bmatrix} 1_m & & \\ & u & {}^t x \\ & y & 1 \end{bmatrix}\right) J_2 \,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y$$

where the integral in x is over $(\mathfrak{o}/\mathfrak{p}^{(m-2)e})^{m-1}$, that in y is over $(\mathfrak{p}^l/\mathfrak{p}^{c_\pi})^{m-1}$, and that in u over

$$K_{m-1}/\{u \in K_{m-1} \mid u_{ij} \in 1 + \mathfrak{p}^{(n-4)e};\, i \neq j\}$$
$$\simeq \mathrm{SL}_{m-1}(\mathfrak{o})/\{u \in \mathrm{SL}_{m-1}(\mathfrak{o}) \mid u_{ij} \in 1 + \mathfrak{p}^{(n-4)e};\, i \neq j\}.$$

By (ii) of the lemma,

$$\pi(k)J_3 = \chi \circ \det(d_k)J_3$$

for k lying in the subgroup relevant to

$$\mathcal{O}_m, \quad \{b \mid b_{ij} \in \mathfrak{p}^{n-2}\}, \quad \{c \mid c_{mj} \in \mathfrak{p}^l;\, c_{ij} \in \mathfrak{o},\, i < m\},$$
$$\Big\{d = \begin{bmatrix} u & {}^t\mathfrak{o}^{m-1} \\ (\mathfrak{p}^l)^{m-1} & * \end{bmatrix} \in K_m \mid u \in K_{m-1}\Big\}.$$

By (2-4),

$$J_3(1_n) = \iiint J_2\left(\begin{bmatrix} 1_m & & \\ & u & {}^t x \\ & y & 1 \end{bmatrix}\right) \mathrm{d}u\, \mathrm{d}x\, \mathrm{d}y = \iiint J_2\left(\begin{bmatrix} u & {}^t x & \\ y & 1 & \\ & & 1_m \end{bmatrix}\right) \mathrm{d}u\, \mathrm{d}x\, \mathrm{d}y$$

is nonzero, where the integral region is the same as above. Finally, we set

$$J_\pi = c \int \pi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right) J_3 \, \mathrm{d}x,$$

where the integral in x is over $\mathcal{O}_m/\{b \mid b_{ij} \in \mathfrak{p}^{n-2}\}$, and c is the nonzero constant taken so that $J_\pi(1_n) = 1$. Another desired property of J_π follows from (ii) of the lemma. Let $\mathbb{K}(\mathfrak{c}_\pi)^*$ be the open compact subgroup consisting of the matrices (1-5), and $J_\pi^* \in \mathbb{S}_{\pi^\vee}(\chi^{-1})$ be the Shalika form (1-4). By (i) of the lemma,

$$\pi^\vee(k) J_\pi^* = \chi \circ \det(d_k) J_\pi^*, k \in \mathbb{K}(\mathfrak{c}_\pi)^*.$$

Now let us compute the Godement–Jacquet zeta integrals of these Shalika forms and specific Schwartz functions. For a finite dimensional vector space V over F, let $\mathscr{S}(V)$ denote the Schwartz space of V. Define $\varphi_{\mathfrak{c}_\pi} \in \mathscr{S}(M_m)$ as follows. In the case where χ is unramified, $\varphi_{\mathfrak{c}_\pi}(x) = \mathrm{Ch}(x;\, R_{\mathfrak{c}_\pi})$ (see (1-2) for the definition of $R_{\mathfrak{c}_\pi}$). Suppose that χ is ramified. Define $\chi_0 \in \mathscr{S}(F)$ by

$$\chi_0(x) = \mathrm{Ch}(x;\, \mathfrak{o}^\times)\chi(x).$$

Define $\phi_\chi^\circ \in \mathscr{S}(M_{m-1})$ by

$$\phi_\chi^\circ(x) = \prod_{1 \le i \ne j \le m-1} \mathrm{Ch}(x_{ij};\, \mathfrak{o}) \prod_{i=1}^{m-1} \chi_0(\varpi^e x_{ii}).$$

Define $\phi_{\chi,l} \in \mathscr{S}(M_m)$ by

$$\phi_{\chi,l}\left(\begin{bmatrix} x & {}^t y \\ z & w \end{bmatrix}\right) = \frac{\mathrm{Ch}(y, z;\, (\mathfrak{p}^{-l})^{m-1} \times \mathfrak{o}^{m-1})\chi_0(\varpi^e w)}{\mathrm{vol}(\mathrm{SL}_{m-1}(\mathfrak{o}))} \int_{\mathrm{SL}_{m-1}(\mathfrak{o})} \phi_\chi^\circ(xu)\, \mathrm{d}u.$$

Observe that

$$\phi_{\chi,l}(vk) = \chi \circ \det(k)\phi_{\chi,l}(v), \quad k \in \Gamma_m(l). \tag{5-1}$$

Define $\varphi_{\mathfrak{c}_\pi} \in \mathscr{S}(M_n)$ by

$$\varphi_{\mathfrak{c}_\pi}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathrm{Ch}\left(\begin{bmatrix} a & \\ c & d \end{bmatrix};\, R_{\mathfrak{c}_\pi}\right)\phi_{\chi,l}(b).$$

We have defined $\varphi_{\mathfrak{c}_\pi}$ so that $J_\pi \varphi_{\mathfrak{c}_\pi}$ is right $\mathbb{K}(\mathfrak{c}_\pi)$-invariant.

For $J \in \mathbb{S}_\pi(\chi)$ and $\varphi \in \mathscr{S}(G_n)$, define the Godement–Jacquet zeta integral

$$Z(s, J, \varphi) = \int_{G_n} J\varphi(g)\, |\det(g)|^s \, \mathrm{d}g,$$

which is absolutely convergent when $\Re(s)$ is sufficiently large. Originally, their zeta integrals are defined for φ and matrix coefficients of π (cf. [4]), but $Z(s, J, \varphi)$ can be understood as one of them by the proof of Proposition 3.1 of [9], as follows. Take an open compact subgroup $K \subset G_n$ such that φ is left K-invariant. The linear form $J \mapsto c_K \int_K J(k)\,\mathrm{d}k$ on $\mathbb{S}_\pi(\chi)$ is smooth and thus belongs to $\pi^\vee$, where $c_K = \mathrm{vol}(K)^{-1}$. Therefore, $f_{K,J}(g) := c_K \int_K J(kg)\,\mathrm{d}k$, $g \in G_n$ is a matrix coefficient of π. When $\Re(s)$ is sufficiently large, we have

$$\begin{aligned} Z(s, J, \varphi) &= c_K \int_{G_n}\Big(\int_K \varphi(kg)\,\mathrm{d}k\Big)\, J(g)\, |\det(g)|^s\,\mathrm{d}g \\ &= c_K \int_K\Big(\int_{G_n} \varphi(kg) J(g)\, |\det(g)|^s\,\mathrm{d}g\Big)\,\mathrm{d}k \\ &= c_K \int_K\Big(\int_{G_n} \varphi(g) J(k^{-1}g)\, |\det(g)|^s\,\mathrm{d}g\Big)\,\mathrm{d}k \\ &= c_K \int_{G_n}\Big(\int_K J(kg)\,\mathrm{d}k\Big)\, \varphi(g)\, |\det(g)|^s\,\mathrm{d}g = Z(s, f_{K,J}, \varphi). \end{aligned}$$

Now, for $k \in \mathbb{Z}$, let

$$B_{m,k} = \{b \in B_m \cap \mathcal{O}_m \mid o(\det(b)) = k\} \quad \text{and} \quad c_k = \sum_{B_{m,k}/B_{m,0}} \acute{J}_\pi(b).$$

Proposition 5.2. *With the above notation*,

$$Z\Big(s + \frac{n-1}{2}, J_\pi, \varphi_{\mathfrak{c}_\pi}\Big) = q^{l(m-1)}\mathrm{vol}(\mathbb{K}(\mathfrak{c}_\pi)) \sum_{i=0}^{\infty} c_i q^{i(-s+1/2)} \times \begin{cases} 1 & \text{if } e = 0, \\ \mathfrak{g}(\chi, \psi_{\varpi^{-e}})^m & \text{otherwise.} \end{cases}$$

Proof. Since the proofs are similar, we only treat the case $e > 0$. We observe what $g \in G_n$ contributes to the zeta integral. Using a complete system of representatives for $\Gamma_n(\mathfrak{c}_\pi)/\mathbb{K}(\mathfrak{c}_\pi)$ (cf. Lemma 2.1 of [13]), we may assume that g is of the form

$$\begin{bmatrix} a + bdc & bd \\ dc & d \end{bmatrix} = \begin{bmatrix} 1_m & b \\ & 1_m \end{bmatrix}\begin{bmatrix} a & \\ & d \end{bmatrix}\begin{bmatrix} 1_m & \\ c & 1_m \end{bmatrix}, \tag{5-2}$$

where a, b, c, d are $m \times m$ block matrices. We claim that d lies in $\Gamma_m(l)$, using the identity

$$\begin{bmatrix} 1_m & b' \\ & 1_m \end{bmatrix} g = \begin{bmatrix} (a + bdc) + b'dc & bd + b'd \\ dc & d \end{bmatrix}. \tag{5-3}$$

If g contributes to the zeta integral, then (5-2) lies in $\mathrm{supp}(\varphi_{\mathfrak{c}_\pi})$; therefore

$$(dc)_{mj} \in \mathfrak{p}^l, \quad (dc)_{ij} \in \mathfrak{p}^e, \quad i \in \{1, \ldots, m-1\},\ j \in \{1, \ldots, m\}, \tag{5-4}$$

and we may write

$$d = \begin{bmatrix} d^\circ & * \\ * & d_{mm} \end{bmatrix} \in \begin{bmatrix} \mathcal{O}_{m-1} & {}^t(\mathfrak{o})^{m-1} \\ (\mathfrak{p}^l)^{m-1} & \mathfrak{o} \end{bmatrix}.$$

By the $\mathbb{K}(\mathfrak{c}_\pi)$-invariance property of $J_\pi \varphi_{\mathfrak{c}_\pi}$, we may assume that d° is an upper triangular matrix. It suffices to show that $\det(d) \in \mathfrak{o}^\times$. Assume that $\det(d) \in \mathfrak{p}$. Then $d_{kk} \in \mathfrak{p}$ for some $k \in \{1, \ldots, m\}$. Using the assumption (5-4), and that d° is an upper trianglar matrix, we find that it is possible to take a b' so that

$$b'_{kk} \in \mathfrak{p}^{-1}, \quad b'_{jj} = 0, \quad j \in \{1, \ldots, m\} \setminus \{k\}, \quad b'dc \in \mathcal{O}_m, \quad b'd \in \mathfrak{o}E_{kk},$$

where E_{kk} indicates the k-th row and k-th column matrix unit. But, since (5-3) also lies in $\operatorname{supp}(\varphi_{\mathfrak{c}_\pi})$, and

$$J_\pi \varphi_{\mathfrak{c}_\pi}\left(\begin{bmatrix} 1_m & b' \\ & 1_m \end{bmatrix} g\right) = \psi(b'_{kk}) J_\pi \varphi_{\mathfrak{c}_\pi}(g)$$

by (2-4), g does not contribute. Hence, the claim. Now, it is easy to see that

$$\begin{bmatrix} & \\ c & \end{bmatrix} = \begin{bmatrix} & \\ & d^{-1} \end{bmatrix} \begin{bmatrix} & \\ dc & \end{bmatrix} \in R_{\mathfrak{c}_\pi}$$

and we may assume that

$$\begin{aligned} b_{ii} &\in \varpi^{-e}\mathfrak{o}^\times, \quad i \in \{1, \ldots, m\}, \\ b_{mj} &\in \mathfrak{o}, b_{jm} \in \mathfrak{p}^{-l}, \quad j \in \{1, \ldots, m-1\}, \end{aligned}$$

by (5-1). Therefore, bdc lies in $\mathcal{O}_m$, and so does a. Now the assertion follows from the argument of Lemma 4.2. □

By the way, the proof for (1-3) is as follows.

Proposition 5.3. *Set* $G_m^k = \{g \in G_m \mid o(\det(g)) = k\}$. *Then*

$$c_k = \operatorname{vol}(K_m)^{-1} \int_{G_m^k} J_\pi\left(\begin{bmatrix} g & \\ & 1_m \end{bmatrix}\right) \mathrm{d}g.$$

Proof. Let $g \in G_m^k \setminus \mathcal{O}_m$. Then, there exists an $x \in \mathcal{O}_m$ such that $\operatorname{tr}(gx) \notin \mathfrak{o}$. Therefore, $J_\pi(\acute{g}) = 0$ by Lemma 4.8, and the identity:

$$\begin{bmatrix} g & \\ & 1_m \end{bmatrix} \begin{bmatrix} 1_m & x \\ & 1_m \end{bmatrix} \begin{bmatrix} g & \\ & 1_m \end{bmatrix}^{-1} = \begin{bmatrix} 1_m & gx \\ & 1_m \end{bmatrix}.$$

Corresponding to the decomposition $G_m^k = G_m^k \cap \mathcal{O}_m \sqcup (G_m^k \setminus \mathcal{O}_m)$, the homogeneous space G_m^k/K_m decomposes into $(G_m^k \cap \mathcal{O}_m)/K_m \sqcup (G_m^k \setminus \mathcal{O}_m)/K_m$. Now the assertion follows from $B_{m,k}/B_{m,0} \simeq (G_m^k \cap \mathcal{O}_m)/K_m$ and the fact J_π is invariant under $\acute{K}_m$. □

For $\varphi \in \mathscr{S}(M_n)$, let $\varphi^\sharp$ be the Fourier transform of φ relevant to ψ

$$\varphi^\sharp(x) = \int_{G_n} \varphi(y)\psi(\operatorname{tr}(yx))\,\mathrm{d}y,$$

where $\mathrm{d}y$ indicates the self-dual Haar measure on M_n. We define

$$\varphi^*_{\mathfrak{c}_\pi}(x) = \varphi^\sharp_{\mathfrak{c}_\pi}(\upsilon_{\mathfrak{c}_\pi}^{-1\,t} x w_n),$$

where $\upsilon_{\mathfrak{c}_\pi}$ is the matrix defined in Section 1. For a function f on G_n, let $f^\vee$ denote the function defined by $f^\vee(g) = f(g^{-1})$. Then $J_\pi^* \varphi_{\mathfrak{c}_\pi}^*$ is right $\mathbb{K}_{\mathfrak{c}_\pi}^*$-invariant, and

$$q^{\mathfrak{c}_\pi s} Z(s, J_\pi^*, \varphi_{\mathfrak{c}_\pi}^*) = Z(s, J_\pi^\vee, \varphi_{\mathfrak{c}_\pi}^\sharp). \tag{5-5}$$

Now, in general,

$$\begin{aligned} Z(s, J^\vee, \varphi^\sharp) &= \int_{G_n} J(g^{-1}) \varphi^\sharp(g) \,|\det(g)|^s \,\mathrm{d}g \\ &= c_K \int_{G_n} J(g^{-1}) \int_K \varphi^\sharp(gk) \,\mathrm{d}k \,|\det(g)|^s \,\mathrm{d}g \\ &= c_K \int_K \int_{G_n} J(k^{-1} g^{-1}) \varphi^\sharp(g) \,|\det(g)|^s \,\mathrm{d}g \,\mathrm{d}k = Z(s, f_{K,J}^\vee, \varphi^\sharp), \end{aligned} \tag{5-6}$$

where K indicates an open compact subgroup of G_n such that φ is left K-invariant (therefore, $\varphi^\sharp$ is right K-invariant). An explicit description for $\varphi_{\mathfrak{c}_\pi}^*$ is as follows. When χ is unramified, it is given by

$$\varphi_{\mathfrak{c}_\pi}^*\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathrm{Ch}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}; R_{\mathfrak{c}_\pi}^*\right) \prod_{j=2}^m \mathrm{Ch}(b_{j1}; \mathfrak{p}^l),$$

where $R_{\mathfrak{c}_\pi}^*$ is the ring (cf. (1-5)). In the case where χ is ramified, define $\phi_\chi^{\circ *} \in \mathscr{S}(M_{m-1})$ by

$$\phi_\chi^{\circ *}(x) = \prod_{1 \le i \ne j \le m-1} \mathrm{Ch}(x_{ij}; \mathfrak{o}) \prod_{i=1}^{m-1} \chi_0^{-1}(x_{ii}).$$

Define $\phi_{\chi,l}^* \in \mathscr{S}(M_m)$ by

$$\phi_{\chi,l}^*\left(\begin{bmatrix} w & y \\ {}^t z & x \end{bmatrix}\right) = \frac{\mathrm{Ch}(y, z; \mathfrak{o}^{m-1} \times (\mathfrak{p}^l)^{m-1}) \chi_0^{-1}(w)}{\mathrm{vol}(\mathrm{SL}_{m-1}(\mathfrak{o}))} \int_{\mathrm{SL}_{m-1}(\mathfrak{o})} \phi_\chi^{\circ *}(xu) \,\mathrm{d}u.$$

Then, the explicit form of $\varphi_{\mathfrak{c}_\pi}^*$ is

$$\varphi_{\mathfrak{c}_\pi}^*\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathfrak{g}(\chi, \psi_{\varpi^{-e}})^m \phi_{\chi,l}^*(d) \mathrm{Ch}\left(\begin{bmatrix} a & b \\ c & \end{bmatrix}; R_{\mathfrak{c}_\pi}^*\right) \prod_{j=2}^m \mathrm{Ch}(b_{j1}; \mathfrak{p}^{-l}).$$

For $k \in \mathbb{Z}$, let

$$B_{m,k}^e = \{b \in B_m \mid o(\det(b)) = k, b_{11} \in \mathfrak{o}, b_{1j} \in \mathfrak{p}^{e-l};\ j > 1, b_{ij} \in \mathfrak{o};\ i > 1\},$$

and

$$c_k^* = \sum_{B_{m,k}^e / B_{m,0}^e} \acute{J}_\pi^*(b).$$

Similar to Proposition 5.3 and Proposition 5.2, we can show (1-6) and the following, respectively.

Proposition 5.4. *With the above notation*,

$$Z\Big(s+\frac{n-1}{2},\, J_\pi^*,\, \varphi^*_{\mathfrak{c}_\pi}\Big) = q^{l(m-1)}\mathrm{vol}(\mathbb{K}(\mathfrak{c}_\pi)^*)\sum_{i=0}^{\infty} c_i^* q^{i(-s+1/2)} \times \begin{cases} 1 & \text{if } e=0,\\ \mathfrak{g}(\chi,\, \psi_{\varpi^{-e}})^m & \text{otherwise.}\end{cases}$$

From (5-5), (5-6), and the functional equation in [4], it follows that

$$\frac{Z\big(\frac{n+1}{2}-s,\, J_\pi^*,\, \varphi^*_{\mathfrak{c}_\pi}\big)}{L(1-s,\pi^\vee)} = \varepsilon_\pi \frac{Z\big(s+\frac{n-1}{2},\, J_\pi,\, \varphi_{\mathfrak{c}_\pi}\big)}{L(s,\pi)}$$

where both the L-functions and the root number ε_π are same as those defined by Whittaker forms (cf. [11]). By Propositions 5.2 and 5.4, the both sides lie in $\mathbb{C}[q^{-s}]$ and $\mathbb{C}[q^s]$, respectively. Hence, both sides are nonzero constant. Thus we have:

Theorem 5.5. *With the above notation and assumptions*,

$$\sum_{i=0}^{\infty} c_i q^{i(-s+1/2)} = L(s,\pi), \quad \sum_{i=0}^{\infty} c_i^* q^{i(-s+1/2)} = \varepsilon_\pi L(s,\pi^\vee).$$

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References

[1] A. Aizenbud, D. Gourevitch, and H. Jacquet, "Uniqueness of Shalika functionals: the Archimedean case", *Pacific J. Math.* **243**:2 (2009), 201–212. MR

[2] I. N. Bernstein and A. N. Zelevinskii, "Representations of the group GL(n, F), where F is a local non-Archimedean field", *Uspehi Mat. Nauk* **31**:3 (1976), 5–70. In Russian; translated in *Russ. Math. Surv.* **31**:3 (1976), 1–68. MR

[3] F. Chen and B. Sun, "Uniqueness of twisted linear periods and twisted Shalika periods", *Sci. China Math.* **63**:1 (2020), 1–22. MR

[4] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Math. **260**, Springer, 1972. MR

[5] H. Grobner and N. Matringe, "On the non-vanishing of Shalika newvectors at the identity", *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.* **28** (558) (2024), 93–106. MR

[6] H. Grobner and A. Raghuram, "On the arithmetic of Shalika models and the critical values of L-functions for GL_{2n}", *Amer. J. Math.* **136**:3 (2014), 675–728. MR

[7] H. Jacquet and S. Rallis, "Uniqueness of linear periods", *Compositio Math.* **102**:1 (1996), 65–123. MR

[8] H. Jacquet and J. Shalika, "Exterior square L-functions", pp. 143–226 in *Automorphic forms, Shimura varieties, and L-functions* (Ann Arbor, MI, 1988), vol. 2, Academic, Boston, 1990. MR

[9] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, "Automorphic forms on GL(3), I", *Ann. of Math.* (2) **109**:1 (1979), 169–212. MR

[10] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, "Conducteur des représentations du groupe linéaire", *Math. Ann.* **256**:2 (1981), 199–214. MR

[11] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, "Rankin–Selberg convolutions", *Amer. J. Math.* **105**:2 (1983), 367–464. MR

[12] A. C. Kable, "Asai L-functions and Jacquet's conjecture", *Amer. J. Math.* **126**:4 (2004), 789–820. MR

[13] S. Kondo and S. Yasuda, "Local L and epsilon factors in Hecke eigenvalues", *J. Number Theory* **132**:9 (2012), 1910–1948. MR

[14] M. Reeder, "Old forms on GL_n", *Amer. J. Math.* **113**:5 (1991), 911–930. MR

[15] Y. Sakellaridis, "A Casselman–Shalika formula for the Shalika model of GL_n", *Canad. J. Math.* **58**:5 (2006), 1095–1120. MR

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INTEGRAL SOLUTIONS TO SYSTEMS OF DIAGONAL EQUATIONS

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We obtain an asymptotic formula for the number of integral solutions to a system of diagonal equations. We obtain an asymptotic formula for the number of solutions with variables restricted to smooth numbers as well. We improve the required number of variables compared to previous results by incorporating recent progress on Waring's problem and the resolution of the main conjecture in Vinogradov's mean value theorem.

1. Introduction

Consider the system of equations defined by

$$\begin{aligned} m_{1,1}x_1^d + \cdots + m_{1,n}x_n^d &= \mu_1, \\ &\vdots \\ m_{R,1}x_1^d + \cdots + m_{R,n}x_n^d &= \mu_R, \end{aligned} \tag{1-1}$$

which we write as $M\boldsymbol{x}^d = \boldsymbol{\mu}$, where $M = [m_{i,j}]_{\substack{1\leqslant i\leqslant R\\ 1\leqslant j\leqslant n}}$ is the coefficient matrix with integer entries and

$$\boldsymbol{x}^d = \begin{bmatrix} x_1^d \\ \vdots \\ x_n^d \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_R \end{bmatrix} \in \mathbb{Z}^R.$$

The system of diagonal equations (1-1) with $\boldsymbol{\mu} = \mathbf{0}$ was first studied by Davenport and Lewis, who established the following.

Theorem 1.1 [11, Lemma 32]. *Let $d \geqslant 3$ and $\boldsymbol{\mu} = \mathbf{0}$. Suppose that all n variables occur explicitly in the equations* (1-1). *Suppose that any linear combination, not identically zero, of the R rows of M contains more than $(2H + 3d - 1)R$ nonzero entries, where $H = \lfloor 3d \log Rd \rfloor$. Suppose the equations* (1-1) *have a nonsingular solution in every p-adic field, and further, if d is even, a real nonsingular solution. Then the equations* (1-1) *have infinitely many solutions in integers.*

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In fact, they obtained an asymptotic formula for the number of solutions. Their main results [11, Theorems 1 and 2] are consequences of this theorem and require

$$n \geqslant \begin{cases} \lfloor 9R^2 d \log(3Rd) \rfloor & \text{if } d \text{ is odd,} \\ \lfloor 48R^2 d^3 \log(3Rd^2) \rfloor & \text{if } d \geqslant 4 \text{ is even,} \end{cases}$$

for the conclusions to hold. By incorporating the breakthrough on Waring's problem by Vaughan [17], Brüdern and Cook [6] improved the number of variables required to

$$n > n_0(d)R,$$

where $n_0(d) = 2d(\log d + O(\log \log d))$, under a suitable "rank condition" on the coefficient matrix M. They also obtained an asymptotic formula for the number of solutions but with variables restricted to smooth numbers, which in turn provided a lower bound for the number of solutions in positive integers.

Since the release of these two papers, there has been great progress regarding Waring's problem (for example, by Wooley [22; 23] and more recently by Wooley and Brüdern [9]) and also the resolution of the main conjecture in Vinogradov's mean value theorem (see the works by Bourgain, Demeter and Guth [2] and by Wooley [25; 27]). The purpose of this paper is to incorporate these recent progress to improve the required number of variables in both the setting of solutions in positive integers as in [11] and in the smooth numbers as in [6].

There have been a number of results regarding pairs of diagonal equations, in which the improvements have been achieved by making use of various developments in the theory of smooth Weyl sums. For example, the work of Parsell [14] on pairs of equations of small degrees, Parsell and Wooley [15] on pairs of quintic equations, and Brüdern and Wooley [7] on pairs of cubic equations. For larger systems of diagonal equations, there are the papers of Brüdern and Wooley [8] on systems of cubic equations, and of Brandes and Parsell [3] and Brandes and Wooley [4] on systems of equations involving different degrees. These works assume that the system is "highly nonsingular" which is to say that any $R \times R$ submatrix of the coefficient matrix is invertible. Our work is instead in line with [11] and [6] which hold for systems of diagonal equations of the same (arbitrary) degree with slightly less restrictive conditions on the underlying coefficient matrix.

For $X \geqslant 1$ and $\mathfrak{B} \subseteq \mathbb{N}$, we introduce the counting function

$$N(\mathfrak{B}; X) = \#\{\boldsymbol{x} \in (\mathfrak{B} \cap [1, X])^n : M\boldsymbol{x}^d = \boldsymbol{\mu}\}.$$

Instead of restricting the singularity of the variety defined by our system of equations, as in the work of Birch [1], we will require a condition on how well the underlying coefficient matrix can be partitioned.

Definition 1.2. For an $R \times n$ matrix A with $n \geqslant R$, we define $\Psi(A)$ to be the largest integer $\mathfrak{T}$ such that there exists

$$\{\mathfrak{D}_1, \dots, \mathfrak{D}_{\mathfrak{T}}\},$$

where each $\mathfrak{D}_i$ is a linearly independent set of R columns of A and $\mathfrak{D}_i \cap \mathfrak{D}_j = \varnothing$ if $i \neq j$.

Remark 1.3. There are at least two ways to obtain lower bounds for $\Psi(M)$: by studying the ranks of submatrices (thanks to a result of Low, Pitman and Wolff [13, Lemma 1]) as in [12], or by algorithmically enumerating sets of R linearly independent columns as in [16]. In these two papers, lower bounds of the form a constant times $\frac{n}{R}$ were obtained for coefficient matrices related to $n \times n$ magic squares.

The following are the main results of this paper.

Theorem 1.4. *Let $d \geqslant 2$ and $T_{\text{int}}(d)$ be as recorded in Table 1. If $\Psi(M) \geqslant T_{\text{int}}(d)+1$, then there exists $\gamma > 0$ for which*

$$N(\mathbb{N}; X) = \mathfrak{S}\mathfrak{I}X^{n-dR} + O(X^{n-dR-\gamma}),$$

where $\mathfrak{S}$ is the singular series defined in (4-1) *and $\mathfrak{I}$ is the singular integral defined in* (4-5). *We remark that $T_{\text{int}}(d) \leqslant \min\{2^d, d(d+1)\}$ for all $d \geqslant 2$.*

Given $1 \leqslant Z \leqslant X$, we denote the Z-smooth numbers by

$$\mathscr{A}(X, Z) = \{x \in [1, X] \cap \mathbb{Z} : \text{prime } p | x \text{ implies } p \leqslant Z\}.$$

Theorem 1.5. *Let $d \geqslant 5$ and $T_{\text{smo}}(d)$ be as recorded in Table 2. If $\Psi(M) \geqslant T_{\text{smo}}(d) + 1$, then for $\eta > 0$ sufficiently small, there exists $\gamma > 0$ such that*

$$N(\mathscr{A}(X, X^{\eta}); X) = \varrho\,(1/\eta)^n\, \mathfrak{S}\mathfrak{I}X^{n-dR} + O(X^{n-dR}(\log X)^{-\gamma}),$$

where $\mathfrak{S}$ is the singular series defined in (4-1), *$\mathfrak{I}$ is the singular integral defined in* (4-5) *and ϱ is Dickman's function. We remark that $T_{\text{smo}}(d) \leqslant \lceil d(\log d + 4.20032)\rceil$ for all $d \geqslant 5$.*

Remark 1.6. Instead of the condition $\Psi(M) \geqslant T_{\text{int}}(d) + 1$ in Theorem 1.4, we may assume that there exists an $R \times (RT_{\text{int}}(d) + 1)$ submatrix of M with the property that after removing any one of its columns it still contains $T_{\text{int}}(d)$ pairwise disjoint $R \times R$ invertible submatrices; the same holds for Theorem 1.5 with $T_{\text{smo}}(d)$ in place of $T_{\text{int}}(d)$. This is essentially the hypothesis assumed in [6], but for simplicity we assume the former condition; to assume the latter condition, one needs to slightly modify the proof of Proposition 3.3.

Remark 1.7. The proofs of Theorems 1.4 and 1.5 show that one may take γ to be any number in the intervals

$$\left(0, \frac{\lambda(d)}{(2R+4)(R+1)}\right) \quad \text{and} \quad \left(0, \frac{1}{2d(2R+4)(R+1)}\right],$$

respectively, where $\lambda(d)$ is defined in Lemma 2.1.

An immediate corollary is a lower bound for $N(\mathbb{N}; X)$ which requires a smaller value of $\Psi(M)$ than in Theorem 1.5.

Corollary 1.8. *Let $d \geqslant 3$ and suppose $\Psi(M) \geqslant T_{\text{smo}}(d) + 1$. Then for $\boldsymbol{\mu} = \mathbf{0}$ such that $\mathfrak{S}\mathfrak{J} > 0$, we have*

$$N(\mathbb{N}; X) \gg X^{n-dR}.$$

We note that for a fixed choice of $\boldsymbol{\mu}$, by standard arguments, $\mathfrak{S} > 0$ if the equations (1-1) have a nonsingular solution in the ring of p-adic integers for every prime p, and $\mathfrak{J} > 0$ if the equations (1-1) have a nonsingular solution in $(\mathbb{R}_{>0})^n$.

The function $\theta(d) \in \{1, 2\}$ that appears in Table 1 is defined in (2-1). The values of $T_{\text{int}}(d)$ are described in Lemma 2.2, and they correspond to the smallest known number of variables s required to produce an asymptotic formula for the number of representations of any sufficiently large natural number as a sum of s d-th powers (compare with [27, Corollary 14.7] for larger powers and [24, Theorem 4.1] for intermediate powers).

For $d \geqslant 13$, the values of $T_{\text{smo}}(d)$ correspond to the best known values of $G(d)$, the least number of variables required to represent every sufficiently large natural number as a sum of d-th powers. Note the distinction that in this problem one asks only for the existence of a solution, and not the asymptotic formula for the number of solutions. For smaller values of d, $T_{\text{smo}}(d)$ is slightly larger than the best known values of $G(d)$, which are $G(7) \leqslant 31$, $G(8) \leqslant 39$, $G(9) \leqslant 47$, $G(10) \leqslant 55$, $G(11) \leqslant 63$ and $G(12) \leqslant 72$, as found in [26]); these values are obtained by

d	2	3	4	5	6	7	8	9	$\geqslant 10$
$T_{\text{int}}(d)$	4	8	15	23	34	47	61	78	$d^2-d+2\lfloor\sqrt{2d+2}\rfloor-\theta(d)$

Table 1. Values of T_{int} (see Theorem 1.4).

d	5	6	7	8	9	10	11	12
$T_{\text{smo}}(d)$	19	25	33	41	49	57	65	73
d	13	14	15	16	17	18	19	$\geqslant 20$
$T_{\text{smo}}(d)$	81	89	97	105	113	121	129	$\lceil d(\log d+4.20032)\rceil$

Table 2. Values of T_{smo} (see Theorem 1.5).

considering solutions to the underlying Diophantine equations for which only some of the variables are restricted to the smooth numbers.

Notation. We use the standard abbreviations $e(z) = e^{2\pi i z}$ and $e_q(z) = e^{2\pi i z/q}$. Given a vector $\boldsymbol{a} = (a_1, \ldots, a_R) \in \mathbb{Z}^R$, by $0 \leqslant \boldsymbol{a} \leqslant q$ we mean $0 \leqslant a_i \leqslant q$ for each $1 \leqslant i \leqslant R$. We also let $|\boldsymbol{\gamma}| = \max_{1 \leqslant i \leqslant R} |\gamma_i|$ for any $\boldsymbol{\gamma} \in \mathbb{R}^R$.

2. Preliminaries

Weyl sums. In this section, we collect two results that are key to proving Theorem 1.4. Both are consequences of the resolution of the main conjecture in Vinogradov's mean value theorem (by Bourgain, Demeter and Guth [2] and by Wooley [25; 27]).

Lemma 2.1. *Let $d \geqslant 2$. Let $\alpha \in \mathbb{R}$ and suppose that there exist $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $\gcd(q, a) = 1$ such that $|\alpha - a/q| \leqslant q^{-2}$ and $q \leqslant X^d$. We define*

$$\lambda(d) = \begin{cases} \dfrac{1}{2^{d-1}} & \text{if } 2 \leqslant d \leqslant 5, \\ \dfrac{1}{d(d-1)} & \text{otherwise.} \end{cases}$$

Then

$$\left| \sum_{1 \leqslant x \leqslant X} e(\alpha x^d) \right| \ll X^{1+\varepsilon} (q^{-1} + X^{-1} + q X^{-d})^{\lambda(d)},$$

for any $\varepsilon > 0$.

Proof. The bound for $2 \leqslant d \leqslant 5$ is the classic Weyl's inequality [18, Lemma 2.4]. The other estimate for larger d is a consequence of the resolution of the main conjecture in Vinogradov's mean value theorem (cf. [5, Lemma 2.4]). □

Let us define

$$\theta(d) = \begin{cases} 1 & \text{if } 2d+2 \geqslant \lfloor \sqrt{2d+2} \rfloor^2 + \lfloor \sqrt{2d+2} \rfloor, \\ 2 & \text{if } 2d+2 < \lfloor \sqrt{2d+2} \rfloor^2 + \lfloor \sqrt{2d+2} \rfloor. \end{cases} \tag{2-1}$$

Lemma 2.2. *Let $d \geqslant 2$ and let s be a real number greater than or equal to*

$$T_{\text{int}}(d) = \min\left\{ 2^d,\, d^2 - d + 2\lfloor \sqrt{2d+2} \rfloor - \theta(d),\, d^2 + 1 - \max_{\substack{1 \leqslant j \leqslant d-1 \\ 2^j \leqslant d^2}} \left\lceil \frac{dj - 2^j}{d+1-j} \right\rceil \right\}.$$

Then

$$\int_0^1 \left| \sum_{1 \leqslant x \leqslant X} e(\alpha x^d) \right|^s \mathrm{d}\alpha \ll X^{s-d+\varepsilon},$$

for any $\varepsilon > 0$.

Proof. The first bound, $s \geqslant 2^d$, is the classical version of Hua's lemma [18, Lemma 2.5], while the other two bounds are consequences of the resolution of the main conjecture in Vinogradov's mean value theorem. The second bound can be found in [27, Corollary 14.7], on noting that the bound for s_0 in the statement is given by

$$s_0 \leqslant \lfloor s_0 \rfloor + 1 \leqslant d^2 - d + 2\lfloor \sqrt{2d+2} \rfloor - \theta(d),$$

as explained in the proof. The third bound essentially follows from [24, Theorem 4.1]; it can be seen in the proof that the integral over the minor arcs satisfies $\ll X^{s-d+\varepsilon}$, while over the major arcs the same estimate follows by combining familiar estimates from the major arc analysis in the theory of Waring's problem (see [18, Section 4]). It can be verified that the values of $T_{\text{int}}(d)$ are precisely as in Table 1 (see the paragraph following [27, Corollary 14.7] and the proof of [24, Theorem 4.1]). □

Smooth Weyl sums. In this section, we record some key estimates regarding the smooth Weyl sums needed to prove Theorem 1.5. Let $d \geqslant 3$. We let

$$f(\alpha; X, Z) = \sum_{x \in \mathscr{A}(X,Z)} e(\alpha x^d).$$

We first need two estimates from [6]. We begin with [6, Lemma 3] which is obtained by combining [17, Theorem 1.8] and [19, Lemma 7.2].

Lemma 2.3 [6, Lemma 3]. *Let $d \geqslant 3$ and $\varepsilon > 0$ be sufficiently small. Suppose $\eta > 0$ is sufficiently small. Then there exists $\gamma = \gamma(d) > 0$ such that given $\alpha \in [0, 1]$ one of the following two alternatives holds*:

(i) $|f(\alpha; X, X^\eta)| < X^{1-\gamma}$.

(ii) *There exist* $0 \leqslant a \leqslant q$, $\gcd(q, a) = 1$ *such that*

$$f(\alpha; X, X^\eta) \ll q^\varepsilon X (q + X^d |q\alpha - a|)^{-1/(2d)} (\log X)^3.$$

The following is [6, Lemma 4], which is a special case of [19, Lemma 8.5].

Lemma 2.4 [6, Lemma 4]. *Let $d \geqslant 3$. Suppose $\eta > 0$ is sufficiently small. Let $A_0 > 0$. Suppose* $\gcd(q, a) = 1$, $1 \leqslant q \leqslant (\log X)^{A_0}$ *and* $|q\alpha - a| \leqslant (\log X)^{A_0} X^{-d}$. *Then*

$$f(\alpha; X, X^\eta) \ll X q^\varepsilon (q + X^d |q\alpha - a|)^{-1/d},$$

for any $\varepsilon > 0$.

Given a real parameter $\mathfrak{L} \geqslant 1$, we define

$$\mathfrak{N}_{\mathfrak{L}} = \bigcup_{1 \leqslant q \leqslant \mathfrak{L}} \bigcup_{\substack{0 \leqslant a \leqslant q \\ \gcd(q,a)=1}} \{\theta \in [0, 1] : |q\theta - a| < \mathfrak{L} X^{-d}\}.$$

We make use of the previous two lemmas to prove the following.

Lemma 2.5. *Let $\delta > 0$, $A = 2d\delta$ and $\mathfrak{L} = (\log X)^A$. Suppose $\eta > 0$ is sufficiently small. If*

$$|f(\alpha; X, X^\eta)| > X(\log X)^{-\delta}$$

holds for $X \geqslant 1$ sufficiently large, then

$$\alpha \in \mathfrak{N}_{\mathfrak{L}}.$$

Proof. Since we are in alternative (ii) of Lemma 2.3, it follows that

$$X(\log X)^{-\delta} < Cq^\varepsilon X(q + X^d|q\alpha - a|)^{-1/(2d)}(\log X)^3,$$

for $\varepsilon > 0$ sufficiently small and some $C > 0$, which in turn implies

$$q^{1/(2d)} < Cq^\varepsilon(\log X)^{\delta+3} \quad \text{and} \quad (X^d|q\alpha - a|)^{1/(2d)} < Cq^\varepsilon(\log X)^{\delta+3}.$$

Therefore, by setting $A_0 = (\delta+3)4d$, we obtain $1 \leqslant q < (\log X)^{4d(\delta+3)}$, $\gcd(q, a) = 1$ and $|q\alpha - a| < (\log X)^{A_0}X^{-d}$. It then follows from Lemma 2.4 that

$$X(\log X)^{-\delta} < C_1q^\varepsilon X(q + X^d|q\alpha - a|)^{-1/d},$$

for some $C_1 = C_1(d, \delta, \varepsilon) > 0$, which in turn implies

$$q^{1/d} < C_1q^\varepsilon(\log X)^\delta \quad \text{and} \quad (X^d|q\alpha - a|)^{1/d} < C_1q^\varepsilon(\log X)^\delta.$$

Therefore, for $\mathfrak{L} = (\log X)^A$ with $A = 2d\delta$, it follows that $\alpha \in \mathfrak{N}_{\mathfrak{L}}$ as desired. □

Finally, we have the following mean value estimate from [9].

Lemma 2.6. *Let $d \geqslant 5$ and s be an integer such that $s \geqslant T_{\mathrm{smo}}(d)$ as recorded in Table 2. Let $\eta > 0$ be sufficiently small and $1 \leqslant Z \leqslant X^\eta$. Then*

$$\int_0^1 |f(\alpha; X, Z)|^s \mathrm{d}\alpha \ll X^{s-d}.$$

Proof. A real number Δ_s is referred to as an admissible exponent (for d) if it has the property that, whenever $\varepsilon > 0$ and η is a positive number sufficiently small in terms of ε, d and s, then whenever $1 \leqslant Z \leqslant X^\eta$ and X is sufficiently large, one has

$$\int_0^1 |f(\alpha; X, Z)|^s \mathrm{d}\alpha \ll X^{s-d+\Delta_s+\varepsilon}.$$

Let us introduce the number

$$\tau(d) = \max_{w\in\mathbb{N}} \frac{d - 2\Delta_{2w}}{4w^2}.$$

Suppose that s is a real number with $s \geqslant 2$, and that the exponents Δ_u are admissible for $2 \leqslant u \leqslant s$. We define

$$\Delta_s^* = \min_{0\leqslant t\leqslant s-2}(\Delta_{s-t} - t\tau(d)),$$

and refer to Δ_s^* as an admissible exponent for minor arcs. Let $d \geqslant 3$, $s \geqslant 2d+3$ and let Δ_s^* be an admissible exponent for minor arcs with $\Delta_s^* < 0$. Then applying [9, Theorem 6.1] with $Q = 1$ provides the bound

$$\int_0^1 |f(\alpha; X, Z)|^s \mathrm{d}\alpha \ll X^{s-d}.$$

We now follow the argument in the proof of [9, Theorem 6.2]. We assume that we have available an admissible exponent Δ_u for each positive number u (which we know we may assume as explained in [9, Section 2], and also see [9, (7.1)] for further information regarding Δ_u when u is even and $d \geqslant 4$). When $d \geqslant 4$, we define

$$G_0(d) = \min_{v \geqslant 2} \left(v + \frac{\Delta_v}{\tau(d)} \right). \tag{2-2}$$

Suppose that $d \geqslant 4$ and $s \geqslant \max\{\lfloor G_0(d) \rfloor + 1, 2d+3\}$. Then there exists a positive number v with $v \geqslant 2$ and an admissible exponent Δ_v for which the exponent Δ_s^* is admissible for minor arcs, where

$$\Delta_s^* = \Delta_v - (s-v)\tau(d) = -\tau(d)(s - G_0(d)) < 0.$$

For $d \geqslant 14$, the value of $T_{\mathrm{smo}}(d)$ is precisely the value of $\lfloor G_0(d) \rfloor + 1$ found in the proofs of [9, Theorems 1.1 and 1.3], which can be seen to be greater than $2d+3$. For smaller d, we follow the proof of [9, Theorem 8.1] and compute $G_0(d)$ using the expression

$$T(d) = \frac{4w^2}{d - 2\Delta_{2w}}$$

for a suitably chosen value of w. Since $\tau(d) \geqslant T(d)^{-1}$, we clearly have

$$G_0(d) \leqslant v' + \Delta_{v'} T(d)$$

for any choice of $v' \geqslant 2$. We use the values of w and the corresponding admissible exponents Δ_{2w} recorded in Vaughan–Wooley [21, §9–15]. Here, the exponents λ_w of [21] are related to Δ_{2w} via the formula $\Delta_{2w} = \lambda_w - 2w + d$. Table 3 shows the chosen values of w and v used to compute $2v + \Delta_{2v} T(d)$.

For $d = 5$ and 6, the necessary data come instead from the appendix of [20] and we choose the values shown in Table 4. One readily observes that

$$T_{\mathrm{smo}}(d) = \lfloor 2v + \Delta_{2v} T(d) \rfloor + 1 \geqslant \lfloor G_0(d) \rfloor + 1$$

for the listed values of d. □

d	w	Δ_{2w}	$T(d)$	v	Δ_{2v}	$2v+\Delta_{2v}T(d)$
7	6	2.0143820	48.46467935	16	0.0105382	32.51073048
8	7	2.3105992	58.00873304	19	0.0473193	40.74493264
9	8	2.6039271	67.50795289	22	0.0727119	48.90863152
10	9	2.8945712	76.94394605	25	0.0895832	56.89288491
11	10	3.1849727	86.39206976	28	0.1020502	64.81632800
12	11	3.4700805	95.65521749	31	0.1118679	72.70074830
13	12	3.7557170	104.94544480	35	0.1010835	80.60825287

Table 3

d	w	Δ_{2w}	$T(d)$	v	Δ_{2v}	$2v+\Delta_{2v}T(d)$
5	4	1.4386563	30.15045927	8	0.0773627	18.33252094
6	5	1.7246965	39.20635362	12	0.0000000	24.00000000

Table 4

3. The Hardy–Littlewood circle method

Let $\mathfrak{B} = \mathbb{N}$ or $\mathscr{A}(X, X^\eta)$. Throughout the remainder of the paper, unless stated otherwise, we assume $d \geqslant 2$ if $\mathfrak{B} = \mathbb{N}$, and $d \geqslant 3$ if $\mathfrak{B} = \mathscr{A}(X, X^\eta)$. Our main tool to study $N(\mathfrak{B}; X)$ is the Hardy–Littlewood circle method and the key input are the estimates regarding the associated exponential sums. In contrast to the exposition in [6], we find it more natural to index our exponential sums by the columns of the corresponding coefficient matrix. For $\boldsymbol{\theta} \in [0, 1]^R$ and $\boldsymbol{c} \in \mathrm{Col}(M)$, we introduce the exponential sum

$$S_{\boldsymbol{c}}(\boldsymbol{\theta}) = S_{\boldsymbol{c}}(\mathfrak{B}; \boldsymbol{\theta}) = \sum_{x \in \mathfrak{B} \cap [1, X]} e(\boldsymbol{c} \cdot \boldsymbol{\theta} x^d).$$

Then

$$N(\mathfrak{B}; X) = \int_{[0,1]^R} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} S_{\boldsymbol{c}}(\mathfrak{B}; \boldsymbol{\theta}) \cdot e\biggl(-\sum_{i=1}^{R} \mu_i \theta_i\biggr)\, \mathrm{d}\boldsymbol{\theta}. \tag{3-1}$$

We set

$$\mathfrak{L} = \begin{cases} X^\delta & \text{if } \mathfrak{B} = \mathbb{N}, \\ (\log X)^A & \text{if } \mathfrak{B} = \mathscr{A}(X, X^\eta), \end{cases} \tag{3-2}$$

where $\delta, A > 0$ are to be chosen in due course. We define the major arcs

$$\mathfrak{M}_{\mathfrak{L}} = \bigcup_{1 \leqslant q \leqslant \mathfrak{L}} \bigcup_{\substack{\boldsymbol{a} \in \mathbb{Z}^R \\ 0 \leqslant \boldsymbol{a} \leqslant q \\ \gcd(q, \boldsymbol{a}) = 1}} \{\boldsymbol{\theta} \in [0, 1]^R : |q\theta_i - a_i| < \mathfrak{L} X^{-d} \ (1 \leqslant i \leqslant R)\},$$

and the minor arcs

$$\mathfrak{m}_{\mathfrak{L}} = [0, 1]^R \setminus \mathfrak{M}_{\mathfrak{L}}.$$

From here on out, we will use the following notation for simplicity.

Definition 3.1. We let T be a natural number such that $\Psi(M) \geqslant T$.

The minor arc estimate. The following lemma allows us to understand when a phase of the form $\boldsymbol{c}\cdot\boldsymbol{\theta}$ belongs to $[0,1]\backslash\mathfrak{N}_{\mathfrak{L}}$. Given a set of vectors $\mathfrak{D} = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_R\}$, we denote by $M(\mathfrak{D}) = [\boldsymbol{c}_1 \cdots \boldsymbol{c}_R]$ the matrix with these vectors as its columns.

Lemma 3.2. *Let $\mathfrak{D} = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_R\} \subseteq \mathrm{Col}(M)$ be a set of R linearly independent vectors. Suppose $X \geqslant 1$ is sufficiently large. If $\boldsymbol{c}_i \cdot \boldsymbol{\theta} \in \mathfrak{N}_{\mathfrak{L}^{1/(R+1)}}$ for all $1 \leqslant i \leqslant R$, then $\boldsymbol{\theta} \in \mathfrak{M}_{\mathfrak{L}}$.*

Proof. We have

$$\begin{bmatrix} q_1 \boldsymbol{c}_1 \cdot \boldsymbol{\theta} \\ \vdots \\ q_R \boldsymbol{c}_R \cdot \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} a_1 + E_1 \\ \vdots \\ a_R + E_R \end{bmatrix}$$

for some $1 \leqslant q_i \leqslant \mathfrak{L}^{1/(R+1)}$ and $1 \leqslant a_i \leqslant q_i$ such that $\gcd(a_i, q_i) = 1$ and $|E_i| < \mathfrak{L}^{1/(R+1)} X^{-d}$ for each $1 \leqslant i \leqslant R$. Then

$$q_1 \cdots q_R M(\mathfrak{D})^t \boldsymbol{\theta} = \begin{bmatrix} q_1 \cdots q_R (a_1 + E_1)/q_1 \\ \vdots \\ q_1 \cdots q_R (a_R + E_R)/q_R \end{bmatrix},$$

and the result follows by multiplying both sides of the equation by the inverse of $M(\mathfrak{D})^t$ on the left and simplifying the resulting equation. □

We are now ready to bound the contribution from the minor arcs.

Proposition 3.3. *Suppose that*

$$T \geqslant \begin{cases} T_{\mathrm{int}}(d) + 1 & \text{if } \mathfrak{B} = \mathbb{N}, \\ T_{\mathrm{smo}}(d) + 1 & \text{if } \mathfrak{B} = \mathscr{A}(X, X^\eta). \end{cases}$$

Suppose $\eta > 0$ is sufficiently small. Then, we may choose $\delta, A > 0$ such that there exists $\gamma > 0$ satisfying

$$\int_{\mathfrak{m}_{\mathfrak{L}}} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} |S_{\boldsymbol{c}}(\mathfrak{B}; \boldsymbol{\theta})| \, \mathrm{d}\boldsymbol{\theta} \ll X^{n-dR} \mathfrak{L}^{-\gamma}.$$

Proof. Let

$$\mathfrak{D}_1, \dots, \mathfrak{D}_T$$

be pairwise disjoint sets of R linearly independent columns of M. We begin by applying Lemma 3.2 with $\mathfrak{D}_T = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_R\}$. Given $\boldsymbol{\theta} \in \mathfrak{m}_{\mathfrak{L}}$, it follows from

Lemma 3.2 that there exists $1 \leqslant i \leqslant R$ such that $\boldsymbol{c}_i \cdot \boldsymbol{\theta} \notin \mathfrak{N}_{\mathfrak{L}'}$ with $\mathfrak{L}' = \mathfrak{L}^{1/(R+1)}$. Extracting the contribution from this column, we have the bound

$$\int_{\mathfrak{m}_{\mathfrak{L}}} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} |S_{\boldsymbol{c}}(\boldsymbol{\theta})| \, \mathrm{d}\boldsymbol{\theta} \leqslant X^{R-1} \sup_{\alpha \in [0,1] \setminus \mathfrak{N}_{\mathfrak{L}'}} \left| \sum_{x \in \mathfrak{B} \cap [1,X]} e(\alpha x^d) \right| \int_{[0,1]^R} \prod_{\boldsymbol{c} \in \mathrm{Col}(M) \setminus \mathfrak{D}_T} |S_{\boldsymbol{c}}(\boldsymbol{\theta})| \, \mathrm{d}\boldsymbol{\theta}.$$

Bounding the contribution from any column which does not belong to $\mathfrak{D}_1, \ldots, \mathfrak{D}_{T-1}$ trivially gives a bound for the integral over the minor arcs of

$$X^{n-(T-1)R-1} \sup_{\alpha \in [0,1] \setminus \mathfrak{N}_{\mathfrak{L}'}} \left| \sum_{x \in \mathfrak{B} \cap [1,X]} e(\alpha x^d) \right| \int_{[0,1]^R} \prod_{\ell=1}^{T-1} \prod_{\boldsymbol{c} \in \mathfrak{D}_\ell} |S_{\boldsymbol{c}}(\boldsymbol{\theta})| \, \mathrm{d}\boldsymbol{\theta}.$$

Applying Hölder's inequality, this is bounded by

$$X^{n-(T-1)R-1} \sup_{\alpha \in [0,1] \setminus \mathfrak{N}_{\mathfrak{L}'}} \left| \sum_{x \in \mathfrak{B} \cap [1,X]} e(\alpha x^d) \right| \prod_{\ell=1}^{T-1} \left(\int_{[0,1]^R} \prod_{\boldsymbol{c} \in \mathfrak{D}_\ell} |S_{\boldsymbol{c}}(\boldsymbol{\theta})|^{T-1} \, \mathrm{d}\boldsymbol{\theta} \right)^{1/(T-1)}.$$

Since the columns in $\mathfrak{D}_\ell$ are linearly independent, by a linear change of variables we obtain

$$\int_{[0,1]^R} \prod_{\boldsymbol{c} \in \mathfrak{D}_\ell} |S_{\boldsymbol{c}}(\boldsymbol{\theta})|^{T-1} \, \mathrm{d}\boldsymbol{\theta} \ll \prod_{i=1}^{R} \int_0^1 \left| \sum_{x \in \mathfrak{B} \cap [1,X]} e(\lambda_i x^d) \right|^{T-1} \mathrm{d}\lambda_i,$$

for each $1 \leqslant \ell \leqslant T-1$. We may now apply the bounds from Lemmas 2.1 and 2.2 or from Lemmas 2.5 and 2.6, depending on $\mathfrak{B}$, to conclude the proof. $\square$

Major arc analysis. We define

$$\mathfrak{M}_{\mathfrak{L}}^{+} = \bigcup_{1 \leqslant q \leqslant \mathfrak{L}} \bigcup_{\substack{\boldsymbol{a} \in \mathbb{Z}^R \\ 0 \leqslant \boldsymbol{a} \leqslant q \\ \gcd(q,\boldsymbol{a})=1}} \left\{ \boldsymbol{\theta} \in [0,1]^R : |q\theta_i - a_i| < q\mathfrak{L}X^{-d} \ (1 \leqslant i \leqslant R) \right\},$$

which clearly satisfies $\mathfrak{M}_{\mathfrak{L}} \subseteq \mathfrak{M}_{\mathfrak{L}}^{+}$. For any $q \in \mathbb{N}$, $a \in \mathbb{Z}$ and $\beta \in \mathbb{R}$, we introduce the standard notation

$$S(q,a) = \sum_{1 \leqslant x \leqslant q} e_q(ax^d) \quad \text{and} \quad I(\beta) = \int_0^1 e(\beta \xi^d) \, \mathrm{d}\xi.$$

Lemma 3.4. *Suppose that $q \in \mathbb{N}$, $a \in \mathbb{Z}$ and $\beta = \alpha - a/q$. Then*

$$\sum_{1 \leqslant x \leqslant X} e(\alpha x^d) = Xq^{-1} S(q,a) I(X^d \beta) + O\left(\frac{q}{\gcd(q,a)} (1 + X^d |\beta|) \right).$$

Proof. The statement with the additional hypothesis $\gcd(q,a)=1$ follows from [18, Theorem 4.1]. Suppose $\gcd(q,a)=g$ and let $q_0=q/g$ and $a_0=a/g$. Then

$$\begin{aligned} q^{-1}S(q,a) = q^{-1}\sum_{1\leqslant x\leqslant q} e_q(ax^d) &= q^{-1}\sum_{1\leqslant x\leqslant q} e_{q_0}(a_0x^d) \\ &= q^{-1}g\sum_{1\leqslant x\leqslant q_0} e_{q_0}(a_0x^d) = q_0^{-1}S(q_0,a_0). \end{aligned}$$

Therefore, we see that we may remove the coprimality condition. □

For the smooth Weyl sums we have the following.

Lemma 3.5. *Suppose that $1\leqslant q\leqslant Z$, $a\in\mathbb{Z}$ and $\beta=\alpha-a/q$. Then*

$$f(\alpha;X,Z)=q^{-1}S(q,a)w(\beta)+O\left(\frac{qX}{\gcd(q,a)\log X}(1+X^d|\beta|)\right),$$

where

$$w(\beta)=\sum_{Z^d<m\leqslant X^d}\frac{1}{d}m^{1/d-1}\varrho\left(\frac{\log m}{d\log Z}\right)e(\beta m)$$

and ϱ is Dickman's function (see [17, p. 53], for example).

Proof. The statement with the additional hypothesis $\gcd(q,a)=1$ is precisely [17, Lemma 5.4]. The coprimality condition may be removed in the same way as in the proof of Lemma 3.4. □

Lemma 3.6. *Let $|\beta|<\mathfrak{L}X^{-d}$ and w be as in Lemma 3.5. Then*

$$w(\beta)=\varrho\left(\frac{d\log X}{d\log Z}\right)XI(X^d\beta)+O\left(\frac{X}{\log Z}+Z\right).$$

Proof. Let us denote

$$P(y)=\sum_{Z^d<m\leqslant y}\frac{1}{d}m^{1/d-1}e(\beta m).$$

Then, by summation by parts, it follows that

$$\begin{aligned} w(\beta) &= \sum_{Z^d<m\leqslant X^d}\frac{1}{d}m^{1/d-1}e(\beta m)\,\varrho\left(\frac{\log m}{d\log Z}\right) \\ &= P(X^d)\,\varrho\left(\frac{d\log X}{d\log Z}\right)+O\left(1+\int_{Z^d}^{X^d}|P(y)|\frac{1}{y\log Z}\mathrm{d}y\right). \end{aligned}$$

Since $|P(y)|\ll y^{1/d}$, we have

$$\int_{Z^d}^{X^d}|P(y)|\frac{1}{y\log Z}dy\ll\frac{1}{\log Z}\int_{Z^d}^{X^d}y^{1/d-1}\mathrm{d}y\ll\frac{X}{\log Z}.$$

Therefore, we obtain

$$w(\beta)=\varrho\left(\frac{d\log X}{d\log Z}\right)\sum_{1\leqslant m\leqslant X^d}\frac{1}{d}m^{1/d-1}e(\beta m)+O\left(\frac{X}{\log Z}+Z\right).$$

By the mean value theorem, we obtain

$$\begin{aligned}\frac{1}{d}\sum_{1\leqslant m\leqslant X^d} & m^{1/d-1}e(\beta m)\\ &=\frac{1}{d}\int_1^{X^d}x^{1/d-1}e(\beta x)\,\mathrm{d}x+O\Bigl(1+\sum_{1\leqslant m\leqslant X^d}m^{1/d-1}(m^{-1}+|\beta|)\Bigr)\\ &=\int_0^X e(\beta t^d)\,\mathrm{d}t+O(1)=X\int_0^1 e(X^d\beta y^d)\,\mathrm{d}y+O(1)\\ &=XI(X^d\beta)+O(1).\end{aligned}$$

□

Let us now combine the above three lemmas in the following convenient manner.

Lemma 3.7. *Let $\eta>0$ be sufficiently small and*

$$C_{\mathfrak{B}}=\begin{cases}1 & \text{if } \mathfrak{B}=\mathbb{N},\\ \varrho(1/\eta) & \text{if } \mathfrak{B}=\mathscr{A}(X,X^\eta).\end{cases}\tag{3-3}$$

Let $\delta, A>0$ be sufficiently small. Suppose that $0\leqslant a\leqslant q\leqslant\mathfrak{L}$, $\beta=\alpha-a/q$ and $|\beta|<\mathfrak{L}X^{-d}$. Then

$$\Bigl|\sum_{x\in\mathfrak{B}\cap[1,X]}e(\alpha x^d)-C_{\mathfrak{B}}Xq^{-1}S(q,a)I(X^d\beta)\Bigr|\ll\begin{cases}\mathfrak{L}^2 & \text{if } \mathfrak{B}=\mathbb{N},\\ \dfrac{X\mathfrak{L}^2}{\log X} & \text{if } \mathfrak{B}=\mathscr{A}(X,X^\eta).\end{cases}$$

We define the truncated singular series

$$\mathfrak{S}(B)=\sum_{1\leqslant q\leqslant B}q^{-n}\sum_{\substack{1\leqslant \boldsymbol{a}\leqslant q\\ \gcd(q,\boldsymbol{a})=1}}\prod_{\boldsymbol{c}\in\mathrm{Col}(M)}S(q,\boldsymbol{a}\cdot\boldsymbol{c})\cdot e_q\Bigl(-\sum_{i=1}^R\mu_i a_i\Bigr)$$

for any $B\geqslant 1$, and the truncated singular integral

$$\mathfrak{I}(B)=\int_{|\boldsymbol{\gamma}|<B}\prod_{\boldsymbol{c}\in\mathrm{Col}(M)}I(\boldsymbol{\gamma}\cdot\boldsymbol{c})\cdot e\Bigl(-\frac{1}{X^d}\sum_{i=1}^R\mu_i\gamma_i\Bigr)\,\mathrm{d}\boldsymbol{\gamma}$$

for any $B>0$.

Proposition 3.8. *Let $\eta>0$ be sufficiently small and $C_{\mathfrak{B}}$ as in* (3-3). *Then*

$$\int_{\mathfrak{M}^+_{\mathfrak{L}}}\prod_{\boldsymbol{c}\in\mathrm{Col}(M)}S_{\boldsymbol{c}}(\boldsymbol{\theta})\cdot e\Bigl(-\sum_{i=1}^R\mu_i\theta_i\Bigr)\,\mathrm{d}\boldsymbol{\theta}=C_{\mathfrak{B}}^nX^{n-dR}\mathfrak{S}(\mathfrak{L})\mathfrak{I}(\mathfrak{L})+O(X^{n-dR}\mathfrak{L}^{-1}).$$

Proof. First, if $\boldsymbol{\theta} \in \mathfrak{M}_{\mathfrak{L}}^{+}$ then there exist $0 \leqslant \boldsymbol{a} \leqslant q$ such that $\gcd(q, \boldsymbol{a}) = 1$ and

$$\left| \boldsymbol{c} \cdot \boldsymbol{\theta} - \frac{\boldsymbol{c} \cdot \boldsymbol{a}}{q} \right| < C \mathfrak{L} X^{-d},$$

where $C > 0$ is a constant depending only on $\boldsymbol{c}$; therefore, $\boldsymbol{c} \cdot \boldsymbol{\theta}$, reduced modulo 1, satisfies the hypotheses of Lemma 3.7 with $\boldsymbol{c} \cdot \boldsymbol{\gamma}$ and $C\mathfrak{L}$ in place of β and $\mathfrak{L}$, respectively. Thus we may apply Lemma 3.7 to $S_{\boldsymbol{c}}(\boldsymbol{\theta})$ for any $\boldsymbol{c} \in \mathrm{Col}(M)$ and $\boldsymbol{\theta} \in \mathfrak{M}_{\mathfrak{L}}^{+}$. The measure of $\mathfrak{M}_{\mathfrak{L}}^{+}$ is at most $\mathfrak{L}^{2R+1} X^{-dR}$ and thus integrating the error term coming from applying Lemma 3.7 to $\prod_{\boldsymbol{c} \in \mathrm{Col}(M)} S_{\boldsymbol{c}}(\boldsymbol{\theta})$ gives a total error of size

$$\begin{cases} O(X^{n-dR-1} \mathfrak{L}^{2R+3}) & \text{if } \mathfrak{B} = \mathbb{N}, \\ O\left(X^{n-dR} \dfrac{\mathfrak{L}^{2R+3}}{\log X} \right) & \text{if } \mathfrak{B} = \mathscr{A}(X, X^{\eta}). \end{cases}$$

In the former case, the saving is $X^{1-(2R+3)\delta}$, which is greater than X^{δ} when $\delta \leqslant 1/(2R+4)$. Similarly, in the latter case, one saves $(\log X)^{1-A(2R+3)}$ which is again sufficient under the condition that $A \leqslant 1/(2R+4)$.

As a result we have

$$\begin{aligned}
&\int_{\mathfrak{M}_{\mathfrak{L}}^{+}} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} S_{\boldsymbol{c}}(\boldsymbol{\theta}) \cdot e\left(-\sum_{i=1}^{R} \mu_i \theta_i \right) \mathrm{d}\boldsymbol{\theta} \\
&= C_{\mathfrak{B}}^{n} X^{n} \sum_{1 \leqslant q \leqslant \mathfrak{L}} q^{-n} \sum_{\substack{1 \leqslant \boldsymbol{a} \leqslant q \\ \gcd(q, \boldsymbol{a}) = 1}} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} S_{\boldsymbol{c}}(\boldsymbol{a}/q) \cdot e_q\left(-\sum_{i=1}^{R} \mu_i a_i \right) \\
&\qquad \times \int_{|\boldsymbol{\gamma}| < \mathfrak{L} X^{-d}} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} I(X^{d} \boldsymbol{c} \cdot \boldsymbol{\gamma}) \cdot e\left(-\sum_{i=1}^{R} \mu_i \gamma_i \right) \mathrm{d}\boldsymbol{\gamma} + O(X^{n-dR} \mathfrak{L}^{-1}) \\
&= C_{\mathfrak{B}}^{n} X^{n} \mathfrak{S}(\mathfrak{L}) \int_{|\boldsymbol{\gamma}| < \mathfrak{L} X^{-d}} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} I(X^{d} \boldsymbol{c} \cdot \boldsymbol{\gamma}) \cdot e\left(-\sum_{i=1}^{R} \mu_i \gamma_i \right) \mathrm{d}\boldsymbol{\gamma} + O(X^{n-dR} \mathfrak{L}^{-1}) \\
&= C_{\mathfrak{B}}^{n} X^{n-dR} \mathfrak{S}(\mathfrak{L}) \int_{|\boldsymbol{\gamma}| < \mathfrak{L}} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} I(\boldsymbol{c} \cdot \boldsymbol{\gamma}) \cdot e\left(-\frac{1}{X^{d}} \sum_{i=1}^{R} \mu_i \gamma_i \right) \mathrm{d}\boldsymbol{\gamma} + O(X^{n-dR} \mathfrak{L}^{-1}) \\
&= C_{\mathfrak{B}}^{n} X^{n-dR} \mathfrak{S}(\mathfrak{L}) \mathfrak{I}(\mathfrak{L}) + O(X^{n-dR} \mathfrak{L}^{-1}),
\end{aligned}$$

which completes the claim. □

4. Singular series and singular integral

Let us denote

$$A(q) = q^{-n} \sum_{\substack{1 \leqslant \boldsymbol{a} \leqslant q \\ \gcd(q, \boldsymbol{a}) = 1}} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} S(q, \boldsymbol{a} \cdot \boldsymbol{c}) \cdot e_q\left(-\sum_{i=1}^{R} \mu_i a_i \right).$$

We define the singular series as

$$\mathfrak{S} = \sum_{q=1}^{\infty} A(q) = \lim_{B\to\infty} \mathfrak{S}(B). \tag{4-1}$$

In the following lemma, we bound the quantity $A(q)$ in order to show that the singular series does indeed converge absolutely.

Lemma 4.1. *Suppose $T > d$ and let $q \in \mathbb{N}$. Then*

$$A(q) \ll q^{-T/d+1+\varepsilon},$$

for any $\varepsilon > 0$.

Proof. By [10, Lemma 6.4], we have

$$\begin{aligned} |S(q, \boldsymbol{a}\cdot\boldsymbol{c})| &= \gcd(q, \boldsymbol{a}\cdot\boldsymbol{c}) \left| S\left(\frac{q}{\gcd(q, \boldsymbol{a}\cdot\boldsymbol{c})}, \frac{\boldsymbol{a}}{\gcd(q, \boldsymbol{a}\cdot\boldsymbol{c})}\right)\right| \\ &\ll \gcd(q, \boldsymbol{a}\cdot\boldsymbol{c}) \left(\frac{q}{\gcd(q, \boldsymbol{a}\cdot\boldsymbol{c})}\right)^{1-1/d}. \end{aligned}$$

We know that there exist pairwise disjoint sets $\mathfrak{D}_1, \ldots, \mathfrak{D}_T$ of R linearly independent columns of M. Applying Hölder's inequality, it follows that

$$\begin{aligned} |A(q)| &\leqslant q^{-TR} \sum_{\substack{1\leqslant \boldsymbol{a}\leqslant q\\ \gcd(q,\boldsymbol{a})=1}} \prod_{\ell=1}^{T} \prod_{\boldsymbol{c}\in\mathfrak{D}_\ell} |S(q, \boldsymbol{a}\cdot\boldsymbol{c})| \\ &\ll q^{-TR} \prod_{\ell=1}^{T} \Biggl(\sum_{\substack{1\leqslant \boldsymbol{a}\leqslant q\\ \gcd(q,\boldsymbol{a})=1}} \prod_{\boldsymbol{c}\in\mathfrak{D}_\ell} |S(q, \boldsymbol{a}\cdot\boldsymbol{c})|^T \Biggr)^{1/T} \\ &\ll q^{-TR/d} \prod_{\ell=1}^{T} \Biggl(\sum_{\substack{1\leqslant \boldsymbol{a}\leqslant q\\ \gcd(q,\boldsymbol{a})=1}} \prod_{\boldsymbol{c}\in\mathfrak{D}_\ell} \gcd(q, \boldsymbol{a}\cdot\boldsymbol{c})^{T/d} \Biggr)^{1/T}. \end{aligned} \tag{4-2}$$

Let us suppose for the time being that we have

$$\sum_{\substack{1\leqslant \boldsymbol{a}\leqslant q\\ \gcd(q,\boldsymbol{a})=1}} \prod_{\boldsymbol{c}\in\mathfrak{D}_\ell} \gcd(q, \boldsymbol{a}\cdot\boldsymbol{c})^{T/d} \ll q^{(R-1)T/d+1+\varepsilon}, \tag{4-3}$$

for every $1 \leqslant \ell \leqslant T$ and any $\varepsilon > 0$. Then, by substituting this estimate into (4-2), we obtain

$$|A(q)| \ll q^{-TR/d} q^{(R-1)T/d+1+\varepsilon} = q^{-T/d+1+\varepsilon},$$

as desired.

We now prove the estimate (4-3). Write $\mathfrak{D}_\ell = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_R\}$. First we have

$$\sum_{\substack{1\leqslant \boldsymbol{a}\leqslant q\\ \gcd(\boldsymbol{a},q)=1}} \prod_{\boldsymbol{c}\in\mathfrak{D}_\ell} \gcd(q, \boldsymbol{a}\cdot\boldsymbol{c})^{T/d} \leqslant \sum_{d_1|q,\dots,d_R|q} (d_1\cdots d_R)^{T/d}\, \#\left\{ \begin{matrix} 1\leqslant \boldsymbol{a}\leqslant q \\ \gcd(\boldsymbol{a},q)=1 \end{matrix} : d_i \,|\, \boldsymbol{a}\cdot\boldsymbol{c}_i \quad (1\leqslant i\leqslant R)\right\}. \tag{4-4}$$

Let $\tilde{d} = \gcd(d_1, \dots, d_R)$. From $d_i \,|\, \boldsymbol{a}\cdot\boldsymbol{c}_i$ $(1\leqslant i\leqslant R)$, it follows that

$$M(\mathfrak{D}_\ell)^t \boldsymbol{a} \equiv \boldsymbol{0} \pmod{\tilde{d}}.$$

By multiplying both sides of this congruence relation by the adjugate of $M(\mathfrak{D}_\ell)^t$, we obtain

$$\det M(\mathfrak{D}_\ell) a_i \equiv 0 \pmod{\tilde{d}}$$

for all $1\leqslant i\leqslant R$. Since $\gcd(\boldsymbol{a}, q) = \gcd(\boldsymbol{a}, \tilde{d}) = 1$, it follows that

$$\tilde{d} \,|\, \det M(\mathfrak{D}_\ell).$$

Thus we obtain that (4-4) is

$$\begin{aligned}
&\ll \sum_{\substack{d_1|q,\dots,d_R|q\\ \gcd(d_1,\dots,d_R)\ll 1}} (d_1\cdots d_R)^{T/d}\, \#\left\{ \begin{matrix} 1\leqslant \boldsymbol{a}\leqslant q \\ \gcd(\boldsymbol{a},q)=1 \end{matrix} : d_i \,|\, \boldsymbol{a}\cdot\boldsymbol{c}_i \quad (1\leqslant i\leqslant R)\right\}\\
&\ll \sum_{\substack{d_1|q,\dots,d_R|q\\ \gcd(d_1,\dots,d_R)\ll 1}} (d_1\cdots d_R)^{T/d} \frac{q^R}{d_1\cdots d_R}\\
&\ll q^R \sum_{\substack{\tilde{d}|q\\ \tilde{d}\ll 1}} \sum_{\substack{v_1|q/\tilde{d},\dots,v_R|q/\tilde{d}\\ \gcd(v_1,\dots,v_R)=1}} (\tilde{d}^R v_1\cdots v_R)^{T/d-1}.
\end{aligned}$$

Since $\gcd(v_1, \dots, v_R) = 1$, we may deduce from $v_1 \,|\, q/\tilde{d}, \dots, v_R \,|\, q/\tilde{d}$ that

$$v_1\cdots v_R \,|\, (q/\tilde{d})^{R-1}.$$

Therefore, the final expression above is

$$\begin{aligned}
&\ll q^R \sum_{\substack{\tilde{d}|q\\ \tilde{d}\ll 1}} \sum_{w|(q/\tilde{d})^{R-1}} w^{T/d-1}\, \#\{(v_1,\dots,v_R)\in\mathbb{N}^R : w = v_1\cdots v_R\}\\
&\ll q^R \sum_{\substack{\tilde{d}|q\\ \tilde{d}\ll 1}} \left(\frac{q}{\tilde{d}}\right)^{(R-1)(T/d-1)+\varepsilon} \ll q^{R+(R-1)(T/d-1)+\varepsilon} = q^{(R-1)T/d+1+\varepsilon},
\end{aligned}$$

for any $\varepsilon > 0$. □

Using this lemma we may extend the truncated singular series.

Lemma 4.2. *Suppose $T > 2d$ and let $\varepsilon > 0$ be sufficiently small. Then*

$$\mathfrak{S} = \mathfrak{S}(B) + O(B^{2-T/d+\varepsilon})$$

for any $B \geqslant 1$. In fact,

$$\mathfrak{S} = \prod_{p \text{ prime}} \chi(p),$$

where

$$\chi(p) = 1 + \sum_{k=1}^{\infty} A(p^k).$$

Proof. The statement is obtained from Lemma 4.1 by writing

$$|\mathfrak{S} - \mathfrak{S}(B)| \leqslant \sum_{q>B} |A(q)| \ll \sum_{q>B} q^{-T/d+1+\varepsilon} \ll B^{2-T/d+\varepsilon}.$$

Since $A(q_1 q_2) = A(q_1)A(q_2)$ for any coprime positive integers q_1 and q_2, we also have

$$\mathfrak{S} = \prod_{p \text{ prime}} \chi(p)$$

as desired. □

Similarly, we define the singular integral as

$$\mathfrak{J} = \int_{\mathbb{R}^R} \prod_{\boldsymbol{c} \in \mathrm{Col}(M)} I(\boldsymbol{\gamma} \cdot \boldsymbol{c}) \cdot e\biggl(-\frac{1}{X^d} \sum_{i=1}^{R} \mu_i \gamma_i\biggr) \,\mathrm{d}\boldsymbol{\gamma} = \lim_{B \to \infty} \mathfrak{J}(B). \tag{4-5}$$

We may also extend the truncated singular integral.

Lemma 4.3. *Suppose $T > d$. Then*

$$\mathfrak{J}(B) = \mathfrak{J} + O(B^{1-T/d})$$

for any $B > 1$.

Proof. We begin with the bound

$$I(\boldsymbol{\gamma} \cdot \boldsymbol{c}) = \int_0^1 e(\boldsymbol{\gamma} \cdot \boldsymbol{c} \xi^d) \,\mathrm{d}\xi \ll \min\{1, |\boldsymbol{\gamma} \cdot \boldsymbol{c}|^{-1/d}\}, \tag{4-6}$$

which for instance can be found in [10, p. 21] or [18, Lemma 2.8]. We know that there exist pairwise disjoint sets $\mathfrak{D}_1, \dots, \mathfrak{D}_T$ of R linearly independent columns of M. It then follows by Hölder's inequality that

$$|\mathfrak{I}-\mathfrak{I}(B)| \leqslant \int_{|\boldsymbol{\gamma}|\geqslant B} \prod_{\boldsymbol{c}\in \mathrm{Col}(M)} \min\{1, |\boldsymbol{\gamma}\cdot \boldsymbol{c}|^{-1/d}\}\, \mathrm{d}\boldsymbol{\gamma} \tag{4-7}$$

$$\leqslant \int_{|\boldsymbol{\gamma}|\geqslant B} \prod_{\ell=1}^{T} \prod_{\boldsymbol{c}\in \mathfrak{D}_\ell} \min\{1, |\boldsymbol{\gamma}\cdot \boldsymbol{c}|^{-1/d}\}\, \mathrm{d}\boldsymbol{\gamma}$$

$$\leqslant \prod_{\ell=1}^{T} \biggl(\int_{|\boldsymbol{\gamma}|\geqslant B} \prod_{\boldsymbol{c}\in \mathfrak{D}_\ell} \min\{1, |\boldsymbol{\gamma}\cdot \boldsymbol{c}|^{-1/d}\}^T \mathrm{d}\boldsymbol{\gamma} \biggr)^{1/T}.$$

By the change of variable $\tilde{\boldsymbol{\gamma}} = M(\mathfrak{D}_\ell)^t \boldsymbol{\gamma}$, we obtain

$$\int_{|\boldsymbol{\gamma}|\geqslant B} \prod_{\boldsymbol{c}\in \mathfrak{D}_\ell} \min\{1, |\boldsymbol{\gamma}\cdot \boldsymbol{c}|^{-1/d}\}^T \mathrm{d}\boldsymbol{\gamma}$$

$$\ll \int_{|\tilde{\boldsymbol{\gamma}}|\gg B} \min\{1, |\tilde{\gamma}_1|^{-1/d}\}^T \cdots \min\{1, |\tilde{\gamma}_R|^{-1/d}\}^T \mathrm{d}\tilde{\boldsymbol{\gamma}}$$

$$\ll \int_{\substack{\tilde{\gamma}_R > \cdots > \tilde{\gamma}_1 \geqslant 0 \\ \tilde{\gamma}_R \gg B}} \min\{1, |\tilde{\gamma}_1|^{-1/d}\}^T \cdots \min\{1, |\tilde{\gamma}_R|^{-1/d}\}^T \mathrm{d}\tilde{\boldsymbol{\gamma}}$$

$$\ll B^{1-T/d}$$

for each $1 \leqslant \ell \leqslant T$. On substituting this estimate into (4-7), it follows that

$$|\mathfrak{I}-\mathfrak{I}(B)| \ll B^{1-T/d}. \qquad \square$$

We may now conclude the proof of our main results.

Proof of Theorems 1.4 and 1.5. Recall our starting point for the circle method (3-1) and that $\mathfrak{M}_{\mathfrak{L}} \subseteq \mathfrak{M}_{\mathfrak{L}}^+$. On combining Propositions 3.3 and 3.8, we have

$$N(\mathfrak{B}; X) = C_{\mathfrak{B}}^n X^{n-dR} \mathfrak{S}(\mathfrak{L}) \mathfrak{I}(\mathfrak{L}) + O(X^{n-dR} \mathfrak{L}^{-\gamma}), \tag{4-8}$$

for some $\gamma > 0$. Lastly, we obtain from Lemmas 4.2 and 4.3 that

$$\mathfrak{S}(\mathfrak{L})\mathfrak{I}(\mathfrak{L}) = \mathfrak{S}\mathfrak{I} + O(\mathfrak{L}^{2-T/d+\varepsilon} + \mathfrak{L}^{1-T/d}),$$

for any $\varepsilon > 0$ sufficiently small. These two equations together give the desired asymptotic formula. $\square$

Remark 4.4. To see the allowable value of γ given in Remark 1.7, observe that the dominant error term in (4-8) comes from Proposition 3.3. Here, we have replaced $\mathfrak{L}$ by $\mathfrak{L}^{1/(R+1)}$ after an application of Lemma 3.2 and then saved a power $\lambda(d)$ or $1/(2d)$ of this via Lemmas 2.1 and 2.5, respectively. Finally, one recalls from the proof of Proposition 3.8 that $\delta, A \leqslant 1/(2R+4)$.

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References

[1] B. J. Birch, "Forms in many variables", *Proc. Roy. Soc. London Ser. A* **265** (1961/62), 245–263.

[2] J. Bourgain, C. Demeter, and L. Guth, "Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three", *Ann. of Math.* (2) **184**:2 (2016), 633–682.

[3] J. Brandes and S. T. Parsell, "Simultaneous additive equations: repeated and differing degrees", *Canad. J. Math.* **69**:2 (2017), 258–283.

[4] J. Brandes and T. D. Wooley, "Optimal mean value estimates beyond Vinogradov's mean value theorem", *Acta Arith.* **200**:2 (2021), 149–182.

[5] T. D. Browning and S. Yamagishi, "Arithmetic of higher-dimensional orbifolds and a mixed Waring problem", *Math. Z.* **299**:1-2 (2021), 1071–1101.

[6] J. Brüdern and R. J. Cook, "On simultaneous diagonal equations and inequalities", *Acta Arith.* **62**:2 (1992), 125–149.

[7] J. Brüdern and T. D. Wooley, "The Hasse principle for pairs of diagonal cubic forms", *Ann. of Math.* (2) **166**:3 (2007), 865–895.

[8] J. Brüdern and T. D. Wooley, "The Hasse principle for systems of diagonal cubic forms", *Math. Ann.* **364**:3-4 (2016), 1255–1274.

[9] J. Brüdern and T. D. Wooley, "On Waring's problem for larger powers", *J. Reine Angew. Math.* **805** (2023), 115–142.

[10] H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, 2nd ed., Cambridge University Press, 2005.

[11] H. Davenport and D. J. Lewis, "Simultaneous equations of additive type", *Philos. Trans. Roy. Soc. London Ser. A* **264** (1969), 557–595.

[12] D. Flores, "A circle method approach to K-multimagic squares", (2024). arXiv 2406.08161

[13] L. Low, J. Pitman, and A. Wolff, "Simultaneous diagonal congruences", *J. Number Theory* **29**:1 (1988), 31–59.

[14] S. T. Parsell, "Pairs of additive equations of small degree", *Acta Arith.* **104**:4 (2002), 345–402.

[15] S. T. Parsell and T. D. Wooley, "On pairs of diagonal quintic forms", *Compositio Math.* **131**:1 (2002), 61–96.

[16] N. Rome and S. Yamagishi, "On the existence of magic squares of powers", preprint, 2024. arXiv 2406.09364

[17] R. C. Vaughan, "A new iterative method in Waring's problem", *Acta Math.* **162** (1989), 1–71.

[18] R. C. Vaughan, *The Hardy–Littlewood method*, 2nd ed., Cambridge Tracts in Mathematics **125**, Cambridge University Press, 1997.

[19] R. C. Vaughan and T. Wooley, “On Waring’s problem: some refinements”, *Proc. London Math. Soc.* (3) **63** (1991), 35–68.

[20] R. C. Vaughan and T. D. Wooley, “Further improvements in Waring’s problem”, *Acta Math.* **174**:2 (1995), 147–240. MR

[21] R. C. Vaughan and T. D. Wooley, “Further improvements in Waring’s problem, IV: Higher powers”, *Acta Arith.* **94**:3 (2000), 203–285.

[22] T. D. Wooley, “Large improvements in Waring’s problem”, *Ann. of Math.* (2) **135**:1 (1992), 131–164.

[23] T. D. Wooley, “New estimates for smooth Weyl sums”, *J. London Math. Soc.* (2) **51**:1 (1995), 1–13.

[24] T. D. Wooley, “The asymptotic formula in Waring’s problem”, *Int. Math. Res. Not.* **2012**:7 (2012), 1485–1504.

[25] T. D. Wooley, “The cubic case of the main conjecture in Vinogradov’s mean value theorem”, *Adv. Math* **294** (2016), 532–561.

[26] T. D. Wooley, “On Waring’s problem for intermediate powers”, *Acta Arith.* **176**:3 (2016), 241–247.

[27] T. D. Wooley, “Nested efficient congruencing and relatives of Vinogradov’s mean value theorem”, *Proc. Lond. Math. Soc.* (3) **118**:4 (2019), 942–1016.

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