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We construct a groupoid equivariant Kasparov class for transversely oriented foliations in all codimensions. In codimension 1 we show that the Chern character of an associated semifinite spectral triple recovers the Connes–Moscovici cyclic cocycle for the Godbillon–Vey secondary characteristic class.

1. Introduction

In this paper we construct a semifinite spectral triple for codimension 1 foliations whose Chern character is a global, non-étale version of the cyclic cocycle, constructed by Connes and Moscovici [2005], representing the Godbillon–Vey class. The construction passes through groupoid equivariant Kasparov theory, and this initial part of the construction works in all codimensions.

Associated to any foliated manifold $(M, \mathcal{F})$ of codimension q is a canonical real rank q vector bundle $N = TM/T\mathcal{F}$ called the normal bundle. One of the foundational results of the theory of foliated manifolds is *Bott's vanishing theorem*, which states that the Pontrjagin classes $p^i(N)$ of the normal bundle N must vanish for all $i > 2q$ [Bott 1970]. This vanishing theorem guarantees the existence of new characteristic classes for M called *secondary characteristic classes*, which have been studied extensively [Bott 1972; Bott and Haefliger 1972; Kamber and Tondeur 1974]. It has been shown in particular that all such classes arise under the image of a characteristic map from the Gelfand–Fuks cohomology of the Lie algebra of formal vector fields [Gelfand and Fuks 1970] to the cohomology of M [Bott 1976; Bott and Haefliger 1972].

The most famous example of a secondary characteristic class is the Godbillon–Vey invariant, first discovered by Godbillon and Vey [1971], which arises in the context of transversely orientable foliations and can be constructed explicitly at the level of differential forms. More specifically, transverse orientability of a codimension q foliated manifold $(M, \mathcal{F})$ amounts to the existence of a nonvanishing section of the top degree line bundle $\Lambda^q N^*$ of the conormal bundle N^* over M. Any identification of N^* with a subbundle of T^*M, obtained say by equipping M

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with a Riemannian metric, identifies such a section with a nonvanishing differential form $\omega \in \Omega^q(M)$ such that

$$\omega(X_1 \wedge \cdots \wedge X_q) = 0 \tag{1.1}$$

whenever any one of the X_j is contained in the space $\Gamma(T\mathcal{F})$ of vector fields which are tangent to the foliation. Since the subbundle $T\mathcal{F} \subset TM$ is integrable, by the Frobenius theorem one is guaranteed the existence of a 1-form $\eta \in \Omega^1(M)$ for which

$$d\omega = \eta \wedge \omega.$$

The differential form $\eta \wedge (d\eta)^q$ is closed, and its class GV in de Rham cohomology is independent of the choices of ω and η. The Godbillon–Vey invariant has been shown to be closely related to measure theory and dynamics; see [Cantwell and Conlon 1984; Duminy 1982; Heitsch and Hurder 1984; Hurder 1986] for example.

Building on work of Winkelnkemper [1983] which associated to any foliated manifold $(M, \mathcal{F})$ its holonomy groupoid $\mathcal{G}$, Connes [1982] initiated the study of foliated manifolds as noncommutative geometries using the convolution algebra $C_c^\infty(\mathcal{G})$. While the convolution algebra $C_c^\infty(\mathcal{G})$ associated to the full holonomy groupoid is necessary when considering leafwise phenomena [Connes and Skandalis 1984], when considering *transverse geometry only* one can simplify matters in the following way. Choose a q-dimensional submanifold T of M which intersects each leaf of $\mathcal{F}$ at least once and which is everywhere transverse to $\mathcal{F}$ (such a T can be found by taking the disjoint union of local transversals in M defined by a covering of foliated charts). Then the restricted groupoid

$$(\mathcal{G})_T^T := \{u \in \mathcal{G} : s(u), r(u) \in T\}$$

inherits a differential topology from $\mathcal{G}$ under which it is an étale Lie groupoid [Crainic and Moerdijk 2001, Lemma 2]. Importantly, the groupoids $(\mathcal{G})_T^T$ and $\mathcal{G}$ are *Morita equivalent* [Crainic and Moerdijk 2001, Lemma 2]. They therefore have the same cyclic (co)homology [Crainic and Moerdijk 2001; 2004], and have Morita equivalent C^*-algebras [Muhly and Williams 2008] so are the same as far as K-theory is concerned also. In treating the transverse geometry of a foliation it has therefore become standard in the literature to use the étale groupoid $(\mathcal{G})_T^T$ in the place of the full holonomy groupoid $\mathcal{G}$ [Connes 1986; Connes and Moscovici 1998; 2001; 2005; Gorokhovsky 1999; 2002; Crainic and Moerdijk 2004; Moscovici and Rangipour 2007].

A reasonable model for any such étale groupoid $(\mathcal{G})_T^T$ is simply the action groupoid $V \rtimes \Gamma$, where V is an oriented manifold (a stand-in for the transversal T), and where Γ is a discrete group of orientation-preserving diffeomorphisms of V

(which is a stand-in for the action of the holonomy groupoid on T). It is in this setting that Connes [1986, Theorem 7.15] shows that all Gelfand–Fuks cohomology classes (hence all secondary characteristic classes) can be represented by cyclic cocycles on $C_c^\infty(V) \rtimes \Gamma$. Connes gives in particular an explicit formula for the cyclic cocycle defined by the Godbillon–Vey invariant when $V = S^1$. If dx denotes the standard volume form on S^1, then associated to any $g \in \Gamma$ is the $\mathbb{R}$-valued group cocycle

$$\ell(g) := \log\left(\frac{d(x \cdot g^{-1})}{dx}\right).$$

Connes shows that the formula

$$\begin{aligned}&\phi_{\mathrm{GV}}(f^0, f^1, f^2)\\ &:= \sum_{g_0g_1g_2=1_\Gamma} \int_{S^1} f^0(x)f^1(x\cdot g_0)f^2(x\cdot g_0g_1)(d\ell(g_1g_2)\ell(g_2)-\ell(g_1g_2)d\ell(g_2))\end{aligned} \tag{1.2}$$

defines a cyclic 2-cocycle on $C_c^\infty(V) \rtimes \Gamma$, and that the class of this 2-cocycle coincides with that defined by the Godbillon–Vey invariant.

More recently, Connes and Moscovici [1998] used a deep link with Hopf symmetry to construct a characteristic map sending Gelfand–Fuks cocycles to cyclic cocycles on the convolution algebra $C_c^\infty(F^+(V)) \rtimes \Gamma$ of the groupoid $F^+(V) \rtimes \Gamma$ associated to the lift of Γ to the oriented frame bundle $F^+(V)$ for V. Connes and Moscovici [2005] show that the formula

$$\tilde{\phi}_{\mathrm{GV}}(a^0, a^1) := \sum_{g_0g_1=1_\Gamma} \int_{F^+(V)} a^0(y)(\delta_1 a^1)(y \cdot g_0)\tilde{\omega}(y), \tag{1.3}$$

where δ_1 is a derivation of $C_c^\infty(F^+(V)) \rtimes \Gamma$ related to $d\ell$ and where $\tilde{\omega}$ is a G-invariant transverse volume form on $F^+(V)$, defines a 1-cocycle on $C_c^\infty(F^+(V)) \rtimes \Gamma$ that represents the Godbillon–Vey invariant. As will be shown in this paper, the derivation δ_1 can be realized in the *non-étale setting* of the *full holonomy groupoid* $\mathcal{G}$ of a foliated manifold, where it arises as a commutator between convolution along $\mathcal{G}$ with a dual Dirac operator on a Hilbert space of sections of an exterior algebra bundle. In noncommutative geometry, the Godbillon–Vey invariant has since been further explored in groupoid cohomology [Crainic and Moerdijk 2004], in cyclic cohomology [Gorokhovsky 1999; 2002], via its pairing with the indices of longitudinal Dirac operators [Moriyoshi and Natsume 1996], and in relation to manifolds with boundary [Moriyoshi and Piazza 2012].

Accompanying his introduction of the formula (1.2) for the cyclic cocycle ϕ_{GV}, Connes [1986, p. 4] remarks that the pairing of ϕ_{GV} with K-theory will not in general be integer-valued, which implies that ϕ_{GV} must not arise as the Chern character of a spectral triple on $C_c^\infty(V) \rtimes \Gamma$. Such constraints do not apply to *semifinite spectral triples*, whose pairings with K-theory need not lie in the integers [Connes and Cuntz 1988; Benameur and Fack 2006; Carey and Phillips 1998].

In this paper we will recover the analogue of (1.3) in the global setting of the full holonomy groupoid $\mathcal{G}$, from a semifinite spectral triple. Bearing in mind the close relationship between semifinite spectral triples and KK-theory [Kaad et al. 2012], this fact can be seen already in the étale case of an action groupoid of the form $V \rtimes \Gamma$ using the formalism of differential forms on jet bundles arising from Gelfand–Fuks cohomology [Connes and Moscovici 2005, Proposition 19]. An entirely novel nuance of our constructions, however, is that they are *global in nature*, applying immediately to foliated manifolds *without* needing to choose a complete transversal and pass through a Morita equivalence. This has the advantage of producing cocycles that are defined in terms of global geometric data for $(M, \mathcal{F})$, which previously has not been attempted.

We now outline the layout of the paper. Section 2 will discuss the background required on Clifford bundles, groupoid actions, semifinite spectral triples, and groupoid equivariant KK-theory. Section 3 will detail the constructions of the KK-classes required. The constructions of this section are very natural for foliations of arbitrary codimension, so will be carried out at this level of generality. Section 4 will consist of the proof of an index theorem in codimension 1 which states that the pairing with K-theory of the semifinite spectral triple obtained using the constructions of Section 3 coincides with the pairing coming from the Connes–Moscovici Godbillon–Vey cyclic cocycle. Finally in Section 5 we describe how, in codimension 1, our constructions can be viewed as a global geometric analogue of the jet bundle approach described by Connes [1986]. In particular this justifies our claim that the index formula we obtain really does represent the Godbillon–Vey invariant.

We remark that while the spectral triple itself can be easily constructed for foliations of arbitrary codimension, it is at this stage unclear whether the corresponding index pairing continues to compute the pairing of the higher codimension Godbillon–Vey invariant with K-theory. We leave this question to future work.

2. Background

Here we recall some basic facts about groupoid actions on spaces, Clifford algebras, semifinite spectral triples, groupoid actions on algebras, and the resulting equivariant Kasparov theory.

We will assume that the reader is familiar with locally compact groupoids and their associated convolution algebras [Connes 1982; Renault 1980]. All Hilbert spaces are assumed to be separable. For such a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the bounded operators on $\mathcal{H}$ and by $\mathcal{K}(\mathcal{H})$ the compact operators on $\mathcal{H}$. Inner products on Hilbert modules and Hilbert spaces are assumed to be conjugate-linear in the left variable and linear in the right.

If X, Y, and Z are sets with maps $f : Y \to X$ and $g : Z \to X$, we denote by $Y \times_{f,g} Z$ the fibred product $\{(y, z) \in Y \times Z : f(y) = g(z)\}$ of Y and Z.

2A. ***Clifford algebras.*** For our constructions we will need some facts regarding Clifford algebras and their representations on exterior algebra bundles. First, if $(V, \langle \cdot, \cdot \rangle)$ is a real inner product space with nondegenerate inner product, we denote by $\mathbb{C}\mathrm{liff}(V)$ the *complex Clifford algebra* of V, which is the complexification of the real Clifford algebra $\mathrm{Cliff}(V, \langle \cdot, \cdot \rangle)$.

There exists a linear isomorphism $\psi_V : \Lambda^* V \to \mathrm{Cliff}(V, \langle \cdot, \cdot \rangle)$ between the exterior algebra and the Clifford algebra of V defined with respect to any orthonormal basis $\{e_1, \dots, e_{\mathrm{rank}(V)}\}$ by

$$\psi_V(e_{i_1} \wedge \cdots \wedge e_{i_r}) := e_{i_1} \cdots e_{i_r}$$

for any multi-index $(i_1, \dots, i_r)$ with $r \le \mathrm{rank}(V)$. The isomorphism ψ_V determines the structure of a Clifford bimodule on $\Lambda^*(V)$, with left action given by

$$c_L(a)w := \psi_V^{-1}(a \cdot \psi_V(w))$$

and right action given by

$$c_R(a)w := \psi_V^{-1}(\psi_V(w) \cdot a)$$

for $a \in \mathrm{Cliff}(V)$ and $w \in \Lambda^*(V)$. We have the following important lemma describing how these representations behave with respect to orthogonal maps.

Lemma 2.1. *Let V and W be finite dimensional inner product spaces and let $\psi_V : \Lambda^* V \to \mathrm{Cliff}(V)$ and $\psi_W : \Lambda^* W \to \mathrm{Cliff}(W)$ be the corresponding linear isomorphisms. Then if $A : V \to W$ is an orthogonal transformation with induced algebra isomorphisms $A_\Lambda : \Lambda^* V \to \Lambda^* W$ and $A_{\mathrm{Cliff}} : \mathrm{Cliff}(V) \to \mathrm{Cliff}(W)$, we have*

$$A_{\mathrm{Cliff}} \circ \psi_V = \psi_W \circ A_\Lambda.$$

Proof. Regard V as a subspace of $\Lambda^* V$ in the usual way, let $\iota : V \to \mathrm{Cliff}(V)$ denote the inclusion map, and consider the map $j := (\psi_W \circ A_\Lambda)|_V : V \to \mathrm{Cliff}(W)$. Since A is orthogonal, we have $j(v)^2 = \|v\|^2 1_{\mathrm{Cliff}(W)}$ and so by the universal property of the Clifford algebra, there is a unique algebra isomorphism $\phi : \mathrm{Cliff}(V) \to \mathrm{Cliff}(W)$ such that $\phi \circ \iota = j$. Given any vector $v \in V$ we see that

$$j(v) = A_{\mathrm{Cliff}} \circ \iota(v)$$

so that $\phi = A_{\text{Cliff}}$. Given an orthonormal basis $\{e_1, \dots, e_{\dim(V)}\}$ for V, and a multi-index $(i_1, \dots, i_k)$ we calculate

$$\begin{aligned} A_{\text{Cliff}} \circ \psi_V(e_{i_1} \wedge \cdots \wedge e_{i_k}) &= A_{\text{Cliff}}(\iota(e_{i_1}) \cdots \iota(e_{i_k})) \\ &= A_{\text{Cliff}}(\iota(e_{i_1})) \cdots A_{\text{Cliff}}(\iota(e_{i_k})) \\ &= \psi_W(A_\Lambda(e_{i_1})) \wedge \cdots \wedge \psi_W(A_\Lambda(e_{i_k})) \\ &= \psi_W \circ A_\Lambda(e_{i_1} \wedge \cdots \wedge e_{i_k}), \end{aligned}$$

where the first line is due to the equality $\psi_V|_V = \iota$, and the second is since A_{Cliff} is an algebra homomorphism. By linearity we obtain the required identity. $\square$

By abuse of notation, we have a linear isomorphism $\psi_V : \Lambda^*(V) \otimes \mathbb{C} \to \mathbb{C}\text{liff}(V)$, which gives, by the same formulae as in the real case, commuting actions c_L and c_R of $\mathbb{C}\text{liff}(V)$ on $\Lambda^*(V) \otimes \mathbb{C}$. Any orthogonal map $A : V \to W$ of inner product spaces has the property that the induced maps $A_{\mathbb{C}\text{liff}} : \mathbb{C}\text{liff}(V) \to \mathbb{C}\text{liff}(W)$ and $A_{\Lambda_{\mathbb{C}}} : \Lambda^*(V) \otimes \mathbb{C} \to \Lambda^*(W) \otimes \mathbb{C}$ satisfy $A_{\mathbb{C}\text{liff}} \circ \psi_V = \psi_W \circ A_{\Lambda_{\mathbb{C}}}$.

If Y is a manifold and $E \to Y$ is a Euclidean vector bundle, we obtain a corresponding Clifford algebra bundle $\text{Cliff}(E)$ and exterior bundle $\Lambda^*(E)$, as well as corresponding complexifications $\mathbb{C}\text{liff}(E) = \text{Cliff}(E) \otimes \mathbb{C}$ and $\Lambda^*(E) \otimes \mathbb{C}$. Operating pointwise, we have an isomorphism $\psi_E : \Lambda^*(E) \otimes \mathbb{C} \to \mathbb{C}\text{liff}(E)$ of vector spaces giving $\Lambda^*(E) \otimes \mathbb{C}$ the structure of a $\mathbb{C}\text{liff}(E)$-bimodule, with left and right actions denoted, again by abuse of notation, by c_L and c_R, respectively. We will denote by $\mathbb{C}\ell(E)$ the continuous sections vanishing at infinity of the bundle $\mathbb{C}\text{liff}(E)$ over Y. This $\mathbb{C}\ell(E)$ is a C^*-algebra and is $\mathbb{Z}_2$-graded by even and odd elements.

2B. ***$\mathcal{G}$-spaces and $\mathcal{G}$-bundles.*** Let $\mathcal{G}$ be a groupoid, with unit space X and range and source maps $r : \mathcal{G} \to X$ and $s : \mathcal{G} \to X$, respectively. We say that $\mathcal{G}$ *acts on* (*the left of*) *a set* Y or that Y *is a* $\mathcal{G}$*-space* if there exists a map $a : Y \to X$ called the *anchor map* and a map $m : \mathcal{G} \times_{s,a} Y \to Y$, denoted $m(u, y) := u \cdot y$, such that

(1) $a(u \cdot y) = r(u)$ for all $(u, y) \in \mathcal{G} \times_{s,a} Y$,

(2) $(uv) \cdot y = u \cdot (v \cdot y)$ for all $(v, y) \in \mathcal{G} \times_{s,a} Y$ and $(u, v) \in \mathcal{G}^{(2)}$,

(3) $a(y) \cdot y = y$ for all $y \in Y$.

If $\mathcal{G}$ and Y are topological or smooth spaces, we require the maps a and m to be continuous or smooth, respectively. The simplest example of a $\mathcal{G}$-space is the unit space X of $\mathcal{G}$.

If $\mathcal{G}$ acts on Y, we denote by $Y \rtimes \mathcal{G}$ the space $Y \times_{a,r} \mathcal{G}$, regarded as a groupoid whose unit space is Y, with range and source maps $r(y, u) := y$ and $s(y, u) := u^{-1} \cdot y$, respectively, and with multiplication defined by

$$(y, u) \cdot (u^{-1} \cdot y, v) := (y, uv)$$

for all $(y, u) \in Y \times_{a,r} \mathcal{G}$ and $(u, v) \in \mathcal{G}^{(2)}$. If $\mathcal{G}$ and Y are topological or smooth spaces, the groupoid $Y \rtimes \mathcal{G}$ is equipped with a topological or smooth structure from its containment as a subspace of the topological or smooth space $Y \times \mathcal{G}$, respectively. While for left $\mathcal{G}$-spaces it is more natural to consider the analogous (and isomorphic) groupoid $\mathcal{G} \ltimes Y$ obtained from the set $\mathcal{G} \times_{s,a} Y$, it will be easier for our purposes to use $Y \rtimes \mathcal{G}$ because, as we will see, our convention in using $\mathcal{G}$-equivariant Kasparov theory consists of forming pullbacks using the range map rather than the source.

We say that a vector bundle $\pi : E \to X$ is *$\mathcal{G}$-equivariant* if E is a $\mathcal{G}$-space, with $\mathcal{G}$-action conventionally denoted $(u, e) \mapsto u_*e$ and with anchor map π, and if for each $u \in \mathcal{G}$ the map $(u, e) \mapsto u_*e$ defined on $E_{s(u)} := \pi^{-1}\{s(u)\}$ is a vector space isomorphism $E_{s(u)} \to E_{r(u)}$. More generally, if $\pi : E \to Y$ is a vector bundle over a $\mathcal{G}$-space Y, we say that E is *$\mathcal{G}$-equivariant* if it is $Y \rtimes \mathcal{G}$-equivariant as a bundle over Y, in which case we will often denote the map $(Y \rtimes \mathcal{G}) \times_{s,\pi} E \to E$, $((y, u), e) \mapsto (y, u)_*e$, by simply $(u, e) \mapsto u_*e$. If $\pi : E \to X$ admits a Euclidean or Hermitian structure, we say that E is a *$\mathcal{G}$-equivariant Euclidean or Hermitian bundle* if for all $(y, u) \in Y \rtimes \mathcal{G}$ the linear isomorphism $E_{u^{-1}\cdot y} \to E_y$ defined by $(u, e) \mapsto u_*e$ is orthogonal or unitary, respectively.

If $\pi : E \to Y$ is a $\mathcal{G}$-equivariant vector bundle over Y, then by functoriality $\Lambda^*(E) \otimes \mathbb{C}$ is also an equivariant bundle over Y, with action of $u \in \mathcal{G}$ denoted by $u_* : \Lambda^*(E|_{Y_{s(u)}}) \otimes \mathbb{C} \to \Lambda^*(E|_{Y_{r(u)}}) \otimes \mathbb{C}$. If moreover E is an equivariant Euclidean bundle, then by functoriality $\mathbb{C}\mathrm{liff}(E)$ is also an equivariant bundle, with action of $u \in \mathcal{G}$ denoted by $u_\diamond : \mathbb{C}\mathrm{liff}(E|_{Y_{s(u)}}) \to \mathbb{C}\mathrm{liff}(E|_{Y_{r(u)}})$. In this case, by Lemma 2.1 we have

$$u_*(c_L(a)e) = c_L(u_\diamond a)(u_*e), \tag{2.2}$$

$$u_*(c_R(a)e) = c_R(u_\diamond a)(u_*e) \tag{2.3}$$

for all $u \in \mathcal{G}$, $a \in \mathbb{C}\mathrm{liff}(E|_{Y_{s(u)}})$, and $e \in \Lambda^*(E|_{Y_{s(u)}})$.

When $(M, \mathcal{F})$ is a foliated manifold with holonomy groupoid $\mathcal{G}$, the normal bundle $N = TM/T\mathcal{F} \to M$ is a $\mathcal{G}$-equivariant bundle. As this fact is fundamental for our constructions, let us briefly review why it is the case. We choose a countable covering of M by foliated charts $\phi_i : U_i \cong T_i \times P_i$, where $T_i \subset \mathbb{R}^q$ and $P_i \subset \mathbb{R}^p$ are open balls, with change-of-chart maps $\varphi_{j,i} := \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ of the form

$$\varphi_{j,i}(t, p) = (h_{j,i}(t), \tilde{\varphi}_{j,i}(t, p)),$$

such that the $h_{i,j}$ are *compatible* in the sense that they satisfy

$$h_{i,k} = h_{i,j} \circ h_{j,k}$$

whenever $U_i \cap U_j \cap U_k \neq \varnothing$. That such a covering can be chosen can be regarded as the definition of the foliation $\mathcal{F}$ on M [Candel and Conlon 2000, Chapter 1.2]. We say that a path $\gamma : [0, 1] \to M$ is *leafwise* if its image is entirely contained in a leaf L of M, and we refer to its endpoints $\gamma(0)$ and $\gamma(1)$ as its *source* and *range*, denoted $s(\gamma)$ and $r(\gamma)$, respectively. Any leafwise path γ whose image is contained in a union $U_0 \cup U_1$ of charts such that $U_0 \cap U_1 \neq \varnothing$, and with $s(\gamma) \in U_0$ and $r(\gamma) \in U_1$, determines a local diffeomorphism $h_\gamma := h_{1,0}$ on a small neighborhood of $T_0 \subset \mathbb{R}^q$. More generally, if the image of a leafwise path γ is covered by a chain of charts $\{U_0, \ldots, U_k\}$ such that for each $0 \leq j < k$ we have $U_j \cap U_{j+1} \neq \varnothing$, on a sufficiently small neighborhood of T_0 we may define a local diffeomorphism

$$h_\gamma := h_{k,k-1} \circ h_{k-1,k-2} \circ \cdots \circ h_{1,0}$$

mapping onto a small neighborhood of T_k. Because of the compatibility of the $h_{i,j}$, the germ of h_γ at $s(\gamma)$ does not depend on the chain of charts chosen in its definition. By definition, the holonomy groupoid $\mathcal{G}$ consists of equivalence classes of leafwise paths γ for which $\gamma_1 \sim \gamma_2$ if and only if γ_1 and γ_2 have the same source and range and the germ at $s(\gamma_1) = s(\gamma_2)$ of h_{γ_1} is equal to that of h_{γ_2}.

In the coordinates defined by a chart U_j, the fibres of N identify with tangent vectors to the transversal neighborhood T_j, and via this identification it follows that for any leafwise path γ in M, the derivative of h_γ furnishes a linear isomorphism

$$dh_\gamma : N_{s(\gamma)} \to N_{r(\gamma)}.$$

It can be seen from the definition of h_γ that $dh_{\gamma_1} \circ dh_{\gamma_2} = dh_{\gamma_1\gamma_2}$ whenever the range of γ_2 is equal to the source of γ_1, where $\gamma_1\gamma_2$ is the path obtained by concatenating γ_1 and γ_2. Since local diffeomorphisms with the same germ at a point have the same derivative at that point, to any $u \in \mathcal{G}$ corresponds a well-defined linear isomorphism $u_* := h_\gamma : N_{s(u)} \to N_{s(u)}$ for any path γ that represents u. Since $dh_{\gamma_1\gamma_2} = dh_{\gamma_1} \circ dh_{\gamma_2}$, we have $(uv)_* = u_* \circ v_*$ for all $(u, v) \in \mathcal{G}^{(2)}$, and so N is indeed a $\mathcal{G}$-equivariant bundle over M.

We remark that in general the normal bundle N of a foliated manifold $(M, \mathcal{F})$ will not admit the structure of a $\mathcal{G}$-equivariant Euclidean bundle. Indeed, the existence of a $\mathcal{G}$-equivariant Euclidean structure for N implies the existence of a $\mathcal{G}$-invariant transverse volume form ω for $(M, \mathcal{F})$, and hence implies the existence of a faithful normal semifinite trace on the von Neumann algebra of $(M, \mathcal{F})$ defined by restricting functions in the weakly dense algebra $C_c(\mathcal{G})$ to M, and then integrating with respect to ω. If the Godbillon–Vey invariant of $(M, \mathcal{F})$ is nonzero, however, then by results of Hurder and Katok [1984, Theorem 2] and, in codimension 1, Connes [1986, Theorem 7.14], the von Neumann algebra of $(M, \mathcal{F})$ contains a type III factor and so admits no nonzero semifinite normal traces. Examples of foliated manifolds with nonzero Godbillon–Vey invariant are known to be plentiful [Thurston 1972].

2C. ***Equivariant KK-theory for locally Hausdorff groupoids.*** Equivariant KK-theory for Hausdorff topological groupoids was first developed by Le Gall [1994]. Since foliated manifolds generally have only locally Hausdorff holonomy groupoids, Le Gall's treatment requires extension for applications to foliation theory. Androulidakis and Skandalis [2019] have developed an equivariant KK-theory for the holonomy groupoids arising from singular foliations, whose topologies are generally even worse than the locally Hausdorff topologies on the holonomy groupoids of regular foliations, and which include all regular foliation groupoids as a subclass.

This section will summarize the required results and definitions of Androulidakis and Skandalis in the setting of locally Hausdorff Lie groupoids, as well as giving the unbounded picture in parallel with work of Pierrot [2006a]. See also [Muhly and Williams 2008; Tu 2004] for useful perspectives on non-Hausdorff groupoid actions which have further informed the exposition.

Let $\mathcal{G}$ be a locally Hausdorff Lie groupoid with locally compact Hausdorff unit space X, and let $\{U_i\}_{i\in I}$ be a countable cover of $\mathcal{G}$ by Hausdorff open sets. For each $i \in I$ we let $r_i := r|_{U_i}$ and $s_i := s|_{U_i}$ be the restrictions of range and source, respectively, to the set U_i.

Definition 2.4. A $C_0(X)$*-algebra* is a C^*-algebra A together with a homomorphism $\theta : C_0(X) \to \mathcal{M}(A)$ into the multiplier algebra of A such that $\theta(C_0(X))A = A$. For $a \in A$ and $f \in C_0(X)$, we will often denote $\theta(f)a$ by $f \cdot a$.

For $x \in X$, the *fibre over* x is the algebra $A_x := A/I_x A$, where I_x is the kernel of the evaluation functional $C_0(X) \ni f \mapsto f(x)$ on $C_0(X)$.

If A and B are $C_0(X)$-algebras, a homomorphism $\phi : A \to B$ is said to be a $C_0(X)$*-homomorphism* if $\phi(f \cdot a) = f \cdot \phi(a)$ for all $f \in C_0(X)$ and $a \in A$. Such a homomorphism induces a family $\phi_x : A_x \to B_x$ of homomorphisms between the fibres.

The simplest nontrivial example of a $C_0(X)$-algebra is $C_0(Y)$, where Y is a locally compact Hausdorff space equipped with a continuous map $p : Y \to X$. The $C_0(X)$-structure of $C_0(Y)$ is given by $\theta(f)g(y) := f(p(y))g(y)$ for all $f \in C_0(X)$ and $g \in C_0(Y)$, and the fibre over $x \in X$ is $C_0(Y)_x = C_0(Y_x)$, where $Y_x := p^{-1}\{x\}$.

Definition 2.5. Let A be a $C_0(X)$-algebra, and let $p : Y \to X$ be a continuous map of locally compact Hausdorff spaces. Then the *pullback* of A by p is the $C_0(Y)$-algebra $p^*A := C_0(Y) \otimes_{p,C_0(X)} A$, where we take the balanced tensor product by regarding the $C_0(X)$-algebras $C_0(Y)$ and A as $C_0(X)$-modules. If there is no ambiguity about the map p, it will often be omitted from the notation, so that $p^*A = C_0(Y) \otimes_{C_0(X)} A$.

It is easy to check that if A is a $C_0(X)$-algebra and $p : Y \to X$ is a continuous map of locally compact Hausdorff spaces, the fibre over $y \in Y$ of p^*A is $A_{p(y)}$. Equipped with the notion of pullbacks, we can define what is meant by a $\mathscr{G}$-algebra.

Definition 2.6. Let A be a $C_0(X)$-algebra. A $\mathscr{G}$-*action* on A is a family $\alpha = \{\alpha^i : s_i^* A \to r_i^* A\}_{i \in I}$ of grading-preserving $C_0(U_i)$-isomorphisms, such that $\alpha^i|_{s|^*_{U_i \cap U_j} A} = \alpha^j|_{s|^*_{U_i \cap U_j} A}$ for all $i, j \in I$, and such that the induced homomorphisms $\alpha_u : A_{s(u)} \to A_{r(u)}$ satisfy $\alpha_{uv} = \alpha_u \circ \alpha_v$. If A admits a $\mathscr{G}$-action α, we call (A, α) a $\mathscr{G}$-*algebra*.

The simplest nontrivial example of a $\mathscr{G}$-algebra is $C_0(Y)$, where Y is a $\mathscr{G}$-space with anchor map $p : Y \to X$, and where $C_0(Y)$ is equipped with the $\mathscr{G}$-action

$$\alpha_u(f)(y) := f(u^{-1} \cdot y)$$

for all $u \in \mathscr{G}$ and $f \in C_0(Y_{r(u)})$.

Now suppose that E is a Hilbert module over a $\mathscr{G}$-algebra A. For $x \in X$, we can consider the fibre $E_x := E \otimes_A A_x$, which is a Hilbert A_x-module, and if $p : Y \to X$ is a continuous map of locally compact Hausdorff spaces, we can consider the pullback $p^*E := E \otimes_A p^*A$, which is a Hilbert p^*A-module. If T is an A-linear operator on E, we let $p^*T := T \otimes 1_{p^*A}$ be its pullback to a p^*A-linear operator on p^*E.

Definition 2.7. Let (A, α) be a $\mathscr{G}$-algebra, and let E be a $\mathbb{Z}_2$-graded Hilbert A-module. A $\mathscr{G}$-*action* on E consists of a family $W = \{W^i : s_i^* E \to r_i^* E\}_{i \in I}$ of grading-preserving isometric Banach space isomorphisms, such that $W^i|_{s|^*_{U_i \cap U_j} E} = W^j|_{s|^*_{U_i \cap U_j} E}$ for all $i, j \in I$, and such that the induced isomorphisms $W_u : E_{s(u)} \to E_{r(u)}$ on the fibres satisfy $W_{uv} = W_u \circ W_v$, $\langle W_u \rho_1, W_u \rho_2 \rangle_{r(u)} = \alpha_u(\langle \rho_1, \rho_2 \rangle_{s(u)})$, and $W_u(\rho \cdot a) = W_u(\rho) \cdot \alpha_u(a)$ for all $(u, v) \in \mathscr{G}^{(2)}$, $a \in A_{s(u)}$, and $\rho, \rho_1, \rho_2 \in E_{s(u)}$. If E admits a $\mathscr{G}$-action W, we call (E, W) a $\mathscr{G}$-*Hilbert* A-*module*.

If $V \to Y$ is a $\mathscr{G}$-equivariant Hermitian vector bundle over a $\mathscr{G}$-space Y, then the continuous sections vanishing at infinity $\Gamma_0(Y; V)$ of V over Y is a $\mathscr{G}$-Hilbert $C_0(Y)$-module, with pointwise inner product and right action by $C_0(Y)$, and with $\mathscr{G}$-action defined by

$$(W_u \rho)(y) := u_* \rho(u^{-1} \cdot y) \tag{2.8}$$

for all $\rho \in \Gamma_0(Y_{r(u)}; V|_{Y_{r(u)}})$. All $\mathscr{G}$-Hilbert module constructions in this paper will arise from some variant of the action (2.8).

Definition 2.9. If B is a $\mathscr{G}$-algebra, and $\pi : A \to \mathscr{L}(E)$ is a representation of a $\mathscr{G}$-algebra (A, α) on a $\mathscr{G}$-Hilbert B-module (E, W), we say that π is *equivariant* if for all $i \in I$ we have

$$\mathrm{Ad}_{W^i}(\pi_i^s(a)) = \pi_i^r(\alpha^i(a))$$

for all $a \in A$. Here $\pi_i^s := 1_{C_b(U_i)} \otimes \pi$ and $\pi_i^r := 1_{C_b(U_i)} \otimes \pi$ are the induced homomorphisms $s_i^* A = C_0(U_i) \otimes_{s, C_0(X)} A \to \mathcal{L}(s_i^* E)$ and $r_i^* A = C_0(U_i) \otimes_{r, C_0(X)} A \to \mathcal{L}(r_i^* E)$, respectively.

The definition of the equivariant KK-groups now follows in the usual way.

Definition 2.10. Let (A, α) and (B, β) be $\mathcal{G}$-C^*-algebras. A *$\mathcal{G}$-equivariant Kasparov A-B-module* is a triple $(A, {}_\pi E_B, F)$, where (E, W) is a $\mathcal{G}$-equivariant Hilbert B-module carrying an equivariant representation $\pi : A \to \mathcal{L}(E)$, and where $F \in \mathcal{L}(E)$ is homogeneous of degree 1 such that for all $a \in A$ one has

(1) $\pi(a)(F - F^*) \in \mathcal{K}(E)$,

(2) $\pi(a)(F^2 - 1) \in \mathcal{K}(E)$,

(3) $[F, \pi(a)] \in \mathcal{K}(E)$,

and such that for all $i \in I$

(4) $\pi_i^r(r_i^*(a))(r_i^* F - W^i \circ s_i^* F \circ (W^i)^{-1}) \in r_i^* \mathcal{K}(E)$.

We say that two $\mathcal{G}$-equivariant Kasparov A-B-modules $(A, {}_\pi E_B, F)$, $(A, {}_{\pi'} E'_B, F')$ are *unitarily equivalent* if there exists a $\mathcal{G}$-equivariant unitary $V : E \to E'$ of degree 0 such that $VFV^* = F'$ and $V\pi(a)V^* = \pi'(a)$ for all $a \in A$. We denote by $\mathbb{E}^{\mathcal{G}}(A, B)$ the set of all unitary equivalence classes of $\mathcal{G}$-equivariant Kasparov A-B-modules.

A *homotopy* in $\mathbb{E}^{\mathcal{G}}(A, B)$ is an element of $\mathbb{E}^{\mathcal{G}}(A, B[0, 1])$, and we denote by $KK^{\mathcal{G}}(A, B)$ the set of homotopy equivalence classes in $\mathbb{E}^{\mathcal{G}}(A, B)$.

The direct sum of $\mathcal{G}$-equivariant Kasparov A-B-modules makes $KK^{\mathcal{G}}(A, B)$ into an abelian group.

We also need *unbounded representatives* of equivariant KK-classes. The definition for such representatives is the natural extension of that due to Pierrot [2006a] to the locally Hausdorff case. We remark here that if $\mathcal{A}$ is a dense $*$-subalgebra of a $C_0(X)$-algebra A, then we will assume that $C_0(X) \cdot \mathcal{A} \subset \mathcal{A}$, which will be true in our examples. We will denote by $\mathcal{A}_x := \mathcal{A}/I_x\mathcal{A}$ the fibre over $x \in X$, where as before I_x is the kernel of the evaluation functional $f \mapsto f(x)$ on $C_0(X)$.

Definition 2.11. Let A and B be $\mathcal{G}$-algebras. An *unbounded $\mathcal{G}$-equivariant Kasparov A-B-module* is a triple $(\mathcal{A}, {}_\pi E, D)$, where (E, W) is a $\mathcal{G}$-Hilbert B-module carrying an equivariant representation π of A in $\mathcal{L}(E)$, and D is a densely defined, odd, unbounded, self-adjoint, and regular operator on E commuting with the right action of B, and where $\mathcal{A}$ is a dense $*$-subalgebra of A preserved by the action of $\mathcal{G}$ such that for all $a \in \mathcal{A}$ one has

(1) $\pi(a)\,\mathrm{dom}(D) \subset \mathrm{dom}(D)$,

(2) $[D, \pi(a)]$ extends to an element of $\mathcal{L}(E)$,

(3) $\pi(a)(1+D^2)^{-1/2} \in \mathcal{K}(E)$,

and such that for all $i \in I$, $a \in \mathcal{A}$, and $f \in C_c(U_i)$ one has

(4) $f \cdot \pi_i^r(r_i^*(a)) \cdot (r_i^* D - W^i \circ s_i^* D \circ (W^i)^{-1})$ extends to an element of $\mathcal{L}(r_i^* E)$,

(5) $\operatorname{dom}((r_i^* D) f) = W^i \operatorname{dom}((s_i^* D) f)$.

That all unbounded equivariant Kasparov modules define classes in $KK^{\mathcal{G}}$ is an easy consequence of the corresponding result by Pierrot for Hausdorff groupoids.

Proposition 2.12. *Let A and B be $\mathcal{G}$-algebras, and let $(\mathcal{A}, {}_\pi E, D)$ be an unbounded $\mathcal{G}$-equivariant Kasparov A-B-module. Then $(A, {}_\pi E, D(1+D^2)^{-1/2})$ is a $\mathcal{G}$-equivariant Kasparov A-B-module.*

Proof. That $(A, {}_\pi E, D(1+D^2)^{-1/2})$ satisfies the first three requirements of Definition 2.10 is a consequence of the corresponding result in the nonequivariant case [Baaj and Julg 1983]. That the fourth requirement is met is a consequence of restricting the corresponding result of Pierrot [2006a, Théorème 6] to each of the Hausdorff open subsets U_i of $\mathcal{G}$. □

We now come to the descent map in equivariant KK-theory, for which we need to discuss groupoid crossed products. We will assume for this that $\mathcal{G}$ comes equipped with a bundle $\Omega^{1/2} \to \mathcal{G}$ of leafwise half-densities, as in [Connes 1994, Chapter 2.8]. Regard a $C_0(X)$-algebra A as the continuous sections vanishing at infinity $\Gamma_0(X; \mathfrak{A})$ of the upper-semicontinuous bundle $\mathfrak{A} \to X$ of C^*-algebras whose fibre over $x \in X$ is A_x [Le Gall 1994; Muhly and Williams 2008]. Thus, a $\mathcal{G}$-algebra (A, α) can be regarded as the continuous sections vanishing at infinity of the $\mathcal{G}$-space $\mathfrak{A}$ over X, where $\alpha_u : A_{s(u)} \to A_{r(u)}$ determines the action of $\mathcal{G}$ on the bundle $\mathfrak{A}$.

Define $\Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ to be the space of finite linear combinations of sections of the bundle $r^*\mathfrak{A} \otimes \Omega^{1/2} \to \mathcal{G}$ which have compact support and are continuous in one of the U_i. The space $\Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ is a $*$-algebra equipped with the convolution product

$$(f * g)_u := \int_{v \in \mathcal{G}^{r(u)}} f_v \alpha_v(g_{v^{-1}u}) \quad \text{and with involution} \quad (f^*)_u := \alpha_u((f_{u^{-1}})^*).$$

The appropriate completion of $\Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ to a reduced C^*-algebra $A \rtimes_r \mathcal{G}$ has been given in [Khoshkam and Skandalis 2004, §3.7].

In a similar manner, if A is a $\mathcal{G}$-algebra we can regard any $\mathcal{G}$-Hilbert A-module E as the continuous sections vanishing at infinity of an upper-semicontinuous bundle $\mathfrak{E} \to X$ whose fibre over $x \in X$ is E_x. We define $\Gamma_c(\mathcal{G}; r^*\mathfrak{E} \otimes \Omega^{1/2})$ to be the space of finite linear combinations of sections of the bundle $r^*\mathfrak{E} \otimes \Omega^{1/2} \to \mathcal{G}$ that have compact support and are continuous in one of the U_i. The formulae

$$\langle \rho^1, \rho^2\rangle_u^{\mathcal{G}} := \int_{v\in\mathcal{G}^{r(u)}} \alpha_v\langle \rho^1_{v^{-1}}, \rho^2_{v^{-1}u}\rangle \quad\text{and}\quad (\rho\cdot f)_u := \int_{v\in\mathcal{G}^{r(u)}} \rho_v\alpha_v(f_{v^{-1}u})$$

defined for $\rho^1, \rho^2, \rho \in \Gamma_c(\mathcal{G}; r^*\mathfrak{E}\otimes\Omega^{1/2})$ and $f\in\Gamma_c(\mathcal{G}; r^*\mathfrak{A}\otimes\Omega^{1/2})$ determine an $A\rtimes_r\mathcal{G}$-valued inner product and right action, respectively, on $\Gamma_c(\mathcal{G}; r^*\mathfrak{E}\otimes\Omega^{1/2})$, and we may complete in the norm arising from $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathcal{G}}$ to obtain a Hilbert $A\rtimes_r\mathcal{G}$-module which we denote by $E\rtimes_r\mathcal{G}$. If T is an A-linear operator on E, we denote by $\mathfrak{Dom}(T)$ the bundle over X whose fibre over $x\in X$ is $\mathrm{dom}(T)\otimes_A A_x$. As in [Pierrot 2006a, Définition 2, Proposition 3] we define $r^*(T)$ on $\Gamma_c(\mathcal{G}; r^*\,\mathfrak{Dom}(T)\otimes\Omega^{1/2})$ by

$$(r^*(T)\rho)_u := T_{r(u)}\rho_u.$$

If $T\in\mathcal{L}(E)$, one can use the norm of T to bound that of $r^*(T)$, and then one can use T^* to show that $r^*(T)\in\mathcal{L}(E\rtimes_r\mathcal{G})$.

Lemma 2.13. *For any densely defined A-linear operator $T:\mathrm{dom}(T)\to E$, we have $r^*(T^*)\subset r^*(T)^*$. Moreover, $\overline{r^*(T^*)} = r^*(T)^*$.*

Proof. Fix $\xi\in\mathrm{dom}(r^*(T^*)) = \Gamma_c(\mathcal{G}; r^*\,\mathfrak{Dom}(T^*)\otimes\Omega^{1/2})$, and assume without loss of generality that ξ has compact support in some Hausdorff open subset U_i of $\mathcal{G}$. For each $u\in\mathcal{G}$, use the fact that $\xi_u\in\mathfrak{Dom}(T^*)_{r(u)}\otimes\Omega_u^{1/2}$ to define a section η of $r^*\mathfrak{E}\otimes\Omega^{1/2}\to\mathcal{G}$ by

$$\eta_u := T^*_{r(u)}\xi_u.$$

Because ξ is continuous with compact support in U_i, so too is η; therefore, $\eta\in\Gamma_c(\mathcal{G}, r^*\mathfrak{E}\otimes\Omega^{1/2})$. For any $\rho\in\mathrm{dom}(r^*(T)) = \Gamma_c(\mathcal{G}; r^*\,\mathfrak{Dom}(T)\otimes\Omega^{1/2})$ we can then calculate

$$\begin{aligned}\langle\xi, r^*(T)\rho\rangle_u^{\mathcal{G}} &= \int_{v\in\mathcal{G}^{r(u)}} \alpha_v(\langle\xi_{v^{-1}}, T_{s(v)}\rho_{v^{-1}u}\rangle)\\ &= \int_{v\in\mathcal{G}^{r(u)}} \alpha_v(\langle T^*_{s(v)}\xi_{v^{-1}}, \rho_{v^{-1}u}\rangle) = \langle\eta,\rho\rangle_u^{\mathcal{G}}\end{aligned}$$

for all $u\in\mathcal{G}$, so that $\xi\in\mathrm{dom}(r^*(T)^*)$. The above calculation also shows that $r^*(T)^*\xi = \eta = r^*(T^*)\xi$, so that we indeed have $r^*(T^*)\subset r^*(T)^*$.

Fix $\xi\in\mathrm{dom}(r^*(T)^*)$. Then we will show that $\xi\in\overline{r^*(T^*)}$. Let $\{\xi^n\}_{n\in\mathbb{N}}\subset\Gamma_c(\mathcal{G}; r^*\,\mathfrak{Dom}(T^*)\otimes\Omega^{1/2})$ be a sequence converging in $E\rtimes_r\mathcal{G}$ to ξ. Then the sequence $\{\langle\xi^n, r^*(T)\rho\rangle^{\mathcal{G}}\}_{n\in\mathbb{N}}$ of elements of $\Gamma_c(\mathcal{G}; r^*\mathfrak{A}\otimes\Omega^{1/2})$ defined for $u\in\mathcal{G}$ by

$$\langle\xi^n, r^*(T)\rho\rangle_u^{\mathcal{G}} = \int_{v\in\mathcal{G}^{r(u)}} \alpha_v(\langle\xi^n_{v^{-1}}, T_{s(v)}\rho_{v^{-1}u}\rangle) = \int_{v\in\mathcal{G}^{r(u)}} \alpha_v(\langle T^*_{s(v)}\xi^n_{v^{-1}}, \rho_{v^{-1}u}\rangle) \tag{2.14}$$

converges in $A \rtimes_r \mathcal{G}$ for all $\rho \in \Gamma_c(\mathcal{G}; r^*\,\mathfrak{Dom}(T) \otimes \Omega^{1/2})$. For each $v \in \mathcal{G}^{r(u)}$ one can on the right-hand side of (2.14) take bump functions ρ with support of decreasing radius about $v^{-1}u$ to show that we have convergence of $\{(r^*(T^*)\xi^n)_{v^{-1}} = T^*_{s(v)}\xi^n_{v^{-1}}\}_{n\in\mathbb{N}}$ to an element of $E_{s(v)}$, and doing this for all $v \in \mathcal{G}^{r(u)}$ and all $u \in \mathcal{G}$ shows that in fact $\{r^*(T^*)\xi^n\}_{n\in\mathbb{N}}$ converges in $E \rtimes_r \mathcal{G}$, implying that $\xi^n \to \xi$ in the graph norm on $\operatorname{dom}(r^*(T^*))$ as claimed. □

Finally, we observe that if A and B are $\mathcal{G}$-algebras, and if (E, W) is a $\mathcal{G}$-Hilbert B-module with an equivariant representation $\pi : A \to \mathcal{L}(E)$, then the formula

$$((\pi \rtimes_r \mathcal{G})(f)\rho)_u := \int_{v\in\mathcal{G}^{r(u)}} \pi(f_v)W_v(\rho_{v^{-1}u})$$

defined for $f \in \Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ and $\rho \in \Gamma_c(\mathcal{G}; r^*\mathfrak{E} \otimes \Omega^{1/2})$ determines a representation $\pi \rtimes_r \mathcal{G} : A \rtimes_r \mathcal{G} \to \mathcal{L}(E \rtimes_r \mathcal{G})$.

Proposition 2.15. *Let A and B be $\mathcal{G}$-algebras and $(\mathcal{A}, {}_\pi E, D)$ a $\mathcal{G}$-equivariant unbounded Kasparov A-B-module. Let $\tilde{\mathcal{A}}$ denote the bundle of $*$-algebras over X whose fibre over $x \in X$ is $\mathcal{A}_x$. Then*

$$(\Gamma_c(\mathcal{G}; r^*\tilde{\mathcal{A}} \otimes \Omega^{1/2}), {}_{\pi\rtimes_r\mathcal{G}}E \rtimes_r \mathcal{G}, r^*(D))$$

is an unbounded Kasparov $A \rtimes_r \mathcal{G}$-$B \rtimes_r \mathcal{G}$-module.

Proof. Since D is odd for the grading of E, $r^*(D)$ is odd for the induced grading of $E \rtimes_r \mathcal{G}$. Symmetry of D gives symmetry of $r^*(D)$, so without loss of generality we may assume that $r^*(D)$ is closed. Self-adjointness of $r^*(D)$ is then a consequence of the self-adjointness of D together with Lemma 2.13.

Regularity of $r^*(D)$ is a consequence of the regularity of D. Indeed, for any $\rho \in \Gamma_c(\mathcal{G}; r^*\,\mathfrak{Dom}(D) \otimes \Omega^{1/2})$ we have

$$((1 + r^*(D)^2)\rho)_u = (1_{r(u)} + D^2_{r(u)})\rho_u.$$

Hence, the range of the operator $(1+r^*(D)^2)$ when restricted to $\Gamma_c(\mathcal{G}; r^*\,\mathfrak{Dom}(D)\otimes \Omega^{1/2})$ is $\Gamma_c(\mathcal{G}; r^*\,\mathfrak{range}(1 + D^2) \otimes \Omega^{1/2})$, where $\mathfrak{range}(1 + D^2)$ denotes the bundle over X whose fibre over $x \in X$ is $\operatorname{range}(1 + D^2) \otimes_A A_x$, which by regularity of D is dense in $E_x = E \otimes_A A_x$. Thus, the range of $(1 + r^*(D)^2)$ contains the dense subspace $\Gamma_c(\mathcal{G}; r^*\,\mathfrak{range}(1 + D^2) \otimes \Omega^{1/2})$ of $E \rtimes_r \mathcal{G}$, and it follows that $r^*(D)$ is regular.

Regarding commutators, a simple calculation tells us that for any $u \in \mathcal{G}$, the vector $([r^*(D), (\pi \rtimes_r \mathcal{G})(f)]\rho)_u$ is equal to

$$\int_{v\in\mathcal{G}^{r(u)}} ([D_{r(u)}, \pi(f_v)] + \pi(f_v)(D_{r(v)} - W_v \circ D_{s(v)} \circ W_{v^{-1}}))(W_v\rho_{v^{-1}u})$$

for all $\rho \in \Gamma_c(\mathcal{G}; r^*\,\mathfrak{Dom}(T)\otimes\Omega^{1/2})$, so properties (2) and (4) in Definition 2.11 imply that the operator $[r^*(D), (\pi \rtimes_r \mathcal{G})(f)]$ extends to an element of $\mathcal{L}(E \rtimes_r \mathcal{G})$, with adjoint $[r^*(D), (\pi \rtimes_r \mathcal{G})(f^*)]$.

The only thing remaining to check is compactness of $(\pi \rtimes_r \mathcal{G})(f)(1+r^*(D)^2)^{-1/2}$ for $f \in \Gamma_c(\mathcal{G}; r^*\tilde{\mathcal{A}}\otimes\Omega^{1/2})$. For any $\rho \in \Gamma_c(\mathcal{G}; r^*\mathfrak{E}\otimes\Omega^{1/2})$ the definition of $r^*(D)$ gives

$$\begin{aligned}((1+r^*(D)^2)^{-1/2}(\pi \rtimes_r \mathcal{G})(f^*)\rho)_u &= (1+D_{r(u)}^2)^{-1/2}\int_{v\in\mathcal{G}^{r(u)}} \pi((f)_v^*)W_v(\rho_{v^{-1}u})\\ &= \int_{v\in\mathcal{G}^{r(u)}} (1+D_{r(v)}^2)^{-1/2}\pi((f)_v^*)W_v(\rho_{v^{-1}u}),\end{aligned}$$

and since $(1+D_{r(v)}^2)^{-1/2}\pi((f)_v^*) \in \mathcal{K}(E)_{r(v)}$ for all $v \in \mathcal{G}^{r(u)}$ by property (3) in Definition 2.11, it follows that $(1+r^*(D)^2)^{-1/2}(\pi \rtimes_r \mathcal{G})(f^*)$ is an element of $\Gamma_c(\mathcal{G}; r^*\mathcal{K}(E)\otimes\Omega^{1/2})$. A similar argument to the one used in [Kasparov 1988, p. 172] then tells us that $(1+r^*(D))^{-1/2}(\pi \rtimes_r \mathcal{G})(f^*)$ can be approximated by finite-rank operators on $E \rtimes_r \mathcal{G}$ so is an element of $\mathcal{K}(E \rtimes_r \mathcal{G})$, and hence so too is its adjoint $(\pi \rtimes_r \mathcal{G})(f)(1+r^*(D)^2)^{-1/2}$. ☐

Let us remark finally that if Y is a locally compact Hausdorff $\mathcal{G}$-space, with corresponding bundle $C_0(\mathfrak{Y}) \to X$ whose fibre over $x \in X$ is $C_0(Y_x)$, then we have an inclusion $\Gamma_c(Y \rtimes \mathcal{G}; \Omega^{1/2}) \ni f \mapsto \tilde{f} \in \Gamma_c(\mathcal{G}; r^*C_0(\mathfrak{Y})\otimes\Omega^{1/2})$ defined by

$$\tilde{f}_u(y) := f(y,u).$$

For ease of notation we will usually just refer to $\tilde{f}$ as f. By density of $C_c(Y_x)$ in $C_0(Y_x)$ for each $x \in X$, the subalgebra $\Gamma_c(Y \rtimes \mathcal{G}; \Omega^{1/2})$ is dense in $C_0(Y) \rtimes_r \mathcal{G}$. We will use this fact in the construction of our Godbillon–Vey spectral triple.

2D. ***Semifinite spectral triples.*** One of the defining features of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is that the operators $a(1+\mathcal{D}^2)^{-1/2}$ are contained in the compact operators $\mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{A}$. These compact operators come equipped with a trace Tr, which is used to measure the rank of projections that appear in the definition of the index, and subsequent index formulae [Connes and Moscovici 1995; Higson 2004].

Semifinite spectral triples are a generalization of spectral triples for which the rank of projections is measured by a different trace. More precisely we require a faithful normal semifinite trace τ on a semifinite von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$. We denote by $\mathcal{K}_\tau(\mathcal{N})$ the norm-closed ideal in $\mathcal{N}$ generated by projections of finite τ-trace, and refer to $\mathcal{K}_\tau(\mathcal{N})$ as the ideal of *τ-compact operators* [Fack and Kosaki 1986].

Definition 2.16. Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra, regarded as an algebra of operators on a Hilbert space $\mathcal{H}$. A *semifinite spectral triple relative to* $(\mathcal{N}, \tau)$ is a triple $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ consisting of a $*$-algebra $\mathcal{A}$ represented in $\mathcal{N}$ by

$\pi : \mathcal{A} \to \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, and a densely defined, unbounded, self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that

(1) $\pi(a)\,\mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$ so that $[\mathcal{D}, \pi(a)]$ is densely defined, and moreover extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$,

(2) $\pi(a)(1+\mathcal{D}^2)^{-1/2} \in \mathcal{K}_\tau(\mathcal{N})$ for all $a \in \mathcal{A}$.

We say that $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ is *even* if $\mathcal{A}$ is even and $\mathcal{D}$ is odd for some $\mathbb{Z}_2$-grading on $\mathcal{H}$, and otherwise we call $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ *odd*.

Connes' original notion of spectral triple defines a subclass of semifinite spectral triples, for which $(\mathcal{N}, \tau) = (\mathcal{B}(\mathcal{H}), \mathrm{Tr})$. Just as the bounded transform of a spectral triple $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ defines a Fredholm module (over the C^*-completion A of $\mathcal{A}$), and hence a class in $KK_*(A, \mathbb{C})$, semifinite spectral triples have a close relationship with KK-theory.

To see this, we first suppose B is a C^*-algebra, X_B is a Hilbert B-module with inner product $\langle \cdot, \cdot \rangle_B$, and τ is a faithful norm lower-semicontinuous semifinite trace on B. We can form the GNS space $L^2(B, \tau)$, or $L^2(X, \tau)$ with inner product $(x \mid y) = \tau(\langle x, y\rangle_B)$. These two Hilbert spaces are related by $X \otimes_B L^2(B, \tau) \cong L^2(X, \tau)$.

Then by results in [Laca and Neshveyev 2004], we obtain a faithful normal semifinite trace Tr_τ, called the *dual trace*, on the weak closure $\mathcal{N} = \mathrm{End}_B(X)'' \subset \mathcal{B}(L^2(X_B, \tau))$ of the adjointable B-linear operators on X_B. The functional Tr_τ satisfies

$$\mathrm{Tr}_\tau(\Theta_{\xi,\eta}) := \tau(\langle \eta, \xi\rangle_B).$$

Proposition 2.17. *Let $(\mathcal{A}, {}_\pi X_B, \mathcal{D})$ be an even or odd unbounded Kasparov A-B-module, and suppose that τ is a faithful norm lower semicontinuous semifinite trace on B. Let $(\mathcal{N}, \mathrm{Tr}_\tau)$ be the semifinite von Neumann algebra obtained from X_B and τ as above. Then (with a slight abuse of notation)*

$$(\mathcal{A}, {}_{\pi\hat{\otimes}1}X_B \hat{\otimes}_B L^2(B, \tau), \mathcal{D} \hat{\otimes} 1) = (\mathcal{A}, {}_\pi L^2(X_B, \tau), \mathcal{D})$$

is an even or odd semifinite spectral triple relative to $(\mathcal{N}, \mathrm{Tr}_\tau)$, respectively.

Proof. Clearly $\mathcal{A} \subset \mathcal{N}$, and the commutant of $\mathcal{N}$ is just B''. Since $\mathcal{D}$ is B-linear, every unitary in B'' preserves the domain of $\mathcal{D} \hat{\otimes} 1$, whence $\mathcal{D} \hat{\otimes} 1$ is affiliated to $\mathcal{N}$. That $[\mathcal{D} \hat{\otimes} 1, \pi(a) \hat{\otimes} 1]$ is bounded for all $a \in \mathcal{A}$ is a consequence of the corresponding fact for the Kasparov module $(\mathcal{A}, {}_\pi X_B, \mathcal{D})$, and that $(\pi(a)\hat{\otimes}1)(1+\mathcal{D}\hat{\otimes}1^2)^{-1/2}$ is τ-compact is true since the algebra $\mathcal{K}(X_B)$ is contained in $\mathcal{K}_\tau(\mathcal{N})$ by construction. □

In fact, a converse to Proposition 2.17 is also true: namely, every semifinite spectral triple can be factorized into a KK-class and a trace [Kaad et al. 2012]. Although we will not need this converse result, it provides a useful way of thinking about semifinite spectral triples.

One of the most useful features of (nice) spectral triples is that their pairing with K-theory can be computed using the local index formula [Connes and Moscovici 1995]. The same is true for (nice) semifinite spectral triples. There are now numerous results generalizing the Connes–Moscovici local index formula for spectral triples to semifinite spectral triples [Benameur and Fack 2006; Carey et al. 2004; 2006a; 2006b; 2008; 2012; 2014].

3. Construction of the Kasparov modules

In this section, $(M, \mathcal{F})$ will denote a transversely orientable foliated manifold of codimension q, with holonomy groupoid $\mathcal{G}$ and normal bundle $N = TM/T\mathcal{F} \to M$. The normal bundle is a $\mathcal{G}$-equivariant vector bundle, as explained at the end of Section 2B, and for $u \in \mathcal{G}$ we let $u_* : N_{s(u)} \to N_{r(u)}$ be the corresponding map $n \mapsto u_* n$. We assume $\mathcal{G}$ to be equipped with a countable cover $\mathcal{U} := \{U_i\}_{i \in I}$ by Hausdorff open subsets. We do not assume K-orientability at any point, working with exterior algebra bundles instead of spinor bundles.

The first of the two constructions, the Connes fibration, will not be featured in the index theorem in the final section. The Kasparov module of the Connes fibration provides a Thom-type isomorphism which does not conceptually affect our final index formulae. We include the Connes fibration for the sake of completeness, and to show that the whole construction does indeed factor through groupoid equivariant KK-theory.

3A. ***The Connes fibration.*** We begin this section with a revision of a construction due to Connes [1986]. Connes starts with an oriented manifold M of dimension n with an action of a discrete group Γ of orientation-preserving diffeomorphisms. Such a setting provides an étale model of the transverse geometry of a transversely oriented foliation.

Connes shows that if $W \to M$ denotes the "bundle of Euclidean metrics" for the tangent bundle TM over M, then one can construct a dual Dirac class in $KK^{\Gamma}_{n(n+1)/2}(C_0(M), C_0(W))$. The manifold W has the advantage that the pullback of TM to W admits a Γ-invariant Euclidean metric, even though one need not exist on M in general. We show that Connes' construction can be carried out directly in the groupoid equivariant setting, as it may be useful for future work in constructing the Godbillon–Vey invariant as a semifinite spectral triple in arbitrary codimension.

We let $\pi_F : F^+(N) \to M$ be the principal $GL^+(q, \mathbb{R})$-bundle of positively oriented frames for the vector bundle $N \to M$, whose fibre $(F^+(N))_x$ over $x \in M$ consists of positively oriented linear isomorphisms $\phi : \mathbb{R}^q \to N_x$. Then $F^+(N)$ is a $\mathcal{G}$-space with anchor map $\pi_F : F^+(N) \to M$ and action defined by

$$u \cdot \phi := u_* \circ \phi : \mathbb{R}^q \to N_{r(u)} \tag{3.1}$$

for $\phi : \mathbb{R}^q \to N_{s(u)}$ in $F^+(N)_{s(u)}$. Observe that this action of $\mathcal{G}$ commutes with the right action of $GL^+(q, \mathbb{R})$ on the principal $GL^+(q, \mathbb{R})$-bundle $F^+(N) \to M$.

The vertical subbundle $\ker(d\pi_F) = VF^+(N) \to F^+N$ of $TF^+(N)$ admits a trivialization $VF^+(N) \to F^+(N) \times \mathfrak{gl}(q, \mathbb{R})$, where $\mathfrak{gl}(q, \mathbb{R}) = M_q(\mathbb{R})$ is the Lie algebra of $GL^+(q, \mathbb{R})$ consisting of all $q \times q$ real matrices. The trivialization is given by the formula

$$F^+(N) \times \mathfrak{gl}(q, \mathbb{R}) \ni (\phi, v) \mapsto v_\phi := \frac{d}{dt}(\phi \cdot \exp(tv))\Big|_{t=0} \in VF^+(N).$$

For $u \in \mathcal{G}$, the differential $u_* : VF^+(N)_{s(u)} \to VF^+(N)_{r(u)}$ of $u \cdot : F^+(N)_{s(u)} \to F^+(N)_{r(u)}$ in the fibres defines on $VF^+(N)$ the structure of a $\mathcal{G}$-equivariant vector bundle. Since the left action of $\mathcal{G}$ commutes with the right action of $GL^+(q, \mathbb{R})$, one has

$$u_* v_\phi = \frac{d}{dt}(u \cdot (\phi \cdot \exp(tv)))\Big|_{t=0} = \frac{d}{dt}((u \cdot \phi) \cdot \exp(tv))\Big|_{t=0} = v_{u \cdot \phi} \tag{3.2}$$

for all $\phi \in (F^+(N))_{s(u)}$, and so with respect to the trivialization $F^+(N) \times \mathfrak{gl}(q, \mathbb{R})$ of $VF^+(N)$ we have

$$u_*(\phi, v) = (u \cdot \phi, v) \tag{3.3}$$

for all $\phi \in F^+(N)$ and $v \in \mathfrak{gl}(q, \mathbb{R})$.

Consider now the quotient $Q := F^+(N)/SO(q, \mathbb{R})$ of $F^+(N)$ by the right action of $SO(q, \mathbb{R})$. The projection $\pi_F : F^+(N) \to M$ descends to a projection $\pi_Q : Q \to M$, which defines a fibre bundle with typical fibre $S_q^+ := GL^+(q, \mathbb{R})/SO(q, \mathbb{R})$, the space of positive definite, symmetric $q \times q$ matrices. Moreover, since the action of $\mathcal{G}$ on $F^+(N)$ commutes with the right action of $SO(q, \mathbb{R})$, it follows that Q is a $\mathcal{G}$-space with anchor map $\pi_Q : Q \to M$, and with action of $u \in \mathcal{G}$ given by

$$u \cdot [\phi] := [u \cdot \phi] = [u_* \circ \phi] \tag{3.4}$$

for all $[\phi] \in Q_{s(u)}$. Following [Benameur and Heitsch 2018; Zhang 2017], we refer to $\pi_Q : Q \to M$ as the Connes fibration.

Definition 3.5. The fibre bundle $\pi_Q : Q \to M$ is a $\mathcal{G}$-space called the *Connes fibration* for the normal bundle N.

Let us consider the geometry of the fibres of $Q \to M$. Since $SO(q, \mathbb{R})$ is compact, the pair $(GL^+(q, \mathbb{R}), SO(q, \mathbb{R}))$ is a Riemannian symmetric pair and hence the space S_q^+ can be equipped with a $GL^+(q, \mathbb{R})$-invariant metric under which it is, by [Helgason 1962, Proposition 3.4], a globally symmetric Riemannian space. The Riemannian space S_q^+ is moreover of noncompact type, so by [Helgason 1962, Theorem 3.1] has everywhere nonpositive sectional curvature. We can find a locally finite open cover $\mathcal{U}$ of M by sets U for which the vertical bundle $VQ|_U \cong U \times TS_q$, and then choosing a partition of unity subordinate to $\mathcal{U}$ allows us to equip the bundle $VQ \to Q$ with a Euclidean structure. We will assume from here on that $VQ \to Q$ is equipped with a Euclidean structure in this way.

Proposition 3.6. *The bundle $VQ \to Q$ is a $\mathcal{G}$-equivariant Euclidean bundle over the $\mathcal{G}$-space Q. Consequently* $\mathrm{Cliff}(VQ)$ *and* $\mathrm{Cliff}(V^*Q)$ *are $\mathcal{G}$-equivariant bundles.*

Proof. Fix $u \in \mathcal{G}$ and suppose that U_s and U_r are open sets in M containing $s(u)$ and $r(u)$, respectively, such that we have local trivializations $N|_{U_s} \cong U \times \mathbb{R}^q$ and $N|_{U_r} \cong U \times \mathbb{R}^q$, with respect to which the holonomy action $u_* : N_{s(u)} \to N_{r(u)}$ is the action on $\mathbb{R}^q$ of an element $\tilde{u} \in GL^+(q, \mathbb{R})$.

We obtain corresponding local trivializations $F^+(N)|_{U_s} \cong U \times GL^+(q, \mathbb{R})$ and $F^+(N)|_{U_r} \cong U \times GL^+(q, \mathbb{R})$ of the local frame bundles over U_s and U_r, in which the holonomy action $u\, \cdot : F^+(N)_{s(u)} \to F^+(N)_{r(u)}$ is left multiplication on $GL^+(q, \mathbb{R})$ by $\tilde{u}$, and taking the quotient by $SO(q, \mathbb{R})$ we get local trivializations $Q|_{U_s} \cong U \times S_q^+$ and $Q|_{U_r} \cong U \times S_q^+$ in which $u\, \cdot : Q_{s(u)} \to Q_{r(u)}$ is the isometry of $S_q^+ = GL^+(q, \mathbb{R})/SO(q, \mathbb{R})$ defined by left multiplication by $\tilde{u} \in GL^+(q, \mathbb{R})$. Thus, $\mathcal{G}$ acts by orientation-preserving isometries between the fibres of Q, inducing an action by special orthogonal transformations on the Euclidean bundle $VQ \to Q$ of vectors tangent to the fibres of $Q \to M$, hence making $VQ \to Q$ a $\mathcal{G}$-equivariant Euclidean bundle over the $\mathcal{G}$-space Q. The final statement follows from functoriality of Clifford algebras with respect to orthogonal maps. □

That the fibres have nonpositive sectional curvature allows us to define a dual Dirac class for Q over M in a similar manner to Connes [1986]. First, let $\mathbb{C}\ell(V^*Q)$ be equipped with the $\mathcal{G}$-structure arising from the action of $\mathcal{G}$ on the equivariant bundle $\mathrm{Cliff}(V^*Q)$ over the $\mathcal{G}$-space Q, denoted for $u \in \mathcal{G}$ by $u_\diamond : \mathrm{Cliff}(V^*_{[\phi]}Q) \to \mathrm{Cliff}(V^*_{u\cdot[\phi]}Q)$ for all $[\phi] \in Q_{s(u)}$. That is, we define for any $u \in \mathcal{G}$ an isomorphism $\alpha_u^1 : \mathbb{C}\ell(V^*Q|_{Q_{s(u)}}) \to \mathbb{C}\ell(V^*Q|_{Q_{s(u)}})$ by

$$\alpha_u^1(a)([\phi]) := u_\diamond a(u^{-1} \cdot [\phi]) \tag{3.7}$$

for all $[\phi] \in Q_{r(u)}$. Also let

$$E^1 := \Lambda^*(V^*Q) \otimes \mathbb{C}$$

be the complexified exterior algebra bundle of the bundle of vertical covectors V^*Q over Q. Here we equip V^*Q with the Euclidean structure coming from its dual VQ, which determines a Hermitian structure on $V^*Q \otimes \mathbb{C}$ and hence on E^1. Observe that

$$X_{E^1} := \Gamma_0(Q; E^1)$$

is a Hilbert $\mathbb{C}\ell(V^*Q)$-module under the inner product

$$\langle \rho^1, \rho^2 \rangle_{\mathbb{C}\ell(V^*Q)}([\phi]) := \psi_{V^*Q}(\rho^1([\phi]))\psi_{V^*Q}(\rho^2([\phi]))$$

and right action

$$(\rho \cdot a)([\phi]) := c_R(a([\phi]))\rho([\phi]),$$

where c_R is the right action of $\mathrm{Cliff}(V^*Q)$ on the Clifford bimodule E^1.

The isometric action of $\mathcal{G}$ on the Euclidean bundle VQ over Q gives rise to a unitary action of $\mathcal{G}$ on E^1, denoted for each $u \in \mathcal{G}$ by $u_* : E^1_{[\phi]} \to E^1_{u\cdot[\phi]}$ for all $[\phi] \in Q_{s(u)}$, and hence determines an isomorphism $W^1_u : \Gamma_0(Q_{s(u)}; E^1|_{Q_{s(u)}}) \to \Gamma_0(Q_{r(u)}; E^1|_{Q_{r(u)}})$ of Banach spaces given by the formula

$$(W^1_u\rho)([\phi]) := u_*\rho(u^{-1} \cdot [\phi])$$

for all $[\phi] \in Q_{r(u)}$. A routine calculation using Lemma 2.1 shows that

$$\langle W^1_u\rho^1, W^1_u\rho^2\rangle_{\mathbb{C}\ell(V^*Q)} = \alpha^1_u(\langle\rho^1, \rho^2\rangle_{\mathbb{C}\ell(V^*Q)}),$$

so (X_{E^1}, W^1) is a $\mathcal{G}$-equivariant Hilbert $\mathbb{C}\ell(V^*Q)$-module.

Choose now a Euclidean metric for N. Such a choice is determined by a section $\sigma : M \to Q$ of $\pi_Q : Q \to M$. For $[\phi_1], [\phi_2]$ in the same fibre Q_x, denote by $h([\phi_1], [\phi_2])$ the geodesic distance between $[\phi_1]$ and $[\phi_2]$ in the fibre, and then for any $[\phi_0] \in Q$ let $h^{[\phi_0]} : Q \to \mathbb{R}$ be the function

$$h^{[\phi_0]}([\phi]) := h([\phi_0], [\phi]).$$

In particular, for $x \in M$ and $[\phi] \in Q_x$, $h^{\sigma(x)}([\phi])$ gives the distance in the fibre between $[\phi]$ and the section σ. Consider now the vertical 1-form

$$Z_{[\phi]} := h^{\sigma(\pi_Q([\phi]))}([\phi])dh^{\sigma(\pi_Q([\phi]))}_{[\phi]},$$

where d denotes the exterior derivative in the fibre. Define an operator B_1 on the dense submodule $X^c_{E^1} := \Gamma_c(Q; E^1)$ of X_{E^1} by the formula

$$(B_1\rho)([\phi]) := c_L(Z_{[\phi]})\rho([\phi]),$$

where c_L is the left representation of $\mathrm{Cliff}(V^*Q)$ on the Clifford bimodule E^1. Since c_L and c_R commute, B_1 commutes with the right action of $\mathbb{C}\ell(V^*Q)$. Finally, we let m be the representation of $C_0(M)$ on X_{E^1} by multiplication, that is,

$$(m(f)\rho)([\phi]) := f(\pi_Q([\phi]))\rho([\phi])$$

for all $f \in C_0(M)$ and $\rho \in X_{E^1}$. Equivariance of the map π_Q tells us that m is an equivariant representation.

Proposition 3.8. *The triple $(C_0(M), {}_mX_{E^1}, B_1)$ is an unbounded $\mathcal{G}$-equivariant Kasparov $C_0(M)$-$\mathbb{C}\ell(V^*Q)$-module and hence defines a class*

$$[B_1] \in KK^{\mathcal{G}}(C_0(M), \mathbb{C}\ell(V^*Q)).$$

Proof. The first thing we need to prove is that B_1 is self-adjoint and regular. Observe first that B_1 is clearly symmetric. For each $[\phi] \in Q$, the localization $(X_{E^1})_{[\phi]}$ of X_{E^1} in the sense of [Pierrot 2006b; Kaad and Lesch 2012] is just the finite-dimensional Hilbert space

$$\mathcal{H}_{[\phi]} := \Lambda^*(V^*_{[\phi]}Q) \otimes \mathbb{C}$$

with the inner product coming from the Hermitian structure on $\Lambda^*(V^*_{[\phi]}Q) \otimes \mathbb{C}$, and the action of the localized operator $(B_1)_{[\phi]}$ on $\mathcal{H}_{[\phi]}$ is

$$(B_1)_{[\phi]}\eta := c_L(Z_{[\phi]})\eta.$$

Since $(B_1)_{[\phi]}$ is then self-adjoint on $\mathcal{H}_{[\phi]}$, it follows from [Pierrot 2006b, Théorème 1.18] that B_1 is self-adjoint and regular.

That $m(f)(1+B_1^2)^{-1/2}$ is a compact operator for all $f \in C_0(M)$ follows from the definition of Clifford multiplication. Indeed, one has $c_L(Z_{[\phi]})^2 = \|Z_{[\phi]}\|^2 = h^{\sigma(\pi_Q([\phi]))}([\phi])^2$ since $dh^{\sigma(\pi_Q([\phi]))}_{[\phi]}$ has norm 1 for all $[\phi]$ as the dual of the tangent to the unique unit-speed geodesic joining $\sigma(\pi_Q([\phi]))$ to $[\phi]$, and so for any $f \in C_0(M)$, one simply has

$$(m(f)(1+B_1^2)^{-1/2}\rho)([\phi]) = \frac{f(\pi_Q([\phi]))}{(1+h^{\sigma(\pi_Q([\phi]))}([\phi])^2)^{1/2}}\rho([\phi]).$$

Since f vanishes at infinity on the base M of $Q \to M$, and because $[\phi] \mapsto (1+h^{\sigma(\pi_Q([\phi]))}([\phi])^2)^{-1/2}$ vanishes at infinity on the fibres of $Q \to M$, the function $[\phi] \mapsto f(\pi_Q([\phi]))(1+h^{\sigma(\pi_Q([\phi]))}([\phi])^2)^{-1/2}$ is an element of $C_0(Q)$, so that $m(f)(1+B_1^2)^{-1/2}$ is indeed a compact operator on the $\mathbb{C}\ell(V^*Q)$-module X_{E^1}.

Concerning commutators, it is clear that B_1 commutes with the representation m of $C_0(M)$. Thus, it only remains to prove that B_1 is appropriately equivariant. The idea of this is essentially the unbounded version of analogous results by Connes [1986, Lemma 5.3] and Kasparov [1988, §5.3], but the details are somewhat technical so we give them here. Fix $u \in \mathcal{G}$ and $\rho \in \Gamma_c(Q_{r(u)}; E^1|_{Q_{r(u)}})$. We calculate

$$\begin{aligned}(B_1 - W^1_u B_1 W^1_{u^{-1}})\rho([\phi]) &= c_L(Z_{[\phi]})\rho([\phi]) - u_*(B_1 W^1_{u^{-1}}\rho)(u^{-1}\cdot[\phi]) \\ &= c_L(Z_{[\phi]})\rho([\phi]) - u_*(c_L(Z_{u^{-1}\cdot[\phi]})(W^1_{u^{-1}}\rho)(u^{-1}\cdot[\phi])) \\ &= c_L(Z_{[\phi]})\rho([\phi]) - u_*(c_L(Z_{u^{-1}\cdot[\phi]})(u_*^{-1}\rho([\phi]))) \\ &= c_L(Z_{[\phi]} - u_*Z_{u^{-1}\cdot[\phi]})\rho([\phi])\end{aligned}$$

where on the third line we have used the identity (2.2). Thus, we must calculate a bound for the norm of the covector $Z_{[\phi]} - u_*Z_{u^{-1}\cdot[\phi]}$.

Denote $\sigma_r := \sigma(r(u))$ and $\sigma_s := \sigma(s(u))$. With this notation, we have

$$Z_{[\phi]} - u_*Z_{u^{-1}\cdot[\phi]} = h^{\sigma_r}([\phi])dh^{\sigma_r}_{[\phi]} - u_*h^{\sigma_s}(u^{-1}\cdot[\phi])dh^{\sigma_s}_{u^{-1}\cdot[\phi]}.$$

For any vector $\gamma \in V_{[\phi]}Q$ we have

$$(u_* dh^{\sigma_s}_{u^{-1}\cdot[\phi]})(\gamma) = dh^{\sigma_s}_{u^{-1}\cdot[\phi]}(u_*^{-1}\gamma) = d(h^{\sigma_s}\circ u^{-1})_{[\phi]}(\gamma),$$

giving $u_* dh^{\sigma_s}_{u^{-1}\cdot[\phi]} = d(h^{\sigma_s}\circ u^{-1})_{[\phi]}$, and since the action of $\mathcal{G}$ is isometric on the fibres we get

$$(h^{\sigma_s}\circ u^{-1})([\phi]) = h(\sigma_s, u^{-1}\cdot[\phi]) = h(u\cdot\sigma_s, [\phi]) = h^{u\cdot\sigma_s}([\phi]).$$

Thus,

$$u_* dh^{\sigma_s}_{u^{-1}\cdot[\phi]} = dh^{u\cdot\sigma_s}_{[\phi]}.$$

We then see that

$$\begin{aligned} h^{\sigma_r}([\phi])dh^{\sigma_r}_{[\phi]} - u_* h^{\sigma_s}(u^{-1}\cdot[\phi])dh^{\sigma_s}_{u^{-1}\cdot[\phi]} &= h^{\sigma_r}([\phi])dh^{\sigma_r}_{[\phi]} - h^{u\cdot\sigma_s}([\phi])dh^{u\cdot\sigma_s}_{[\phi]} \\ &= \tfrac{1}{2} d((h^{\sigma_r})^2 - (h^{u\cdot\sigma_s})^2)_{[\phi]} \\ &= \tfrac{1}{2} d((h^{\sigma_r} - h^{u\cdot\sigma_s})(h^{\sigma_r} + h^{u\cdot\sigma_s}))_{[\phi]}. \end{aligned}$$

By the argument [Kasparov 1988, Lemma 5.3], we have

$$\|dh^{\sigma_r}_{[\phi]} - dh^{u\cdot\sigma_s}_{[\phi]}\| \le 2h(\sigma_r, u\cdot\sigma_s)(h^{\sigma_r}([\phi]) + h^{u\cdot\sigma_s}([\phi]))^{-1},$$

which we use to estimate

$$\begin{aligned} &\|h^{\sigma_r}([\phi])dh^{\sigma_r}_{[\phi]} - u_* h^{\sigma_s}(u^{-1}\cdot[\phi])dh^{\sigma_s}_{u^{-1}\cdot[\phi]}\|^2 \\ &\quad\le \tfrac{1}{4}\|(dh^{\sigma_r}_{[\phi]} - dh^{u\cdot\sigma_s}_{[\phi]})(h^{\sigma_r}([\phi]) + h^{u\cdot\sigma_s}([\phi]))\|^2 \\ &\qquad\qquad + \tfrac{1}{4}\|(h^{\sigma_r}([\phi]) - h^{u\cdot\sigma_s}([\phi]))(dh^{\sigma_r}_{[\phi]} + dh^{u\cdot\sigma_s}_{[\phi]})\|^2 \\ &\quad\le h(\sigma_r, u\cdot\sigma_s)^2 + (h(\sigma_r, [\phi]) - h(u\cdot\sigma_s, [\phi]))^2 \\ &\quad= h(\sigma_r, u\cdot\sigma_s)^2 + h(\sigma_r, [\phi])^2 + h(u\cdot\sigma_s, [\phi])^2 - 2h(\sigma_r, [\phi])h(u\cdot\sigma_s, [\phi]) \\ &\quad\le 2h(\sigma_r, u\cdot\sigma_s)^2, \end{aligned}$$

where the last line is a consequence of the cosine inequality for spaces of nonpositive sectional curvature [Helgason 1962, Corollary 13.2].

Thus, for all $[\phi] \in Q_{r(u)}$, we have that $\|Z_{[\phi]} - u_* Z_{u^{-1}\cdot[\phi]}\|^2 \le 2h(\sigma(r(u)), u\cdot\sigma(s(u)))^2$ independently of $[\phi] \in Q_{r(u)}$, implying that $B_1 - W^1_u B_1 W^1_{u^{-1}}$ extends to a bounded operator on $(X_{E^1})_{r(u)}$. Moreover, $u \mapsto h(\sigma(r(u)), u\cdot\sigma(s(u)))$ is continuous and hence bounded on compact Hausdorff sets, so for any element U_i of the cover $\mathcal{U} = \{U_i\}_{i\in I}$ of $\mathcal{G}$ by Hausdorff open subsets, and for any $\varphi \in C_c(U_i)$ and $f \in C_0(M)$, we have that

$$\varphi\cdot m^r_i(r_i^*(f))\cdot(r_i^* B_1 - (W^1)^i \circ s_i^* B_1 \circ ((W^1)^i)^{-1}) \in \mathcal{L}(r_i^* X_{E^1}).$$

It follows that $(C_0(M), {}_m X_{E^1}, B_1)$ is an unbounded equivariant Kasparov $C_0(M)$-$\mathbb{C}\ell(V^*Q)$-module. □

3B. ***The foliation of the Connes fibration.*** Before we can construct a second Kasparov module and the semifinite spectral triple associated to it, we need a closer study of the geometry and groupoid representation theory at our disposal.

Definition 3.9. Assume M to be equipped with a Riemannian metric g, with Levi-Civita connection ∇^{LC}. Then N identifies with the subbundle $N = T\mathcal{F}^{\perp}$ of TM, and we use the notation $X = X_{\mathcal{F}} + X_N$ for the corresponding decomposition of vector fields into their normal and leafwise components. Define a connection $\nabla^{\flat}$ on N by the formula

$$\nabla^{\flat}_X(Y) = [X_{\mathcal{F}}, Y]_N + \nabla^{\mathrm{LC}}_{X_N}(Y)_N, \quad X \in \Gamma^{\infty}(M; TM),\ Y \in \Gamma^{\infty}(M; N).$$

We refer to $\nabla^{\flat}$ as a *torsion-free Bott connection*.

The terminology "torsion-free" in Definition 3.9 refers to the fact that for any such connection $\nabla^{\flat}$ the associated torsion tensor

$$T_{\nabla^{\flat}}(X, Y) := \nabla^{\flat}_X(Y_N) - \nabla^{\flat}_Y(X_N) - [X, Y]_N$$

vanishes for all $X, Y \in \Gamma^{\infty}(M; TM)$. This fact follows from an easy calculation using the corresponding property of the Levi-Civita connection. Bott connections more generally are characterized by the formula $\nabla^{\flat}_{X_{\mathcal{F}}}(Y) = [X_{\mathcal{F}}, Y]_N$ for all smooth sections Y of N and $X_{\mathcal{F}}$ of $T\mathcal{F}$, and are the key ingredient for the Chern–Weil proof of Bott's vanishing theorem [1970]. For us, the use of the Levi-Civita connection in Definition 3.9 serves a purpose that will become apparent in Section 5.

Let us now come back to the frame bundle $\pi_F : F^+(N) \to M$. The total space of this bundle carries a foliation $\mathcal{F}_F$ by the orbits of the action of $\mathcal{G}$ on $F^+(N)$ defined in (3.1), whose normal we denote by $N_F := TF^+(N)/T\mathcal{F}_F$. The foliation $\mathcal{F}_F$ is everywhere transverse to the fibres of $F^+(N)$, and its tangent bundle $T\mathcal{F}_F$ projects fibrewise-isomorphically onto $T\mathcal{F}$. It is readily verified using a calculation in foliated coordinates that the connection form $\alpha^{\flat} \in \Omega^1(F^+(N); \mathfrak{gl}(q, \mathbb{R}))$ associated to any Bott connection $\nabla^{\flat}$ on N contains $T\mathcal{F}_F$ in its kernel $HF^+(N) := \ker(\alpha^{\flat})$. Consequently we find that the normal bundle to the foliation $\mathcal{F}_F$ admits a decomposition

$$N_F = VF^+(N) \oplus (HF^+(N)/T\mathcal{F}_F). \tag{3.10}$$

The normal bundle N_F is again a $\mathcal{G}$-equivariant bundle, and with respect to the splitting (3.10) we write

$$u_* = \begin{pmatrix} \tilde{a}(u) & \tilde{c}(u) \\ 0 & \tilde{d}(u) \end{pmatrix}$$

for the action of $u \in \mathcal{G}$ on N_F. Note that the zero appearing in the bottom-left corner is a consequence of the fact that by (3.1), $\mathcal{G}$ acts via diffeomorphisms between the fibres $GL^+(q, \mathbb{R})$ of $F^+(N) \to M$, and so preserves the bundle $VF^+(N) \to M$ of vectors tangent to the fibres.

Now we are not so interested in the frame bundle $F^+(N)$ as the Connes fibration Q. Since the action of $\mathcal{G}$ on $F^+(N)$ commutes with the right action of $SO(q, \mathbb{R})$, however, we find that we also obtain a foliation on the total space of $\pi_Q : Q \to M$.

To be more specific, let $k : F^+(N) \to Q$ be the quotient map. Then $T\mathcal{F}_Q := dk(T\mathcal{F}_F)$ is an integrable subbundle of TQ, which determines a foliation $\mathcal{F}_Q$ of Q. Since $\pi_Q \circ k = \pi_F$, we see that $d\pi_Q$ maps $T\mathcal{F}_Q$ isomorphically onto $T\mathcal{F}$ making $\pi_Q : Q \to M$ a foliated bundle. The normal bundle N_Q of $\mathcal{F}_Q$ also admits a splitting

$$N_Q = VQ \oplus (HQ/T\mathcal{F}_Q),$$

where HQ is the isomorphic image under dk of the horizontal subbundle $HF^+(N) \subset TF^+(N)$. For convenience, we will denote $HQ/T\mathcal{F}_Q$ simply by H. Thus,

$$N_Q = VQ \oplus H.$$

Now, $d\pi_Q$ maps the fibres of HQ isomorphically onto those of TM, and maps the fibres of $T\mathcal{F}_Q$ isomorphically onto those of $T\mathcal{F}$. It follows that $d\pi_Q$ induces an isomorphism of the fibres of $H = HQ/T\mathcal{F}_Q$ onto those of $N = TM/T\mathcal{F}$. We can then equip H with a Euclidean metric in the following way [Connes 1986, §5].

Proposition 3.11. *For $h_1, h_2 \in H_{[\phi]}$ and with $\cdot$ denoting the Euclidean inner product in $\mathbb{R}^q$, the formula*

$$m^H_{[\phi]}(h_1, h_2) := \phi^{-1}(d\pi_Q(h_1)) \cdot \phi^{-1}(d\pi_Q(h_2))$$

determines a well-defined Euclidean metric on the bundle $H \to Q$.

Proof. Suppose we were to choose a different representation $\phi' = \phi \circ A$ of $[\phi]$, where A is some matrix in $SO(q, \mathbb{R})$. Then by the invariance of the Euclidean inner product under special orthogonal transformations we have

$$\begin{aligned}(\phi')^{-1}(d\pi_Q(h_1)) \cdot (\phi')^{-1}(d\pi_Q(h_2)) &= (A^{-1}\phi^{-1}(d\pi_Q(h_1))) \cdot (A^{-1}\phi^{-1}(d\pi_Q(h_2))) \\ &= \phi^{-1}(d\pi_Q(h_1)) \cdot \phi^{-1}(d\pi_Q(h_2)),\end{aligned}$$

giving well-definedness. That we have defined a metric follows from the linearity of the maps ϕ and $d\pi_Q$, and the fact that the Euclidean inner product is a metric on $\mathbb{R}^q$. ☐

Remarkably, holonomy translations are orthogonal with respect to this Euclidean structure of H.

Proposition 3.12. *The normal bundle $N_Q \to Q$ of the foliation $\mathcal{F}_Q$ of Q is a $\mathcal{G}$-equivariant vector bundle over the $\mathcal{G}$-space Q. Moreover, with respect to the* *splitting $N_Q = VQ \oplus H$, for $u \in \mathcal{G}$ and $[\phi] \in Q_{s(u)}$ the holonomy action $u_* : (N_Q)_{[\phi]} \to (N_Q)_{u\cdot[\phi]}$ has the form*

$$u_* = \begin{pmatrix} a(u) & c(u) \\ 0 & d(u) \end{pmatrix}, \tag{3.13}$$

with $a(u) : V_{[\phi]}Q \to V_{u\cdot[\phi]}Q$ and $d(u) : H_{[\phi]} \to H_{u\cdot[\phi]}$ orthogonal and orientation-preserving.

Proof. The holonomy groupoid for the foliation $\mathcal{F}_Q$ of Q is precisely the groupoid $Q \rtimes \mathcal{G}$, under which the normal bundle $N_Q \to Q$ is therefore equivariant. Thus, $N_Q \to Q$ is a $\mathcal{G}$-equivariant vector bundle over the $\mathcal{G}$-space Q.

Proposition 3.6 tells us that $a(u) : V_{[\phi]}Q \to V_{u\cdot[\phi]}Q$ is orthogonal and orientation-preserving, and that the vertical bundle is preserved under holonomy translation, which accounts for the 0 appearing in the bottom-left corner of (3.13). Since $\pi_Q : Q \to M$ is the anchor map for the $\mathcal{G}$-space Q, it is $\mathcal{G}$-equivariant, implying that the identification $d\pi_Q$ of fibres of H with those of N is also $\mathcal{G}$-equivariant.

That $d(u) : H_{[\phi]} \to H_{u\cdot[\phi]}$ is orientation-preserving is then a consequence of the fact that $d(u)$ may be identified with the orientation-preserving action of u on the fibres of N. That $d(u)$ is orthogonal is a consequence of the following calculation for $h_1, h_2 \in H_{[\phi]}$:

$$\begin{aligned} &m^H_{u\cdot[\phi]}(d(u)h_1, d(u)h_2) \\ &\qquad = (u_* \circ \phi)^{-1}((d\pi_Q \circ d(u))(h_1)) \cdot (u_* \circ \phi)^{-1}((d\pi_Q \circ d(u))(h_1)) \\ &\qquad = (\phi^{-1} \circ u_*^{-1})((u_* \circ d\pi_Q)(h_1)) \cdot (\phi^{-1} \circ u_*^{-1})((u_* \circ d\pi_Q)(h_2)) \\ &\qquad = \phi^{-1}(d\pi_Q(h_1)) \cdot \phi^{-1}(d\pi_Q(h_2)) = m^H_{[\phi]}(h_1, h_2), \end{aligned}$$

where for the second equality, we have used the equivariance of the anchor map $d\pi_Q$ between H and N. □

The triangular shape of the matrix in Proposition 3.12 is what is referred to as an *almost isometric* or *triangular structure* by Connes [1986] and Connes and Moscovici [1995], respectively.

The map $c(u) : H_{[\phi]} \to V_{u\cdot[\phi]}Q$, for $u \in \mathcal{G}$ and $[\phi] \in Q_{s(u)}$, is where the interesting representation theory is encoded. Currently, however, the range of $c(u)$ is too high in dimension to be of much use, and these extra dimensions need to be "traced out". Observing that there is indeed a canonical trace $\mathrm{tr}_{F^+(N)} : VF^+(N) \to \mathbb{R}$ induced fibrewise by the usual matrix trace on $\mathfrak{gl}(q, \mathbb{R}) = M_q(\mathbb{R})$, we now check that we can apply this map to VQ also.

Lemma 3.14. *The map $\mathrm{tr}_{F^+(N)} : VF^+(N) \to \mathbb{R}$ descends to a well-defined map $\mathrm{tr}_Q : VQ \to \mathbb{R}$ for which $\mathrm{tr}_Q \circ a(u) = \mathrm{tr}_Q$ for all $u \in \mathcal{G}$.*

Proof. For $A \in GL^+(q, \mathbb{R})$, we denote by $R_A : F^+(N) \to F^+(N)$ the map $\phi \mapsto \phi \cdot A$. By definition, the action of $A \in SO(q, \mathbb{R})$ on $VF^+(N)$ is then given for $\phi \in F^+(N)$ and $v_\phi \in V_\phi F^+(N)$ by

$$v_\phi \cdot A := (dR_A)_\phi(v_\phi).$$

We compute

$$\begin{aligned}(dR_A)_\phi(v_\phi) = \frac{d}{dt}(\phi \cdot \exp(tv) \cdot A)\Big|_{t=0} &= \frac{d}{dt}((\phi \cdot A) \cdot (A^{-1} \exp(tv)A))\Big|_{t=0} \\ &= (A^{-1}vA)_{\phi \cdot A},\end{aligned}$$

from which we deduce that the action of $A \in SO(q, \mathbb{R})$ in the trivialization $VF^+(N) = F^+(N) \times \mathfrak{gl}(q, \mathbb{R})$ is given by

$$(\phi, v) \cdot A = (\phi \cdot A, A^{-1}vA)$$

for all $\phi \in F^+(N)$, $v \in \mathfrak{gl}(q, \mathbb{R})$. Now, $\mathrm{tr}_{F^+(N)} : F^+(N) \times \mathfrak{gl}(q, \mathbb{R}) \to \mathbb{R}$ is by definition

$$\mathrm{tr}_{F^+(N)}(\phi, v) := \mathrm{tr}(v),$$

with tr denoting the usual matrix trace on $q \times q$ matrices, and with the range $\mathbb{R}$ of $\mathrm{tr}_{F^+(N)}$ carrying the trivial action of $SO(q, \mathbb{R})$. Then since the matrix trace is invariant under conjugation, we see that $\mathrm{tr}_{F^+(N)}$ is equivariant,

$$\mathrm{tr}_{F^+(N)}((\phi, v) \cdot A) = \mathrm{tr}(A^{-1}vA) = \mathrm{tr}(v) = \mathrm{tr}_{F^+(N)}(\phi, v) \cdot A,$$

and so descends to a well-defined map $\mathrm{tr}_Q : VQ \to \mathbb{R}$.

For the second assertion, note that since u commutes with the quotient map $Q : F^+N \to Q$ and since u_* acts as the identity on the fibres of $VF^+(N) = F^+(N) \times \mathbb{R}^{q^2}$ by (3.3), we have

$$\mathrm{tr}_Q \circ a(u) \circ dQ = \mathrm{tr}_Q \circ dQ \circ \mathrm{id} = \mathrm{tr}_Q \circ dQ.$$

Since dQ is surjective, we conclude that

$$\mathrm{tr}_Q \circ a(u) = \mathrm{tr}_Q$$

as claimed. □

Remark 3.15. Note that what makes Lemma 3.14 possible is the fact that the map $v \mapsto \mathrm{tr}(v)$ on $\mathfrak{gl}(q, \mathbb{R})$ is invariant under conjugation by invertible matrices. Thus, in fact we could replace tr with any other invariant polynomial on $\mathfrak{gl}(q, \mathbb{R})$, paralleling the Chern–Weil construction of characteristic classes, and still obtain a well-defined (but no longer necessarily linear) map on the vertical tangent bundle of the Connes fibration. This observation is due to M. T. Benameur.

Let us put Lemma 3.14 to use in simplifying the groupoid representation theory. For $u \in \mathcal{G}$ and $[\phi] \in Q_{s(u)}$, define

$$\delta(u) := \mathrm{tr}_Q \circ c(u) : H_{[\phi]} \to \mathbb{R}.$$

This $\delta(u)$ is linear, and so can be regarded as an element of $H^*_{[\phi]}$. We also define

$$\theta(u) := d(u^{-1})^t : H^*_{[\phi]} \to H^*_{u\cdot[\phi]},$$

the action on the covector bundle for H coming from the transpose of $d(u^{-1}) : H_{u\cdot[\phi]} \to H_{[\phi]}$. We have the following "$ax+b$ group"-type transformation laws.

Lemma 3.16. *For all $u, v \in \mathcal{G}^{(2)}$, we have*

$$\theta(uv) = \theta(u)\theta(v), \quad \textit{and} \quad \delta(uv) = \delta(v) + \theta(v^{-1})\delta(u).$$

Proof. These identities follow from the triangular structure of the matrices (3.13) and Lemma 3.14. Specifically, since $\mathcal{G}$ acts on N_Q we have

$$\begin{pmatrix} a(uv) & c(uv) \\ 0 & d(uv) \end{pmatrix} = \begin{pmatrix} a(u) & c(u) \\ 0 & d(u) \end{pmatrix} \begin{pmatrix} a(v) & c(v) \\ 0 & d(v) \end{pmatrix} = \begin{pmatrix} a(u)a(v) & a(u)c(v)+c(u)d(v) \\ 0 & d(u)d(v) \end{pmatrix},$$

from which we immediately deduce that $d(uv) = d(u)d(v)$ and hence $\theta(uv) = \theta(u)\theta(v)$. We also calculate

$$\begin{aligned} \delta(uv) &= \mathrm{tr}_Q \circ c(uv) = \mathrm{tr}_Q \circ a(u) \circ c(v) + \mathrm{tr}_Q \circ c(u) \circ d(v) \\ &= \mathrm{tr}_Q \circ c(v) + \mathrm{tr}_Q \circ c(u) \circ d(v) = \delta(v) + \theta(v^{-1})\delta(u), \end{aligned}$$

using Lemma 3.14 for the third equality, giving the desired identities. □

3C. *The Vey Kasparov module.* We now go about constructing a second Kasparov module, referred to in this paper as the Vey Kasparov module since it appears to be analogous to the Vey homomorphism considered in previous work [Hurder 1986; Duminy 1982]. Our first job in constructing a second Kasparov module is to endow the total space H^* of the horizontal covector bundle $\pi_{H^*} : H^* \to Q$ with an action of $\mathcal{G}$ that encodes both θ and δ from Lemma 3.16.

Proposition 3.17. *For $u \in \mathcal{G}$ and $\eta \in H^*|_{Q_{s(u)}}$, the formula*

$$u \cdot \eta := \theta(u)\eta + \delta(u^{-1})$$

determines the structure of a $\mathcal{G}$-space on H^* with anchor map $\pi_Q \circ \pi_{H^*} : H^* \to M$.

Proof. It is clear that $(\pi_Q \circ \pi_{H^*})(u \cdot \eta) = r(u)$ for all $u \in \mathcal{G}$ and $\eta \in H^*|_{Q_{s(u)}}$, and since by Lemma 3.16 θ is the identity on units and δ is zero on units, we get $(\pi_Q \circ \pi_{H^*})(\eta) \cdot \eta = \eta$ for all η. Thus, it remains only to check that $(uv)\cdot\eta = u\cdot(v\cdot\eta)$ for all $(u, v) \in \mathcal{G}^{(2)}$ and $\eta \in H^*|_{Q_{s(v)}}$. For this we simply have

$$(uv) \cdot \eta = \theta(uv)\eta + \delta(v^{-1}u^{-1}) = \theta(u)(\theta(v)\eta + \delta(v^{-1})) + \delta(u^{-1}) = u \cdot (v \cdot \eta),$$

with the second equality being a consequence of Lemma 3.16. □

We can now construct another dual Dirac class in much the same way as we did for the Connes fibration. Consider the bundle $VH^* := \ker(d\pi_{H^*})$ of vertical tangent vectors over the horizontal covector bundle $\pi_{H^*} : H^* \to Q$, and denote by $\pi_H : H \to Q$ the projection for the horizontal bundle. Since the fibres of H^* are vector spaces, we have $V_\eta H^*_{[\phi]} \cong H^*_{[\phi]}$ for all $[\phi] \in Q$ and $\eta \in H^*_{[\phi]}$. Thus, the dual space $V^*_\eta H^*_{[\phi]}$ is a copy of $H_{[\phi]}$ and so we can write V^*H^* as the fibred product

$$V^*H^* \cong H^* \times_{\pi_{H^*},\pi_H} H,$$

regarded as a vector bundle over H^* by using the projection onto the first factor. Since H is a $\mathcal{G}$-equivariant Euclidean bundle over Q via the map d in Proposition 3.12, for all $u \in \mathcal{G}$, $\eta \in H^*|_{Q_{s(u)}}$, and $h \in H|_{Q_{s(u)}}$, the formula

$$u_*(\eta, h) := (u \cdot \eta, d(u)h) = (\theta(u)\eta + \delta(u^{-1}), d(u)h)$$

defines on V^*H^* the structure of a $\mathcal{G}$-equivariant Euclidean bundle over the $\mathcal{G}$-space H^*. Then by functoriality $\mathrm{Cliff}(V^*H^*)$ is a $\mathcal{G}$-equivariant bundle over H^*, and we denote the action of $u \in \mathcal{G}$ on $k \in \mathrm{Cliff}(V^*H^*|_{H^*_{[\phi]}})$ by $k \mapsto u_\diamond k$ for all $[\phi] \in Q_{s(u)}$. Using these facts together with Proposition 3.17, the following result is clear.

Proposition 3.18. *The formula*

$$\alpha^2_u(\zeta)(\eta) := u_\diamond\zeta(u^{-1} \cdot \eta) = u_\diamond\zeta(\theta(u^{-1})\eta + \delta(u)),$$

defined for $\zeta \in \mathbb{C}\ell(V^*H^*)$, $u \in \mathcal{G}$, and $\eta \in H^*_{[\phi]}$ with $[\phi] \in Q_{r(u)}$, determines the structure of a $\mathcal{G}$-algebra on $\mathbb{C}\ell(V^*H^*)$.

We now come to the definition of an appropriate Hilbert module. Let

$$E^2 := \Lambda^*(V^*H^*) \otimes \mathbb{C}$$

be the complexified exterior algebra bundle of V^*H^* over H^*, and define

$$X_{E^2} := \Gamma_0(H^*; E^2),$$

which is a Hilbert $\mathbb{C}\ell(V^*H^*)$-module whose structure as such is determined in the same way as for X_{E^1} using the identification of E^2 with $\mathrm{Cliff}(V^*H^*)$ as vector bundles.

By equivariance of V^*H^* over H^* and functoriality, for $u \in \mathcal{G}$, $[\phi] \in Q_{s(u)}$, and $\eta \in H^*_{[\phi]}$ we obtain a unitary holonomy transport map $u_* : E^2_\eta \to E^2_{u\cdot\eta}$ and an isomorphism $W^2_u : \Gamma_0(H^*_{[\phi]}; E^2|_{H^*_{[\phi]}}) \to \Gamma_0(H^*_{u\cdot[\phi]}; E^2|_{H^*_{u\cdot[\phi]}})$ of Banach spaces defined by

$$(W^2_u\zeta)(\eta) := u_*\zeta(u^{-1} \cdot \eta) = u_*\zeta(\theta(u^{-1})\eta + \delta(u)).$$

Using Lemma 2.1, we observe that

$$\begin{aligned}\langle W_u^2\zeta_1, W_u^2\zeta_2\rangle_{\mathbb{C}\ell(V^*H^*)_{r(u)}}(\eta) &= u_\diamond\langle\zeta_1(\theta(u^{-1})\eta+\delta(u)), \zeta_2(\theta(u^{-1})\eta+\delta(u))\rangle \\ &= \alpha_u^2(\langle\zeta_1, \zeta_2\rangle_{\mathbb{C}\ell(V^*H^*)_{s(u)}})(\eta)\end{aligned}$$

for all $u \in \mathcal{G}$, $[\phi] \in Q_{r(u)}$, and $\eta \in H^*_{[\phi]}$, so (X_{E^2}, W^2) is a $\mathcal{G}$-Hilbert $\mathbb{C}\ell(V^*H^*)$-module.

We define an unbounded operator B_2 on the dense submodule $X^c_{E^2} = \Gamma_c(H^*; E^2)$ of X_{E^2} by the formula

$$(B_2\zeta)(\eta) := c_L(\eta)\zeta(\eta),$$

where for $c_L(\eta)$ we regard $\eta \in H^*$ as a vertical covector in $V^*H^* = H^* \times_{\pi_{H^*},\pi_H} H$ using the Euclidean metric on H.

Finally, we take m^2 to be the representation of $C_0(Q)$ on X_{E^2} defined by

$$m^2(f)\zeta(\eta) := f(\pi_{H^*}(\eta))\zeta(\eta).$$

Using the fact that π_{H^*} is an equivariant map and that $\pi_{H^*}(\eta+\eta') = \pi_{H^*}(\eta) = [\phi]$ for all $[\phi] \in Q$ and $\eta, \eta' \in H^*_{[\phi]}$, a routine calculation shows that m^2 is an equivariant representation.

Proposition 3.19. *The triple $(C_0(Q), {}_{m^2}X_{E^2}, B_2)$ is an unbounded $\mathcal{G}$-equivariant Kasparov $C_0(Q)$-$\mathbb{C}\ell(V^*H^*)$-module, defining a class*

$$[B_2] \in KK^{\mathcal{G}}(C_0(Q), \mathbb{C}\ell(V^*H^*)).$$

Proof. The proof is essentially the same as the proof of Proposition 3.8. The only part that must be changed is checking the equivariance condition. For any $u \in \mathcal{G}$, $[\phi] \in Q_{r(u)}$, and $\eta \in H^*_{[\phi]}$, we have

$$\begin{aligned}(W_u^2 B_2 W_{u^{-1}}^2)\zeta(\eta) &= u_*(B_2 W_{u^{-1}}^2\zeta)(\theta(u^{-1})\eta+\delta(u)) \\ &= u_*(c_L(\theta(u^{-1})\eta+\delta(u))(W_{u^{-1}}^2\zeta)(\theta(u^{-1})\eta+\delta(u))) \\ &= u_*(c_L(\theta(u^{-1})\eta+\delta(u))(u_*^{-1}\zeta(\theta(u)(\theta(u^{-1})\eta+\delta(u))+\delta(u^{-1})))) \\ &= u_*(c_L(\theta(u^{-1})\eta+\delta(u))(u_*^{-1}\zeta(\eta))) \\ &= c_L(\eta-\delta(u^{-1}))\zeta(\eta)\end{aligned}$$

where the last line follows from the identity $\theta(u)\delta(u) = -\delta(u^{-1})$ arising from Lemma 3.16, together with the identity (2.2). We then have

$$B_2 - W_u^2 B_2 W_{u^{-1}}^2 = c_L(\delta(u^{-1})),$$

which defines a bounded operator on $(X_{E^2})_{r(u)}$. The rest of the proof is then the same as in Proposition 3.8. ☐

4. The index theorem

4A. ***Some simplifications in codimension 1.*** There are important simplifications in the codimension 1 case. Observe that for a codimension 1, transversely orientable foliation $\mathcal{F}$ of M, the conormal bundle $N^* \to M$ is trivialized by a choice of orientation, which is given by a choice of a transverse volume form ω. Such a choice determines a dual section ω^* of $N \to M$ and hence a map $t : N \to \mathbb{R}$ defined by the equality $n = t(n)\omega^*$ for $n \in N$. Thus,

$$N = M \times \mathbb{R}.$$

The action of $u \in \mathcal{G}$ on N will then be denoted by

$$u_*(s(u), n) := (r(u), \Delta(u)n), \tag{4.1}$$

with $\Delta : \mathcal{G} \to \mathbb{R}^*_+$ a multiplicative homomorphism. Observe that under the correspondence $\omega \mapsto \omega^*$, this $\Delta(u)$ is precisely the Radon–Nikodym derivative of the transverse volume form ω with respect to the holonomy translation u. The principal $\mathbb{R}^*_+$-bundle $F^+(N)$ of positively oriented frames for N, which coincides with the Connes fibration Q since $SO(1, \mathbb{R}) = 1$, is then also trivial under the map $\phi \mapsto (\pi_Q(\phi), t \circ \phi)$:

$$Q = M \times \mathbb{R}^*_+.$$

The action of u on the fibres of Q, defined by (3.1) since $q = 1$, is induced by the same homomorphism $\Delta(u)$:

$$u \cdot (s(u), b) := (r(u), \Delta(u)b).$$

We will assume for ease of calculation that

$$Q = M \times \mathbb{R}$$

using the logarithm map on the fibres, so that the action of a groupoid element $u \in \mathcal{G}$ on Q is now given by

$$u \cdot (s(u), c) = (r(u), c + \log \Delta(u)).$$

The horizontal and vertical bundles are both trivial line bundles, so

$$N_Q = VQ \oplus H = Q \times (\mathbb{R} \oplus \mathbb{R}).$$

Here we regard the horizontal bundle $H = Q \times \mathbb{R}$ as a Euclidean bundle with metric m arising from Q defined as in Proposition 3.11 by

$$m^H_{(x,c)}(h_1, h_2) := (e^{-c}h_1) \cdot (e^{-c}h_2) = e^{-2c}h_1h_2.$$

We use the metric m^H to identify H with its dual H^* by mapping $h \in H$ to the functional $m^H(h, \cdot)$. More precisely, we identify $h \in H_{(x,c)} = \mathbb{R}$ with $\eta_h := e^{-2c}h \in H^*_{(x,c)}$. We then find that the resulting metric on H^* is

$$m^{H^*}_{(x,c)}(\eta_h, \eta_{h'}) := m^H_{(x,c)}(h, h') = e^{-2c}hh' = e^{2c}\eta_h\eta_{h'}.$$

Under this identification, the map $\theta(u) : H^*_{(s(u),c)} \to H^*_{(r(u),c+\log\Delta(u))}$ is precisely $\eta \mapsto \Delta(u^{-1})\eta$.

With no need to trace over the vertical fibres in the codimension 1 case, we can then write the triangular structure of a holonomy transformation $u \in \mathcal{G}$ as

$$u_* = \begin{pmatrix} 1 & \delta(u) \\ 0 & \Delta(u) \end{pmatrix}.$$

This action of u_* on $VQ \oplus H \subset TQ$ is the differential of the action of u on Q. It follows then that $\delta(u)$ is the derivative with respect to the transverse coordinate in M of the map $c \mapsto c + \log\Delta(u)$ on the fibres of Q. Since the normal bundle N over M has been trivialized, we can write this derivative as the scalar $\delta(u) = \partial \log\Delta(u)$, with ∂ denoting the derivative with respect to the transverse coordinate. Thus,

$$u_* = \begin{pmatrix} 1 & \partial\log\Delta(u) \\ 0 & \Delta(u) \end{pmatrix}.$$

Let us now consider the Kasparov module $[B_2]$. The right-hand algebra in this case is $\mathbb{C}\ell(V^*H^*)$, and since for each $(x, c, \eta) \in H^*$ we can identify vertical tangent vectors in $V_{(x,c,\eta)}H^*$ with vectors in $H^*_{(x,c)}$, it follows that we can identify vertical covectors in $V^*_{(x,c,\eta)}H^*$ with linear functionals $H^*_{(x,c)} \to \mathbb{R}$. Observe then that there is a nonvanishing section κ of $V^*H^* \to H^*$ defined by

$$\kappa(x, c, \eta) := e^c\eta \quad \text{for } (x, c, \eta) \in H^*.$$

One has

$$\kappa(r(u), c + \log\Delta(u), \Delta(u^{-1})\eta) = e^{c+\log\Delta(u)}\Delta(u^{-1})\eta = e^c\eta = \kappa(s(u), c, \eta),$$

so κ is invariant under the action of $\mathcal{G}$ and therefore defines a trivialization $V^*H^* \cong H^* \times \mathbb{R}$ for which the action of $\mathcal{G}$ is given by

$$u_*(s(u), c, \eta, s) = (r(u), c + \log\Delta(u), \Delta(u^{-1})\eta, s) \quad \text{for } c \in Q,\ s \in \mathbb{R},\ \eta \in H^*_{(s(u),c)}.$$

It follows that we can take $\mathbb{C}\ell(V^*H^*)$ to be $C_0(H^*) \otimes \mathrm{Cliff}(\mathbb{R})$, where $\mathcal{G}$ acts trivially on $\mathrm{Cliff}(\mathbb{R})$. That is, for all $f \otimes e \in C_0(H^*) \otimes \mathrm{Cliff}(\mathbb{R})$ we have

$$\alpha^2_u(f \otimes e)(r(u), c, \eta) = f(s(u), c - \log\Delta(u), \Delta(u)\eta + \partial\log\Delta(u)) \otimes e, \qquad \eta \in H^*_{(r(u),c)}.$$

We therefore define an action α of $\mathcal{G}$ on $C_0(H^*)$ by

$$\alpha_u(f)(r(u), c, \eta) := f(s(u), c - \log\Delta(u), \Delta(u)\eta + \partial\log\Delta(u))$$

for $f \in C_0(H^*)$, so that $\alpha^2_u(f \otimes e) = \alpha_u(f) \otimes e$ for all $u \in \mathcal{G}$ and $e \in \mathbb{C}\mathrm{liff}(\mathbb{R})$.

The same remarks carry over to the exterior bundle $\Lambda^* V^* H^*$, so that $\Gamma_0(H^*; \Lambda^*(V^*H^*) \otimes \mathbb{C})$ is just $C_0(H^*) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})$, on which the representation W^2 of $\mathcal{G}$ is defined by the same formula as α^2:

$$W^2_u(\rho \otimes e)(r(u), c, \eta) = \rho(s(u), c - \log\Delta(u), \Delta(u)\eta + \partial\log\Delta(u)) \otimes e$$

for all $\rho \otimes e \in C_0(H^*) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})$. We thus define an action W of $\mathcal{G}$ on the Hilbert $C_0(H^*)$-module $C_0(H^*)$ by

$$W_u(\rho)(r(u), c, \eta) := \rho(s(u), c - \log\Delta(u), \Delta(u)\eta + \partial\log\Delta(u))$$

for all $\rho \in C_0(H^*)$, and we see that $W^2_u(\rho \otimes e) = W_u(\rho) \otimes e$ for all $u \in \mathcal{G}$ and $e \in \mathbb{C}\mathrm{liff}(\mathbb{R})$.

Finally, the operator B_2 acts on $C_0(H^*) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})$ by

$$(B_2\rho \otimes e)(x, c, \eta) := e^c \eta\rho(x, c, \eta) \otimes c_L(e_1)e, \quad e \in \mathbb{C}\mathrm{liff}(\mathbb{R}), \; \eta \in H^*_{(x,c)},$$

where c_L is the left Clifford multiplication and e_1 is a fixed element of $\mathbb{C}\mathrm{liff}(\mathbb{R})$ with square 1. We can now proceed with the construction of a spectral triple from this data and the proof of the index theorem relating the spectral triple to the Godbillon–Vey invariant.

4B. ***The spectral triple.*** Applying the descent map to the equivariant Kasparov module $(C_0(Q), {}_{m^2}X_{E^2}, B_2)$ of Proposition 3.19 in codimension 1 gives us by Proposition 2.15 a Kasparov module

$$(\Gamma_c(Q \rtimes \mathcal{G}, \Omega^{1/2}), X_{E^2} \rtimes_r \mathcal{G}, r^* B_2) \tag{4.2}$$

which defines a class in $KK(C_0(Q) \rtimes_r \mathcal{G}, \mathbb{C}\ell(V^*H^*) \rtimes_r \mathcal{G})$. By the remarks of the previous section, we actually have

$$\mathbb{C}\ell(V^*H^*) \rtimes_r \mathcal{G} = (C_0(H^*) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})) \rtimes_r \mathcal{G} = (C_0(H^*) \rtimes_r \mathcal{G}) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})$$

since $\mathcal{G}$ acts trivially on $\mathbb{C}\mathrm{liff}(\mathbb{R})$. Thus, the module (4.2) can be replaced [Connes 1994, Proposition 13, Appendix A, Chapter 4] by the odd Kasparov $C_0(Q) \rtimes_r \mathcal{G}$-$C_0(H^*) \rtimes_r \mathcal{G}$-module

$$(\Gamma_c(Q \rtimes \mathcal{G}, \Omega^{1/2}), C_0(H^*) \rtimes_r \mathcal{G}, \mathcal{B}) \tag{4.3}$$

where we define $\mathcal{B}$ on $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2}) \subset C_0(H^*) \rtimes_r \mathcal{G}$ by

$$(\mathcal{B}\rho)_u(x, c, \eta) := (\mathcal{B}_{r(u)}\rho_u)(x, c, \eta) := e^c \eta\rho_u(x, c, \eta), \quad \eta \in \mathbb{R}.$$

Here we are using density of $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$ in $C_0(H^*) \rtimes_r \mathcal{G}$ and density of $\Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ in $C_0(Q) \rtimes_r \mathcal{G}$ as in the final paragraph of Section 2C.

The $\mathcal{G}$-invariant transverse volume forms of interest on Q and H^*, respectively, are

$$d\nu_Q = e^{-c}\omega \wedge dc, \qquad d\nu_{H^*} = \omega \wedge dc \wedge d\eta, \tag{4.4}$$

and we denote by τ_Q and τ_{H^*} the corresponding traces on $\Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ and $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$ defined by integration against $d\nu_Q$ and $d\nu_{H^*}$.

Putting the trace τ_{H^*} together with the odd Kasparov module (4.3), by Proposition 2.17 we obtain an odd semifinite spectral triple

$$(\mathcal{A}, \mathcal{H}, \mathcal{B})$$

relative to $(\mathcal{N}, \tau)$ where:

(1) $\mathcal{A} = \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ acts by convolution operators on

(2) $\mathcal{H}$, the Hilbert space completion of $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$ in the inner product

$$(\rho_1 \mid \rho_2) = \tau_{H^*}(\rho_1^* * \rho_2),$$

(3) $\mathcal{B}$ is regarded as an operator on $\mathcal{H}$ with domain $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$,

(4) $\mathcal{N}$ is the weak closure of $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$ in the bounded operators on $\mathcal{H}$,

(5) τ is the normal extension of τ_{H^*} to $\mathcal{N}$.

We now apply the semifinite local index formula to $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ to prove the codimension 1 Godbillon–Vey index theorem.

4C. *The index theorem.* We will apply the residue cocycle of [Carey et al. 2014, Definition 3.2] to prove the following theorem.

Theorem 4.5. *Let $(M, \mathcal{F})$ be a foliated manifold of codimension 1. The Chern character of the semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ given in Section 4B is, up to a factor of $(2\pi i)^{1/2}$, the global, non-étale analogue of Godbillon–Vey cyclic cocycle of Connes and Moscovici* [2005, Proposition 19].

To apply the local index formula of [Carey et al. 2014], we need to check the summability and smoothness of the spectral triple.

Lemma 4.6. *The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ is smoothly summable of spectral dimension $p = 1$ and has isolated spectral dimension.*

Proof. We first check finite summability. For $s \in \mathbb{R}$, $a \in \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$, and $\rho \in \mathcal{H}$, we calculate

$$
\begin{aligned}
(a(1+\mathcal{B}^2)^{-s/2}\rho)_u(x,c,\eta) &= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(W_v(1+\mathcal{B}^2_{s(v)})^{-s/2}\rho_{v^{-1}u})(x,c,\eta) \\
&= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(1+e^{2(c-\log\Delta(v))}(\Delta(v)\eta+\partial\log\Delta(v))^2)^{-s/2}(W_v\rho_{v^{-1}u})(x,c,\eta) \\
&= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(1+e^{2c}\Delta(v^{-1})^2(\Delta(v)\eta+\partial\log\Delta(v))^2)^{-s/2}(W_v\rho_{v^{-1}u})(x,c,\eta) \\
&= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(1+e^{2c}(\eta-\partial\log\Delta(v^{-1}))^2)^{-s/2}(W_v\rho_{v^{-1}u})(x,c,\eta),
\end{aligned}
$$

where on the last line we have used Lemma 3.16 in simplifying $\Delta(v^{-1})\partial\log\Delta(v) = -\partial\log\Delta(v^{-1})$. So $a(1+\mathcal{B}^2)^{-s/2}$ is the half-density on $H^* \rtimes \mathcal{G}$ defined by

$$
((x,c,\eta),u) \mapsto a_u(x,c)(1+e^{2c}(\eta-\partial\log\Delta(u^{-1}))^2)^{-s/2},
$$

compactly supported in the u and (x,c) variables. Thus,

$$
\begin{aligned}
\tau_{H^*}(a(1+\mathcal{B}^2)^{-s/2}) &= \int_{M\times\mathbb{R}\times\mathbb{R}} a(x,c)(1+e^{2c}\eta^2)^{-s/2}\omega\wedge dc\wedge d\eta \\
&= \int_Q a(x,c)\,d\nu_Q \int_{\mathbb{R}} (1+t^2)^{-s/2}\,dt,
\end{aligned}
$$

where we've made the substitution $t = e^c\eta$. It is then clear that $\tau_{H^*}(a(1+\mathcal{B}^2)^{-s/2})$ is finite for all $s > 1$. For smoothness, we fix $a \in \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ and calculate

$$
\begin{aligned}
([\mathcal{B}^2, a]\rho)_u(x,c,\eta) &= e^{2c}\eta^2 \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(W_v\rho_{v^{-1}u})(x,c,\eta) \\
&\qquad - \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(W_v\mathcal{B}^2_{s(v)}\rho_{v^{-1}u})(x,c,\eta) \\
&= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^{2c}(\eta^2-\Delta(v^{-1})^2(\Delta(v)\eta+\partial\log\Delta(v))^2)(W_v\rho_{v^{-1}u})(x,c,\eta) \\
&= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^{2c}(2\eta\partial\log\Delta(v^{-1}) - (\partial\log\Delta(v^{-1}))^2)(W_v\rho_{v^{-1}u})(x,c,\eta)
\end{aligned}
$$

so that $[\mathcal{B}^2, a]$ is convolution by the half-density on $H^* \rtimes \mathcal{G}$ defined by

$$
((x,c,\eta),u) \mapsto a_u(x,c)e^{2c}(2\eta\partial\log\Delta(u^{-1}) - (\partial\log\Delta(u^{-1}))^2).
$$

We also calculate

$$
\begin{aligned}
([\mathcal{B}^2,[\mathcal{B},a]]\rho)_u(x,c,\eta) &= e^{2c}\eta^2([\mathcal{B},a]\rho)_u(x,c,\eta)-([\mathcal{B},a]\mathcal{B}^2\rho)_u(x,c,\eta)\\
&= e^{2c}\eta^2\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^c\partial\log\Delta(v^{-1})(W_v\rho_{v^{-1}u})(x,c,\eta)\\
&\qquad -\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^c\partial\log\Delta(v^{-1})(W_v\mathcal{B}^2_{s(v)}\rho_{v^{-1}u})(x,c,\eta)\\
&= e^{2c}\eta^2\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^c\partial\log\Delta(v^{-1})(W_v\rho_{v^{-1}u})(x,c,\eta)\\
&\qquad -\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^{3c}\partial\log\Delta(v^{-1})\Delta(v^{-1})^2(\Delta(v)\eta+\partial\log\Delta(v))^2\\
&\qquad\qquad\qquad\times(W_v\rho_{v^{-1}u})(x,c,\eta)\\
&= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^{3c}(2\eta\partial\log\Delta(v^{-1})-(\partial\log\Delta(v^{-1}))^2)\\
&\qquad\qquad\qquad\times\partial\log\Delta(v^{-1})(W_v\rho_{v^{-1}u})(x,c,\eta),
\end{aligned}
$$

so that $[\mathcal{B}^2,[\mathcal{B},a]]$ is the half-density on $H^*\rtimes\mathcal{G}$ defined by

$$((x,c,\eta),u)\mapsto a_u(x,c)e^{3c}(2\eta\partial\log\Delta(u^{-1})-(\partial\log\Delta(u^{-1}))^2)\partial\log\Delta(u^{-1}).$$

More generally, setting $T^{(0)}:=T$ and then inductively defining $T^{(k)}:=[\mathcal{B}^2,T^{(k-1)}]$, we see that $[\mathcal{B},a]^{(k)}$ is the half-density on $H^*\rtimes\mathcal{G}$ defined by

$$((x,c,\eta),u)\mapsto a_u(x,c)e^{(2k+1)c}(2\eta\partial\log\Delta(u^{-1})-(\partial\log\Delta(u^{-1}))^2)^k\partial\log\Delta(u^{-1}).$$

Now these computations show that for $a\in\mathcal{A}$ and $k\in\mathbb{N}$, the operators $a^{(k)}$ and $[\mathcal{B},a]^{(k)}$ are half-densities on $H^*\rtimes\mathcal{G}$, with compact support in the $((x,c),u)\in Q\rtimes\mathcal{G}$ variables equal to that of a, and growing like η^k in the fibre variable $\eta\in H^*_{(x,c)}$ for all $(x,c)\in Q$. Hence, both $a^{(k)}(1+\mathcal{B}^2)^{-k/2}$ and $[\mathcal{B},a]^{(k)}(1+\mathcal{B}^2)^{-k/2}$ are bounded with compact support in the $Q\rtimes\mathcal{G}$ directions. Hence, for all $a\in\mathcal{A}$ the operator

$$(1+\mathcal{B}^2)^{-k/2-s/4}(a^{(k)})^*a^{(k)}(1+\mathcal{B}^2)^{-k/2-s/4}$$

is trace class whenever the real part of s is greater than 1, and similarly with a replaced by $[\mathcal{B},a]$. Thus, $\mathcal{A}\cup[\mathcal{B},\mathcal{A}]\subset B_2^\infty(\mathcal{B},1)$ in the notation of [Carey et al. 2014]. Thus, $\mathcal{A}^2$, the span of products from $\mathcal{A}$, satisfies $\mathcal{A}^2\cup[\mathcal{B},\mathcal{A}^2]\subset B_1^\infty(\mathcal{B},1)$, showing that the semifinite spectral triple over $\mathcal{A}^2$ is smoothly summable.

The last step to establish smooth summability is to observe that $\mathcal{A}$ has a (left) approximate unit for the inductive limit topology by [Muhly and Williams 2008, Proposition 6.8]. This ensures that any compactly supported section in $\mathcal{A}=\Gamma_c(Q\rtimes\mathcal{G};\Omega^{1/2})$ can be approximated by products while preserving summability.

Finally the computations also show that $(\mathcal{A},\mathcal{H},\mathcal{B})$ has isolated spectral dimension, as in [Carey et al. 2014, Definition 3.1], since for all multi-indices k of length

$m \geq 0$ we have proved that

$$\tau_{H^*}(a_0[\mathcal{B}, a_1]^{(k_1)} \cdots [\mathcal{B}, a_m]^{(k_m)}(1+\mathcal{B}^2)^{-|k|-m/2-s})$$

has a meromorphic continuation in a neighborhood of $s = 0$. □

Finally we can prove Theorem 4.5.

Proof of Theorem 4.5. Since the spectral dimension $p = 1$ and since the parity of the spectral triple is 1, the only nonzero term in the residue cocycle is ϕ_1 as defined in [Carey et al. 2014, Definition 3.2]. For any $a \in \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ we have

$$\begin{aligned}
([\mathcal{B}, a]\rho)_u(x, c, \eta) &= \mathcal{B}_{r(u)} \int_{v \in \mathcal{G}^{r(u)}} a_v(x, c)(W_v \rho_{v^{-1}u})(x, c, \eta) \\
&\qquad - \int_{v \in \mathcal{G}^{r(u)}} a_v(x, c)(W_v \mathcal{B}_{s(v)} \rho_{v^{-1}u})(x, c, \eta) \\
&= \int_{v \in \mathcal{G}^{r(u)}} a_v(x, c)(\mathcal{B}_{r(v)} - W_v \mathcal{B}_{s(v)} W_{v^{-1}})(W_v \rho_{v^{-1}u})(x, c, \eta) \\
&= \int_{v \in \mathcal{G}^{r(u)}} a_v(x, c) e^c \partial \log \Delta(v^{-1})(W_v \rho_{v^{-1}u})(x, c, \eta) \\
&= (\delta_1(a)\rho)_u(x, c, \eta),
\end{aligned}$$

where δ_1 is the derivation of $\Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ defined by

$$\delta_1(a)_u(x, c) := e^c \partial \log \Delta(u^{-1}) a_u(x, c).$$

The derivation δ_1 is the non-étale analogue of that given in [Connes and Moscovici 2005, p. 39]. Thus, for $a_0, a_1 \in \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$, we calculate

$$\begin{aligned}
\phi_1(a_0, a_1) &= 2(2\pi i)^{1/2} \operatorname{res}_{z=0} \tau_{H^*}(a_0[\mathcal{B}, a_1](1+\mathcal{B}^2)^{-1/2-z}) \\
&= 2(2\pi i)^{1/2} \tau_Q(a_0 \delta_1(a_1)) \operatorname{res}_{z=0} \int_{\mathbb{R}} (1+t^2)^{-1/2-z}\, dh \\
&= 2(2\pi i)^{1/2} \tau_Q(a_0 \delta_1(a_1)) \operatorname{res}_{z=0} \frac{\Gamma\big(\frac{1}{2}\big)\Gamma(z)}{2\Gamma\big(\frac{1}{2}+z\big)} \\
&= (2\pi i)^{1/2} \tau_Q(a_0 \delta_1(a_1)).
\end{aligned}$$

This is, up to the factor $(2\pi i)^{1/2}$, the non-étale analogue of the Godbillon–Vey cyclic cocycle from [Connes and Moscovici 2005, Proposition 19]. □

5. Relation with Connes' approach

We will outline in this section how our construction can, in codimension 1, be reconciled with Connes' approach to realize Gelfand–Fuks cocycles as cyclic cocycles for a convolution algebra. In doing so we will be able to justify why our spectral triple represents the Godbillon–Vey invariant. Let us first briefly recall Connes' approach [1986, Theorem 7.15].

5A. ***Connes' approach.*** Connes considers a discrete group Γ of orientation preserving diffeomorphisms of an oriented manifold V of dimension n. Associated to V and any $k \in \mathbb{N} \cup \{0\}$ is its *positively oriented k-th order jet bundle* $J_k^+(V)$, whose fibre over $x \in V$ consists of equivalence classes of local diffeomorphisms $\varphi : U \to V$, where U is an open neighborhood of 0 in $\mathbb{R}^n$, for which $\varphi(0) = x$. Two such diffeomorphisms φ and ψ are said to have the same *k-jet at* 0, denoted $j_0^k(\varphi) = j_0^k(\psi)$, if in any local coordinate system about x all partial derivatives of φ and ψ of order less than or equal to k coincide at 0. All the bundles $J_k^+(V)$, for $k \geq 1$, carry canonical right actions of $GL^+(n, \mathbb{R})$, and we write $\underline{J}_k^+(V) := J_k^+(V)/SO(n, \mathbb{R})$. The $\underline{J}_k^+(V)$ have contractible fibres [Bott 1976, p. 132].

It is then well-known [Bott 1976, (3.2)] that associated to any ($SO(n, \mathbb{R})$-relative) Gelfand–Fuks cocycle ϖ is $k \in \mathbb{N}$ and a canonical, diffeomorphism-invariant differential form, also denoted ϖ, on the quotient $\underline{J}_k^+(V)$. Connes uses the exterior derivative on V to manufacture ϖ into a cyclic cocycle ϖ_c for the algebra $C_c^\infty(\underline{J}_k^+(V) \rtimes \Gamma)$. Letting W denote the bundle of metrics over V, Connes invokes [Kasparov 1995] to deduce the existence of a class $j_{1,k} \in KK^\Gamma(C_0(W), C_0(\underline{J}_k^+(V)))$, whose descent $j_{1,k}^\Gamma \in KK(C_0(W \rtimes \Gamma), C_0(\underline{J}_k^+(V) \rtimes \Gamma))$ implements an isomorphism on K-theory via the Kasparov product. Letting $j_{0,1}^\Gamma \in KK(C_0(V \rtimes \Gamma), C_0(W \rtimes \Gamma))$ denote the bundle of metrics Kasparov module constructed in [Connes 1986, §5], Connes obtains a linear map

$$K_*(C_0(V) \rtimes \Gamma) \ni a \mapsto \varpi_c(a \otimes_{C_0(V)\rtimes\Gamma} j_{0,1}^\Gamma \otimes_{C_0(W)\rtimes\Gamma} j_{1,k}^\Gamma) \in \mathbb{R}$$

defined by ϖ. In the case when $V = S^1$, one has $J_2^+(S^1) = S^1 \times \mathbb{R}_+^* \times \mathbb{R}$, and if dx is the standard volume form on S^1, then the Godbillon–Vey invariant is represented by the invariant differential form [Connes 1986, Lemma 7.7]

$$\varpi = \frac{1}{y^3} dx \wedge dy \wedge dy_1, \quad (x, y, y_1) \in S^1 \times \mathbb{R}_+^* \times \mathbb{R}. \tag{5.1}$$

The associated cyclic cocycle ϖ_c is the trace on $C_c^\infty(J_2^+(S^1)) \rtimes \Gamma$ obtained by integration with respect to ϖ, and an involved calculation [Connes 1986, Theorem 7.3] shows that the linear map thus obtained on $K_0(C_0(V) \rtimes \Gamma)$ is the cyclic cocycle given by (1.2). Alternatively, one can obtain by essentially the same method a map

$$K_1(C_0(W) \rtimes \Gamma) \ni a \mapsto \varpi_c(a \otimes_{C_0(W)\rtimes\Gamma} j_{1,2}^\Gamma) \in \mathbb{R} \tag{5.2}$$

on the K-theory of $C_0(W) \rtimes \Gamma$. In the next subsection we will indicate how the index pairing induced by the spectral triple in Theorem 4.5 can be thought of as a non-étale version of the map in (5.2).

5B. ***The case of a general foliation.*** In the setting of a transversely orientable foliated manifold $(M, \mathscr{F})$ of codimension q, one has access to the *transverse jet bundles* $J_k^+(\mathscr{F})$ [Morita 2001, pp. 113–114]. The fibre $J_k^+(\mathscr{F})_x$ over $x \in M$ now consists of the k-jets $j_0^k(\varphi)$ of orientation-preserving local diffeomorphisms φ sending an open neighborhood of $0 \in \mathbb{R}^q$ to a local *transversal* about $x = \varphi(0)$. Note in particular that $J_1^+(\mathscr{F})$ is the same thing as the oriented transverse frame bundle $F^+(N)$.

We have natural projections $\pi_k : J_k^+(\mathscr{F}) \to M$ and $\pi_{k+1,k} : J_{k+1}^+(\mathscr{F}) \to J_k^+(\mathscr{F})$ defined respectively by forgetting all partial derivatives, and by forgetting all partial derivatives of order $k+1$. Moreover the holonomy groupoid $\mathscr{G}$ acts naturally on each $J_k^+(\mathscr{F})$, and its orbits define a foliation $\mathscr{F}_k$ of $J_k^+(\mathscr{F})$. There is a canonical right action of $GL^+(q, \mathbb{R})$ on all of the $J_k^+(\mathscr{F})$, $k \geq 1$, which commutes with the action of $\mathscr{G}$.

Proposition 5.3. *Let $S^2(\mathbb{R}^q)$ denote the symmetric polynomials of degree* 2 *over the vector space $\mathbb{R}^q$. Then the bundle $J_2^+(\mathscr{F})$ is an affine bundle over $J_1^+(\mathscr{F})$, modeled on the vector bundle $\pi_1^*(N) \otimes S^2(\mathbb{R}^q)$. Moreover, if $(M, \mathscr{F})$ is of codimension* 1, *a choice of Bott connection $\nabla^\flat$ on N determines an affine isomorphism of $J_2^+(\mathscr{F})$ with the total space of the bundle $H^* = (\ker(\alpha^\flat / T\mathscr{F}_1))^*$ over $J_1^+(\mathscr{F})$.*

Proof. That $J_2^+(\mathscr{F})$ is an affine bundle over $J_1^+(\mathscr{F})$ modeled on $\pi_1^*(N) \otimes S^2(\mathbb{R}^q)$ can be seen using local coordinates. Indeed, if $U \subset \mathbb{R}^q$ is an open neighborhood of 0 and T_β is a local transversal in M, then the 2-jet at 0 of any diffeomorphism $\varphi : U \to T$ is distinguished from its 1-jet at 0 by the partial derivatives

$$\frac{\partial^2 \varphi^i}{\partial y^j \partial y^k}\bigg|_0, \quad i, j, k = 1, \ldots, q,$$

which are elements of $N_x \otimes S^2(\mathbb{R}^q)$ (here we have identified $\mathbb{R}^q$ with its dual in the natural way). The chain rule implies that whenever $c_{\alpha\beta} : T_\beta \to T_\alpha$ is the diffeomorphism defined by a transverse change of coordinates, then the corresponding coordinate change in the fibre $J_2^+(\mathscr{F})_{j_0^1(\varphi)}$ is given by the affine transformation

$$\left(\frac{\partial^2 \varphi^i}{\partial y^j \partial y^k}\bigg|_0\right)_{i,j,k=1,\ldots,q} \mapsto \left(\frac{\partial c^i_{\alpha\beta}}{\partial y^l}\bigg|_{\varphi(0)} \frac{\partial^2 \varphi^l}{\partial y^j \partial y^k}\bigg|_0 + \frac{\partial^2 c^i_{\alpha\beta}}{\partial y^j \partial y^k}\bigg|_{\varphi(0)}\right)_{i,j,k=1,\ldots,q},$$

where we have used the Einstein summation convention.

Suppose now that ∇ is a torsion-free connection on N. Then on any sufficiently small transversal $\mathscr{Y}_x$ about $x \in M$, ∇ restricts to a torsion-free affine connection and is therefore associated with an exponential map $\exp_x^\nabla$ which sends an open neighborhood of $0 \in N_x = T_x\mathscr{Y}_x$ diffeomorphically onto $\mathscr{Y}_x$. We therefore obtain a section $\sigma_\nabla : J_1^+(\mathscr{F}) \to J_2^+(\mathscr{F})$ defined by

$$\sigma_\nabla(j_0^1(\varphi)) := j_0^2(\exp_x^\nabla \circ j_0^1(\varphi)), \tag{5.4}$$

where we consider $j^1_0(\varphi)$ as a frame $\mathbb{R}^q \to N_x$, which by the chain rule is equivariant under the right action of $GL^+(q,\mathbb{R})$. Therefore, ∇ determines an affine isomorphism of $J^+_2(\mathcal{F})$ with the vector bundle $\pi_1^*(N) \otimes S^2(\mathbb{R}^q)$ on which it is modeled. In particular, if $(M,\mathcal{F})$ is codimension 1, ∇ determines an affine isomorphism of $J^+_2(\mathcal{F})$ with $\pi_1^*(N)$. Therefore, if $\nabla = \nabla^\flat$ is a Bott connection, we attain the stated isomorphism of $J^+_2(\mathcal{F})$ with $H \cong \pi_1^*(N)$, and hence with $H^* \cong H$ via the tautological Euclidean structure of Proposition 3.11. □

Now any Gelfand–Fuks class ϖ admits a canonical representative in the $\mathcal{G}$-invariant forms on $\underline{J}^+_k(\mathcal{F})$ for sufficiently large k [Morita 2001, pp. 117–119]. Let us briefly describe this canonical representative in the case of the Godbillon–Vey invariant of a codimension-1 foliation. Let $\varphi_t : U \to \mathcal{Y}$ be a 1-parameter family of diffeomorphisms of some open neighborhood U of $0 \in \mathbb{R}$ onto a local transversal $\mathcal{Y}$ in M, determining tangent vectors

$$X_k := \frac{d}{dt}\bigg|_{t=0} j^{k+1}_0(\varphi_t)$$

in $J^+_{k+1}(\mathcal{F})$, for all $k \in \mathbb{N} \cup \{0\}$. Let $\varphi := \varphi_0$, and define a 1-parameter family of coordinates $x_t := \varphi^{-1} \circ \varphi_t$. Then we define the *$k$-th tautological 1-form ω^k* on $J^+_{k+1}(\mathcal{F})$ by the formula

$$\omega^k(X_k) = \frac{d}{dt}\bigg|_{t=0} \left(\frac{d^k x_t}{dy^k}\bigg|_{y=0} \right) \tag{5.5}$$

for all $k \in \mathbb{N} \cup \{0\}$. Such tautological forms were introduced (in the nonfoliated case) by Kobayashi [1961]. Notice that if $u \in \mathcal{G}$ is represented by a diffeomorphism $h : \mathcal{Y} \to \mathcal{Y}'$ of local transversals, then

$$(h^*\omega^k)(X_k) = \omega^k \left(\frac{d}{dt}\bigg|_{t=0} j^{k+1}_0(h\circ\varphi_t) \right) = \frac{d}{dt}\bigg|_{t=0} \left(\frac{d^k(\varphi^{-1}\circ h^{-1}\circ h\circ\varphi_t)}{dy^k} \right) = \omega^k(X_k),$$

so the ω^k are $\mathcal{G}$-invariant. Pulling back via the projections, let us assume that ω^0, ω^1, and ω^2 are all defined on $J^+_3(\mathcal{F})$. Then, using the $\mathcal{G}$ invariance of these forms together with [Morita 2001, Proposition 3.23], it can be shown that the ω^i satisfy the structure equations

$$d\omega^0 = -\omega^1 \wedge \omega^0, \qquad d\omega^1 = -\omega^2 \wedge \omega^0. \tag{5.6}$$

Now by (5.5) the form ω^0 on $J^+_1(\mathcal{F})$ is, by definition, simply the solder form on the transverse frame bundle. Therefore, the first of the equations (5.6) says that ω^1 on $J^+_2(\mathcal{F})$ behaves like a torsion-free connection form: indeed, the pullback $\sigma^*_\nabla\omega^1$ of ω^1 by the section $\sigma_\nabla : J^+_1(\mathcal{F}) \to J^+_2(\mathcal{F})$ defined by any torsion-free connection ∇ on N is precisely the connection form corresponding to ∇. Therefore, the second of the equations (5.6) is a formula for the "curvature" of the "connection form" ω^1.

In particular, the Godbillon–Vey form on $J_3^+(\mathcal{F})$ is simply the (negative of) the "connection" wedged with its "curvature" [Bott 1972, Lemma 10.9]:

$$\mathrm{gv} = -\omega^1 \wedge d\omega^1 = \omega^0 \wedge \omega^1 \wedge \omega^2. \tag{5.7}$$

It is by no means obvious that the form in (5.7) is related to the form $d\nu_{H^*}$ on H^* considered in (4.4). The next result tells us that in fact these forms are the same, and therefore justifies our claim that the cocycle obtained as the index formula from Theorem 4.5 truly does represent the Godbillon–Vey invariant.

Theorem 5.8. *Let $(M, \mathcal{F})$ be a transversely oriented foliation of codimension 1, with transverse volume form $\omega \in \Omega^1(M)$. Suppose moreover we are gifted with a torsion-free Bott connection on N, giving an identification $J_2^+(\mathcal{F}) \cong H^*$ as in Proposition 5.3. Then the form* gv *of* (5.7) *on $J_3^+(\mathcal{F})$ descends to a form on $J_2^+(\mathcal{F}) \cong H^*$, which, in the trivialization $H^* = M \times \mathbb{R} \times \mathbb{R}$ determined by ω as in Section 4A, coincides with the form $d\nu_{H^*} = \omega \wedge dc \wedge d\eta$ of* (4.4).

Proof. Associated to the transverse volume form ω is a nonvanishing normal vector field $Z \in \Gamma^\infty(M; N)$ characterized by the equation $\omega(Z) \equiv 1$. Fix $x \in M$ and let $\mathcal{Y}_x$ be a local transversal through x. Then the torsion-free Bott connection $\nabla^\flat$ on N restricts to an affine connection on $\mathcal{Y}_x$, and so determines an exponential map $\exp^{\nabla^\flat} : U \to \mathcal{Y}_x$ which is a local diffeomorphism defined on an open neighborhood U of $0 \in T_x\mathcal{Y}_x$. Rescaling ω if necessary, we can always assume that $Z_x \in U$ and we obtain a coordinate $u_0 : \mathcal{Y}_x \to \mathbb{R}$ defined by the equation

$$u_0(x')Z_x = (\exp^{\nabla^\flat})^{-1}(x'), \quad x' \in \mathcal{Y}_x.$$

Now fix a local diffeomorphism φ from an open neighborhood of $0 \in \mathbb{R}$ to $\mathcal{Y}_x$. The coordinate u_0 on $\mathcal{Y}_x$ identifies φ with a local diffeomorphism $\tilde{\varphi} := u_0 \circ \varphi$ of $\mathbb{R}$, so the 3-jet $j_0^3(\varphi)$ is determined by the polynomial

$$\tilde{\varphi}(0) + \frac{d\tilde{\varphi}}{dy}\bigg|_0 y + \frac{d^2\tilde{\varphi}}{dy^2}\bigg|_0 y^2 + \frac{d^3\tilde{\varphi}}{dy^3}\bigg|_0 y^3$$

where we use y to denote the standard coordinate in $\mathbb{R}$. We thus define coordinates $u_i(j_0^3(\varphi)) := \frac{d^i\tilde{\varphi}}{dy^i}\big|_0$ for $i = 1, 2, 3$ for $j_0^3(\varphi) \in J_3^+(\mathcal{Y}_x)$. Suppose now that φ_t is a 1-parameter family of local diffeomorphisms from an open neighborhood of $0 \in \mathbb{R}$ to $\mathcal{Y}_x$, with $\varphi_0 = \varphi$. The coordinate representation $\tilde{\varphi}_t := u_0 \circ \varphi_t$ of the family φ_t determines a curve

$$j_0^3(\tilde{\varphi}_t) = \left(\tilde{\varphi}_t(0), \frac{d\tilde{\varphi}_t}{dx}\bigg|_0, \frac{d^2\tilde{\varphi}_t}{dx^2}\bigg|_0 \frac{d^3\tilde{\varphi}_t}{dx^3}\bigg|_0\right)$$

in $J_3^+(\mathbb{R})$; hence, we can write the tangent vector $X = \frac{d}{dt}\big|_0 j_0^3(\varphi_t)$ on $J_3^+(\mathcal{Y}_x)$ determined by the curve $j_0^3(\varphi_t)$ in the form

$$X = \frac{d\tilde{\varphi}_t}{dt}\bigg|_0 \frac{\partial}{\partial u^0} + \frac{d}{dt}\bigg|_0 \left(\frac{d\tilde{\varphi}_t}{dy}\bigg|_0\right)\frac{\partial}{\partial u^1} + \frac{d}{dt}\bigg|_0 \left(\frac{d^2\tilde{\varphi}_t}{dy^2}\bigg|_0\right)\frac{\partial}{\partial u^2} + \frac{d}{dt}\bigg|_0 \left(\frac{d^3\tilde{\varphi}_t}{dy^3}\bigg|_0\right)\frac{\partial}{\partial u^3}.$$

Setting $h_t := \varphi^{-1} \circ \varphi_t$ we have $\tilde{\varphi}_t = \tilde{\varphi} \circ h_t$. Then using the chain rule together with the fact that $h_0 = \mathrm{id}_{\mathbb{R}}$, we compute

$$\begin{aligned}
\frac{d\tilde{\varphi}_t}{dt}\bigg|_0 &= \frac{d}{dt}\bigg|_0 (\tilde{\varphi} \circ h_t) = u_1 \frac{dh_t}{dt}\bigg|_0,\\
\frac{d}{dt}\bigg|_0 \left(\frac{d\tilde{\varphi}_t}{dy}\bigg|_0\right) &= \frac{d}{dt}\bigg|_0 \left(\frac{d(\tilde{\varphi} \circ h_t)}{dy}\bigg|_0\right) = u_2 \frac{dh_t}{dt}\bigg|_0 + u_1 \frac{d}{dt}\bigg|_0 \left(\frac{dh_t}{dy}\bigg|_0\right),\\
\frac{d}{dt}\bigg|_0 \left(\frac{d^2\tilde{\varphi}_t}{dy^2}\bigg|_0\right) &= \frac{d}{dt}\bigg|_0 \left(\frac{d^2(\tilde{\varphi} \circ h_t)}{dy^2}\bigg|_0\right)\\
&= u_3 \frac{dh_t}{dt}\bigg|_0 + 2u_2 \frac{d}{dt}\bigg|_0 \left(\frac{dh_t}{dy}\bigg|_0\right) + u_1 \frac{d}{dt}\bigg|_0 \left(\frac{d^2h_t}{dy^2}\bigg|_0\right).
\end{aligned}$$

Therefore, by (5.5) we find that[1]

$$du_0 = u_1\omega^0, \qquad du_1 = u_2\omega^0 + u_1\omega^1, \qquad du_2 = u_3\omega^0 + 2u_2\omega^1 + u_1\omega^2, \tag{5.9}$$

and we deduce that

$$\omega^0 \wedge \omega^1 \wedge \omega^2 = \frac{1}{u_1^3} du_0 \wedge du_1 \wedge du_2, \tag{5.10}$$

which is a well-defined form on $J_2^+(\mathcal{Y}_x)$.

Now we come to transporting the form of (5.10) on $J_2^+(\mathcal{Y}_x) \subset J_2^+(\mathcal{F})$ to the total space of the bundle H^* over $J_1^+(\mathcal{F})$ as in Proposition 5.3. The transverse volume form $\omega \in \Omega^1(M)$ determines a trivialization

$$J_1^+(\mathcal{F}) \ni \phi_x \mapsto (x, \omega_x(\phi_x(1))) =: (x, t) \in M \times \mathbb{R}_+^*,$$

where we think of $\phi_x = d\varphi_0$ as a frame $\mathbb{R} \to N_x$. The transverse vector field $Z \in \Gamma^\infty(M; N)$ corresponding to ω determines a trivialization

$$N \ni hZ_x \mapsto (x, h) \in M \times \mathbb{R}, \quad x \in M,$$

[1] The formulae in (5.9) that we compute here differ slightly from the analogous equations of Kobayashi [1961, §4] and Connes and Moscovici [2005, p. 45], for whom the summands containing ω^0 in the second and third equations have factors of 2 and 3, respectively. The reader can easily verify using elementary calculus that our own computations do *not* give rise to these factors. In the absence of any explicit computations provided by Kobayashi and Connes–Moscovici, it is difficult to determine why these additional factors appear in their equations. In any case, these additional factors have no impact on the coordinate expression we obtain for the Godbillon–Vey differential form.

of N and therefore a corresponding trivialization $H \cong J_1^+(\mathscr{F}) \times \mathbb{R} \cong M \times \mathbb{R}_+^* \times \mathbb{R}$ of $H \cong \pi_1^* N$. Unlike the coordinates u_i used for the transversal $\mathscr{Y}_x$ in the first part of the proof, the trivialization $H \cong M \times \mathbb{R}_+^* \times \mathbb{R}$ is global, and we must show that on $\mathscr{Y}_x$, we have equalities $du_0 = \omega$, $u_1 = t$, and $u_2 = h$.

Now $du_0(Z) \equiv 1$ by definition of the coordinate u_0 on $\mathscr{Y}_x$, so $du_0 = \omega$. For u_1 we see that

$$u_1 = \frac{d(u_0 \circ \varphi)}{dy} = du_0 \circ d\varphi(1) = \omega_x(d\varphi(1)) = t$$

by definition of the variable t. Finally, in the trivial bundle $J_2^+(\mathbb{R}) = \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}$ that is the image of $J_2^+(\mathscr{Y}_x)$ under the coordinates (u_0, u_1, u_2), the u_2 variable identifies with the tangent variable for $\mathbb{R}$ in the manner of Proposition 5.3. Viewed as coordinates on $J_2^+(\mathscr{Y}_x)$ and $T\mathscr{Y}_x$, respectively, we then have $u_2 = h$ and therefore

$$\mathrm{gv} = \frac{1}{u_1^3}\, du_0 \wedge du_1 \wedge du_2 = -\frac{1}{t^3}\, \omega \wedge dt \wedge dh.$$

Finally, as in Section 4A, we make the substitutions $t = e^c$ giving $dt = e^c dc$, and $h = e^{2c}\eta$ giving $dh = e^{2c} d\eta$. Thus, on H^*, we find that

$$\mathrm{gv} = \omega \wedge dc \wedge d\eta = d\nu_{H^*}$$

as claimed. □

6. Concluding remarks

It is tempting to view the higher-codimension version of the codimension-1 Kasparov module and spectral triple as analogous data representing the Godbillon–Vey invariant in higher codimension. Sadly, despite the naturality of the constructions presented here, it is far from clear that such an interpretation is warranted. Without an identification of the Chern character of these spectral triples with the Godbillon–Vey class, they must remain an interesting construction.

One final remark on the constructions presented here: they all pass to real algebras and real KK-theory. All our constructions are Real [Kasparov 1980] for the obvious variations of complex conjugation, in part because of our systematic use of the exterior algebra rather than the spinor bundle. This means that we can at all stages retain contact with homology of manifolds with real coefficients.

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