Pacific Journal of Mathematics

A CRITERION FOR MODULES OVER GORENSTEIN LOCAL RINGS TO HAVE RATIONAL POINCARÉ SERIES

ANJAN GUPTA

Volume 305 No. 1 March 2020

PACIFIC JOURNAL OF MATHEMATICS
Vol. 305, No. 1, 2020
dx.doi.org/10.2140/pjm.2020.305.165

A CRITERION FOR MODULES OVER GORENSTEIN LOCAL RINGS TO HAVE RATIONAL POINCARÉ SERIES

ANJAN GUPTA

We prove that modules over an Artinian Gorenstein local ring R have rational Poincaré series sharing a common denominator if $R/\mathrm{socle}(R)$ is a Golod ring. If R is a Gorenstein local ring with square of the maximal ideal being generated by at most two elements, we show that modules over R have rational Poincaré series sharing a common denominator. By a result of Şega, it follows that R satisfies the Auslander–Reiten conjecture. We provide a different proof of a result of Rossi and Şega (*Adv. Math.* 259 (2014), 421–447) concerning rationality of Poincaré series of modules over compressed Gorenstein local rings. We also give a new proof of the fact that modules over Gorenstein local rings of codepth at most 3 have rational Poincaré series sharing a common denominator, which is originally due to Avramov, Kustin and Miller (*J. Algebra* 118:1 (1988), 162–204).

1. Introduction

Let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Let M be a finitely generated module over R. The Poincaré series of M over R is a formal power series in $\mathbb{Z}[|t|]$ defined as

$$P_M^R(t) = \sum_{i\geq 0} \beta_i^R(M)t^i \in \mathbb{Z}[|t|],$$

where $\beta_i^R(M) = \dim_k \mathrm{Tor}_i^R(M, k)$ denotes the i-th Betti number of M. We say that a formal power series $P(t) \in \mathbb{Z}[|t|]$ is a rational function if there exists a polynomial $g(t) \in \mathbb{Z}[t]$ such that $g(t)P(t)$ is a polynomial in $\mathbb{Z}[t]$. An example due to Anick [1982] shows that the Poincaré series $P_k^R(t)$ is not a rational function in general. Bøgvad [1983] observed that $P_k^R(t)$ may not be a rational function even if R is a Gorenstein ring.

Following Roos [2005, Definition 2.1], we say that a ring R is *good* if there exists a polynomial $d_R(t) \in \mathbb{Z}[t]$ such that $d_R(t)P_M^R(t) \in \mathbb{Z}[t]$ for every finitely

MSC2010: 13D02, 13D40, 13H10.
Keywords: Poincaré series, Golod rings, DG algebras, fibre products, connected sums.

generated R-module M and *bad* otherwise. Roos proved that bad rings exist [2005, Theorem 2.4]. Nevertheless there are an abundance of good rings, e.g., regular local rings, local complete intersections; see [Gulliksen 1974, Corollary 4.2]. We refer to [Avramov et al. 1988; 1994] for more examples of good rings and a detailed account of applications of rationality of Poincaré series.

We use $\mu(-)$ to denote the minimal number of generators. The embedding dimension of R ($= \dim_k \mathfrak{m}/\mathfrak{m}^2$) is denoted by $\operatorname{edim}(R)$. Let $\hat{R}$ denote the $\mathfrak{m}$-adic completion of R. By Cohen's structure theorem, there is a regular local ring Q with maximal ideal $\mathfrak{n}$ and a surjective ring homomorphism $\eta : Q \to \hat{R}$ such that $\ker \eta = I \subset \mathfrak{n}^2$. The map η is called a minimal Cohen presentation of R. The Loewy length of R is defined as $\operatorname{ll}(R) = \max\{i : \mathfrak{m}^i \neq 0\}$ if R is Artinian and infinity otherwise.

We recall a few more examples of good rings collected from existing literature. Precise references are given with each of the examples.

Examples. Let R be a Gorenstein local ring such that $\operatorname{edim}(R) = n \geq 2$, $\operatorname{ll}(R) = s$ and $\mu(I) = r$. If R satisfies one of the conditions (1)–(3) below, then $P_k^R(t) = (1+t)^n/d_R(t)$ for some polynomial $d_R(t) \in \mathbb{Z}[t]$ and $d_R(t)P_M^R(t) \in \mathbb{Z}[t]$ for every finitely generated R-module M.

(1) R is a compressed Artinian Gorenstein ring (see Definition 4.3) and $s \geq 2$, $s \neq 3$. If $\eta : Q \to R$ is a minimal Cohen presentation of R, then the polynomial $d_R(t)$ is given by $1 - t(P_R^Q(t) - 1) + t^{n+1}(1+t)$; see [Rossi and Şega 2014, Theorem 5.1].

(2) R is an Artinian Gorenstein ring and $\mu(\mathfrak{m}^2) = 1$. The polynomial $d_R(t)$ is given by $1 - nt + t^2$; see [Sally 1980, Theorem 2] and [Croll et al. 2018, Theorem 5.4]

(3) R is not a complete intersection and $\operatorname{codepth}(R) = \operatorname{edim}(R) - \operatorname{depth}(R) \leq 3$. The polynomial $d_R(t)$ is equal to $1 - rt^2 - rt^3 + t^5$; see [Wiebe 1969, Satz 9] and [Avramov et al. 1988, Theorem 6.4].

The main objective of the present article is to give a criterion for Gorenstein local rings to be good, which provides a common method to prove the good property in each of the above examples. As a new application we show that if R is an Artinian Gorenstein ring and $\mu(\mathfrak{m}^2) = 2$, then R is a good ring.

We recall a few definitions. Let $\phi : R \to S$ be a surjective homomorphism of local rings and k be the common residue field of R and S. From the standard change of rings spectral sequence of Tor, Serre proved the following term-wise inequality of power series:

$$P_k^S(t) \prec \frac{P_k^R(t)}{1 - t(P_S^R(t) - 1)}.$$

The homomorphism ϕ is called a Golod homomorphism if the above inequality is an equality. The most widespread method to show that a ring R is good is to use a result of Levin (Theorem 2.2) which states that a ring R is good if there is a surjective Golod homomorphism from a complete intersection onto R.

Let $\eta : Q \twoheadrightarrow \hat{R}$ be a minimal Cohen presentation of R. We say that R is a Golod ring if η is a Golod homomorphism. Let $\operatorname{edim}(R) = n$ and K^R denote the Koszul complex of R on a minimal set of generators of maximal ideal $\mathfrak{m}$. It follows that R is a Golod ring whenever one has

$$P_k^R(t) = \frac{(1+t)^n}{1 - \sum_{i=1}^n \dim_k \mathrm{H}_i(K^R) t^{i+1}}.$$

We refer to [Avramov 1998, §3] for more details on Golod rings and Golod homomorphisms. The main result of the present article is the following:

Theorem I. *Let R be an Artinian Gorenstein local ring of embedding dimension $n \geq 2$ such that $R/\operatorname{socle}(R)$ is a Golod ring. Let $\eta : Q \to R$ be a minimal Cohen presentation, $\mathfrak{n}$ denote the maximal ideal of Q and $I = \ker(\eta) \subset \mathfrak{n}^2$. Then the following hold.*

(1) *For any $f \in I \setminus \mathfrak{n}I$, the induced map $Q/(f) \twoheadrightarrow R$ is a Golod homomorphism.*

(2) *Let $d_R(t) = 1 - t(P_R^Q(t) - 1) + t^{n+1}(1+t)$. Then for any R-module M we have $d_R(t) P_M^R(t) \in \mathbb{Z}[t]$.*

It is worth noting that if R is an Artinian Gorenstein ring, $\operatorname{edim}(R) \geq 2$ and $R/\operatorname{socle}(R)$ is a Golod ring, then with the notation used in the above theorem,

$$P_k^R(t) = \frac{(1+t)^n}{d_R(t)}$$

by a result of Rossi and Şega [2014, Proposition 6.2]. Therefore, statement (2) is an immediate consequence of statement (1) and the result of Levin.

The following is proved as an application of Theorem I.

Theorem II. *Let R be an Artinian Gorenstein local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let M be a finitely generated R-module. Assume that $\operatorname{edim}(R) = n$ and $\mu(\mathfrak{m}^2) \leq 2$. Then the following hold.*

(1) *If $n = 1$, then $P_k^R(t) = \frac{1}{1-t}$ and $(1-t)P_M^R(t) \in \mathbb{Z}[t]$.*

(2) *If $n \geq 2$, then $P_k^R(t) = \frac{1}{1-nt+t^2}$ and $(1+t)^n(1-nt+t^2)P_M^R(t) \in \mathbb{Z}[t]$.*

(3) *If $\operatorname{Ext}^i(M, M) = 0$ for all $i \geq 1$, then M is a free R-module.*

Statement (3) follows from statements (1) and (2) by an argument of Şega [2003]. It implies that R satisfies the Auslander–Reiten conjecture [1975].

The rings considered in Theorem II are called *stretched* when $\mu(\mathfrak{m}^2) = 1$ and *almost stretched* when $\mu(\mathfrak{m}^2) = 2$ (see Definition 3.11). Stretched Cohen–Macaulay local rings were introduced by Sally [1980]. She proved that $P_k^R(t)$ is rational for such a ring R [1980, Theorem 2]. Later Elias and Valla introduced almost stretched Cohen–Macaulay local rings. They proved that if R is an almost stretched Gorenstein local ring and the residue field k of R has characteristic zero, then $P_k^R(t)$ is rational [Elias and Valla 2009, Theorem 1.1]. In a recent article [Croll et al. 2018, Corollary 5.6], stretched Cohen–Macaulay local rings are shown to be good. Using Theorem II, we prove that stretched Cohen Macaulay and almost stretched Gorenstein rings are good without any assumption on residue fields. We also prove that such rings satisfy the Auslander–Reiten conjecture.

Now we briefly describe the organisation of the article. In Section 2, we prove Theorem I. The proof extensively uses a characterisation theorem for Golod algebras (see Theorem 2.1) and chain derivations on acyclic closures whose construction dates back to the work of Gulliksen. The connected sum of Gorenstein local rings was introduced in [Ananthnarayan et al. 2012] (see Definition 3.2). In Section 3, we provide a criterion for connected sum decompositions of Gorenstein local rings. We show that if R is an Artinian Gorenstein local ring with maximal ideal $\mathfrak{m}$ and $\mu(\mathfrak{m}^2) \leq 2$, then R decomposes as a connected sum unless $\mu(\mathfrak{m}) \leq 2$ (Corollary 3.5). We use this decomposition to show that quotients of such a ring R by nonzero powers of maximal ideal $\mathfrak{m}$ are Golod rings (Lemma 3.8). This fact is crucially used in the proof of Theorem II. Finally, Section 4 contains new proofs of Examples (1) and (3) using Theorem I. We identify a certain quotient C of the Koszul algebra K^R of an Artinian Gorenstein ring R such that R is a surjective image of a complete intersection under a Golod homomorphism whenever C is a Golod DG algebra (Section 2B). We show that for the ring considered in Examples (1) and (3), this quotient is a Golod algebra. We make it a point to advertise here that our versions are slightly stronger than the earlier ones in both examples since we constructed Golod homomorphisms from hypersurfaces given by any choice of generator belonging to a minimal generating set of the defining ideal.

We conclude with a remark that our approach only constructs Golod homomorphisms from hypersurface rings. We hope that the present approach can be generalised further to find criteria for existence of Golod homomorphisms from complete intersections of higher codimension.

All rings in this article are Noetherian local rings with $1 \neq 0$. All modules are nonzero and finitely generated. Throughout this article, the expression "local ring $(R, \mathfrak{m}, k)$" refers to a commutative Noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. When information on the residue field is not necessary, we denote a local ring R with maximal ideal $\mathfrak{m}$ simply by $(R, \mathfrak{m})$.

2. The main result

Let $(R, \mathfrak{m}, k)$ be a local ring. A DG algebra (A, ∂) over the ring R consists of a nonnegatively graded strictly skew-commutative R-algebra $A = \oplus_{i \geq 0} A_i$ such that

$A_0 = R/I$ for some ideal I of R and an R-linear differential map ∂ of degree -1 satisfying the Leibniz rule. A DG-algebra homomorphism $\phi : A \to B$ is a chain map of complexes which induces a ring homomorphism $\phi^\# : A^\# \to B^\#$ between underlying skew-commutative rings $A^\#$ and $B^\#$ after forgetting the differential maps on A and B. The DG-algebra B is called a semifree extension of A if $B^\#$ is a free module over $A^\#$.

The DG algebra (A, ∂) is augmented if it is equipped with a surjective DG algebra homomorphism $\epsilon : A \twoheadrightarrow k$. If $\tilde{\epsilon} : \mathrm{H}(A) \to k$ is the induced map on homology, we set $\mathrm{I}A = \ker \epsilon$, $\mathrm{IH}(A) = \ker \tilde{\epsilon}$ and $\mathrm{IZ}(A) = \mathrm{I}A \cap \mathrm{Z}(A)$. We say that the DG algebra (A, ∂) is minimal if $\partial(A) \subset \mathfrak{m}A$. A minimal DG algebra A is augmented naturally with the surjective map $\epsilon = q \circ pr$ where $pr : A \twoheadrightarrow A_0$ is the projection and $q : A_0 \twoheadrightarrow k$ is the natural quotient map. We refer to [Avramov 1998] for more information on DG algebras and related terminologies.

2A. ***Tate resolutions.*** Tate described a method to construct a DG algebra resolution of the residue field k over R. The method involves an iterated process of adjoining exterior variables to kill cycles of even degrees and divided powers variables to kill cycles of odd degrees starting from R. In literature, this construction is known as Tate resolution. Later Gulliksen proved that if the number of variables added at each step of killing cycles of a certain degree is the minimum possible, the resulting Tate resolution becomes a minimal free resolution of the residue field k. In this case, the DG algebra is called the acyclic closure of k over R which is unique up to isomorphism of DGΓ algebras. We refer the reader to [Avramov 1998, §6] and [Gulliksen and Levin 1969, Chapter 1] for more details.

In this article, by Tate resolution we mean a surjective R-linear quasi-isomorphism $\epsilon : R\langle X\rangle \twoheadrightarrow k$ where $R\langle X\rangle$ is the acyclic closure of k. The adjoined set of variables $X = \{X_i : i \geq 1\}$ is ordered such that $1 \leq \deg(X_i) \leq \deg(X_j)$ for $i < j$. Note that by construction the acyclic closure $R\langle X\rangle$ is a semifree extension of R.

An R-linear derivation of degree n on the acyclic closure $R\langle X\rangle$ is an R-linear map $\eta : R\langle X\rangle \to R\langle X\rangle$ of degree n satisfying the following properties:

(1) $\eta(R) = 0$ (R-linearity).

(2) η satisfies the Leibniz rule; that is, $\eta(uv) = \eta(u)v + (-1)^{n \deg(u)} x \eta(v)$ for $u, v \in R\langle X\rangle$.

(3) $\eta(X_i^{(i)}) = \eta(X_i) X_i^{(i-1)}$, with $X_i^{(i)}$ being the i-th divided power of a variable X_i of even positive degree.

The derivation η is called a chain derivation if it commutes with the differential ∂ of $R\langle X\rangle$ in the graded sense, i.e., $\eta \circ \partial = (-1)^n \partial \circ \eta$.

Gulliksen and Levin [1969, Theorem 1.6.2] constructed a sequence of R-linear chain derivations η_j on the acyclic closure $R\langle X\rangle$ such that $\eta_j(X_j) = 1$ and $\eta_j(X_i) = 0$ for $i < j$.

2B. ***Golod algebras.*** An augmented DG algebra A over R with augmentation map $\epsilon : A \twoheadrightarrow k$ is called a Golod algebra if A admits a trivial Massey operation, i.e., there are a graded k-basis $\mathfrak{b}_R = \{h_\lambda\}_{\lambda\in\Lambda}$ of $\mathrm{IH}(A)$, a function $\mu : \bigsqcup_{i=1}^{\infty} \mathfrak{b}_R^i \to A$ such that $\mu(h_\lambda) \in \mathrm{IZ}(A)$ with $\mathrm{cls}(\mu(h_\lambda)) = h_\lambda$, and setting $\bar{a} = (-1)^{i+1}a$ for $a \in A_i$ one has

$$\partial\mu(h_{\lambda_1}, \dots, h_{\lambda_p}) = \sum_{j=1}^{p-1} \overline{\mu(h_{\lambda_1}, \dots, h_{\lambda_j})}\mu(h_{\lambda_{j+1}}, \dots, h_{\lambda_p}).$$

The following is proved in [Levin 1976, Theorem 1.5] and also follows from [Levin 1985, Theorem 1.1].

Theorem 2.1. *Let $f : (R, \mathfrak{m}) \twoheadrightarrow (S, \mathfrak{n})$ be a surjective homomorphism of local rings with common residue field k and $\epsilon : R\langle X\rangle \twoheadrightarrow k$ be a DG algebra resolution of k over R. Set $A = R\langle X\rangle \otimes_R S$. Consider A augmented with the augmentation $\epsilon \otimes_R S$. Then the following are equivalent*:

(1) *The DG algebra A is a Golod algebra.*

(2) *The map f is a Golod homomorphism.*

(3) *The induced maps* $\mathrm{Tor}^R(k, k) \to \mathrm{Tor}^S(k, k)$ *and* $\mathrm{Tor}^R(\mathfrak{n}, k) \to \mathrm{Tor}^S(\mathfrak{n}, k)$ *are injective.*

We recall the following result of Levin recorded in [Avramov et al. 1988, Proposition 5.18].

Theorem 2.2. *Let $(R, \mathfrak{m}, k)$ be a local ring and $\phi : P \twoheadrightarrow R$ be a surjective Golod homomorphism from a local complete intersection P of embedding dimension n onto R. Then there exists a polynomial $d_R(t) \in \mathbb{Z}[t]$ such that for any finitely generated R-module M, we have $d_R(t)P_M^R(t) \in \mathbb{Z}[t]$. Further, $d_R(t)P_k^R(t) = (1+t)^n$.*

We are now equipped to prove the main result.

2C. ***Proof of Theorem I.***

Proof. Let the maximal ideal of R be $\mathfrak{m}$ and $k = R/\mathfrak{m}$ denote the residue field of R. We know that $\mathrm{H}_1(K^R) \cong I/\mathfrak{n}I$. Therefore, the minimal generators of I are in one-to-one correspondence with the generators of $\mathrm{H}_1(K^R)$. Let the maximal ideal $\mathfrak{n}$ of Q be minimally generated by $y_1, \dots, y_n$. The Koszul complex of Q is $K^Q = Q\langle X_i : \partial(X_i) = y_i, 1 \le i \le n\rangle$. Let $f = \sum_{i=1}^n a_i y_i$ and $P = Q/(f)$. Note that $K^P = K^Q \otimes_Q P$ and $K^R = K^Q \otimes_Q R$ are Koszul complexes of P and R, respectively. Set $z = \sum_{i=1}^n a_i X_i \in K_1^Q$. Then its residue class $\bar{z}$ is a cycle in $\mathrm{Z}_1(K^P)$. Let $V = K^P\langle T : \partial(T) = \bar{z}\rangle$ be the extension of K^P by adjoining a divided powers variable T of degree 2 to kill the cycle $\bar{z}$. By [Avramov 1998, Theorem 7.3.3], the natural augmentation $V \twoheadrightarrow k$ is the Tate resolution of k over P.

Set $U = V \otimes_R R = K^R\langle T : \partial(T) = \bar{z}\rangle$. Since $f \in I \setminus \mathfrak{n}I$, we have $\bar{z} \in \mathrm{Z}_1(K^R) \setminus B_1(K^R)$. Therefore, we can adjoin variables to U to obtain the acyclic closure $\mathfrak{X}$ of the residue field k over R. The augmentation map $\epsilon : \mathfrak{X} \twoheadrightarrow k$ is the Tate resolution of k over R.

By Theorem 2.1, to show that $P \twoheadrightarrow R$ is a Golod homomorphism, we need to prove that the induced maps $\operatorname{Tor}^P(k, k) \to \operatorname{Tor}^R(k, k)$ and $\operatorname{Tor}^P(\mathfrak{m}, k) \to \operatorname{Tor}^R(\mathfrak{m}, k)$ are injective. Both V and $\mathfrak{X}$ are minimal algebras. Therefore, the first map is $U \otimes_R k \to \mathfrak{X} \otimes_R k$ which is obviously injective since $\mathfrak{X}$ is a semifree extension of U. The second map is $i_* : \mathrm{H}(\mathfrak{m}U) \to \mathrm{H}(\mathfrak{m}\mathfrak{X})$ which is induced by the inclusion $\mathfrak{i} : \mathfrak{m}U \to \mathfrak{m}\mathfrak{X}$. We prove that $\mathfrak{i}_*$ is an injective map.

We have an R-linear chain derivation $\upsilon : \mathfrak{X} \to \mathfrak{X}$ of degree -2 such that $\upsilon(T) = 1$. Set $\bar{R} = R/\operatorname{socle}(R)$. Note that $\operatorname{socle}(R) \subset \mathfrak{m}^2$, so $K^{\bar{R}} = \bar{R} \otimes_R K^R$ is the Koszul complex of $\bar{R}$. Now $K^{\bar{R}}$ can be extended to the acyclic closure $\mathfrak{Y}$ over $\bar{R}$. Let $\mathfrak{j} : K^{\bar{R}} \to \mathfrak{Y}$ denote the inclusion. The augmentation $\epsilon_{\mathfrak{Y}} : \mathfrak{Y} \twoheadrightarrow k$ is an algebra homomorphism over K^R. The acyclic closure $\mathfrak{X}$ is semifree over K^R. Therefore, the augmentation $\epsilon_{\mathfrak{X}} : \mathfrak{X} \twoheadrightarrow k$ lifts to a DG algebra homomorphism $\beta : \mathfrak{X} \to \mathfrak{Y}$ over K^R [Avramov 1998, Proposition 2.1.9]. Let $\alpha : K^R \to K^{\bar{R}}$ denote the quotient map. By abuse of notation we denote restriction of a map by the same symbol. We have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{m}K^R & \xrightarrow{\ \mathfrak{i}\ } & \mathfrak{m}\mathfrak{X} \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\ \mathfrak{m}K^{\bar{R}} & \xrightarrow{\ \mathfrak{j}\ } & \mathfrak{m}\mathfrak{Y} \end{array}$$

A cycle y in $\mathfrak{m}U$ can be written as $y = \sum_{k=0}^{m} a_k T^{(m-k)},\ a_i \in \mathfrak{m}K^R$. Suppose $\mathfrak{i}(y)$ is in the boundary of $\mathfrak{m}\mathfrak{X}$. We prove by induction on m that y is in the boundary of $\mathfrak{m}U$.

First assume that $m = 0$. Then $y \in \mathfrak{m}K^R$. Since $\mathfrak{i}(y)$ is in the boundary of $\mathfrak{m}\mathfrak{X}$, $\mathfrak{j} \circ \alpha(y)$ is in the boundary of $\mathfrak{m}\mathfrak{Y}$ by the commutative diagram. Now $\bar{R}$ is a Golod ring, so $\mathfrak{j}$ induces an injective map $\mathfrak{j}_* : \mathrm{H}(\mathfrak{m}K^{\bar{R}}) \to \mathrm{H}(\mathfrak{m}\mathfrak{Y})$. Therefore, $\alpha(y)$ is in the boundary of $\mathfrak{m}K^{\bar{R}}$. This implies that $y = sy_1 + \partial(y_2)$ where $y_1 \in K^R,\ y_2 \in \mathfrak{m}K^R$ and $\operatorname{socle}(R) = (s)$.

We know from [Levin and Avramov 1978, Lemma 1.2] that $\operatorname{socle}(R)K_i^R \subset (0 : \mathfrak{m}^2)\,\mathrm{B}_i(K^R)$ for $1 \le i \le n-1$. If $\deg(y) = \deg(y_1) < n$, then $sy_1 \in (0 : \mathfrak{m}^2)\,\mathrm{B}(K^R)$ and consequently $y \in \mathfrak{m}\,\mathrm{B}(K^R) \subset \mathfrak{m}\,\mathrm{B}(U)$. On the other hand if $\deg(y) = n$, then $y_2 = 0$ and $y = sy_1 = asX_1 \cdots X_n,\ a \in R$. Since $\mathrm{H}(K^R)$ is a Poincaré algebra [Avramov and Golod 1971], there is a $z' \in \mathrm{Z}_{n-1}(K^R)$ such that $\bar{z}z' = sX_1 \cdots X_n$. We conclude $y = a\bar{z}z' = \partial(aTz') \in \mathfrak{m}\,\mathrm{B}(U)$. Therefore, the induction step for $m = 0$ follows.

Now we assume that $m > 0$. Note that $a_0 = \upsilon^m(\mathfrak{i}(y))$. Since $\mathfrak{i}(y) \in \mathfrak{m}\,\mathrm{B}(\mathfrak{X})$ and the chain derivation υ commutes with the differential of $\mathfrak{X}$, we have $a_0 \in \mathfrak{m}K^R \cap \mathrm{B}(\mathfrak{m}\mathfrak{X})$. We consider two cases.

Suppose $\deg(a_0) = n$. Then $a_0 \in \mathrm{Z}_n(\mathfrak{m}K^R) = \mathrm{socle}(R)K^R_n$. Therefore, $a_0 = asX_1 \cdots X_n,\ a \in R$. One observes $a_0 T^{(m)} = a\bar{z}z'T^{(m)} = \partial(az'T^{(m+1)}) \in \mathfrak{m}\,\mathrm{B}(U)$. Therefore, $\mathfrak{i}\big(\sum_{k=0}^{m-1} a_k T^{(m-k)}\big) = \mathfrak{i}(y) - a_0 T^{(m)}$ is in the boundary of $\mathfrak{m}\mathfrak{X}$. Consequently, $\sum_{k=0}^{m-1} a_k T^{(m-k)}$ is in the boundary of $\mathfrak{m}U$ by the induction hypothesis. We conclude that y is in the boundary of $\mathfrak{m}U$.

Suppose $\deg(a_0) < n$. Then by the argument in the induction step $m = 0$, one has $a_0 = \partial(y_3)$ for $y_3 \in \mathfrak{m}K^R$. We can write

$$y=\partial(y_3)T^{(m)}+\sum_{k=0}^{m-1}a_k T^{(m-k)}=\partial(y_3 T^{(m)})+\Bigg[(-1)^{\deg(y_3)+1}y_3\bar{z}T^{(m-1)}+\sum_{k=0}^{m-1}a_k T^{(m-k)}\Bigg].$$

The first summand is in $\mathfrak{m}\,\mathrm{B}(U)$. This implies that the second summand is in the boundary of $\mathfrak{m}\mathfrak{X}$ and therefore also in the boundary of $\mathfrak{m}U$ by induction hypothesis. We conclude that y is in the boundary of $\mathfrak{m}U$. This completes the induction step. Hence $\mathfrak{i}_*$ is an injective map and statement (1) follows.

The ring $\bar{R}$ is Golod. The Poincaré series of R is computed in [Rossi and Şega 2014, Proposition 6.2] as

$$P_k^R(t) = \frac{(1+t)^n}{1 - t(P_R^Q(t) - 1) + t^{n+1}(1+t)}$$

so statement (2) follows from Theorem 2.2. □

3. Stretched and almost stretched rings

Our aim in this section is to prove that stretched and almost stretched Gorenstein rings are good. The key step is to show that rings of these types decompose as connected sums. If the residue field is infinite, then Lemma 3.1 follows from [Eakin and Sathaye 1976, Theorem 1]. The proof of the lemma was suggested by the anonymous referee.

Lemma 3.1. *Let $(R, \mathfrak{m}, k)$ be a local ring such that $\mu(\mathfrak{m}^2) \le 2$. Then there exists an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathfrak{m}^2 = x\mathfrak{m}$. Furthermore, if $\mathrm{ll}(R) \ge 3$, then $x^2 \notin \mathfrak{m}^3$.*

Proof. If $\mathfrak{m}^2 = x\mathfrak{m} + \mathfrak{m}^3$ and $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, then by Nakayama's lemma we have $m^2 = x\mathfrak{m}$. Therefore, to prove the first assertion, it is enough to assume that $\mathfrak{m}^3 = 0$, i.e., $\mathfrak{m}^2$ is a k-vector space. If $\mathfrak{m}^2 = 0$, then $\mathfrak{m}^2 = x\mathfrak{m} = 0$ for all $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. If $\mu(\mathfrak{m}^2) = 1$, then for any $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $x\mathfrak{m} \neq 0$, we have $\mathfrak{m}^2 = x\mathfrak{m}$. Therefore, we only need to consider the case when $\mu(\mathfrak{m}^2) = 2$, i.e., $\mathfrak{m}^2$ is a vector space of dimension two.

Let $x_1, \ldots, x_n$ be a minimal generating set of $\mathfrak{m}$. Let r be such that $x_i\mathfrak{m} \neq 0$ for all i with $1 \le i \le r$ but $x_i\mathfrak{m} = 0$ for $i > r$. Assume, by way of contradiction, that $\mathfrak{m}^2 \neq x\mathfrak{m}$ for all $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Thus, if $i \le r$ then $x_i\mathfrak{m}$ is a one-dimensional vector space. We may also assume $\mathfrak{m}^2 = x_1\mathfrak{m} + x_2\mathfrak{m}$. Clearly $x_1\mathfrak{m} \neq x_2\mathfrak{m}$ since otherwise $\mathfrak{m}^2 = x_1\mathfrak{m}$, a contradiction to our assumption.

If $i \leq r$, $j \leq r$ and $x_i x_j \neq 0$, then one observes that $x_i\mathfrak{m} = x_j\mathfrak{m}$. This is true because $x_i\mathfrak{m}$ and $x_j\mathfrak{m}$ are both one-dimensional vector spaces and they share the nonzero element $x_i x_j$. Since $x_1\mathfrak{m} \neq x_2\mathfrak{m}$, we must therefore have $x_1x_2 = 0$. Now $x_1\mathfrak{m} \neq 0$ and $x_2\mathfrak{m} \neq 0$, so there exist i, j with $i \leq r$, $j \leq r$ such that $x_1x_i \neq 0$ and $x_2x_j \neq 0$. This implies $x_1\mathfrak{m} = x_i\mathfrak{m}$ and $x_2\mathfrak{m} = x_j\mathfrak{m}$. In particular, $x_i\mathfrak{m} \neq x_j\mathfrak{m}$ and hence $x_ix_j = 0$. We have $(x_2 + x_i)x_1 = x_1x_i \neq 0$ and $(x_2 + x_i)x_j = x_2x_j \neq 0$, hence $(x_2 + x_i)\mathfrak{m} = x_1\mathfrak{m}$ and $(x_2 + x_i)\mathfrak{m} = x_j\mathfrak{m} = x_2\mathfrak{m}$. This yields $x_1\mathfrak{m} = x_2\mathfrak{m}$, a contradiction. Therefore, the first part of the lemma follows.

If $x^2 \in \mathfrak{m}^3$, then $\mathfrak{m}^3 = x^2\mathfrak{m} \subset \mathfrak{m}^4$. By Nakayama's lemma, $\mathfrak{m}^3 = 0$ which implies $\mathrm{ll}(R) \leq 2$. Therefore, $x^2 \notin \mathfrak{m}^3$ if $\mathrm{ll}(R) \geq 3$. □

We recall definitions of fibre products and connected sums [Ananthnarayan et al. 2012].

Definition 3.2. Let $(R, \mathfrak{m}_R, k)$ and $(S, \mathfrak{m}_S, k)$ be local rings with a common residue field k. Let $\pi_R : R \to k$ and $\pi_S : S \to k$ be natural quotient maps from R and S onto k, respectively. The fibre product of R and S is defined as the ring $R \times_k S = \{(r, s) \in R \times S : \pi_R(r) = \pi_S(s)\}$. The ring $R \times_k S$ is local with maximal ideal $\mathfrak{m}_R \oplus \mathfrak{m}_S$.

Now assume that both R and S are Artinian Gorenstein local rings with one-dimensional socles $\mathrm{socle}(R) = \langle \delta_R \rangle$ and $\mathrm{socle}(S) = \langle \delta_S \rangle$. Then the connected sum of R and S is defined as

$$R\#S = \frac{R \times_k S}{\langle (\delta_R, -\delta_S) \rangle}.$$

We say a Gorenstein local ring Q is decomposable as a connected sum if there are rings R and S such that $Q = R\#S$, $l(R) < l(Q)$ and $l(S) < l(Q)$. Here $l(-)$ denotes the length function.

Define a left module structure on the polynomial ring $T = k[Y_1, \ldots, Y_n]$ over the ring $S = k[X_1, \ldots, X_n]$, $X_i = Y_i^{-1}$, by defining the action of X_i on a monomial $M \in T$ as the usual multiplication if $X_iM \in T$ and zero otherwise. Macaulay's inverse system establishes a one-to-one correspondence between local Artinian Gorenstein algebras

$$R = \frac{k[X_1, \ldots, X_n]}{I}$$

such that $I \subset (X_1, \ldots, X_n)$ and polynomials F in the ring T up to a unit multiple. The correspondence is given by $I = \operatorname{ann} F$; see [Eisenbud 1995, Theorem 21.6]. If Gorenstein local rings R and S correspond to $F \in k[Y_1, \ldots, Y_m]$ and $G \in k[Y_{m+1}, \ldots, Y_n]$, respectively, then the connected sum $R\#S$ corresponds to $F + G \in k[Y_1, \ldots, Y_n]$; see [Ananthnarayan 2009, Remark 4.24].

The following result follows from [Ananthnarayan et al. 2019, Proposition 4.1] and can be proved easily for Artinian k-algebras using Macaulay's inverse system.

Theorem 3.3. *Let $(Q, \mathfrak{n}, k)$ be a regular local ring and $I \subset \mathfrak{n}^2$ be an ideal such that $R = Q/I$ is an Artinian Gorenstein local ring. Let $\mathfrak{n}$ be minimally generated by $x_1, \dots, x_m, y_1, \dots, y_n$ such that $(x_1, \dots, x_m)(y_1, \dots, y_n) \subset I$. Let $\max\{i : (x_1, \dots, x_m)^i \not\subset I\} = s$ and $\max\{i : (y_1, \dots, y_n)^i \not\subset I\} = t$. Then there are ideals I_1 and I_2 in Q containing $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$, respectively, such that the following hold.*

(1) *The rings $S = Q/I_1$ and $T = Q/I_2$ are Gorenstein rings. Further, $\operatorname{edim}(S) = n$, $\operatorname{edim}(T) = m$, $\operatorname{ll}(S) = t$, and $\operatorname{ll}(T) = s$.*

(2) *$R = S\#_k T$.*

The next theorem is the key to decomposing an Artinian Gorenstein local ring $(R, \mathfrak{m}, k)$ with $\mu(\mathfrak{m}^2) \leq 2$ as a connected sum. When $\operatorname{edim}(R) = 2$ and $\operatorname{char}(k) = 0$, the theorem follows from [Elias and Valla 2008, Theorem 4.1].

Theorem 3.4. *Let $(R, \mathfrak{m}, k)$ be a local Artinian Gorenstein ring. Let $\operatorname{edim}(R) = n$, $\operatorname{ll}(R) \geq 3$ and $\dim_k \mathfrak{m}^2/\mathfrak{m}^3 = m < n$. Assume that $\mathfrak{m}$ admits a generator x_1 such that $\mathfrak{m}^2 = x_1\mathfrak{m}$. Then there exists a minimal generating set $\{x_1, x_2, \dots, x_n\}$ of $\mathfrak{m}$ extending x_1 such that the following hold.*

(1) $\mathfrak{m}^2 = (x_1^2, x_1x_2, \dots, x_1x_m)$.

(2) $(x_1, \dots, x_m)(x_{m+1}, \dots, x_n) = 0$.

(3) $(x_{m+1}, \dots, x_n)^2 = \operatorname{socle}(R)$.

The ring R decomposes as a connected sum $R = S\#T$ such that $\operatorname{edim}(S) = m$, $\operatorname{edim}(T) = n - m$, $\operatorname{ll}(S) = \operatorname{ll}(R)$ and $\operatorname{ll}(T) = 2$.

Proof. Since $\operatorname{ll}(R) \geq 3$, we have $x_1^2 \notin \mathfrak{m}^3$. Therefore, we can choose a minimal generating set $\{x_1, x_2, \dots, x_n\}$ of $\mathfrak{m}$ such that $\mathfrak{m}^2 = (x_1^2, x_1x_2, \dots, x_1x_m)$. Statement (1) follows.

We have $x_1x_j = \alpha_{1j}x_1^2 + \alpha_{2j}x_1x_2 + \cdots + \alpha_{mj}x_1x_m$, $\alpha_{ij} \in R$ for $1 \leq i \leq m$ and $m+1 \leq j \leq n$. This gives $x_1(x_j - \alpha_{1j}x_1 - \alpha_{2j}x_2 - \cdots - \alpha_{mj}x_m) = 0$. Replacing $x_j - \alpha_{1j}x_1 - \alpha_{2j}x_2 - \cdots - \alpha_{mj}x_m$ by x_j, we assume that $x_1x_j = 0$ for $m+1 \leq j \leq n$.

If $m = 1$, property (2) is satisfied. We assume that $\mu(\mathfrak{m}^2) = m \geq 2$. Since $\mathfrak{m}^2$ is minimally generated by $\{x_1^2, x_1x_2, \dots, x_1x_m\}$ and $x_1(x_{m+1}, \dots, x_n) = 0$, we have

$$(0 : x_1) \subset \mathfrak{m}(x_1, \dots, x_m) + (x_{m+1}, \dots, x_n).$$

This implies that the residue classes of elements $x_{m+1}, \dots, x_n$ form a k-basis of $((0 :_R x_1) + \mathfrak{m}^2)/\mathfrak{m}^2$. Therefore, $\dim_k(((0 :_R x_1) + \mathfrak{m}^2)/\mathfrak{m}^2) = n - m$.

Claim 1: $\mathfrak{m}[(0 :_R x_1) \cap \mathfrak{m}^2] = \operatorname{socle}(R)$.

Proof. Note $\mathfrak{m}^2(0:_R x_1) = x_1\mathfrak{m}(0:_R x_1) = 0$. This implies $\mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2] \subset \operatorname{socle}(R)$. Therefore, it is enough to prove that $\mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2] \neq 0$. We have $\mathfrak{m}^{i+1} = x_1\mathfrak{m}^i$ for $i \geq 1$. This means that $\mu(\mathfrak{m}^{i+1}) \leq \mu(\mathfrak{m}^i)$, $i \geq 1$. Let $t = \max\{i : \mu(\mathfrak{m}^i) = m\}$. Then $t \geq 2$. Since $m \geq 2$, we have $\operatorname{ll}(R) \geq t+1$. The map $\mathfrak{m}^t/\mathfrak{m}^{t+1} \xrightarrow{\cdot x_1} \mathfrak{m}^{t+1}/\mathfrak{m}^{t+2}$ is not injective since $\dim_k(\mathfrak{m}^t/\mathfrak{m}^{t+1}) > \dim_k(\mathfrak{m}^{t+1}/\mathfrak{m}^{t+2})$. Therefore, we find $y \in \mathfrak{m}^t \setminus \mathfrak{m}^{t+1}$ such that $yx_1 \in \mathfrak{m}^{t+2}$. Note that $\mathfrak{m}^{t+2} = x_1\mathfrak{m}^{t+1}$. It follows that $yx_1 = x_1 m$ for some $m \in \mathfrak{m}^{t+1}$. Consequently $x_1(y-m) = 0$. Clearly, $y - m \in [(0:_R x_1)\cap\mathfrak{m}^2]$ and $y - m \notin \mathfrak{m}^{t+1}$. Since $\operatorname{socle}(R) \subset \mathfrak{m}^{t+1}$, we have $y - m \notin \operatorname{socle}(R)$. Therefore, $[(0:_R x_1)\cap\mathfrak{m}^2] \not\subset \operatorname{socle}(R)$. We conclude that $\mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2] \neq 0$ and the claim is proved. □

Claim 2: $\dim_k\big([(0:_R x_1)\cap\mathfrak{m}^2]/(\mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2])\big) = m-1$.

Proof. We know that $(0:_R x_1) = \operatorname{Hom}_R(R/(x_1), R)$. We have $\mathfrak{m}^2 = x_1\mathfrak{m} \subset (x_1)$. By Matlis duality $l(0:_R x_1) = l(R/x_1R) = l(R/\mathfrak{m}^2) - l((x_1R+\mathfrak{m}^2)/\mathfrak{m}^2) = 1+n-1 = n$. Now we have

$$\begin{aligned} l[(0:_R x_1)\cap\mathfrak{m}^2] &= l(0:_R x_1) + l(\mathfrak{m}^2) - l[(0:_R x_1)+\mathfrak{m}^2] \\ &= l(0:_R x_1) - l\left[\frac{(0:_R x_1)+\mathfrak{m}^2}{\mathfrak{m}^2}\right] = n-(n-m) = m. \end{aligned}$$

Therefore,

$$\dim_k \frac{[(0:_R x_1)\cap\mathfrak{m}^2]}{\mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2]} = l\left(\frac{[(0:_R x_1)\cap\mathfrak{m}^2]}{\mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2]}\right) = l[(0:_R x_1)\cap\mathfrak{m}^2]-1 = m-1. \ \square$$

Claim 3: The pairing

$$\frac{(x_2,\ldots,x_m)}{\mathfrak{m}(x_2,\ldots,x_m)} \times \frac{[(0:_R x_1)\cap\mathfrak{m}^2]}{\mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2]} \to \operatorname{socle}(R)$$

given by $(\bar{x},\bar{y}) \to xy$ is well defined and nondegenerate.

Proof. We have $\mathfrak{m}(x_2,\ldots,x_m)(0:_R x_1) = 0$ since $\mathfrak{m}^2 = x_1\mathfrak{m}$. This implies that $(x_2,\ldots,x_m)[(0:_R x_1)\cap\mathfrak{m}^2] \subset \operatorname{socle}(R)$. Therefore, the above pairing exists. Note that $\mathfrak{m}^2(x_{m+1},\ldots,x_n) = 0$. As a result, if $y \in [(0:_R x_1)\cap\mathfrak{m}^2]$ and $y(x_2,\ldots,x_m) = 0$, we have $y \in \operatorname{socle}(R) = \mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2]$. This implies that the map

$$\frac{[(0:_R x_1)\cap\mathfrak{m}^2]}{\mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2]} \to \operatorname{Hom}\left(\frac{(x_2,\ldots,x_m)}{\mathfrak{m}(x_2,\ldots,x_m)}, \operatorname{socle}(R)\right)$$

induced by the above pairing is injective. We have

$$\dim_k \frac{(x_2,\ldots,x_m)}{\mathfrak{m}(x_2,\ldots,x_m)} = \dim_k \frac{[(0:_R x_1)\cap\mathfrak{m}^2]}{\mathfrak{m}[(0:_R x_1)\cap\mathfrak{m}^2]} = m-1.$$

Therefore, the above map is an isomorphism and consequently the pairing is nondegenerate. □

Note that $\mathfrak{m}(x_2, \dots, x_m)(x_{m+1}, \dots, x_n) = 0$ so

$$(x_2, \dots, x_m)(x_{m+1}, \dots, x_n) \subset \operatorname{socle}(R).$$

The multiplication by x_j defines a map $(x_2, \dots, x_m)/(\mathfrak{m}(x_2, \dots, x_m)) \to \operatorname{socle}(R)$ for $m+1 \le j \le n$. Since the pairing in Claim 3 is nondegenerate, we have $\hat{x}_j \in [(0:_R x_1)\cap \mathfrak{m}^2]$ such that $x_j x_i = \hat{x}_j x_i$ for $2 \le i \le m$ and $m+1 \le j \le n$. We also have $x_1 x_j = x_1 \hat{x}_j = 0$ for $m+1 \le j \le n$. It follows that $(x_j - \hat{x}_j)(x_1, \dots, x_m) = 0,\ m+1 \le j \le n$. Therefore, replacing $(x_j - \hat{x}_j)$ by x_j we have $(x_1, \dots, x_m)(x_{m+1}, \dots, x_n) = 0$ and property (2) is satisfied.

Let $\operatorname{socle}(R) = \langle \delta \rangle$ and $K = (x_{m+1}, \dots, x_n)$. We have $\mathfrak{m}^2 K = x_1 K \mathfrak{m} = 0$, so $K^2 \subset \mathfrak{m}K \subset \operatorname{socle}(R)$. Note that $K^2 \neq 0$ for otherwise each of the $x_j,\ m+1 \le j \le n$, is in $\operatorname{socle}(R) \subset \mathfrak{m}^2$, a contradiction. Therefore, $K^2 = \mathfrak{m}K = \operatorname{socle}(R)$ and the property (3) is satisfied.

The last statement follows from Theorem 3.3. □

The following is a consequence of Lemma 3.1 and Theorem 3.4.

Corollary 3.5. *Let $(R, \mathfrak{m})$ be an Artinian Gorenstein local ring such that $\operatorname{ll}(R) \ge 3$ and $\mu(\mathfrak{m}^2) \le \min\{2, \operatorname{edim}(R) - 1\}$. Then $R = S\#T$ where $(S, \mathfrak{p})$ and $(T, \mathfrak{q})$ are Gorenstein local rings*, $\operatorname{edim}(S) = \mu(\mathfrak{m}^2)$, $\operatorname{ll}(S) = \operatorname{ll}(R)$ *and* $\operatorname{ll}(T) = 2$.

Lemma 3.6. *Let $(R, \mathfrak{m})$ be an Artinian Gorenstein local ring such that $\operatorname{edim}(R) \ge 2$ and $\operatorname{ll}(R) = s$. Then the quotient ring $R/\mathfrak{m}^i$ is not a Gorenstein ring for $2 \le i \le s$.*

Proof. If possible assume that $R/\mathfrak{m}^i$ is a Gorenstein ring for some i satisfying $2 \le i \le s$. Then the injective hull of k over $R/\mathfrak{m}^i$ is $E_{R/\mathfrak{m}^i}(k) \cong R/\mathfrak{m}^i$. We know that $E_{R/\mathfrak{m}^i}(k) \cong \operatorname{Hom}_R(R/\mathfrak{m}^i, R) = (0:_R \mathfrak{m}^i)$. Consequently $(0:_R \mathfrak{m}^i) = (x)$, a principal ideal for some $x \in R$. Now $\mathfrak{m}^{s-i+1} \subset (0:_R \mathfrak{m}^i)$ and $\mathfrak{m}^{s-i+1} \not\subset \mathfrak{m}(0:_R \mathfrak{m}^i)$ for otherwise $\mathfrak{m}^{s-i+1} \subset \mathfrak{m}(0:_R \mathfrak{m}^i) \subset (0:_R \mathfrak{m}^{i-1})$ which implies $\mathfrak{m}^s = \mathfrak{m}^{s-i+1}\mathfrak{m}^{i-1} = 0$, a contradiction. Since $(0:_R \mathfrak{m}^i)$ is principal, we have $(0:_R \mathfrak{m}^i) = \mathfrak{m}^{s-i+1} = (x)$.

Apply Macaulay's theorem characterising Hilbert function [Bruns and Herzog 1993, Theorem 4.2.10] to the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$. We obtain $\mu(\mathfrak{m}^{n+1}) \le \mu(\mathfrak{m}^n)^{\langle n \rangle}$ for all $n \ge 1$. We already have $\mu(\mathfrak{m}^{s-i+1}) = 1$. This implies that $\mathfrak{m}^j$ is a principal ideal for j satisfying the inequality $s - i + 1 \le j \le s$ and so $l(0:_R \mathfrak{m}^i) = l(\mathfrak{m}^{s-i+1}) = i$. By Matlis duality, $l(R/\mathfrak{m}^i) = l\operatorname{Hom}(R/\mathfrak{m}^i, R) = l(0:_R \mathfrak{m}^i) = i$. We have $\sum_{j=0}^{i-1}[l(\mathfrak{m}^j/\mathfrak{m}^{j+1}) - 1] = l(R/\mathfrak{m}^i) - i = 0$ and each summand is nonnegative. This shows that $\operatorname{edim}(R) = l(\mathfrak{m}/\mathfrak{m}^2) = 1$, a contradiction. □

The following theorem is proved in [Dress and Krämer 1975, Satz 2].

Theorem 3.7. *Let $(S, \mathfrak{m}_S, k)$ and $(T, \mathfrak{m}_T, k)$ be two local rings and $R = S \times_k T$. Then,*

$$\frac{1}{P_k^R(t)} = \frac{1}{P_k^S(t)} + \frac{1}{P_k^T(t)} - 1.$$

If M is an S-module, then

$$\frac{1}{P_M^R(t)}=\frac{P_k^S(t)}{P_M^S(t)}\left(\frac{1}{P_k^S(t)}+\frac{1}{P_k^T(t)}-1\right)=\frac{P_k^S(t)}{P_M^S(t)P_k^R(t)}.$$

The following is a consequence of Lemma 3.1 and Theorem 3.4.

Lemma 3.8. *Let $(R,\mathfrak{m},k)$ be an Artinian Gorenstein local ring such that $\mu(\mathfrak{m})=n$, $\operatorname{ll}(R)\geq 2$ and $\mu(\mathfrak{m}^2)\leq 2$. Let i be an integer satisfying $2\leq i\leq \operatorname{ll}(R)$. Then $R/\mathfrak{m}^i$ is a Golod ring and $P_k^{R/\mathfrak{m}^i}(t)=\frac{1}{1-nt}$.*

Proof. Fix i satisfying $2\leq i\leq \operatorname{ll}(R)$ and set $\bar{R}=R/\mathfrak{m}^i$. The result is clear when $n=1$ or $\operatorname{ll}(R)=2$. First we assume that $n=2$ and $\operatorname{ll}(R)>2$. A result of Scheja [1964] states that a codepth 2 local ring is either a Gorenstein (equivalently complete intersection) or a Golod ring. The ring $\bar{R}$ cannot be a Gorenstein ring by Lemma 3.6 so $\bar{R}$ is a Golod ring. Since R is a complete intersection, the defining ideal of $\bar{R}$ is minimally generated by three elements. It follows that $\kappa^{\bar{R}}(t)=\sum_{i\geq 0}\dim_k \mathrm{H}_i(K^{\bar{R}})t^i=1+3t+2t^2$ and

$$(1-t(\kappa^{\bar{R}}(t)-1))=1-t(1+3t+2t^2-1)=(1+t)^2(1-2t).$$

We have

$$P_k^{\bar{R}}(t)=\frac{(1+t)^2}{1-t(\kappa^{\bar{R}}(t)-1)}=\frac{1}{1-2t}.$$

Now we assume that $n>2$ and $\operatorname{ll}(R)>2$. By Corollary 3.5, it follows that $R=S\#T$ where $(S,\mathfrak{p})$ and $(T,\mathfrak{q})$ are Gorenstein local rings, $\operatorname{edim}(S)=\mu(\mathfrak{m}^2)\leq 2$, $\operatorname{ll}(S)=\operatorname{ll}(R)\geq 3$ and $\operatorname{ll}(T)=2$. One has $\bar{R}=\bar{S}\times_k\bar{T}$ where $\bar{S}=S/\mathfrak{p}^i$ and $\bar{T}=T/\mathfrak{q}^2$. Both $\bar{S}$ and $\bar{T}$ are Golod rings. The ring $\bar{R}$ is a Golod ring because a fibre product of Golod rings is Golod; see [Lescot 1983, Theorem 4.1].

$P_k^{\bar{S}}(t)=1/(1-\operatorname{edim}(\bar{S})t)$ by the case $n=2$ and $P_k^{\bar{T}}(t)=1/(1-\operatorname{edim}(\bar{T})t)$ ([Avramov 1998, Example 4.2.2]). The formula for the Poincaré series follows from Theorem 3.7. □

The following is a well known fact.

Lemma 3.9. *Let $(R,\mathfrak{m},k)$ be a local ring, x be a nonzero divisor of R and $S=R/(x)$. If there exists a polynomial $d(t)\in \mathrm{Z}[t]$ such that $d(t)P_M^S(t)\in\mathbb{Z}[t]$ for all S-modules M, then $d(t)P_M^R(t)\in\mathbb{Z}[t]$ for all R-modules M. Now assume further that $x\in\mathfrak{m}\setminus\mathfrak{m}^2$, then $P_k^S(t)=P_k^R(t)/(1+t)$. The ring S is Golod if and only if R is so.*

Proof. Let M be an R-module and N be the first syzygy of M. We have the exact sequence

$$0\to N\to R^{\mu(M)}\to M\to 0.$$

This implies that $P_M^R(t) = \mu(M) + tP_N^R(t)$. Therefore, it is enough to show that $d(t)P_N^R(t) \in \mathbb{Z}[t]$. Let $F_* \twoheadrightarrow N$ be the minimal free resolution of N over R. Note that x is also a nonzero divisor of N. This implies that $S \otimes_R F_* \twoheadrightarrow S \otimes_R N$ is a minimal free resolution of $S \otimes_R N = \bar{N}$ as an S module. As a result, we have $P_N^R(t) = P_{\bar{N}}^S(t)$. Therefore, the first part of the lemma follows from the hypothesis.

The assertions regarding Poincaré series and Golod property follow from Propositions 3.3.5 (1) and 5.2.4 in [Avramov 1998], respectively. □

The following is due to Şega [2003, Proposition 1.5].

Proposition 3.10. *Let R be a local ring such that there is a $d_R(t) \in \mathbb{Z}[t]$ satisfying $d_R(t)P_M^R(t) \in \mathbb{Z}[t]$ for each finitely generated R-module M. Let $d_R(t) = p(t)q(t)r(t)$ where $p(t)$ is 1 or irreducible, $q(t)$ has nonnegative coefficients, $r(t)$ is 1 or irreducible and has no positive real root among its complex roots of minimal absolute value. Then the following hold for each pair of R-modules M, N.*

(1) *If $\operatorname{Tor}_i^R(M, N) = 0$ for $i \gg 0$, either M or N has finite projective dimension.*
(2) *If $\operatorname{Ext}_R^i(M, N) = 0$ for $i \gg 0$, either M has a finite projective dimension or N has a finite injective dimension.*

We are ready to prove Theorem II.

3A. *Proof of Theorem II.*

Proof. The case $n = 1$ is easy. We skip the details.

Now we assume that $n \geq 2$. The quotient map $R \twoheadrightarrow \frac{R}{\operatorname{socle}(R)}$ is a Golod homomorphism and

$$P_k^{R/\operatorname{socle}(R)}(t) = \frac{P_k^R(t)}{1 - t^2 P_k^R(t)}$$

[Levin and Avramov 1978, Theorem 2]. Therefore, we have

$$P_k^R(t) = \frac{P_k^{R/\operatorname{socle}(R)}(t)}{1 + t^2 P_k^{R/\operatorname{socle}(R)}(t)} = \frac{1}{1/(P_k^{R/\operatorname{socle}(R)}(t)) + t^2} = \frac{1}{1 - nt + t^2}.$$

The last equality follows from Lemma 3.8. The same lemma states that $R/\operatorname{socle}(R)$ is a Golod ring. By Theorem I, the ring R is a surjective image of a complete intersection under a Golod homomorphism. The second part of statement (2) is a consequence of Theorem 2.2.

Now we prove (3). If the projective dimension $\operatorname{pd}_R(M)$ of M is finite, then $\operatorname{Ext}_R^{\operatorname{pd}_R(M)}(M, M) \neq 0$. Therefore, it is enough to show that $\operatorname{pd}_R(M) < \infty$. When $n \leq 2$, R is a complete intersection. The statement follows from [Avramov and Buchweitz 2000, Theorem 4.2]. When $n \geq 3$, the polynomial $(1 - nt + t^2)$ is irreducible.

The statement follows from Proposition 3.10 (take $p(t) = (1 - nt + t^2)$, $q(t) = (1+t)^n$ and $r(t) = 1$). Here one uses the fact that a module over a Gorenstein ring has a finite projective dimension if and only if it has a finite injective dimension. □

Stretched and almost stretched rings were introduced by Sally [1980] and Elias and Valla [2008], respectively.

Definition 3.11. An Artinian local ring R with maximal ideal $\mathfrak{m}$ is called stretched if $\mathfrak{m}^2$ is a principal ideal and almost stretched if $\mathfrak{m}^2$ is minimally generated by two elements.

Let R be a Cohen–Macaulay local ring of dimension d with maximal ideal $\mathfrak{m}$. Then R is called stretched (almost stretched) if there exists a minimal reduction $\underline{x} = x_1, \dots, x_d$ of $\mathfrak{m}$ such that $R/(\underline{x})$ is a stretched (almost stretched) Artinian ring. Here by minimal reduction we mean that $\underline{x}$ satisfies $\mathfrak{m}^{r+1} = (x_1, \dots, x_d)\mathfrak{m}^r$ for some nonnegative integer r.

Stretched Cohen–Macaulay local rings were shown to be good in [Croll et al. 2018, Corollary 5.6]. We outline a different method. In statement (2) of the following corollary, we find a more efficient common denominator of Poincaré series of modules over such rings.

Corollary 3.12. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional stretched Cohen–Macaulay local ring and M be an R-module. Let $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$ and $r = \dim_k \operatorname{Ext}^d_R(k, R)$ denote the type of R. Then the following hold.*

(1) *If $r = n - d$, then R is a Golod ring, $P^R_k(t) = (1+t)^d/(1-(n-d)t)$ and $(1-(n-d)t)P^R_M(t) \in \mathbb{Z}[t]$.*

(2) *If $r \neq n-d$, then $(1+t)^{n-d-r+1}(1-(n-d)t+t^2)P^R_M(t) \in \mathbb{Z}[t]$ and $P^R_k(t) = (1+t)^d/(1-(n-d)t+t^2)$.*

(3) *If $\operatorname{Ext}^i(M, M \oplus R) = 0$ for all $i \geq 1$, then M is a free R-module.*

Proof. To prove statements (1) and (2), it is enough to assume that R is a stretched Artinian ring, i.e., $d = 0$ (see Lemma 3.9). We have $\operatorname{edim}(R) = n$ and $\dim_k \operatorname{socle}(R) = r$. Let $\mathfrak{m} = (x_1, \dots, x_n)$ and $\operatorname{ll}(R) = s$. The ideals $\mathfrak{m}^i$, $i \geq 2$, are principal ideals. If $\operatorname{socle}(R) \subset \mathfrak{m}^2$, then $\operatorname{socle}(R) = \mathfrak{m}^s$ is a principal ideal, so R is a Gorenstein ring. Both statements (1) and (2) follow from Theorem II.

Otherwise assume that $x_1 \in \operatorname{socle}(R) \setminus \mathfrak{m}^2$. One observes that $(x_1) \cap (x_2, \dots, x_n) = 0$ and $(x_1) + (x_2, \dots, x_n) = \mathfrak{m}$. For any two ideals I, J in a ring R, we know that $R = R/I \times_{R/I+J} R/J$. Therefore, it follows that $R = R/(x_1) \times_k R/(x_2, \dots, x_n)$.

The maximal ideal of the ring $R/(x_2, \dots, x_n)$ is generated by the residue class of x_1, so its square is zero since $x_1 \in \operatorname{socle}(R)$. On the other hand, the ring $R/(x_1)$ is a stretched Artinian ring. If the socle of $R/(x_1)$ is contained in the square of its maximal ideal, it is a Gorenstein ring. Otherwise we decompose $R/(x_1)$ again as before.

After a finite number of steps, we have $R = S \times_k T$ where $(S, \mathfrak{m}_S)$ is a stretched Artinian Gorenstein ring and $(T, \mathfrak{m}_T)$ is a local ring with $\mathfrak{m}_T^2 = 0$. Clearly $r = 1+\operatorname{edim}(T)$. This implies that $\operatorname{edim}(T) = r-1$ and $\operatorname{edim}(S) = \operatorname{edim}(R) - \operatorname{edim}(T) = n - r + 1$. By Theorem II, we have

$$P_k^S(t) = \begin{cases} \frac{1}{1-t(n-r+1)+t^2} & \text{when } n \geq r+1, \\ \frac{1}{1-t} & \text{when } n = r. \end{cases}$$

On the other hand T is a Golod ring and $P_k^T(t) = \frac{1}{1-(r-1)t}$. The rational expression of $P_k^R(t)$ follows by the following computation:

$$\begin{aligned} \frac{1}{P_k^R(t)} &= \frac{1}{P_k^S(t)} + \frac{1}{P_k^T(t)} - 1 \\ &= \begin{cases} (1-t) + 1 - (r-1)t - 1 & \text{when } n = r, \\ (1 - t(n-r+1) + t^2) + 1 - (r-1)t - 1 & \text{when } n \geq r+1, \end{cases} \\ &= \begin{cases} 1 - nt & \text{when } n = r, \\ 1 - nt + t^2 & \text{when } n \geq r+1. \end{cases} \end{aligned}$$

If $n = r$, then $\operatorname{edim}(S) = 1$. This implies that S is a Golod ring. Therefore, R is also a Golod ring since a fibre product of Golod rings is Golod [Lescot 1983, Theorem 4.1].

Now we find a polynomial $d_R(t) \in \mathbb{Z}[t]$ such that $d_R(t) P_M^R(t) \in \mathbb{Z}[t]$ for any R-module M. The second syzygy of M is a direct sum of two modules, one is over S and another over T; see [Dress and Krämer 1975, Remark 3]. Therefore, it suffices to assume that $M = M_1 \oplus M_2$ where M_1 and M_2 are modules over S and T, respectively. By Theorem 3.7 we have

$$\frac{P_M^R(t)}{P_k^R(t)} = \frac{1}{P_k^R(t)}(P_{M_1}^R(t) + P_{M_2}^R(t)) = \frac{P_{M_1}^S(t)}{P_k^S(t)} + \frac{P_{M_2}^T(t)}{P_k^T(t)}.$$

We observe that

$$\frac{P_{M_2}^T(t)}{P_k^T(t)} = \frac{1}{P_k^T(t)}(1 + t P^T_{\operatorname{Syz}_1^T(M_2)}(t)) = 1 - (r-1)t + t\frac{P^T_{\operatorname{Syz}_1^T(M_2)}(t)}{P_k^T(t)}.$$

Since the square of the maximal ideal of T is zero, the first syzygy $\operatorname{Syz}_1^T(M_2)$ is a k-vector space. Therefore, $P_{M_2}^T(t)/P_k^T(t)$ is a polynomial in $\mathbb{Z}[t]$.

If $n = r$, we have $\operatorname{edim}(S) = 1$. By (1) of Theorem II, $P_{M_1}^S(t)/P_k^S(t)$ is a polynomial. Hence we conclude that $(1-nt)P_M^R(t) = P_M^R(t)/P_k^R(t) \in \mathbb{Z}[t]$ if $n = r$.

If $n \geq r+1$, we have $\operatorname{edim}(S) = n - r + 1$. By (2) of Theorem II, we have

$$(1+t)^{(n-r+1)} \frac{P_{M_1}^S(t)}{P_k^S(t)} \in \mathbb{Z}[t].$$

Hence we conclude that

$$(1+t)^{(n-r+1)}(1-nt+t^2)P_M^R(t) = (1+t)^{(n-r+1)}\frac{P_M^R(t)}{P_k^R(t)} \in \mathbb{Z}[t]$$

if $n \geq r+1$.

Therefore, both statements (1) and (2) follow. To prove statement (3), it suffices to show that $\mathrm{pd}_R(M) < \infty$. If $n-d=2$, then R is either a complete intersection or a Golod ring; see [Scheja 1964]. In both cases, rings are known to satisfy statement (3); see, for instance, [Avramov and Buchweitz 2000, Theorem 4.2] when R is a complete intersection and [Jorgensen and Şega 2004, Proposition 1.4] when R is a Golod ring. If $n-d>2$, then we see at once that $\mathrm{pd}_R(M) < \infty$ from Proposition 3.10. Here one observes that if the injective dimension of $M \oplus R$ is finite then R is Gorenstein and both projective and injective dimensions of M are finite. □

The following result follows from Lemma 3.9 and Theorem II.

Corollary 3.13. *Let $(R, \mathfrak{m}, k)$ be an almost stretched Gorenstein local ring of dimension d and embedding dimension n. Let M be a finitely generated R-module. Then the following hold.*

(1) *If $n-d=1$, then $P_k^R(t) = (1+t)^d/(1-t)$ and $(1-t)P_M^R(t) \in \mathbb{Z}[t]$.*

(2) *If $n-d \geq 2$, then $(1+t)^{n-d}(1-(n-d)t+t^2)P_M^R(t) \in \mathbb{Z}[t]$ and $P_k^R(t) = (1+t)^d/(1-(n-d)t+t^2)$.*

(3) *If $\mathrm{Ext}^i(M, M) = 0$ for all $i \geq 1$, then M is a free R-module.*

4. Revisiting known results

In this section, we provide proofs of Examples (1) and (3) in the introduction. As the section title suggests, these examples were found by other authors. Our proofs are different and obtained using Theorem I. Further, our versions are slightly stronger; see Remark 4.7. We recall the following from [Levin and Avramov 1978, Theorem 1].

Theorem 4.1. *Let $(R, \mathfrak{m})$ be an Artinian Gorenstein local ring of embedding dimension n and K^R be the Koszul complex on a minimal set of generators of the maximal ideal $\mathfrak{m}$. Set $\mathrm{socle}(R) = (s)$, $\bar{R} = R/sR$ and $K^{\bar{R}} = \bar{R} \otimes_R K^R$, the Koszul complex of $\bar{R}$. Define a DG algebra structure on*

$$K^R \oplus \frac{K^R}{\mathfrak{m}K^R}[-1]$$

with multiplication $(k, \bar{l})(k', \bar{l'}) = (kk', \overline{lk'} + (-1)^{\deg(k)}\overline{kl'})$ and differential $\partial(k, \bar{l}) = (\partial(k) + sl, 0)$. Then the chain map

$$K^R \oplus \frac{K^R}{\mathfrak{m}K^R}[-1] \to K^{\bar{R}}, \qquad (k, \bar{l}) \mapsto \bar{k},$$

is a quasi-isomorphism. If

$$\overline{H} = \frac{\mathrm{H}(K^R)}{\mathrm{H}_n(K^R)} \quad \textit{and} \quad \overline{K} = \frac{K^R \otimes_R k}{K^R_n \otimes_R k}[-1],$$

then $\mathrm{H}(K^{\overline{R}}) = \overline{H} \ltimes \overline{K}$.

Lemma 4.2. *Let* $(R, \mathfrak{m})$ *be an Artinian Gorenstein local ring of embedding dimension* $n \geq 2$. *Let* K^R *denote the Koszul complex on a minimal set of generators of* $\mathfrak{m}$ *and* C *denote the quotient of* K^R *defined by*

$$C_i = \begin{cases} K^R_i & \textit{for } 0 \leq i \leq n-2, \\ K^R_{n-1}/\mathrm{B}_{n-1}(K^R) & \textit{for } i = n-1, \\ 0 & \textit{for } i = n. \end{cases}$$

Then C *has a DG algebra structure. Assume that* C *is a Golod DG algebra with natural augmentation. Then* $R/\operatorname{socle}(R)$ *is a Golod ring and* R *satisfies assertions* (1) *and* (2) *of Theorem I.*

Proof. The fact that C is a DG algebra is straightforward because $K^R_n \oplus \mathrm{B}_{n-1}(K^R)$ is a DG ideal of the Koszul algebra K^R. Let $q : K^R \twoheadrightarrow C$ be the quotient map and $\operatorname{socle}(R) = (s)$. Let $G : K^R/(\mathfrak{m}K^R) \to K^R$ and $H : C/\mathfrak{m}C \to C$ denote chain maps induced by multiplications by s on K^R, C respectively. We have $q \circ G = H \circ \bar{q}$ where $\bar{q} : K^R/(\mathfrak{m}K^R) \twoheadrightarrow C/(\mathfrak{m}C)$ is the map induced by q. Let

$$\mathfrak{c}(G) = K^R \oplus \frac{K^R}{\mathfrak{m}K^R}[-1] \quad \text{and} \quad \mathfrak{c}(H) = C \oplus \frac{C}{\mathfrak{m}C}[-1]$$

be the cones of G and H respectively. Define $\alpha : \mathfrak{c}(G) \twoheadrightarrow \mathfrak{c}(H)$ by $\alpha(k, \bar{l}) = (q(k), \bar{q}(\bar{l}))$. Both $\mathfrak{c}(G)$ and $\mathfrak{c}(H)$ have DG algebra structure and α is a surjective DG algebra homomorphism. The kernel of α is the complex

$$0 \to \frac{K^R_n}{\mathfrak{m}K^R_n} \xrightarrow{\cdot s} K^R_n \xrightarrow{\partial} \mathrm{B}_{n-1}(K^R) \to 0$$

which is exact. Thus, α are a quasi-isomorphism of DG algebras. By Theorem 4.1, $\mathfrak{c}(G)$ is quasi-isomorphic to $K^{R/\operatorname{socle}(R)}$. Therefore, to show that $R/\operatorname{socle}(R)$ is a Golod ring it is enough to prove that $\mathfrak{c}(H)$ is a Golod algebra.

Since C is a Golod algebra, there are a k-basis $\mathfrak{b}_C = \{h_\lambda\}_{\lambda \in \Lambda}$ of $\mathrm{H}_{\geq 1}(C)$ and a function (trivial Massey operation) $\mu : \bigsqcup_{i=1}^{\infty} \mathfrak{b}^i_C \to C$ such that $\mu(h_\lambda) \in \mathrm{Z}_{\geq 1}(C)$ with $\operatorname{cls}(\mu(h_\lambda)) = h_\lambda$ for all $\lambda \in \Lambda$ and

$$\partial\mu(h_{\lambda_1}, \ldots, h_{\lambda_p}) = \sum_{j=1}^{p-1} \overline{\mu(h_{\lambda_1}, \ldots, h_{\lambda_j})}\mu(h_{\lambda_{j+1}}, \ldots, h_{\lambda_p}).$$

By (2) of [Avramov 1998, Lemma 4.1.6], $\{x \in C : \partial(x) \in \mathfrak{m}^2 C\} \subset \mathfrak{m}C$. Since $\mu(h_\lambda) \in \mathfrak{m}C$, by induction on p we conclude that $\mu(h_{\lambda_1}, \ldots, h_{\lambda_p}) \in \mathfrak{m}C$.

By [Levin and Avramov 1978, Lemma 1.2], $\operatorname{socle}(R)K_i^R \subset (0:\mathfrak{m}^2)\,\mathrm{B}_i(K^R)$ for $1 \le i \le n-1$ and so we have $sC \subset (0:\mathfrak{m}^2)\,\mathrm{B}(C)$. Let $z \in \mathrm{Z}(C)$ be such that $(z,0) = \partial(y_1, \bar{y}_2)$ for $(y_1, \bar{y}_2) \in \mathfrak{c}(H)$. Then $z = \partial(y_1) + sy_2 \in \mathrm{B}(C)$. Therefore, the inclusion $C \hookrightarrow \mathfrak{c}(H)$ induces an injective map $\mathrm{H}(C) \hookrightarrow \mathrm{H}(\mathfrak{c}(H))$. By abuse of notation we write $(c,0)$ as c. It follows that $\mathfrak{b}_C = \{h_\lambda\}_{\lambda\in\Lambda}$ is a linearly independent set in $\mathrm{H}_{\ge 1}(\mathfrak{c}(H))$ and $\mu(h_{\lambda_1}, \dots, h_{\lambda_p}) \in \mathfrak{m}\,\mathfrak{c}(H)$, $p \ge 1$, satisfy the properties above.

We extend $\mathfrak{b}_C$ to a basis $\mathfrak{b}_{\mathfrak{c}(H)} = \{h_\lambda\}_{\lambda\in\Lambda\sqcup\Lambda'}$ of $\mathrm{H}_{\ge 1}(\mathfrak{c}(H))$. Let $h_{\lambda'}$, $\lambda' \in \Lambda'$, be the homology class of $(c_{\lambda'}, \bar{d}_{\lambda'}) \in \mathrm{Z}_{\ge 1}(\mathfrak{c}(H))$. Now $\partial(c_{\lambda'}, \bar{d}_{\lambda'}) = 0$ implies that $\partial(c_{\lambda'}) + sd_{\lambda'} = 0$. We have $sd_{\lambda'} = \partial(e_{\lambda'})$ for some $e_{\lambda'} \in (0:\mathfrak{m}^2)C$. We write $(c_{\lambda'}, \bar{d}_{\lambda'}) = (c_{\lambda'} + e_{\lambda'}, 0) + (-e_{\lambda'}, \bar{d}_{\lambda'})$. Note that $c_{\lambda'} + e_{\lambda'} \in \mathrm{Z}(C)$. Therefore, after subtracting suitable R-linear combinations of h_λ, $\lambda \in \Lambda$, from each $h_{\lambda'}$, $\lambda' \in \Lambda'$, if necessary, we may assume that $h_{\lambda'}$ is a homology class of some cycle in $(0:\mathfrak{m}^2)C \oplus \frac{C}{\mathfrak{m}C}$.

We define $\mu(h_{\lambda'})$, $\lambda' \in \Lambda'$, to be an element in $(0:\mathfrak{m}^2)C \oplus \frac{C}{\mathfrak{m}C}$ whose homology class is $h_{\lambda'}$. We extend μ from $\mathfrak{b}_C^i$ to a Massey operation on $\mathfrak{b}^i_{\mathfrak{c}(H)}$, $i > 1$, such that $\mu : \mathfrak{b}^i_{\mathfrak{c}(H)} \setminus \mathfrak{b}^i_C \to (0:\mathfrak{m}^2)C$, $i > 1$, by induction on i. Note that $\overline{\mu(h_\lambda)}\mu(h_{\lambda'}) \in sC \in (0:\mathfrak{m}^2)\,\mathrm{B}(C)$ for $(\lambda, \lambda') \notin \Lambda \times \Lambda$. We choose $\mu(h_\lambda, h_{\lambda'}) \in (0:\mathfrak{m}^2)C$ such that $\partial(\mu(h_\lambda, h_{\lambda'})) = \overline{\mu(h_\lambda)}\mu(h_{\lambda'})$. Now assume $\mu(h_{\delta_1}, \dots, h_{\delta_i}) \in (0:\mathfrak{m}^2)C$, with $(h_{\delta_1}, \dots, h_{\delta_i}) \in \mathfrak{b}^i_{\mathfrak{c}(H)} \setminus \mathfrak{b}^i_C$ satisfying the desired relations, are constructed for all $i \le p$. We choose $(h_{\delta_1}, \dots, h_{\delta_{p+1}}) \in \mathfrak{b}^{p+1}_{\mathfrak{c}(H)} \setminus \mathfrak{b}^{p+1}_C$ and observe that $\sum_{j=1}^p \overline{\mu(h_{\delta_1}, \dots, h_{\delta_j})}\mu(h_{\delta_{j+1}}, \dots, h_{\delta_{p+1}})$ is an element in sC. Therefore, we can choose $\mu(h_{\delta_1}, \dots, h_{\delta_{p+1}}) \in (0:\mathfrak{m}^2)C$ such that $\partial(\mu(h_{\delta_1}, \dots, h_{\delta_{p+1}})) = \sum_{j=1}^p \overline{\mu(h_{\delta_1}, \dots, h_{\delta_j})}\mu(h_{\delta_{j+1}}, \dots, h_{\delta_{p+1}})$. Thus by induction μ extends to a trivial Massey operation on $\mathfrak{b}_{\mathfrak{c}(H)}$. Therefore, $\mathfrak{c}(H)$ is a Golod algebra and the result follows. $\square$

We recall the definition of compressed Gorenstein local rings.

Definition 4.3. Let $(R, \mathfrak{m}, k)$ be an Artinian Gorenstein local ring of Loewy length s and embedding dimension $n \ge 2$. Set

$$\varepsilon_i = \min\left\{\binom{n-1+s-i}{n-1}, \binom{n-1+i}{n-1}\right\}$$

for all i with $0 \le i \le s$. Then it is shown in [Rossi and Şega 2014, Proposition 4.2] that $l(R) \le \sum_{i=0}^n \varepsilon_i$. The ring R is called a compressed Gorenstein ring if equality holds.

We provide a different proof of the result of Rossi and Şega [2014, Theorem 5.1] in Theorem 4.4.

Theorem 4.4. *Let R be a compressed Gorenstein local ring such that* $\operatorname{edim}(R) = n \ge 2$ *and* $\operatorname{ll}(R) = s$, $s \ge 2$, $s \ne 3$. *Then $R/\operatorname{socle}(R)$ is a Golod ring. Consequently, R satisfies assertions* (1) *and* (2) *of Theorem I.*

Proof. We follow notation as set in the proof of Theorem I and Lemma 4.2. Let $t = \max\{i : I \subset \mathfrak{n}^i\}$. It is proved in [Rossi and Şega 2014, Proposition 4.2] that $t = \lceil \frac{s+1}{2} \rceil$, the least integer not less than $\frac{s+1}{2}$. By [Rossi and Şega 2014, Lemma 1.4], the map $\mathrm{H}_{\geq 1}(R/\mathfrak{m}^t \otimes_Q K^Q) \to \mathrm{H}_{\geq 1}(R/\mathfrak{m}^{t-1} \otimes_Q K^Q)$ induced by the surjection $R/\mathfrak{m}^t \twoheadrightarrow R/\mathfrak{m}^{t-1}$ is a zero map. This implies that $\mathrm{Z}_{\geq 1}(K^R) \subset \mathrm{B}_{\geq 1}(K^R) + \mathfrak{m}^{t-1} K^R_{\geq 1}$ and therefore $\mathrm{Z}_{\geq 1}(C) \subset \mathrm{B}_{\geq 1}(C) + \mathfrak{m}^{t-1} C$. Thus we find a basis $\mathfrak{b} = \{h_\lambda\}_{\lambda \in \Lambda}$ of $\mathrm{H}_{\geq 1}(C)$ represented by cycles in $\mathfrak{m}^{t-1} C$.

Lemma 4.4 in [Rossi and Şega 2014] proves that the map $\psi : \mathrm{H}_{<n}(\mathfrak{m}^{r+1} K^R) \to \mathrm{H}_{<n}(\mathfrak{m}^r K^R)$ induced by the inclusion $\mathfrak{m}^{r+1} \hookrightarrow \mathfrak{m}^r$ is zero for $r = s + 1 - t$. Since, $s \geq 2$, $s \neq 3$, we have $t - 1 \leq r \leq r + 1 \leq 2(t-1)$. This implies that the map $\mathrm{H}_{<n}(\mathfrak{m}^{2(t-1)} K^R) \to \mathrm{H}_{<n}(\mathfrak{m}^{t-1} K^R)$ is also zero since it factors through ψ. Therefore, we have $\mathrm{Z}_{<n}(\mathfrak{m}^{2t-2} K^R) \subset \mathrm{B}(\mathfrak{m}^{t-1} K^R)$ which implies $\mathrm{Z}(\mathfrak{m}^{2t-2} C) \subset \mathrm{B}(\mathfrak{m}^{t-1} C)$. It is worth pointing out that both the lemmas used here are independent of all other results in [Rossi and Şega 2014].

We construct inductively a trivial Massey operation $\mu : \bigsqcup_{i=1}^\infty \mathfrak{b}^i \to \mathfrak{m}^{t-1} C$. Define $\mu(h_\lambda)$ to be a cycle in $\mathfrak{m}^{t-1} C$ such that the homology class of $\mu(h_\lambda)$ is h_λ. Now $\overline{\mu(h_\lambda)} \mu(h_{\lambda'}) \in \mathrm{Z}(\mathfrak{m}^{2t-2} C) \subset \mathrm{B}(\mathfrak{m}^{t-1} C)$, and so we choose $\mu(h_\lambda, h_{\lambda'}) \in \mathfrak{m}^{t-1} C$ such that $\partial(\mu(h_\lambda, h_{\lambda'})) = \overline{\mu(h_\lambda)} \mu(h_{\lambda'})$. The method carries over to the next steps of construction. Thus, C is a Golod DG algebra and the result follows from Lemma 4.2. □

Lemma 4.5. *Let $(R, \mathfrak{m}, k)$ be an Artinian Gorenstein local ring but not a complete intersection. Let $\eta : Q \twoheadrightarrow R$ be a minimal Cohen presentation of R and $I = \ker(\eta)$. Assume that $\mu(I) = r$ and $\mathrm{edim}(R) = n \leq 3$. Then $R/\operatorname{socle}(R)$ is a Golod ring, so R satisfies both assertions* (1) *and* (2) *of Theorem I. If $d_R(t) = 1 - rt^2 - rt^3 + t^5$, then for any finitely generated R-module M, we have $d_R(t) P^R_M(t) \in Z[t]$. The Poincaré series of k is given by*

$$P^R_k(t) = \frac{(1+t)^n}{1 - rt^2 - rt^3 + t^5}.$$

Proof. As before, we follow notation as set in the proof of Theorem I and Lemma 4.2. By [Wiebe 1969, Satz 7], we have $\mathrm{H}_1(K^R)^2 = 0$ giving $\mathrm{H}_1(C)^2 = 0$. For a proof written in English, we refer to [Bruns and Herzog 1993, Corollary 3.4.8]. Now C is a DG algebra of length 2. Therefore, any basis of $\mathrm{H}_{\geq 1}(C)$ admits a trivial Massey operation and so C is a Golod algebra. The first part of the result follows from Lemma 4.2.

We compute the denominator. The Koszul complex of R is of length 3. We see $\dim_k \mathrm{H}_0(K^R) = 1$, $\dim_k \mathrm{H}_1(K^R) = \mu(I) = r$, $\dim_k \mathrm{H}_3(K^R) = \dim_k \operatorname{socle}(R) = 1$ and so $\dim_k \mathrm{H}_2(K^R) = r$ since $\sum_{i \geq 0} (-1)^i \dim_k \mathrm{H}_i(K^R)$ must be zero. With the notation used in Theorem I, we have $P^Q_R(t) = \sum_{i=0}^3 \dim_k \mathrm{H}_i(K^R) = 1 + rt + rt^2 + t^3$. Therefore, we have

$$d_R(t) = 1 - t(P^Q_R(t) - 1) + t^{n+1}(1+t) = 1 - t^2(r + rt + t^2) + t^4(1+t) = 1 - rt^2 - rt^3 + t^5.$$

The formula for $P^R_k(t)$ follows from in [Rossi and Şega 2014, Proposition 6.2]. □

If $(R, \mathfrak{m}, k)$ is a Gorenstein local ring satisfying the hypothesis of the theorem below, then the rational expression of $P_k^R(t)$ was computed by Wiebe [1969, Satz 9]. It was proved in [Avramov et al. 1988, Theorem 6.4] that if $(R, \mathfrak{m})$ is any Artinian ring such that $\operatorname{edim}(R) - \operatorname{depth}(R) \leq 3$, then all R-modules have rational Poincaré series. We prove a weaker version.

Theorem 4.6. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring but not a complete intersection such that $\operatorname{edim}(R) - \operatorname{depth}(R) = n \leq 3$. Let $\eta : Q \twoheadrightarrow \hat{R}$ be a minimal Cohen presentation of R, $\ker \eta = I$ and $\mu(I) = r$. Then for any $f \in I \setminus \mathfrak{n}I$, the induced map $Q/(f) \twoheadrightarrow \hat{R}$ is a Golod homomorphism.*

Let $d_R(t) = 1 - rt^2 - rt^3 + t^5$. Then $P_k^R(t) = (1+t)^{\operatorname{edim}(R)}/d_R(t)$ and for any R-module M we have $d_R(t) P_M^R(t) \in \mathbb{Z}[t]$.

Proof. Let $\dim(R) = \operatorname{depth}(R) = d$ and the maximal ideal $\mathfrak{m}$ be minimally generated by $x_1, \ldots, x_e$ such that $x_1, \ldots, x_d$ form an R-sequence. Then $S = R/(x_1, \ldots, x_d)$ is an Artinian Gorenstein local ring and $\operatorname{edim}(S) = e - d = n \leq 3$. Let $K^R = R\langle X_i : \partial(X_i) = x_i, 1 \leq i \leq n\rangle$ and $K^S = S\langle X_i : \partial(X_i) = x_i, d+1 \leq i \leq n\rangle$ denote the Koszul complexes. The quotient map $q : K^R \twoheadrightarrow K^S$ is a quasi-isomorphism; see [Avramov 1998, Lemma 4.1.6].

To prove that the map $Q/(f) \twoheadrightarrow \hat{R}$ is a Golod homomorphism for any $f \in I \setminus \mathfrak{n}I$, it is equivalent to show that for any cycle $z \in \mathrm{Z}_1(K^R) \setminus \mathrm{B}_1(K^R)$, the semifree extension $K^R\langle T \mid \partial(T) = z\rangle$ is a Golod algebra. Now one observes that q extends to a surjective quasi-isomorphism

$$\tilde{q} : K^R\langle T \mid \partial(T) = z\rangle \twoheadrightarrow K^S\langle T \mid \partial(T) = q(z)\rangle;$$

see, for instance, [Gulliksen and Levin 1969, Proposition 1.3.5]. Lemma 4.5 applies to S. We conclude that the image of $\tilde{q}$ is a Golod algebra. Therefore, $K^R\langle T \mid \partial(T) = z\rangle$ is a Golod algebra.

The statement about Poincaré series is an immediate consequence of Lemmas 3.9 and 4.5. □

Remark 4.7. In both Theorems 4.4 and 4.6, we constructed a Golod homomorphism from a hypersurface ring which is a quotient of an arbitrary generator belonging to a minimal generating set of the defining ideal. Thus both theorems are slightly stronger than their earlier versions.

Acknowledgements

Work on this article started as a result of discussions with Liana M. Şega during the conference in honour of Craig Huenke held in Ann Arbor in July 2016. I would like to thank her for clarifications on her results about the Auslander–Reiten conjecture. I sincerely thank the anonymous referee for suggesting the proof of Lemma 3.1. I owe many thanks to Matteo Varbaro for his inspiration and encouragement throughout. This research is supported by the INdAM-COFUND-2012 Fellowship cofounded by Marie Curie actions, Italy.

References

[Ananthnarayan 2009] H. Ananthnarayan, *Approximating Artinian rings by Gorenstein rings and 3-standardness of the maximal ideal*, Ph.D. thesis, University of Kansas, 2009, available at https://search.proquest.com/docview/304917043.

[Ananthnarayan et al. 2012] H. Ananthnarayan, L. L. Avramov, and W. F. Moore, "Connected sums of Gorenstein local rings", *J. Reine Angew. Math.* **667** (2012), 149–176. MR Zbl

[Ananthnarayan et al. 2019] H. Ananthnarayan, E. Celikbas, J. Laxmi, and Z. Yang, "Decomposing Gorenstein rings as connected sums", *J. Algebra* **527** (2019), 241–263. MR Zbl

[Anick 1982] D. J. Anick, "A counterexample to a conjecture of Serre", *Ann. of Math.* (2) **115**:1 (1982), 1–33. MR Zbl

[Auslander and Reiten 1975] M. Auslander and I. Reiten, "On a generalized version of the Nakayama conjecture", *Proc. Amer. Math. Soc.* **52** (1975), 69–74. MR Zbl

[Avramov 1994] L. L. Avramov, "Local rings over which all modules have rational Poincaré series", *J. Pure Appl. Algebra* **91**:1-3 (1994), 29–48. MR Zbl

[Avramov 1998] L. L. Avramov, "Infinite free resolutions", pp. 1–118 in *Six lectures on commutative algebra* (Bellaterra, 1996), edited by J. M. Giral et al., Progr. Math. **166**, Birkhäuser, Basel, 1998. MR Zbl

[Avramov and Buchweitz 2000] L. L. Avramov and R.-O. Buchweitz, "Support varieties and cohomology over complete intersections", *Invent. Math.* **142**:2 (2000), 285–318. MR Zbl

[Avramov and Golod 1971] L. L. Avramov and E. S. Golod, "The homology of algebra of the Koszul complex of a local Gorenstein ring", *Mat. Zametki* **9** (1971), 53–58. MR Zbl

[Avramov et al. 1988] L. L. Avramov, A. R. Kustin, and M. Miller, "Poincaré series of modules over local rings of small embedding codepth or small linking number", *J. Algebra* **118**:1 (1988), 162–204. MR Zbl

[Bøgvad 1983] R. Bøgvad, "Gorenstein rings with transcendental Poincaré-series", *Math. Scand.* **53**:1 (1983), 5–15. MR Zbl

[Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1993. MR Zbl

[Croll et al. 2018] A. Croll, R. Dellaca, A. Gupta, J. Hoffmeier, V. Mukundan, L. M. Şega, G. Sosa, P. Thompson, and D. R. Tracy, "Detecting Koszulness and related homological properties from the algebra structure of Koszul homology", *Nagoya Math. J.* (online publication June 2018).

[Dress and Krämer 1975] A. Dress and H. Krämer, "Bettireihen von Faserprodukten lokaler Ringe", *Math. Ann.* **215** (1975), 79–82. MR Zbl

[Eakin and Sathaye 1976] P. Eakin and A. Sathaye, "Prestable ideals", *J. Algebra* **41**:2 (1976), 439–454. MR Zbl

[Eisenbud 1995] D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, 1995. MR Zbl

[Elias and Valla 2008] J. Elias and G. Valla, "Structure theorems for certain Gorenstein ideals", *Michigan Math. J.* **57** (2008), 269–292. MR Zbl

[Elias and Valla 2009] J. Elias and G. Valla, "A family of local rings with rational Poincaré series", *Proc. Amer. Math. Soc.* **137**:4 (2009), 1175–1178. MR Zbl

[Gulliksen 1974] T. H. Gulliksen, "A change of ring theorem with applications to Poincaré series and intersection multiplicity", *Math. Scand.* **34** (1974), 167–183. MR Zbl

[Gulliksen and Levin 1969] T. H. Gulliksen and G. Levin, *Homology of local rings*, Queen's Paper in Pure and Applied Mathematics **20**, Queen's University, Kingston, ON, 1969. MR Zbl

[Jorgensen and Şega 2004] D. A. Jorgensen and L. M. Şega, "Nonvanishing cohomology and classes of Gorenstein rings", *Adv. Math.* **188**:2 (2004), 470–490. MR Zbl

[Lescot 1983] J. Lescot, "La série de Bass d'un produit fibré d'anneaux locaux", pp. 218–239 in *Paul Dubreil and Marie-Paule Malliavin algebra seminar, 35th year* (Paris, 1982), edited by M.-P. Malliavin, Lecture Notes in Math. **1029**, Springer, 1983. MR Zbl

[Levin 1976] G. Levin, *Lectures on Golod homomorphisms*, Matematiska institutionen, Stockholms universitet **15**, University of Stockholm, 1976. Zbl

[Levin 1985] G. Levin, "Modules and Golod homomorphisms", *J. Pure Appl. Algebra* **38**:2-3 (1985), 299–304. MR Zbl

[Levin and Avramov 1978] G. L. Levin and L. L. Avramov, "Factoring out the socle of a Gorenstein ring", *J. Algebra* **55**:1 (1978), 74–83. MR Zbl

[Roos 2005] J.-E. Roos, "Good and bad Koszul algebras and their Hochschild homology", *J. Pure Appl. Algebra* **201**:1-3 (2005), 295–327. MR Zbl

[Rossi and Şega 2014] M. E. Rossi and L. M. Şega, "Poincaré series of modules over compressed Gorenstein local rings", *Adv. Math.* **259** (2014), 421–447. MR Zbl

[Sally 1980] J. D. Sally, "The Poincaré series of stretched Cohen–Macaulay rings", *Canadian J. Math.* **32**:5 (1980), 1261–1265. MR Zbl

[Scheja 1964] G. Scheja, "Über die Bettizahlen lokaler Ringe", *Math. Ann.* **155** (1964), 155–172. MR Zbl

[Şega 2003] L. M. Şega, "Vanishing of cohomology over Gorenstein rings of small codimension", *Proc. Amer. Math. Soc.* **131**:8 (2003), 2313–2323. MR Zbl

[Wiebe 1969] H. Wiebe, "Über homologische Invarianten lokaler Ringe", *Math. Ann.* **179** (1969), 257–274. MR Zbl

Received January 30, 2018. Revised September 21, 2019.

ANJAN GUPTA
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL
BHOPAL
INDIA
agmath@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2020 is US $520/year for the electronic version, and $705/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 305 No. 1 March 2020