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# A COMBINATORIAL IDENTITY FOR THE JACOBIAN OF $t$-SHIFTED INVARIANTS

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**Let $\mathfrak{g}$ be a simple Lie algebra. There are classical formulas for the Jacobians of the generating invariants of the Weyl group of $\mathfrak{g}$ and of the images under the Harish-Chandra projection of the generators of the centre $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$. We present a modification of these formulas related to Takiff Lie algebras.**

## Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Let $\Delta \subset \mathfrak{h}^*$ be the corresponding root system with $\Delta^+ \subset \Delta$ being the subset of positive roots. Set $n = \operatorname{rk} \mathfrak{g} = \dim \mathfrak{h}$. Define $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. For $\alpha \in \Delta^+$, let $\{f_\alpha, h_\alpha, e_\alpha\} \subset \mathfrak{g}$ be an $\mathfrak{sl}_2$-triple with $e_\alpha \in \mathfrak{g}_\alpha$. Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the Weyl group of $\mathfrak{g}$.

In a basis $\{h_1, \dots, h_n\}$ for $\mathfrak{h}$, the Jacobian $\boldsymbol{J}$ of $P_1, \dots, P_n \in \mathcal{S}(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}^*]$ is a polynomial with the following property:

$$dP_1 \wedge dP_2 \wedge \cdots \wedge dP_n = \boldsymbol{J}\, dh_1 \wedge \cdots \wedge dh_n.$$

Up to the sign, $\boldsymbol{J}$ is independent of the order of the elements $P_i$ ; up to a nonzero scalar, $\boldsymbol{J}$ is independent of the choice of a basis for $\mathfrak{h}$. We set

$$J(\{P_i\}) = J(P_1, \dots, P_n) = \boldsymbol{J}.$$

Suppose $\{\hat{P}_1, \dots, \hat{P}_n\} \subset \mathcal{S}(\mathfrak{h})^W$ is a set of homogeneous generating invariants. Then

$$J(\{\hat{P}_i\}) = \prod_{\alpha \in \Delta^+} h_\alpha \quad \text{under a suitable normalisation of the elements } \hat{P}_i$$

by a classical argument, which is presented, for example, in [Humphreys 1990, Section 3.13].

Set $d_i := \deg \hat{P}_i$. Let $\mathcal{U}(\mathfrak{g}) = \bigcup_{d=0}^{\infty} \mathcal{U}_d(\mathfrak{g})$ be the canonical filtration on the enveloping algebra $\mathcal{U}(\mathfrak{g})$, let $\mathcal{Z}(\mathfrak{g})$ stand for the centre of $\mathcal{U}(\mathfrak{g})$. Then $\mathcal{Z}(\mathfrak{g})$ has a set $\{\mathcal{P}_i \mid 1 \leqslant i \leqslant n\}$ of algebraically independent generators such that $\mathcal{P}_i \in \mathcal{U}_{d_i}(\mathfrak{g})$.

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Let $P_i \in \mathcal{S}(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}^*]$ be the image of $\mathcal{P}_i$ under the Harish-Chandra projection. Define $\hat{P}_i \in \mathbb{C}[\mathfrak{h}^*]$ by $\hat{P}_i(x) = P_i(x-\rho)$ for $x \in \mathfrak{h}^*$. Then $\hat{P}_i \in \mathcal{S}(\mathfrak{h})^W$; see, e.g., [Dixmier 1974, Section 7.4], and

$$J(\{P_i\}) = \prod_{\alpha\in\Delta^+} (h_\alpha + \rho(h_\alpha)) \text{ under a suitable normalisation of the elements } \mathcal{P}_i.$$

For any complex Lie algebra $\mathfrak{l}$, let $\varpi\colon \mathcal{S}(\mathfrak{l}) \to \mathcal{U}(\mathfrak{l})$ be the canonical symmetrisation map. Let $\mathcal{S}(\mathfrak{l})^{\mathfrak{l}}$ denote the ring of symmetric $\mathfrak{l}$-invariants. Since $\varpi$ is an isomorphism of $\mathfrak{l}$-modules, it provides an isomorphism of vector spaces $\mathcal{S}(\mathfrak{l})^{\mathfrak{l}} \cong \mathcal{Z}(\mathfrak{l})$.

Suppose next that $\mathcal{P}_i = \varpi(H_i)$ is the symmetrisation of $H_i$ and that $H_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is a homogeneous generator of degree $d_i$. Let $T : \mathfrak{g} \to \mathfrak{g}[t]$ be the $\mathbb{C}$-linear map sending each $x \in \mathfrak{g}$ to $xt$. Here $\mathfrak{g}[t]$ is the current algebra associated with $\mathfrak{g}$, where $[\xi t^a, \eta t^b] = [\xi, \eta] t^{a+b}$ for $\xi, \eta \in \mathfrak{g}$, $a, b \in \mathbb{Z}_{\geqslant 0}$. The map $T$ extends uniquely to the commutative algebras homomorphism

$$T : \mathcal{S}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g}[t]).$$

Set $H_i^{[1]} = T(H_i)$ and $\mathcal{P}_i^{[1]} = \varpi(H_i^{[1]})$. Here $\mathcal{P}_i^{[1]} \in \mathcal{U}(t\mathfrak{g}[t])$.

The triangular decomposition of $\mathfrak{g}$ extends to $\mathfrak{g}[t]$ as $\mathfrak{g}[t] = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t]$. Let $P_i^{[1]} \in \mathcal{S}(t\mathfrak{h}[t])$ be the image of $\mathcal{P}_i^{[1]}$ under the Harish-Chandra projection. For $h_j \in \mathfrak{h}$, let $D_{h_j}$ be the unique derivation of $\mathcal{S}(t\mathfrak{h}[t])$ such that

$$D_{h_j}(yt^k) = kt^{k-1} \partial_{h_j} y$$

if $k \geqslant 1$ and $y \in \mathfrak{h}$. We define the Jacobian $J(\{P_i^{[1]}\}) = J(P_1^{[1]}, \dots, P_n^{[1]})$ by

$$J(\{P_i^{[1]}\}) = \det(D_{h_j} P_i^{[1]})|_{t=1}.$$

Note that $J(\{P_i^{[1]}\}) \in \mathcal{S}(\mathfrak{h})$.

**Theorem 1.** *Under a suitable normalisation of the elements $H_i$, we have the identity*

$$J(\{P_i^{[1]}\}) = \prod_{\alpha\in\Delta^+} (h_\alpha + \rho(h_\alpha) + 1).$$

Our proof of Theorem 1 interprets the zero set of $J(\{P_i^{[1]}\})$ in terms of the *Takiff Lie algebra* $\mathfrak{q} = \mathfrak{g}[u]/(u^2)$ and then uses the *extremal projector* associated with $\mathfrak{g}$; see Section 2A for the definition.

Takiff [1971, Corollary 11.3] proved that $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a polynomial ring whose Krull dimension equals $2 \operatorname{rk} \mathfrak{g}$. This has started a serious investigation of these Lie algebras and their generalisations [Chari and Greenstein 2009; 2011; Khare and Ridenour 2012; Greenstein and Mazorchuk 2017], see also [Panyushev and Yakimova 2020] and references therein. Verma modules and an analogue of the Harish-Chandra homomorphism for $\mathfrak{q}$ were defined and studied in [Geoffriau 1995; Wilson 2011]. We remark that $\mathfrak{q}$-modules appearing in this paper are essentially different.

## 1. Several combinatorial formulas

Keep the notation of the introduction. In particular, $H_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ stands for a homogeneous generator of degree $d_i$, $P_i$ is the image of $\mathcal{P}_i = \varpi(H_i)$ under the Harish-Chandra projection, $\hat{P}_i \in \mathcal{S}(\mathfrak{h})^W$ is the $(-\rho)$-shift of $P_i$, i.e., $\hat{P}_i(x) = P_i(x-\rho)$ for $x \in \mathfrak{h}^*$, and $P_i^{[1]}$ is the image of $\varpi(T(H_i))$ under the Harish-Chandra projection related to $\mathfrak{g}[t]$. Also let $P_i^\circ$ be the highest degree component of $P_i$. Then $P_i^\circ = H_i|_{\mathfrak{h}}$. By the Chevalley restriction theorem, the polynomials $P_i^\circ$ with $1 \leqslant i \leqslant n$ generate $\mathcal{S}(\mathfrak{h})^W$. It is clear that $J(\{P_i^\circ\}) = J(\{\hat{P}_i\})$. Let a choice of $H_1, \ldots, H_n$ be such that $J(\{P_i^\circ\}) = \prod_{\alpha\in\Delta^+} h_\alpha$.

**Lemma 1.1.** *The highest degree component of $J(\{P_i^{[1]}\})$ is equal to $\prod_{\alpha\in\Delta^+} h_\alpha$.*

*Proof.* The highest degree component of $P_i^{[1]}$ is $T(P_i^\circ) \in \mathcal{S}(\mathfrak{h}t)$. Each monomial of $T(P_i^\circ)$ is of the form $(x_1 t)\cdots(x_{d_i} t)$ with $x_j \in \mathfrak{h}$ for each $j$. By the construction, $D_{h_j} T(P_i^\circ)|_{t=1} = \partial_{h_j} P_i^\circ$. The result follows. □

In order to prove the next lemma, we need a well-known equality, namely $\prod_{i=1}^n d_i = |W|$.

**Lemma 1.2.** *For the constant term of the Jacobian $J(\{P_i^{[1]}\})$, we have the formula $J(\{P_i^{[1]}\})(0) = \prod_{\alpha\in\Delta^+}(\rho(h_\alpha)+1)$.*

*Proof.* Clearly $P_i^{[1]}|_{t=1} = P_i$. Since $H_i$ is a homogeneous polynomial of degree $d_i$, the linear in $\mathfrak{h}$ part of $P_i^{[1]}$ has degree $d_i$ in $t$. It follows that

$$J(\{P_i^{[1]}\})(0) = \left(\prod_{i=1}^n d_i\right) J(\{P_i\})(0) = |W| \prod_{\alpha\in\Delta^+} \rho(h_\alpha).$$

We complete the proof with a formula of Kostant which states

$$\prod_{\alpha\in\Delta^+} \frac{\rho(h_\alpha)+1}{\rho(h_\alpha)} = |W^\vee| = |W|. \tag{1-1}$$ □

**Remark 1.3.** The Kostant formula (1-1) is a particular case of another combinatorial statement. Let $W(t) = \sum_{w\in W} t^{l(w)}$ be the Poincaré polynomial of $W$. Then

$$W^\vee(t) = \prod_{\alpha\in\Delta^+} \frac{t^{(\rho,\alpha^\vee)+1}-1}{t^{(\rho,\alpha^\vee)}-1};$$

see equation (34) in [Humphreys 1990, Section 3.20]. Since $\rho(h_\alpha) = (\rho,\alpha^\vee)$, evaluating at $t=1$ one gets exactly (1-1).

**Example 1.4.** Take $\mathfrak{g} = \mathfrak{sl}_2$ with the usual basis $\{e, h, f\}$, where $[h,e] = 2e$. Then $H = H_1 = 2ef + \frac{1}{2}h^2$, $H^{[1]} = 2(et)(ft) + \frac{1}{2}(ht)^2$, and

$$\mathcal{P}^{[1]} = \varpi(H^{[1]}) = (et)(ft) + (ft)(et) + \tfrac{1}{2}(ht)^2 = \tfrac{1}{2}(ht)^2 + ht^2 + 2(ft)(et).$$

Therefore $P^{[1]} = P_1^{[1]} = \frac{1}{2}(ht)^2 + ht^2$. Computing the partial derivative and evaluating at $t = 1$, we obtain $J(\{P^{[1]}\}) = h + 2 = h + \rho(h) + 1$.

## 2. Takiff Lie algebras and branching

Theorem 1 can be interpreted as a statement in representation theory of Takiff Lie algebras

$$\mathfrak{q} = \mathfrak{g} \ltimes \mathfrak{g}^{\mathrm{ab}} \cong \mathfrak{g}[u]/(u^2).$$

The first factor, the nonabelian copy of $\mathfrak{g}$, acts on $\mathfrak{g}^{\mathrm{ab}} = \mathfrak{g}u$ as a subalgebra of $\mathfrak{gl}(\mathfrak{g})$. Therefore there is the canonical embedding $\mathfrak{q} \subset \mathfrak{gl}(\mathfrak{g}) \ltimes \mathfrak{g}^{\mathrm{ab}}$. Set $\ell = \dim \mathfrak{g} + 1$. In its turn, $\mathfrak{gl}(\mathfrak{g}) \ltimes \mathfrak{g}^{\mathrm{ab}}$ can be realised as a subalgebra of $\mathfrak{gl}(\mathfrak{g} \oplus \mathbb{C}) \cong \mathfrak{gl}_\ell(\mathbb{C})$. The Lie algebra $\mathfrak{gl}_\ell(\mathbb{C})$ is equipped with the standard triangular decomposition. Let $\mathfrak{b}_\ell \subset \mathfrak{gl}_\ell(\mathbb{C})$ be the corresponding positive Borel. Recall that we have chosen a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$. We fix an embedding $\mathfrak{gl}(\mathfrak{g}) \ltimes \mathfrak{g}^{\mathrm{ab}} \subset \mathfrak{gl}(\mathfrak{g} \oplus \mathbb{C})$ such that $\mathfrak{b}^- \ltimes \mathfrak{g}^{\mathrm{ab}}$ lies in the opposite Borel $\mathfrak{b}_\ell^-$ and $\mathfrak{b} \subset \mathfrak{b}_\ell$.

Let $\psi : \mathcal{S}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{q})$ be a derivation defined uniquely by $\psi(\xi) = \xi u$ for any $\xi \in \mathfrak{g}$. Set $\mathcal{R}_i = \varpi(\psi(H_i))$. The elements $\mathcal{R}_1, \ldots, \mathcal{R}_n \in \mathcal{U}(\mathfrak{q})$ are not necessarily central. If we assume that $d_1 = 2$, then $\mathcal{R}_1 \in \mathcal{Z}(\mathfrak{q})$. However, since both maps, $\psi$ and $\varpi$, are homomorphisms of $\mathfrak{g}$-modules, each $\mathcal{R}_i$ commutes with $\mathfrak{g}$. Note that the elements $\mathcal{R}_i$ have degree 1 in $\mathfrak{g}u$. They are crucial for further considerations and our next goal is to relate them to $\mathcal{P}_i^{[1]} \in \mathcal{U}(t\mathfrak{g}[t])$.

The map $\psi$ is also well defined for the tensor algebra of $\mathfrak{g}$, but not for $\mathcal{U}(\mathfrak{g})$, because of the following obstacle:

$$\begin{aligned}\psi(\xi_1\xi_2 - \xi_2\xi_1) &= (\xi_1 u)\xi_2 + \xi_1(\xi_2 u) - (\xi_2 u)\xi_1 - \xi_2(\xi_1 u)\\ &= [\xi_1 u, \xi_2] + [\xi_1, \xi_2 u] = 2[\xi_1, \xi_2]u \neq [\xi_1, \xi_2]u.\end{aligned}$$

The remedy is to pass to the current algebras $\mathfrak{g}[t]$ and $\mathfrak{q}[t]$.

**Lemma 2.1.** *There is a well-defined $\mathbb{C}$-linear map $\mathcal{T} : \mathcal{U}(t\mathfrak{g}[t]) \to \mathcal{U}(\mathfrak{q}[t])$ such that*

- $\mathcal{T}(\xi t^k) = k(\xi u)t^{k-1}$ *for each* $\xi \in \mathfrak{g}$,
- $\mathcal{T}(ab) = \mathcal{T}(a)b + a\mathcal{T}(b)$ *for all* $a, b \in \mathcal{U}(t\mathfrak{g}[t])$, *i.e.,* $\mathcal{T}$ *is a derivation.*

*Proof.* Take $\xi, \eta \in \mathfrak{g}$ and $k, m \geqslant 1$. Then

$$\begin{aligned}&\mathcal{T}(\xi t^k \eta t^m - \eta t^m \xi t^k)\\ &\qquad = k(\xi u)t^{k-1}\eta t^m + m\xi t^k(\eta u)t^{m-1} - m(\eta u)t^{m-1}\xi t^k - k\eta t^m(\xi u)t^{k-1}\\ &\qquad = k[\xi u, \eta]t^{k-1+m} + m[\xi, \eta u]t^{k+m-1}\\ &\qquad = (k+m)([\xi, \eta]u)t^{k+m-1} = \mathcal{T}([\xi, \eta]t^{k+m}).\end{aligned}$$

□

For a monomial $\boldsymbol{\xi} = \xi_1 \cdots \xi_d$ with $\xi_i \in \mathfrak{g}$, we have $\mathcal{T} \circ \varpi \circ T(\boldsymbol{\xi})|_{t=1} = \varpi \circ \psi(\boldsymbol{\xi})$. Hence, by the construction, $\mathcal{R}_i = \mathcal{T}(\mathcal{P}_i^{[1]})|_{t=1}$.

A word of caution: in $\mathcal{U}(\mathfrak{b}^-)(\mathfrak{n}^+u)$ and similar expressions, $(\mathfrak{n}^+u)$ stands for the subspace $\mathfrak{n}^+u \subset \mathfrak{g}^{\mathrm{ab}}$ and *not* for an ideal generated by $\mathfrak{n}^+u$. The same applies to $(\mathfrak{b}u)$, $(\mathfrak{g}u)$, etc.

**Lemma 2.2.** *Let $M(\lambda) = \mathcal{U}(\mathfrak{b}_\ell^-)v_\lambda$ with $\lambda \in \mathbb{C}^\ell$ be a Verma module of $\mathfrak{gl}(\mathfrak{g}\oplus\mathbb{C})$. Set $\mu = \lambda|_{\mathfrak{h}}$. There exists a nontrivial linear combination $\mathcal{R} = \sum c_i \mathcal{R}_i$ such that $\mathcal{R}v_\lambda \in \mathfrak{n}^-\mathcal{U}(\mathfrak{b}^-)(\mathfrak{n}^+u)v_\lambda$ if and only if $J(\{P_i^{[1]}\})(\mu) = 0$.*

*Proof.* We have

$$\mathcal{P}_i^{[1]} \in P_i^{[1]} + \mathfrak{n}^-[t]\mathcal{U}(\mathfrak{g}[t])\mathfrak{n}^+[t].$$

Accordingly, $\mathcal{R}_i = \mathcal{T}(P_i^{[1]})|_{t=1} + \mathcal{X}$, where $\mathcal{X}$ is the image of the second summand of $\mathcal{P}_i^{[1]}$. Let $X = x_1 \cdots x_r$ be a monomial appearing in $\mathcal{X}$. If $x_r \in \mathfrak{n}^+$, then $Xv_\lambda = 0$. Assume that $Xv_\lambda \neq 0$. Then necessarily $x_r \in \mathfrak{n}^+u$ and $x_1, \ldots, x_{r-1} \in \mathfrak{g}$. If $x_i \in \mathfrak{n}^+$ for some $i \leqslant (r-1)$, then we replace $X$ by $x_1 \cdots x_{i-1}[x_i, x_{i+1}, \ldots, x_r]$. Note that here $[x_i, x_r] \in \mathfrak{n}^+u$. Applying this procedure as often as possible one replaces $X$ by an element of $\mathcal{U}(\mathfrak{b}^-)(\mathfrak{n}^+u)$ without altering $Xv_\lambda$. Since $X$ is an invariant of $\mathfrak{h}$, the new element lies in $\mathfrak{n}^-\mathcal{U}(\mathfrak{b}^-)(\mathfrak{n}^+u)$. Summing up,

$$\mathcal{R}_i v_\lambda \in \mathcal{T}(P_i^{[1]})|_{t=1} v_\lambda + \mathfrak{n}^-\mathcal{U}(\mathfrak{b}^-)(\mathfrak{n}^+u)v_\lambda. \tag{2-1}$$

Further

$$\mathcal{T}(P_i^{[1]})|_{t=1} = \sum_{j=1}^{n} (D_{h_j} P_i^{[1]})|_{t=1} h_j u,$$

where $D_{h_j}$ are the same as in the introduction. Recall that they have been used in order to define $J(\{P_i^{[1]}\})$. Hence $J(\{P_i^{[1]}\})(\mu) = 0$ if and only if there is a nonzero vector $\bar{c} = (c_1, \ldots, c_n)$ such that $\mathcal{A}(\mu) = \sum c_i \mathcal{T}(P_i^{[1]})|_{t=1,\mu} = 0$. This shows that if $J(\{P_i^{[1]}\})(\mu) = 0$, then $\mathcal{R}v_\lambda \in \mathfrak{n}^-\mathcal{U}(\mathfrak{b}^-)(\mathfrak{n}^+u)v_\lambda$.

Suppose now that $\mathcal{R}v_\lambda \in \mathfrak{n}^-\mathcal{U}(\mathfrak{b}^-)(\mathfrak{n}^+u)v_\lambda \subset \mathcal{U}(\mathfrak{n}^-)(\mathfrak{n}^+u)v_\lambda$. Then

$$\mathcal{A}(\mu)v_\lambda \in \mathcal{U}(\mathfrak{n}^-)(\mathfrak{n}^+u)v_\lambda.$$

We are working with a Verma module of $\mathfrak{gl}_\ell(\mathbb{C})$; the subspaces $\mathfrak{n}^+u$ and $\mathfrak{h}u$ are contained in $\mathfrak{n}_\ell^-$, and $\mathcal{A}(\mu) \in \mathfrak{h}u$. Hence $\mathcal{A}(\mu)$ must be an element of $\mathcal{U}(\mathfrak{n}^-)(\mathfrak{n}^+u)$. At the same time, $\mathfrak{h}u \cap \mathfrak{n}^+u = 0$. Therefore $\sum_{i=1}^{n} c_i (D_{h_j} P_i^{[1]})|_{t=1,\mu} = 0$ for each $j$ and thus $J(\{P_i^{[1]}\})(\mu) = 0$. □

For $\gamma \in \mathfrak{h}^*$, let $M(\lambda|_{\mathfrak{q}})_\gamma$ be the corresponding weight subspace of $\mathcal{U}(\mathfrak{q})v_\lambda \subset M(\lambda)$. Since $\mathfrak{h}u \subset \mathfrak{n}_\ell^-$, either $M(\lambda|_{\mathfrak{q}})_\gamma = 0$ or $\dim M(\lambda|_{\mathfrak{q}})_\gamma = \infty$. We also have $(\mathfrak{h}u)v_\lambda \neq 0$. Because of these facts, the $\mathfrak{q}$-modules $M(\lambda)$ and $\mathcal{U}(\mathfrak{q})v_\lambda$ do not fit in the framework of the highest weight theory developed in [Geoffriau 1995; Wilson 2011]. Nevertheless, they may have some nice features.

Lemma 2.2 relates $J(\{P_i^{[1]}\})$ to a property of the branching $\mathfrak{q} \downarrow \mathfrak{g}$ in a particular case of the $\mathfrak{q}$-module $\mathcal{U}(\mathfrak{q})v_\lambda$. In order to get a better understanding of this branching problem, we employ a certain projector introduced by Asherova, Smirnov, and Tolstoy in [Asherova et al. 1971].

**2A.** ***The extremal projector.*** Recall that $\{f_\alpha, h_\alpha, e_\alpha\} \subset \mathfrak{g}$ is the $\mathfrak{sl}_2$-triple corresponding to $\alpha \in \Delta^+$. Set $N = |\Delta^+|$ and choose a numbering of positive roots, $\alpha_1, \dots, \alpha_N$. Define

$$p_\alpha = 1 + \sum_{k=1}^{\infty} f_\alpha^k e_\alpha^k \frac{(-1)^k}{k!(h_\alpha + \rho(h_\alpha) + 1) \cdots (h_\alpha + \rho(h_\alpha) + k)}.$$

Each $p_\alpha$, as well as any product of finitely many of them, is regarded as an element of the algebra of formal series of monomials

$$f_{\alpha_1}^{r_1} \cdots f_{\alpha_N}^{r_N} e_{\alpha_N}^{k_N} \cdots e_{\alpha_1}^{k_1} \text{ such that } (k_1 - r_1)\alpha_1 + \cdots + (k_N - r_N)\alpha_N = 0$$

with coefficients in the field of fractions of the commutative algebra $\mathcal{U}(\mathfrak{h})$. A total order on $\Delta^+$ is said to be *normal* (or *convex*) if either $\alpha < \alpha+\beta < \beta$ or $\beta < \alpha+\beta < \alpha$ for each pair of positive roots $\alpha, \beta$ such that $\alpha+\beta \in \Delta$. There is a bijection between the normal orders and the reduced decompositions of the longest element of $W$.

Choose now a normal order $\alpha_1 < \alpha_2 < \cdots < \alpha_N$, and define

$$p = p_{\alpha_1} \cdots p_{\alpha_N}$$

accordingly. The element $p$ is known as the *extremal projector* [Asherova et al. 1971]. It is independent of the choice of a normal order and $p^2 = p$. For proofs and more details on this operator, see, e.g., [Molev 2007, Section 9.1]. Most importantly, it has the property that

$$e_\alpha p = p f_\alpha = 0 \tag{2-2}$$

for each $\alpha \in \Delta^+$.

The nilpotent radical $\mathfrak{n}_\ell \subset \mathfrak{b}_\ell$ acts on $M(\lambda)$ locally nilpotently. Recall that $\mathfrak{n}^+ \subset \mathfrak{n}_\ell$. Let $v \in M(\lambda)$ be an eigenvector of $\mathfrak{h}$ of weight $\gamma \in \mathfrak{h}^*$. The element $pv$ is a finite sum of vectors $q_j v_j$, where $q_j \in \operatorname{Quot}\mathcal{U}(\mathfrak{h}) \cong \mathbb{C}(\mathfrak{h}^*)$ and $v_j \in M(\lambda)$. If all the appearing denominators are nonzero at $\gamma$, then $pv$ is a well-defined vector of $M(\lambda)$ of the same weight $\gamma$.

## 3. Proof of Theorem 1

Let $\lambda$, $\mu$, and $M(\lambda)$ be as in Lemma 2.2. Keep in mind that $\lambda$ and $\mu$ are arbitrary elements of $\mathbb{C}^\ell$ and $\mathbb{C}^n$. Since each $\mathcal{R}_i$ commutes with $\mathfrak{g}$, each $\mathcal{R}_i v_\lambda$ is a highest weight vector of $\mathfrak{g}$.

We use the extremal projector $p$ associated with $\mathfrak{g}$. If $p$ can be applied to a highest weight vector $v$, then $pv = v$. Suppose that $p$ is defined at $\mu$. Then, in view of (2-1) and (2-2),

$$\mathcal{R}_i v_\lambda = p\mathcal{R}_i v_\lambda = p\mathcal{T}(P_i^{[1]})|_{t=1} v_\lambda.$$

Assume that $J(\{P_i^{[1]}\})(\mu)=0$. Then there is a nontrivial linear combination $\mathcal{R}=\sum c_i\mathcal{R}_i$ such that

$$\mathcal{R}v_\lambda \in \mathfrak{n}^-\mathcal{U}(\mathfrak{b}^-)(\mathfrak{n}^+u)v_\lambda;$$

see Lemma 2.2. Here $p\mathcal{R}v_\lambda=0$ and hence $\mathcal{R}v_\lambda=0$ as well.

Since we are considering a Verma module of $\mathfrak{gl}_\ell(\mathbb{C})$, this implies that

$$\mathcal{R}\in\mathcal{U}(\mathfrak{gl}_\ell(\mathbb{C}))\mathfrak{b}_\ell\cap\mathcal{U}(\mathfrak{q})=\mathcal{U}(\mathfrak{q})\mathfrak{b}.$$

Hence the symbol $\mathrm{gr}(\mathcal{R})$ of $\mathcal{R}$ lies in the ideal of $\mathcal{S}(\mathfrak{q})$ generated by $\mathfrak{b}$.

The decomposition $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{b}$ defines a bigrading on $\mathcal{S}(\mathfrak{g})$. Let $H_i^\bullet$ be the bihomogeneous component of $H_i$ having the highest degree with respect to $\mathfrak{n}^-$. According to [Panyushev and Yakimova 2012, Section 3], $H_i^\bullet\in\mathfrak{b}\mathcal{S}^{d_i-1}(\mathfrak{n}^-)$ and the polynomials $H_1^\bullet,\dots,H_n^\bullet$ are algebraically independent. We have

$$\psi(H_i^\bullet)\in(\mathfrak{b}u)\mathcal{S}^{d_i-1}(\mathfrak{n}^-)\oplus\mathfrak{b}(\mathfrak{n}^-u)\mathcal{S}^{d_i-2}(\mathfrak{n}^-).$$

Write this as $\psi(H_i^\bullet)\in H_{i,1}+\mathfrak{b}(\mathfrak{n}^-u)\mathcal{S}^{d_i-2}(\mathfrak{n}^-)$. Then the polynomials $H_{i,1}$ with $1\leqslant i\leqslant n$ are still algebraically independent. As can be easily seen, $\psi(H_i)\in H_{i,1}+\mathfrak{b}\mathcal{S}(\mathfrak{q})$.

Set $d=\max_{i:\,c_i\neq 0}d_i$. Then

$$\mathrm{gr}(\mathcal{R})=\sum_{i:\,d_i=d}c_i\psi(H_i)$$

and it lies in $(\mathfrak{b})\lhd\mathcal{S}(\mathfrak{q})$ if and only if $\sum_{i:\,d_i=d}c_iH_{i,1}=0$. Since at least one $c_i$ in this linear combination is nonzero, we get a contradiction. The following is settled: if $p$ is defined at $\mu$, then $J(\{P_i^{[1]}\})(\mu)\neq 0$.

Now we know that the zero set of $J(\{P_i^{[1]}\})$ lies in the union of hyperplanes $h_\alpha+\rho(h_\alpha)=-k$ with $k\geqslant 1$. At the same time, this zero set is an affine subvariety of $\mathbb{C}^n$ of codimension one. By Lemma 1.1, the highest degree component of $J(\{P_i^{[1]}\})$ is equal to $\prod_{\alpha\in\Delta^+}h_\alpha$. Therefore the zero set of $J(\{P_i^{[1]}\})$ is the union of $N$ hyperplanes and $J(\{P_i^{[1]}\})$ is the product of $N$ linear factors of the form $(h_\alpha+\rho(h_\alpha)+k_\alpha)$. Moreover, each $\alpha\in\Delta^+$ must appear in exactly one linear factor of $J(\{P_i^{[1]}\})$.

By Lemma 1.2, $J(\{P_i^{[1]}\})(0)=\prod_{\alpha\in\Delta^+}(\rho(h_\alpha)+1)$. Hence we have

$$\prod_{\alpha\in\Delta^+}\frac{\rho(h_\alpha)+k_\alpha}{\rho(h_\alpha)+1}=1.$$

Since $\rho(h_\alpha),k_\alpha\geqslant 1$, each $k_\alpha$ is equal to 1. □

## 4. Conclusion

The elements $\mathcal{R}_i$ are rather natural $\mathfrak{g}$-invariants in $\mathcal{U}(\mathfrak{q})$ of degree one in $\mathfrak{g}u$. Note that because $\mathfrak{g}u$ is an abelian ideal of $\mathfrak{q}$, there is no ambiguity in defining the degree in $\mathfrak{g}u$.

The involvement of these elements in the branching rules $\mathfrak{q} \downarrow \mathfrak{g}$ remains unclear. However, combining Lemma 2.2 with Theorem 1, we obtain the following statement.

**Corollary 4.1.** *In the notation of Lemma 2.2, there is a nontrivial linear combination $\mathcal{R} = \sum c_i \mathcal{R}_i$ such that $\mathcal{R} v_\lambda \in \mathfrak{n}^- \mathcal{U}(\mathfrak{q}) v_\lambda$ if and only if $\mu(h_\alpha) = -\rho(h_\alpha) - 1$ for some $\alpha \in \Delta^+$.* □

As the theory of finite-dimensional representations suggests, it is unusual for a highest weight vector of $\mathfrak{g}$ to belong to the image of $\mathfrak{n}^-$. The proof of Theorem 1 shows that $\mathcal{R} v_\lambda \neq 0$ for the linear combination of Corollary 4.1.

**Remark 4.2.** The subspace $\mathcal{V}[1] = (\mathcal{U}(\mathfrak{g})(\mathfrak{g}u))^{\mathfrak{g}} \subset \mathcal{U}(\mathfrak{q})$ is a $\mathcal{Z}(\mathfrak{g})$-module. From a well-known description of $(\mathfrak{g} \otimes \mathcal{S}(\mathfrak{g}))^{\mathfrak{g}}$, one can deduce that $\mathcal{V}[1]$ is freely generated by $\mathcal{R}_1, \dots, \mathcal{R}_n$ as a $\mathcal{Z}(\mathfrak{g})$-module. There are other choices of generators in $\mathcal{V}[1]$ and it is not clear, whether one can get nice formulas for the corresponding Jacobians.

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