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SUMS OF ALGEBRAIC TRACE FUNCTIONS TWISTED BY ARITHMETIC FUNCTIONS

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We obtain new bounds for short sums of isotypic trace functions associated to some sheaf modulo prime p of bounded conductor, twisted by the Möbius function and also by the generalised divisor function. These trace functions include Kloosterman sums and several other classical number theoretic objects. Our bounds are nontrivial for intervals of length of at least $p^{1/2+\varepsilon}$ with an arbitrary fixed $\varepsilon > 0$, which is shorter than the required length of at least $p^{3/4+\varepsilon}$ in the case of the Möbius function and of at least $p^{2/3+\varepsilon}$ in the case of the divisor function required in recent results of É. Fouvry, E. Kowalski and P. Michel (2014) and E. Kowalski, P. Michel and W. Sawin (2018), respectively.

1. Background and motivation

For a prime p and arbitrary integers m, n, we define the s-dimensional Kloosterman sums

$$K_{s,p}(n) = \sum_{\substack{x_1,\dots,x_s=1 \\ x_1\cdots x_s \equiv n \bmod p}}^{p-1} \boldsymbol{e}\left(\frac{x_1+\ldots+x_s}{p}\right),$$

where for a real z we denote

$$\boldsymbol{e}(z) = e^{2\pi i z}.$$

The classical Deligne bound yields the estimate

$$|K_{s,p}(n)| \le s p^{(s-1)/2}; \tag{1-1}$$

see [Iwaniec and Kowalski 2004, Equation (11.58)] and the follow-up discussion.

Since the bound (1-1) is essentially optimal, it is natural to study cancellations between Kloosterman sums in various families of pairs (n, p) of parameters, with and without some weights attached.

Studying cancellations for fixed m and n and varying n is related to the Linnik conjecture [1963], and thus to the groundbreaking results of Kuznetsov [1980]; see also [Iwaniec and Kowalski 2004, Chapter 16]. We refer to [Andersen and

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Duke 2018; Blomer and Milićević 2015; Drappeau 2017; Kıral 2016; Sarnak and Tsimerman 2009] for some recent developments and applications.

Recently, the dual question about cancellations between Kloosterman sums $K_s(n;\, p)$, and more general functions (see below) has attracted quite a lot of attention due to its applications to several other problems; see [Blomer et al. 2017a; 2017b; Fouvry et al. 2004; 2014a; 2015; 2017; Kowalski et al. 2017; 2018; Liu et al. 2018; 2019; Shparlinski 2019; Shparlinski and Zhang 2016; Wu and Xi 2016; Xi 2017].

Furthermore, it turns out that Kloosterman sums are representatives of a much richer class of *isotypic trace functions* $\mathcal{K}(n)$ which are associated with isotypic trace sheaves $\mathcal{F}$ modulo p of bounded conductor; we refer to [Fouvry et al. 2014a; 2014b] for precise definitions and properties of trace functions.

For our purposes we do not need to know any specific deep properties of trace functions, it is quite enough to use the facts which are summarised, for example, in [Fouvry et al. 2014a; 2014b]. We also note that this class of functions includes

- normalised Kloosterman sums $p^{-(s-1)/2}K_{s,p}(n)$;
- traces of Frobenius of elliptic curves modulo p;
- exponential functions of the form $\boldsymbol{e}(\psi(n)/p)$ with a rational function $\psi(Z) \in \mathbb{Q}(Z)$, and similar values of multiplicative characters $\chi(\psi(n))$, as well as their products (excluding for the exceptional function $\boldsymbol{e}(an/p)\chi(n)$ with $a \in \mathbb{Z}$);

see, for example, [Fouvry et al. 2014a, Remark 1.4].

Here, for a positive integer N, a prime p, and an isotypic trace function $\mathcal{K}(n)$ modulo p we consider the sum

$$\mathsf{M}_p(\mathcal{K}, N) = \sum_{n \le N} \mu(n)\mathcal{K}(n) \tag{1-2}$$

with the Möbius function, which is given by $\mu(n) = 0$ if an integer m is divisible by a prime square and $\mu(n) = (-1)^r$ if m is a product of r distinct primes.

We note that some of the motivation to consider the sums (1-2) comes from the program devised by Sarnak [2012] to establish the pseudorandomness of the Möbius function and in particular, to show that it is not correlated with other arithmetic sequences. Generally, the goal is to improve the bound

$$\mathsf{M}_p(\mathcal{K}, N) = O(N) \tag{1-3}$$

which trivially holds for many trace functions.

In particular, Fouvry, Kowalski and Michel [Fouvry et al. 2014a, Theorem 1.7], have given a bound on $\mathsf{M}_p(\mathcal{K}, N)$ for a wide class of isotypic trace functions $\mathcal{K}(n)$ of bounded conductor with a power saving against the trivial bound (1-3) assuming that

$$N \ge p^{3/4+\varepsilon} \tag{1-4}$$

for some fixed $\varepsilon > 0$. Furthermore, it is natural to expect that [Blomer et al. 2017b, Theorem 1.8], which gives a nontrivial bound for sums of $\mathcal{K}(\ell)$ over prime arguments ℓ, can also be extended to the sums (1-2) and thus improve the bound of [Fouvry et al. 2014a, Theorem 1.7]; however it is not obvious how to extend the range (1-4).

Here we use a different approach to obtain a nontrivial bound on the sums (1-2) in a shorter range

$$N \geq p^{1/2+\varepsilon}. \tag{1-5}$$

On the other hand, the saving now is only logarithmic. We remark that Fouvry, Kowalski and Michel [Fouvry et al. 2014a, Remark 1.9] mention the possibility of a nontrivial bound in the range (1-5) via the method of Bourgain, Sarnak and Ziegler [Bourgain et al. 2013, Theorem 2]. We however obtain an explicit bound on the saving, which seems to be stronger than the one achievable via the approach of [Bourgain et al. 2013]. More precisely, using a version of [Bourgain et al. 2013, Theorem 2] seems to lead to a saving which is about a square-root of our saving.

Furthermore, for a fixed integer $\nu \geq 1$ we also consider the sums

$$\mathsf{T}_{p,\nu}(\mathcal{K}, N) = \sum_{n \leq N} \tau_\nu(n)\mathcal{K}(n)$$

with the generalised divisor function $\tau_\nu(n)$, which is defined as the number of ordered representations $n = d_1 \dots d_\nu$ with integer numbers $d_1, \dots, d_\nu \geq 1$. Using the bound

$$\sum_{n \leq z} \tau_\nu(n) = O(z(\log z)^{\nu-1}) \tag{1-6}$$

for any real $z \geq 2$, see [Iwaniec and Kowalski 2004, Equation (1.80)], we see that

$$\mathsf{T}_{p,\nu}(\mathcal{K}, N) = O(N(\log N)^{\nu-1}),$$

with the implied constant which depends only on ν.

As in the case of $\mathsf{M}_p(\mathcal{K}, N)$, we also give nontrivial bounds on the sums $\mathsf{T}_{p,\nu}(\mathcal{K}, N)$ under the condition (1-5). We remark that our treatment of the sums $\mathsf{T}_{p,\nu}(\mathcal{K}, N)$ follows the same pattern as for the sums $\mathsf{M}_p(\mathcal{K}, N)$ but is more involved in the case of the divisor function since only its average values admit good bounds (see (1-6) or (8-1) below), while individual values can be quite large.

In the case of $\nu = 2$ and normalised s-dimensional Kloosterman sums $\mathcal{K}(n) = p^{-(s-1)/2}K_{s,p}(n)$, a nontrivial bound on $\mathsf{T}_{p,2}(\mathcal{K}, N)$, with a power saving, has recently been given by Kowalski, Michel and Sawin [Kowalski et al. 2018] under the condition

$$N \geq p^{2/3+\varepsilon}.$$

Here we also extend this range to (1-5).

We also remark, that at least in the case of Kloosterman sums, one can use the results and ideas of Liu, Shparlinski and Zhang [Liu et al. 2019] to obtain a power saving bounds for analogues of $\mathsf{M}_p(\mathcal{K}, N)$ and $\mathsf{T}_{p,\nu}(\mathcal{K}, N)$ modulo prime powers $q = p^k$, in a much shorter range, namely, for $N \ge q^{\varepsilon}$.

2. Our approach and main results

To estimate $\mathsf{M}_p(\mathcal{K}, N)$ and $\mathsf{T}_{p,\nu}(\mathcal{K}, N)$ we employ the method of [Korolëv 2018], also used in [Gong et al. 2017]. This is then combined with some results of Fouvry, Kowalski and Michel [Fouvry et al. 2014a]; see Lemma 4.1 below.

Before we formulate our results we need to recall that the notations $F \ll G$ and $F = O(G)$ are equivalent to $|F| \le cG$ for some constant $c > 0$, which throughout the paper may occasionally depend on the integer parameters ν and the conductor of the trace function $\mathcal{K}$.

Following [Fouvry et al. 2014a], we say that $\mathcal{K}(n)$ is a *nonexceptional* function modulo p if it is not proportional to a function of the form $\boldsymbol{e}(an/p)\chi(n)$ with an integer a and a multiplicative character χ modulo p.

Theorem 2.1. *For any fixed real* $\varepsilon > 0$, *if a prime* p *and an integer* N *satisfy* (1-5), *then for any nonexceptional isotypic trace function* $\mathcal{K}$ *associated to some sheaf* $\mathcal{F}$ *modulo* p *we have*

$$\mathsf{M}_p(\mathcal{K}, N) \ll \varepsilon^{-1} N \frac{\log\log p}{\log p}.$$

We also have a similar result for $\mathsf{T}_{p,\nu}(\mathcal{K}, N)$.

Theorem 2.2. *For any fixed integer* $\nu \ge 2$ *and real* $\varepsilon > 0$, *if a prime* p *and an integer* N *satisfy* (1-5), *then for any nonexceptional isotypic trace function* $\mathcal{K}$ *associated to some sheaf* $\mathcal{F}$ *modulo* p *we have*

$$\mathsf{T}_{p,\nu}(\mathcal{K}, N) \ll \varepsilon^{-\nu} N \frac{(\log\log p)^{\nu}}{\log p}.$$

We recall our convention that in Theorems 2.1 and 2.2 the implied constants may depend on the conductor of $\mathcal{K}$.

3. Some applications

As an application of Theorem 2.1, we now consider the correlation of the Möbius function $\mu(n)$ with the characteristic function $\chi_{\alpha,\beta}(\varphi(n))$ of angles

$$\varphi(n) = \arccos\left(\frac{1}{2p^{1/2}} K_p(n)\right)$$

of regular Kloosterman sums $K_p(n) = K_{2,p}(n)$ being contained in a given interval $[\alpha, \beta] \subseteq [0, \pi]$. Indeed, a combination of Theorem 2.1 with a result of Katz [1988] which says that

$$\mathcal{K} : n \mapsto \operatorname{sym}_k \varphi(n),$$

where

$$\operatorname{sym}_k \varphi = \frac{\sin((k+1)\varphi)}{\sin \varphi},$$

is a nonexceptional trace function for every integer $k \geq 1$, instantly yields the following result:

Corollary 3.1. *For any fixed real $\varepsilon > 0$, if a prime p and an integer N satisfy* (1-5), *then for any continuous function $f : [0, \pi] \to \mathbb{C}$ we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(\varphi(n)) = 0.$$

We can now approximate $\chi_{\alpha,\beta}$ by a sequence of continuous functions to obtain the desired noncorrelation statement.

Similarly, Theorem 2.2 implies that under the condition (1-5), the angles $\varphi(n)$, $n = 1, \dots, N$, are asymptotically distributed with the *Sato–Tate density*

$$\lambda_{\mathrm{ST}}(\alpha) = \frac{2}{\pi} \int_0^{\alpha} \sin^2 \vartheta \, d\vartheta,$$

when counted with weights given by the generalised divisor function $\tau_\nu(n)$.

This is immediate from the following analogue of Corollary 3.1, which follows from Theorem 2.2 and [Katz 1988].

Corollary 3.2. *For any fixed real $\varepsilon > 0$, if a prime p and an integer N satisfy* (1-5), *then for any continuous function $f : [0, \pi] \to \mathbb{C}$ we have*

$$\lim_{N \to \infty} \frac{1}{T_\nu(N)} \sum_{n \leq N} \tau_\nu(n) f(\varphi(n)) = \frac{2}{\pi} \int_0^{\alpha} f(\vartheta) \sin^2 \vartheta \, d\vartheta,$$

where

$$T_\nu(N) = \sum_{n \leq N} \tau_\nu(n).$$

Again, approximating $\chi_{\alpha,\beta}$ by a sequence of continuous functions we obtain the desired distribution result.

4. Correlations of trace functions

We recall the following bound which is combination of [Fouvry et al. 2014a, Proposition 6.2 and Theorem 6.3]; see also [Fouvry et al. 2004; 2015; 2017] for several other variations of this result.

Lemma 4.1. *For any nonexceptional isotypic trace function $\mathcal{K}$ associated to some sheaf $\mathcal{F}$ modulo a prime p of bounded conductor, there exists a set $\mathcal{E}_{\mathcal{F}} \subseteq \mathbb{F}_p$ of cardinality $\#\mathcal{E}_{\mathcal{F}} \ll 1$, such that uniformly over $a \in \mathbb{F}_p \setminus \mathcal{E}_{\mathcal{F}}$ we have*

$$\sum_{n=1}^{p} \mathcal{K}(n)\overline{\mathcal{K}(an)}\boldsymbol{e}(hn/p) \ll p^{1/2}.$$

Now using a standard reduction between complete and incomplete sums, see [Iwaniec and Kowalski 2004, Section 12.2], we immediately derive a bound on a sum of products of $\mathcal{K}(an)\mathcal{K}(bn)$, $i = 1, \dots, \nu$, with n running over an interval $[1, N]$.

Corollary 4.2. *For any nonexceptional isotypic trace function $\mathcal{K}$ associated to some sheaf $\mathcal{F}$ modulo a prime p of bounded conductor, there exists a set $\mathcal{E}_{\mathcal{F}} \subseteq \mathbb{F}_p$ of cardinality $\#\mathcal{E}_{\mathcal{F}} \ll 1$, such that uniformly over $a, b \in \mathbb{F}_p$ with $a/b \notin \mathbb{F}_p \setminus \mathcal{E}_{\mathcal{F}}$ and an arbitrary integer $N \le p$ we have*

$$\sum_{n=1}^{N} \mathcal{K}(an)\overline{\mathcal{K}(bn)} \ll p^{1/2} \log p.$$

5. Integers avoiding some prime divisors

Given two real numbers $y \ge x > 2$ and an integer $N \ge 1$ we denote by $\mathcal{A}_0(N; x, y)$ the set of positive integers $n \le N$ that do not have a prime divisor in the half-open interval $(x, y]$. We need the following upper bound on the cardinality $\#\mathcal{A}_0(N; x, y)$, which follows instantly from the so called *fundamental lemma* of combinatorial sieve; see, for example, [Friedlander and Iwaniec 2010, Lemma 6.8] or [Tenenbaum 2015, Part I, Theorem 4.4].

Lemma 5.1. *Uniformly over integers N and real x, y with $N \ge y \ge x \ge 2$, we have*

$$\#\mathcal{A}_0(N; x, y) \ll N \frac{\log x}{\log y}.$$

For $N \ge y \ge 2$, we use $\Psi(N, y)$ to denote the set of positive integers $n \le N$ whose prime divisors are at most y.

The following bound on the cardinality $\#\Psi(N, y)$ is well-known; see, for example, [Tenenbaum 2015, Part III, Theorem 5.1].

Lemma 5.2. *For any N and y with $N \ge y \ge 2$, we have*

$$\#\Psi(N, y) \ll N \exp\left(-\frac{\log N}{2 \log y}\right).$$

6. Sums of the divisor function over integers without small prime divisors

For $N \ge y \ge 2$, we use $\Phi(N, y)$ to denote the set containing 1 and all integers $n \le N$ whose prime divisors are at least y.

Suppose that $N \ge y \ge 2$. For an integer $\nu \ge 1$, we set

$$S_\nu(N; y) = \sum_{n \in \Phi(N,y)} \tau_\nu(n).$$

Since $\tau_1(n) = 1$, by Lemma 5.1 taken with $x = 2$, we have

$$S_1(N, y) = \#\Phi(N, y) \ll \frac{N}{\log y}. \tag{6-1}$$

Lemma 6.1. *For a fixed integer $\nu \ge 2$, for any N and y with $N \ge y \ge 2$, we have*

$$S_\nu(N; y) \ll \frac{N(\log N)^{\nu-1}}{(\log y)^\nu}.$$

Proof. If $\sqrt{N} < y \le N$, then the desired estimate follows from prime number theorem, since in this case one has

$$S_\nu(N; y) = \sum_{\sqrt{N} < p \le N} \tau_\nu(p) \ll \frac{N}{\log N} \ll N\frac{(\log N)^{\nu-1}}{(\log y)^\nu}.$$

Thus, in the following discussion we assume that $2 \le y \le \sqrt{N}$.

We now establish the bound by induction on ν. The bound (6-1) treats the case where $\nu = 1$.

Assume that the result holds for $S_{\nu-1}(N; y)$. We now derive it for $\nu \ge 2$. First, we write

$$\begin{aligned} S_\nu(N; y) &= \sum_{d \in \Phi(N,y)} \sum_{m \in \Phi(N/d,y)} \tau_{\nu-1}(m) = \sum_{d \in \Phi(N,y)} S_{\nu-1}(N/d; y) \\ &\le \sum_{d \in \Phi(N/y,y)} S_{\nu-1}(N/d; y) + S_{\nu-1}(N; y). \end{aligned}$$

By the induction assumption, we obtain

$$\begin{aligned} S_\nu(N; y) &\le \sum_{d \in \Phi(N/y,y)} \frac{N(\log(N/d))^{k-2}}{d(\log y)^{k-1}} + \frac{N(\log N)^{\nu-2}}{(\log y)^{k-1}} \\ &\le \frac{N(\log N)^{\nu-2}}{d(\log y)^{k-1}} \sum_{d \in \Phi(N/y,y)} \frac{1}{d} + \frac{N(\log N)^{\nu-2}}{(\log y)^{k-1}}. \end{aligned}$$

Using (6-1) by partial summation one easily derives that for any $Z \ge 1$

$$\sum_{d \in \Phi(Z,y)} \frac{1}{d} \le \frac{\log Z}{\log y}$$

and the result follows. □

7. Proof of Theorem 2.1

We can certainly assume that
$$N < p,$$
because for larger values of N the result of Fouvry, Kowalski and Michel [Fouvry et al. 2014a, Theorem 1.7] is stronger.

We fix some real x and y and for an integer $r \ge 0$ we denote by $\mathcal{A}_r(N; x, y)$ the set of positive integers $n \le N$ which have exactly r prime divisors (counted with multiplicities) in the half-open interval $\mathcal{I} = (x, y]$.

In particular, the cardinality of $\mathcal{A}_0(N; x, y)$ has been estimated in Lemma 5.1. Let R be the largest value of r for which $\mathcal{A}_r(N; x, y) \ne \varnothing$. In particular, we have the trivial bound
$$R \ll \log N. \tag{7-1}$$

We now write
$$\mathsf{M}_p(\mathcal{K}, N) = \sum_{r=0}^{R} U_r, \tag{7-2}$$
where
$$U_r = \sum_{n \in \mathcal{A}_r(N;x,y)} \mu(n)\mathcal{K}(n).$$

Note that any trace function $\mathcal{K}(n)$ of the type we consider is bounded pointwise by its conductor, that is, for a trace function $\mathcal{K}(n)$ of bounded conductor we have
$$\mathcal{K}(n) \ll 1. \tag{7-3}$$
In particular using (7-3) and estimating U_0 trivially as
$$U_0 \ll \#\mathcal{A}_0(N; x, y) \ll N \frac{\log x}{\log y},$$
where have also recalled Lemma 5.1, we obtain
$$\mathsf{M}_p(\mathcal{K}, N) \ll \sum_{r=1}^{R} |U_r| + N \frac{\log x}{\log y}. \tag{7-4}$$

Clearly, every square-free integer $n \in \mathcal{A}_r(N; x, y)$ has exactly r representations as $n = \ell m$ with a prime $\ell \in \mathcal{I}$ and integer $m \in \mathcal{A}_{r-1}(N/\ell; x, y)$. Hence, for $r = 1, \ldots, R$,
$$U_r = \frac{1}{r} \sum_{\ell \in \mathcal{I}} \sum_{\substack{m \in \mathcal{A}_{r-1}(N/\ell;x,y) \\ \gcd(\ell,m)=1}} \mu(\ell m)\mathcal{K}(\ell m).$$
Throughout the proof, ℓ always denotes a prime number.

Changing the order of summation and using the multiplicativity of the Möbius function, we now write

$$U_r = \frac{1}{r} V_r, \tag{7-5}$$

where, using $\mu(\ell) = -1$ we have

$$\begin{aligned} V_r &= \sum_{m\in\mathcal{A}_{r-1}(N/x;x,y)} \mu(m) \sum_{\substack{\ell\in\mathcal{I}\cap[1,N/m]\\ \gcd(\ell,m)=1}} \mu(\ell)\mathcal{K}(\ell m) \\ &= -\sum_{m\in\mathcal{A}_{r-1}(N/x;x,y)} \mu(m) \sum_{\substack{\ell\in\mathcal{I}\cap[1,N/m]\\ \gcd(\ell,m)=1}} \mathcal{K}(\ell m). \end{aligned}$$

Let us define the integer K by the inequality

$$x2^K \le \lfloor y\rfloor + 1 < x2^{K+1}. \tag{7-6}$$

We now partition the interval $\mathcal{I}$ into at most $K+1 = O(\log y)$ intervals

$$\mathcal{I}_k = (x_k,\, y_k] \tag{7-7}$$

with

$$x_k = 2^k\lceil x\rceil \quad\text{and}\quad y_k = \min\{2x_k,\, \lfloor y\rfloor\},\quad k = 0,\ldots,K.$$

Thus

$$|V_r| \le \sum_{k=0}^{K} |V_{k,r}|, \tag{7-8}$$

where

$$V_{k,r} = \sum_{m\in\mathcal{A}_{r-1}(N/x_k;x,y)} \mu(m) \sum_{\substack{\ell\in\mathcal{I}_k\cap[1,N/m]\\ \gcd(\ell,m)=1}} \mathcal{K}(\ell m).$$

Clearly for each $m \in \mathcal{A}_{r-1}(N/x_k;\, x,\, y)$, there are at most $r-1$ primes $\ell \in \mathcal{I}$ with $\gcd(\ell, m) > 1$. Hence, using (7-3), at the cost of the error term $O(rN/x_k)$ we can discard the co-primality condition $\gcd(\ell, m) = 1$ and write

$$V_{k,r} \ll W_{k,r} + rN/x_k, \tag{7-9}$$

where

$$W_{k,r} = \sum_{m\in\mathcal{A}_{r-1}(N/x_k;x,y)} \Bigl|\sum_{\ell\in\mathcal{I}_k\cap[1,N/m]} \mathcal{K}(\ell m)\Bigr|.$$

By the Cauchy inequality, we obtain

$$W_{k,r}^2 \le \#\mathcal{A}_{r-1}(N/x_k;\, x,\, y) \sum_{m\in\mathcal{A}_{r-1}(N/x_k;x,y)} \Bigl|\sum_{\ell\in\mathcal{I}_k\cap[1,N/m]} \mathcal{K}(\ell m)\Bigr|^2. \tag{7-10}$$

We now also use the trivial estimate

$$\#\mathcal{A}_{r-1}(N/x_k; x, y) \le N/x_k$$

(more precise estimates are available but do not improve the final result) and extend the summation to all positive integer $m \le N/x_k$, which yields

$$W_{k,r}^2 \le N x_k^{-1} \sum_{m \le N/x_k} \Bigg| \sum_{\ell \in \mathcal{I}_k \cap [1, N/m]} \mathcal{K}(\ell m) \Bigg|^2.$$

Finally, squaring out and changing the order of summation, we obtain

$$W_{k,r}^2 \le N x_k^{-1} \sum_{\ell_1, \ell_2 \in \mathcal{I}_k} \sum_{m \le N/\max\{x_k, \ell_1, \ell_2\}} \mathcal{K}(\ell_1 m)\overline{\mathcal{K}(\ell_2 m)}. \tag{7-11}$$

For at most $O(y_k)$ pairs (ℓ_1, ℓ_2) with $\ell_1/\ell_2 \in \mathcal{E}_{\mathcal{F}}$, where the set $\mathcal{E}_{\mathcal{F}}$ is as in Lemma 4.1, we estimate the inner sum trivially as $N x_k^{-1}$, For the remaining $O(y_k^2)$ pairs (ℓ_1, ℓ_2) we recall that $N \le p$ and apply Corollary 4.2 with $\nu = 2$. Therefore,

$$\begin{aligned} W_{k,r}^2 &\ll N x_k^{-1}(y_k N x_k^{-1} + y_k^2 p^{1/2} \log p) \\ &\ll N^2 x_k^{-1} + N p^{1/2} x_k \log p. \end{aligned} \tag{7-12}$$

Recalling the definition $x_k = 2^k \lceil x \rceil$ we obtain

$$W_{k,r} \ll N x^{-1/2} 2^{-k/2} + N^{1/2} p^{1/4} x^{1/2} 2^{k/2} (\log p)^{1/2}. \tag{7-13}$$

Substituting (7-13) into (7-9), we obtain

$$V_{k,r} \ll N x^{-1/2} 2^{-k/2} + N^{1/2} p^{1/4} x^{1/2} 2^{k/2} (\log p)^{1/2} + r N x^{-1} 2^{-k}.$$

Thus, substituting this bound into (7-8) we derive

$$\begin{aligned} V_r &\ll \sum_{k=0}^{K} \big(N x^{-1/2} 2^{-k/2} + N^{1/2} p^{1/4} x^{1/2} 2^{k/2} (\log p)^{1/2} + r N x^{-1} 2^{-k} \big) \\ &\ll N x^{-1/2} + N^{1/2} p^{1/4} x^{1/2} 2^{K/2} (\log p)^{1/2} + r N/x. \end{aligned}$$

Therefore, recalling the definition of K given by (7-6), we obtain

$$V_r \ll N x^{-1/2} + N^{1/2} p^{1/4} y^{1/2} (\log p)^{1/2} + r N/x,$$

and thus substituting this bound into (7-5) we obtain

$$U_r \ll \frac{1}{r} \big(N x^{-1/2} + N^{1/2} p^{1/4} y^{1/2} (\log p)^{1/2} + r N/x \big). \tag{7-14}$$

Using the bound (7-14) in (7-4), we see that

$$\mathsf{M}_p(\mathcal{K}, N) \ll \sum_{r=1}^{R} \frac{1}{r}\big(Nx^{-1/2} + N^{1/2}p^{1/4}y^{1/2}(\log p)^{1/2} + rN/x\big) + N\frac{\log x}{\log y}$$
$$\ll Nx^{-1/2}\log R + N^{1/2}p^{1/4}y^{1/2}(\log p)^{1/2}\log R + NR/x + N\frac{\log x}{\log y}.$$

We choose $x = (\log p)^4$, $y = p^{\varepsilon/3}$ and recall that by (7-1) we have $\log R \ll \log\log p$. The result now follows.

8. Proof of Theorem 2.2

We fix some real x and y and define the sets $\mathcal{A}_r(N; x, y)$ and the integer R as in the proof of Theorem 2.1. In particular, the bound (7-1) still holds.

Instead of (7-2), we now have

$$\mathsf{T}_{p,\nu}(\mathcal{K}, N) = \sum_{r=0}^{R} U_r,$$

where we define

$$U_r = \sum_{n\in\mathcal{A}_r(N;x,y)} \tau_\nu(n)\mathcal{K}(n).$$

As in the proof of Theorem 2.1, we estimate the sums U_0 and U_r, with $r \geq 1$, separately. We note that now estimating U_0 takes slightly more care than the corresponding bound on U_0 in the proof of Theorem 2.1.

We observe that any $n \in \mathcal{A}_0(N; x, y)$ can be uniquely expressed in the form $n = uv$ or $n = u$ where $u \in \Psi(N, x)$ has no prime divisors greater than x and $v \in \Phi^*(N, y)$, where

$$\Phi^*(N, y) = \Phi(N, y) \setminus \{1\}.$$

Recalling (7-3), we see that

$$|U_0| \ll U_{0,1} + U_{0,2},$$

where

$$U_{0,1} = 2\sum_{u\in\Psi(N,x)} \tau_\nu(u) \quad \text{and} \quad U_{0,2} = \sum_{\substack{u\in\Psi(N,x),\, v\in\Phi^*(N,y)\\ uv\leq N}} \tau_\nu(uv).$$

We estimate $U_{0,1}$ rather crudely. Namely, using the bound

$$\sum_{n\leq z} \tau_\nu^2(n) \ll z(\log z)^{\nu^2-1}, \tag{8-1}$$

which holds for any real $z \geq 2$; see [Iwaniec and Kowalski 2004, Equation (1.80)], by the Cauchy inequality and Lemma 5.2 we see that

$$U_{0,1} \leq \left(\#\Psi(N,x) \sum_{n\leq N} \tau_\nu^2(n)\right)^{1/2} \ll N \exp\left(-\frac{\log N}{4\log x}\right)(\log N)^{(\nu^2-1)/2}. \tag{8-2}$$

Next, recalling the multiplicativity of $\tau_\nu(n)$ and using the notations and the bound of Lemma 6.1, we find

$$\begin{aligned} U_{0,2} &\leq \sum_{u\in\Psi(N/y,x)} \tau_\nu(u) \sum_{v\in\Phi^*(N/u,y)} \tau_\nu(v) \leq \sum_{u\in\Psi(N/y,x)} \tau_\nu(u) S_\nu(Nu^{-1}; y) \\ &\ll \frac{N(\log N)^{\nu-1}}{(\log y)^\nu} \sum_{u\leq N/y} \frac{\tau_\nu(u)}{u} \\ &\ll \frac{N(\log N)^{\nu-1}}{(\log y)^\nu} \prod_{\ell\leq x} \sum_{j=0}^{\infty} \frac{1}{\ell^j}\binom{j+\nu-1}{j}, \end{aligned}$$

where, as in the proof of Theorem 2.1, ℓ always denotes a prime number.

Clearly

$$\sum_{j=0}^{\infty} \frac{1}{\ell^j}\binom{j+\nu-1}{j} = \left(\sum_{i=0}^{\infty} \frac{1}{\ell^i}\right)^\nu = \left(1-\frac{1}{\ell}\right)^{-\nu}.$$

Hence, by the Mertens theorem, see [Iwaniec and Kowalski 2004, Equation (2.16)],

$$\begin{aligned} U_{0,2} &\ll \frac{N(\log N)^{\nu-1}}{(\log y)^\nu} \prod_{\ell\leq x}\left(1-\frac{1}{\ell}\right)^{-\nu} \\ &\ll \frac{N(\log N)^{\nu-1}(\log x)^\nu}{(\log y)^\nu}. \end{aligned} \tag{8-3}$$

Therefore, combining (8-2) and (8-3), we see that instead of (7-4) we now have

$$\begin{aligned} \mathsf{T}_{p,\nu}(\mathcal{K}, N) \ll \sum_{r=1}^{R} |U_r| + N\exp\left(-\frac{\log N}{4\log x}\right)(\log N)^{(\nu^2-1)/2} \\ + \frac{N(\log N)^{\nu-1}(\log x)^\nu}{(\log y)^\nu}. \end{aligned} \tag{8-4}$$

Then writing $n = \ell m \in \mathcal{A}_r(N; x, y)$ with a prime $\ell \in \mathcal{I}$ and integer $m \in \mathcal{A}_{r-1}(N/\ell; x, y)$ we have

$$U_r = U_{r,1} + U_{r,2},$$

where $U_{r,1}$, $U_{r,2}$ denote the contributions coming from n with the conditions $\gcd(m, \ell) = 1$ and $\gcd(m, \ell) > 1$, respectively. We estimate $U_{r,2}$ rather crudely.

Namely, using the bound (1-5), we obtain

$$
\begin{aligned}
|U_{r,2}| &\le \sum_{x<\ell\le y} \sum_{\substack{m\le N/\ell \\ \ell\mid m}} \tau_\nu(\ell m) \le 2 \sum_{x<\ell\le y} \sum_{s\le N/\ell^2} \tau_\nu(\ell^2 s) \\
&\le 2 \sum_{x<\ell\le y} \tau_\nu(\ell^2) \sum_{s\le N/\ell^2} \tau_\nu(s) \ll N(\log N)^{\nu-1} \sum_{\ell>x} \frac{1}{\ell^2} \ll \frac{N(\log N)^{\nu-1}}{x\log x}.
\end{aligned}
$$

Furthermore, any $n \in \mathcal{A}_r(N; x, y)$ from the sum $U_{r,1}$ has exactly r representations of the form $n = \ell m$ where $\ell \in \mathcal{I}$ and $m \in \mathcal{A}_{r-1}(N/\ell; x, y)$ with $\gcd(m, \ell) = 1$. Therefore, since $\tau_\nu(\ell) = \nu$, we obtain

$$
\begin{aligned}
U_{r,1} &= \frac{1}{r} \sum_{x<\ell\le y} \sum_{\substack{m\in\mathcal{A}_{r-1}(N/\ell;x,y) \\ \gcd(m,\ell)=1}} \tau_\nu(\ell m)\mathcal{K}(\ell m) \\
&= \frac{\nu}{r} \sum_{x<\ell\le y} \sum_{\substack{m\in\mathcal{A}_{r-1}(N/\ell;x,y) \\ \gcd(m,\ell)=1}} \tau_\nu(m)\mathcal{K}(\ell m).
\end{aligned}
$$

Thus, defining the intervals $\mathcal{I}_k$, $k = 0, \dots, K$ as in (7-7), we have the inequality

$$
|U_{r,1}| \le \frac{\nu}{r} \sum_{k=0}^{K} |V_{k,r}|,
$$

where

$$
V_{k,r} = \sum_{x_k<\ell\le y_k} \sum_{\substack{m\in\mathcal{A}_{r-1}(N/\ell;x,y) \\ \gcd(m,\ell)=1}} \tau_\nu(m)\mathcal{K}(\ell m).
$$

If we drop the condition $\gcd(m, \ell) = 1$ in the above inner sum then we see that it introduces an error

$$
\sum_{x_k<\ell\le y_k} \sum_{\substack{s\in\mathcal{A}_{r-2}(N/\ell^2;x,y) \\ \gcd(m,\ell)=1}} \tau_\nu(\ell s)\mathcal{K}(\ell^2 s) \ll \sum_{x_k<\ell\le y_k} \sum_{s\le N/\ell^2} \tau_\nu(\ell s) \ll \frac{N(\log N)^{\nu-1}}{x_k\log x_k}.
$$

Thus, we have the following analogue of (7-9)

$$
|V_{k,r}| \ll |W_{k,r}| + \frac{N(\log N)^{\nu-1}}{x_k\log x_k}, \tag{8-5}
$$

where

$$
\begin{aligned}
W_{k,r} &= \sum_{x_k<\ell\le y_k} \sum_{m\in\mathcal{A}_{r-1}(N/\ell;x,y)} \tau_\nu(m)\mathcal{K}(\ell m) \\
&= \sum_{m\in\mathcal{A}_{r-1}(N/x_k;x,y)} \tau_\nu(m) \sum_{\ell\in\mathcal{I}_k\cap[1,N/m]} \mathcal{K}(\ell m).
\end{aligned}
$$

Now, by the Cauchy inequality and also recalling (8-1), we obtain

$$|W_{k,r}|^2 \le \frac{N}{x_k}(\log N)^{\nu^2-1} \sum_{m \le N/x_k} \Bigg| \sum_{\ell \in \mathcal{I}_k \cap [1,N/m]} \mathcal{K}(\ell m)\Bigg|^2 .$$

We now proceed exactly as in the proof of Theorem 2.1 except that we have an extra factor of $(\log N)^{\nu^2-1}$ in analogues of the bounds (7-11)–(7-13). Hence recalling (8-5) and using very crude estimates

$$\nu - 1 < (\nu^2-1)/2 < \nu^2 - 1 < \nu^2$$

(using more accurate bounds does not change the final result), instead of (7-14) we obtain

$$U_r \ll \frac{1}{r}\big(Nx^{-1/2} + N^{1/2}p^{1/4}y^{1/2}(\log p)^{1/2} + rNx^{-1}\big)(\log N)^{\nu^2}.$$

Using this bound in (8-4), we see that

$$\begin{aligned} \mathsf{T}_{p,\nu}(\mathcal{K}, N) &\ll (\log N)^{\nu^2} \sum_{r=1}^{R} \frac{1}{r}\left(Nx^{-1/2} + N^{1/2}p^{1/4}y^{1/2}(\log p)^{1/2} + rN/x\right) \\ &\qquad + N \exp\left(-\frac{\log N}{4\log x}\right)(\log N)^{(\nu^2-1)/2} + \frac{N(\log N)^{\nu-1}(\log x)^{\nu}}{(\log y)^{\nu}} \\ &\ll \big(Nx^{-1/2}\log R + N^{1/2}p^{1/4}y^{1/2}(\log p)^{1/2}\log R + NR/x\big)(\log N)^{\nu^2} \\ &\qquad + N \exp\left(-\frac{\log N}{4\log x}\right)(\log N)^{(\nu^2-1)/2} + \frac{N(\log N)^{\nu-1}(\log x)^{\nu}}{(\log y)^{\nu}}. \end{aligned}$$

We now choose $x = (\log p)^{2(\nu^2+1)}$, $y = p^{\varepsilon/3}$ and recall that by (7-1) we have $\log R \ll \log\log p$. The result follows.

9. Comments

Examining [Fouvry et al. 2014a, Proposition 6.2 and Theorem 6.3], one can easily see that the dependence of implied constants on the conductor of $\mathcal{K}$ in the assumptions of Theorems 2.1 and 2.2 is polynomial.

It is easy to see that our approach also applies to the sums

$$\sum_{n \le N} \mu(n)\boldsymbol{e}(ag^n/q) \quad \text{and} \quad \sum_{n \le N} \mu(n)\chi(ag^n+1)$$

with some integer g of multiplicative order t modulo a q; see [Banks et al. 2004, Theorem 5.1]. Using the techniques and results from [Banks et al. 2012; Bourgain 2005] one is likely to be able to improve [Banks et al. 2004, Theorem 5.1], however our approach leads to nontrivial bounds in a wider range of parameters N and t.

One can also obtain similar results for sums with the divisor function $\tau_\nu(n)$ instead of the Möbius function and in fact with many other multiplicative functions. Certainly Corollaries 3.1 and 3.2 can be extended to angles

$$\varphi(n) = \arccos\left(\frac{1}{sp^{(s-1)/2}} K_{s,p}(n)\right)$$

of multidimensional Kloosterman sums and of many other trace functions, for which results on nondegenerality of the functions $\operatorname{sym}_k \varphi(n)$ can be established.

However, in order to obtain quantitative bounds on the rate of convergence, one needs to work out the explicit dependence of our results, and thus the results of [Fouvry et al. 2014a], on the conductor of the trace function $\mathcal{K}$, which we pose as an open question.

We also note that for a wide class of trace functions, including Kloosterman sums, a very broad extension of Lemma 4.1 has been given in [Fouvry et al. 2015]. To formulate this we consider the actions

$$\gamma(n) = \frac{an+b}{cn+d}$$

of matrices

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{F}_p).$$

Then for a wide class of trace functions $\mathcal{K}(n)$, under some natural *normality* conditions of the matrices $\gamma_1, \ldots, \gamma_m$ we have

$$\sum_{n=1}^{p} \prod_{j=1}^{m} \mathcal{K}(\gamma_j(n)) \boldsymbol{e}(hn/p) \ll p^{1/2}; \tag{9-1}$$

see [Fouvry et al. 2015, Corollary 1.6]. In the most interesting case when $\mathcal{K}(n)$ is given by Kloosterman sums $\mathcal{K}(n) = p^{-(s-1)/2} K_{s,p}(n)$, these normality conditions reduce to the constraint that at least one matrix γ in the sequence $\gamma_1, \ldots, \gamma_m$ appears an odd number of times.

Combining (9-1) (which we need only for linear transformations $n \mapsto an+b$) with the argument of [Ostafe and Shparlinski 2012] one can obtain nontrivial bounds on sums of trace functions $\mathcal{K}(n)$ over integers n whose sum of binary digits is given. More precisely, let $\sigma(n)$ denote the sum of binary digits of n. For any integers $0 \le s \le r$, we define $\mathcal{G}_s(r)$ as the set of integers with r binary digits such that the sum of the digits is equal to s, that is,

$$\mathcal{G}_s(r) = \{0 \le n < 2^r \mid \sigma(n) = s\} \quad \text{and} \quad \#\mathcal{G}_s(r) = \binom{r}{s}.$$

Then, the bound (9-1) implies an analogue of bounds of [Ostafe and Shparlinski 2012, Theorems 1 and 2] for a wide class of trace functions $\mathcal{K}(n)$. In particular, as in [Ostafe and Shparlinski 2012] we see that if $2^r = p^{1+o(1)}$ then for any $\delta > 0$ there exists some $\eta > 0$ such that for $r/2 \ge s \ge (\rho_0 + \delta)r$ we have

$$\sum_{n \in \mathcal{G}_s(r)} \mathcal{K}(n) \ll \binom{r}{s}^{1-\eta}, \tag{9-2}$$

where $\rho_0 = 0.11002786\ldots$ is the root of the equation

$$H(\vartheta) = 1/2, \qquad 0 \le \vartheta \le 1/2,$$

with the *binary entropy function*

$$H(\gamma) = \frac{-\gamma \log \gamma - (1-\gamma)\log(1-\gamma)}{\log 2}.$$

In particular, the bound (9-2) holds for sums with Kloosterman sums $\mathcal{K}(n) = p^{-(s-1)/2} K_{s,p}(n)$.

It is also interesting to consider sums of trace functions over integers with other digit restrictions. For example, for integers with fixed binary digits at s prescribed positions; see [Dietmann et al. 2017] for some relevant results.

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