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**LIUVILLE-TYPE THEOREMS  
FOR WEIGHTED  $p$ -HARMONIC 1-FORMS  
AND WEIGHTED  $p$ -HARMONIC MAPS**

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# LIUVILLE-TYPE THEOREMS FOR WEIGHTED $p$ -HARMONIC 1-FORMS AND WEIGHTED $p$ -HARMONIC MAPS

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**In this paper, we obtain Bochner–Weitzenböck formulas for the weighted Hodge Laplacian operator acting on differential forms and more generally on vector bundle-valued weighted  $p$ -harmonic forms. Applying these formulas, we prove Liouville-type theorems for weighted  $L^q$   $p$ -harmonic 1-forms and for weighted  $p$ -harmonic maps in a weighted complete noncompact manifold with nonnegative Bakry–Émery Ricci curvature, where  $q = 2p - 2$  or  $q = p$ .**

## 1. Introduction

The celebrated Liouville theorem states that every positive harmonic function on  $\mathbb{R}^n$  is constant. There have been a lot of effort over the years to generalize the classical Liouville theorem into complete noncompact Riemannian manifolds. Huber [1957] proved that any negative subharmonic function on a complete surface with nonnegative curvature is constant. Yau [1975] proved that any positive harmonic function on a noncompact Riemannian manifold with nonnegative Ricci curvature is constant. See also [Greene and Wu 1979; Hildebrandt 1982; Karp 1982] for further related results. Moreover, Yau [1976] obtained an  $L^p$ -Liouville type theorem. More precisely, he proved that, for  $1 < p < \infty$ , any  $L^p$  harmonic function on a complete Riemannian manifold is constant. Given a harmonic function  $f$  on a Riemannian manifold  $M$ , we note that the differential  $df$  is obviously a harmonic 1-form on  $M$ . In the case where  $M$  is a complete noncompact Riemannian manifold, it is natural to consider  $L^2$  harmonic forms on  $M$  because  $L^2$ -Hodge theory remains valid in complete noncompact manifolds as classical Hodge theory works well in compact manifolds. It turned out that the theory of  $L^2$  harmonic 1-forms is useful to investigate the geometry and topology at infinity. For example, Li and Tam [1992] proved that if the space of  $L^2$  harmonic 1-forms on a complete Riemannian manifold  $M$  is trivial, then  $M$  must have at most one nonparabolic end. Cao, Shen,

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and Zhu [Cao et al. 1997] also obtained an interesting topological result which says that if  $M$  is a complete Riemannian manifold with all ends of infinite volume supporting a Sobolev inequality and if the space of  $L^2$  harmonic 1-forms is trivial, then  $M$  must have only one end. Their argument using the space of  $L^2$  harmonic 1-forms to study the geometry and topology at infinity has been extended in various ways. We refer the readers to [Dung and Seo 2012; 2017; Li and Wang 2002; 2004; Lin 2015; Pigola et al. 2005; Seo 2010; 2014; Vieira 2016; Yun 2002] for recent developments on this topic.

In this paper, we study Liouville-type properties on  $p$ -harmonic 1-forms and  $p$ -harmonic maps in weighted manifolds. Given a smooth Riemannian manifold  $(M, g)$  and a smooth function  $f : M \rightarrow \mathbb{R}$ , a *weighted manifold* (or a smooth metric measure space, also known as a manifold with density) is a triple  $M_f := (M, g, e^{-f} dv_g)$ , where  $dv_g$  is the volume form induced by the metric  $g$ . Since the geometry of weighted manifolds were developed by Bakry and Émery [1985], it has been intensively studied by many authors (for instance, see [Lott 2003; Lott and Villani 2009; Sturm 2006a; 2006b; Wei and Wylie 2009]). Moreover, it turned out that the study of weighted manifolds is closely related with that of self-shrinkers and gradient Ricci solitons.

An important geometric quantity on a weighted manifold  $M_f$  known as *Bakry–Émery Ricci curvature* is defined by

$$\text{Ric}_f^M = \text{Ric} + \text{Hess}(f),$$

where  $\text{Hess}(f)$  denotes the Hessian of  $f$ . Obviously, the Bakry–Émery Ricci curvature is a generalization of Ricci curvature. In a weighted manifold, there is a useful elliptic differential operator, the so-called  *$f$ -Laplacian*,  $\Delta_f$  which is defined by

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle.$$

The  $f$ -Laplacian is a natural generalization of the Laplace–Beltrami operator  $\Delta$  as it is self-adjoint with respect to the weighted measure  $e^{-f} dv_g$ , i.e.,

$$\int_M v \Delta_f u e^{-f} dv_g = \int_M (u \Delta_f v) e^{-f} dv_g$$

and

$$\int_M (v \Delta_f u) e^{-f} dv_g = - \int_M \langle \nabla u, \nabla v \rangle e^{-f} dv_g$$

for  $u, v \in C_0^\infty(M)$ .

On the other hand, for a smooth map  $\varphi : (M^n, g, e^{-f} dv_g) \rightarrow (N^m, h)$  from an  $f$ -weighted manifold into a Riemannian manifold, and for a bounded domain

$\Omega \subset M$ , the  $f$ -weighted  $p$ -energy  $\mathcal{E}_{f,p}(\varphi; \Omega)$  with  $p > 1$  of  $\varphi$  over  $\Omega$  is defined by

$$(1-1) \quad \mathcal{E}_{f,p}(\varphi; \Omega) = \frac{1}{p} \int_{\Omega} |d\varphi|^p e^{-f} dv_g,$$

where  $|d\varphi|$  denotes the Hilbert–Schmidt norm of  $d\varphi$  induced by the metrics  $g$  and  $h$ . Namely, if  $\{e_i\}$  is a local frame on  $M$ ,  $|d\varphi|$  is given by

$$(1-2) \quad |d\varphi|^2 = \sum_{i=1}^n \langle d\varphi(e_i), d\varphi(e_i) \rangle$$

so that

$$|d\varphi|^2 = \text{tr}_g \varphi^* h = \langle g, \varphi^* h \rangle.$$

A smooth map  $\varphi : (M, g, e^{-f} dv_g) \rightarrow (N, h)$  is called  $f$ -weighted  $p$ -harmonic if it is a critical point of the  $f$ -weighted  $p$ -energy functional  $\mathcal{E}_{f,p}(\varphi; \Omega)$  for any bounded domain  $\Omega \subset M$ . It can be easily shown that when  $\varphi$  is  $C^2$ -regular, the Euler–Lagrange equation for the  $f$ -weighted  $p$ -energy  $\mathcal{E}_{f,p}$  is the  $f$ -weighted  $p$ -harmonic map equation

$$(1-3) \quad \tau_{f,p}(\varphi) = -\delta_f(|d\varphi|^{p-2} d\varphi) = |d\varphi|^{p-2} \tau_f(\varphi) + d\varphi(\nabla |d\varphi|^{p-2}) = 0.$$

Here  $\delta_f = \delta + i_{\nabla f}$  is the adjoint operator of the exterior derivative  $d$  with respect to the measure  $e^{-f} dv_g$ ,  $i_{\nabla f}$  denotes the interior product with the vector  $\nabla f$ ,  $\tau_f(\varphi) = \tau(\varphi) - i_{\nabla f} d\varphi$  and  $\tau(\varphi)$  is the classical tension field of  $\varphi$ . In the case where  $p = 2$  and  $f$  is a constant function, Schoen and Yau [1976] obtained the following well-known Liouville-type theorem for harmonic maps between complete Riemannian manifolds.

**Theorem [Schoen and Yau 1976].** *Let  $M$  be a complete Riemannian manifold of nonnegative Ricci curvature and let  $N$  be a complete Riemannian manifold of nonpositive sectional curvature. Then, for any constant function  $f$ , every harmonic map  $u : M \rightarrow N$  with finite 2-energy  $\mathcal{E}_{f,2}(u)$  must be constant.*

Recently, Rimoldi and Veronelli [2013] generalized Schoen and Yau’s Liouville-type theorem for harmonic maps into  $f$ -weighted 2-harmonic maps between complete Riemannian manifolds. More precisely, they showed that if

$$u : (M^n, g, e^{-f} dv_g) \rightarrow (N^m, h)$$

is an  $f$ -weighted 2-harmonic map from a complete Riemannian manifold  $M$  with nonnegative Bakry–Émery Ricci curvature into a complete Riemannian manifold with nonpositive sectional curvature and if the  $f$ -weighted 2-energy  $\mathcal{E}_{f,2}(u)$  is finite, then the harmonic map  $u$  must be constant. See also [Hua et al. 2017; Nakauchi 1998; Takeuchi 1991; Zhang and Wang 2016] for related previous results. In this paper, we extend their result into  $f$ -weighted  $p$ -harmonic maps.

The organization of this paper is the following. In [Section 2](#) we derive a Bochner–Weitzenböck formula for the weighted Hodge Laplacian  $\Delta_f$  on differential forms. Applying this formula, we are able to show a Liouville-type property of weighted  $L^q$   $p$ -harmonic 1-forms on a complete noncompact weighted manifold with nonnegative Bakry–Émery Ricci curvature (see [Theorem 2.4](#) for  $q = 2p - 2$  and [Theorem 2.5](#) for  $q = p$ ). In [Section 3](#) we obtain a Bochner–Weitzenböck formula for vector bundle-valued weighted  $p$ -harmonic forms ([Lemma 3.1](#)), which is an extension of our previous results in [Section 2](#). In [Section 4](#) we prove Liouville-type theorems for weighted  $p$ -harmonic maps. In fact, we prove that if  $u$  is a weighted  $p$ -harmonic map from a complete noncompact weighted manifold with nonnegative Bakry–Émery Ricci curvature into a Riemannian manifold with nonpositive sectional curvature and if  $u$  has finite weighted  $q$ -energy, then  $u$  must be constant (see [Theorem 4.1](#) for  $q = 2p - 2$  and [Theorem 4.2](#) for  $q = p$ ).

## 2. Weighted $p$ -harmonic forms

Let  $(M^n, g)$  be an  $n$ -dimensional complete noncompact Riemannian manifold and let  $f : M \rightarrow \mathbb{R}$  be a smooth function on  $M$ . We consider differential forms on the  $f$ -weighted manifold  $(M, g, e^{-f} dv_g)$  and derive a Bochner–Weitzenböck formula for the weighted Hodge Laplacian. Recall that the formal adjoint of the exterior derivative  $d$  with respect to the measure  $e^{-f} dv_g$  is given by the formula

$$\delta_f = \delta + i_{\nabla f}.$$

Then the  $f$ -Hodge Laplacian  $\Delta_f$  on differential forms is defined by

$$\Delta_f = -(d\delta_f + \delta_f d).$$

**Lemma 2.1** (Bochner–Weitzenböck formula). *Let  $(M, g, e^{-f} dv_g)$  be an  $f$ -weighted manifold. If  $\omega$  is a differential 1-form on  $M$ , then*

$$(2-1) \quad \frac{1}{2} \Delta_f |\omega|^{2p-2} = \langle |\omega|^{p-2} \omega, \Delta_f (|\omega|^{p-2} \omega) \rangle + |\nabla (|\omega|^{p-2} \omega)|^2 + |\omega|^{2p-4} \mathbf{Ric}_f^M(\omega^\sharp, \omega^\sharp).$$

Here  $\omega^\sharp$  is the dual vector field to  $\omega$ .

*Proof.* It is well-known (see [[Chang and Sung 2011](#)]) that

$$\begin{aligned} \frac{1}{2} \Delta |\omega|^{2p-2} &= \frac{1}{2} \Delta (|\omega|^{p-2} \omega)^2 \\ &= \langle |\omega|^{p-2} \omega, \Delta (|\omega|^{p-2} \omega) \rangle + |\nabla (|\omega|^{p-2} \omega)|^2 + |\omega|^{2p-4} \mathbf{Ric}(\omega^\sharp, \omega^\sharp). \end{aligned}$$

Using the definition of the  $f$ -weighted Laplacian  $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ , we have

$$\frac{1}{2} \Delta_f |\omega|^{2p-2} = \frac{1}{2} \Delta |\omega|^{2p-2} - \frac{1}{2} \langle \nabla f, \nabla |\omega|^{2p-2} \rangle.$$

Since  $\text{Ric}_f^M = \text{Ric} + \text{Hess}(f)$  and  $\Delta_f = \Delta - di_{\nabla f} - i_{\nabla f}d$ , we get

$$\begin{aligned} \frac{1}{2}\Delta_f|\omega|^{2p-2} &= \langle |\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega) \rangle + |\nabla(|\omega|^{p-2}\omega)|^2 + |\omega|^{2p-4}\text{Ric}_f^M(\omega^\sharp, \omega^\sharp) \\ &\quad + \langle |\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega) \rangle + \langle |\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega) \rangle \\ &\quad - |\omega|^{2p-4}\text{Hess}(f)(\omega^\sharp, \omega^\sharp) - \frac{1}{2}\langle \nabla f, \nabla(|\omega|^{2p-2}) \rangle. \end{aligned}$$

We claim that

$$(2-2) \quad \langle |\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega) \rangle + \langle |\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega) \rangle - |\omega|^{2p-4}\text{Hess}(f)(\omega^\sharp, \omega^\sharp) - \frac{1}{2}\langle \nabla f, \nabla(|\omega|^{2p-2}) \rangle = 0.$$

Let  $\{e_1, \dots, e_n\}$  be a local geodesic frame at a point  $p$  in  $M$  and  $\{\theta_1, \dots, \theta_n\}$  its dual coframe. Let  $\{\theta_{ij}\}$  be the connection 1-form vanishing at the point  $p$ . Writing  $\omega = \omega^i \theta_i$  with Einstein convention, we have

$$|\omega|^{2p-4}\text{Hess}(f)(\omega^\sharp, \omega^\sharp) = |\omega|^{2p-4}\omega^i \omega^j f_{ij}.$$

Since

$$di_{\nabla f}(|\omega|^{p-2}\omega) = f_i \omega^i d|\omega|^{p-2} + |\omega|^{p-2} \omega^i f_{ij} \theta_j + |\omega|^{p-2} f_i \omega^i_{;j} \theta_j,$$

we have

$$\begin{aligned} \langle |\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega) \rangle &= |\omega|^{p-2} f_i \omega^i \langle \omega, d|\omega|^{p-2} \rangle + |\omega|^{2p-4} \omega^i \omega^j f_{ij} + |\omega|^{2p-4} \omega^j f_i \omega^i_{;j}. \end{aligned}$$

Here the semicolon means the covariant differentiation. Moreover,

$$\begin{aligned} d(|\omega|^{p-2}\omega) &= d|\omega|^{p-2} \wedge \omega + |\omega|^{p-2} d\omega \\ &= d|\omega|^{p-2} \wedge \omega + |\omega|^{p-2} d\omega^i \wedge \theta_i \end{aligned}$$

which gives

$$(2-3) \quad \begin{aligned} i_{\nabla f}d(|\omega|^{p-2}\omega) &= d|\omega|^{p-2}(\nabla f)\omega - \omega^i f_i d|\omega|^{p-2} + |\omega|^{p-2} \omega^i_{;j} f_j \theta_i - |\omega|^{p-2} \omega^i_{;j} f_i \theta_j. \end{aligned}$$

Thus

$$\begin{aligned} \langle |\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega) \rangle &= |\omega|^p d|\omega|^{p-2}(\nabla f) - |\omega|^{p-2} \omega^i f_i \langle \omega, d|\omega|^{p-2} \rangle \\ &\quad + |\omega|^{2p-4} \omega^i_{;j} \omega^j f_j - |\omega|^{2p-4} \omega^i_{;j} f_j \omega^j. \end{aligned}$$

Next we have

$$\begin{aligned} \frac{1}{2}\langle \nabla f, \nabla|\omega|^{2p-2} \rangle &= \frac{1}{2}\langle \nabla f, \nabla(|\omega|^p \cdot |\omega|^{p-2}) \rangle \\ &= \frac{1}{2}|\omega|^p \langle \nabla f, \nabla|\omega|^{p-2} \rangle + \frac{1}{2}|\omega|^{p-2} \langle \nabla f, \nabla|\omega|^p \rangle. \end{aligned}$$

Since

$$\nabla|\omega|^2 = 2\omega^i \omega^i_{;j} e_j \quad \text{and} \quad \nabla|\omega|^p = \nabla(|\omega|^2)^{p/2} = \frac{p}{2}|\omega|^{p-2} \nabla|\omega|^2,$$

we get

$$\frac{1}{2}|\omega|^{p-2}\langle\nabla f, \nabla|\omega|^p\rangle = \frac{p}{4}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle = \frac{p}{2}|\omega|^{2p-4}f_j\omega^i\omega^i_{;j}$$

and

$$|\omega|^{2p-4}f_j\omega^i_{;j}\omega^i = \frac{1}{2}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle.$$

Thus the left-hand side of (2-2) becomes

$$\begin{aligned} |\omega|^p d|\omega|^{p-2}\langle\nabla f, \nabla|\omega|^p\rangle + |\omega|^{2p-4}f_j\omega^i_{;j}\omega^i - \frac{1}{2}|\omega|^p\langle\nabla f, \nabla|\omega|^{p-2}\rangle - \frac{1}{2}|\omega|^{p-2}\langle\nabla f, \nabla|\omega|^p\rangle \\ = \frac{1}{2}|\omega|^p\langle\nabla f, \nabla|\omega|^{p-2}\rangle + \frac{2-p}{4}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle. \end{aligned}$$

Since

$$\nabla|\omega|^{p-2} = \nabla(|\omega|^2)^{(p-2)/2} = \frac{p-2}{2}(|\omega|^2)^{(p-2)/2-1}\nabla|\omega|^2 = \frac{p-2}{2}|\omega|^{p-4}\nabla|\omega|^2,$$

the left-hand side of (2-2) vanishes, which completes the proof of Lemma 2.1.  $\square$

As a consequence of Lemma 2.1, we have the following.

**Corollary 2.2.** *Let  $\omega$  be a differential 1-form on a weighted manifold  $(M, g, e^{-f}dv_g)$ . Then*

$$|\omega|^{p-1}\Delta_f|\omega|^{p-1} \geq \langle|\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\omega|^{2p-4}\text{Ric}_f^M(\omega^\sharp, \omega^\sharp).$$

*Proof.* Since

$$(2-4) \quad \frac{1}{2}\Delta_f|\omega|^{2p-2} = |\omega|^{p-1}\Delta_f|\omega|^{p-1} + |\nabla|\omega|^{p-1}|^2,$$

it follows from Lemma 2.1 that

$$\begin{aligned} |\omega|^{p-1}\Delta_f|\omega|^{p-1} + |\nabla|\omega|^{p-1}|^2 \\ = \langle|\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\nabla(|\omega|^{p-2}\omega)|^2 + |\omega|^{2p-4}\text{Ric}_f^M(\omega^\sharp, \omega^\sharp). \end{aligned}$$

From the generalized Kato type inequality, we have

$$|\nabla|\omega|^{p-1}|^2 = |\nabla|\omega|^{p-2}\omega|^2 \leq |\nabla(|\omega|^{p-2}\omega)|^2.$$

Thus we get

$$|\omega|^{p-1}\Delta_f|\omega|^{p-1} \geq \langle|\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\omega|^{2p-4}\text{Ric}_f^M(\omega^\sharp, \omega^\sharp). \quad \square$$

Let  $\phi : M \rightarrow \mathbb{R}$  be a harmonic function. Since

$$d(d\phi) = 0 \quad \text{and} \quad \Delta\phi = \delta(d\phi) = 0,$$

the differential  $d\phi$  is a harmonic 1-form. Similarly, if  $\phi : M \rightarrow \mathbb{R}$  is a  $p$ -harmonic function, then

$$\Delta_p\phi = \text{div}(|\nabla\phi|^{p-2}\nabla\phi) = 0,$$

which is equivalent to the equation

$$\delta(|d\phi|^{p-2}d\phi) = 0.$$

In fact, this is the Euler–Lagrange equation of the  $p$ -energy functional  $\mathcal{E}_p(\phi) = \frac{1}{p} \int_M |d\phi|^p dv_g$ . Using this observation, one can define a  $p$ -harmonic form  $\omega$  on  $M$  as follows [Chang and Sung 2011]:

$$d\omega = 0 \quad \text{and} \quad \delta(|\omega|^{p-2}\omega) = 0,$$

which shows that, for any  $p$ -harmonic function  $\phi$  on  $M$ , its differential  $d\phi$  is a  $p$ -harmonic 1-form. Motivated by this notion of  $p$ -harmonic differential forms in [Chang and Sung 2011] and weighted harmonic forms in [Vieira 2013], we give the definition of weighted  $p$ -harmonic forms on a weighted manifold.

**Definition 2.3.** A differential form  $\omega$  on  $M$  is  $f$ -weighted  $p$ -harmonic if  $\omega$  satisfies

$$d\omega = 0 \quad \text{and} \quad \delta_f(|\omega|^{p-2}\omega) = 0.$$

When  $f$  is constant, we note that the above definition of  $f$ -weighted  $p$ -harmonic forms is equivalent to the definition of  $p$ -harmonic forms in the sense of [Chang and Sung 2011]. Consider an  $f$ -weighted  $L_f^{2p-2}$   $p$ -harmonic 1-form  $\omega$  on a weighted manifold  $M_f$  with nonnegative Bakry–Émery Ricci curvature, where the  $L_f^{2p-2}$  norm of  $\omega$  is given by

$$\int_M |\omega|^{2p-2} e^{-f} dv_g < \infty.$$

Then we have the following Liouville-type theorem for weighted  $p$ -harmonic 1-forms.

**Theorem 2.4.** *Let  $(M, g, e^{-f} dv_g)$  be a complete noncompact  $f$ -weighted manifold with nonnegative Bakry–Émery Ricci tensor,  $\text{Ric}_f^M \geq 0$ . Suppose that  $f$  is a bounded function. If  $\omega$  is an  $f$ -weighted  $L_f^{2p-2}$   $p$ -harmonic 1-form on  $M$  for  $p > 1$ , then  $\omega$  vanishes.*

*Proof.* Since  $\omega$  is an  $f$ -weighted  $p$ -harmonic 1-form, we have

$$\delta_f(|\omega|^{p-2}\omega) = 0.$$

Thus Corollary 2.2 together with curvature condition implies

$$(2-5) \quad |\omega|^{p-1} \Delta_f |\omega|^{p-1} \geq \langle |\omega|^{p-2}\omega, \delta_f d(|\omega|^{p-2}\omega) \rangle.$$

Fix a point  $p \in M$  and choose a cut-off function  $\eta$  satisfying

$$(2-6) \quad 0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } B_p(r), \quad \text{supp}(\eta) \subset B_p(2r), \quad \text{and} \quad |\nabla \eta| \leq \frac{1}{r}.$$

Here  $B_p(r)$  denotes the geodesic ball of radius  $r$  centered at  $p$ . Multiplying (2-5) by  $\eta^2$  and integrating it over  $M$  with respect to the measure  $e^{-f} dv_g$ , we obtain

$$\begin{aligned} & \int_M \eta^2 |\omega|^{p-1} \Delta_f |\omega|^{p-1} e^{-f} dv_g \\ &= - \int_M \eta^2 |\nabla |\omega|^{p-1}|^2 e^{-f} dv_g - 2 \int_M \eta |\omega|^{p-1} \langle \nabla \eta, \nabla |\omega|^{p-1} \rangle e^{-f} dv_g \\ &\leq - \int_M \eta^2 |\nabla |\omega|^{p-1}|^2 e^{-f} dv_g + \frac{1}{2} \int_M \eta^2 |\nabla |\omega|^{p-1}|^2 e^{-f} dv_g \\ &\hspace{25em} + 2 \int_M |\omega|^{2p-2} |\nabla \eta|^2 e^{-f} dv_g \\ &= -\frac{1}{2} \int_M \eta^2 |\nabla |\omega|^{p-1}|^2 e^{-f} dv_g + 2 \int_M |\omega|^{2p-2} |\nabla \eta|^2 e^{-f} dv_g. \end{aligned}$$

Moreover

$$\begin{aligned} & \int_M \eta^2 \langle |\omega|^{p-2} \omega, \delta_f d(|\omega|^{p-2} \omega) \rangle e^{-f} dv_g \\ &= \int_M \langle d(\eta^2 |\omega|^{p-2} \omega), d(|\omega|^{p-2} \omega) \rangle e^{-f} dv_g \\ &= 2 \int_M \eta |\omega|^{p-2} \langle d\eta \wedge \omega, d(|\omega|^{p-2} \omega) \rangle e^{-f} dv_g + \int_M \eta^2 |d(|\omega|^{p-2} \omega)|^2 e^{-f} dv_g \\ &\geq - \int_M \eta^2 |d(|\omega|^{p-2} \omega)|^2 e^{-f} dv_g - \int_M |\nabla \eta|^2 |\omega|^{2p-2} e^{-f} dv_g \\ &\hspace{25em} + \int_M \eta^2 |d(|\omega|^{p-2} \omega)|^2 e^{-f} dv_g \\ &= - \int_M |\nabla \eta|^2 |\omega|^{2p-2} e^{-f} dv_g. \end{aligned}$$

Therefore

$$\frac{1}{2} \int_M \eta^2 |\nabla |\omega|^{p-1}|^2 e^{-f} dv_g \leq 3 \int_M |\nabla \eta|^2 |\omega|^{2p-2} e^{-f} dv_g \leq \frac{3}{r^2} \int_M |\omega|^{2p-2} e^{-f} dv_g.$$

Since  $\omega$  is an  $f$ -weighted  $L^{2p-2}$  harmonic 1-form, we obtain

$$\nabla |\omega|^{p-1} = 0$$

by letting  $r \rightarrow \infty$ . Hence  $|\omega|^{p-1}$  is constant. Since  $\text{Ric}_f^M \geq 0$  and  $f$  is bounded, the  $f$ -volume of  $(M, g)$  is infinite (see [Wei and Wylie 2009], for example). Therefore we see that  $\omega = 0$ .  $\square$

Using the Bochner–Weitzenböck formula, we can also prove the following.

**Theorem 2.5.** *Let  $(M, g, e^{-f} dv_g)$  be a complete noncompact  $f$ -weighted manifold with nonnegative Bakry–Émery Ricci tensor. Suppose that  $f$  is a bounded function. For  $p \geq 2$ , if  $\omega$  is an  $f$ -weighted  $L_f^p$   $p$ -harmonic 1-form on  $M$ , then  $\omega = 0$ .*

*Proof.* Since  $\delta_f(|\omega|^{p-2}\omega) = 0$ , [Corollary 2.2](#) and the curvature condition implies

$$(2-7) \quad \begin{aligned} |\omega|\Delta_f|\omega|^{p-1} &\geq \langle \omega, \Delta_f(|\omega|^{p-2}\omega) \rangle + |\omega|^{2p-4} \text{Ric}_f^M(\omega^\sharp, \omega^\sharp) \\ &\geq \langle \omega, \delta_f d(|\omega|^{p-2}\omega) \rangle. \end{aligned}$$

Fix a point  $p \in M$  and choose a cut-off function  $\eta$  satisfying [\(2-6\)](#). Multiplying [\(2-7\)](#) by  $\eta^2$  and integrating it over  $M$  with respect to the measure  $e^{-f} dv_g$ , we obtain

$$(2-8) \quad \int_M \eta^2 |\omega|\Delta_f|\omega|^{p-1} e^{-f} dv_g \geq \int_M \eta^2 \langle \omega, \delta_f d(|\omega|^{p-2}\omega) \rangle e^{-f} dv_g.$$

Then the left-hand side of [\(2-8\)](#) is given by

$$(2-9) \quad \begin{aligned} &\int_M \eta^2 |\omega|\Delta_f|\omega|^{p-1} e^{-f} dv_g \\ &= - \int_M \eta^2 \langle \nabla|\omega|, \nabla|\omega|^{p-1} \rangle e^{-f} dv_g - 2 \int_M \eta |\omega| \langle \nabla\eta, \nabla|\omega|^{p-1} \rangle e^{-f} dv_g \\ &= -(p-1) \int_M \eta^2 |\omega|^{p-2} |\nabla|\omega||^2 e^{-f} dv_g \\ &\quad - 2(p-1) \int_M \eta |\omega|^{p-1} \langle \nabla\eta, \nabla|\omega| \rangle e^{-f} dv_g. \end{aligned}$$

Note that

$$|\omega|^{p-2} |\nabla|\omega||^2 = \frac{4}{p^2} |\nabla|\omega|^{p/2}|^2$$

and

$$(2-10) \quad |\omega|^{p-1} \nabla|\omega| = |\omega|^{p/2} \cdot |\omega|^{p/2-1} \nabla|\omega| = \frac{2}{p} |\omega|^{p/2} \nabla|\omega|^{p/2}.$$

Substituting these two identities into [\(2-9\)](#), we obtain

$$(2-11) \quad \begin{aligned} &\int_M \eta^2 |\omega|\Delta_f|\omega|^{p-1} e^{-f} dv_g \\ &= - \frac{4(p-1)}{p^2} \int_M \eta^2 |\nabla|\omega|^{p/2}|^2 e^{-f} dv_g \\ &\quad - \frac{4(p-1)}{p} \int_M \eta |\omega|^{p/2} \langle \nabla\eta, \nabla|\omega|^{p/2} \rangle e^{-f} dv_g \\ &\leq - \frac{4(p-1)}{p^2} \int_M \eta^2 |\nabla|\omega|^{p/2}|^2 e^{-f} dv_g \\ &\quad + \frac{2(p-1)}{p} \left\{ \varepsilon \int_M \eta^2 |\nabla|\omega|^{p/2}|^2 e^{-f} dv_g + \frac{1}{\varepsilon} \int_M |\omega|^p |\nabla\eta|^2 e^{-f} dv_g \right\}, \end{aligned}$$

where we used Young's inequality in the last inequality for arbitrary  $\varepsilon > 0$ .

On the other hand, applying the divergence theorem with respect to the measure  $e^{-f} dv_g$ , the right-hand side of (2-8) becomes

$$\int_M \eta^2 \langle \omega, \delta_f d(|\omega|^{p-2} \omega) \rangle e^{-f} dv_g = \int_M \langle d(\eta^2 \omega), d(|\omega|^{p-2} \omega) \rangle e^{-f} dv_g.$$

Since

$$|d(\varphi \omega)| = |d\varphi \wedge \omega| \leq |d\varphi| |\omega|$$

for any smooth function  $\varphi : M \rightarrow \mathbb{R}$  and any closed 1-form  $\omega$  (see Lemma 13 in [Pigola et al. 2008]), using (2-10) and Young's inequality again gives

$$\begin{aligned} (2-12) \quad & |\langle d(\eta^2 \omega), d(|\omega|^{p-2} \omega) \rangle| \leq |d(\eta^2 \omega)| |d(|\omega|^{p-2} \omega)| \\ & \leq |d\eta^2| |\omega|^2 |d|\omega|^{p-2}| \\ & = 2\eta |\omega|^2 |\nabla \eta| |\nabla |\omega|^{p-2}| \\ & = 2(p-2)\eta |\nabla \eta| |\omega|^{p-1} |\nabla |\omega|| \\ & = \frac{4(p-2)}{p} \eta |\nabla \eta| |\omega|^{p/2} |\nabla |\omega|^{p/2}| \\ & \leq \frac{2(p-2)}{p} \left( \delta \eta^2 |\nabla |\omega|^{p/2}|^2 + \frac{1}{\delta} |\nabla \eta|^2 |\omega|^p \right) \end{aligned}$$

for any  $\delta > 0$ . Therefore

$$\begin{aligned} (2-13) \quad \int_M \eta^2 \langle \omega, \delta_f d(|\omega|^{p-2} \omega) \rangle e^{-f} dv_g & \geq -\frac{2(p-2)}{p} \delta \int_M \eta^2 |\nabla |\omega|^{p/2}|^2 e^{-f} dv_g \\ & \quad - \frac{2(p-2)}{p} \frac{1}{\delta} \int_M |\nabla \eta|^2 |\omega|^p e^{-f} dv_g. \end{aligned}$$

Combining (2-8), (2-11) and (2-13), we obtain

$$\begin{aligned} & \left( \frac{4(p-1)}{p^2} - \frac{2(p-1)}{p} \varepsilon - \frac{2(p-2)}{p} \delta \right) \int_M \eta^2 |\nabla |\omega|^{p/2}|^2 e^{-f} dv_g \\ & \leq \left( \frac{2(p-1)}{p} \frac{1}{\varepsilon} + \frac{2(p-2)}{p} \frac{1}{\delta} \right) \int_M |\nabla \eta|^2 |\omega|^p e^{-f} dv_g. \end{aligned}$$

Choose  $\varepsilon$  and  $\delta$  sufficiently small so that

$$\frac{4(p-1)}{p^2} - \frac{2(p-1)}{p} \varepsilon - \frac{2(p-2)}{p} \delta > 0.$$

Since  $\omega$  is an  $L_f^p$   $p$ -harmonic 1-form, as  $r$  tends to infinity, we see

$$\nabla |\omega|^{p/2} = 0,$$

which implies that  $\omega \equiv 0$  as in the proof of Theorem 2.4. □

**Remark 2.6.** In Theorems 2.4 and 2.5, the boundedness on the weighted function  $f$  is only needed to guarantee that the weighted volume of  $(M, g, e^{-f} dv_g)$  is infinite. In fact, we prove that any  $f$ -weighted  $L_f^{2p-2}$   $p$ -harmonic 1-form with  $p > 1$  or  $L_f^p$   $p$ -harmonic 1-form with  $p \geq 2$  on a complete noncompact  $f$ -weighted manifold with nonnegative Bakry–Émery Ricci tensor has constant length, which implies that  $\omega$  is  $f$ -harmonic. Thus applying the standard Bochner formula for  $f$ -harmonic 1-forms (see Lemma 2.1 with  $p = 2$ , [Lott 2003] or [Vieira 2013]), one can see that  $\omega$  is parallel without the assumption that  $f$  is bounded. This result leads to applications in gradient steady Ricci solitons or, more generally, to applications in weighted manifolds with infinite weighted volumes (see [Vieira 2013]). Recall that a gradient steady Ricci soliton is a manifold  $(M, g)$  together with a smooth function  $f$  satisfying  $\text{Ric}_f^M = 0$ .

Furthermore, if we assume that  $\text{Ric}_f^M$  is nonnegative and positive at a point, it is easy to see, from Corollary 2.2, that  $\omega$  vanishes without assuming the boundedness of  $f$ . This property leads to applications in gradient shrinking Ricci solitons satisfying  $\text{Ric}_f^M = \lambda g$  for some positive constant  $\lambda$  as follows.

**Corollary 2.7.** *Let  $(M, g, e^{-f} dv_g)$  be a complete gradient shrinking Ricci soliton satisfying  $\text{Ric} + \text{Hess}(f) = \lambda g$  with  $\lambda > 0$ , constant. Then if  $\omega$  is an  $L_f^{2p-2}$  ( $p > 1$ ) or  $L_f^p$  ( $p \geq 2$ )  $p$ -harmonic 1-form on  $M$ , then  $\omega = 0$ .*

*Proof.* The proof follows from the argument in Remark 2.6. □

In case of gradient steady Ricci solitons, we also have the following same vanishing property.

**Corollary 2.8.** *Let  $(M, g, e^{-f} dv_g)$  be a complete gradient steady Ricci soliton satisfying  $\text{Ric} + \text{Hess}(f) = 0$ . Then if  $\omega$  is an  $L_f^{2p-2}$  ( $p > 1$ ) or  $L_f^p$  ( $p \geq 2$ )  $p$ -harmonic 1-form on  $M$ , then  $\omega = 0$ .*

*Proof.* For  $q = 2p - p$  or  $q = p$ , applying the same argument as in the proofs of Theorems 2.4 and 2.5, we see that  $|\omega| \equiv C$  for some constant  $C$ . Thus

$$\int_M |\omega|^q e^{-f} dv_g = C^q \text{Vol}_f(M),$$

where  $\text{Vol}_f(M)$  denotes the  $f$ -weighted volume of  $M$ .

On the other hand, it is well-known that the scalar curvature of a gradient steady Ricci soliton is nonnegative and  $|\nabla f|$  is bounded by a positive constant (see [Cao 2010] for example). Moreover, Munteanu and Wang [2011] proved that the first eigenvalue of  $f$ -Laplacian  $\Delta_f$  on the nontrivial gradient steady Ricci solitons is positive. Therefore, applying the result by Vieira [2013], we get  $\text{Vol}_f(M) = \infty$ . This shows that  $\omega = 0$ . □

### 3. Vector bundle-valued weighted $p$ -harmonic forms

In this section, we extend the notions discussed in Section 2 including the Bochner–Weitzenböck formula to vector bundles over a weighted manifold.

Let  $\pi : E \rightarrow M$  be a vector bundle of rank  $m$  over a smooth oriented Riemannian manifold  $(M^n, g)$ . We denote by  $\Gamma(E)$  the vector space of smooth sections of  $E$  over  $M$ . A Riemannian structure on the bundle  $E$  is a pair  $(\nabla^E, \rho)$ , where  $\rho$  is a Riemannian metric on  $E$ ,  $\nabla^E$  a connection and  $\nabla^E \rho = 0$ . Denoting  $\rho = \langle \cdot, \cdot \rangle$ , the condition  $\nabla^E \rho = 0$  means that, for each  $X \in \Gamma(TM)$  and  $s_1, s_2 \in \Gamma(E)$ , we have

$$X \cdot \langle s_1, s_2 \rangle = \langle \nabla^E_X s_1, s_2 \rangle + \langle s_1, \nabla^E_X s_2 \rangle.$$

The curvature of the connection  $\nabla^E$  is the map  $R^E : \Lambda^2 TM \otimes \Gamma(E) \rightarrow \Gamma(E)$  defined by

$$R^E(X, Y)s = -\nabla^E_X \nabla^E_Y s + \nabla^E_Y \nabla^E_X s + \nabla^E_{[X, Y]}s.$$

Let  $\omega$  be an  $l$ -form on  $M$  with values in the vector bundle  $\pi : E \rightarrow M$ . Then, choosing a (local) frame  $s_1, \dots, s_m$  on  $E$ , for each  $X_1, \dots, X_l \in \Gamma(TM)$ , we can write

$$\omega(X_1, \dots, X_l) = \sum_{\alpha=1}^m a_\alpha s_\alpha$$

for some local smooth functions  $a_\alpha$  on  $M$ . For the Levi–Civita connection  $D^M = D$  on  $(M, g)$ , the induced connection  $\nabla$  on  $\Gamma(\Lambda^l T^*M \otimes E)$ , the space of smooth  $l$ -forms on  $M$  with values in the vector bundle  $\pi : E \rightarrow M$ , is given by

$$(\nabla_X \omega)(X_1, \dots, X_l) = \nabla^E_X(\omega(X_1, \dots, X_l)) - \sum_{i=1}^l \omega(X_1, \dots, D_X X_i, \dots, X_l)$$

and its associated curvature is given by

$$\begin{aligned} (R(X, Y)\omega)(X_1, \dots, X_l) &= R^E(X, Y)(\omega(X_1, \dots, X_l)) - \sum_{i=1}^l \omega(X_1, \dots, \widehat{R^M(X, Y)X_i}, \dots, X_l). \end{aligned}$$

For the induced connection  $\nabla$ , the exterior differential operator

$$d : \Gamma(\Lambda^l T^*M \otimes E) \rightarrow \Gamma(\Lambda^{l+1} T^*M \otimes E)$$

is given by

$$(d\omega)(X_1, \dots, X_{l+1}) = \sum_{i=1}^{l+1} (-1)^{i+1} (\nabla_{X_i} \omega)(X_1, \dots, \widehat{X_i}, \dots, X_{l+1}),$$

where the symbol covered by  $\widehat{X}_i$  is omitted. The codifferential operator  $\delta$  is given by

$$(\delta\omega)(X_1, \dots, X_{l-1}) = - \sum_{i=1}^n (\nabla_{e_i}\omega)(e_i, X_1, \dots, X_{l-1}),$$

where  $\{e_i\}$  is a local frame on  $M$ . Finally the Laplacian  $\Delta$  and the  $f$ -weighted Laplacian  $\Delta_f$  are defined on  $E$ -valued differential forms by

$$\Delta = -(d\delta + \delta d) \quad \text{and} \quad \Delta_f = -(d\delta_f + \delta_f d),$$

respectively.

For a vector bundle  $\pi : E \rightarrow M$  over a weighted manifold  $(M, g, e^{-f} dv_g)$ , we have the following Bochner–Weitzenböck formula for differential 1-forms on  $M$  with values in  $E$ .

**Lemma 3.1** (Bochner–Weitzenböck formula). *Let  $\pi : E \rightarrow M$  be a vector bundle of rank  $m$  over a smooth oriented Riemannian manifold  $(M, g)$ , and let  $f : M \rightarrow \mathbb{R}$  be a smooth function. If  $\omega$  is an  $E$ -valued 1-form on  $M$ , then*

$$(3-1) \quad \frac{1}{2} \Delta_f |\omega|^{2p-2} = \langle |\omega|^{p-2} \omega, \Delta_f (|\omega|^{p-2} \omega) \rangle + |\nabla (|\omega|^{p-2} \omega)|^2 \\ + |\omega|^{2p-4} \sum_{i=1}^n \langle \omega(\text{Ric}_f^M(e_i)), \omega(e_i) \rangle \\ - |\omega|^{2p-4} \sum_{i,j} \langle R^E(e_i, e_j) \omega(e_i), \omega(e_j) \rangle,$$

where  $\{e_i\}$  is a local frame on  $M$  and  $\text{Ric}_f^M(e_i)$  is a vector given by

$$\text{Ric}_f^M(e_i) = \sum_{j=1}^n \text{Ric}_f^M(e_i, e_j) e_j = \sum_{j=1}^n [\text{Ric}^M(e_i, e_j) + \text{Hess}(f)(e_i, e_j)] e_j.$$

*Proof.* It is well-known (see [Eells and Lemaire 1983]) that

$$(3-2) \quad \frac{1}{2} \Delta |\omega|^{2p-2} = \langle |\omega|^{p-2} \omega, \Delta (|\omega|^{p-2} \omega) \rangle + |\nabla (|\omega|^{p-2} \omega)|^2 \\ + |\omega|^{2p-4} \sum_{i=1}^n \langle \omega(\text{Ric}^M(e_i)), \omega(e_i) \rangle \\ - |\omega|^{2p-4} \sum_{i,j} \langle R^E(e_i, e_j) \omega(e_i), \omega(e_j) \rangle.$$

By definition of weighted Laplacian  $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ , we have

$$\frac{1}{2} \Delta_f |\omega|^{2p-2} = \frac{1}{2} \Delta |\omega|^{2p-2} - \frac{1}{2} \langle \nabla f, \nabla |\omega|^{2p-2} \rangle.$$

Since

$$\text{Ric}_f^M = \text{Ric}^M + \text{Hess}(f) \quad \text{and} \quad \Delta_f = \Delta - di_{\nabla f} - i_{\nabla f} d,$$

we get

$$\begin{aligned}
& \frac{1}{2} \Delta_f |\omega|^{2p-2} \\
&= \langle |\omega|^{p-2} \omega, \Delta_f (|\omega|^{p-2} \omega) \rangle + \langle |\omega|^{p-2} \omega, di_{\nabla f} (|\omega|^{p-2} \omega) \rangle \\
&\quad + \langle |\omega|^{p-2} \omega, i_{\nabla f} d (|\omega|^{p-2} \omega) \rangle + |\nabla (|\omega|^{p-2} \omega)|^2 \\
&\quad + |\omega|^{2p-4} \sum_{i=1}^n \langle \omega(\text{Ric}_f^M(e_i)), \omega(e_i) \rangle - |\omega|^{2p-4} \langle \omega(\text{Hess}(f)(e_i)), \omega(e_i) \rangle \\
&\quad - |\omega|^{2p-4} \sum_{i,j} \langle R^E(e_i, e_j) \omega(e_i), \omega(e_j) \rangle - \frac{1}{2} \langle \nabla f, \nabla (|\omega|^{2p-2}) \rangle.
\end{aligned}$$

We claim that

$$\begin{aligned}
(3-3) \quad & \langle |\omega|^{p-2} \omega, di_{\nabla f} (|\omega|^{p-2} \omega) \rangle + \langle |\omega|^{p-2} \omega, i_{\nabla f} d (|\omega|^{p-2} \omega) \rangle \\
& - |\omega|^{2p-4} \langle \omega(\text{Hess}(f)(e_i)), \omega(e_i) \rangle - \frac{1}{2} \langle \nabla f, \nabla (|\omega|^{2p-2}) \rangle = 0.
\end{aligned}$$

Let  $\{e_1, \dots, e_n\}$  be a local geodesic frame at a point  $p$  in  $M$ , and  $\{\theta_1, \dots, \theta_n\}$  be its dual coframe. Let  $\{\theta_{ij}\}$  be the connection 1-form vanishing at the point  $p$ . Let  $\{s_1, \dots, s_m\}$  be a local frame on  $E$  such that

$$\nabla^E s_\alpha|_p = 0.$$

Then  $\omega$  can be expressed as

$$\omega = \sum_{\alpha=1}^m \sum_{i=1}^n a_{i\alpha} \theta_i \otimes s_\alpha$$

so that

$$\omega(e_j) = \sum_{\alpha} a_{j\alpha} s_\alpha \quad \text{and} \quad |\omega|^2 = \sum_{i,\alpha} a_{i\alpha}^2.$$

Since

$$\begin{aligned}
di_{\nabla f} (|\omega|^{p-2} \omega) &= di_{\nabla f} (|\omega|^{p-2} a_{i\alpha} \theta_i \otimes s_\alpha) \\
&= d (|\omega|^{p-2} f_i a_{i\alpha} s_\alpha) \\
&= f_i a_{i\alpha} d |\omega|^{p-2} \otimes s_\alpha + |\omega|^{p-2} a_{i\alpha} f_{ij} \theta_j \otimes s_\alpha \\
&\quad + |\omega|^{p-2} f_i a_{i\alpha; j} \theta_j \otimes s_\alpha + |\omega|^{p-2} f_i a_{i\alpha} \nabla^E s_\alpha,
\end{aligned}$$

we have

$$\begin{aligned}
(3-4) \quad & \langle |\omega|^{p-2} \omega, di_{\nabla f} (|\omega|^{p-2} \omega) \rangle = |\omega|^{p-2} \langle a_{j\alpha} \theta_j \otimes s_\alpha, di_{\nabla f} (|\omega|^{p-2} \omega) \rangle \\
&= |\omega|^{p-2} a_{j\alpha} f_i a_{i\alpha} d |\omega|^{p-2} (e_j) \\
&\quad + |\omega|^{2p-4} a_{j\alpha} a_{i\alpha} f_{ij} + |\omega|^{2p-4} a_{j\alpha} f_i a_{i\alpha; j}.
\end{aligned}$$

Moreover

$$\begin{aligned} d(|\omega|^{p-2}\omega) &= a_{i\alpha}(d|\omega|^{p-2} \wedge \theta_i) \otimes s_\alpha + |\omega|^{p-2}(da_{i\alpha} \wedge \theta_i) \otimes s_\alpha \\ &\quad + |\omega|^{p-2}a_{i\alpha}\theta_{ij} \wedge \theta_j \otimes s_\alpha - |\omega|^{p-2}a_{i\alpha}\theta_i \wedge \nabla^E s_\alpha \\ &= a_{i\alpha}(d|\omega|^{p-2} \wedge \theta_i) \otimes s_\alpha + |\omega|^{p-2}a_{i\alpha;j}(\theta_j \wedge \theta_i) \otimes s_\alpha \end{aligned}$$

which gives

$$\begin{aligned} i_{\nabla f}d(|\omega|^{p-2}\omega) &= d|\omega|^{p-2}(\nabla f)\omega - a_{i\alpha}f_i d|\omega|^{p-2} \otimes s_\alpha \\ &\quad + |\omega|^{p-2}a_{i\alpha;j}f_j\theta_i \otimes s_\alpha - |\omega|^{p-2}a_{i\alpha;j}f_i\theta_j \otimes s_\alpha. \end{aligned}$$

Thus

$$\begin{aligned} (3-5) \quad \langle |\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega) \rangle &= |\omega|^{p-2}\langle a_{j\beta}\theta_j \otimes s_\beta, i_{\nabla f}d(|\omega|^{p-2}\omega) \rangle \\ &= |\omega|^p d|\omega|^{p-2}(\nabla f) - |\omega|^{p-2}a_{i\alpha}a_{j\alpha}f_i d|\omega|^{p-2}(e_j) \\ &\quad + |\omega|^{2p-4}a_{i\alpha;j}a_{i\alpha}f_j - |\omega|^{2p-4}a_{i\alpha;j}f_i a_{j\alpha} \\ &= |\omega|^p \langle \nabla f, \nabla |\omega|^{p-2} \rangle - |\omega|^{p-2}a_{i\alpha}a_{j\alpha}f_i d|\omega|^{p-2}(e_j) \\ &\quad + \frac{1}{2}|\omega|^{2p-4} \langle \nabla f, \nabla |\omega|^2 \rangle - |\omega|^{2p-4}a_{i\alpha;j}f_i a_{j\alpha}. \end{aligned}$$

Note that

$$(3-6) \quad \langle \omega(\text{Hess}(f)(e_i)), \omega(e_i) \rangle = f_{ij} \langle \omega(e_j), \omega(e_i) \rangle = f_{ij}a_{j\alpha}a_{i\alpha}.$$

From (3-4), (3-5), and (3-6), it follows that

$$\begin{aligned} \langle |\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega) \rangle + \langle |\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega) \rangle \\ - |\omega|^{2p-4} \langle \omega(\text{Hess}(f)(e_i)), \omega(e_i) \rangle \\ = |\omega|^p \langle \nabla f, \nabla |\omega|^{p-2} \rangle + \frac{1}{2}|\omega|^{2p-4} \langle \nabla f, \nabla |\omega|^2 \rangle. \end{aligned}$$

We observe that

$$\begin{aligned} \frac{1}{2} \langle \nabla f, \nabla |\omega|^{2p-2} \rangle &= \frac{1}{2} \langle \nabla f, \nabla (|\omega|^p \cdot |\omega|^{p-2}) \rangle \\ &= \frac{1}{2} |\omega|^p \langle \nabla f, \nabla |\omega|^{p-2} \rangle + \frac{1}{2} |\omega|^{p-2} \langle \nabla f, \nabla |\omega|^p \rangle. \end{aligned}$$

Since

$$\nabla |\omega|^p = \nabla (|\omega|^2)^{p/2} = \frac{p}{2} |\omega|^{p-2} \nabla |\omega|^2,$$

we have

$$\frac{1}{2} |\omega|^{p-2} \langle \nabla f, \nabla |\omega|^p \rangle = \frac{p}{4} |\omega|^{2p-4} \langle \nabla f, \nabla |\omega|^2 \rangle.$$

Thus the left-hand side of (3-3) becomes

$$\frac{1}{2} |\omega|^p \langle \nabla f, \nabla |\omega|^{p-2} \rangle + \frac{2-p}{4} |\omega|^{2p-4} \langle \nabla f, \nabla |\omega|^2 \rangle.$$

Using

$$\begin{aligned} \nabla|\omega|^{p-2} &= \nabla(|\omega|^2)^{(p-2)/2} \\ &= \frac{p-2}{2}(|\omega|^2)^{(p-2)/2-1}\nabla|\omega|^2 \\ &= \frac{p-2}{2}|\omega|^{p-4}\nabla|\omega|^2, \end{aligned}$$

we see that the left-hand side of (3-3) vanishes, which completes the proof of Lemma 3.1.  $\square$

As in the proof of Corollary 2.2, we can easily show the following by using Lemma 3.1.

**Corollary 3.2.** *Let  $\pi : E \rightarrow M$  be a vector bundle of rank  $m$  over a smooth oriented Riemannian manifold  $(M, g)$ , and let  $f : M \rightarrow \mathbb{R}$  be a smooth function. If  $\omega$  is an  $E$ -valued 1-form on  $M$ , then*

$$\begin{aligned} |\omega|^{p-1}\Delta_f|\omega|^{p-1} &\geq \langle |\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega) \rangle + |\omega|^{2p-4} \sum_{i=1}^n \langle \omega(\text{Ric}_f^M(e_i)), \omega(e_i) \rangle \\ &\quad - |\omega|^{2p-4} \sum_{i,j} \langle R^E(e_i, e_j)\omega(e_i), \omega(e_j) \rangle. \end{aligned}$$

#### 4. Weighted $p$ -harmonic maps

In this section, we obtain some Liouville-type theorems for weighted  $p$ -harmonic maps as an application of the Bochner–Weitzenböck formula stated in Section 3. The following theorem shows that the same result holds for  $f$ -weighted  $p$ -harmonic maps with  $L_f^{2p-2}$ -finite energy for  $p > 1$  as in the case of  $f$ -weighted  $L_f^{2p-2}$   $p$ -harmonic 1-forms.

**Theorem 4.1.** *Let  $u : (M, g, e^{-f} dv_g) \rightarrow (N, h)$  be an  $f$ -weighted  $p$ -harmonic map from an oriented complete noncompact  $f$ -weighted manifold into a Riemannian manifold for  $p > 1$ . Suppose that  $f$  is bounded. Assume that the Bakry–Émery Ricci curvature of  $M$  is nonnegative,  $\text{Ric}_f^M \geq 0$ , and the sectional curvature of  $N$  is nonpositive,  $K^N \leq 0$ . If  $u$  has finite  $f$ -weighted  $(2p-2)$ -energy, i.e.,*

$$\int_M |du|^{2p-2} e^{-f} dv_g < \infty,$$

then  $u$  must be a constant map.

*Proof.* Let  $du = \omega$ . Then  $\omega$  is an  $f$ -weighted  $p$ -harmonic 1-form with values in the pull-back bundle  $u^{-1}TN$ . In particular,

$$\delta_f(|\omega|^{p-2}\omega) = 0.$$

From [Corollary 3.2](#) together with curvature conditions, it follows that

$$(4-1) \quad |\omega|^{p-1} \Delta_f |\omega|^{p-1} \geq \langle |\omega|^{p-2} \omega, \delta_f d(|\omega|^{p-2} \omega) \rangle.$$

From this, we can see that the same argument as in the proof of [Theorem 2.4](#) shows  $\omega = 0$ .  $\square$

From [Corollary 3.2](#), it follows that

$$(4-2) \quad |\omega| \Delta_f |\omega|^{p-1} \geq \langle \omega, \Delta_f (|\omega|^{p-2} \omega) \rangle + |\omega|^{p-2} \sum_{i=1}^n \langle \omega(\text{Ric}_f^M(e_i)), \omega(e_i) \rangle \\ - |\omega|^{p-2} \sum_{i,j} \langle R^E(e_i, e_j) \omega(e_i), \omega(e_j) \rangle.$$

Applying the same argument as in [Theorem 2.5](#) to (4-2), we are able to prove the following theorem.

**Theorem 4.2.** *Let  $u : (M, g, e^{-f} dv_g) \rightarrow (N, h)$  be an  $f$ -weighted  $p$ -harmonic map from an oriented complete noncompact  $f$ -weighted manifold into a Riemannian manifold. Suppose that  $f$  is bounded, and  $\text{Ric}_f^M \geq 0$  and  $K^N \leq 0$ . For  $p \geq 2$ , if  $u$  has finite  $f$ -weighted  $p$ -energy, then  $u$  must be a constant map.*

**Remark 4.3.** In [Theorems 4.1](#) and [4.2](#), without the boundedness of  $f$ , if we assume that  $\text{Ric}_f$  is nonnegative and positive at a point, we can conclude that any  $f$ -weighted  $p$ -harmonic map  $u : (M, g, e^{-f} dv_g) \rightarrow (N, h)$  with finite  $f$ -weighted  $(2p-2)$ -energy or  $p$ -energy for  $p > 1$  from an oriented complete noncompact  $f$ -weighted manifold into a Riemannian manifold of nonpositive sectional curvature,  $K^N \leq 0$ , must be constant.

Applying the argument in [Remark 4.3](#) to gradient shrinking Ricci solitons, we have the following as in the case of  $L_f^p$   $p$ -harmonic 1-forms.

**Corollary 4.4.** *Let  $(M, g, e^{-f} dv_g)$  be a complete noncompact gradient shrinking Ricci soliton satisfying  $\text{Ric} + \text{Hess}(f) = \lambda g$  with  $\lambda > 0$ , constant. If  $u : (M, g, e^{-f} dv_g) \rightarrow (N, h)$  is an  $f$ -weighted  $p$ -harmonic map into a Riemannian manifold of nonpositive sectional curvature  $K^N \leq 0$  with finite  $f$ -weighted  $(2p-2)$ -energy or  $p$ -energy for  $p > 1$ , then  $u$  must be a constant map.*

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