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THE GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE 5-DIMENSIONAL DEFOCUSING CONFORMAL INVARIANT NLW WITH RADIAL INITIAL DATA IN A CRITICAL BESOV SPACE

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We obtain the global well-posedness and scattering for the radial solution to the defocusing conformal invariant nonlinear wave equation with initial data in the critical Besov space $\dot{B}^3_{1,1} \times \dot{B}^2_{1,1}(\mathbb{R}^5)$. This is the 5-dimensional analogue of Dodson's result (2019), which was the first on the global well-posedness and scattering of the energy subcritical nonlinear wave equation without the uniform boundedness assumption on the critical Sobolev norms employed as a substitute of the missing conservation law with respect to the scaling invariance of the equation. The proof is based on exploiting the structure of the radial solution, developing the Strichartz-type estimates and incorporation of Dodson's strategy (2019), where we also avoid a logarithm-type loss by employing the inhomogeneous Strichartz estimates.

1. Introduction

We consider the solutions u to

$$(1\text{-}1)\qquad \begin{cases} \partial_{tt}u - \Delta u + \mu |u|^p u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ (u(0), \partial_t u(0)) = (u_0, u_1), & x \in \mathbb{R}^d, \end{cases}$$

where $\mu = \pm 1$, $d \geq 1$, and $p > 0$. If $\mu = 1$, (1-1) is described as defocusing, otherwise focusing. There is a natural scaling symmetry for (1-1), i.e., if we let $u_\lambda(t,x) = \lambda^{2/p} u(\lambda t, \lambda x)$ for $\lambda > 0$, then u_λ is also a solution to (1-1) with initial data $(\lambda^{2/p}u_0(\lambda x), \lambda^{(2/p)+1}u_1(\lambda x))$ preserving the $\dot{H}^{s_p} \times \dot{H}^{s_p - 1}(\mathbb{R}^d)$ norm of the initial data, where we define the critical regularity as $s_p = \frac{d}{2} - \frac{2}{p}$. At least, the solutions to (1-1) formally conserve the energy

$$(1\text{-}2)\qquad E(u(t), \partial_t u(t)) := \frac{1}{2}\int_{\mathbb{R}^5} |\nabla_x u(t)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^5} |\partial_t u(t)|^2\,dx + \frac{\mu}{p+2}\int_{\mathbb{R}^5} |u(t)|^{p+2}\,dx,$$

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which is also invariant under the scaling if $s_p = 1$. In view of this, we say the Cauchy problem (1-1) is energy critical when $s_p = 1$, subcritical for $s_p < 1$ and supercritical when $s_p > 1$.

Lindblad and Sogge [1995] proved the local theory of the Cauchy problem (1-1) in the minimal regularity spaces. In fact, if $d \geq 2$ and $p \geq (d+3)/(d-1)$, the Cauchy problem (1-1) with initial data in the critical spaces $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^d)$ is locally well-posed. The global theory for the Cauchy problem (1-1) with $\mu = 1$ and $s_p \leq 1$ has been studied extensively. While for the focusing case, even the solution with smooth initial data may blow up at finite time. For more related results see [Sogge 1995].

We will consider global existence and scattering of the solutions to (1-1). In general, a solution u is said to be scattering if it is a global solution and approaches a linear solution as $t \to \pm\infty$. In the cases of $d \geq 2$ and $p \geq (d+3)/(d-1)$, the solution to (1-1) with small initial datum in the critical Sobolev spaces is globally well-posed and scattering; see [Lindblad and Sogge 1995].

For the defocusing energy critical wave equation (1-1), Grillakis [1990] first established the global existence theory for classical solution when $d = 3$. The results for other dimensions are proved in [Grillakis 1992; Shatah and Struwe 1993]. Scattering results for large energy data are proved in [Bahouri and Gérard 1999; Bahouri and Shatah 1998; Nakanishi 1999] by establishing variants of the Morawetz estimates [1968]

$$\iint_{\mathbb{R}^{1+d}} \frac{|u|^{\frac{2d}{d-2}}}{|x|}\, dx\, dt \leq C_d E(u_0, u_1), \tag{1-3}$$

where C_d is a constant depending on d. While in focusing energy critical cases, the Morawetz estimates (1-3) fails. The scattering results do not hold in general, since (1-1) has a ground state

$$W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}.$$

In the cases of $3 \leq d \leq 5$, Kenig and Merle [2008] proved the scattering result for solution with initial data such that $E(u_0, u_1) < E(W, 0)$ and $\|u_0\|_{\dot{H}^1(\mathbb{R}^d)} < \|W\|_{\dot{H}^1(\mathbb{R}^d)}$. In their proofs, the main ingredient is the concentration compactness/rigidity theorem method introduced by [Kenig and Merle 2006]. This method is powerful and plays an important role in study of many other nonlinear dispersive equations. We refer to [Killip and Vişan 2013; Koch et al. 2014; Kenig 2015].

For the defocusing subcritical equation (1-1), the global existence has been proved for solution with initial data in the energy space $\dot{H}^1 \times L^2(\mathbb{R}^d)$ by Ginibre and Velo [1985; 1989]. However, there are no scattering results even for solutions with initial datum in $(\dot{H}^1 \cap \dot{H}^{s_p}) \times (L^2 \cap \dot{H}^{s_p-1})(\mathbb{R}^d)$.

Recently, Dodson [2019] proved scattering results for the defocusing cubic wave equation with the initial datum belonging to the space $\dot{B}^2_{1,1} \times \dot{B}^1_{1,1}(\mathbb{R}^3)$, which is a subspace of $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)$. We remark that this is the first work that gives scattering results for large data in the critical Sobolev space without any a priori bound on the critical norm of the solution. Dodson's strategy consists of three steps:

(1) By establishing some new Strichartz-type estimates, one can show that the solution is in the energy space $\dot{H}^1 \times L^2(\mathbb{R}^3)$ up to some free evolutions. Then this decomposition enables one to prove the global well-posedness of the solution.

(2) To obtain the scattering result, a conformal transformation is applied to show that the energy part of the solution has finite energy in hyperbolic coordinates. Then from the conformal invariance of the equation and a Morawetz-type inequality, one can deduce that $\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^5)} \le C(\|(u_0,u_1)\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)}, \delta_1)$, where the parameter δ_1 relies on the scaling and spatial profiles of the initial data.

(3) Finally, one can remove the dependence of δ_1 by employing the profile decomposition, which completes the proof.

Let $S(t)(f, g)$ be the solution of Cauchy problem to the free wave equation

$$\begin{cases} \partial_{tt} v - \Delta v = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^5, \\ (v, \partial_t v)|_{t=0} = (f(x), g(x)), & x \in \mathbb{R}^5. \end{cases} \tag{1-4}$$

For the sake of statement, we introduce the following notation as

$$\dot{S}(t)(f,g) \triangleq \partial_t S(t)(f,g), \quad \text{and} \quad \vec{S}(t)(f,g) \triangleq \big(S(t)(f,g), \dot{S}(t)(f,g)\big).$$

We consider the Cauchy problem of nonlinear wave equation

$$\begin{cases} \partial_{tt} u - \Delta u + |u|u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ (u(0), \partial_t u(0)) = (u_0, u_1), & x \in \mathbb{R}^d, \end{cases} \tag{1-5}$$

Our main result can be stated as:

Theorem 1.1. *For any radial initial data $(u_0, u_1) \in \dot{B}^3_{1,1} \times \dot{B}^2_{1,1}(\mathbb{R}^5)$, the solution u to (1-5) is globally well-posed and scattering, i.e., there exists $(u_0^\pm, u_1^\pm) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^5)$ such that*

$$\lim_{t\to\pm\infty} \|(u(t), \partial_t u(t)) - \vec{S}(t)(u_0^\pm, u_1^\pm)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)} \to 0. \tag{1-6}$$

Furthermore, there is a function $A : [0,\infty) \to [0,\infty)$, such that

$$\|u\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} \le A(\|(u_0,u_1)\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)}). \tag{1-7}$$

Remark 1.2. (1) This theorem extends the results of [Dodson 2019] to the 5-dimension case. The proof will utilize the strategy given in [Dodson 2019], but it is highly nontrivial.

(2) Unlike the 3-dimensional case, the dispersive estimate (see (2-20)) gives a decay in time of order -2, which may cause a logarithmic failure when one estimates

$$\|u\|_{L_t^{4/3}L_x^4(J\times\mathbb{R}^5)} + \sup_{t\in J}\bigl(t^{3/4}\|u\|_{L_x^4(\mathbb{R}^5)}\bigr),$$

where $0\in J$ is a local time interval. We circumvent this difficulty by using the inhomogeneous Strichartz estimates in [Taggart 2010] and prove the global well-posedness of u.

(3) For the scattering result, by reductions, we need to bound the $L^3_{t,x}$ of w on the light cone $\{|x|\le t+\frac{1}{2}\}$. We will define the hyperbolic energy by rewriting (1-5) as the form

$$\partial_{tt}(r^2w)-\partial_{rr}(r^2w)=-2w-r^2|w|w.$$

Observing that the additional term $2w$ and the nonlinear term $r^2|w|w$ enjoy the same sign, we can bound the $L^3_{t,x}$ norm of w by applying a Morawetz-type inequality, if we assume the hyperbolic energy of w is bounded.

(4) To certify the above assumption, we will make full use of (2-19) for radial solution and the sharp Hardy inequality. In contrast to the 3-dimensional case, some terms in (2-19) seem more difficult to dealt with. However, the integration domains of these terms are symmetric about the radius r, which is also consistent with the Huygens principle. This fact allows us apply the Hardy–Littlewood maximal functions to verify the assumption.

Now, we give the outline of the proof. By the Strichartz estimates and a standard fix point argument, for initial data (u_0,u_1), there exists a maximal time interval $I\subset\mathbb{R}$ such that there exists a unique solution u (see Definition 2.9 in Section 2) to (1-5) on $I\times\mathbb{R}^5$. We consider the global well-posedness by developing some Strichartz-type estimates (3-30). Utilizing the standard blowup criterion, we can show the global well-posedness of u.

Next, we claim the following proposition:

Proposition 1.3. *For every radial initial data $(u_0,u_1)\in\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)$, let u be the corresponding solution to (1-5). Then there exists a parameter δ_1 depending the initial data (u_0,u_1) and a function $A:[0,\infty)^2\to[0,\infty)$ such that*

$$\|u\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)}\le A(\|(u_0,u_1)\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)},\delta_1). \tag{1-8}$$

We prove Theorem 1.1 by employing this and establishing Proposition 4.2, where the proof provides an alternate proof of Lemma 6.2 in [Dodson 2019].

Finally, we need to prove Proposition 1.3. From the partition $u=v+w$, it suffices to show the boundedness of $L^3_{t,x}$ norm of w. We prove the hyperbolic energy of w is uniformly bounded. Then, a Morawetz-type inequality yields that the $L^3_{t,x}$ norm of w is bounded in the cone, which finishes the proof.

This paper is organized as follows: Section 2 gives some tools from harmonic analysis and basic properties for the wave equation. In Section 3 we give the decomposition of u and prove its global well-posedness. The existence of the function A in (1-7) is shown in Section 4 based on the Proposition 1.3. Finally, in Section 5, we complete the proof by showing Proposition 1.3.

We end the introduction with some notations used throughout this paper. We use $\mathcal{S}(\mathbb{R}^d)$ to denote the space of Schwartz functions on $\mathbb{R}^d$. For $1 \le p \le \infty$, we define $L^p(\mathbb{R}^d)$ by the spaces of Lebesgue measurable functions with finite $L^p(\mathbb{R}^d)$-norm, which is defined by

$$\|f\|_{L^p(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |f(x)|^p \, dx \Big)^{1/p}, \quad \text{for } 1 \le p < \infty,$$

and $\|f\|_{L^\infty(\mathbb{R}^d)} = \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|$. We let ℓ^p be the spaces of complex number sequences $\{a_n\}_{n\in\mathbb{Z}}$ such that $\{a_n\}_{n\in\mathbb{Z}} \in l^p$ if and only if

$$\|\{a_n\}\|_{\ell^p_n(\mathbb{Z})} \triangleq \Big(\sum_n |a_n|^p \Big)^{1/p} < \infty, \quad \text{for } 1 \le p < \infty$$

and $\|\{a_n\}\|_{\ell^\infty_n(\mathbb{Z})} := \sup_n |a_n| < \infty$. We use $X \lesssim Y$ to mean that there exists a constant $C > 1$ such that $X \le CY$, where the dependence of C on the parameters will be clear from the context. We use $X \sim Y$ to denote $X \lesssim Y$ and $Y \lesssim X$. $A \ll B$ denotes there is a sufficiently large number C such that $A \le C^{-1}B$.

2. Basic tools and some elementary properties for the wave equation

In this section, we recall some tools from harmonic analysis and useful results for the wave equation.

2A. ***Some tools from harmonic analysis.*** Recall the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\xi} \, dx,$$

which can be extended to Schwartz distributions naturally. We will make frequent use of the Littlewood–Paley projection operators. Specifically, we let φ be a radial smooth function supported on the ball $|\xi| \le 2$ and equal to 1 on the ball $|\xi| \le 1$. For $j \in \mathbb{Z}$, we define the Littlewood–Paley projection operators by

$$\begin{aligned} \widehat{P_{\le j} f}(\xi) &:= \varphi(\xi/2^j)\widehat{f}(\xi), \\ \widehat{P_{> j} f}(\xi) &:= (1-\varphi(\xi/2^j))\widehat{f}(\xi), \\ \widehat{P_j f}(\xi) &:= (\varphi(\xi/2^j) - \varphi(\xi/2^{j-1}))\widehat{f}(\xi). \end{aligned}$$

The Littlewood–Paley operators commute with derivative operators and are bounded on the general Sobolev spaces. These operators also obey the following standard Bernstein estimates:

Lemma 2.1 (Bernstein estimates). *For* $1 \le r \le q \le \infty$ *and* $s \ge 0$,

$$\begin{aligned} \big\| |\nabla|^{\pm s} P_j f \big\|_{L^r_x(\mathbb{R}^d)} &\sim 2^{\pm js} \| P_j f \|_{L^r_x(\mathbb{R}^d)}, \\ \big\| |\nabla|^{s} P_{\le j} f \big\|_{L^r_x(\mathbb{R}^d)} &\lesssim 2^{js} \| P_{\le j} f \|_{L^r_x(\mathbb{R}^d)}, \\ \| P_{>j} f \|_{L^r_x(\mathbb{R}^d)} &\lesssim 2^{-js} \big\| |\nabla|^{s} P_{>j} f \big\|_{L^r_x(\mathbb{R}^d)}, \\ \| P_{\le j} f \|_{L^q(\mathbb{R}^d)} &\lesssim 2^{(d/r-d/q)j} \| P_{\le j} f \|_{L^r_x(\mathbb{R}^d)}, \end{aligned}$$

where the fractional derivative operator $|\nabla|^\sigma$ *is defined by* $\widehat{|\nabla|^\sigma f}(\xi) = |\xi|^\sigma \widehat{f}(\xi)$, *for* $\sigma \in \mathbb{R}$.

Definition 2.2 (homogeneous Besov spaces). Let s be a real number and let $1 \le p, r \le \infty$. We denote the homogeneous Besov norm by

$$\| f \|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} := \big\| \{ 2^{js} \| P_j f \|_{L^p(\mathbb{R}^d)} \} \big\|_{\ell^r_j(\mathbb{Z})}, \tag{2-1}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. Then the Besov space $\dot{B}^s_{p,r}(\mathbb{R}^d)$ is the completion of the Schwartz function under this norm.

We shall give the following radial Sobolev-type inequalities, which are analogous to the 3-dimensional cases established in [Dodson 2019]. We denote radial derivative by $\partial_r f(x) = \big(\frac{x}{|x|} \cdot \nabla\big) f(x)$ for any function f defined on $\mathbb{R}^5$.

Lemma 2.3 (radial Sobolev-type inequalities in Besov spaces). *For any radial function* $f \in \mathcal{S}(\mathbb{R}^5)$, *we have*

$$\| |x|^2 f \|_{L^\infty(\mathbb{R}^5)} \lesssim \| f \|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^5)}. \tag{2-2}$$

Let $(u_0, u_1) \in \dot{B}^3_{1,1} \times \dot{B}^2_{1,1}(\mathbb{R}^5)$ *be a radial function*; *then we have*

$$\begin{aligned} \Big\| \frac{1}{|x|^2} \partial_r u_0(x) \Big\|_{L^1_x(\mathbb{R}^5)} + \Big\| \frac{1}{|x|} \partial_{rr} u_0(x) \Big\|_{L^1_x(\mathbb{R}^5)} \\ + \Big\| \frac{1}{|x|^3} u_0(x) \Big\|_{L^1_x(\mathbb{R}^5)} + \big\| |x|^3 \partial_r u_0 \big\|_{L^\infty_x(\mathbb{R}^5)} \lesssim \| u_0 \|_{\dot{B}^3_{1,1}(\mathbb{R}^5)}, \end{aligned} \tag{2-3}$$

$$\Big\| \frac{1}{|x|} \partial_r u_1(x) \Big\|_{L^1_x(\mathbb{R}^5)} + \Big\| \frac{1}{|x|^2} u_1(x) \Big\|_{L^1_x(\mathbb{R}^5)} + \big\| |x|^3 u_1(x) \big\|_{L^\infty_x(\mathbb{R}^5)} \lesssim \| u_1 \|_{\dot{B}^2_{1,1}(\mathbb{R}^5)}. \tag{2-4}$$

Proof. We first consider (2-2). Since f is radial, using polar coordinates, we have

$$P_j f(|x|) = P_j f(x) = \int_{\mathbb{R}^5} \widehat{P_j f}(\xi) e^{ix\xi}\, d\xi = \int_0^\infty \int_{\mathbb{S}^4} \widehat{P_j f}(r) r^4 e^{irx\omega}\, d\sigma(\omega)\, dr. \tag{2-5}$$

Recall the decay estimates of Fourier transform of the surface measure on the sphere

$$\widehat{d\sigma_{\mathbb{S}^4}}(\xi) \le C(1+|\xi|)^{-2},$$

which, with Hölder's inequality, yields

$$|P_j f(|x|)| \lesssim \int_0^\infty |\widehat{P_j f}(r)| r^2 |x|^{-2}\, dr \lesssim |x|^{-2} 2^{\frac{1}{2}j} \|P_j f\|_{L^2}. \tag{2-6}$$

Then the inequality (2-2) follows from (2-6) and the definition of the Besov space.

Next, we consider (2-3) and (2-4). By the density of Schwartz functions in $\dot B^3_{1,1} \times \dot B^2_{1,1}(\mathbb{R}^5)$, we may assume that $u_0, u_1 \in \mathcal{S}(\mathbb{R}^5)$. We claim it suffices to show

$$\left\| \frac{1}{|x|^2} \partial_r u_0(x) \right\|_{L^1_x(\mathbb{R}^5)} + \left\| \frac{1}{|x|} |\Delta u_0(x)| \right\|_{L^1_x(\mathbb{R}^5)} + \left\| \frac{1}{|x|^3} u_0(x) \right\|_{L^1_x(\mathbb{R}^5)} \lesssim \|u_0\|_{\dot B^3_{1,1}(\mathbb{R}^5)}, \tag{2-7}$$

$$\left\| \frac{1}{|x|} \partial_r u_1(x) \right\|_{L^1_x(\mathbb{R}^5)} + \left\| \frac{1}{|x|^2} u_1(x) \right\|_{L^1_x(\mathbb{R}^5)} \lesssim \|u_1\|_{\dot B^2_{1,1}(\mathbb{R}^5)}. \tag{2-8}$$

To see this, by using the fact $\Delta f = \partial_{rr} f + \frac{4}{r}\partial_r f$ for radial function $f(x)$ on $\mathbb{R}^5$, we have

$$\left\| \frac{1}{|x|} \partial_{rr} u_0(x) \right\|_{L^1_x(\mathbb{R}^5)} \lesssim \left\| \frac{1}{|x|^2} \partial_r u_0(x) \right\|_{L^1_x(\mathbb{R}^5)} + \left\| \frac{1}{|x|} |\Delta u_0(x)| \right\|_{L^1_x(\mathbb{R}^5)} \lesssim \|u_0\|_{\dot B^3_{1,1}(\mathbb{R}^5)} \tag{2-9}$$

From the fundamental theorem of calculus and polar coordinates, for $y \in \mathbb{R}^5 \backslash \{0\}$,

$$|y|^3 |\partial_r u_0(y)| \lesssim \int_{|y|}^\infty \int_{\mathbb{S}^4} r^3 |\partial_{rr} u_0(r)|\, d\sigma(\omega)\, dr \lesssim \left\| \frac{1}{|x|} \partial_{rr} u_0(x) \right\|_{L^1_x(\mathbb{R}^5)} \lesssim \|u_0\|_{\dot B^3_{1,1}(\mathbb{R}^5)} \tag{2-10}$$

and

$$|y|^3 |u_1(y)| \lesssim \int_{|y|}^\infty \int_{\mathbb{S}^4} r^3 |\partial_{rr} u_1(r)|\, d\sigma(\omega)\, dr \lesssim \left\| \frac{1}{|x|} \partial_r u_1(x) \right\|_{L^1_x(\mathbb{R}^5)} \lesssim \|u_1\|_{\dot B^2_{1,1}(\mathbb{R}^5)}. \tag{2-11}$$

Hence, we are reduced to proving (2-7) and (2-8). We just give the estimate for the first term on the left-hand side of (2-7), since others can be handled similarly. For $j \in \mathbb{Z}$, utilizing Bernstein's estimates and polar coordinates, we obtain

$$\begin{aligned}
\Big\| \frac{1}{|x|^2} \partial_r P_j u_0(x) \Big\|_{L^1_x(\mathbb{R}^5)} &\lesssim \int_0^\infty \int_{\mathbb{S}^4} r^2 \partial_r P_j u_0(r)\, d\sigma(\omega)\, dr \\
&\lesssim \int_0^{2^{-j}} \frac{1}{r^2} |\partial_r (P_j u_0)| r^4\, dr + \int_{2^{-j}}^\infty \frac{1}{r^2} |\partial_r (P_j u_0)| r^4\, dr \\
&\lesssim 2^{-3j} \|\partial_r (P_j u_0)\|_{L^\infty_r(\mathbb{R}_+)} + 2^{2j} \|\partial_r (P_j u_0)\|_{L^1_x(\mathbb{R}^5)} \\
&\lesssim 2^{3j} \|P_j u_0\|_{L^1_x(\mathbb{R}^5)}.
\end{aligned} \tag{2-12}$$

Thus, we have $\|(1/|x|^2)\partial_r u_0(x)\|_{L^1_x(\mathbb{R}^5)} \lesssim \|u_0\|_{\dot{B}^3_{1,1}(\mathbb{R}^5)}$. □

As a direct consequence of Lemma 2.3, we have

$$\begin{aligned}
\big\| |x|^{1/2} \partial_r u_0(x) \big\|_{L^2_x(\mathbb{R}^5)} + \big\| |x|^{-1/2} u_0(x) \big\|_{L^2_x(\mathbb{R}^5)} + \big\| |x|^{1/2} u_1(x) \big\|_{L^2_x(\mathbb{R}^5)} \\
\lesssim \|(u_0, u_1)\|_{\dot{B}^3_{1,1} \times \dot{B}^2_{1,1}(\mathbb{R}^5)}.
\end{aligned} \tag{2-13}$$

Lemma 2.4. *Suppose $\chi(x) \in C_c^\infty(\mathbb{R}^5)$. Let $R = 2^k$ be a dyadic number for $k \in \mathbb{Z}$ and denote $\chi_R(x) = \chi(\frac{x}{R})$. Then we have*

$$\|\chi_R(x) f\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^5)} \lesssim \|f\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^5)}, \tag{2-14}$$

$$\|\chi_R(x) g\|_{\dot{B}^{-1/2}_{2,1}(\mathbb{R}^5)} \lesssim \|g\|_{\dot{B}^{-1/2}_{2,1}(\mathbb{R}^5)}, \tag{2-15}$$

where the bound is independent of R. Furthermore, if $\chi(x) = 1$ on $|x| \le 1$, then for $(f, g) \in \dot{B}^{1/2}_{2,1} \times \dot{B}^{-1/2}_{2,1}(\mathbb{R}^5)$, we have

$$\lim_{R\to\infty} \|(1 - \chi_R(x)) f\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^5)} + \|(1 - \chi_R(x)) g\|_{\dot{B}^{-1/2}_{2,1}(\mathbb{R}^5)} = 0. \tag{2-16}$$

Proof. By scaling, to prove the inequalities (2-14) and (2-15), it suffices to prove the cases for $R = 1$, which follows from a similar proof of Lemma 2.2 in [Dodson 2019]. On the other hand, (2-16) follows from (2-14), (2-15), and the fact that $C_c^\infty \times C_c^\infty(\mathbb{R}^5)$ is dense in $\dot{B}^{1/2}_{2,1} \times \dot{B}^{-1/2}_{2,1}(\mathbb{R}^5)$. □

Finally, we need the following chain rule estimates for later use.

Lemma 2.5 (C^1-fractional chain rule [Christ and Weinstein 1991]). *Suppose $G \in C^1(\mathbb{C})$, $s \in (0, 1]$, and $1 < q, q_1, q_2 < \infty$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\big\| |\nabla|^s G(u) \big\|_{L^q(\mathbb{R}^d)} \lesssim \big\| G'(u) \big\|_{L^{q_1}(\mathbb{R}^d)} \big\| |\nabla|^s u \big\|_{L^{q_2}(\mathbb{R}^d)}. \tag{2-17}$$

2B. ***Fundamental properties of the wave equations.*** Throughout the paper, by abuse of notations, we often write $u(t) = u(t, x)$ for simplicity and $u(t, r) = u(t, x)$ when $u(t, \cdot)$ is radially symmetric.

Recall the explicit formula for solution to the linear wave equation in 5 dimensions,

$$\begin{aligned} S(t)(f, g)(x) &= \cos(t|\nabla|) f(x) + \frac{\sin(t|\nabla|)}{|\nabla|} g(x) \\ &= \frac{1}{3\omega_5} \partial_t \left[\frac{1}{t} \partial_t \right] \left(t^3 \int_{|y|=1} f(x+ty)\, d\sigma(y) \right) \\ &\qquad + \frac{1}{3\omega_5} \frac{1}{t} \partial_t \left(t^3 \int_{|y|=1} g(x+ty)\, d\sigma(y) \right), \end{aligned} \tag{2-18}$$

where ω_5 is the surface area of the unit sphere in $\mathbb{R}^5$. When (f, g) is radially symmetric, for $t > 0$, (2-18) can be rewritten as

$$\begin{aligned} S(t)(f, g)(r) &= \frac{1}{2r^2} \big[(r-t)^2 f(r-t) + (r+t)^2 f(r+t) \big] \\ &\quad - \frac{t}{2r^3} \int_{|r-t|}^{r+t} s f(s)\, ds + \frac{1}{4r^3} \int_{|r-t|}^{r+t} s(s^2 + r^2 - t^2) g(s)\, ds. \end{aligned} \tag{2-19}$$

See also [Rammaha 1987; Lindblad and Sogge 1996; Colzani et al. 2002] for the radial solutions to general dimensions linear wave equation. From the explicit formula (2-18), we can obtain the following dispersive estimate.

Proposition 2.6 (dispersive estimate).

$$\|S(t)(u_0, u_1)\|_{L^\infty(\mathbb{R}^5)} \lesssim \frac{1}{t^2} \big[\|\nabla^3 u_0\|_{L^1(\mathbb{R}^5)} + \|\nabla^2 u_1\|_{L^1(\mathbb{R}^5)} \big]. \tag{2-20}$$

Proof. We give the proof for completeness. A similar proof for the 3-dimensional case can be found in [Killip and Visan 2011]. By (2-18), the free solution $S(t)(u_0, u_1)$ can be rewritten as

$$\begin{aligned} &\frac{1}{\omega_5} \int_{|y|=1} u_0(x+ty)\, d\sigma(y) + \frac{5t}{3\omega_5} \int_{|y|=1} y(\nabla u_0)(x+ty)\, d\sigma(y) \\ &\quad + \frac{t^2}{3\omega_5} \int_{|y|=1} y(\nabla^2 u_0)(x+ty) y\, d\sigma(y) + \frac{t}{\omega_5} \int_{|y|=1} u_1(x+ty)\, d\sigma(y) \\ &\qquad + \frac{t^2}{3\omega_5} \int_{|y|=1} y(\nabla u_1)(x+ty)\, d\sigma(y), \end{aligned} \tag{2-21}$$

which, with the fundamental theorem of calculus, yields (2-20). For instance, using polar coordinates, we can estimate the first term of (2-21) as

$$\begin{aligned}(2\text{-}22)\quad \Big|\frac{1}{\omega_5}\int_{|y|=1} u_0(x+ty)\,d\sigma(y)\Big| &= \Big|\frac{1}{\omega_5}\int_t^\infty\int_s^\infty\int_\tau^\infty\int_{|y|=1}\frac{d^3}{d\rho^3}[u_0(x+\rho y)]\,d\sigma(y)\,d\rho\,d\tau\,ds\Big| \\ &\lesssim \int_t^\infty\int_s^\infty\int_\tau^\infty\int_{|y|=1}|\nabla^3 u_0|(x+\rho y)\,d\sigma(y)\,d\rho\,d\tau\,ds \\ &\lesssim \int_t^\infty\int_s^\infty\frac{1}{\tau^4}\,d\tau\,ds\|\nabla^3 u_0\|_{L^1(\mathbb{R}^5)} \\ &\lesssim \frac{1}{t^2}\|\nabla^3 u_0\|_{L^1(\mathbb{R}^5)}.\end{aligned}$$

The other terms can be dealt with similarly. □

We recall the Strichartz estimates of the wave equation in $\mathbb{R}^5$. Let $I\subset\mathbb{R}$ be an interval. We denote the spacetime norm $L_t^qW_x^{s,r}(I\times\mathbb{R}^5)$ of a function $u(t,x)$ on $I\times\mathbb{R}^5$ by

$$\|u\|_{L_t^qW_x^{s,r}(I\times\mathbb{R}^5)} := \big\|\|u(t,x)\|_{W_x^{s,r}(\mathbb{R}^5)}\big\|_{L_t^q(I)},$$

for $s\in\mathbb{R}$, $1\le q,\ r\le\infty$. We denote that a pair (q,r) of exponents is admissible, if

$$(2\text{-}23)\qquad 2\le q\le\infty,\quad 2\le r<\infty,\quad\text{and}\quad \frac{1}{q}+\frac{2}{r}\le 1.$$

Moreover, we say (q,r) is wave acceptable, provided

$$(2\text{-}24)\qquad 1\le q<\infty,\quad 2\le r\le\infty,\quad \frac{1}{q}<4\Big(\frac{1}{2}-\frac{1}{r}\Big),$$

or $(q,r)=(\infty,2)$.

Proposition 2.7 (Strichartz estimates [Lindblad and Sogge 1995; Ginibre and Velo 1995; Keel and Tao 1998]). *Let $(u_0,u_1)\in\dot H^{1/2}\times\dot H^{-1/2}(\mathbb{R}^5)$ and (q,r), $(\tilde q,\tilde r)$ be two admissible pairs. If u is a weak solution to the wave equation $\partial_{tt}u-\Delta u=F(t,x)$ with initial data (u_0,u_1), then we have*

$$\begin{aligned}(2\text{-}25)\quad \big\||\nabla|^\rho u\big\|_{L_t^qL_x^r(I\times\mathbb{R}^5)}+\sup_{t\in I}\|(u,u_t)\|_{\dot H^{1/2}\times\dot H^{-1/2}(\mathbb{R}^5)} \\ \lesssim \|(u_0,u_1)\|_{\dot H^{1/2}\times\dot H^{-1/2}(\mathbb{R}^5)}+\big\||\nabla|^{-\mu}F\big\|_{L_t^{\tilde q'}L^{\tilde r'}(I\times\mathbb{R}^5)},\end{aligned}$$

provided that

$$(2\text{-}26)\qquad \rho=\frac{1}{q}+\frac{5}{r}-2\quad\textit{and}\quad \mu=\frac{1}{\tilde q}+\frac{5}{\tilde r}-2.$$

Proposition 2.8 (inhomogeneous Strichartz estimates [Taggart 2010]). *Suppose that the exponent pairs (q_1, r_1) and $(\tilde{q}_1, \tilde{r}_1)$ are wave acceptable, and satisfy the scaling condition*

$$\frac{1}{q}+\frac{1}{\tilde{q}}=2-2\Big(\frac{1}{r_1}+\frac{1}{\tilde{r}_1}\Big)$$

and the conditions

$$\frac{1}{q}+\frac{1}{\tilde{q}}<1, \qquad \frac{1}{2}\le\frac{\tilde{r}_1}{r_1}\le 2.$$

Let $r\ge r_1$, $\tilde{r}\ge\tilde{r}_1$, $\rho\in\mathbb{R}$ be such that

$$\rho+5\Big(\frac{1}{2}-\frac{1}{r}\Big)-\frac{1}{q}=1-\Big(\tilde{\rho}+5\Big(\frac{1}{2}-\frac{1}{\tilde{r}}\Big)-\frac{1}{\tilde{q}}\Big).$$

If $F(t,x)$ is in $L_t^{\tilde{q}'}(\mathbb{R};\dot{B}^{-\tilde{\rho}}_{\tilde{r}',2}(\mathbb{R}^5))$ and u is a weak solution to the inhomogeneous wave equation

$$-\partial_t^2 u+\Delta u=F(t,x), \qquad u(0)=u_t(0)=0, \tag{2-27}$$

then

$$\|u\|_{L_t^q(\mathbb{R};\dot{B}^{\rho}_{r,2}(\mathbb{R}^5))}\lesssim\|F(t,x)\|_{L_t^{\tilde{q}'}(\mathbb{R};\dot{B}^{-\tilde{\rho}}_{\tilde{r}',2}(\mathbb{R}^5))}. \tag{2-28}$$

Next, we recall the well-posedness theory and the perturbation theory of the Cauchy problem (1-5).

Definition 2.9 (solution). Let I be a time interval such that $0\in I$. We say function $u: I\times\mathbb{R}^5\to\mathbb{R}$ is a (strong) solution to the Cauchy problem (1-5) in I if it satisfies $(u,u_t)(0)=(u_0,u_1)$,

$$(u,u_t)\in C(I;\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5))\cap L^3_{t,x}(I\times\mathbb{R}^5),$$

and the integral equation

$$u(t)=S(t)(u_0,u_1)-\int_0^t S(t-\tau)(0,|u|u(\tau))\,d\tau \tag{2-29}$$

for all $t\in I$.

Theorem 2.10 (local well-posedness [Lindblad and Sogge 1995; Rodriguez 2017]). *Let $(u_0,u_1)\in\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)$ with*

$$\|(u_0,u_1)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)}\le A.$$

There exists $\delta=\delta(A)>0$ such that, if

$$\|S(t)(u_0,u_1)\|_{L^3_{t,x}([0,T]\times\mathbb{R}^5)}\le\delta, \qquad \textit{for some } T>0, \tag{2-30}$$

then there exists a unique solution u to (1-5) in $[0,T]\times\mathbb{R}^5$, such that

$$\sup_{0\le t\le T}\|(u,u_t)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)}+\|u\|_{L^3_{t,x}([0,T]\times\mathbb{R}^5)}\le C(A). \tag{2-31}$$

In addition, if $A > 0$ is small enough, we can take $T = \infty$.

We define $T_+(u_0, u_1) := \sup I$, where I is the maximal interval of existence of the solution u.

Lemma 2.11 (standard blowup criterion). *Suppose u is the solution to the Cauchy problem (1-5) with initial data $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^5)$ and $T_+(u_0, u_1) < \infty$. Then we have*

$$\|u\|_{L^3_{t,x}([0,T_+(u_0,u_1))\times\mathbb{R}^5)} = \infty. \tag{2-32}$$

The proof is standard and similar to the energy critical case in [Kenig 2015].

We end this section by recalling the stability lemma for the Cauchy problem (1-5), which plays an important role in the Theorem 4.1.

Theorem 2.12 (perturbation theory [Rodriguez 2017]). *Let $I \subset \mathbb{R}$ be a time interval with $0 \in I$. Let $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^5)$ and some constants $M, A, A' > 0$ be given. Let $\tilde{u}$ be defined on $I \times \mathbb{R}^5$ and satisfy*

$$\sup_{t\in I}\|(\tilde{u}, \partial_t\tilde{u})\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)} \le A, \tag{2-33}$$

$$\|\tilde{u}\|_{L^3_{t,x}(I\times\mathbb{R}^5)} \le M. \tag{2-34}$$

Assume that $\partial_{tt}\tilde{u} - \Delta\tilde{u} = -|\tilde{u}|\tilde{u} + e$ on $I \times \mathbb{R}^5$,

$$\big\|\big(u_0 - \tilde{u}(0), u_1 - \partial_t\tilde{u}(0)\big)\big\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)} \le A' \tag{2-35}$$

and that

$$\|e\|_{L^{3/2}_{t,x}(I\times\mathbb{R}^5)} + \big\|S(t)\big[(\tilde{u}(0), \partial_t\tilde{u}(0)) - (u_0, u_1)\big]\big\|_{L^3_{t,x}(I\times\mathbb{R}^5)} < \varepsilon. \tag{2-36}$$

Then, there exist $\beta > 0$ and $\varepsilon_0 = \varepsilon_0(M, A, A') > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, there exists a solution u to (1-5) in I such that $(u(0), \partial_t u(0)) = (u_0, u_1)$, with

$$\|u\|_{L^3_{t,x}(I\times\mathbb{R}^5)} \le C(M, A, A'), \tag{2-37}$$

$$\sup_{t\in I}\|(\partial_t\tilde{u}(t), \tilde{u}(t)) - (u, \partial_t u(t))\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)} \le C(M, A, A')(A' + \varepsilon^\beta). \tag{2-38}$$

3. Decomposition of the solution and global well-posedness

In this section, we will prove that for any given initial data $(u_0, u_1) \in \dot{B}^3_{1,1} \times \dot{B}^2_{1,1}(\mathbb{R}^5)$, the corresponding solution u to (1-5) is globally well-posed. To prove this, we first show the solution u belongs to some suitable Strichartz-type spaces on a local time interval. Then, we split it into two parts: $u = v + w$. Based on the inhomogeneous Strichartz estimates (2-28), we will derive a decay property for v and prove that w is in the energy space $\dot{H}^1 \times L^2(\mathbb{R}^5)$. We remark that the constants in "$\lesssim$" in this section depend upon $\|(u_0, u_1)\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)}$.

For the sake of simplicity, we denote $F(u) = |u|u$. Recall that u_λ is also a solution to the Cauchy problem (1-5) with initial data $(\lambda^2 u_0(\lambda x), \lambda^3 u_1(\lambda x))$, where

$$u_\lambda(t, x) = \lambda^2 u(\lambda t, \lambda x). \tag{3-1}$$

Given $(u_0, u_1) \in \dot{B}^3_{1,1} \times \dot{B}^2_{1,1}(\mathbb{R}^5)$, for any $\eta > 0$, there exists $j_0 = j_0(u_0, u_1, \eta) < \infty$ such that

$$\sum_{j \geq j_0} 2^{2j} \|P_j u_0\|_{L^1(\mathbb{R}^5)} < \eta. \tag{3-2}$$

Replace u by u_λ for $\lambda = 2^{-j_0}$, then we have

$$\sum_{j \geq 0} 2^{3j} \|P_j u_0\|_{L^1(\mathbb{R}^5)} + \sum_{j \geq 0} 2^{2j} \|P_j u_1\|_{L^1(\mathbb{R}^5)} < \eta. \tag{3-3}$$

Lemma 3.1. *Let $\epsilon_0 > 0$ be a small constant and $\eta \ll \epsilon_0$. If the initial data $(u_0, u_1) \in \dot{B}^3_{1,1} \times \dot{B}^2_{1,1}(\mathbb{R}^5)$ satisfies (3-3) and u is the solution to (1-5) with initial data (u_0, u_1) given by Theorem 2.10, then there exists*

$$\delta = \delta(\epsilon_0, \|(u_0, u_1)\|_{\dot{B}^3_{1,1} \times \dot{B}^2_{1,1}}) > 0$$

such that

$$\|u\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} \leq \sum_{j\in\mathbb{Z}} \|P_j u\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} \lesssim \epsilon_0, \tag{3-4}$$

$$\|u\|_{L^\infty_t([-\delta,\delta]; \dot{B}^{1/2}_{2,1}(\mathbb{R}^5))} \lesssim \sum_{j\in\mathbb{Z}} \|P_j u\|_{L^\infty_t \dot{H}^{1/2}([-\delta,\delta]\times\mathbb{R}^5)} \lesssim 1. \tag{3-5}$$

Proof. By Strichartz's estimates in Proposition 2.7 and (3-3), we obtain

$$\|S(t) P_{\geq 0}(u_0, u_1)\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} \leq \tfrac{1}{2}\epsilon_0. \tag{3-6}$$

On the other hand, for every $t \in \mathbb{R}$, by Bernstein, we have

$$\|S(t) P_{\leq 0}(u_0, u_1)\|_{L^3_x(\mathbb{R}^5)} \lesssim 1. \tag{3-7}$$

Hence, taking δ small enough, we have,

$$\|S(t)(u_0, u_1)\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} \leq \epsilon_0. \tag{3-8}$$

Then, by the Strichartz estimates, we have

$$\|u\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} \lesssim \|S(t)(u_0, u_1)\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} + \|u\|^2_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)}, \tag{3-9}$$

from which, by a standard continuity argument, we deduce that

$$\|u\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} \lesssim \epsilon_0. \tag{3-10}$$

Let

$$a_k = \|P_k u\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} + 2^{\frac{1}{2}k}\|P_k u\|_{L^\infty_t L^2_x([-\delta,\delta]\times\mathbb{R}^5)} + 2^{\frac{1}{4}k}\|P_k u\|_{L^6_t L^{12/5}_x([-\delta,\delta]\times\mathbb{R}^5)}, \tag{3-11}$$

$$b_k = 2^{\frac{1}{2}k}\|P_k u_0\|_{L^2_x} + 2^{-\frac{1}{2}k}\|P_k u_1\|_{L^2_x}. \tag{3-12}$$

By Young's inequality, it suffices to show there is a recurrence relation

$$a_k \lesssim b_k + \epsilon_0 \sum_j 2^{-\frac{1}{4}|j-k|} a_j. \tag{3-13}$$

To prove this, making use of the Strichartz estimates, we have

$$a_k \lesssim b_k + 2^{\frac{1}{4}k}\|P_k F(u)\|_{L^2_t L^{4/3}_x([0,\delta]\times\mathbb{R}^5)}. \tag{3-14}$$

First, we consider the low frequency part of the second term in the right-hand side of (3-14). By Lemma 2.5 and Hölder, we have

$$\begin{aligned} \|P_k F(u_{\le k})\|_{L^2_t L^{4/3}_x([-\delta,\delta]\times\mathbb{R}^5)} &\lesssim 2^{-\frac{1}{2}k}\|P_k|\nabla_x|^{\frac{1}{2}} F(u_{\le k})\|_{L^2_t L^{4/3}_x([-\delta,\delta]\times\mathbb{R}^5)} \\ &\lesssim 2^{-\frac{1}{2}k}\|u\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} \big\||\nabla|^{1/2}(P_{\le k}u)\big\|_{L^6_t L^{12/5}_x([-\delta,\delta]\times\mathbb{R}^5)} \\ &\lesssim \|u\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} \sum_{j\le k} 2^{-\frac{1}{2}(k-j)}\|P_j u\|_{L^6_t L^{12/5}_x([-\delta,\delta]\times\mathbb{R}^5)}, \end{aligned} \tag{3-15}$$

from which it follows that

$$\begin{aligned} 2^{\frac{k}{4}}\|P_k F(u_{\le k})\|_{L^2_t L^{4/3}_x([-\delta,\delta]\times\mathbb{R}^5)} &\lesssim \epsilon_0 \sum_{j\le k} 2^{-\frac{1}{4}(k-j)} 2^{\frac{1}{4}j}\|P_j u\|_{L^6_t L^{12/5}_x([-\delta,\delta]\times\mathbb{R}^5)} \\ &\lesssim \epsilon_0 \sum_{j\le k} 2^{-\frac{1}{4}(k-j)} a_j. \end{aligned} \tag{3-16}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} &2^{\frac{1}{4}k}\|P_k(F(u) - F(u_{\le k}))\|_{L^2_t L^{4/3}_x([-\delta,\delta]\times\mathbb{R}^5)} \\ &\lesssim \|u\|_{L^3_{t,x}([-\delta,\delta]\times\mathbb{R}^5)} 2^{\frac{1}{4}k}\|P_{\ge k+1}u\|_{L^6_t L^{12/5}_x([-\delta,\delta]\times\mathbb{R}^5)} \\ &\lesssim \epsilon_0 \sum_{j\ge k+1} 2^{-\frac{1}{4}(j-k)} 2^{\frac{1}{4}j}\|P_j u\|_{L^6_t L^{12/5}_x([-\delta,\delta]\times\mathbb{R}^5)} \lesssim \epsilon_0 \sum_{j\ge k+1} 2^{-\frac{1}{4}(j-k)} a_j. \end{aligned} \tag{3-17}$$

Then the recurrence relation (3-13) follows from (3-16) and (3-17). □

Note that by the inequality (3-5), the inequalities (3-16) and (3-17) yield that

$$\sum_{k\in\mathbb{Z}} 2^{\frac{1}{4}k}\|P_k(F(u))\|_{L^2_t L^{4/3}_x([0,\delta]\times\mathbb{R}^5)} \lesssim \epsilon_0. \tag{3-18}$$

As an application of Lemma 3.1 and the radial Sobolev inequality (2-2), we will see that the solution u possesses some space decay property in the region $\{|x| \geq |t| + C\}$ for some large constant $C > 0$. Let $\chi(x)$ be a smooth cutoff function such that $\chi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\chi(x) = 0$ for $|x| \geq 1$. By Lemma 2.4, there exists a dyadic integer $R = R(u_0, u_1, \epsilon_0)$ such that $\|(1 - \chi(\frac{x}{R}))(u_0, u_1)\|_{\dot{B}^{1/2}_{2,1} \times \dot{B}^{-1/2}_{2,1}(\mathbb{R}^5)} \leq \epsilon_0$. Then by scaling, we have

$$\big\|(1 - \chi(2x))\big((2R)^2 u_0(2Rx), (2R)^3 u_1(2Rx)\big)\big\|_{\dot{B}^{1/2}_{2,1} \times \dot{B}^{-1/2}_{2,1}(\mathbb{R}^5)} \leq \epsilon_0. \tag{3-19}$$

By abuse of notations, we will still use (u_0, u_1) to represent the initial data $\big((2R)^2 u_0(2Rx), (2R)^3 u_1(2Rx)\big)$. Then we have,

$$\|(1 - \chi(2x))(u_0, u_1)\|_{\dot{B}^{1/2}_{2,1} \times \dot{B}^{-1/2}_{2,1}(\mathbb{R}^5)} \leq \epsilon_0. \tag{3-20}$$

In addition, by Lemma 3.1, we have

$$\|u\|_{L^3_{t,x}([-\delta/(2R),\, \delta/(2R)] \times \mathbb{R}^5)} \lesssim \epsilon_0, \tag{3-21}$$

$$\|u\|_{L^\infty_t([-\delta/(2R),\, \delta/(2R)];\, \dot{B}^{1/2}_{2,1}(\mathbb{R}^5))} \lesssim 1. \tag{3-22}$$

Lemma 3.2. *Let $J \subset \mathbb{R}$ be an interval such that u is a solution to (1-5) on J. Then*

$$\|u\|_{L^3_{t,x}(\{(t,x) \in J \times \mathbb{R}^5:\, |x| \geq |t| + \frac{1}{2}\})} + \sup_{t \in J} \big\| |x|^2 u(t,x) \big\|_{L^\infty_x(\{x \in \mathbb{R}^5:\, |x| \geq |t| + \frac{1}{2}\})} \lesssim \epsilon_0. \tag{3-23}$$

Proof. Let $U(t,x)$ be the solution to (1-5) with initial data $(1 - \chi(2x))(u_0(x), u_1(x))$. Employing Theorem 2.10 and arguing by similar arguments in Lemma 3.1, one can deduce (3-14) when u is replaced by U and $[-\delta, \delta]$ is replaced by $\mathbb{R}$. Thus

$$\|U\|_{L^3_{t,x}(\mathbb{R} \times \mathbb{R}^5)} + \|U\|_{L^\infty_t \dot{B}^{1/2}_{2,1}(\mathbb{R} \times \mathbb{R}^5)} \lesssim \epsilon_0. \tag{3-24}$$

Due to the finite propagation speed property of the wave equation (1-5), we have $u(t,x) = U(t,x)$ when $|x| \geq |t| + \frac{1}{2}$. Then (3-23) follows from (3-24) and the radial Sobolev inequality (2-2). □

Next, we want to show the following local properties, which will play an important role in Section 3B. Unlike the case of three dimensions in [Dodson 2019], we will make use of the inhomogeneous Strichartz estimates (2-28) to conquer the difficulties caused by the higher order decay of time.

Lemma 3.3. *If ϵ_0 is sufficiently small and δ is as given in Lemma 3.1, then, for $3 < r < 4$, we have*

$$\sup_{-\frac{2\delta}{2R} < t < \frac{2\delta}{2R}} t^{(2r-5)/r} \|u\|_{L^r_x(\mathbb{R}^5)} + \|u\|_{L^{5/4}_t L^{25/6}_x([0,\, \delta/(2R)] \times \mathbb{R}^5)} \lesssim 1. \tag{3-25}$$

We remark that for $3 < r < \infty$, the space $L^{r/(2r-5)}_t L^r_x(\mathbb{R} \times \mathbb{R}^5)$ is $\dot{H}^{1/2}$-critical but not admissible.

Proof. First, we consider the estimates for the linear part. Utilizing dispersive estimate (2-20), we have

$$\|S(t)P_j(u_0,u_1)\|_{L_x^\infty(\mathbb{R}^5)} \lesssim \frac{1}{t^2}\big[2^{3j}\|P_ju_0\|_{L^1(\mathbb{R}^5)} + 2^{2j}\|P_ju_1\|_{L^1(\mathbb{R}^5)}\big]. \tag{3-26}$$

By Bernstein, we have

$$\begin{aligned}\|S(t)P_j(u_0,u_1)\|_{L_x^2(\mathbb{R}^5)} &\lesssim \|P_ju_0\|_{L_x^2(\mathbb{R}^5)} + 2^{-j}\|P_ju_1\|_{L_x^2(\mathbb{R}^5)}\\ &\lesssim 2^{\frac{5}{2}j}\|P_ju_0\|_{L^1(\mathbb{R}^5)} + 2^{\frac{3}{2}j}\|P_ju_1\|_{L^1(\mathbb{R}^5)}.\end{aligned} \tag{3-27}$$

Interpolating this inequality with the estimate (3-26) yields that,

$$\|S(t)P_j(u_0,u_1)\|_{L_x^r(\mathbb{R}^5)} \lesssim t^{-2(1-\frac{2}{r})}2^{-\frac{j}{r}}\big[2^{3j}\|P_ju_0\|_{L^1(\mathbb{R}^5)} + 2^{2j}\|P_ju_1\|_{L^1(\mathbb{R}^5)}\big]. \tag{3-28}$$

On the other hand, for $r \geq \frac{5}{2}$, employing Bernstein's estimates, we have

$$\begin{aligned}\|S(t)P_j(u_0,u_1)\|_{L_x^r(\mathbb{R}^5)} &\lesssim 2^{5(\frac{1}{2}-\frac{1}{r})j}\|S(t)P_j(u_0,u_1)\|_{L_x^2(\mathbb{R}^5)}\\ &\lesssim 2^{(2-\frac{5}{r})j}\big[2^{3j}\|P_ju_0\|_{L^1(\mathbb{R}^5)} + 2^{2j}\|P_ju_1\|_{L^1(\mathbb{R}^5)}\big].\end{aligned} \tag{3-29}$$

This estimate and (3-28) yield that, for $r \geq \frac{5}{2}$,

$$\begin{aligned}\sup_{t\in\mathbb{R}} t^{(2r-5)/r}\|S(t)(u_0,u_1)\|_{L_x^r(\mathbb{R}^5)} + \|S(t)(u_0,u_1)\|_{L_t^{r/(2r-5)}L_x^r(\mathbb{R}\times\mathbb{R}^5)}&\\ \lesssim \|(u_0,u_1)\|_{\dot B^3_{1,1}\times\dot B^2_{1,1}(\mathbb{R}^5)}&.\end{aligned} \tag{3-30}$$

By the reversal property of the wave equation, it suffices to prove (3-25) for $t \geq 0$. Using the inhomogeneous Strichartz estimates (2-28), Lemmas 2.5, 3.1, and Hölder, we have

$$\begin{aligned}&\Big\|\int_0^t S(t-\tau)(0,F(u(\tau)))\,d\tau\Big\|_{L_t^{5/4}L_x^{25/6}([0,\frac{\delta}{2R}]\times\mathbb{R}^5)}\\ &\quad\lesssim \big\||\nabla|^{1/4}F(u)\big\|_{L_t^{30/29}L_x^{300/197}([0,\frac{\delta}{2R}]\times\mathbb{R}^5)}\\ &\quad\lesssim \big\||\nabla|^{1/2}u\big\|^{1/2}_{L_t^\infty L_x^2([0,\frac{\delta}{2R}]\times\mathbb{R}^5)}\|u\|^{1/2}_{L^3_{t,x}([0,\frac{\delta}{2R}]\times\mathbb{R}^5)}\|u\|_{L_t^{5/4}L_x^{25/6}([0,\frac{\delta}{2R}]\times\mathbb{R}^5)}.\end{aligned} \tag{3-31}$$

This estimates together with the estimate (3-30) yields

$$\|u\|_{L_t^{5/4}L_x^{25/6}([0,\frac{\delta}{2R}]\times\mathbb{R}^5)} \lesssim 1, \tag{3-32}$$

provided $0 < \epsilon_0 \ll \min(1, \|(u_0,u_1)\|^{-1}_{\dot B^3_{1,1}\times\dot B^2_{1,1}(\mathbb{R}^5)})$.

Let $c \in (0,1)$ to be chosen later. First, employing the dispersive estimate (2-20), Lemma 2.5, and interpolation, for $r \in (3,4)$, we have

$$(3\text{-}33)\quad \sup_{t\in[0,\frac{\delta}{2R}]} t^{\frac{2r-5}{r}} \Big\| \int_{(1-c)t}^{t} S(t-\tau)(0, F(u(\tau)))\,d\tau \Big\|_{L^r(\mathbb{R}^5)}$$
$$\lesssim \sup_{t\in[0,\frac{\delta}{2R}]} t^{\frac{2r-5}{r}} \int_{(1-c)t}^{t} \frac{1}{(t-\tau)^{2-\frac{4}{r}}} \big\| |\nabla|^{2-\frac{6}{r}} F(u(\tau)) \big\|_{L^{r'}(\mathbb{R}^5)}\,d\tau$$
$$\lesssim \sup_{t\in[0,\frac{\delta}{2R}]} t^{\frac{2r-5}{r}} \int_{(1-c)t}^{t} \frac{1}{(t-\tau)^{2-\frac{4}{r}}} \|u(\tau)\|_{L^r(\mathbb{R}^5)}^{\frac{r-1}{2r-5}} \times \|u(\tau)\|_{\dot H_x^{1/2}(\mathbb{R}^5)}^{4-\frac{12}{r}} \|u(\tau)\|_{L_x^{5/2}(\mathbb{R}^5)}^{(\frac{12}{r}-2-\frac{r-1}{2r-5})}\,d\tau$$
$$\lesssim \Big(\sup_{t\in[0,\frac{\delta}{2R}]} t^{\frac{2r-5}{r}} \|u(t)\|_{L^r(\mathbb{R}^5)} \Big)^{\frac{r-1}{2r-5}} \|u\|_{L_t^\infty \dot H_x^{1/2}([0,\frac{\delta}{2R}]\times\mathbb{R}^5)}^{4-\frac{12}{r}} \times \|u\|_{L_t^\infty L_x^{5/2}([0,\frac{\delta}{2R}]\times\mathbb{R}^5)}^{(\frac{12}{r}-2-\frac{r-1}{2r-5})} \cdot \int_{(1-c)t}^{t} \frac{1}{(t-\tau)^{2-\frac{4}{r}}} t^{1-\frac{4}{r}}\,d\tau$$
$$\lesssim c^{4/r-1} \Big(\sup_{t\in[0,\frac{\delta}{2R}]} t^{\frac{2r-5}{r}} \|u(t)\|_{L^r(\mathbb{R}^5)} \Big)^{\frac{r-1}{2r-5}}.$$

For the remainder part, we utilize the dispersive estimate (2-20), Lemma 2.5, the Hölder inequality, and interpolation to obtain

$$(3\text{-}34)\quad t^{\frac{2r-5}{r}} \Big\| \int_0^{(1-c)t} S(t-\tau)(0, F(u(\tau)))\,d\tau \Big\|_{L^r(\mathbb{R}^5)}$$
$$\lesssim t^{\frac{2r-5}{r}} \int_0^{(1-c)t} \frac{1}{(t-\tau)^{2-\frac{4}{r}}} \big\| |\nabla|^{2-\frac{6}{r}} F(u(\tau)) \big\|_{L_x^{r'}(\mathbb{R}^5)}\,d\tau$$
$$\lesssim c^{\frac{4}{r}-2} \big\| |\nabla|^{2-\frac{6}{r}} F(u) \big\|_{L_{t,x}^{r'}(\mathbb{R}\times\mathbb{R}^5)}$$
$$\lesssim c^{\frac{4}{r}-2} \|u\|_{L_t^{r/(2r-5)} L_x^r(\mathbb{R}\times\mathbb{R}^5)} \|u\|_{L_t^\infty \dot H_x^{1/2}(\mathbb{R}\times\mathbb{R}^5)}^{4-\frac{12}{r}} \|u\|_{L_{t,x}^3(\mathbb{R}\times\mathbb{R}^5)}^{\frac{12}{r}-3}$$
$$\lesssim c^{\frac{4}{r}-2} \epsilon_0^{\frac{159}{7r}-\frac{39}{7}},$$

where in the last step we used the fact that

$$\|u\|_{L_t^{r/(2r-5)} L_x^r([0,\frac{\delta}{2R}]\times\mathbb{R}^5)} \lesssim \|u\|_{L_{t,x}^3([0,\frac{\delta}{2R}]\times\mathbb{R}^5)}^{\frac{75}{7}(\frac{1}{r}-\frac{6}{25})} \|u\|_{L_t^{5/4} L_x^{25/6}([0,\frac{\delta}{2R}]\times\mathbb{R}^5)}^{\frac{25}{7}(1-\frac{3}{r})} \lesssim \epsilon_0^{\frac{75}{7}(\frac{1}{r}-\frac{6}{25})}.$$

Hence, by (3-30) and (3-32)–(3-34), taking $c > 0$ small enough and using a continuity method, there exists $\epsilon_0 = \epsilon_0(\|(u_0,u_1)\|_{\dot B^3_{1,1}\times\dot B^2_{1,1}}) > 0$ such that

$$(3\text{-}35)\qquad \sup_{t\in[0,\frac{\delta}{2R}]} t^{\frac{2r-5}{r}} \|u\|_{L_x^r} \lesssim 1, \qquad \text{for } 3 < r < 4. \qquad \square$$

We denote $\delta_1 = \frac{\delta}{2R}$ for simplicity. Let $\psi \in C_0^\infty(\mathbb{R}^5)$ be supported in $|x| \le \frac{\delta_1}{10}$ and $\psi(x) = 1$ when $|x| \le \frac{\delta_1}{20}$. Then we can assume that $|(\nabla\psi)(x)| \lesssim \frac{1}{\delta_1}$. For $t \ge \delta_1$, we split $u(t,x) = v(t,x) + w(t,x)$, where

$$v(t) = S(t)(\psi u_0, \psi u_1) - \int_0^{\delta_1/10} S(t-\tau)(0, \psi F(u(\tau)))\, d\tau. \tag{3-36}$$

We will prove that v has a decay property and w has finite energy.

3A. ***The decay part of the solution u.***

Lemma 3.4. *For $t \ge \delta_1$, we have*

$$\|v(t)\|_{L_x^\infty(\mathbb{R}^5)} \lesssim \delta_1^{-1/2} t^{-2}. \tag{3-37}$$

In addition, we have

$$\|v\|_{L_t^\infty(\mathbb{R};\dot{B}_{2,1}^{1/2}(\mathbb{R}^5))} + \|v\|_{L_{t,x}^3(\mathbb{R}\times\mathbb{R}^5)} \lesssim \delta_1^{-1/2}. \tag{3-38}$$

Proof. We first estimate the linear part of v. By the Huygens principle, the radial Sobolev inequality (2-2) and Lemma 2.4, we have, for $t \ge \delta_1$

$$\|S(t)(\psi u_0, \psi u_1)\|_{L_x^\infty(\mathbb{R}^5)} \lesssim \frac{1}{t^2}\|(u_0,u_1)\|_{\dot{B}_{2,1}^{1/2}\times\dot{B}_{2,1}^{-1/2}} \lesssim \frac{1}{t^2}. \tag{3-39}$$

For the second part of v and $t \ge \delta_1$, using the Radial Sobolev inequality (2-2), the Huygens principle and the Strichartz estimates, we obtain

$$\begin{aligned}
&\Big\|\int_0^{\delta_1/10} S(t-\tau)(0, \psi F(u(\tau)))\, d\tau\Big\|_{L_x^\infty(\mathbb{R}^5)} \\
&\lesssim \frac{1}{t^2}\Big\|\int_0^{\delta_1/10} S(t-\tau)(0, \tilde{\chi} F(u(\tau)))\, d\tau\Big\|_{\dot{B}_{2,1}^{1/2}(\mathbb{R}^5)} \\
&\lesssim \frac{1}{t^2}\Big\|\int_0^{\delta_1/10} \frac{\sin(\tau|\nabla|)}{|\nabla|}(\psi F(u(\tau)))\, d\tau\Big\|_{\dot{B}_{2,1}^{1/2}} \\
&\qquad + \frac{1}{t^2}\Big\|\int_0^{\delta_1/10} \frac{\cos(\tau|\nabla|)}{|\nabla|}(\psi F(u(\tau)))\, d\tau\Big\|_{\dot{B}_{2,1}^{1/2}}
\end{aligned} \tag{3-40}$$

$$\lesssim \frac{1}{t^2}\sum_{j<0} 2^{-\frac{1}{2}j}\big\|P_j\big[\psi F(u(\tau))\big]\big\|_{L_t^1L_x^2([0,\delta_1/10]\times\mathbb{R}^5)} \tag{3-41}$$

$$+ \frac{1}{t^2}\sum_{j\ge0} 2^{\frac{1}{4}j}\big\|P_j\big[\psi F(u(\tau))\big]\big\|_{L_t^2L_x^{4/3}([0,\delta_1/10]\times\mathbb{R}^5)} \tag{3-42}$$

For the low frequency part (3-41), by Bernstein's estimates and Hölder's inequality, we have

$$t^2 \times (3\text{-}41) \lesssim \sum_{j<0} 2^{\frac{j}{3}}\|F(u)\|_{L_t^1L_x^{3/2}([0,\delta_1/10]\times\mathbb{R}^5)} \lesssim \delta_1^{\frac{1}{3}}\|u\|^2_{L_{t,x}^3([0,\delta_1/10]\times\mathbb{R}^5)} \lesssim 1. \tag{3-43}$$

For (3-42), it suffices to estimate

$$\sum_{j\geq 0} 2^{\frac{1}{4}j} \big\| [P_j, \psi] F(u(\tau)) \big\|_{L_t^2 L_x^{4/3}([0,\delta_1/10]\times\mathbb{R}^5)} \tag{3-44}$$

$$+ \sum_{j\geq 0} 2^{\frac{1}{4}j} \big\| \psi P_j F(u(\tau)) \big\|_{L_t^2 L_x^{4/3}([0,\delta_1/10]\times\mathbb{R}^5)}. \tag{3-45}$$

For (3-44), by commutator estimates, Young's inequality, the Sobolev embedding and Lemma 3.1, we have

$$\begin{aligned} \text{(3-44)} &\lesssim \sum_{j\geq 0} 2^{-\frac{3}{4}j} \delta_1^{-1} \| Q_j F(u(\tau)) \|_{L_\tau^2 L_x^{4/3}([0,\delta_1/10]\times\mathbb{R}^5)} \\ &\lesssim \delta_1^{-1} \sum_{j\geq 0} 2^{-\frac{3}{4}j} 2^{\frac{1}{4}j} \| F(u(\tau)) \|_{L_\tau^2 L_x^{5/4}([0,\delta_1/10]\times\mathbb{R}^5)} \\ &\lesssim \delta_1^{-\frac{1}{2}} \|u\|^2_{L_t^\infty L_x^{5/2}([0,\delta_1/10]\times\mathbb{R}^5)} \lesssim \delta_1^{-\frac{1}{2}}, \end{aligned} \tag{3-46}$$

where

$$Q_j f(x) = 2^{6j} \int_{\mathbb{R}^5} |y| \phi(2^j y) | f |(x-y)\, dy$$

and in the first inequality we used the mean value theorem. For (3-45), by the estimate (3-18), we have

$$\text{(3-45)} \lesssim \sum_{j\geq 0} 2^{\frac{1}{4}j} \| P_j F(u) \|_{L_\tau^2 L_x^{4/3}} \lesssim \epsilon_0. \tag{3-47}$$

Hence, by (3-39)–(3-47), we have $\|v(t)\|_{L_x^\infty(\mathbb{R}^5)} \lesssim \delta_1^{-\frac{1}{2}} t^{-2}$.

Now we consider (3-38). For simplicity, we write

$$\|v\|_{S(\mathbb{R})} := \|v\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} + \|v\|_{L_t^\infty(\mathbb{R};\dot{B}^{1/2}_{2,1}(\mathbb{R}^5))}.$$

For the linear part, by the Strichartz estimates and Lemma 2.4, we have

$$\|S(t)(\psi u_0, \psi u_1)\|_{S(\mathbb{R})} \lesssim \|(u_0, u_1)\|_{\dot{B}^{1/2}_{2,1}\times \dot{B}^{-1/2}_{2,1}(\mathbb{R}^5)} \lesssim 1. \tag{3-48}$$

By the Strichartz estimates and repeating the arguments that deal with (3-40),

$$\begin{aligned} &\Big\| \int_0^{\delta_1/10} S(t-\tau)(0, \psi F(u(\tau)))\, d\tau \Big\|_{S(\mathbb{R})} \\ &\quad \lesssim \Big\| \int_0^{\delta_1/10} \frac{\sin(\tau|\nabla|)}{|\nabla|} (\psi F(u(\tau)))\, d\tau \Big\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^5)} \\ &\qquad\qquad + \Big\| \int_0^{\delta_1/10} \frac{\cos(\tau|\nabla|)}{|\nabla|} (\psi F(u(\tau)))\, d\tau \Big\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^5)} \lesssim \delta_1^{-\frac{1}{2}}. \end{aligned}$$

This completes the proof. □

3B. *The energy part of the solution u.*

Lemma 3.5. *We have*

$$\|\vec{w}(\delta_1)\|_{\dot{H}^1\times L^2(\mathbb{R}^5)} \lesssim \delta_1^{-1/2}. \tag{3-49}$$

Proof. By the definition of w, for $t \geq \delta_1$, we have

$$\begin{aligned} w(t) &= S(t)\big((1-\psi)u_0, (1-\psi)u_1\big) \\ &\quad - \int_0^{\frac{\delta_1}{10}} S(t-\tau)(0, (1-\psi)F(u(\tau)))\,d\tau - \int_{\frac{\delta_1}{10}}^t S(t-\tau)(0, F(u(\tau)))d\tau. \end{aligned} \tag{3-50}$$

First, we consider the contribution of the third term of (3-50). Taking $r = \frac{50}{13}$ in (3-25), by interpolation, we have

$$\begin{aligned} &\|u\|^2_{L_t^2 L_x^4([\delta_1/10,\delta_1]\times\mathbb{R}^5)} \\ &\quad = \int_{\delta_1/10}^{\delta_1} \|u\|_{L_x^{50/13}(\mathbb{R}^5)} \|u\|_{L_x^{25/6}(\mathbb{R}^5)}\,dt \\ &\quad \lesssim \delta_1^{-\frac{1}{2}} \sup_{t\in[\frac{\delta_1}{10},\delta_1]} \Big[t^{\frac{7}{10}} \|u(t)\|_{L_x^{50/13}(\mathbb{R}^5)}\Big] \|u\|_{L_t^{5/4} L_x^{25/6}([\frac{\delta_1}{10},\delta_1]\times\mathbb{R}^5)} \lesssim \delta_1^{-\frac{1}{2}}. \end{aligned} \tag{3-51}$$

From this inequality and Strichartz, we have

$$\begin{aligned} &\Big\| \int_{\delta_1/10}^{\delta_1} S(\delta_1-\tau)(0, F(u(\tau)))d\tau \Big\|_{\dot{H}_x^1(\mathbb{R}^5)} \\ &\quad + \Big\| \partial_t \Big[\int_{\delta_1/10}^{t} S(t-\tau)(0, F(u(\tau)))\,d\tau \Big]_{t=\delta_1} \Big\|_{L_x^2(\mathbb{R}^5)} \\ &\qquad \lesssim \|F(u)\|_{L_\tau^1 L_x^2([\frac{\delta_1}{10},\delta_1]\times\mathbb{R}^5)} \lesssim \|u\|^2_{L_\tau^2 L_x^4([\frac{\delta_1}{10},\delta_1]\times\mathbb{R}^5)} \lesssim \delta_1^{-\frac{1}{2}}. \end{aligned} \tag{3-52}$$

By Strichartz, radial Sobolev inequality (2-2) and Hölder, the second term of (3-50) can be estimated as

$$\begin{aligned} &\Big\| \int_0^{\delta_1/10} \vec{S}(t-\tau)\big(0, (1-\psi)F(u(\tau))\big)\,d\tau\Big|_{t=\delta_1} \Big\|_{\dot{H}^1\times L^2(\mathbb{R}^5)} \\ &\quad \lesssim \|(1-\psi)F(u(\tau))\|_{L_\tau^1 L_x^2([0,\frac{\delta_1}{10}]\times\mathbb{R}^5)} \\ &\quad \lesssim \delta_1^{\frac{1}{2}} \|(1-\psi)u\|^{\frac{1}{2}}_{L_{t,x}^\infty([0,\frac{\delta_1}{10}]\times\mathbb{R}^5)} \|u\|^{\frac{3}{2}}_{L_{t,x}^3([0,\frac{\delta_1}{10}]\times\mathbb{R}^5)} \\ &\quad \lesssim \delta_1^{-\frac{1}{2}} \|u\|^{\frac{1}{2}}_{L_t^\infty \dot{B}^{1/2}_{2,1}} \lesssim \delta_1^{-\frac{1}{2}}. \end{aligned} \tag{3-53}$$

Hence, it remains to estimate

$$\|(1-\psi)(u_0,u_1)\|_{\dot H^1\times L^2(\mathbb{R}^5)}. \tag{3-54}$$

For u_0, by radial Sobolev inequality (2-2) and polar coordinates,

$$\|(\nabla\psi)u_0\|^2_{L^2(\mathbb{R}^5)} \lesssim \delta_1^{-1}\int_{\delta_1/100}^{2\delta_1}\int_{\mathbb{S}^4} u_0^2(r)\,d\sigma(\omega)r^4\,dr \lesssim \|u_0\|^2_{\dot B^{1/2}_{2,1}(\mathbb{R}^5)} \lesssim 1. \tag{3-55}$$

By the inequality (2-13), we have

$$\|(1-\psi)\partial_r u_0\|_{L^2_x(\mathbb{R}^5)} \lesssim \delta_1^{-\frac{1}{2}}. \tag{3-56}$$

For u_1, by the inequality (2-8) and polar coordinates, one can deduce that

$$\|(1-\psi)u_1\|^2_{L^2_x(\mathbb{R}^5)} \lesssim \int_{\frac{\delta_1}{10}}^{\infty}\int_{\mathbb{S}^4}|u_1(r)|^2r^4 d\sigma(\omega)\,dr \lesssim \int_{\frac{\delta_1}{10}}^{\infty} r^{-2}\,dr \lesssim \delta_1^{-1}. \tag{3-57}$$

This inequality together with (3-55) and (3-56) implies that

$$\|(1-\psi)(u_0,u_1)\|_{\dot H^1\times L^2(\mathbb{R}^5)} \lesssim \delta_1^{-\frac{1}{2}}. \tag{3-58}$$

This completes the proof. □

3C. *Global well-posedness.* In this subsection, we prove that the solution u is globally well-posed. We emphasize that the constants in "$\lesssim$" in this subsection depend only upon δ_1 and $\|(u_0,u_1)\|_{\dot B^3_{1,1}\times \dot B^2_{1,1}(\mathbb{R}^5)}$.

Theorem 3.6. *Let u be the solution to (1-5) with initial data $(u_0,u_1)\in \dot B^3_{1,1}\times \dot B^2_{1,1}$. Then u is globally well-posed and such that for any compact interval $J\subset\mathbb{R}$,*

$$\|u\|_{L^3_{t,x}(J\times\mathbb{R}^5)} < C\big(J, \|(u_0,u_1)\|_{\dot B^3_{1,1}\times\dot B^2_{1,1}}, \delta_1\big). \tag{3-59}$$

Proof. By Lemma 2.11, it suffices to show (3-59). By the time reversibility of the wave equation, we just need consider the part of $t\ge 0$.

For $t\ge\delta_1$, by $u(t)=w(t)+v(t)$ and the formula (3-36) of $v(t)$, we have

$$w_{tt}-\Delta w = -|u|u. \tag{3-60}$$

We define the energy of w as (1-2) by

$$E(w(t)) = \frac12\int_{\mathbb{R}^5}|\nabla_t w(t)|^2\,dx + \frac12\int_{\mathbb{R}^5}|\nabla_x w(t)|^2\,dx + \frac13\int_{\mathbb{R}^5}|w(t)|^3\,dx. \tag{3-61}$$

By the estimates (3-22), (3-38), Lemma 3.5 and interpolation, we have

$$E(w(\delta_1)) \lesssim 1. \tag{3-62}$$

Now, we consider

$$\begin{aligned}(3\text{-}63)\quad \Big|\frac{d}{dt}E(w(t))\Big| &= \Big|\int_{\mathbb{R}^5}(|w|w-|u|u)w_t\,dx\Big| \\ &\lesssim \|v\|_{L_x^6(\mathbb{R}^5)}\|w_t\|_{L_x^2(\mathbb{R}^5)}\|w\|_{L_x^3(\mathbb{R}^5)}+\|w_t\|_{L_x^2(\mathbb{R}^5)}\|v\|_{L_x^6(\mathbb{R}^5)}\|v\|_{L_x^3(\mathbb{R}^5)}.\end{aligned}$$

By interpolation and the dispersive estimate (3-37) of v, we have

$$\begin{aligned}(3\text{-}64)\quad \|v\|_{L_x^6(\mathbb{R}^5)}\|w_t\|_{L_x^2(\mathbb{R}^5)}\|w\|_{L^3(\mathbb{R}^5)} &\lesssim \frac{1}{t}\|v\|_{L_x^3(\mathbb{R}^5)}^{\frac{1}{2}}\|w_t\|_{L_x^2(\mathbb{R}^5)}\|w\|_{L_x^3(\mathbb{R}^5)} \\ &\lesssim \frac{1}{t}E(w(t))^{\frac{5}{6}}\|v\|_{L_x^3(\mathbb{R}^5)}^{\frac{1}{2}} \\ &\lesssim \frac{1}{t}\big[E(w)(t)+\|v\|_{L_x^3(\mathbb{R}^5)}^3\big],\end{aligned}$$

$$\begin{aligned}(3\text{-}65)\quad \|w_t\|_{L_x^2(\mathbb{R}^5)}\|v\|_{L_x^6(\mathbb{R}^5)}\|v\|_{L_x^3(\mathbb{R}^5)} &\lesssim \frac{1}{t}E(w(t))^{\frac{1}{2}}\|v\|_{L_x^3(\mathbb{R}^5)}^{\frac{3}{2}} \\ &\lesssim \frac{1}{t}\big[E(w)(t)+\|v\|_{L_x^3(\mathbb{R}^5)}^3\big].\end{aligned}$$

Substituting (3-64) and (3-65) into (3-63), we obtain

$$(3\text{-}66)\qquad \frac{d}{dt}E(w(t)) \le C\frac{1}{t}(E(w(t))+\|v\|_{L_x^3(\mathbb{R}^5)}^3).$$

Hence,

$$(3\text{-}67)\qquad \frac{d}{dt}[t^{-C}(E(w(t)))] \le t^{-C-1}\|v\|_{L_x^3(\mathbb{R}^5)}^3.$$

This estimate and the inequality (3-38) imply that

$$(3\text{-}68)\qquad E(w(t)) \le C_1(1+|t|)^{C_2}.$$

Thus, for any compact interval $J \subset \mathbb{R}$, we have

$$(3\text{-}69)\qquad \|u\|_{L_{t,x}^3(J\times\mathbb{R}^5)} \le \|v\|_{L_{t,x}^3(J\times\mathbb{R}^5)}+\|w\|_{L_{t,x}^3(J\times\mathbb{R}^5)} < \infty. \qquad \square$$

4. Scattering

In this section we prove Theorem 1.1 by assuming Proposition 1.3, that is, removing the dependence of δ_1 in (1-8). From the arguments in Section 3, we have $\delta_1 = \delta/(2R)$, where δ and R depending the scaling and spatial profile of the initial data, respectively. We give the heuristic idea of the proof by analyzing the effect of the parameters δ and R on the critical norm $L_{t,x}^3(\mathbb{R}\times\mathbb{R}^5)$.

Note that the critical norm $L_{t,x}^3(\mathbb{R}\times\mathbb{R}^5)$ of the solution to the nonlinear equation (1-5) is invariant under the scaling transform. Hence the parameter δ may not be the main difficulty in proving Theorem 1.1. On the other hand, the latter parameter R relies on the spatial profile of the initial data. For example, let R be the parameter corresponding to the initial data (u_0, u_1) with compact support. The linear evolution $\vec{S}(t)$ for $t \in \mathbb{R}$ does not change the critical norm $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ by the Plancherel theorem, but owing to the Huygens principle for the odd-dimension linear wave equations, it does change the spatial support of the initial datum. Thus, for the initial data $\vec{S}(t_0)(u_0, u_1)$, the spatial parameter (may be chosen as $R + t_0$) is likely very huge, when t_0 is large enough. However, the $\dot{B}^3_{1,1} \times \dot{B}^2_{1,1}$ norm may become huge under the evolution of $\vec{S}(t)$. Indeed, if $\|(u_0, u_1)\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)} = 1$, then $\|\vec{S}(t_0)(u_0, u_1)\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)} \to \infty$ as $t_0 \to \infty$.[1] Hence, if

$$\bigl\|\vec{S}(t_0)(u_0, u_1)\bigr\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)} \lesssim 1,$$

then t_0 remains bounded. Taking account of this fact, one may conquer the difficulties caused by the parameter R.

To finish the proof of Theorem 1.1, we need the following theorem of profile decomposition.

4A. *Profile decomposition.* Now, we recall the linear profile decomposition from [Ramos 2012] in the radial case. We refer to [Bahouri and Gérard 1999] for the profile decompositions in the energy critical spaces.

Theorem 4.1 (profile decomposition). *Let $C > 0$ be a fixed number and let $(u_0^n, u_1^n)_n$ be a sequence in $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^5)$, with*

$$\|(u_0^n, u_1^n)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)} \le C. \tag{4-1}$$

Then there exist a subsequence of (u_0^n, u_1^n) (still denoted by (u_0^n, u_1^n)), a sequence

$$(\phi_0^j, \phi_1^j)_{j\in\mathbb{N}} \subset \dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5),$$

a sequence

$$(R_{0,n}^N, R_{1,n}^N)_{N\ge 1} \subset \dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)$$

and a sequence of parameters $(t_j^n, \lambda_j^n) \subset \mathbb{R} \times (0, \infty)$ such that for each $N \ge 1$,

$$S(t)(u_0^n, u_1^n) = \sum_{j=1}^{N} (\lambda_j^n)^2 \bigl[S(\lambda_j^n(t - t_j^n))(\phi_0^j, \phi_1^j)\bigr](\lambda_j^n x) + S(t)(R_{0,n}^N, R_{1,n}^N) \tag{4-2}$$

with

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S(t)(R_{0,n}^N, R_{1,n}^N)\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} = 0. \tag{4-3}$$

[1] By interpolation and density, it suffices to show that, for $f \in \mathcal{S}(\mathbb{R}^5)$, $\lim_{t\to\infty} \|e^{it|\nabla|} f\|_{\dot{B}^{-2}_{\infty,\infty}(\mathbb{R}^5)} = 0$, which follows from the dispersive estimate (2-20) and Bernstein's estimates.

In addition, the parameters (t_j^n, λ_j^n) satisfy the orthogonality property: for any $j \neq k$,

$$\lim_{n\to\infty}\left(\frac{\lambda_j^n}{\lambda_k^n}+\frac{\lambda_k^n}{\lambda_j^n}+(\lambda_k^n)^{\frac12}(\lambda_j^n)^{\frac12}|t_j^n-t_k^n|\right)=\infty. \tag{4-4}$$

Furthermore, for every $N \geq 1$,

$$\begin{aligned}\|(u_0^n, u_1^n)\|^2_{\dot H^{1/2}\times\dot H^{-1/2}(\mathbb{R}^5)} \\ =\sum_{j=1}^N \|(\phi_0^j,\phi_1^j)\|^2_{\dot H^{1/2}\times\dot H^{-1/2}(\mathbb{R}^5)}+\|(R_{0,n}^N, R_{1,n}^N)\|^2_{\dot H^{1/2}\times\dot H^{-1/2}(\mathbb{R}^5)}+o_n(1).\end{aligned} \tag{4-5}$$

4B. *End of the proof of the main theorem.* Now, we apply the strategy in [Dodson 2019] to finish the proof of Theorem 1.1, that is, remove the δ_1 in Proposition 1.3.

We prove Theorem 1.1 by contradiction. We assume u is the solution to (1-5) with the initial data such that $(u_0, u_1) \in \dot B^3_{1,1}\times\dot B^2_{1,1}(\mathbb{R}^5)$. For $M \geq 0$, let

$$f(M)=\sup\{\|u\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} : \|(u_0,u_1)\|_{\dot B^3_{1,1}\times\dot B^2_{1,1}(\mathbb{R}^5)}\leq M\}. \tag{4-6}$$

Then by Proposition 1.3, f is well defined. Also, by Bernstein and Theorem 2.10, $f(M) < \infty$ when M is small enough. From the definition, $f(M)$ is nondecreasing as M increases.

If Theorem 1.1 fails, then there exist $M_0 < \infty$ and a sequence $\{(u_0^n, u_1^n)\}_{n\in\mathbb{N}} \subset \dot B^3_{1,1}\times\dot B^2_{1,1}(\mathbb{R}^5)$, such that

$$\|(u_0^n,u_1^n)\|_{\dot B^3_{1,1}\times\dot B^2_{1,1}(\mathbb{R}^5)}\leq M_0, \tag{4-7}$$

and the solution u^n to (1-5) with $(u^n(0), (\partial_t u^n)(0)) = (u_0^n, u_1^n)$ satisfying

$$\|u^n\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)}\to\infty, \tag{4-8}$$

as $n \to \infty$. By Theorem 4.1, we have

$$S(t)(u_0^n,u_1^n)=\sum_{j=1}^N(\lambda_j^n)^2\big[S(\lambda_j^n(t-t_j^n))(\phi_0^j,\phi_1^j)\big](\lambda_j^n x)+S(t)(R_{0,n}^N,R_{1,n}^N). \tag{4-9}$$

In the proof of Theorem 4.1, Ramos [2012] actually proved that

$$\vec S(t+t_j^n\lambda_j^n)\left[(\lambda_j^n)^{-2}u_0^n\left(\frac{\cdot}{\lambda_j^n}\right),(\lambda_j^n)^{-3}u_1^n\left(\frac{\cdot}{\lambda_j^n}\right)\right]\Big|_{t=0}\rightharpoonup(\phi_0^j,\phi_1^j), \tag{4-10}$$

weakly in $\dot H^{1/2}\times\dot H^{-1/2}(\mathbb{R}^5)$ as $n\to\infty$. From this we can prove the following:

Proposition 4.2. *For fixed $j \in \mathbb{N}$, if $(\phi_0^j, \phi_1^j) \neq 0$, then $|t_j^n\lambda_j^n|$ is bounded as $n \to \infty$.*

Proof. First, if $t_j^n\lambda_j^n$ is unbounded for $n\in\mathbb{N}$, then by taking a subsequence of n (still denoted by n), we assume that $|t_j^n\lambda_j^n|\to\infty$, as $n\to\infty$. In light of the heuristic analysis at the beginning of this section, we have

$$\vec{S}(t_j^n\lambda_j^n)\Big[(\lambda_j^n)^{-2}u_0^n\Big(\frac{\cdot}{\lambda_j^n}\Big),\,(\lambda_j^n)^{-3}u_1^n\Big(\frac{\cdot}{\lambda_j^n}\Big)\Big]\to(0,0). \tag{4-11}$$

in $L_x^3\times W_x^{-1,3}(\mathbb{R}^5)$. In fact, using (4-7) and the estimate (3-30) in Section 3, we have

$$\begin{aligned}\Big\|S(t_j^n\lambda_j^n)\Big[(\lambda_j^n)^{-2}u_0^n\Big(\frac{\cdot}{\lambda_j^n}\Big),\,(\lambda_j^n)^{-3}u_1^n\Big(\frac{\cdot}{\lambda_j^n}\Big)\Big]\Big\|_{L_x^3(\mathbb{R}^5)}&\\ \lesssim|t_j^n\lambda_j^n|^{-\frac13}\Big\|\Big((\lambda_j^n)^{-2}u_0^n\Big(\frac{\cdot}{\lambda_j^n}\Big),\,(\lambda_j^n)^{-3}u_1^n\Big(\frac{\cdot}{\lambda_j^n}\Big)\Big)\Big\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)}&\\ \lesssim|t_j^n\lambda_j^n|^{-\frac13}\|(u_0^n,u_1^n)\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)}\to0,&\end{aligned} \tag{4-12}$$

as $n\to\infty$. Similarly, by the dispersive estimate (2-20), Bernstein and interpolation, we have

$$\begin{aligned}\Big\|\dot{S}(t_j^n\lambda_j^n)\Big[(\lambda_j^n)^{-2}u_0^n\Big(\frac{\cdot}{\lambda_j^n}\Big),\,(\lambda_j^n)^{-3}u_1^n\Big(\frac{\cdot}{\lambda_j^n}\Big)\Big]\Big\|_{\dot{W}_x^{-1,3}(\mathbb{R}^5)}&\\ \lesssim|t_j^n\lambda_j^n|^{-\frac13}\|(u_0^n,u_1^n)\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)}\to0,&\end{aligned} \tag{4-13}$$

as $n\to\infty$. Hence, from the weak convergence relation (4-10), (4-12) and (4-13) imply that $(\phi_0^j,\phi_1^j)=(0,0)$. □

For simplicity, we assume that every (ϕ_0^j,ϕ_1^j) in (4-9) is nontrivial. By Proposition 4.2, $t_j^n\lambda_j^n$ is bounded for each fixed j, and therefore after taking a suitable subsequence of n (still denoted by n), we can assume $t_j^n\lambda_j^n\to t_j\in\mathbb{R}$ as $n\to\infty$. Hence, if we denote $(\varphi_0^j,\varphi_1^j)=\vec{S}(-t_j)(\phi_0^j,\phi_1^j)$, then

$$\vec{S}(-\lambda_j^nt_j^n)(\phi_0^j,\phi_1^j)-(\varphi_0^j,\varphi_1^j)\to0, \tag{4-14}$$

in $\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^5)$ as $n\to\infty$. Let

$$\begin{cases}\tilde{R}_{0,n}^N=R_{0,n}^N+\sum_{j=1}^N(\lambda_j^n)^2\big[S(-\lambda_j^nt_j^n)(\phi_0^j,\phi_1^j)\big](\lambda_j^nx)-(\lambda_j^n)^2(\varphi_0^j,\varphi_1^j)(\lambda_j^nx),\\ \tilde{R}_{1,n}^N=R_{1,n}^N+\sum_{j=1}^N(\lambda_j^n)^3\big[\dot{S}(-\lambda_j^nt_j^n)(\phi_0^j,\phi_1^j)\big](\lambda_j^nx)-(\lambda_j^n)^3[(\varphi_0^j,\varphi_1^j)](\lambda_j^nx),\end{cases} \tag{4-15}$$

then

$$\lim_{N\to\infty}\limsup_{n\to\infty}\|S(t)(\tilde{R}_{0,n}^N,\tilde{R}_{1,n}^N)\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)}=0. \tag{4-16}$$

Taking $t = 0$ in (4-9), by (4-14) and (4-15), we have

$$(u_0^n, u_1^n) = \sum_{j=1}^{N} \big((\lambda_j^n)^2 \big(\varphi_0^j(\lambda_j^n x), (\lambda_j^n)^3 \varphi_1^j(\lambda_j^n x)\big)\big) + (\tilde{R}_{0,n}^N, \tilde{R}_{1,n}^N). \tag{4-17}$$

In addition, by the orthogonality (4-4) and Proposition 4.2, we have for each $j \neq k$

$$\lim_{n\to\infty} \frac{\lambda_j^n}{\lambda_k^n} + \frac{\lambda_k^n}{\lambda_j^n} = \infty, \tag{4-18}$$

as $n \to \infty$. Thus, for fixed $j \in \mathbb{N}$, we have

$$\left((\lambda_j^n)^{-2} u_0^n\left(\frac{x}{\lambda_j^n}\right), (\lambda_j^n)^{-3} u_1^n\left(\frac{x}{\lambda_j^n}\right)\right) \rightharpoonup (\varphi_0^j, \varphi_1^j) \tag{4-19}$$

weakly in $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^5)$ as $n \to \infty$. By Fatou's lemma, this fact and the inequality (4-7) imply

$$\|(\varphi_0^j, \varphi_1^j)\|_{\dot{B}^3_{1,1} \times \dot{B}^2_{1,1}(\mathbb{R}^5)} \le M_0. \tag{4-20}$$

On the other hand, (4-5) and (4-14) yield that

$$\sum_{j\ge 1} \|(\varphi_0^j, \varphi_1^j)\|^2_{\dot{H}^{1/2}\times \dot{H}^{-1/2}(\mathbb{R}^5)} \lesssim \sup_{n\ge 1} \|(u_0^n, u_1^n)\|^2_{\dot{H}^{1/2}\times \dot{H}^{-1/2}(\mathbb{R}^5)} \lesssim C_0^2. \tag{4-21}$$

Hence, for fixed $\epsilon > 0$, there exists a finite integer N_0 such that

$$\sum_{j \ge N_0+1} \left\|(\varphi_0^j, \varphi_1^j)\right\|^2_{\dot{H}^{1/2}\times \dot{H}^{-1/2}(\mathbb{R}^5)} \le \epsilon. \tag{4-22}$$

By the local well-posedness theory, if $\epsilon > 0$ is small enough, then the solution v^j to (1-5) with the initial data $(\varphi_0^j, \varphi_1^j)$ is globally well-posed and

$$\|v^j\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} \lesssim \|(\varphi_0^j, \varphi_1^j)\|_{\dot{H}^{1/2}\times \dot{H}^{-1/2}(\mathbb{R}^5)}, \quad \text{for every } j \ge N_0 + 1. \tag{4-23}$$

For $1 \le j \le N_0$, as a consequence of Proposition 1.3, the solution to (1-5) with the initial data $(\varphi_0^j, \varphi_1^j)$ is globally well-posed and such that

$$\|v^j\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} \lesssim_{M_0, j} 1. \tag{4-24}$$

By the orthogonality property (4-18), for any $j \neq k$,

$$\lim_{n\to\infty} \iint_{\mathbb{R}\times\mathbb{R}^5} |(\lambda_j^n)^2 v^j(\lambda_j^n t, \lambda_j^n x)|^2 \, |(\lambda_k^n)^2 v^k(\lambda_k^n t, \lambda_k^n x)| \, dx\, dt = 0. \tag{4-25}$$

This together with the estimates (4-22)–(4-24) implies

$$\sup_{N \ge N_0+1} \lim_{n\to\infty} \Big\| \sum_{1\le j\le N} (\lambda_j^n)^2 v^j(\lambda_j^n t, \lambda_j^n x) \Big\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} \tag{4-26}$$

is bounded. Similarly, as a consequence of the trivial estimate

$$\Big|F\Big(\sum_{j=1}^{N}(\lambda_j^n)^2 v^j(\lambda_j^n t,\lambda_j^n x)\Big)-\sum_{j=1}^{N}F((\lambda_j^n)^2 v^j(\lambda_j^n t,\lambda_j^n x))\Big| \lesssim \sum_{1\le j,k\le N,\, j\ne k} |(\lambda_j^n)^2 v^j(\lambda_j^n t,\lambda_j^n x))|\,|(\lambda_k^n)^2 v^k(\lambda_k^n t,\lambda_k^n x)|$$

and the orthogonality property (4-18), we have

$$\lim_{n\to\infty}\Big\|F\Big(\sum_{j=1}^{N}(\lambda_j^n)^2 v^j(\lambda_j^n t,\lambda_j^n x)\Big)-\sum_{j=1}^{N}F((\lambda_j^n)^2 v^j(\lambda_j^n t,\lambda_j^n x))\Big\|_{L^{3/2}_{t,x}(\mathbb{R}\times\mathbb{R}^5)}=0. \tag{4-27}$$

Let u_N^n be an approximate solution to (1-5) defined by

$$u_N^n=\sum_{j=1}^{N}(\lambda_j^n)^2 v^j(\lambda_j^n x)+S(t)(\tilde{R}_{0,n}^N,\tilde{R}_{1,n}^N).$$

Then, recall the property (4-16) for $(\tilde{R}_{0,n}^N,\tilde{R}_{1,n}^N)$ and the fact that (4-26) is uniformly bounded for $N\ge N_0+1$, we obtain

$$\limsup_{N\to\infty}\lim_{n\to\infty}\|u_N^n\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)}\lesssim 1. \tag{4-28}$$

Moreover, combining (4-27), the property (4-16) for $(\tilde{R}_{0,n}^N,\tilde{R}_{1,n}^N)$, and Hölder's inequality, we have

$$\limsup_{N\to\infty}\lim_{n\to\infty}\Big\|F(u_N^n)-\sum_{j=1}^{N}F\big((\lambda_j^n)^2 v^j(\lambda_j^n t,\lambda_j^n x)\big)\Big\|_{L^{3/2}_{t,x}(\mathbb{R}\times\mathbb{R}^5)}=0. \tag{4-29}$$

Utilizing Theorem 2.12, by (4-17), (4-28) and (4-29), we have that for n large enough, the solution u^n to (1-5) with initial data (u_0^n,u_1^n) is global and such that

$$\lim_{n\to\infty}\|u^n\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} \tag{4-30}$$

is bounded, which contradicts the hypothesis (4-8) of u^n. Thus, we have proved Theorem 1.1.

5. Hyperbolic coordinates and spacetime estimates

In this section, we will finish the proof of Proposition 1.3. We first reduce Proposition 1.3 to estimating the $L^3_{t,x}$ norm of w on the region Ω_2, which will be defined below. Without loss of generality, we assume that $\delta_1<\frac{1}{4}$. As in Theorem 3.6, we also note the constants in "$\lesssim$" in this section may be different in each step and are dependent on δ_1 and $\|(u_0,u_1)\|_{\dot{B}^3_{1,1}\times\dot{B}^2_{1,1}(\mathbb{R}^5)}$.

5A. ***Reduction of the proof of Proposition 1.3.*** Now we consider the $L^3_{t,x}$ norm of u on $\mathbb{R}_+ \times \mathbb{R}^5$. First, we split time-spatial region $\mathbb{R}_+ \times \mathbb{R}^5$ as the union

$$\mathbb{R}_+ \times \mathbb{R}^5 = \Omega_1 \cup \Omega_2 \cup \Omega_3, \tag{5-1}$$

where

$$\Omega_1 = \left\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^5 : |x| \geq t + \tfrac{1}{2}\right\},$$
$$\Omega_2 = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^5 : (t + (1-\delta_1))^2 - |x|^2 \geq 1\}.$$

Since $\delta_1 < \frac{1}{4}$, there exists a large constant $C > 0$, such that

$$\Omega_3 \subset \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^5 : t + |x| \leq C\}.$$

Recalling the estimate (3-23) in Section 3, we obtain $\|u\|_{L^3_{t,x}(\Omega_1)} \lesssim 1$. For the bounded region Ω_3, Theorem 3.6 yields $\|u\|_{L^3_{t,x}(\Omega_3)} \lesssim 1$. Hence, we just need to consider the $L^3_{t,x}$ norm of u on the region Ω_2. By the estimate (3-38) for v, we are reduced to showing $\|w\|_{L^3_{t,x}(\Omega_2)} \lesssim 1$.

5B. ***Hyperbolic coordinates.*** For the radial solution $u(t,x)$ to (1-5), if we denote $u(t,r) = u(t,x)$ for $r = |x|$, then

$$\partial_{tt}(r^2 u) - \partial_{rr}(r^2 u) = -2u - r^2|u|u. \tag{5-2}$$

Denote $\bar{u}(t,r) = u(t-(1-\delta_1), r)$ and denote $\bar{v}, \bar{w}$ similarly. Let $(t,r) = (e^\tau \cosh s, e^\tau \sinh s)$; then $dr\,dt = e^{2\tau} d\tau\, ds$. We denote the hyperbolic transforms by

$$\tilde{u} = \frac{e^{2\tau}\sinh^2 s}{s^2}\bar{u}(e^\tau \cosh s, e^\tau \sinh s), \tag{5-3}$$

$$\tilde{v} = \frac{e^{2\tau}\sinh^2 s}{s^2}\bar{v}(e^\tau \cosh s, e^\tau \sinh s), \tag{5-4}$$

$$\widetilde{w} = \frac{e^{2\tau}\sinh^2 s}{s^2}\overline{w}(e^\tau \cosh s, e^\tau \sinh s). \tag{5-5}$$

Hence, we have

$$\partial_{\tau\tau}(s^2\tilde{u}) - \partial_{ss}(s^2\tilde{u}) = -\frac{2s^2}{\sinh^2 s}\tilde{u} - \frac{s^4}{\sinh^2 s}|\tilde{u}|\tilde{u}, \tag{5-6}$$

$$\partial_{\tau\tau}(s^2\tilde{v}) - \partial_{ss}(s^2\tilde{v}) = -\frac{2s^2}{\sinh^2 s}\tilde{v}, \tag{5-7}$$

$$\partial_{\tau\tau}(s^2\widetilde{w}) - \partial_{ss}(s^2\widetilde{w}) = -\frac{2s^2}{\sinh^2 s}\widetilde{w} - \frac{s^4}{\sinh^2 s}|\tilde{u}|\tilde{u}. \tag{5-8}$$

Define the hyperbolic energy of $\widetilde{w}$ by

$$E_h(\widetilde{w})(\tau)=\int_0^\infty\left[\frac{1}{2}|(s^2\widetilde{w})_\tau|^2+\frac{1}{2}|(s^2\widetilde{w})_s|^2+\frac{|s^2\widetilde{w}|^2}{\sinh^2 s}+\frac{1}{3}\frac{|s^2\widetilde{w}|^3}{\sinh^2 s}\right]ds. \tag{5-9}$$

5C. *The hyperbolic energy for some $\tau_0\geq 0$.* First, we want to prove $E_h(\widetilde{w})(\tau)$ is bounded for some $\tau_0\geq 0$. We claim that it suffices to show the boundedness of

$$\int_0^\infty\left[|(s^2\widetilde{w})_\tau(\tau_0,s)|^2+|(s^2\widetilde{w})_s(\tau_0,s)|^2\right]ds \tag{5-10}$$

for some $\tau_0>0$.

To prove this claim, we need the sharp Hardy inequality,

$$\left(\frac{d-2}{2}\right)^2\int_{\mathbb{R}^d}\frac{|f(x)|^2}{|x|^2}\,dx\leq\int_{\mathbb{R}^d}|\nabla f|^2(x)\,dx. \tag{5-11}$$

By polar coordinates, we rewrite this inequality for radial functions,

$$\left(\frac{d-2}{2}\right)^2\int_0^\infty|f(r)|^2r^{d-3}\,dr\;\leq\int_0^\infty|\partial_r f(r)|^2(r)r^{d-1}\,dr. \tag{5-12}$$

Then, this inequality and integration by parts imply that

$$\begin{aligned}\int_0^\infty&|(s^2\widetilde{w}(\tau_0))_s|^2\,ds\\&=\int_0^\infty s^4\widetilde{w}_s^2(\tau_0)\,ds+4\int_0^\infty s^2\widetilde{w}(\tau_0)s\widetilde{w}_s(\tau_0)\,ds+4\int_0^\infty s^2\widetilde{w}^2(\tau_0)\,ds\\&=\int_0^\infty s^4\widetilde{w}_s^2(\tau_0)\,ds-2\int_0^\infty s^2\widetilde{w}(\tau_0)^2\,ds\geq\frac{1}{9}\int_0^\infty s^4\widetilde{w}_s^2(\tau_0)\,ds.\end{aligned} \tag{5-13}$$

In addition, by Hölder and Sobolev in polar coordinates, we have

$$\begin{aligned}\int_0^\infty\frac{|s^2\widetilde{w}(\tau_0)|^3}{\sinh^2 s}\,ds&=\int_0^\infty\frac{s^2}{\sinh^2 s}|\widetilde{w}(\tau_0)|^3s^4\,ds\\&\lesssim\left(\int_0^\infty|\widetilde{w}(\tau_0)|^{\frac{10}{3}}s^4\,ds\right)^{\frac{9}{10}}\lesssim\left(\int_0^\infty|\widetilde{w}_s(\tau_0)|^2s^4\,ds\right)^{\frac{3}{2}}.\end{aligned} \tag{5-14}$$

By Hardy's inequality, we have

$$\int_0^\infty\frac{|s^2\widetilde{w}(\tau_0)|^2}{\sinh^2 s}\,ds\lesssim\int_0^\infty\frac{1}{s^2}|\widetilde{w}(\tau_0)|^2s^4\,ds\lesssim\int_0^\infty|\widetilde{w}_s(\tau_0)|^2s^4\,ds. \tag{5-15}$$

Hence the claim follows.

5C1. ***The hyperbolic energy for*** $s > s_0 > 0$. For $\tau \in [0, 1]$ and sufficiently large $s_0 > 0$, we can assume that $e^{\tau - s_0} < \frac{1}{2} - \delta_1$. By the finite speed of propagation, $v(t, r)$ are supported in the region $\{(t, r) \in \mathbb{R} \times \mathbb{R}_+ : r - t \lesssim \delta_1/5\}$. Then, for $\tau \in [0, 1]$ and $s \geq s_0$, we have

$$e^\tau \sinh s - [e^\tau \cosh s - (1 - \delta_1)] = 1 - \delta_1 - e^{\tau - s} > \frac{1}{2} > \frac{\delta_1}{5},$$

which leads to $v(e^\tau \cosh s - (1 - \delta_1), e^\tau \sinh s) = 0$. Hence, for $\tau \in [0, 1]$, we have

$$\int_{s_0}^{\infty} \tfrac{1}{2}|(s^2\widetilde{w})_\tau(\tau, s)|^2 + \tfrac{1}{2}|(s^2\widetilde{w})_s(\tau, s)|^2\, ds = \int_{s_0}^{\infty} \tfrac{1}{2}|(s^2\tilde{u})_\tau(\tau, s)|^2 + \tfrac{1}{2}|(s^2\tilde{u})_s(\tau, s)|^2\, ds. \tag{5-16}$$

Since u is a radial solution to (1-5), we have, by (2-19),

$$\begin{aligned} r^2 u(t, r) = {} & \frac{1}{2}[(r - t)^2 u_0(r - t) + (r + t)^2 u_0(r + t)] \\ & - \frac{1}{2} t r^{-1} \int_{r-t}^{r+t} s u_0(s)\, ds + \frac{1}{4r} \int_{r-t}^{r+t} s(s^2 + r^2 - t^2) u_1(s)\, ds \\ & + \frac{1}{4r} \int_0^t \int_{r-t+s}^{r+t-s} \rho(\rho^2 + r^2 - (t - s)^2)|u|u(s, \rho)\, d\rho\, ds, \end{aligned} \tag{5-17}$$

for $r \geq t \geq 0$. Hence, by the hyperbolic transform (5-3), we have

$$s^2\tilde{u}(\tau, s)$$

$$= \frac{1}{2}\big[(1-\delta_1-e^{\tau-s})^2 u_0(1-\delta_1-e^{\tau-s}) + (e^{\tau+s}-(1-\delta_1))^2 u_0(e^{\tau+s}-(1-\delta_1))\big] \tag{5-18}$$

$$-\frac{1}{2}(e^\tau \cosh s-(1-\delta_1))(e^\tau \sinh s)^{-1} \int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho u_0(\rho)\, d\rho \tag{5-19}$$

$$+\frac{1}{4}\int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho \frac{\rho^2+(e^{\tau+s}-(1-\delta_1))(1-\delta_1-e^{\tau-s})}{e^\tau \sinh s} u_1(\rho)\, d\rho \tag{5-20}$$

$$+\frac{1}{4}\int_{1-\delta_1}^{e^\tau \cosh s} \int_{t-e^{\tau-s}}^{e^{\tau+s}-t} \rho \frac{\rho^2+(e^{\tau+s}-t)(t-e^{\tau-s})}{e^\tau \sinh s} |\bar{u}|\bar{u}(t, \rho)\, d\rho\, dt. \tag{5-21}$$

For (5-18), by a direct calculation, we obtain

$$\begin{aligned} (\partial_\tau + \partial_s)(5\text{-}18) = {} & 2(e^{\tau+s} - (1 - \delta_1))e^{\tau+s} u_0(e^{\tau+s} - (1 - \delta_1)) \\ & + (e^{\tau+s} - (1 - \delta_1))^2 u_0'(e^{\tau+s} - (1 - \delta_1))e^{\tau+s}, \end{aligned} \tag{5-22}$$

$$\begin{aligned} (\partial_\tau - \partial_s)(5\text{-}18) = {} & 2(1 - \delta_1 - e^{\tau-s})e^{\tau-s} u_0(1 - \delta_1 - e^{\tau-s}) \\ & + ((1 - \delta_1 - e^{\tau-s})^2 u_0'(1 - \delta_1 - e^{\tau-s})e^{\tau-s}. \end{aligned} \tag{5-23}$$

Using the estimate (2-13) in Section 2 and polar coordinates, we deduce that

$$\int_{s_0} \left|(e^{\tau+s}-(1-\delta_1))e^{\tau+s}u_0(e^{\tau+s}-(1-\delta_1))\right|^2 ds \lesssim \int_0^\infty |u_0(r)|^2 r^3\,dr \lesssim 1, \tag{5-24}$$

$$\int_{s_0} \left|(e^{\tau+s}-(1-\delta_1))^2 e^{\tau+s}u_0'(e^{\tau+s}-(1-\delta_1))\right|^2 ds \lesssim \int_0^\infty |\partial_r u_0(r)|^2 r^5\,dr \lesssim 1. \tag{5-25}$$

By the radial Sobolev inequality (2-2), we have $|u_0(r)| \lesssim r^{-2}$. This estimate and the inequality (2-3) imply

$$\begin{aligned}\int_{s_0}^\infty &\left|(1-\delta_1-e^{\tau-s})e^{\tau-s}u_0(1-\delta_1-e^{\tau-s})\right|^2 ds \\ &+ \int_{s_0}^\infty \left|((1-\delta_1-e^{\tau-s})^2 u_0'(1-\delta_1-e^{\tau-s})e^{\tau-s}\right|^2 ds \\ &\lesssim \int_{s_0}^\infty e^{-2s}\,ds \lesssim 1.\end{aligned} \tag{5-26}$$

We now take the derivatives of (5-19) with respect to τ and s,

$$\partial_\tau(\text{5-19}) = \frac{1-\delta_1}{2e^\tau \sinh s}\int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho u_0(\rho)\,d\rho + I_1 + I_2 \tag{5-27}$$

$$\partial_s(\text{5-19}) = \partial_s\left(\frac{e^\tau \cosh s-(1-\delta_1)}{2e^\tau \sinh s}\right)\int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho u_0(\rho)\,d\rho + I_1 - I_2, \tag{5-28}$$

where

$$I_1 = \frac{e^\tau \cosh s-(1-\delta_1)}{2e^\tau \sinh s} e^{\tau+s}(e^{\tau+s}-(1-\delta_1))u_0(e^{\tau+s}-(1-\delta_1)), \tag{5-29}$$

$$I_2 = \frac{e^\tau \cosh s-(1-\delta_1)}{2e^\tau \sinh s} e^{\tau-s}(1-\delta_1-e^{\tau-s})u_0(1-\delta_1-e^{\tau-s}). \tag{5-30}$$

For the first term in the right-hand side of (5-27), by the inequality (2-3), we have

$$\int_{s_0}^\infty \left|e^{-s}\int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho u_0(\rho)\,d\rho\right|^2 ds \lesssim \int_{s_0}^\infty e^{-2s}\,ds \lesssim 1. \tag{5-31}$$

By similar estimates, one can find that the contribution of the first term in the right-hand side of (5-28) to (5-16) is finite. For I_1, a change of variables and the inequality (2-13) yield

$$\int_{s_0}^\infty |I_1|^2\,ds \lesssim \int_{s_0}^\infty |e^{2s}u_0(e^{\tau+s}-(1-\delta_1))|^2\,ds \lesssim \int_{\frac{1}{2}e^{s_0}}^\infty \rho^3|u_0(\rho)|^2\,d\rho \lesssim 1. \tag{5-32}$$

For I_2, by (2-2), we obtain

$$\int_{s_0}^{\infty} |I_2|\, ds \lesssim \int_{s_0}^{\infty} |e^{-s}|^2\, ds \lesssim 1. \tag{5-33}$$

Next, we consider the contribution of (5-20) to (5-16). For simplicity, we consider

$$\tfrac{1}{2}(\partial_\tau - \partial_s)\text{(5-20)}$$
$$= e^{\tau-s}(1-\delta_1-e^{\tau-s})^2 u_1(1-\delta_1-e^{\tau-s}) \tag{5-34}$$
$$+ \frac{e^{\tau-s}(e^{\tau+s}-(1-\delta_1))^2}{(e^{\tau+s}-e^{\tau-s})^2}\int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho u_1(\rho)\, d\rho \tag{5-35}$$
$$+ \frac{e^{\tau-s}}{(e^{\tau+s}-e^{\tau-s})^2}\int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho^3 u_1(\rho)\, d\rho, \tag{5-36}$$

and

$$\tfrac{1}{2}(\partial_\tau + \partial_s)\text{(5-20)}$$
$$= e^{\tau+s}(e^{\tau+s}-(1-\delta_1))^2 u_1(e^{\tau+s}-(1-\delta_1)) \tag{5-37}$$
$$+ \frac{e^{\tau+s}(1-\delta_1-e^{\tau-s})^2}{(e^{\tau+s}-e^{\tau-s})^2}\int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho u_1(\rho)\, d\rho \tag{5-38}$$
$$+ \frac{e^{\tau+s}}{(e^{\tau+s}-e^{\tau-s})^2}\int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho^3 u_1(\rho)\, d\rho. \tag{5-39}$$

Using the estimates (2-4) and (2-13), we can easily estimate the contributions of (5-34)–(5-38) to (5-16). Let $\mathbb{1}_J(y)$ be the characteristic function of an interval $J \subset \mathbb{R}$. For (5-39), by the inequality (2-13) and a change of variables, we see that

$$\begin{aligned}
\int_{s_0}^{\infty}\Big|e^{-s}\int_{1-\delta_1-e^{\tau-s}}^{e^{\tau+s}-(1-\delta_1)} \rho^3 u_1(\rho)\, d\rho\Big|^2 ds &\lesssim \int_{s_0}^{\infty}\Big|e^{-s}\int_0^{2e^s} \rho^3|u_1(\rho)|\, d\rho\Big|^2 ds \\
&\lesssim \int_0^{\infty}\Big|\frac{1}{\eta}\int_0^{2\eta} \rho^3|u_1(\rho)|\, d\rho\Big|^2 \frac{1}{\eta}\, d\eta \\
&\lesssim \int_0^{\infty}\Big|\frac{1}{\eta}\int_0^{2\eta} \rho^{\frac{5}{2}}|u_1(\rho)|\, d\rho\Big|^2 d\eta \\
&\lesssim \int_0^{\infty}\big|\mathcal{M}\big(\mathbb{1}_{[0,\infty)}(\rho)\rho^{\frac{5}{2}}u_1(\rho)\big)\big|^2(\eta)\, d\eta \\
&\lesssim \int_0^{\infty} |u_1(\rho)|^2\rho^5\, d\rho \lesssim 1,
\end{aligned} \tag{5-40}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal function and we used the fact that $\mathcal{M}$ is bounded in L^2.

Next, we consider the contribution of (5-21) to the energy (5-16). Also, for simplicity, we consider

$$(\partial_\tau+\partial_s)(5\text{-}21)$$

$$= e^{\tau+s}\int_{1-\delta_1}^{e^\tau\cosh s}(e^{\tau+s}-t)^2(|\bar u|\bar u)(t,e^{\tau+s}-t)\,dt \tag{5-41}$$

$$+\frac{e^{\tau+s}}{(e^{\tau+s}-e^{\tau-s})^2}\int_{1-\delta_1}^{e^\tau\cosh s}\int_{t-e^{\tau-s}}^{e^{\tau+s}-t}\rho(t-e^{\tau-s})^2(|\bar u|\bar u)(t,\rho)\,d\rho\,dt \tag{5-42}$$

$$-\frac{e^{\tau+s}}{(e^{\tau+s}-e^{\tau-s})^2}\int_{1-\delta_1}^{e^\tau\cosh s}\int_{t-e^{\tau-s}}^{e^{\tau+s}-t}\rho^3(|\bar u|\bar u)(t,\rho)\,d\rho\,dt, \tag{5-43}$$

and

$$(\partial_\tau-\partial_s)(5\text{-}21)$$

$$= e^{\tau-s}\int_{1-\delta_1}^{e^\tau\cosh s}(t-e^{\tau-s})^2|\bar u|\bar u(t,t-e^{\tau-s})\,dt \tag{5-44}$$

$$-\frac{e^{\tau-s}}{(e^{\tau+s}-e^{\tau-s})^2}\int_{1-\delta_1}^{e^\tau\cosh s}\int_{t-e^{\tau-s}}^{e^{\tau+s}-t}\rho(e^{\tau+s}-t)^2|\bar u|\bar u(t,\rho)\,d\rho\,dt \tag{5-45}$$

$$+\frac{e^{\tau-s}}{(e^{\tau+s}-e^{\tau-s})^2}\int_{1-\delta_1}^{e^\tau\cosh s}\int_{t-e^{\tau-s}}^{e^{\tau+s}-t}\rho^3|\bar u|\bar u(t,\rho)\,d\rho dt. \tag{5-46}$$

By the definition of $\bar u$, the inequality (3-23) and Hölder, the contribution of (5-41) can be estimated as

$$\begin{aligned}\int_{s_0}^\infty |e^{\tau+s}\int_{1-\delta_1}^{e^\tau\cosh s}(e^{\tau+s}-t)^2|\bar u|^2(t,e^{\tau+s}-t)\,dt|^2\,ds \\ \lesssim \int_{s_0}^\infty\int_{1-\delta_1}^{e^\tau\cosh s}|\bar u|^4(t,e^{\tau+s}-t)e^{7s}\,dt\,ds \\ \lesssim \int_{\frac14 e^{s_0}}^\infty\int_{1-\delta_1}^{\rho}|\bar u|^4(t,\rho)\rho^6\,dt\,d\rho \lesssim \int_0^\infty\int_{\rho>t+\frac12}|u|^4(t,\rho)\rho^6\,d\rho\,dt \\ \lesssim \|u\|^3_{L^3(\{|x|>|t|+\frac12\})}\sup_{t\ge 0}\big\||x|^2u(t,x)\big\|_{L^\infty_x(\{|x|>t+\frac12\})}\lesssim 1.\end{aligned} \tag{5-47}$$

Now, we consider (5-42) and (5-43). By Hölder, a change of variables, and the inequality (3-23), we have

$$\begin{aligned}\int_{s_0}^\infty\Big|\int_{1-\delta_1}^{e^\tau\cosh s}e^{-s}\int_{t-e^{\tau-s}}^{e^{\tau+s}-t}(t^2\rho+\rho^3)|\bar u|^2(t,\rho)\,d\rho\,dt\Big|^2ds \\ \lesssim \int_{s_0}^\infty\Big|\int_{1-\delta_1}^{e^\tau\cosh s}\mathcal{M}\big(\mathbb{1}_{[t-\frac12+\delta_1,\infty)}(\rho)\rho^3|\bar u|^2(t,\rho)\big)(e^\tau\sinh s)\,dt\Big|^2ds\end{aligned} \tag{5-48}$$

$$\lesssim \int_{s_0}^{\infty}\int_{1-\delta_1}^{e^\tau \cosh s} \big[\mathcal{M}\big(\mathbb{1}_{[t-\frac12+\delta_1,\infty)}(\rho)\rho^3|\bar u|^2(t,\rho)\big)(e^\tau \sinh s)\big]^2 e^\tau \cosh s\, dt\, ds$$
$$\lesssim \int_{1-\delta_1}^{\infty}\int_{e^{s_0}}^{\infty} \big[\mathcal{M}\big(\mathbb{1}_{[t-\frac12+\delta_1,\infty)}(\rho)\rho^3|\bar u|^2(t,\rho)\big)(r)\big]^2 dr\, dt$$
$$\lesssim \int_{1-\delta_1}^{\infty}\int_{r>t-\frac12+\delta_1} r^6|\bar u|^4(t,r)\, dr\, dt$$
$$\lesssim \int_{0}^{\infty}\int_{\rho>t+\frac12} |u|^4(t,\rho)\rho^6\, d\rho\, dt$$
$$\lesssim \|u\|^3_{L^3(\{|x|>|t|+\frac12\})}\sup_{t\ge 0}\big\||x|^2u(t,x)\big\|_{L^\infty_x(\{|x|>t+\frac12\})}$$
$$\lesssim 1.$$

Thus, the contribution of (5-42) and (5-43) to (5-16) is finite.

For (5-44), by the fact that $e^{\tau-s_0} < \frac12-\delta_1$, the definition of $\bar u$, and the inequality (3-23), we have

$$\text{(5-49)}\quad \int_{s_0}^{\infty} e^{-2s}\Big|\int_{1-\delta_1}^{e^\tau \cosh s}(t-e^{\tau-s})^2|\bar u|^2(t,t-e^{\tau-s})\, dt\Big|^2 ds \lesssim \int_{s_0}^{\infty} e^{-2s}\Big|\int_{1-\delta_1}^{e^\tau \cosh s} t^{-2}\, dt\Big|^2 \lesssim 1.$$

Similarly, for (5-45) and (5-46), by the fact that $e^{\tau-s_0} < \frac12-\delta_1$ and the inequality (3-23), we can obtain that

$$\text{(5-50)}\quad \int_{s_0}^{\infty} e^{-6s}\Big|\int_{1-\delta_1}^{e^\tau \cosh s}\int_{t-e^{\tau-s}}^{e^{\tau+s}-t}[\rho^3+\rho(e^{\tau+s}-t)^2]|\bar u|^2(t,\rho)\, d\rho\, dt\Big|^2 ds$$
$$\lesssim \int_{s_0}^{\infty} e^{-2s}\Big|\int_{1-\delta_1}^{e^\tau \cosh s}\int_{t-e^{\tau-s}}^{e^{\tau+s}-t}\rho|\bar u|^2(t,\rho)\, d\rho\, dt\Big|^2 ds$$
$$\lesssim \int_{s_0}^{\infty} e^{-2s}\Big|\int_{1-\delta_1}^{e^\tau \cosh s}\int_{t-e^{\tau-s}}^{e^{\tau+s}-t}\rho^{-3}\, d\rho\, dt\Big|^2 ds$$
$$\lesssim \int_{s_0}^{\infty} e^{-2s}\Big|\int_{1-\delta_1}^{e^\tau \cosh s} t^{-2}\, dt\Big|^2 ds$$
$$\lesssim 1.$$

Hence, combining (5-24)–(5-50), we have

$$\text{(5-51)}\quad \int_{s_0}^{\infty} \tfrac12|(s^2\widetilde w)_\tau|^2+\tfrac12|(s^2\widetilde w)_s|^2\, ds \lesssim 1.$$

5C2. ***The hyperbolic energy for*** $0 \le s \le s_0$. By the hyperbolic transform (5-5), we have

$$(s^2\widetilde{w})_\tau(\tau,s) = 2e^{2\tau}\sinh^2 s\,\overline{w}(e^\tau\cosh s, e^\tau\sinh s) + e^{3\tau}\sinh^2 s\cosh s\,\overline{w}_t(e^\tau\cosh s, e^\tau\sinh s) + e^{3\tau}\sinh^3 s\,\overline{w}_r(e^\tau\cosh s, e^\tau\sinh s), \tag{5-52}$$

$$(s^2\widetilde{w})_s(\tau,s) = 2e^{2\tau}\sinh s\cosh s\,\overline{w}(e^\tau\cosh s, e^\tau\sinh s) + e^{3\tau}\sinh^3 s\,\overline{w}_t(e^\tau\cosh s, e^\tau\sinh s) + e^{3\tau}\sinh^2 s\cosh s\,\overline{w}_r(e^\tau\cosh s, e^\tau\sinh s). \tag{5-53}$$

Hence

$$\int_0^1\int_0^{s_0} |(s^2\widetilde{w})_\tau|^2 + |(s^2\widetilde{w})_s|^2\,ds\,d\tau$$

$$\lesssim \int_0^1\int_0^{s_0} e^{4\tau}\sinh^2 s(\sinh^2 s+\cosh^2 s)|\overline{w}|^2(e^\tau\cosh s, e^\tau\sinh s)\,ds\,d\tau \tag{5-54}$$

$$+\int_0^1\int_0^{s_0} e^{6\tau}\sinh^4 s(\sinh^2 s+\cosh^2 s)[\overline{w}_t^2+\overline{w}_r^2](e^\tau\cosh s, e^\tau\sinh s)\,ds\,d\tau. \tag{5-55}$$

Taking $C_0 = e^{1+s_0}$, by a change of variables, the Hardy inequality and the inequality (3-68), we obtain

$$\text{(5-54)} \lesssim \iint_{|x|+|t|\le C_0} \frac{1}{|x|^2}|\overline{w}|^2(t,x)\,dx\,dt \lesssim \sup_{0<t<C_0} \|\nabla_x\overline{w}\|_{L^2_x(\mathbb{R}^5)} \lesssim 1. \tag{5-56}$$

Similarly, for (5-55), by a change of variables, we have

$$\text{(5-55)} \lesssim \iint_{|x|+|t|\le C_0} |\nabla_{t,x}\overline{w}|^2(t,x)\,dx\,dt \lesssim \sup_{0<t<C_0} \|\nabla_{t,x}\overline{w}\|_{L^2_x(\mathbb{R}^5)} \lesssim 1. \tag{5-57}$$

Then, by the mean value theorem, there exists $\tau_0 \in [0,1]$, such that

$$\int_0^{s_0} |(s^2\widetilde{w})_\tau|^2(\tau_0,s) + |(s^2\widetilde{w})_s|^2(\tau_0,s)\,ds \lesssim 1. \tag{5-58}$$

This estimate along with (5-51) implies

$$\int_0^{\infty} |(s^2\widetilde{w})_\tau|^2(\tau_0,s) + |(s^2\widetilde{w})_s|^2(\tau_0,s)\,ds \lesssim 1. \tag{5-59}$$

5D. ***Uniform boundedness of the hyperbolic energy of*** $\widetilde{w}$. We are now going to show that $E_h(\widetilde{w})(\tau)$ is uniformly bounded for $\tau \in \mathbb{R}_+$.

A simple calculation gives

$$\frac{d}{d\tau}E_h(\widetilde{w}(\tau)) = \int \frac{|s^2\widetilde{w}|s^2\widetilde{w} - |s^2\tilde{u}|s^2\tilde{u}}{\sinh^2 s} s^2\widetilde{w}_\tau \, ds. \tag{5-60}$$

Utilizing the decay property (3-37) of v, we have, for $\tau, s \ge 0$,

$$(e^\tau \cosh s - (1-\delta_1))^2 v(e^\tau \cosh s - (1-\delta_1), e^\tau \sinh s) \lesssim \delta_1^{-\frac{1}{2}}. \tag{5-61}$$

The Huygens principle implies that $v(e^\tau \cosh s - (1-\delta_1), e^\tau \sinh s) = 0$ unless $1 - \frac{6}{5}\delta_1 \le e^{\tau - s} \le 1 - \frac{4}{5}\delta_1$. Thus, for $\tau, s \ge 0$, we have

$$\begin{aligned} \frac{s^2|\tilde{v}(\tau,s)|}{\sinh^2 s} &= e^{2\tau}\bigl|v(e^\tau \cosh s - (1-\delta_1), e^\tau \sinh s)\bigr| \\ &\lesssim \frac{e^{2\tau}\mathbb{1}_{\{s\ge 0:\ e^{\tau-s}\le 1-\frac{4}{5}\delta_1\}}}{(e^\tau \cosh s - (1-\delta_1))^2} \lesssim e^{-\tau}. \end{aligned} \tag{5-62}$$

Hence, by Hölder and interpolation, we have

$$\begin{aligned} &\int_0^\infty \frac{1}{\sinh^2 s}|s^2\widetilde{w}_\tau||s^2\tilde{v}|^2\,ds \\ &\lesssim \Bigl(\int_0^\infty |s^2\widetilde{w}_\tau|^2\,ds\Bigr)^{\frac{1}{2}} \Bigl(\int_0^\infty \frac{1}{\sinh^2 s}|s^2\tilde{v}|^3\,ds\Bigr)^{\frac{1}{2}} \Bigl\|\frac{s^2}{\sinh^2 s}\tilde{v}(\tau,s)\Bigr\|_{L_s^\infty}^{\frac{1}{2}} \\ &\lesssim e^{-\tau/2}E_h(\widetilde{w}(\tau))^{\frac{1}{2}} \Bigl(\int_0^\infty \frac{1}{\sinh^2 s}|s^2\tilde{v}|^3\,ds\Bigr)^{\frac{1}{2}}, \end{aligned} \tag{5-63}$$

$$\begin{aligned} &\int_0^\infty \frac{1}{\sinh^2 s}|s^2\widetilde{w}_\tau||s^2\tilde{v}||s^2\widetilde{w}|\,ds \\ &\lesssim \Bigl(\int_0^\infty |s^2\widetilde{w}_\tau|^2\,ds\Bigr)^{\frac{1}{2}} \Bigl(\int_0^\infty \frac{1}{\sinh^2 s}|s^2\widetilde{w}|^3\,ds\Bigr)^{\frac{1}{3}} \\ &\qquad\qquad \times \Bigl(\int_0^\infty |s^2\tilde{v}(\tau,s)|^6 \frac{1}{\sinh^8 s}\,ds\Bigr)^{\frac{1}{6}} \\ &\lesssim e^{-\tau/2}E_h(\widetilde{w}(\tau))^{\frac{5}{6}} \Bigl(\int_0^\infty \frac{1}{\sinh^2 s}|s^2\tilde{v}|^3\,ds\Bigr)^{\frac{1}{6}}. \end{aligned} \tag{5-64}$$

Combining (5-63) with (5-64) and employing Hölder again, we have

$$\frac{d}{d\tau}E_h(\widetilde{w}(\tau) \lesssim e^{-\tau/2}\Bigl[E_h(\widetilde{w}(\tau) + \int_0^\infty \frac{1}{\sinh^2 s}|s^2\tilde{v}|^3\,ds\Bigr]. \tag{5-65}$$

On the other hand, by a change of variables, we have

$$\int_0^\infty \int_0^\infty \frac{1}{\sinh^2 s} |s^2 \tilde{v}|^3 \, ds \, d\tau \le \int_{\delta_1}^\infty \int_0^\infty |v(t,r)|^3 r^4 \, dr \, dt \le \|v\|^3_{L^3_{t,x}([\delta_1,\infty)\times\mathbb{R}^5)} \lesssim 1. \tag{5-66}$$

Hence, by Gronwall's inequality, (5-65) and (5-66) yield that $E_h(\widetilde{w})(\tau)$ is uniformly bounded in $\mathbb{R}_+$.

5E. ***Conclusion of the proof of Proposition 1.3.*** We complete the proof by studying the Morawetz action in hyperbolic coordinates.

Proposition 5.1. *Let $\widetilde{w}$ be defined in (5-5), then*

$$\iint_{\Omega_2} |w(t,r)|^3 r^4 \, dt \, dr = \int_0^\infty \int_0^\infty \frac{|s^2 \widetilde{w}|^3}{\sinh^2 s} \, ds \, d\tau \lesssim 1. \tag{5-67}$$

Proof. Define the Morawetz action by

$$M(\tau) = \int_0^\infty (s^2 \widetilde{w})_\tau (s^2 \widetilde{w})_s \, ds = \int_0^\infty \widetilde{w}_\tau \Big(\widetilde{w}_s + \frac{2}{s}\widetilde{w}\Big) s^4 \, ds. \tag{5-68}$$

One can easily find that $|M(\tau)| \le E_h(\widetilde{w})(\tau)$. By (5-8), we have

$$\begin{aligned} \frac{d}{d\tau} M(\tau) &= -\int_0^\infty \frac{2s^2 \widetilde{w}}{\sinh^2 s} (s^2 \widetilde{w})_s \; ds - \int_0^\infty \frac{s^4}{\sinh^2 s} |\tilde{u}| \tilde{u} s^2 \widetilde{w}_s \; ds \\ &= -2 \int_0^\infty \frac{|s^2 \widetilde{w}|^2}{\sinh^2 s} \frac{\cosh s}{\sinh s} \, ds - \frac{2}{3} \int_0^\infty \frac{|s^2 \widetilde{w}|^3}{\sinh^2 s} \frac{\cosh s}{\sinh s} \, ds \\ &\quad + \int_0^\infty \frac{s^4}{\sinh^2 s} (|\widetilde{w}|\widetilde{w} - |\tilde{u}|\tilde{u}) s^2 \widetilde{w}_s \; ds. \end{aligned} \tag{5-69}$$

By Hölder, the estimate (5-62), and the fact that $E_h(\widetilde{w}(\tau))$ is uniformly bounded for $\tau \ge 0$, we have

$$\begin{aligned} &\Big| \int_0^\infty \int_0^\infty \frac{s^4}{\sinh^2 s} (|\widetilde{w}|\widetilde{w} - |\tilde{u}|\tilde{u}) s^2 \widetilde{w}_s \; ds \, d\tau \Big| \\ &\quad \lesssim \int_0^\infty \int_0^\infty \frac{1}{\sinh^2 s} |s^2 \widetilde{w}_\tau| |s^2 \tilde{v}|^2 \, ds \, d\tau \\ &\qquad + \int_0^\infty \int_0^\infty \frac{1}{\sinh^2 s} |s^2 \widetilde{w}_\tau| |s^2 \tilde{v}| |s^2 \widetilde{w}| \, ds \, d\tau \end{aligned} \tag{5-70}$$

$$\lesssim \int_0^\infty e^{-\tau} E_h(\widetilde{w}(\tau))^{\frac{1}{2}} \left(\int_0^\infty \frac{1}{\sinh^2 s} |s^2 \tilde{v}|^3 \, ds \right)^{\frac{1}{2}} \tau + \int_0^\infty e^{-\tau} E_h(\widetilde{w}(\tau))^{\frac{5}{6}} \left(\int_0^\infty \frac{1}{\sinh^2 s} |s^2 \tilde{v}|^3 \, ds \right)^{\frac{1}{6}} d\tau \lesssim \|v\|^3_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^5)} \lesssim 1.$$

This together with the equality (5-69) and the fact $M(\tau)$ is uniformly bounded for $\tau \geq 0$, implies that

$$\int_0^\infty \int_0^\infty \frac{|s^2 \widetilde{w}|^3}{\sinh^2 s} \frac{\cosh s}{\sinh s} \, ds \, d\tau \lesssim 1. \tag{5-71}$$

Thus, we have

$$\int_0^\infty \int_0^\infty \frac{|s^2 \widetilde{w}|^3}{\sinh^2 s} \, ds \, d\tau \lesssim 1. \tag{5-72}$$

This yields (5-67) by the definition of $\widetilde{w}$. □

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