A GENERALIZATION FOR THE EXPECTED VALUE OF THE EARTH MOVER’S DISTANCE

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The earth mover’s distance (EMD), also called the first Wasserstein distance, can be naturally extended to compare arbitrarily many probability distributions, rather than only two, on the set \([n] = \{1, \ldots, n\}\). We present the details for this generalization, along with a highly efficient algorithm inspired by combinatorics; it turns out that in the special case of three distributions, the EMD is half the sum of the pairwise EMDs. Extending the methods of Bourn and Willenbring (2020), we compute the expected value of this generalized EMD on random tuples of distributions, using a generating function which coincides with the Hilbert series of the Segre embedding. We then use the EMD to analyze a real-world data set of grade distributions.

1. Introduction

We generalize a result appearing in [5], in which the authors compute the expected value of the earth mover’s distance (EMD) between two probability distributions by means of a generating function. The EMD can be viewed as the solution to a problem in transport theory, first considered in [20] by French geometer Gaspard Monge in 1781. (Although the term “earth mover” seems to have been coined only in the 1990s, it is pointed out in Villani’s monumental reference [23] that the title of Monge’s original treatise translates, more or less, as “On the theory of material extracted from the earth and input to a new construction.” Monge, then, truly was the original earth mover.) Nearly 200 years later, in [15], Monge’s name was given to a critical property of certain cost arrays for which his problem can be solved by a greedy algorithm. Throughout the late 1980s and 1990s, in [2] and [4], this Monge property was generalized to higher-dimensional arrays. (See also [18] for a more recent treatment.) It is an essential fact in this paper (whose proof is reserved for Section 8 at the end) that the multidimensional cost array associated with our generalized EMD has this Monge property.

Section 2, written for those readers unfamiliar with the EMD, presents a simple example and points out all the relevant details which will reappear in our generalization.

We begin in Section 3 by defining an earth mover’s “distance” EMD\(_d\) between \(d\) distributions; the classical EMD treated in [5] coincides with EMD\(_2\). We actually find that on three distributions, EMD\(_3\) equals half the sum of the three EMD\(_2\) values, although no such relationship holds for \(d > 3\).

In Section 4, en route to constructing a generating function, we define a discrete version of EMD\(_d\) which compares histograms instead of probability distributions, and we describe an efficient computational method using a generalization of the RSK correspondence from combinatorics.

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In Section 5, we encode the values of the discrete $EMD_d$ in a generating function, which we manipulate in order to extract the expected value. Translating this discrete result back into the continuous setting, we prove the main theorem of this paper (Theorem 7), which is a recursive formula to compute the expected value of $EMD_d$. We then apply our theory in Section 6 to analyze a real-world data set of grade distributions.

Finally, in Section 7, we interpret our generating function in terms of the Segre embedding in algebraic geometry. This leads us to realize the $EMD_2$ situation as an infinite-dimensional representation of the Lie algebra $\mathfrak{u}(n, n)$, whose action corresponds to manipulating the two distributions compared by our EMD.

Since the appearance of [5], the problem of finding the expected value of $EMD_2$ has been solved from an analytical approach in [11]. The setup has also been specialized in [19] to a data set of distributions with a fixed average value.

We believe that the result in this paper — a method to evaluate the “closeness” of arbitrarily many distributions — has great potential as a tool in data analysis. In evaluating teaching and assessment practices at the university level, for instance, we can now assign a single value to an entire course by evaluating the EMD between the individual sections, and then track the behavior of that course’s EMD for different groups of instructors, different course coordinators, fall vs. spring semesters, and other variables. We can even assign EMD values to individual exams and other assessments using the grade distributions in various sections; or in the other direction, we can compare different courses to each other, both within and outside a given department. In all of these settings, we believe that the generalized EMD can contribute to an interesting cluster analysis of the kind proposed in [5].

2. EMD between 2 distributions: summary and an example

For readers unfamiliar with the classical EMD, we summarize the idea here. Consider two probability distributions on the finite set of integers $[n] = \{1, \ldots, n\}$. (More vividly, in place of a “probability distribution,” imagine $n$ bins of earth whose combined mass is one unit, located at $1, \ldots, n$ on the number line.) Intuitively, the EMD between the two distributions measures the “cheapest” way to move earth between the bins so as to equalize the distributions, where the “cost” of moving one unit of earth is the distance of the move. For example, the cost of moving 0.25 units of earth from bin 2 to bin 5 is $0.25 \cdot (5 - 2) = 0.75$. To make this precise, we define the cost function $C : [n] \times [n] \rightarrow \mathbb{Z}_{\geq 0}$, where $C(i, j)$ is the cost of moving one unit of earth from bin $i$ to bin $j$. In this case, clearly $C(i, j) = |i - j|$. Any solution which equalizes the two distributions — whether or not it is the optimal solution — can be encoded in an $n \times n$ matrix $J$. Necessarily, the row sums of $J$ will correspond to one distribution, and the column sums to the other distribution, so the entries of $J$ must sum to 1. To find the total cost of earth moved, we simply multiply each matrix entry $J_{ij}$ by its distance $|i - j|$ from the main diagonal, and sum over all entries.

As a trivial example, in the case of two equal distributions, we can take $J$ to be the matrix whose diagonal entries are the values of the distributions, with zeros elsewhere. Since all nonzero entries of $J$ have coordinates of the form $(i, i)$, they contribute zero cost, and so the total cost is likewise zero, just as we would expect for two identical distributions.
To interpret this process of multiplying entries by their distance from the diagonal, our next example describes exactly how the entries of $J$ give (possibly ambiguous, but equivalent) step-by-step instructions to equalize the two distributions. This treatment is admittedly overkill in the case of two distributions, but it will provide the best intuition when we generalize to $d$ distributions in the next section. The less-than-rigorous descriptions below will be formalized in the next section in terms of the taxicab metric.

**Example.** Consider the two distributions $\mu_1 = (0.3, 0.3, 0.4)$ and $\mu_2 = (0.1, 0, 0.9)$. Hence $n = 3$. Then one matrix (among infinitely many) with the prescribed row and column sums is

$$J = \begin{bmatrix}
0.1 & 0 & 0.2 \\
0 & 0 & 0.3 \\
0 & 0 & 0.4
\end{bmatrix}.$$ 

The nonzero entries of $J$ correspond to moving earth as follows:

- $J_{1,1} = 0.1$. Note that the coordinates $(1, 1)$ are already equal to each other, so we do not have to move the 0.1 units of earth at all.
- $J_{1,3} = 0.2$. Now the coordinates $(1, 3)$ are not equal; in order to make them equal with as little cost as possible, we have three valid options, all of which have cost 2:
  - In the first coordinate, we could add 2 to make the change $1 \rightarrow 3$. This corresponds to moving the 0.2 units of earth in $\mu_1$, from bin 1 to bin 3.
  - In the second coordinate, we could subtract 2 to make the change $3 \rightarrow 1$. This corresponds to moving the 0.2 units of earth in $\mu_2$, from bin 3 to bin 1.
  - We could add 1 to the first coordinate ($1 \rightarrow 2$) and subtract 1 from the second coordinate ($3 \rightarrow 2$). This corresponds to moving 0.2 units of earth in $\mu_1$ from bin 1 to bin 2, and then moving 0.2 units of earth in $\mu_2$ from bin 3 to bin 2.
- $J_{2,3} = 0.3$. The cheapest ways to equalize the coordinates $(2, 3)$ are the following two options, each with cost 1:
  - In the first coordinate, we could add 1 to make the change $2 \rightarrow 3$. This corresponds to moving the 0.3 units of earth in $\mu_1$, from bin 2 to bin 3.
  - In the second coordinate, we could subtract 1 to make the change $3 \rightarrow 2$. This corresponds to moving the 0.3 units of earth in $\mu_2$, from bin 3 to bin 2.
- $J_{3,3} = 0.4$. Since the coordinates $(3, 3)$ are already equal, we do not have to move the 0.4 units of earth at all.

Now, depending upon which of the above options we choose at each step, this process can result in any of six distinct pairs of final distributions $\mu'_1$ and $\mu'_2$. But within each possible pair, as the reader can check, we always finish with $\mu'_1 = \mu'_2$, as desired. Furthermore, the total cost of all the earth moved is independent of our choices, since all options above minimized the cost at each step. (Also note that the
cost at each step was always equal to \(|i - j|\), coinciding with the cost function \(C\) we defined earlier.) In this case, the total cost of the earth moved was

\[
0.1(0) + 0.2(2) + 0.3(1) + 0.4(0) = 0.7.
\]

The EMD between \(\mu_1\) and \(\mu_2\) is, by definition, the infimum (actually the minimum) of the set of total costs, taken over all possible matrices \(J\) with the prescribed row and column sums. In this example, although not obvious at first glance, 0.7 is in fact the least possible cost, and so \(EMD(\mu_1, \mu_2) = 0.7\). This turns out to be a consequence of the fact that the support of \(J\) lies in a chain: in other words, if we put the product order \(\leq\) on \([n] \times [n]\), we see that

\[
(1, 1) \leq (1, 3) \leq (2, 3) \leq (3, 3);
\]

this pairwise comparability is what we mean by a chain in \([n] \times [n]\). This fact — that support in a chain implies minimality — is equivalent, on a deeper level (see [15]), to the fact that our cost function \(C\), if considered as an \(n \times n\) array, has the “Monge property” alluded to in the introduction; in this case, the greedy algorithm to solve the earth mover’s problem (known as the “northwest corner rule”; see [4]) eliminates one row or column at each step, meaning the support of the solution matrix \(J\) is always a chain.

There is one phenomenon here in the \(d = 2\) case which will not generalize to \(d > 2\): in the above example, we could have removed any ambiguity by deciding that we would move earth within \(\mu_1\) exclusively, so that both final distributions would equal \(\mu_2\). Therefore, we could interpret the problem as finding the cheapest way to transport material from a “source” or “supply vector” \((\mu_1)\) to a “sink” or “demand vector” \((\mu_2)\). For \(d > 2\), however, the optimal solution at each step may require moving earth in any or all of the distributions, and so we lose the binary supply-demand interpretation of the problem.

Having presented the big picture, without details, in the \(d = 2\) case, we now proceed to build up the general case for arbitrary \(d\). Throughout the next section, the reader can verify that the definitions and results coincide with those found in this simple example where \(d = 2\).

3. Extending EMD to \(d\) distributions

3.1. Definitions and notation. Let \(\mathcal{P}_n\) denote the set of probability distributions on \([n]\). Assume the uniform probability measure on the \(d\)-fold product \(\mathcal{P}_n \times \cdots \times \mathcal{P}_n\), defined by its embedding into \(\mathbb{R}^{dn}\). Our goal is to compare \(d\) elements of \(\mathcal{P}_n\), written as the \(d\)-tuple \(\mu := (\mu_1, \ldots, \mu_d)\). We should keep in mind that each \(\mu_i\) is itself an \(n\)-tuple whose components sum to 1. Throughout this paper, we write the sum of a vector’s components using absolute value bars, so in this case, \(|\mu_i| = 1\). We will denote the \(k\)-th component of \(\mu_i\) by \(\mu_i(k)\), which is just the value of the distribution \(\mu_i\) at \(k \in [n]\). To each \(\mu\) there corresponds the set \(\mathcal{J}_\mu\) of joint distribution arrays, defined as follows.

For an array \(J\), we will write \(J(m_1, \ldots, m_d)\) for the entry at position \((m_1, \ldots, m_d)\). Now, we define \(\mathcal{J}_\mu\) as the set containing all those arrays \(J \in \mathbb{R}_{\geq 0}^{n \times \cdots \times n}\) whose sums within the coordinate hyperplanes coincide with \(\mu\). Specifically, fixing \(m_i = k\), we must have

\[
\sum_{m_1, \ldots, \hat{m}_i, \ldots, m_d = 1}^{n} J(m_1, \ldots, k_{m_i}, \ldots, m_d) = \mu_i(k).
\]
Given some $\mu = (\mu_1, \mu_2, \mu_3)$, every array in $J_\mu$ satisfies the above relations, where each arrow represents the sum of the entries in the designated plane.

In other words, summing all the entries whose positions in the array have $k$ as their $i$-th coordinate, we obtain the $k$-th component of $\mu_i$. In the familiar $d = 2$ case, $i = 1$ gives us the row sums, and $i = 2$ the column sums. For $d = 3$, see Figure 1 for an illustration.

Any array $J \in J_\mu$ can be regarded as a solution to the earth mover’s problem for $n$ bins, determined by the distributions in $\mu$. This means we need a $d$-dimensional analog of the “cost” function from Section 2, and the natural candidate arises from the taxicab metric on $[n]^d$. Specifically, at each position in a $d$-dimensional array, we want the cost to be the taxicab distance to the main diagonal, i.e., to the nearest position in the array whose coordinates are all equal. (This “equality of coordinates” property of the main diagonal, as we recall from Section 2, corresponded to zero earth being moved.) Roughly speaking, this cost is the fewest number of $\pm 1$ we need to add in order to equalize all the coordinates. For example, the most efficient way to equalize the coordinates of the position $(5, 4, 5, 5, 5, 7, 5)$ is to add 1 to the 4, and then to subtract 2 from the 7, for a total cost of 3. This is precisely the taxicab distance to the main diagonal, specifically to the position $(5, 5, 5, 5, 5, 5, 5)$. Just as in the example from the previous section, this distance-finding exercise corresponds to moving earth:

- When we added 1 to the 2nd coordinate to make the change $4 \to 5$, we moved a unit of earth in the 2nd distribution $\mu_2$ from bin 4 to bin 5.
- When we subtracted 2 from the 6th coordinate to make the change $7 \to 5$, we moved a unit of earth in the 6th distribution $\mu_6$ from bin 7 to bin 5.

**Example.** Consider the three distributions

$$
\mu_1 = (0.5, 0.1, 0.4), \quad \mu_2 = (0.5, 0.2, 0.3), \quad \mu_3 = (0.7, 0.2, 0.1).
$$

Then one array in $J_\mu$ is, for instance,

$$
J = \begin{bmatrix}
[.5, 0, 0] & [0, 0, 0] & [0, 0, 0] \\
[0, 0, 0] & [1, 0, 0] & [0, 0, 0] \\
[0, 0, 0] & [.1, 0, 0] & [0, .2, .1] \\
\end{bmatrix},
$$

(2)
flattened so that the first coordinate specifies the row, the second coordinate specifies one of the three main columns, and the third coordinate specifies the position inside the triple at that position. The nonzero entries are

\[ J(1, 1, 1) = 0.5, \quad J(2, 2, 1) = 0.1, \quad J(3, 2, 1) = 0.1, \quad J(3, 3, 2) = 0.2, \quad J(3, 3, 3) = 0.1. \]

This information tells us how to arrive at the solution corresponding to \( J \):

- The cost of \((1, 1, 1)\) is 0 since it is already on the main diagonal, so we do not move the 0.5 at all.
- The cost of \((2, 2, 1)\) is 1, since in the 3rd coordinate we must make the change \(1 \rightarrow 2\). This means that in the 3rd distribution \(\mu_3\), we move 0.1 from bin 1 to bin 2. Currently \(\mu'_3 = (0.6, 0.3, 0.1)\).
- The cost of \((3, 2, 1)\) is 2, since we equalize the coordinates most efficiently by subtracting 1 from the 1st coordinate \((3 \rightarrow 2)\) and adding 1 to the 3rd coordinate \((1 \rightarrow 2)\). Hence, we move 0.1 from bin 3 to bin 2 in \(\mu_1\), and from bin 1 to bin 2 in \(\mu_3\). Now \(\mu'_1 = (0.5, 0.2, 0.3)\) and \(\mu'_3 = (0.5, 0.4, 0.1)\).
- The cost of \((3, 3, 2)\) is 1, by adding 1 to the 3rd coordinate. This corresponds to moving 0.2 from bin 2 to bin 3 in \(\mu_3\). Now \(\mu'_3 = (0.5, 0.2, 0.3)\).
- The cost of \((3, 3, 3)\) is 0, so we do not move the 0.1 anywhere.

Note that our final result is that all three distributions are the same, as desired:

\[ \mu'_1 = \mu'_2 = \mu'_3 = (0.5, 0.2, 0.3). \]

Also note that we rigged this example, unlike that in Section 2, so that none of the steps would present more than one optimal option, although in general there certainly might exist several different solutions for the same array \( J \). But of course in each case the total cost is the same.

The natural computation now is to find that total cost, by multiplying the amount of earth moved at each step by the number of bins it was moved; in other words, multiply each entry in \( J \) by the cost of its position, then add these products together:

\[ 0.5(0) + 0.1(1) + 0.1(2) + 0.2(1) + 0.1(0) = 0.5. \]

This completes the example.

Of course, a priori there is no reason why the particular array \( J \) in the example should give the least costly way to equalize the three distributions. When we finally define our generalized EMD, it will be defined as the least possible cost for any \( J \in \mathcal{J}_\mu \). First, however, we should record a formula for the cost of an array position, to improve upon the somewhat sloppy method by inspection we have used so far.

The formula for the \( d \)-dimensional taxicab distance from a point to a line is derived in [7]. In our case, the line of interest is the main diagonal, which passes through \((1, \ldots, 1)\) in the direction \((1, \ldots, 1)\). This distance, and therefore our cost function \( C \), turns out to be

\[ C(m_1, \ldots, m_d) = \min_{i \in \{d\}} \left\{ \sum_{j \neq i} |m_i - m_j| \right\}. \quad (3) \]
This cost function $C$ can also naturally be thought of as an $n \times \cdots \times n$ array, so we will occasionally refer to the “cost array” in this paper.

There is also a more direct way to compute $C$, which will be convenient later. Let $\mathbf{m} := (m_1, \ldots, m_d)$, and let $\tilde{\mathbf{m}}$ denote the vector whose components are those of $\mathbf{m}$ rearranged in ascending order; e.g., if $\mathbf{m} = (7, 4, 5, 3, 1)$, then $\tilde{\mathbf{m}} = (1, 3, 4, 5, 7)$.

**Proposition 1.** Equation (3) can be computed as $C(\mathbf{m}) = \sum_{i=1}^{\lfloor d/2 \rfloor} \tilde{m}_{d-i+1} - \tilde{m}_i$.

As an example before the proof, take $\mathbf{m} = (7, 4, 5, 3, 1)$ as above. Then by the proposition, to compute $C(\mathbf{m})$, we instead look at $\tilde{\mathbf{m}}$ and sum up the pairwise differences working outside-in:

$$
\begin{align*}
\tilde{m} = (1, 3, 4, 5, 7), \\
\text{therefore } C(\mathbf{m}) &= 6 + 2 \\
&= 8.
\end{align*}
$$

**Proof.** For fixed $i \in [d]$, let $k$ be such that $\tilde{m}_k = m_i$. Then we have

$$
\sum_{j \neq i} |m_i - m_j| = (\tilde{m}_2 - \tilde{m}_1) + 2(\tilde{m}_3 - \tilde{m}_2) + 3(\tilde{m}_4 - \tilde{m}_3) + \cdots + (k-1)(\tilde{m}_k - \tilde{m}_{k-1})
$$

$$
+ (\tilde{m}_d - \tilde{m}_{d-1}) + 2(\tilde{m}_{d-1} - \tilde{m}_{d-2}) + 3(\tilde{m}_{d-2} - \tilde{m}_{d-3}) + \cdots + (d-k)(\tilde{m}_{k+1} - \tilde{m}_k),
$$

which is minimized when $k = \lfloor (d+1)/2 \rfloor$. Making this evaluation in the displayed sum, we find that the sum telescopes: when $d$ is even, we obtain

$$
-\tilde{m}_1 - \tilde{m}_2 - \cdots - \tilde{m}_{\lfloor (d+1)/2 \rfloor} + \tilde{m}_{\lfloor (d+1)/2 \rfloor + 1} + \cdots + \tilde{m}_d,
$$

and when $d$ is odd, we obtain

$$
-\tilde{m}_1 - \tilde{m}_2 - \cdots - \tilde{m}_{\lfloor (d+1)/2 \rfloor - 1} + \tilde{m}_{\lfloor (d+1)/2 \rfloor + 1} + \cdots + \tilde{m}_d.
$$

In either case, this simplifies as

$$
C(\mathbf{m}) = \sum_{i=1}^{\lfloor d/2 \rfloor} \tilde{m}_{d-i+1} - \tilde{m}_i. \quad \Box
$$

**Remark.** A recent paper [18] defines a different cost function than ours for the earth mover’s problem, namely $C'(\mathbf{m}) := \max\{m_i\} - \min\{m_i\}$. We can see from Proposition 1 that $C'$ agrees with our $C$ for $d = 2$ and $d = 3$, but not for $d > 3$. For example, letting $\mathbf{m} = (1, 1, 2, 2)$, we have $C(\mathbf{m}) = 2$ but $C'(\mathbf{m}) = 1$. For our purposes, we have chosen our $C$ because it counts every earth–movement required to equalize the distributions. For example, keeping $\mathbf{m} = (1, 1, 2, 2)$, consider the distributions $\mu_1 = \mu_2 = (1, 0)$ and $\mu_3 = \mu_4 = (0, 1)$. Then one solution is given by the array whose only nonzero entry is a 1 at position $\mathbf{m}$. Intuitively, we want the EMD of these four distributions to be 2, not 1, since we must first move a unit of earth by 1 bin, and then move another unit by 1 bin.

Having built up the necessary intuition and formulas, we are finally ready to make our main definition:
Definition. Let $\mu$ be a $d$-tuple of probability distributions, as above. Then the generalized earth mover’s distance is defined as

$$EMD_d(\mu) := \min_{J \in \mathcal{J}_\mu} \sum_{m \in [n]^d} C(m) J(m).$$

(4)

Remark. For $d > 2$, the term “coefficient” would perhaps be more fitting than “distance,” but in this paper we retain “distance” as a nod to tradition.

3.2. Existence of a greedy algorithm. As mentioned in the first two sections, finding the right-hand side of (4) is equivalent to finding the optimal solution to a $d$-dimensional transport problem. It is shown in [4] that there exists a greedy algorithm to find this solution in $O(d^2n)$ time, precisely when the cost array $C$ has the Monge property mentioned in the introduction:

Definition. A $d$-dimensional array $A$ has the Monge property if, for all $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$, we have

$$A(\min\{x_1, y_1\}, \ldots, \min\{x_d, y_d\}) + A(\max\{x_1, y_1\}, \ldots, \max\{x_d, y_d\}) \leq A(x) + A(y).$$

Remark. If we regard an array as a function on a lattice, then the Monge property is equivalent to submodularity. Characterized by their “diminishing returns” property, submodular functions have found a vast expanse of modern applications, especially in machine learning (see [3]).

We now state the crucial proposition, whose proof we will give in Section 8.

Proposition 2. The cost array $C$ defined in (3) has the Monge property.

This proposition, then, guarantees the existence of a greedy algorithm to compute $EMD_d$. (This justifies our writing “min” instead of “inf” in our definition; we also could have used a compactness argument as in [5].) The greedy algorithm described in [4] is a generalization of the two-dimensional “northwest corner rule.” Just as in the $d = 2$ case (see Section 2), for generic $d$ this algorithm arrives at its solution in the form of an array $J \in \mathcal{J}_\mu$ whose support is a chain, i.e., pairwise comparable under the product order on $[n]^d$. (In [5, Proposition 4], the “straightening” procedure that converts the support of any $J$ into a chain, without increasing the total cost, is valid precisely because the cost array $C(i, j) = |i - j|$ has the Monge property.) Rather than describe this greedy algorithm, which is already well-known (see [4] or [18]), our goal is instead to find the expected value of $EMD_d$. To this end, the importance of the algorithm is the following:

Corollary 3. The minimum in (4) occurs for some $J \in \mathcal{J}_\mu$ whose support is a chain in $[n]^d$.

Since there is nothing special about the condition $|\mu_i| = 1$ from the perspective of transport problems, Corollary 3 also holds in a discrete setting using integer compositions in place of probability distributions. We will take this discrete approach in the next section, where we use a highly efficient combinatorial method to find the optimal array $J$ for any $\mu$. 


4. A discrete approach

We follow the method from [5], with a view toward constructing a generating function in the next section. In place of $P_n$, we temporarily turn our attention to $C(s, n)$, the set of (weak) integer compositions of some positive integer $s$ into $n$ parts. That is, elements of $C(s, n)$ are $n$-tuples of nonnegative integers whose sum is $s$ (whereas before, the elements of $P_n$ were $n$-tuples of nonnegative real numbers whose sum was 1); we can also think of compositions as histograms. The cost function $C$, however, remains the same as before, since it still describes distances among the $n$ bins in each of the $d$ compositions.

In this section, $\mu = (\mu_1, \ldots, \mu_d)$ denotes a sequence of compositions $\mu_i \in C(s, n)$. It is tempting simply to adjust the definition of $J_\mu$ so that arrays in the set must have nonnegative integer entries summing to $s$; then we could just reuse the definition (4) to obtain a definition for the discrete EMD$_d$. Although this is one viable approach, nevertheless, in light of Corollary 3, we need only consider those arrays whose support is a chain; therefore we will work with the following set of arrays from this point forward:

$$J^s_{(n^d)} := \left\{ J \in (\mathbb{Z}_{\geq 0})^{n \times \cdots \times n} \mid \sum_{m \in [n]^d} J(m) = s, \text{ and the support of } J \text{ is a chain in } [n]^d \right\}. $$

We will now show that each $d$-tuple of compositions $\mu \in C(s, n) \times \cdots \times C(s, n)$ corresponds to a unique array $J_\mu \in J^s_{(n^d)}$. Therefore, by the end of the next subsection, we will have a direct computational definition for the discrete version of the EMD, which avoids taking the minimum over a set of arrays as we must in the definition (4) of the continuous EMD. Once we have this definition for the discrete EMD, we will be able to recover the continuous version by scaling all the $\mu_i$ by $1/s$. Near the end of the paper, we will do exactly this, and then let $s \to \infty$, in order to translate discrete results back into the continuous setting.

4.1. Generalized RSK correspondence. The authors of [5] use the Robinson–Schensted–Knuth correspondence to great effect in order to determine a unique optimal matrix $J_{(\mu_1, \mu_2)}$ for an ordered pair of compositions $(\mu_1, \mu_2)$. We now apply this same idea to $d$ compositions in order to uniquely determine an optimal $d$-dimensional array. This will allow us to calculate the discrete EMD$_d$ directly (and even more efficiently, in many cases, than by using the greedy algorithm mentioned above).

For nonexperts, we summarize here a special case of the correspondence. (For full details, see Chapter 4 of [12].) The Robinson–Schensted–Knuth (RSK) correspondence furnishes a bijection:

$$\left\{ \text{ordered pairs of semistandard Young tableaux of the same shape, with entries in } [n] \right\} \leftrightarrow \left\{ n \times n \text{ matrices with nonnegative integer entries} \right\}. $$

For our purposes, we will restrict our attention to the special case of one-row tableaux, since any composition in $C(s, n)$ corresponds uniquely to a one-row tableau containing $s$ boxes with entries from $[n]$. As an example, consider the two compositions,

$$\mu_1 = (1, 2, 3, 4), \quad \mu_2 = (5, 0, 2, 3),$$
in $C(10, 4)$. We associate to each composition $\mu_i$ a one-row tableau $T(\mu_i)$, which we fill so that the entry $k$ appears $\mu_i(k)$ times:

$$T(\mu_1) = \begin{array}{cccccc}
1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4 & 4
\end{array}, \quad T(\mu_2) = \begin{array}{cccccc}
1 & 1 & 1 & 1 & 3 & 3 & 4 & 4 & 4 & 4
\end{array}.$$

Regarding these tableaux as the two rows of a $2 \times s$ matrix, we then have

$$M(\mu_1, \mu_2) = \begin{bmatrix}
1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4 & 4 \\
1 & 1 & 1 & 1 & 3 & 3 & 4 & 4 & 4 & 4
\end{bmatrix}.$$ 

Finally, we fill in an $n \times n$ array $J(\mu_1, \mu_2)$ whose $(i, j)$ entry equals the number of times the column $\begin{bmatrix} i \\ j \end{bmatrix}$ appears in $M(\mu_1, \mu_2)$. For example, $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ appears three times, so we write a 3 in position $(4, 4)$. Filling in the rest of the array, we obtain the correspondence

$$(\mu_1, \mu_2) \leftrightarrow (T(\mu_1), T(\mu_2)) \leftrightarrow M(\mu_1, \mu_2) \leftrightarrow J(\mu_1, \mu_2) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 \\
0 & 0 & 1 & 3
\end{bmatrix}.$$ 

Note that we can also reverse the procedure, starting with the array $J(\mu_1, \mu_2)$, translating its entries into a two-row matrix, and finally recovering the original pair of tableaux (and hence the pair of compositions). Therefore this is indeed a bijection. In the context of the EMD, the array $J(\mu_1, \mu_2)$ has two significant properties:

- The row and column sums coincide with the original compositions $\mu_1$ and $\mu_2$, so $J(\mu_1, \mu_2)$ is a solution to the discrete earth mover’s problem for $\mu_1$ and $\mu_2$.
- Since both rows of $M(\mu_1, \mu_2)$ are nondecreasing, the support of $J(\mu_1, \mu_2)$ is a chain in $[n] \times [n]$.

In summary, we have the following bijective correspondence in the case $d = 2$:

$$C(s, n) \times C(s, n) \leftrightarrow J^s_{(n^2)}.$$ 

This RSK correspondence extends naturally to $d$-tuples of compositions in $C(s, n)$. (For experts, details about the existence of this multivariate RSK generalization can be found in [6].) Given $\mu = (\mu_1, \ldots, \mu_d)$, the tableaux corresponding to the $\mu_i$ uniquely determine a $d \times s$ matrix $M_\mu$, which in turn determines a unique $n \times \cdots \times n$ array $J_\mu \in J^s_{(n^d)}$. This correspondence is again bijective, establishing the following special case of the generalized RSK correspondence:

$$C(s, n) \times \cdots \times C(s, n) \leftrightarrow J^s_{(n^d)}, \quad \mu \mapsto J_\mu.$$ 

This correspondence leads us to the following definition of the discrete EMD:

**Definition.** For positive integers $d$, $n$, and $s$, with $d \geq 2$, let $\mu = (\mu_1, \ldots, \mu_d)$ with each $\mu_i \in C(s, n)$. Let $J_\mu$ be the unique array corresponding to $\mu$, as in (5). Let $C$ be the cost function on $[n]^d$ as in (3). Then we define the *discrete generalized earth mover’s distance* to be

$$\text{EMD}_d^s(\mu) := \sum_{m \in [n]^d} C(m) J_\mu(m).$$
where we write the superscript $s$ to distinguish this discrete version from the continuous version. 

This definition in (6) is largely conceptual; in practice, we can calculate \( \text{EMD}^s_d(\mu) \) directly from the matrix \( M_\mu \), since the support of \( J_\mu \) is determined by the columns of \( M_\mu \):

**Theorem 4.** Let \( \mu = (\mu_1, \ldots, \mu_d) \) with each \( \mu_i \in C(s, n) \), and let \( C \) be the cost function in (3). Let \( M_\mu \) be the unique \( d \times s \) array corresponding to \( \mu \) via the generalized RSK correspondence, as described above, and let \( M_\mu(\bullet, j) \) denote the \( j \)-th column vector in \( M_\mu \). Then

\[
\text{EMD}^s_d(\mu) = \sum_{j=1}^s C(M_\mu(\bullet, j)).
\]

**Proof.** Consider the definition of \( \text{EMD}^s_d \) in (6). By definition, \( J_\mu(m) \) equals the number of occurrences of \( m \) as a column vector of the matrix \( M_\mu \). Therefore \( J_\mu(m) = 0 \) unless \( m \) is one of those column vectors, and so we can simply sum over the \( s \) column vectors to obtain the result. \( \square \)

**Remark.** This construction via RSK is more efficient than the aforementioned greedy algorithm for computing \( \text{EMD}^s_d \) when \( d \) or \( n \) is sufficiently large: rather than filling a \( d \)-dimensional array in \( O(d^2 n) \) time, we need to consider only the \( s \) column vectors of a \( d \times s \) matrix.

**Example.** As an example for \( d = 3 \), consider the three compositions

\[
\mu_1 = (4, 0, 1), \quad \mu_2 = (1, 2, 2), \quad \mu_3 = (0, 5, 0)
\]

in \( C(5, 3) \). These correspond to the tableaux \([1 1 1 1 3], [1 2 2 3 3], \text{ and } [2 2 2 2 2]\), respectively. Stacking these tableaux vertically gives us the matrix

\[
M_\mu = \begin{bmatrix}
1 & 1 & 1 & 1 & 3 \\
1 & 2 & 2 & 3 & 3 \\
2 & 2 & 2 & 2 & 2
\end{bmatrix}.
\]

Now using **Theorem 4** on the five columns of \( M_\mu \), we compute that

\[
\text{EMD}^5_3(\mu) = C(1, 1, 2) + C(1, 2, 2) + C(1, 2, 2) + C(1, 3, 2) + C(3, 3, 2)
\]

\[
= 1 + 1 + 1 + 2 + 1
\]

\[
= 6.
\]

Before advancing to the main problem of the paper, we show that \( \text{EMD}_3 \) can actually be expressed in terms of the classical \( \text{EMD}_2 \). (In the following proposition, we suppress the superscript \( s \) because the result holds for both the discrete and the continuous version of \( \text{EMD} \): the equality is independent of \( s \), and therefore still holds after dividing both sides by \( s \) and letting \( s \to \infty \).)

**Proposition 5.** The value of \( \text{EMD}_3 \) is half the sum of the three pairwise \( \text{EMD}_2 \) values:

\[
\text{EMD}_3(\mu_1, \mu_2, \mu_3) = \frac{1}{2}(\text{EMD}_2(\mu_1, \mu_2) + \text{EMD}_2(\mu_1, \mu_3) + \text{EMD}_2(\mu_2, \mu_3))
\]
Proof. Let \( \mu = (\mu_1, \mu_2, \mu_3) \) as usual. In each column \( j \) of the matrix \( M_\mu \), call the three entries \( a_j, b_j, c_j \), labeled so that \( a_j \leq b_j \leq c_j \). Each of the three pairs \( (a_j, b_j) \), \( (a_j, c_j) \), and \( (b_j, c_j) \) corresponds naturally to one of the pairs \( (\mu_1, \mu_2) \), \( (\mu_1, \mu_3) \), and \( (\mu_2, \mu_3) \), where row \( i \) corresponds to \( \mu_i \). Therefore by Theorem 4, we have

\[
\text{EMD}_2(\mu_1, \mu_2) + \text{EMD}_2(\mu_1, \mu_3) + \text{EMD}_2(\mu_2, \mu_3) = \sum_{j=1}^{s} C(a_j, b_j) + C(a_j, c_j) + C(b_j, c_j)
= \sum_{j} (b_j - a_j) + (c_j - a_j) + (c_j - b_j)
= \sum_{j} 2c_j - 2a_j
= 2 \sum_{j} c_j - a_j
= 2 \sum_{j} C(a_j, b_j, c_j)
= 2 \cdot \text{EMD}_3(\mu_1, \mu_2, \mu_3).
\]

This relationship does not generalize to \( d > 3 \), because in general, the telescoping summand in the proof does not reduce in terms of a higher-dimensional cost function. For example, when \( d = 4 \), the analog of the third line above is \( \sum j 3d_j + c_j - b_j - 3a_j \), or \( \sum j C(a_j, b_j, c_j, d_j) + 2(d_j - a_j) \).

5. Expected value of \( \text{EMD}_d \)

Again we will follow and extend the methods used in [5] for arbitrary values of \( d \). First we define a generating function, in which the exponents encode the values of \( \text{EMD}_d^s \), and which we will then differentiate in order to sum up all of these values. This will allow us to compute expected value for \( \text{EMD}_d^s \) by reading off the coefficients from this derivative.

A generating function is a formal power series whose terms encode combinatorial information, without any regard to convergence. As a famous example, the Fibonacci numbers \( F_i \) may be encoded in a formal power series \( \sum_{i=0}^{\infty} F_i x^i = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots \), whose generating function has the closed form \( x/(1-x-x^2) \). The generating function we will define, however, will contain two variables (one to keep track of composition size \( s \), and one to keep track of \( \text{EMD}_d^s \) values), and unlike the example above, we will have to settle for a recursive definition, because writing down the closed form quickly becomes intractable even for \( n > 3 \).

5.1. Generating function for the discrete setting. Because we are about to make a recursive definition, we must momentarily consider \( d \)-tuples \( \mu \) consisting of compositions with different numbers of bins — i.e., different values \( n_i \) such that each \( \mu_i \in C(s, n_i) \). Therefore, \( n = (n_1, \ldots, n_d) \) will denote this vector of bin numbers. Note that \( \text{EMD}_d^s \) is still defined on \( C(s, n_1) \times \cdots \times C(s, n_d) \), since we can append zeros to the end of each \( n_i \)-tuple as needed; in this way, as in [5], we regard the compositions as sharing a common bin number (the maximum of the \( n_i \)). We do not, however, consider any statistical meaning...
behind the EMD on these padded compositions; in this paper, they are a formal means to an end, and our ultimate result (Theorem 7) once again assumes equal bin numbers.

In order to encode an inclusion-exclusion argument, we define an indicator vector \( e(A) \) for a subset \( A \subseteq [d] \). Namely,

\[
e(A) := \sum_{i \in A} e_i
\]

is the vector whose \( i \)-th component is 1 if \( i \in A \) and 0 otherwise. For example, if \( d = 5 \), and \( A = \{2, 4, 5\} \), then \( e(A) = (0, 1, 0, 1, 1) \).

For fixed \( n = (n_1, \ldots, n_d) \), we define a generating function in two indeterminates \( z \) and \( t \):

\[
H_n(z, t) := \sum_{s=0}^{\infty} \left( \sum_{\mu \in \mathcal{C}(s, n_1) \times \cdots \times \mathcal{C}(s, n_d)} z^{\text{EMD}_d^s(\mu)} \right) t^s.
\]

(7)

The coefficient of \( z^r t^s \) is the number of elements \( \mu \in \mathcal{C}(s, n_1) \times \cdots \times \mathcal{C}(s, n_d) \) such that \( \text{EMD}_d^s(\mu) = r \). A recursive definition of this generating function, for the \( d = 2 \) case, is derived in [5, Theorem 3]. Our generalization for \( d > 2 \) follows:

**Proposition 6.** Fix \( n = (n_1, \ldots, n_d) \). The generating function \( H_n := H_n(z, t) \) has the following recursive definition, where the sum is over all nonempty subsets \( A \subseteq [d] \):

\[
H_n = \frac{\sum_{A} (-1)^{|A|} H_{n-e(A)}}{1 - z^{C(n)} t},
\]

where \( H_{(1^d)} = 1/(1-t) \), and \( H_{n-e(A)} = 0 \) if \( n - e(A) \) contains \( a \) 0.

We reserve the proof for Section 9.

**Example.** We write out this recursive definition in a concrete case, where \( d = 3 \) and \( n = (5, 2, 2) \). It is easiest to arrange the terms of the numerator according to the size of the subset \( A \). First, for \( |A| = 1 \), we add together all possible \( H_n' \), where \( n' \) equals \( n \) with exactly 1 coordinate decreased; then for \( |A| = 2 \), we subtract all possible \( H_n'' \) where \( n'' \) equals \( n \) with exactly 2 coordinates decreased; finally, for \( |A| = 3 \), we add the one possible \( H_n''' \) where \( n''' \) equals \( n \) with all 3 coordinates decreased. As for the denominator, \( C \) is the same cost function we defined in (3), meaning that \( C(5, 2, 2) = 5 - 2 = 3 \). Then the recursion for \( H_n \) looks like this:

\[
H_{(5,2,2)} = \frac{H_{(4,2,2)} + H_{(5,1,2)} + H_{(5,2,1)} - H_{(4,1,2)} - H_{(4,2,1)} - H_{(5,1,1)} + H_{(4,1,1)}}{1 - z^3 t}.
\]

Having seen an example, we now study the important (and very well-studied) specialization that results from setting \( z = 1 \). In this case, the coefficient of \( t^s \) in \( H_n(1, t) \) is the total number of \( d \)-tuples \( \mu \), which is \( \prod_{i=1}^{d} |\mathcal{C}(s, n_i)| = \prod_{i=1}^{d} (s+n_i-1) / (n_i-1) \):

\[
H_n(1, t) = \sum_{s=0}^{\infty} \prod_{i=1}^{d} \left( \frac{s+n_i-1}{n_i-1} \right) t^s.
\]

(8)
It is shown in [9] that the closed form of (8), after adjusting the index to match our setup, and writing $|n| := n_1 + \cdots + n_d$, is

$$H_n(1, t) = \frac{W_n(t)}{(1-t)^{|n|-d+1}},$$

(9)

where the numerator $W_n(t)$ is a polynomial whose coefficients are the “Simon Newcomb” numbers. (For more on this natural generalization of Eulerian numbers to multisets, see [1; 9; 21].) Specifically, denoting the coefficient of $t^i$ in $W_n$ by the symbol $[t^i]W_n$, we have

$$[t^i]W_n = \# \text{ permutations of the multiset } \{1^{n_1-1}, \ldots, d^{n_d-1}\} \text{ containing } i \text{ descents}.$$  

It follows that the evaluation $W_n(1)$ is equal to the total number of permutations of the multiset $\{1^{n_1-1}, \ldots, d^{n_d-1}\}$, a fact we will need later:

$$W_n(1) = \frac{(\sum_{i=1}^d (n_i - 1))!}{\prod_{i=1}^d (n_i - 1)!} = \frac{(|n|-d)!}{\prod(n_i - 1)!}. \tag{10}$$

5.2. A partial derivative. Next, in order to transfer the EMD values from the exponents of $z$ into coefficients, we take the partial derivative of $H_n$ with respect to $z$. Applying the quotient rule to our definition of $H_n$ in Proposition 6, we obtain the following, where the sum still ranges over nonempty subsets $A \subseteq [d]$:

$$\frac{\partial H_n}{\partial z} = \frac{(1-z^{C(n)t}) \left( \sum_A (-1)^{|A|-1} \cdot \frac{\partial H_{n-e(A)}}{\partial z} \right) + C(n) \cdot z^{C(n)-1} \cdot t \cdot \left( \sum_A (-1)^{|A|-1} \cdot H_{n-e(A)} \right)}{(1-z^{C(n)t})^2}.$$  

Now that the exponents have been changed into coefficients of $z$, we can set $z = 1$:

$$H'_n := \frac{\partial H_n}{\partial z} \bigg|_{z=1} = \sum_{s=0}^{\infty} \left( \sum_{\mu \in C(s,n_1) \times \cdots \times C(s,n_d)} \text{EMD}_{d}^{s}(\mu) \right) t^s 

\left( 1-t \right) \left( \sum_A (-1)^{|A|-1} \cdot H'_{n-e(A)} \right) + t \cdot C(n) \left( \sum_A (-1)^{|A|-1} \cdot H_{n-e(A)} \right) 

\frac{1}{(1-t)^2} \tag{11}$$

At this point, $z$ has played out its role, and so from now on we will write $H_n$ in place of $H_n(1, t)$.

Note that the coefficient of $t^s$ in $H'_n$ is the sum of the values EMD$^s_{d}(\mu)$ for all valid $d$-tuples $\mu$. This means that our goal is now in sight: to find the expected value of EMD$^s_{d}$ for fixed $n$, we need to divide the sum of all possible EMD$^s_{d}$ values (i.e., the coefficient of $t^s$ in $H'_n$) by the total number of possible inputs $\mu$ (i.e., the coefficient of $t^s$ in $H_n$). Therefore, once we find a way to simplify (11), we will be able to compute the result

$$E(\text{EMD}^s_{d}) = \frac{[t^s]H'_n}{[t^s]H_n} = \frac{[t^s]H'_n}{\prod_{i=1}^d \left( s+n_i-1 \right)} \cdot \frac{1}{\prod_{i=1}^d \left( s+n_i-1 \right)}. \tag{12}$$

where $[t^s]$ again denotes the coefficient of $t^s$ in a series.

In order to make the expression (11) for $H'_n$ more tractable to program, we will now focus only on the numerators of $H_n$ and $H'_n$. We have already defined the numerator of $H_n$ as $W_n(t)$ in the previous
subsection. We will let $N_n(t)$ denote the numerator of $H'_n$. By using software and observing patterns for small $n$, we anticipate that the denominator of $H'_n$ has exponent $|n| - d + 2$, and so we now set both

$$W_n := (1 - t)^{|n| - d + 1} H_n \quad \text{and} \quad N_n(t) := (1 - t)^{|n| - d + 2} H'_n.$$  

Therefore, we can clear denominators in (11) by multiplying both sides by $(1 - t)^{|n| - d + 2}$. Proceeding carefully and clearing the remaining denominators using (13), the pattern becomes clear:

$$N_n = \sum_A (-1)^{|A| - 1}(1 - t)^{|A| - 1} N_{n-e(A)} + t \cdot C(n) \cdot (1 - t)^{|n| - d} \cdot (1 - t) \cdot H_n$$

$$= \sum_A (t - 1)^{|A| - 1} N_{n-e(A)} + t \cdot C(n) \cdot W_n.$$  

(14)

This provides us with a quick recursive code to obtain $N_n$, after which we need only divide by $(1 - t)^{|n| - d + 2}$ to recover $H'_n$. The rest is just a matter of extracting coefficients in order to apply the result in (12).

5.3. Expected value for continuous version of $\text{EMD}_{d}$. Now that we have a way to determine the expected value for the discrete EMD, we aim to find a formula for the expected value in the continuous setting.

Starting with the expected value from (12), we scale by $1/s$ to normalize, and then let $s$ grow asymptotically:

$$\mathcal{E}_n := \mathbb{E}(\text{EMD}_d) = \lim_{s \to \infty} \frac{1}{s} \cdot \mathbb{E}(\text{EMD}'_d)$$

$$= \lim_{s \to \infty} \frac{1}{s} \cdot [t^s]H'_n \int_1^d \left( \frac{s+n_i-1}{n_i-1} \right).$$

First we focus on the $[t^s]H'_n$ part, namely the coefficient of $t^s$ in $H'_n = N_n(t)/(1 - t)^{|n| - d + 2}$. Now, the coefficient of $t^s$ in the series $1/(1 - t)^{|n| - d + 2}$ is just

$$\left( \frac{s+|n| - d + 1}{|n| - d + 1} \right) = \frac{s^{|n| - d + 1}}{(|n| - d + 1)!} + \text{lower-order terms in } s.$$  

Meanwhile, $N_n(t)$ is a polynomial, with some finite degree $b$. Now, as $s \to \infty$, we have $s - b \to \infty$, and so the coefficient of $t^s$ in $H'_n$ is asymptotic to $s^{|n| - d + 1}/(|n| - d + 1)!$ multiplied by the sum of the coefficients of $N_n(t)$. But this sum is just $N_n(1)$, and so we have

$$[t^s]H'_n \sim N_n(1) \cdot \frac{s^{|n| - d + 1}}{(|n| - d + 1)!}.$$  

Accounting for the $1/s$, we currently have the following:

$$\mathcal{E}_n = \lim_{s \to \infty} N_n(1) \cdot \frac{s^{|n| - d}}{(|n| - d + 1)! \prod_{i=1}^d \left( \frac{s+n_i-1}{n_i-1} \right)}.$$  

Now, since

$$\prod_i \left( \frac{s+n_i-1}{n_i-1} \right) \sim \prod_i \frac{s^{n_i-1}}{(n_i-1)!} = \frac{s^{|n| - d}}{\prod_i (n_i-1)!}.$$
this becomes
\[
\mathcal{E}_n = N_n(1) : \prod_{i=1}^{d} (n_i - 1)! / (|n| - d + 1)!.
\] (15)

But when we evaluate \( N_n(1) \) from (14), the terms with \((t - 1)\) all disappear; hence we need only consider subsets \( A \subseteq [d] \) with one element, meaning we are now summing from 1 to \( d \):
\[
N_n(1) = \sum_{i=1}^{d} N_{n-e(i)}(1) + C(n) W_n(1).
\]

Substituting for \( W_n(1) \) using (10), we have
\[
N_n(1) = \sum_{i=1}^{d} N_{n-e(i)}(1) + \frac{C(n) \cdot (|n| - d)!}{\prod_{i=1}^{d} (n_i - 1)!}.
\]

Finally, returning to (15) and plugging this all in for \( N_n(1) \), we conclude with the recursive definition
\[
\mathcal{E}_n = \left[ \sum_{i=1}^{d} N_{n-e(i)}(1) + \frac{C(n) \cdot (|n| - d)!}{\prod_{i=1}^{d} (n_i - 1)!} \right] \cdot \frac{\prod_{i=1}^{d} (n_i - 1)!}{(|n| - d + 1)!}
\]
\[
= \frac{\sum_{i=1}^{d} (n_i - 1) \mathcal{E}_{n-e(i)} + C(n)}{|n| - d + 1}.
\] (16)

where \( \mathcal{E}_{n-e(i)} = 0 \) if \( n - e(i) \) contains a 0.

We record this as the main theorem of this paper. We state the result only in the case of equal bin numbers, writing \((n^d) = (n, \ldots, n)\), so that the statistical meaning of \( \text{EMD}_d \) is well-defined.

**Theorem 7.** *The expected value of \( \text{EMD}_d \) on \( \mathcal{P}_n \times \cdots \times \mathcal{P}_n \) is \( \mathcal{E}(n^d) \) as defined in (16).*

**Remark.** Recall from Proposition 5 the special relationship between \( \text{EMD}_3 \) and \( \text{EMD}_2 \), namely, \( \text{EMD}_3 \) equals half the sum of the three pairwise \( \text{EMD}_2 \) values. This leads us to anticipate that
\[
\mathcal{E}(n^3) = \mathbb{E}(\text{EMD}_3) = \mathbb{E} \left( \frac{1}{2} (\text{EMD}_2 + \text{EMD}_2 + \text{EMD}_2) \right)
\]
\[
= \mathbb{E} \left( \frac{3}{2} \cdot \text{EMD}_2 \right)
\]
\[
= \frac{3}{2} \cdot \mathbb{E}(\text{EMD}_2)
\]
\[
= \frac{3}{2} \cdot \mathcal{E}(n^2).
\]

We confirm this in Mathematica; see Table 1.

**5.4. Unit normalized EMD.** It is often convenient to unit normalize the value of \( \text{EMD}_d \) so that its value falls between 0 and 1. To this end, we claim that for a given \( n \), the maximum value of \( \text{EMD}_d \) is \([d/2](n-1)\). To see this, observe that the maximum value of the discrete \( \text{EMD}_d^s(\mu) \) occurs when in every column of the matrix \( M_\mu \) (given by the RSK correspondence), half the entries are 1 and the other half are \( n \); if \( d \) is odd, then “half” means \([d/2]\), with the leftover entry being irrelevant by Proposition 1. For such a \( \mu \), then, \( \text{EMD}_d^s(\mu) \) equals the cost \([d/2](n-1)\) multiplied by \( s \) (the number of columns). After
A GENERALIZATION FOR THE EXPECTED VALUE OF THE EARTH MOVER’S DISTANCE

\[ E(n^2) = \frac{1}{2} E(n^3). \]

\[
\begin{array}{|c|c|c|c|}
\hline
n & E(n^2) & E(n^3) & E(n^3)/E(n^2) \\
\hline
2 & 0.3333 & 0.5000 & 1.5 \\
3 & 0.5333 & 0.8000 & 1.5 \\
4 & 0.6857 & 1.0286 & 1.5 \\
5 & 0.8127 & 1.2191 & 1.5 \\
6 & 0.9235 & 1.3853 & 1.5 \\
7 & 1.0230 & 1.5345 & 1.5 \\
8 & 1.1139 & 1.6709 & 1.5 \\
9 & 1.1982 & 1.7972 & 1.5 \\
10 & 1.2770 & 1.9155 & 1.5 \\
\hline
\end{array}
\]

Table 1. Mathematica verification that \( E(n^3) = \frac{3}{2} E(n^2) \).

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
n & d = 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
2 & 0.3333 & 0.5000 & 0.4000 & 0.5000 & 0.4286 & 0.5000 & 0.4444 & 0.5000 & 0.4545 \\
3 & 0.2667 & 0.4000 & 0.3175 & 0.3968 & 0.3388 & 0.3952 & 0.3505 & 0.3943 & 0.3579 \\
4 & 0.2286 & 0.3429 & 0.2711 & 0.3389 & 0.2888 & 0.3370 & 0.2985 & 0.3358 & 0.3046 \\
5 & 0.2032 & 0.3048 & 0.2405 & 0.3006 & 0.2560 & 0.2986 & 0.2644 & 0.2974 & 0.2697 \\
6 & 0.1847 & 0.2771 & 0.2184 & 0.2730 & 0.2322 & 0.2710 & 0.2398 & 0.2698 & 0.2445 \\
7 & 0.1705 & 0.2557 & 0.2014 & 0.2517 & 0.2141 & 0.2498 & 0.2210 & 0.2486 & 0.2253 \\
8 & 0.1591 & 0.2387 & 0.1878 & 0.2348 & 0.1996 & 0.2329 & 0.2060 & 0.2318 & 0.2101 \\
9 & 0.1498 & 0.2247 & 0.1767 & 0.2209 & 0.1877 & 0.2190 & 0.1937 & 0.2180 & 0.1975 \\
10 & 0.1419 & 0.2128 & 0.1673 & 0.2092 & 0.1778 & 0.2074 & 0.1834 & 0.2064 & 0.1870 \\
\hline
\end{array}
\]

Table 2. The unit normalized expected value \( \hat{E}(n^d) \).

\[
\text{dividing by } s \text{ to pass to the continuous setting, we see that the maximum value of } EMD_d \text{ is } \lfloor d/2 \rfloor (n-1),
\]
as claimed. Therefore we present definitions for the unit normalized EMD\(_d\) and its expected value:

\[
\hat{EMD}_d(\mu) := \frac{EMD_d(\mu)}{\lfloor d/2 \rfloor (n-1)} \quad \text{and} \quad \hat{E}(n^d) := \frac{E(n^d)}{\lfloor d/2 \rfloor (n-1)}.
\]

Using Theorem 7 and the definition in (17), we record the unit normalized expected value \( \hat{E}(n^d) \) in Table 2, for the first few values of \( n \) and \( d \). We observe a curious phenomenon in this table when we fix \( n \) and let \( d \) increase: the unit normalized expected value alternately increases (\( d \) changing from even to odd) and decreases (\( d \) changing from odd to even). This suggests that an even number of distributions are more likely to be “closer” together (in the sense of the EMD) than an odd number of distributions. We can explain this dependence on parity from both a geometric and statistical perspective.

Geometrically, in odd dimensions, the taxicab distance from a point to the main diagonal remains unchanged as we vary the median of the point’s coordinates, and the variability of the median increases as the distance increases (i.e., as the other coordinates become more spread out). For example, in a 3-dimensional array with side lengths \( n \), the greatest possible distance from the main diagonal is
Figure 2. A plot of $\widehat{E}(3d)$ (vertical axis), for values of $d$ up to 100 (horizontal axis). The disparity between even and odd values of $d$ is significant when $d$ is small, but becomes negligible for $d$ sufficiently large.

$\left\lfloor \frac{3}{2} \right\rfloor (n - 1) = n - 1$; this is attained by all those positions whose coordinates (up to reordering) are $(1, __, n)$, where the blank can take any value in $[n]$. In a 4-dimensional array, however, there are far fewer positions at maximal distance $\left\lfloor \frac{4}{2} \right\rfloor (n - 1) = 2(n - 1)$ from the main diagonal: namely, those positions whose coordinates (up to reordering) are $(1, 1, n, n)$. In effect, the free “odd-man-out” coordinate does not exist in an even number of dimensions, so there are fewer array positions farther away from the main diagonal compared to an odd number of dimensions. This remains true when we restrict to arrays with support in a chain, and so we expect to see the even–odd disparity reflected in the EMD$_d$ data.

Likewise, statistically, three distributions achieve maximum EMD$_3$ value when one of them is $(1, 0, \ldots, 0)$, another is $(0, \ldots, 0, 1)$, and the third is anything at all. But the maximum EMD$_d$ value is attained only when two distributions are $(1, 0, \ldots, 0)$ and the other two are $(0, \ldots, 0, 1)$. For $d$ odd, the variability in the free “odd-man-out” distribution is greater for higher EMD$_d$ values, and so it is no surprise that there is a greater proportion of higher EMD$_d$ values when $d$ is odd.

Although this even-odd disparity is significant when $d$ is relatively small (as in Table 2), the difference becomes negligible as $d$ increases, and the values of $\widehat{E}(n^d)$ do stabilize. We plot these values in Figure 2, fixing $n = 3$ and letting $d$ grow from 2 to 100 on the horizontal axis. On the far left side of the plot, the disparity between even and odd values of $d$ is indeed striking — and the variability is much greater when $d$ is even than when $d$ is odd — but we see that this difference quickly becomes negligible, with the sequence of expected values converging to approximately 0.39.

6. Real-world data

As a basic example, we apply our generalized discrete EMD to four mathematics courses at the University of Wisconsin–Milwaukee, during the fall 2019, spring 2020, and fall 2020 semesters. (Needless to say, these three semesters are of added interest because of the drastic changes brought on by the Covid-19 pandemic in spring 2020.) This data is contained in the “Section attrition and grade report,” published by
Table 3. Discrete EMD applied to four mathematics courses at the University of Wisconsin–Milwaukee.

<table>
<thead>
<tr>
<th>course</th>
<th>fall 2019</th>
<th>spring 2020</th>
<th>fall 2020</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sections</td>
<td>EMD</td>
<td>sections</td>
</tr>
<tr>
<td>MATH 105</td>
<td>29</td>
<td>0.2059</td>
<td>13</td>
</tr>
<tr>
<td>MATH 231</td>
<td>10</td>
<td>0.1095</td>
<td>6</td>
</tr>
<tr>
<td>MATH 232</td>
<td>7</td>
<td>0.1308</td>
<td>7</td>
</tr>
<tr>
<td>MATH 233</td>
<td>7</td>
<td>0.1650</td>
<td>6</td>
</tr>
</tbody>
</table>

Figure 3. The unit normalized values of $\text{EMD}_{100}^d$ for four mathematics courses, tracked during three consecutive semesters.

The results in the table are also plotted in Figure 3, from which we can make one immediate observation: the EMD of each of the calculus courses increased from one semester to the next, whereas the EMD of the college algebra course decreased. In fact, in the semester before the pandemic, college algebra had a higher EMD than any of the calculus courses, but its EMD was lower than all of the calculus courses by the second semester of the pandemic. Of course, the EMD is only one statistic among many, and we would need to examine other measures (particularly the grade-point averages) and more previous semesters before drawing definitive conclusions. But we can say that the grade distributions of the various
calculus sections (within each course) had grown farther apart since the shift to online learning in spring 2020, while the distributions of the various sections of college algebra had become closer together. This raises further questions:

- To what extent has the shift to online learning caused the recent trends in EMD?
- To what extent are these trends a reflection of instructors’ and/or course coordinators’ and/or students’ response to the online format?
- Are students’ grades trending higher or lower (or neither) in each course since the pandemic began?
- How does EMD typically behave in spring vs. fall semesters before and after the pandemic began?

Finally, we compare these real-world EMD values to the expected value defined in Theorem 7. From Table 2, for \( n = 5 \), we observe that the approximate range of \( \hat{E}(5^d) \) is from 0.25 to 0.3 for the section numbers \( d \) in our data set. The actual EMD values lie significantly below this range (with the sole exception of Calculus II in fall 2020), especially considering that the expected value would be slightly higher in the discrete setting than it is in the continuous setting. We should not be too surprised by this, of course, since college grades are (hopefully) not assigned at random, and therefore not all grade distributions are equally likely.

7. Connection to algebraic geometry and representation theory

In this section, as a sort of appendix, we shift away from statistics and observe the connection between the EMD and the Segre embedding from algebraic geometry. As a result, we show how the entire EMD setting can be realized as a representation of the Lie algebra \( u(n, n) \).

7.1. A determinantal variety. In the \( d = 2 \) case, as indicated in [5], the series \( H_{(n,n)} := H_{(n,n)}(1, t) \), from (8), is in fact the Hilbert series of the determinantal variety

\[
D_n^{\leq 1} := \{ M \in M_n(\mathbb{C}) \mid \text{rank } M \leq 1 \}
\]

consisting of \( n \times n \) complex matrices with rank at most 1. To see this, it suffices to show that for a given nonnegative integer \( s \), the number of elements in \( C(s, n) \times C(s, n) \) equals the dimension of \( \mathbb{C}[D_n^{\leq 1}]^s \), the space of homogeneous degree-\( s \) polynomial functions on \( D_n^{\leq 1} \). To this end, let \( w_{ij} \) be the coordinate functions on a generic \( n \times n \) matrix. Since all 2 \( \times \) 2 minors vanish on any matrix in \( D_n^{\leq 1} \), we observe that

\[
\mathbb{C}[D_n^{\leq 1}] \cong \mathbb{C}[w_{11}, \ldots, w_{nn}] / \mathcal{I},
\]

where \( \mathcal{I} \) is the determinantal ideal generated by the quadratics of the form \( w_{ij}w_{i'j'} - w_{i'j}w_{ij} \) for \( i < i' \) and \( j < j' \). It follows that a basis for \( \mathbb{C}[D_n^{\leq 1}]^s \) is given by the set of monomials

\[
B^s = \left\{ \prod_{k=1}^s w_{i_k,j_k} \mid \{i_k, j_k\} \text{ is a chain in } [n] \times [n] \right\}.
\]

Now we have an obvious bijective correspondence between the sets \( B^s \) and \( J_{(n,n)}^s \):

\[
\prod_{i,j \in [n]} w_{ij} \leftrightarrow J \in J_{(n,n)}^s.
\]
But $\mathcal{J}_{s(n,n)}$ is in bijective correspondence with $\mathcal{C}(s, n) \times \mathcal{C}(s, n)$, as is clear from our RSK correspondence in (5). This proves our claim that $H_{(n,n)}$ is the Hilbert series of $D_n^{\leq 1}$.

7.2. The Segre embedding. We extend the previous result to $d > 2$, writing $(n^d)$ for $(n, \ldots, n)$ as before. The specialization $H_{(n^d)} := H_{(n^d)}(1, t)$ of our generating function from (8) is the Hilbert series of the image of the Segre embedding:

$$\mathbb{P}(\mathbb{C}^n) \times \cdots \times \mathbb{P}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n), \quad ([v^{(1)}], \ldots, [v^{(d)}]) \mapsto [v^{(1)} \otimes \cdots \otimes v^{(d)}]$$

(see [14; 21]). That is, $H_{(n^d)}$ is the Hilbert series of the coordinate ring of the simple tensors. (In the case $d = 2$, the set of simple tensors in $\mathbb{C}^n \otimes \mathbb{C}^n$ is identified with $D_n^{\leq 1}$, coinciding with the previous subsection.)

To sketch this generalization of the $d = 2$ case, which we presented in detail above, we let $m$ range over all multi-indices $(m_1, \ldots, m_d) \in [n] \times \cdots \times [n]$. Now consider a simple tensor $v^{(1)} \otimes \cdots \otimes v^{(d)}$. We can expand this tensor in the standard basis as

$$\sum_{m} \frac{(v^{(1)}_{m_1} \cdots v^{(d)}_{m_d})}{w_m} e_{m_1} \otimes \cdots \otimes e_{m_d},$$

where $v^{(k)}_\ell$ is the $\ell$-th coordinate of the vector $v^{(k)}$, the $w_m$ are coordinate functions. For any two multi-indices $m$ and $m'$, we see that the quadratic $w_m w_{m'}$ is invariant under the exchange of indices componentwise between $m$ and $m'$. Intuitively, then, we can again mod out by the determinantal ideal generated by all $2 \times 2$ minors, just as we did in the $d = 2$ case above.

The upshot is that a basis for the coordinate ring of the simple tensors is given by those monomials $w_{m_1} \cdots w_{m_d}$ such that the set $\{m_i\}$ of multi-indices forms a chain. We therefore have the generalization of (18), and we conclude that $H_{(n^d)}$ is the Hilbert series of the simple tensors in $\mathbb{C}^n \times \cdots \times \mathbb{C}^n$.

7.3. Representation theory. Returning to the $d = 2$ case, the coordinate ring $\mathbb{C}[D_n^{\leq 1}]$ is an infinite-dimensional vector space known as the first Wallach representation of the Lie algebra $u(n, n)$ of the indefinite unitary group (see [10]). We will show how the action on $\mathbb{C}[D_n^{\leq 1}]$ corresponds to manipulating the two compositions in our EMD$_2^s$ setting.

Consider the polynomial ring $\mathbb{C}[x, y] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. On one hand, $\mathbb{C}[x, y]$ admits an action of $G = \text{GL}_1(\mathbb{C})$, the multiplicative group of nonzero complex numbers, via

$$(g \cdot f)(x, y) = f(g^{-1}x, gy)$$

for $g \in G$ and $f \in \mathbb{C}[x, y]$. Note that the invariants under the $G$-action are those polynomials in which the degree of each term is the same with respect to $x$ as it is with respect to $y$; in other words, $\mathbb{C}[x, y]^G$ is generated by the monomials $x_i y_j$. (This is a special case of the first fundamental theorem of invariant theory; see [13, Section 5.2.1].) But since the kernel of the ring homomorphism $w_{ij} \mapsto x_i y_j$ is precisely the determinantal ideal $\mathcal{I}$, we have $\mathbb{C}[x, y]^G \simeq \mathbb{C}[D_n^{\leq 1}]$. (This is a special case of the second fundamental theorem of invariant theory; see [13, Lemma 5.2.4].)

As a result of Howe duality in Type A — the delicate details of which are expounded in [16; 17] — the space $\mathbb{C}[x, y]$ is also a module under the action of the Lie algebra $u(n, n)$ by differential operators. Upon complexification, this gives rise to an action by the Lie algebra $\mathfrak{gl}_{2n}$ of $2n \times 2n$ complex matrices. In
particular, the invariant subring $\mathbb{C}[x, y]^G \simeq \mathbb{C}[D_n^{\leq 1}]$ is the irreducible, infinite-dimensional $\mathfrak{gl}_{2n}$-module with highest weight $(-1, \ldots, -1, 0, \ldots, 0)$ in standard coordinates, where there are $n$ entries of $-1$.

Explicitly, $\mathfrak{gl}_{2n}$ acts via differential operators on $\mathbb{C}[x, y]$, of the following four forms (see [8, Section 2], for the action in full detail):

1. $x_i \frac{\partial}{\partial x_j}$ (Euler operators; technically the action includes the extra term $+\delta_{ij}$);
2. $y_i \frac{\partial}{\partial y_j}$ (Euler operators);
3. $\frac{\partial^2}{\partial x_i \partial y_j}$ ("raising operators");
4. $x_i y_j$ ("lowering operators").

Note that all these operators preserve the difference between the degree with respect to $x$ and the degree with respect to $y$. Therefore the $\mathfrak{gl}_{2n}$-action preserves $\mathbb{C}[x, y]^G$, which we observed is generated by the elements $x_i y_j$.

This $\mathfrak{gl}_{2n}$-action can be described in terms of our EMD$_2^\xi$ setting in this paper. First, observe that any degree-$s$ monic monomial in $\mathbb{C}[x, y]^G$ corresponds uniquely to an ordered pair of compositions $(\mu, \nu) \in \mathcal{C}(s, n) \times \mathcal{C}(s, n)$, via

$$(\mu, \nu) \leftrightarrow x_1^{\mu_1} \cdots x_n^{\mu_n} y_1^{\nu_1} \cdots y_n^{\nu_n}.$$ 

Now we can see how each type (1)–(4) of differential operator has an interpretation in the EMD$_2^\xi$ context. Up to scaling by coefficients, we observe the following:

1. The Euler operator $x_i \partial/\partial x_j$ corresponds to moving 1 unit in $\mu$ from bin $j$ to bin $i$, since the exponent of $x_j$ decreases by 1 and the exponent of $x_i$ increases by 1.
2. The Euler operator $y_i \partial/\partial y_j$ corresponds to moving 1 unit in $\nu$ from bin $j$ to bin $i$.
3. The raising operator $\partial^2/\partial x_i \partial y_j$ corresponds to removing 1 unit from each composition: from bin $i$ in $\mu$ and from bin $j$ in $\nu$.
4. The lowering operator $x_i y_j$ corresponds to adding 1 unit to each composition: to bin $i$ in $\mu$ and to bin $j$ in $\nu$.

It will be interesting to study further whether this connection to representation theory might be exploited in applications of EMD$_2^\xi$.

8. Proof of Proposition 2

The methods in this paper depended heavily upon the fact that we need to consider only those arrays $J$ whose support is a chain. This followed from the statement in Proposition 2 — yet to be proved — that our cost array $C$ has the Monge property. Before proving this here, we state three useful lemmas, the first of which is proved in [2; 22]:

**Lemma 8.** An $n \times \cdots \times n$ array $A$ has the Monge property if and only if every two-dimensional plane of $A$ has the Monge property.
To make this explicit, we choose any two distinct indices \( i, j \) from \( \{1, \ldots, d\} \), and then fix the remaining \( d - 2 \) coordinates at the values \( \tilde{m}_1, \ldots, \tilde{m}_{i-1}, \tilde{m}_{i+1}, \ldots, \tilde{m}_{j-1}, \tilde{m}_{j+1}, \ldots, \tilde{m}_d \in [n] \). Then we will write \( \tilde{m}_{k,j}^{i,j} := (\tilde{m}_1, \ldots, \tilde{m}_{i-1}, k, \tilde{m}_{i+1}, \ldots, \tilde{m}_{j-1}, \ell, \tilde{m}_{j+1}, \ldots, \tilde{m}_d) \). In other words, \( \tilde{m}_{k,j}^{i,j} \) is the vector in which the \( i \)-th coordinate is \( k \), the \( j \)-th coordinate is \( \ell \), and the remaining coordinates are the fixed values \( \tilde{m}_1, \ldots, \tilde{m}_d \). Now we can naturally define the two-dimensional subarray \( A_{i,j}^{k,l} \) in which

\[
A_{i,j}^{k,l}(k, \ell) := A(\tilde{m}_{k,l}^{i,j}).
\]  

(19)

Then Lemma 8 states that \( A \) has the Monge property if and only if \( A_{i,j}^{k,l} \) has the Monge property for every choice of distinct \( i \) and \( j \).

This reduction to the two-dimensional case is extremely useful because of the following characterization of two-dimensional Monge arrays, proved in [22]:

**Lemma 9.** Let \( A \) be an \( n \times n \) array. Then \( A \) has the Monge property if and only if

\[
A(k, \ell) + A(k + 1, \ell + 1) \leq A(k + 1, \ell) + A(k, \ell + 1)
\]

for all \( k, \ell \in [n - 1] \).

In other words, choose a position \( (k, \ell) \) and then consider the \( 2 \times 2 \) subarray consisting of \( A(k, \ell) \) and its three neighbors to the east, south, and southeast. The condition displayed in the lemma means that the sum of the upper-left and lower-right entries must never be greater than the sum of the lower-left and upper-right entries.

We will need one final lemma, specific to the cost function \( C \) in this paper. Recall from Proposition 1 that if we let \( \hat{m} \) denote a vector \( m \) with its coordinates rearranged in ascending order, then

\[
C(m) = -\tilde{m}_1 - \cdots - \tilde{m}_{\lfloor (d+1)/2 \rfloor} + \tilde{m}_{\lfloor (d+1)/2 \rfloor + 1} + \cdots + \tilde{m}_d \quad (d \text{ even}),
\]

or

\[
C(m) = -\tilde{m}_1 - \cdots - \tilde{m}_{\lfloor (d+1)/2 \rfloor - 1} + \tilde{m}_{\lfloor (d+1)/2 \rfloor + 1} + \cdots + \tilde{m}_d \quad (d \text{ odd}).
\]

The index \( \lfloor (d+1)/2 \rfloor \) gave a kind of “median” of the coordinates in \( m \); from now on, however, we will work instead with

\[
M := \left\lceil \frac{d+1}{2} \right\rceil + 1 = \left\lfloor \frac{d+2}{2} \right\rfloor.
\]

Intuitively, this index \( M \) gives the next-greatest coordinate after the “median.” The picture is the following, where the vertical lines divide the coordinates into two equal sets (with one leftover coordinate in the middle if \( d \) is odd):

\[
d \text{ even : } \hat{m} = (\tilde{m}_1, \ldots, \tilde{m}_{M-1}, | \tilde{m}_M, \ldots, \tilde{m}_d),
\]

\[
d \text{ odd : } \hat{m} = (\tilde{m}_1, \ldots, | \tilde{m}_{M-1}, \tilde{m}_M, \ldots, \tilde{m}_d).
\]

With this indexing in mind, we state our final lemma, which records the effect on \( C(m) \) of adding 1 to a single coordinate \( m_i \). Recall from earlier that \( e(i) \) denotes the vector whose coordinates are all 0 except for a 1 in the \( i \)-th component.

**Lemma 10.** Adding 1 to a single coordinate \( m_i \) of \( m \) has one of three effects on \( C(m) \): it either increases by 1, decreases by 1, or remains the same. The effect depends on the value of \( m_i \) relative to the other coordinates of \( m \):

1. \( C(m + e(i)) = C(m) + 1 \) if \( m_i \geq \tilde{m}_M \).
(2) \(C(m + e(i)) = C(m) - 1\) if:
(a) \(d\) is even and \(m_i < \tilde{m}_M\); or
(b) \(d\) is odd and \(m_i < \tilde{m}_{M-1}\).

(3) \(C(m + e(i)) = C(m)\) if \(d\) is odd and \(m_i = \tilde{m}_{M-1} < \tilde{m}_M\).

Proof. We prove each of the three cases; the reader may find it helpful to keep an eye on the two possible “pictures” of \(\tilde{m}\) displayed before this lemma, along with the two possible sums for \(C(m)\) displayed just before that.

(1) Assume \(m_i \geq \tilde{m}_M\). Then \(m_i + 1 > \tilde{m}_M\), and so in the sum defining \(C(m)\), we must have positive \(m_i\) replaced by positive \((m_i + 1)\). Hence \(C(m)\) has increased by 1.

(2) (a) Assume \(d\) is even and \(m_i < \tilde{m}_M\). Then \(m_i + 1 \leq \tilde{m}_M\), and so in the sum defining \(C(m)\), we must have negative \(m_i\) replaced by negative \((m_i + 1)\). Hence \(C(m)\) has decreased by 1.
(b) Assume \(d\) is odd and \(m_i < \tilde{m}_{M-1}\). Then \(m_i + 1 \leq \tilde{m}_{M-1}\), and so we must have negative \(m_i\) replaced by negative \((m_i + 1)\). Hence \(C(m)\) has decreased by 1.

(3) Assume \(d\) is odd and \(m_i = \tilde{m}_{M-1} < \tilde{m}_M\); note that \(\tilde{m}_{M-1}\) does not appear in the sum defining \(C(m)\). Then \(\tilde{m}_{M-2} < m_i + 1 \leq \tilde{m}_M\), and so \(m_i + 1\) still does not appear in the sum defining \(C(m + e(i))\). Hence \(C(m)\) remains unchanged.

We are now ready for the proof, in which we show that an arbitrary two-dimensional subarray of \(C\) has the Monge property.

Proof of Proposition 2. Let \(i, j\) be two distinct indices in \([1, \ldots, d]\). Fix the remaining coordinates \(\tilde{m}_1, \ldots, \tilde{m}_d\) as above, and let \(C^{i,j}\) be the corresponding two-dimensional subarray of \(C\) defined in (19). Now let \(m_i, m_j \in [n - 1]\). By Lemmas 8 and 9, it will suffice to show that

\[C^{i,j}(m_i, m_j) + C^{i,j}(m_i + 1, m_j + 1) \leq C^{i,j}(m_i + 1, m_j) + C^{i,j}(m_i, m_j + 1).\]

But this condition can be rewritten as the following, where we simply write \(\tilde{m}\) for \(\tilde{m}^{i,j}_{m_i, m_j}\):

\[C(\tilde{m}) + C(\tilde{m} + e(i) + e(j)) \leq C(\tilde{m} + e(i)) + C(\tilde{m} + e(j)).\] \hspace{1cm} (20)

To show that this condition holds true, we need to examine six possible cases, indicated in Table 4.

We now examine each of these cases. All simplifications are directly justified by the results in Lemma 10.

<table>
<thead>
<tr>
<th>Case</th>
<th>(C(\tilde{m} + e(i)) = C(\tilde{m}) + 1)</th>
<th>(C(\tilde{m} + e(i)) = C(\tilde{m}) - 1)</th>
<th>(C(\tilde{m} + e(i)) = C(\tilde{m}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>(C(\tilde{m} + e(j)) = C(\tilde{m}) + 1)</td>
<td>Case 2</td>
<td>(C(\tilde{m} + e(j)) = C(\tilde{m}) - 1)</td>
</tr>
<tr>
<td>Case 3</td>
<td>(C(\tilde{m} + e(j)) = C(\tilde{m}))</td>
<td>Case 5</td>
<td>Case 6</td>
</tr>
</tbody>
</table>

Table 4. Six possible cases, depending on whether adding 1 to \(m_i\) and \(m_j\) (independently) causes \(C\) to increase, decrease, or remain the same.
Case 1. In this case, the right-hand side of (20) is $2 \cdot C(\bar{m}) + 2$. For the left-hand side, we know in general that $C(\bar{m} + e(i) + e(j)) = C((\bar{m} + e(i)) + e(j))$, which by Lemma 10 can be no greater than $C(\bar{m}) + 2$. Hence the inequality in (20) must hold.

Case 2. In this case, the right-hand side of (20) is $2 \cdot C(\bar{m})$. As for the second term on the left-hand side, by Lemma 10, we must have $m_i \geq \bar{m}_M$; meanwhile, $m_j$ is strictly less than either $\bar{m}_M$ (if $d$ is even) or $\bar{m}_{M-1}$ (if $d$ is odd), and so neither inequality is affected by adding 1 to $m_i$. Therefore we have

$$C(\bar{m} + e(i) + e(j)) = C((\bar{m} + e(i)) + e(j))$$
$$= C((\bar{m} + e(i)) - 1$$
$$= C(\bar{m}) + 1 - 1$$
$$= C(\bar{m}).$$

Hence we have an equality in (20).

Case 3. Similar to Case 2, the two additions are independent of each other. The right-hand side of (20) is $2 \cdot C(\bar{m}) + 1$. In this case, we must have $d$ odd; also, $m_i \geq \bar{m}_M$, along with $m_j = \bar{m}_{M-1} < \bar{m}_M$. Then

$$C(\bar{m} + e(i) + e(j)) = C((\bar{m} + e(i)) + e(j))$$
$$= C((\bar{m} + e(i))$$
$$= C(\bar{m}) + 1.$$

Again we obtain an equality in (20).

Case 4. The right-hand side of (20) is $2 \cdot C(\bar{m}) - 2$. If $d$ is even, then both $m_i$ and $m_j$ are strictly less than $\bar{m}_M$, and if $d$ is odd, then both are strictly less than $\bar{m}_{M-1}$. Either way, after adding 1 to $m_i$, the same inequality still holds for $m_j$, and so again we have

$$C(\bar{m} + e(i) + e(j)) = C((\bar{m} + e(i)) + e(j))$$
$$= C((\bar{m} + e(i)) - 1$$
$$= C(\bar{m}) - 1 - 1$$
$$= C(\bar{m}) - 2,$$

and we get an equality in (20).

Case 5. The right-hand side of (20) is $2 \cdot C(\bar{m}) - 1$. In this case, $d$ must be odd, with

$$m_i < \bar{m}_{M-1} = m_j < \bar{m}_M.$$ 

After adding 1 to $m_j$, we still have $m_i$ less than the $(M-1)$-th component in the new rearranged vector, and so the effects of the two additions are independent. We obtain

$$C(\bar{m} + e(i) + e(j)) = C((\bar{m} + e(i)) + e(j))$$
$$= C((\bar{m} + e(j)) - 1$$
$$= C(\bar{m}) - 1$$

and so we have an equality in (20).
Case 6. This is the slightly surprising case, in which the two additions are not independent of each other. The right-hand side of (20) is \(2 \cdot C(\bar{m})\), and we know that \(d\) must be odd, with \(m_i = m_j = \bar{m}_{M-1} < \bar{m}_M\). After adding 1 to \(m_i\), we obtain a vector \(m'\) in which \(m'_j = m_j\) is now strictly less than \(\bar{m}'_M - 1\), and so now adding 1 to \(m_j\) results in an overall decrease by 1. Hence we have

\[
C(\bar{m} + e(i) + e(j)) = C((\bar{m} + e(i)) + e(j)) \\
= C((\bar{m} + e(i)) - 1 \\
= C(\bar{m}) - 1.
\]

Hence the left-hand side of (20) is less than the right-hand side, and the condition is still satisfied.

We have exhausted all possible cases, and so since (20) holds in each of them, the two-dimensional array \(C^{i,j}\) has the Monge property. Since \(i\) and \(j\) were arbitrary, every two-dimensional subarray of \(C\) has the Monge property, and so by Lemma 8, we conclude that \(C\) itself has the Monge property. \(\square\)

9. Proof of Proposition 6

We prove here the recursive definition for our generating function \(H_n\) from Section 5.1. Recall that the formal definition is

\[
H_n(z, t) := \sum_{s=0}^{\infty} \left( \sum_{\mu \in \mathcal{C}(s, n_1) \times \cdots \times \mathcal{C}(s, n_d)} z^{\text{EMD}_d(\mu)} \right) t^s,
\]

and Proposition 6 states the recursion as follows, where the sum is over all nonempty subsets \(A \subseteq [d]\):

\[
H_n = \sum_{A} \frac{(-1)^{|A|-1} \cdot H_n-e(A)}{1 - z^{C(n)} t},
\]

where \(H_{(1^d)} = 1/(1-t)\), and \(H_{n-e(A)} = 0\) if \(n - e(A)\) contains a 0.

Proof of Proposition 6. Each \(\mu\) corresponds to a unique monomial

\[
\mu \leftrightarrow \prod_{m} w_m^{J_\mu(m)}, \quad (21)
\]

where \(\mu \leftrightarrow J_\mu\) is the RSK correspondence in (5). The variables \(w_m\) are indexed by multi-indices \(m \in [n_1] \times \cdots \times [n_d]\). Note that the degree of this monomial equals \(s\) (the sum of the entries of \(J_\mu\)). Now making the substitution

\[
w_m \mapsto z^{C(m)} t, \quad (22)
\]

the above correspondence gives us the map

\[
\mu \leftrightarrow \prod_{m} w_m^{J_\mu(m)} \mapsto z^{\text{EMD}_d(\mu)} t^s.
\]

Therefore the generating function \(H_n\) is just the image of the formal sum \(H_n^s\) of all monomials of the form (21), under the substitution (22); as \(s\) ranges over all nonnegative integers, there is one monomial in \(H_n^s\) for each possible \(\mu \in \mathcal{C}(s, n_1) \times \cdots \times \mathcal{C}(s, n_d)\).
Since \( m \leq n \) for all \( m \), every array \( J_\mu \) is allowed to contain \( n \) in its support without violating the chain condition. This means that every monomial in \( H_n^* \) is allowed to contain the variable \( w_n \), and so we may factor out the sum of all possible powers of \( w_n \), rewriting as
\[
H_n^* = \sum_{\mu} \left( \prod_{m} w_{J_\mu(m)} \right) = \sum_{r=0}^{\infty} w_n^r \cdot f(w_{m \neq n}) = \frac{f(w_{m \neq n})}{1 - w_n},
\]
where \( f \) is an infinite formal sum of monomials in the variables \( w_m \) where \( m \neq n \). Now we focus on rewriting this numerator \( f \). Suppose we subtract 1 from exactly one of the coordinates of \( n \); the possible results are \( n - e(i) \) for \( i = 1, \ldots, d \). Now, on one hand, any monomial in \( f \) containing the variable \( w_{n - e(i)} \) appears in \( H_{n - e(i)}^* \). But on the other hand, note that all of these \( n - e(i) \) are mutually incomparable under the product order, and so at most one of them can be in the support of some \( J_\mu \), because of the chain condition. Therefore any monomial in \( f \) contains at most one of the variables \( w_{n - e(i)} \). But the sum \( \sum_{i=1}^{d} H_{n - e(i)}^* \) still overcounts the monomials appearing in \( f \), since the same monomial may appear in several distinct summands.

In other words, we want \( f \) to be the formal sum of the union (without multiplicity) of the monomials which appear in the summands \( H_{n - e(i)}^* \). We can achieve this by using the inclusion–exclusion principle: subtract those monomials which appear in at least 2 of the summands, then add back the monomials which appear in at least 3 of the summands, then subtract those appearing in at least 4 summands, and so on, until we arrive at those monomials appearing in all \( d \) of the summands. We can write this inclusion–exclusion as an alternating sum over nonempty subsets \( A \subseteq [d] \), adding when \( |A| \) is odd and subtracting when \( |A| \) is even:
\[
f = \sum_A (-1)^{|A|-1} \cdot H_{n - e(A)}^*.
\]
Finally, applying the substitution (22), we obtain
\[
H_n = H_n^* \bigg|_{w_m = z^{C(m)} t} = \frac{\sum_A (-1)^{|A|-1} \cdot H_{n - e(A)}^* \bigg|_{w_m = z^{C(m)} t}}{1 - w_n} = \frac{\sum_A (-1)^{|A|-1} \cdot H_{n - e(A)}^* \bigg|_{w_m = z^{C(n)} t}}{1 - z^{C(n)} t},
\]
proving the recursion.

As for the base case \( H_{(1, \ldots, 1)} = 1/(1 - t) \), there is only one element in \( C(s, 1) \), and so since every \( n_i = 1 \), the inside sum in (7) has only one term; moreover, this unique \( \mu \) is just \( d \) copies of the same trivial composition of \( s \) into 1 part, meaning that \( \text{EMD}_d^*(\mu) = 0 \). Hence \( H_{(1, \ldots, 1)}(z, t) = \sum z^0 t^s = \sum t^s \), whose closed form is \( 1/(1 - t) \). Likewise, since \( C(s, 0) \) is empty, we must have \( H = 0 \) if any of the \( n_i \) become 0.

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### References


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