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COHOMOLOGICAL KERNELS OF PURELY INSEPARABLE FIELD EXTENSIONS

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Let F be a field of characteristic p and L a finite purely inseparable extension. The kernel $H_p^{n+1}(L/F) = \ker(H_p^{n+1}F \to H_p^{n+1}L)$ has been described by Sobiech, and independently when $p = 2$ by Aravire, Laghribi, and O'Ryan. When L has exponent 1, the kernel is the sum of the kernels of the simple subextensions, but when L has larger exponent it is significantly more complex. This paper determines $H_{p^m}^{n+1}(L/F)$ for $L = F(\sqrt[p^e]{x})$ and all $m, n, e \geq 1$. Whereas the results when $m = 1$ used the theory of differential forms, the results for $m > 1$ require the de Rham Witt complex. The $m > 1$ case clarifies why the "messy" generators in the $m = 1$ case arose, as they emerge from relations in the de Rham Witt complex. As a corollary, when L is modular over F and m exceeds the exponent of L, the kernel $H_{p^m}^2(L/F)$ is the sum of the kernels of the simple subextensions.

Introduction

We assume throughout F is a field of characteristic p. We consider finite extensions L of F that are purely inseparable and modular. This means that L is a tensor product of simple purely inseparable extensions, that is, for some $x_1, x_2, \ldots, x_s \in F$ which are p-independent, $L = F(\alpha_1, \alpha_2, \ldots, \alpha_s)$, where $\alpha_i^{p^{r_i}} = x_i$ for $r_i \in \mathbb{N}$. The problem of interest is to determine $\ker(i_{L/F} : H_{p^m}^{n+1}F \to H_{p^m}^{n+1}L)$ for $m, n, s \geq 1$. When $m = 1$ this problem has a comprehensive solution due to Sobiech [2018], which coincides with the independent work of Aravire, Laghribi, and O'Ryan [Aravire et al. 2019] when $p = 2$. Therefore our interest here is in the case $m > 1$. This paper treats the case where $s = 1$ and all m, n, and as a corollary obtains the result in the case for all m and s where $n + 1 = 2$. The general case of $n + 1 \geq 3$ is significantly more complicated and requires an analysis of the kernel of de Rham Witt modules $\ker : W_m\Omega_F^n \to W_m\Omega_{F(\sqrt[p^e]{x})}^n$. This will be addressed in subsequent work.

As in [Sobiech 2018; Aravire et al. 2019], the results depend upon the theory of differential forms in characteristic p. We assume familiarity with the differential

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modules Ω_F^n and the exact sequence

$$0 \to \nu_F(n) \to \Omega_F^n \xrightarrow{\wp} \Omega_F^n / d\Omega_F^{n-1} \to H_p^{n+1} F \to 0$$

which defines the two groups $\nu_F(n)$ and $H_p^{n+1}F$. Kato [1982] showed that $\nu_F(n) \cong K_n^M F / p K_n^M F$ (Milnor K-theory) and when $n = 1$ the group $H_p^2 F \cong \mathrm{Br}_2 F$ is the p-torsion in the Brauer group of F (first observed in different notation by Witt). The map $\wp$ is $\wp = \Phi - 1$, where

$$\Phi\left(a \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}\right) = a^p \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \pmod{d\Omega_F^{n-1}}$$

is the Frobenius map. A good reference for this background is [Gille and Szamuely 2006, Chapter 9].

A finite extension L/F is *modular* if there exist p-independent elements

$$x_1, \ldots, x_s \in F \quad \text{with } L = F(\sqrt[p^{r_1}]{x_1}, \ldots, \sqrt[p^{r_s}]{x_s}).$$

If $r = \max\{r_i \mid 1 \le i \le s\}$ then r is called the *exponent* of L (over F.) The exponent 1 case has a particularly nice answer and motivated this work (see [Aravire and Baeza 2003; 2013] for $p = 2$, and [Sobiech 2018] for $p \ge 2$.) Here we have $L = F(\sqrt[p]{x_1}, \ldots, \sqrt[p]{x_s})$ and the result is

$$\ker(H_p^{n+1} F \to H_p^{n+1} L) = \sum_{i=1}^{s} \overline{\Omega_F^{n-1} \wedge dx_i} = \sum_{i=1}^{s} \ker\bigl(H_p^{n+1} F \to H_p^{n+1} F(\sqrt[p]{x_i})\bigr),$$

where the bar denotes cohomology classes. We remark that in the separable case counterexamples to results like this abound and have an interesting history. For example it is known that the analogue of this result fails for separable triquadratic extensions where there can be "nonobvious" elements in the kernel (see [Amitsur et al. 1979; Elman et al. 1980]) and also for separable elementary abelian extensions of degree p^2 when p is odd (see [Tignol 1987]). These results played a key role in finding indecomposable division algebras (see [Tignol 1987; Jacob 1991]) and provide motivation for looking for generalizations.

In [Sobiech 2018; Aravire et al. 2019], the kernel $\ker(H_p^{n+1} F \to H_p^{n+1} L)$ is computed for arbitrary modular L, but it does not have a simple characterization. In fact, even when $s = 1$ the kernel has a messy description (see Theorem 4.1 below.) The main result of this paper is that this can be remedied if one looks instead at $H_{p^m}^{n+1} F$, where m is the exponent of the extension L. When $L = F(\sqrt[p^m]{x})$, Theorem 4.2 gives the direct analogue of the exponent 1 case,

$$\ker(H_{p^m}^{n+1} F \to H_{p^m}^{n+1} L) = \overline{W_m \Omega_F^{n-1} \wedge d[x]_m}.$$

In this description, for $x \in F$, we denote $[x]_m = (x, 0, \ldots, 0) \in W_m F$ the multiplicative representative in the ring of Witt vectors $W_m F$, and $W_m \Omega_F^n$ is the de Rham Witt module discussed below (see [Illusie 1979] for details). With this, by considering the image under the injection $V^{m-1} : H_p^{n+1}F \to H_{p^m}^{n+1}F$, one obtains a simplification of the of the cases where m is less than the exponent of L and a better understanding of the origin of the formulae that emerged in [Sobiech 2018; Aravire et al. 2019] when $m = 1$. As a consequence, when $n+1 = 2$ one obtains a purely cohomological approach to an old result of A. A. Albert [1939] (see Corollary 4.3 below). Suppose $x_1, x_2, \dots, x_s \in F$ are p-independent and set $L = F(\alpha_1, \alpha_2, \dots, \alpha_s)$, where $\alpha_i^{p^{r_i}} = x_i$. Set $\widetilde{m} := \max\{r_i \mid 1 \le i \le s\}$ and define $L_i := F(\alpha_i)$. If $m \ge \widetilde{m}$ then

$$\ker(H^2_{p^m}F \to H^2_{p^m}L) = \sum_{i=1}^{s} p^{m-r_i} \cdot \overline{W_m F \cdot d[x_i]_m} = \sum_{i=1}^{s} \ker(H^2_{p^m}F \to H^2_{p^m}L_i).$$

When interpreted in terms of p-algebras this gives Albert's result that if an algebra is split in such a modular extension then it is similar to a corresponding tensor product of cyclic algebras. Of course, the main interest in Theorem 4.2 is that one has results when $n+1 > 2$.

In order to treat the case of $m > 1$ it is necessary to use the de Rham Witt complex. Although the use of the differential modules Ω_F^n has become fairly standard in work in characteristic p, the de Rham Witt W_mF-modules, $W_m\Omega_F^n$, are not so familiar. Details about $W_m\Omega_F^n$ can be found in [Illusie 1979] and also in Section 2 of [Aravire et al. 2018], which gives a detailed introduction and whose results are referenced here. Here is a rough guide about how to think about them.

Loosely speaking one gets intuition by thinking about these modules as $\Omega^n_{W_mF}$, but in fact they are a quotient module. The relations defining the quotient are introduced for several reasons: one is to make sure that the shift operator $V : W_mF \to W_{m+1}F$ generalizes to $V : W_m\Omega_F^n \to W_{m+1}\Omega_F^n$ (we have $W_m\Omega_F^0 = W_mF$), and a second is to assure that the derivative $dx^p = px^{p-1}\,dx$ interacts correctly with the other arithmetic in $W_m\Omega_F^n$. One particularly important relation is for all $x \in W_{m-s}F$ with $m > s \ge 1$ and $a \in F$, we have $V^s(x)\,d[a]_m = V^s(x[a]_{m-s}^{p^s-1})\,dV^s[a]_{m-s} \in W_m\Omega_F^1$ (see Lemma 3.1 below.) This relation generalizes a defining relation for $W_m\Omega_F^n$ and provides the key as to why $\bar{\omega} \in \ker\bigl(H_p^{n+1}F \to H_p^{n+1}F(\sqrt[p^m]{x})\bigr)$ satisfies $V^{m-1}(\bar{\omega}) \in \overline{W_m\Omega_F^{n-1} \wedge d[x]_m}$.

The Frobenius $\Phi : W_mF \to W_mF$ defined by $\Phi(a_1, \dots, a_m) = (a_1^p, \dots, a_m^p)$ extends to $W_m\Omega_F^n$ but in its most general form shifts levels down, $\Phi : W_m\Omega_F^n \to W_{m-1}\Omega_F^n$. When $m = 1$ (where the Frobenius is also known as the inverse Cartier operator), the shift is remedied by working (mod $d\Omega_F^{n-1}$), that is, we have $\Phi : \Omega_F^n \to \Omega_F^n/d\Omega_F^{n-1}$ as described above. The Frobenius on the de Rham Witt modules can also be set up to avoid the shift and this version is $\Phi : W_m\Omega_F^n \to W_m\Omega_F^n/dV^{m-1}\Omega_F^{n-1}$.

To study the cohomology groups $H_{p^m}^{n+1}F$, Izhboldin [1991] defined groups $Q^n(F, m)$ using generators and relations, mimicking the arithmetic in $W_m\Omega_F^n$ (there is a natural surjection $W_m\Omega_F^n \to H_{p^m}^{n+1}F$, see [Aravire et al. 2018, Section 2] for discussion). One of his main results is the exact sequence [Izhboldin 1991, Theorem C],

$$0 \to K_n^M F / p^m K_n^M F \to Q^n(F,m) \xrightarrow{\wp} Q^n(F,m) \to H_{p^m}^{n+1} F \to 0.$$

However for explicit computations, such as those given in this paper, since the $Q^n(F,m)$ are not $W_m F$-modules we will use the following exact sequence to study the group $H_{p^m}^{n+1} F$,

$$W_m \Omega_F^n \xrightarrow{\wp} \frac{W_m \Omega_F^n}{d W_m \Omega_F^{n-1}} \to H_{p^m}^{n+1} F \to 0$$

described in [Aravire et al. 2018, Theorem 2.31]. Here, as in the case $m = 1$, $\wp = \Phi - 1$. Two particularly important properties of Φ in this context are that $\Phi \circ V = V \circ \Phi = p\,\cdot$ (note that the level reduction for Φ is used here), and if $\omega \in W_m \Omega_F^n$ then $\bar{\omega} = \overline{\Phi(\omega)} \in H_{p^m}^{n+1} F$.

Finally, in order to build up results by induction on m it is necessary to use the analogue of the exact sequence of Witt vectors

$$0 \to W_{m-s} F \xrightarrow{V^s} W_m F \xrightarrow{R^s} W_s F \to 0,$$

where

$$V^s(a_1, \dots, a_{m-s}) = (0, \dots, 0, a_1, \dots, a_{m-s})$$

and

$$R^s(a_1, \dots, a_m) = (a_1, \dots, a_s) \in W_s F.$$

In the case of the de Rham Witt Complex, because of the relations, the map V^s fails to be injective so one does not have an exact analogue, but we do have the following exact sequence [Illusie 1979, Proposition 3.2.],

$$\left(V^s W_{m-s} \Omega_F^n + d V^s W_{m-s} \Omega_F^{n-1}\right) \to W_m \Omega_F^n \xrightarrow{R^s} W_s \Omega_F^n \to 0,$$

where the first map is subgroup inclusion (of the images.) This sequence will suffice to enable the induction required for this paper, for fortunately $dV^s W_{m-s} \Omega_F^{n-1}$, which can be problematic in other contexts, vanishes in the cohomology group $H_{p^m}^{n+1} F$.

1. Background on differential forms and the $m = 1$ case

In this and in the next section we compute in the module Ω_F^n. Although technically the Frobenius $\Phi : \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1}$ we abuse notation and use a lift so that we can write $\Phi : \Omega_F^n \to \Omega_F^n$. We use notation introduced in [Kato 1982] as follows: We fix $\{b_i \mid i \in I\}$ to be a p-basis for F with a well-ordered index set I. We let $\Sigma_F(n)$ be the set of all increasing functions $\gamma : \{1, 2, \dots, n\} \to I$ and we order $\Sigma_F(n)$

lexicographically so that $\gamma < \beta$ if an only if there is some $j \in \{1, 2, \ldots, n\}$ such that $\gamma(j) < \beta(j)$ while $\gamma(i) \le \beta(i)$ for all $i \le j$. For $\gamma \in \Sigma_F(n)$ we denote by

$$\frac{db_\gamma}{b_\gamma} := \frac{db_{\gamma(1)}}{b_{\gamma(1)}} \wedge \cdots \wedge \frac{db_{\gamma(n)}}{b_{\gamma(n)}}.$$

It is well-known that $\{{b_\gamma}^{-1} db_\gamma \mid \gamma \in \Sigma_F(n)\}$ is a F-basis for Ω_F^n and therefore every element $\omega \in \Omega_F^n$ can be expressed uniquely as a (finite) sum $\omega = \sum_{\gamma \in \Sigma_F(n)} a_\gamma {b_\gamma}^{-1} db_\gamma$. With this fixed basis we can define

$$\Phi\biggl(\sum_{\gamma \in \Sigma_F(n)} a_\gamma \frac{db_\gamma}{b_\gamma}\biggr) := \sum_{\gamma \in \Sigma_F(n)} a_\gamma^p \frac{db_\gamma}{b_\gamma}.$$

In prior work on this problem, [Sobiech 2018; Aravire et al. 2019] and other authors use the notation $\omega^{[p]}$ to denote what is defined here as $\Phi(\omega)$. We have chosen to use the Φ notation here for these basis dependent expressions because it will be consistent with how we use Φ in the de Rham Witt complex. In all applications, because the computations only matter (mod $d\Omega_F^{n-1}$) the choice of p-basis will be immaterial.

We need more technicalities about p-bases and these are laid out next.

Notation 1.1. We suppose that $\{b_i \mid i \in I\}$ is an ordered p-basis for F. As I is well-ordered we use $1 \in I$ as its least element. In case I has a greatest element, we use $\mathcal{M} \in I$ as its greatest element. We denote by $F_{<i} := F^p(b_k \,|\, k < i)$. We denote by $\Omega_{F,<\gamma}^n$ the F-subspace of Ω_F^n spanned by all ${b_\beta}^{-1} db_\beta$ with $\beta < \gamma$. Whenever $\omega = \sum_{\gamma \in \Sigma_F(n)} a_\gamma {b_\gamma}^{-1} db_\gamma$ and $\beta \in \Sigma_F(n)$ is the maximum index with $a_\beta \neq 0$, we call $a_\beta {b_\beta}^{-1} db_\beta$ the maximal summand of ω.

With this notation we can give a fundamental tool, referred to as Kato's lemma [1982, Lemma 2].

Lemma 1.2 (Kato's lemma). *Suppose F has characteristic p and $F^* = F^{*(p-1)}$. Let $y \in F$ and $\gamma \in \Sigma_F(n)$. If $(y^p - y){b_\gamma}^{-1} db_\gamma \in \Omega_{F,<\gamma}^n + d\Omega_F^{n-1}$ then $y{b_\gamma}^{-1} db_\gamma = v + {x_1}^{-1} dx_1 \wedge \cdots \wedge {x_n}^{-1} dx_n$, where each $x_i \in F^p(b_j \,|\, j \le \gamma(i))$ and $v \in \Omega_{F,<\gamma}^n$.*

It is useful to have a weaker version of Kato's lemma that does not include the hypothesis that $F^* = F^{*(p-1)}$. This has been noted by many authors. Here is one way to do this; it appears as [Arason et al. 2007, Lemma 3.2] and also appears as [Sobiech 2018, Lemma 3.2]. We recall that $\mathrm{dlog} : K_n^M F \to \Omega_F^n$ is the homomorphism defined on generators by $\mathrm{dlog}(\ell(t_1) \otimes \cdots \otimes \ell(t_n)) = {t_1}^{-1} dt_1 \wedge \cdots \wedge {t_n}^{-1} dt_n$.

Lemma 1.3. *Suppose F has characteristic p. Let $y \in F$ and $\gamma \in \Sigma_F(n)$. If $(y^p - y){b_\gamma}^{-1} db_\gamma \in \Omega_{F,<\gamma}^n + d\Omega_F^{n-1}$ then $y{b_\gamma}^{-1} db_\gamma = v + \mathrm{dlog}(\tau)$, where $v \in \Omega_{F,<\gamma}^n$ and $\tau \in K_n^M F$.*

We need one more technical result dependent upon this notation.

Lemma 1.4 [Arason et al. 2007, Lemma 3.1]. *For F and $\{b_i \mid i \in I\}$ as above, assume*

$$\sum_{\gamma \leq \beta} c_\gamma \frac{db_\gamma}{b_\gamma} \in d\Omega_F^{n-1}$$

where $c_\beta \neq 0$. Then there exist $M_{ij} \in F_{<\beta(i)}$, for $1 \leq i \leq n$ and $1 \leq j \leq p-1$, such that

$$c_\beta = \sum_{i=1}^{n} \sum_{j=1}^{p-1} M_{ij} b^j_{\beta(i)}.$$

We next consider a purely inseparable simple extension $L = F(\alpha)$ where $\alpha^{p^r} = b_1$. It is well-known that if $b_1, b_2, \ldots$ is an ordered p-basis for F, then $b_2, b_3, \ldots, \alpha$ is a p-basis for $F(\alpha)$ for which we select the ordering with α labeled as the last element of this basis (so its index is $\mathcal{M}$ and the index set for the L-basis is $(I - \{1\}) \cup \{\mathcal{M}\}$.) In this setting we will use $\Sigma'_F(i)$ to denote those elements $\gamma \in \Sigma_F(i)$ where the $\gamma(1) > 1 \in I$. We note then that $\Sigma_L(n+1) = \Sigma'_F(n+1) \cup \Sigma'_F(n) \cdot \alpha$, where $\Sigma'_F(n) \cdot \alpha$ denotes the elements $\beta \in \Sigma_L(n+1)$ with $\beta(1) > 1 \in I$ and $\beta(n+1) = \mathcal{M}$. We note that elements $\beta \in \Sigma'_F(n) \cdot \alpha$ are in one-to-one correspondence with elements of $\delta \in \Sigma'_F(n)$ and it is convenient to view them this way. In fact we label them as $\beta = \delta \cup \{\mathcal{M}\}$ where according to the notation just given, $(\delta \cup \{\mathcal{M}\})(i) = \delta(i)$ for $i \leq n$ and $(\delta \cup \{\mathcal{M}\})(n+1) = \mathcal{M}$. We also remark that in proving results using these p-bases we may assume they are finite. This is because one can use elements and the expressions that describe the situation over a finitely generated field over the prime field, from which the result follows for all fields.

This next lemma is needed to prove Lemma 1.6, which is the main tool we develop in this section.

Lemma 1.5. *Suppose F has characteristic p, $r \geq 1$, and $L = F(\alpha)$, where $\alpha^{p^r} = b_1 \in F - F^p$ with $b_2, b_3, \ldots, \alpha$ the ordered p-basis for L given above* (so the index of α is the largest). *Suppose $\omega \in \Omega_L^n$ and*

$$d\omega = \sum_{\beta \in \Sigma'_F(n+1)} c_\beta \frac{db_\beta}{b_\beta} + \sum_{\delta \in \Sigma'_F(n)} c_\delta \frac{db_\delta}{b_\delta} \wedge \frac{d\alpha}{\alpha} \quad \textit{for } c_\beta, c_\delta \in L.$$

Suppose that the maximal summand of $d\omega$ comes from an index $\tilde{\beta} \in \Sigma'_F(n+1)$ (that is, $c_{\tilde{\beta}} \neq 0$, and $\tilde{\beta} > \delta \cup \{\mathcal{M}\}$ for all $\delta \in \Sigma'_F(n)$ with $c_\delta \neq 0$.) *Then the maximal summand $c_{\tilde{\beta}} b_{\tilde{\beta}}^{-1} db_{\tilde{\beta}}$ of $d\omega$ is the leading summand of some $d\omega_0$ where $\omega_0 \in L_0 \cdot \Omega_F^n$ and $L_0 = F(\alpha^p)$.*

Proof. As $\Omega_L^{n+1} = L \cdot \Omega_F^{n+1} \oplus L \cdot \Omega_F^n \wedge \alpha^{-1} d\alpha$ and $L = \bigoplus_{i=0}^{p-1} \alpha^i L_0$, we find

$$\Omega_L^{n+1} = \left(\bigoplus_{i=0}^{p-1} \alpha^i L_0 \cdot \Omega_F^{n+1} \right) \oplus \left(\bigoplus_{i=0}^{p-1} \alpha^i L_0 \cdot \Omega_F^n \wedge \frac{d\alpha}{\alpha} \right).$$

We write $\omega = \sum_{i=0}^{p-1} \alpha^i \omega_i + \sum_{i=0}^{p-1} \alpha^i \omega_i' \wedge \alpha^{-1} d\alpha$ for $\omega_i \in L_0 \cdot \Omega_F^n$ and $\omega_i' \in L_0 \cdot \Omega_F^{n-1}$. Then

$$d\omega = \sum_{i=0}^{p-1} \alpha^i d\omega_i + \sum_{i=0}^{p-1} \alpha^i ((-1)^n i\omega_i + d\omega_i') \wedge \frac{d\alpha}{\alpha}.$$

According to Lemma 1.4, as the maximum summand of $d\omega$ is $c_{\tilde{\beta}} b_{\tilde{\beta}}{}^{-1} db_{\tilde{\beta}}$, we have $c_{\tilde{\beta}} = \sum_{i=1}^{n} \sum_{j=1}^{p-1} M_{ij} b_{\beta(i)}^{j}$, where $M_{ij} \in L_{<\beta(i)}$ for $1 \le i \le n$ and $1 \le j \le p-1$. As $L_{<\beta(i)} := L^p(b_k \,|\, k < \beta(i))$, and as the index of the basis element α is $\mathcal{M}$, the largest in the ordering of the p-basis for L, we have $\beta(i)$ less than this index and it follows that each $L_{<\beta(i)} \subseteq L_0$. It follows that $c_{\tilde{\beta}} \in L_0$. By the directness of the sum in which the summands of $d\omega$ occur, as none of the summands of $d\omega$ coming from $\alpha^i d\omega_i$ or $\alpha^i((-1)^n i\omega_i + d\omega_i') \wedge \alpha^{-1} d\alpha$ where $1 \le i < p$ have coefficients that lie in L_0, this maximum summand must be the maximum summand of $d\omega_0$ or $d\omega_0'$. But by hypothesis, this leading summand cannot involve $\alpha^{-1} d\alpha$ so it must come from $d\omega_0$. So $\omega_0 \in L_0 \cdot \Omega_F^n$ is the desired element. This proves the lemma. □

This next result is used repeatedly. It is our key application of Kato's lemma.

Lemma 1.6. *Suppose F has characteristic p and $r \ge 1$. Let $L = F(\alpha)$, where $\alpha^{p^r} = b_1 \in F - F^p$ with $b_2, b_3, \ldots, \alpha$ a p-basis for L as above* (so the index of α is the largest). *Suppose $\eta \in \Omega_L^{n-1} \wedge \alpha^{-1} d\alpha \subseteq \Omega_L^n$ and $\wp(\eta) \in L \cdot \Omega_F^n + d\Omega_L^{n-1}$. Then $\eta = \eta_0 + \mathrm{dlog}(\tau) \in \Omega_L^n$ for some $\eta_0 \in L \cdot \Omega_F^n$ and $\tau \in K_n^M L$.*

Proof. As noted above we may assume F has a finite p-basis. We assume $\eta = \left(\sum_{\delta \in \Sigma_F'(n-1)} y_\delta b_\delta{}^{-1} db_\delta\right) \wedge \alpha^{-1} d\alpha$, where $0 \ne y_\delta \in L$ and proceed by induction on the size of the largest δ index (which can also be viewed as $\delta \cup \{\mathcal{M}\} \in \Sigma_L(n)$.) By our hypothesis we can write

$$(*) \qquad \sum_{\delta \in \Sigma_F'(n-1)} \wp(y_\delta) \frac{db_\delta}{b_\delta} \wedge \frac{d\alpha}{\alpha} = \sum_{\beta \in \Sigma_F'(n)} z_\beta \frac{db_\beta}{b_\beta} + d\omega \in \Omega_L^n$$

for $z_\beta \in L$ and $\omega \in \Omega_L^{n-1}$. We must analyze two cases, the first where the maximum of the indices $\delta \cup \{\mathcal{M}\}$ and β in both sums in $(*)$ occurs among the β and the second where it occurs among the $\delta \cup \{\mathcal{M}\}$ (note that these two sets of indices are disjoint).

In the first case we assume the maximum occurs at $\tilde{\beta}$ among the $\beta \in \Sigma_F'(n)$ and so that $z_{\tilde{\beta}} \tilde{\beta}^{-1} d\tilde{\beta}$ is the maximum summand of $\sum_{\beta \in \Sigma_F'(n)} z_\beta b_\beta{}^{-1} db_\beta$. The maximality assumption then shows that $d\omega$ has maximum summand $-z_{\tilde{\beta}} \tilde{\beta}^{-1} d\tilde{\beta}$, where $\tilde{\beta} \in \Sigma_F'(n)$. According to Lemma 1.5 there is some $\omega_0 \in L_0 \cdot \Omega_F^{n-1} \subset \Omega_L^{n-1}$ such that $d\omega$ and $d\omega_0$ have the same maximum summand. Since $L_0 = F(\alpha^p)$ we know $d\omega_0 \in L_0 \cdot \Omega_F^n \subset \Omega_L^n$ (note we are viewing $d\omega_0 \in \Omega_L^n$ and therefore $d\alpha^p = 0 \in \Omega_L^1$). Expressing $d\omega_0 = -z_{\tilde{\beta}} \tilde{\beta}^{-1} d\tilde{\beta} + \sum_{\beta \in \Sigma_F'(n), \beta < \tilde{\beta}} z_\beta' \cdot b_\beta{}^{-1} db_\beta$, where $z'_\beta \in L_0$, it follows that

$$\sum_{\delta\in\Sigma'_F(n-1)} \wp(y_\delta)\frac{db_\delta}{b_\delta}\wedge\frac{d\alpha}{\alpha} = z_{\tilde\beta}\frac{d\tilde\beta}{\tilde\beta} + \sum_{\beta\in\Sigma'_F(n),\beta<\tilde\beta} z_\beta\frac{db_\beta}{b_\beta} + d\omega + d\omega_0 - d\omega_0$$
$$= \sum_{\beta\in\Sigma'_F(n),\beta<\tilde\beta} (z_\beta + z'_\beta)\frac{db_\beta}{b_\beta} + d(\omega-\omega_0) \in \Omega^n_L.$$

Iterating this process shows we can assume the maximum of the $\delta\cup\{\mathcal{M}\}$, β in $(*)$ occurs at some $\tilde\delta\cup\{\mathcal{M}\}$ among the $\tilde\delta\in\Sigma'_F(n-1)$, that is, we are in the second case.

We now express $\eta = \eta' + y_{\tilde\delta}b_{\tilde\delta}^{-1}db_{\tilde\delta}\wedge\alpha^{-1}d\alpha$ where $\tilde\delta\cup\{\mathcal{M}\}$ is the maximal index. Then our assumption about equation $(*)$ shows that

$$\wp(y_{\tilde\delta})\frac{db_\delta}{b_\delta}\wedge\frac{d\alpha}{\alpha} \in \Omega^n_{L,<\tilde\delta\cup\{M\}} + d\Omega^{n-1}_L,$$

so by Lemma 1.3 we can express $y_{\tilde\delta}b_{\tilde\delta}^{-1}db_{\tilde\delta}\wedge\alpha^{-1}d\alpha = v + \operatorname{dlog}(\tau)$ for $v\in\Omega^{n-1}_{L,<\tilde\delta\cup\{M\}}$ and $\tau\in K^M_n L$. We decompose $v = v_1 + v_2$, where $v_1\in L\cdot\Omega^n_F$ and $v_2\in\Omega^{n-1}_{L,<\tilde\delta}\wedge\alpha^{-1}d\alpha$. Since $\eta' = \eta - y_{\tilde\delta}b_{\tilde\delta}^{-1}db_{\tilde\delta}\wedge\alpha^{-1}d\alpha = \eta-(v_1+v_2)-\operatorname{dlog}(\tau)$ we can replace η by $\eta - v_1 - \operatorname{dlog}(\tau) = \eta' + v_2 \in \Omega^{n-1}_{L,<\tilde\delta}\wedge\alpha^{-1}d\alpha$. We have removed the maximal summand of type $\tilde\delta\cup\{\mathcal{M}\}$ from η and the original hypotheses still hold. This reduction completes the proof. □

In the next definition we use $\wp = \Phi - 1 : \Omega^n_F \to \Omega^n_F$. We recall our convention that $\Phi : \Omega^n_F \to \Omega^n_F$ is defined relative to a fixed p-basis of F. However, as the groups considered in this definition all contain $d\Omega^{n-1}_F$ they are independent of this choice of p-basis. Again, we remark that the purpose of the convention for Φ is to facilitate expressing our computations and in the end no ambiguities are introduced.

Definition 1.7. (i) We define $B_1\Omega^n_F := d\Omega^{n-1}_F$ and recursively for $r\ge 1$,

$$B_{r+1}\Omega^n_F = \Phi(B_r\Omega^n_F) + B_r\Omega^n_F.$$

(ii) Assume $x\in F - F^p$ and $n\ge 0$. We define $\Gamma^n_F := \wp(\Omega^n_F) + d\Omega^{n-1}_F + \Omega^{n-1}_F\wedge dx$.

The groups $B_{r+1}\Omega^n_F$ were first defined in [Illusie 1979] and played a significant role in [Izhboldin 1991] where Izhboldin showed that his group $Q^n(F,1)\cong\Omega^n_F/\bigl(\bigcup_{r=1}^\infty B_r\Omega^n_F\bigr)$. In particular, if $\omega\in B_r\Omega^n_F$ for some r, then $\bar\omega = 0\in H^{n+1}_p F$. In view of this we note as $B_{r+1}\Omega^n_F = \Phi(B_r\Omega^n_F) + B_r\Omega^n_F$ that $\wp(\Omega^n_F) + d\Omega^{n-1}_F = \wp(\Omega^n_F) + B_r\Omega^n_F$ for all r.

The recursive definition shows that $B_r\Omega^n_F$ is generated by elements $\Phi^{i-1}(d\omega)$, where $1\le i\le r$ and $\omega\in\Omega^{n-1}_F$. One readily checks $x^{p^r}B_r\Omega^n_F\subseteq B_r\Omega^n_F$. Suppose $L = F(\sqrt[p^e]{x})$. This means that $\overline{x^j B_r\Omega^n_F}\subseteq\ker(H^{n+1}_p F\to H^{n+1}_p L)$ whenever

$1 \le r \le e$. The generators for $x^j B_r \Omega_F^n$ can be expressed as

$$x^j \Phi^i\left(ds \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_n}{t_n}\right) = x^j s^{p^{i-1}} \frac{ds}{s} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_n}{t_n},$$
$$\text{where} \quad 1 \le i \le r \quad \text{and} \quad s, t_2, \ldots, t_n \in F.$$

The theorem of Aravire, Laghribi, O'Ryan, and Sobiech [Sobiech 2018, Theorem 3.16; Aravire et al. 2019, Theorem 4.1] shows that these give a generating set for $\ker(H_p^{n+1}F \to H_p^{n+1}L)$. We next state their result in this notation. For simplicity we give it in the case were $L = F(\sqrt[p^{e_1}]{x_1}, \ldots, \sqrt[p^{e_s}]{x_s})$ for p-independent elements $x_1, \ldots, x_s \in F$ and the e_i all are a constant value e. (Their general result does not require constant values.)

Theorem 1.8 (Aravire–Laghribi–O'Ryan–Sobiech). *Suppose* $L=F(\sqrt[p^e]{x_1},\ldots,\sqrt[p^e]{x_s})$ *for* p*-independent elements* $x_1, \ldots, x_s \in F$. *Then*

$$\ker(H_p^{n+1}F \to H_p^{n+1}L) = \sum_{1<r\le e}\ \sum_{0<j_k<p^r} \overline{x_1^{j_1} \cdots x_s^{j_s} B_r \Omega_F^n}.$$

We close this section by considering the simple purely inseparable extensions of degree p. When $p = 2$ this next result is implicit in [Aravire and Baeza 2003] and the general case is implicit in the work of Sobiech. It is pulled out here because it provides a model for the results considered in the next section.

Lemma 1.9. *Suppose* $\omega_i \in \Omega_F^n$ *and* $\sum_{i=1}^{p-1} x^i \Phi(\omega_i) \in \Gamma_F^n$. *Then*

$$\sum_{i=1}^{p-1} i^{-1} x^i \Phi(d\omega_i) \in \Gamma_F^{n+1}.$$

Proof. We express $\sum_{i=1}^{p-1} x^i \Phi(\omega_i) = \wp(\omega_0) + d\hat{\theta} + \hat{\psi} \wedge dx$ for $\omega_0 \in \Omega_F^n$ and $\hat{\theta}, \hat{\psi} \in \Omega_F^{n-1}$. We consider $\big(\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i\big) \wedge \alpha^{-1} d\alpha \in \Omega_L^{n+1}$, where $L = F(\alpha)$ with $\alpha^p = x \in F$. As $i_{L/F}(\hat{\psi} \wedge dx) = 0 \in \Omega_L^{n+1}$, we have

$$(**)\quad \wp\left(\left(\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i\right) \wedge \frac{d\alpha}{\alpha}\right) = \left(\wp(\omega_0) - \sum_{i=1}^{p-1} x^i \Phi(\omega_i) + \sum_{i=1}^{p-1} \alpha^i \omega_i\right) \wedge \frac{d\alpha}{\alpha}$$
$$= \left(\sum_{i=1}^{p-1} \alpha^i \omega_i - d\hat{\theta}\right) \wedge \frac{d\alpha}{\alpha} \in \Omega_L^{n+1}.$$

As $\big(\sum_{i=1}^{p-1} \alpha^i \omega_i\big) \wedge \alpha^{-1} d\alpha = d\big(\sum_{i=1}^{p-1} i^{-1} \alpha^i \omega_i\big) - \sum_{i=1}^{p-1} i^{-1} \alpha^i d\omega_i$, $d\hat{\theta} \wedge \alpha^{-1} d\alpha = d(\hat{\theta} \wedge \alpha^{-1} d\alpha)$ and each $\omega_i \in \Omega_F^n$ we obtain $\wp\big(\big(\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i\big) \wedge \alpha^{-1} d\alpha\big) \in L \cdot \Omega_F^{n+1} + d\Omega_L^n$. Hence by Lemma 1.6 applied to $L = F(\alpha)$ we see that

$$\left(\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i\right) \wedge \frac{d\alpha}{\alpha} = \eta + \operatorname{dlog}(\tau), \quad \text{where } \eta \in L \cdot \Omega_F^{n+1} \text{ and } \tau \in K_{n+1}^M L.$$

We have $\wp\big(\omega_0 - \sum_{i=1}^{p-1}\alpha^i\omega_i\big)\wedge\alpha^{-1}d\alpha = \wp(\eta) + d\theta$ for $\theta\in\Omega_L^n$. We can write $\eta = \eta_0 + \sum_{i=1}^{p-1}\alpha^i\eta_i$, where $\eta_i\in\Omega_F^{n+1}$ and

$$\theta = \theta_0 + \sum_{i=1}^{p-1}\alpha^i\theta_i + \left(\theta_0' + \sum_{i=1}^{p-1}\alpha^i\theta_i'\right)\wedge\frac{d\alpha}{\alpha}\in\Omega_L^n \quad \text{for } \theta_i\in\Omega_F^n \text{ and } \theta_i'\in\Omega_F^{n-1}.$$

With this, expanding $\wp\big(\omega_0 - \sum_{i=1}^{p-1}\alpha^i\omega_i\big)\wedge\alpha^{-1}d\alpha = \wp(\eta) + d\theta$ in Ω_L^{n+1} using the first equality in $(**)$ gives $2p$ equations:

$$\begin{aligned}
0 &= \wp(\eta_0) + \sum_{i=1}^{p-1}x^i\Phi(\eta_i) + d\theta_0,\\
0 &= \alpha^i(-\eta_i + d\theta_i) \quad \text{for } 1\le i\le p-1,\\
\left(\wp(\omega_0) - \sum_{i=1}^{p-1}x^i\Phi(\omega_i)\right)\wedge\frac{d\alpha}{\alpha} &= d\theta_0'\wedge\frac{d\alpha}{\alpha},\\
\alpha^i\omega_i\wedge\frac{d\alpha}{\alpha} &= \alpha^i(i\theta_i + d\theta_i')\wedge\frac{d\alpha}{\alpha} \quad \text{for } 1\le i\le p-1.
\end{aligned}$$

Each of these can replaced by an equality in Ω_F^n by adding $\Omega_F^n\wedge dx$ or $\Omega_F^{n-1}\wedge dx$ terms. We find

$$\begin{aligned}
\sum_{i=1}^{p-1}x^i\Phi(\eta_i) &= -\wp(\eta_0) - d\theta_0 + \psi_{00}\wedge dx,\\
\eta_i &= d\theta_i + \psi_{0i}\wedge dx \quad \text{for } 1\le i < p-1,\\
\sum_{i=1}^{p-1}x^i\Phi(\omega_i) &= \wp(\omega_0) - d\theta_0' + \psi_{10}\wedge dx,\\
\omega_i &= i\theta_i + d\theta_i' + \psi_{1i}\wedge dx \quad \text{for } 1\le i < p-1.
\end{aligned}$$

The second and fourth equations give $\eta_i = i^{-1}d\omega_i + \psi_i\wedge dx$ for some $\psi_i\in\Omega_F^n$. By the first equation $\sum_{i=1}^{p-1}i^{-1}x^i\Phi(d\omega_i)\in\wp(\Omega_F^{n+1}) + d\Omega_F^n + \Omega_F^n\wedge dx = \Gamma_F^{n+1}$. □

We now have the tools to the determine the kernel of $H_p^{n+1}F\to H_p^{n+1}L$ where $L = F(\alpha)$ for $\alpha^p = x\in F$. The result is due to Sobiech [2018, Proposition 3.6] and when $p = 2$ the result is due to Aravire and Baeza [2003, Lemma 2.18]. We give the proof because it illustrates in a much less technical fashion the technique of proof of the main Theorem 4.2 (as well as the base of the induction for that result) and also illustrates how Lemma 1.9 is used.

Theorem 1.10 (Aravire–Baeza–Sobiech). *If $L = F(\alpha)$ for $\alpha^p = x\in F$ then* $\ker(H_p^{n+1}F\to H_p^{n+1}L) = \Omega_F^{n-1}\wedge dx$.

Proof. Here we have

$$\Omega^n_{F(\alpha)} = \bigoplus_{i=0}^{p-1} \alpha^i \cdot i_{L/F}(\Omega^n_F) \oplus \bigoplus_{i=0}^{p-1} \alpha^i \cdot i_{L/F}(\Omega^{n-1}_F) \wedge \frac{d\alpha}{\alpha},$$

where we recall $i_{L/F}(\Omega^n_F) \cong \Omega^n_F/(\Omega^{n-1}_F \wedge dx)$.

Assume $\omega \in \Omega^n_F$ with $\bar{\omega} \in \ker(H^{n+1}_p F \to H^{n+1}_p L)$. We then have $i_{L/F}(\omega) = \wp(\eta) + d\theta$, where $\eta \in \Omega^n_L$ and $\theta \in \Omega^{n-1}_L$. We express

$$\eta = \sum_{i=0}^{p-1} \alpha^i \eta_i + \sum_{i=0}^{p-1} \alpha^i \eta'_i \wedge \frac{d\alpha}{\alpha}, \quad \text{where } \eta_i \in \Omega^n_F \text{ and } \eta'_i \in \Omega^{n-1}_F,$$

$$\theta = \sum_{i=0}^{p-1} \alpha^i \theta_i + \sum_{i=0}^{p-1} \alpha^i \theta'_i \wedge \frac{d\alpha}{\alpha}, \quad \text{where } \theta_i \in \Omega^{n-1}_F \text{ and } \theta'_i \in \Omega^{n-2}_F.$$

We then have

$$\begin{aligned} i_{L/F}(\omega) = \wp(\eta_0) + \sum_{i=1}^{p-1}(x^i\Phi(\eta_i) - \alpha^i\eta_i) + \wp(\eta'_0) \wedge \frac{d\alpha}{\alpha} + \sum_{i=1}^{p-1}(x^i\Phi(\eta'_i) - \alpha^i\eta'_i) \wedge \frac{d\alpha}{\alpha} \\ + \sum_{i=0}^{p-1}\alpha^i d\theta_i + \sum_{i=1}^{p-1} i\alpha^i \cdot \theta_i \wedge \frac{d\alpha}{\alpha} + \sum_{i=0}^{p-1}\alpha^i d\theta'_i \wedge \frac{d\alpha}{\alpha} \\ = \left(\wp(\eta_0) + \sum_{i=1}^{p-1} x^i\Phi(\eta_i) + d\theta_0\right) - \left(\sum_{i=1}^{p-1}\alpha^i(\eta_i - d\theta_i)\right) \\ + \left(\left(\wp(\eta'_0) + \sum_{i=1}^{p-1} x^i\Phi(\eta'_i) + d\theta'_0\right) - \left(\sum_{i=1}^{p-1}\alpha^i(\eta'_i - i\theta_i - d\theta'_i)\right)\right) \wedge \frac{d\alpha}{\alpha}. \end{aligned}$$

By the linear independence properties of the decomposition of Ω^n_L we obtain two sets of p equations. From the first set we find for $1 \le i \le p-1$ that $\eta_i = d\theta_i + \psi_{i1}$ for some $\psi_{i1} \in \Omega^{n-1}_F \wedge dx$. Applying d to the second set we have $i d\theta_i = d\eta'_i + d\psi_{i2}$ for some $\psi_{i2} \in \Omega^{n-2}_F \wedge dx$. Together these give $\eta_i = i^{-1}d\eta'_i + \psi_i$, where $\psi_i \in \Omega^{n-1}_F \wedge dx$. From the second set we have $\sum_{i=1}^{p-1} x^i \Phi(\eta'_i) \in \wp(\Omega^{n-1}_F) + d\Omega^{n-2}_F + \Omega^{n-2}_F \wedge dx$ and so by Lemma 1.9 we know $\sum_{i=1}^{p-1} i^{-1}x^i\Phi(d\eta'_i) \in \wp(\Omega^n_F) + d\Omega^{n-1}_F + \Omega^{n-1}_F \wedge dx$. But as each $\Phi(\psi_i) \in \Omega^{n-1}_F \wedge dx$, substituting $d\eta'_i = i\eta - i\psi_i$ shows

$$\sum_{i=1}^{p-1} i^{-1} i \cdot x^i \Phi(\eta_i) \in \wp(\Omega^n_F) + d\Omega^{n-1}_F + \Omega^{n-1}_F \wedge dx.$$

Finally from the first equation we obtain $i_{L/F}(\omega) = \wp(\eta_0) + \sum_{i=1}^{p-1} x^i\Phi(\eta_i) + d\theta_0$ so $\omega \in \wp(\Omega^n_F) + d\Omega^{n-1}_F + \Omega^{n-1}_F \wedge dx$ proving the theorem. □

2. Generalizing lemmas on differential forms

The main goal of this section is to generalize Lemma 1.9 (see Theorem 2.2 below) so that it can be used to compute $\ker(H^{n+1}_{p^m}F \to H^{n+1}_{p^m}L)$ when $L = F(\alpha)$ for $\alpha^{p^e} = x \in F - F^p$. All notational conventions from the preceding section remain in force and we recall $\Gamma^n_F = \wp(\Omega^n_F) + d\Omega^{n-1}_F + \Omega^{n-1}_F \wedge dx$. Extending this we define

$$\widetilde{\Gamma}^n_F := \sum_{i=1}^{p-1} x^i B_2\Omega^n_F + \sum_{j=1,\,(j,p)=1}^{p^2-1} x^j \Phi^2(\Omega^n_F) + \Gamma^n_F.$$

The next lemma will help simplify computations involving $\sum_{i=0}^{p^{e-1}-1} x^i B_e\Omega^n_F$.

Lemma 2.1. *Suppose $e \geq 2$.*

(i) $\sum_{i=0}^{p^{e-1}-1} x^i B_e\Omega^n_F \subseteq \sum_{\ell=2}^{e} \sum_{i=1,\,(i,p)=1}^{p^{\ell-1}-1} x^i B_\ell\Omega^n_F + \Gamma^n_F$.

(ii) $\sum_{i=0}^{p^{e-1}-1} x^i B_e\Omega^n_F \subseteq \sum_{i=1}^{p-1} x^i B_2\Omega^n_F + \sum_{j=1,\,(j,p)=1}^{p^2-1} x^j \Phi^2(\Omega^n_F) + \Gamma^n_F = \widetilde{\Gamma}^n_F$.

(iii) $d\big(\sum_{i=1}^{p^e-1} x^i B_e\Omega^n_F + \sum_{j=1,\,(j,p)=1}^{p^2-1} x^j \Phi^2(\Omega^n_F)\big) \subseteq \Omega^n_F \wedge dx$.

Proof. We note for $\chi \in \Omega^n_L$ and $s \geq 1$ that $\Phi^s(\chi) = \wp\big(\sum_{j=0}^{s-1} \Phi^j(\chi)\big) + \chi$. For the first statement, we know $B_e\Omega^n_F$ is generated by elements $\Phi^r(d\omega)$ where $0 \leq r \leq e-1$ and $\omega \in \Omega^{n-1}_F$. Consider a generator $x^{i_0 p^s}\Phi^r(d\omega)$ of $\sum_{i=1}^{p^{e-1}-1} x^i B_e\Omega^n_F$, where $(i_0, p) = 1$ and $s \geq 0$. If $s > r$ then $x^{i_0 p^s}\Phi^r(d\omega) = \Phi^r(d(x^{i_0 p^{s-r}}\omega)) \in \wp(\Omega^n_F) + d\Omega^{n-1}_F$. If $s < r$ then $x^{i_0 p^s}\Phi^r(d\omega) = \Phi^{r-s}(x^{i_0}\Phi^s(d\omega)) \in \wp(\Omega^n_F) + x^{i_0}\Phi^s(d\Omega^{n-1}_F)$. If $s = r$ then $x^{i_0 p^s}\Phi^r(d\omega) = \Phi^r(x^{i_0} d\omega) \in \wp(\Omega^n_F) + x^{i_0} d\Omega^{n-1}_F$. However,

$$x^{i_0} d\Omega^{n-1}_F \subseteq d\Omega^{n-1}_F + \Omega^{n-1}_F \wedge dx.$$

The first statement follows from this.

For the second statement we show for $0 \leq r < e$ that $x^i\Phi^r(d\omega) \in \widetilde{\Gamma}^n_F$, where by (i) we may assume $(i, p) = 1$. If $r = 0$ and $i > 0$ then

$$x^i d\omega = d(x^i\omega) - (-1)^n i x^{i-1}\omega \wedge dx \in \Gamma^n_F.$$

If $r = 1$ and $i = i_0 + i_1 p$ then

$$x^i\Phi(d\omega) = x^{i_0}\Phi(x^{i_1} d\omega) = x^{i_0}\Phi\big(d(x^{i_1}\omega) - (-1)^n i_1 x^{i_1-1}\omega \wedge dx\big).$$

But $\Phi(\Omega^{n-1}_F \wedge dx) \subseteq \Omega^{n-1}_F \wedge dx$ so we are done in this case. For $r \geq 2$ and $i = i_0 + i_1 p + i_2 p^2$, where $0 < i_0 < p$ and $0 \leq i_1 < p$, we have $x^i\Phi^r(d\omega) = x^{i_0+i_1 p}\Phi^2(x^{i_2}\Phi^{r-2}(d\omega))$. As $i_0 \neq 0$ we have $x^i\Phi^r(d\omega) \in \sum_{j=1,\,(j,p)=1}^{p^2-1} x^j\Phi^2(\Omega^n_F)$. From this the second statement follows. The final statement is clear as $d\Phi(\omega) = 0$ for all $\omega \in \Omega^n_F$. □

The next result extends Lemma 1.9. The proof follows the same format except that one computes in a purely inseparable extension of degree p^r instead of degree p.

Theorem 2.2. *Let $x \in F - F^p$ and $r, e \geq 1$. Suppose*

$$\sum_{i=1,\,(i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) \in \Gamma_F^n + \sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n,$$

where $\omega_i \in \Omega_F^n$. Then

$$\sum_{i=1,\,(i,p)=1}^{p^r-1} i^{-1} x^i \Phi^r(d\omega_i) \in \Gamma_F^{n+1} + \sum_{i=0}^{p^{r-1}-1} x^i B_r \Omega_F^{n+1}.$$

Proof. By hypothesis we can express

$$\sum_{i=1,\,(i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) = \wp(\omega_0) + d\hat{\theta} + \hat{\psi} \wedge dx + \tilde{\Theta},$$

where $\omega_0 \in \Omega_F^n$, $\hat{\theta}, \hat{\psi} \in \Omega_F^{n-1}$ and $\tilde{\Theta} \in \sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n$. We set $L = F(\alpha)$ with $\alpha^{p^r} = x \in F - F^p$ and define

$$\hat{\omega} := \left(\omega_0 - \sum_{j=0}^{r-1} \sum_{i=1,\,(i,p)=1}^{p^r-1} \alpha^{ip^j} \Phi^j(\omega_i) \right) \wedge \frac{d\alpha}{\alpha} \in \Omega_L^{n+1}.$$

As $i_{L/F}(\hat{\psi} \wedge dx) = 0 \in \Omega_L^{n+1}$ we have

$$\begin{aligned} \wp(\hat{\omega}) &= \left(\wp(\omega_0) - \sum_{i=1,\,(i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) + \sum_{i=1,\,(i,p)=1}^{p^r-1} \alpha^i \omega_i \right) \wedge \frac{d\alpha}{\alpha} \\ &= \left(\sum_{i=1,\,(i,p)=1}^{p^r-1} \alpha^i \omega_i - d\hat{\theta} - \tilde{\Theta} \right) \wedge \frac{d\alpha}{\alpha} \in \Omega_L^{n+1}. \end{aligned}$$

We note $\alpha^i \omega_i \wedge \alpha^{-1} d\alpha = d(i^{-1} \alpha^i \omega_i) - (-1)^n i^{-1} \alpha^i d\omega_i$ which shows

$$\left(\sum_{i=1,\,(i,p)=1}^{p^r-1} \alpha^i \omega_i \right) \wedge \frac{d\alpha}{\alpha} \in L \cdot \Omega_F^{n+1} + d\Omega_L^n.$$

Also, $d\hat{\theta} \wedge \alpha^{-1} d\alpha = d(\hat{\theta} \wedge \alpha^{-1} d\alpha) \in d\Omega_L^n$. We find that

$$\left(\sum_{i=1,\,(i,p)=1}^{p^r-1} \alpha^i \omega_i - d\hat{\theta} \right) \wedge \frac{d\alpha}{\alpha} \in L \cdot \Omega_F^{n+1} + d\Omega_L^n$$

and from this we have

$$\left(\wp(\hat{\omega}) - \tilde{\Theta} \wedge \frac{d\alpha}{\alpha} \right) \in L \cdot \Omega_F^{n+1} + d\Omega_L^n.$$

By Lemma 2.1(i), $\Theta \in \sum_{\ell=2}^{e} \sum_{i=1,\,(i,p)=1}^{p^{\ell-1}-1} x^i B_\ell \Omega_F^n + \Gamma_F^n$ and we can write

$$\Theta = \sum_{\ell=1}^{e-1} \sum_{i=1,\,(i,p)=1}^{p^\ell-1} x^i \Phi^\ell(d\hat{\theta}_{\ell i}) + \gamma$$

for $\hat{\theta}_{\ell i} \in \Omega_F^{n-1}$ and $\gamma \in \Gamma_F^n$. We recall for $\chi \in \Omega_L^n$ and $s \geq 1$ that $\Phi^s(\chi) = \wp\big(\sum_{j=0}^{s-1} \Phi^j(\chi)\big) + \chi$. If $r > \ell$ we have

$$(*_A) \qquad x^i \Phi^\ell(d\hat{\theta}_{\ell i}) \wedge \frac{d\alpha}{\alpha} = \Phi^\ell(\alpha^{ip^{r-\ell}} d\hat{\theta}_{\ell i}) \wedge \frac{d\alpha}{\alpha} = \Phi^\ell\Big(d\Big(\alpha^{ip^{r-\ell}} \hat{\theta}_{\ell i} \wedge \frac{d\alpha}{\alpha}\Big)\Big)$$
$$= \wp\Big(\psi_{\ell i} \wedge \frac{d\alpha}{\alpha}\Big) + d\Big(\alpha^{ip^{r-\ell}} \hat{\theta}_{\ell i} \wedge \frac{d\alpha}{\alpha}\Big),$$

where

$$\psi_{\ell i} := \sum_{j=0}^{\ell-1} \Phi^j(d(\alpha^{ip^{r-\ell}} \hat{\theta}_{\ell i})) = \sum_{j=0}^{\ell-1} \alpha^{ip^{r+j-\ell}} \Phi^j(d\hat{\theta}_{\ell i}) = \sum_{j'=r-\ell}^{r-1} \alpha^{ip^{j'}} \Phi^{j'+\ell-r}(d\hat{\theta}_{\ell i}).$$

When $1 \leq r \leq \ell$ and $(i, p) = 1$ we have

$$(*_B) \quad x^i \Phi^\ell(d\hat{\theta}_{\ell i}) \wedge \frac{d\alpha}{\alpha} = \Phi^r(\alpha^i \Phi^{\ell-r}(d\hat{\theta}_{\ell i})) \wedge \frac{d\alpha}{\alpha} = \Phi^r\Big(\alpha^i \Phi^{\ell-r}(d\hat{\theta}_{\ell i}) \wedge \frac{d\alpha}{\alpha}\Big)$$
$$= \wp\Big(\psi_{\ell i} \wedge \frac{d\alpha}{\alpha}\Big) + d\big(i^{-1}\alpha^i \Phi^{\ell-r}(d\hat{\theta}_{\ell i})\big),$$

where

$$\psi_{\ell i} := \sum_{j=0}^{r-1} \Phi^j(\alpha^i \Phi^{\ell-r}(d\hat{\theta}_{\ell i})) = \sum_{j=0}^{r-1} \alpha^{ip^j} \Phi^j(\Phi^{\ell-r}(d\hat{\theta}_{\ell i})) = \sum_{j=0}^{r-1} \alpha^{ip^j} \Phi^{j+\ell-r}(d\hat{\theta}_{\ell i})$$

in this case.

We define Ψ by

$$(*_C) \qquad \Psi := \sum_{\ell=1}^{e-1} \sum_{\substack{i=1\\(i,p)=1}}^{p^\ell-1} \psi_{\ell i} = \sum_{\ell=1}^{e-1} \sum_{\substack{i=1\\(i,p)=1}}^{p^\ell-1} \sum_{j=\max\{0,r-\ell\}}^{r-1} \alpha^{ip^j} \Phi^{j+\ell-r}(d\hat{\theta}_{\ell i}).$$

Using the latter sum we can express $\Psi = \sum_{j=0}^{r-1} \sum_{i=0\,(i,p)=1}^{p^{r-j}-1} \alpha^{ip^j} \hat{\psi}_{ji}$, where $\hat{\psi}_{ji} \in \Omega_F^n$ for $0 \leq j \leq r-1$ and $0 \leq i \leq p^{r-j}$ are obtained by combining the like α^{ip^j}-summands. It is possible, however, as we only have a bound of $i < p^\ell$ (not the bound of $i < p^{r-j}$) that we have $ip^j = i_0 p^j + kp^r$ for some $k > 0$ and $i_0 < p^{r-j}$. In this case we have $\alpha^{ip^j} \Phi^{j+\ell-r}(d\hat{\theta}_{\ell i}) = \alpha^{i_0 p^j} x^k \Phi^{j+\ell-r}(d\hat{\theta}_{\ell i})$ appearing in the sum used to obtain $\hat{\psi}_{ji_0}$. This shows that for each $\hat{\psi}_{ji}$ we have $d\hat{\psi}_{ji} \in \Omega_F^n \wedge dx$.

The two expressions for $x^i \Phi^\ell(d\hat{\theta}_{\ell i}) \wedge \alpha^{-1} d\alpha$, $(*_A)$ and $(*_B)$, compared to the definition of Ψ in $(*_C)$, show $\tilde{\Theta} \wedge \alpha^{-1} d\alpha - \wp(\Psi \wedge \alpha^{-1} d\alpha) \in d\Omega_L^{n-1}$. As

$$\wp(\hat{\omega}) - \tilde{\Theta} \wedge \frac{d\alpha}{\alpha} \in L \cdot \Omega_F^{n+1} + d\Omega_L^n,$$

taken together these show

$$\wp\Big(\hat{\omega} - \Psi \wedge \frac{d\alpha}{\alpha}\Big) \in L \cdot \Omega_F^{n+1} + d\Omega_L^{n-1}.$$

Lemma 1.6 applied to $L = F(\alpha)$ gives $\hat{\omega} - \Psi \wedge \alpha^{-1} d\alpha = \operatorname{dlog}(\tau) + \eta$, where $\tau \in K_{n+1}^M L$ and $\eta \in L \cdot \Omega_F^{n+1}$. From this we have,

$$\begin{aligned}
&\wp\Big(\hat{\omega} - \Psi \wedge \frac{d\alpha}{\alpha}\Big) \\
&\quad = \wp\Big(\hat{\omega} - \sum_{j=0}^{r-1} \sum_{\substack{i=1\\(i,p)=1}}^{p^{r-j}-1} \alpha^{ip^j} \hat{\psi}_{ji} \wedge \frac{d\alpha}{\alpha}\Big) \\
&\quad = \Big(\wp(\omega_0) - \sum_{\substack{i=1\\(i,p)=1}}^{p^r-1} x^i \Phi^r(\omega_i) + \sum_{\substack{i=1\\(i,p)=1}}^{p^r-1} \alpha^i \omega_i - \sum_{j=0}^{r-1} \sum_{\substack{i=1\\(i,p)=1}}^{p^{r-j}-1} \wp(\alpha^{ip^j} \hat{\psi}_{ji})\Big) \wedge \frac{d\alpha}{\alpha} \\
&\quad = \Big(\wp(\omega_0) - \sum_{i=1,\,(i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) + \sum_{i=1,\,(i,p)=1}^{p^r-1} \alpha^i \omega_i + \sum_{i=0}^{p^r-1} \alpha^i \Psi_i\Big) \wedge \frac{d\alpha}{\alpha} \\
&\quad = \wp(\eta) + d\theta,
\end{aligned}$$

where the $\Psi_i \in \Omega_F^n$ arise to fit the equation by combining all $\wp(\alpha^{ip^j} \hat{\psi}_{ji})$ summands and we have $\eta \in L \cdot \Omega_F^{n+1}$ and $\theta \in \Omega_L^n$. The one thing we need to know about the Ψ_i is that, because we know this for the $\hat{\psi}_{ji}$, we have $d\Psi_i \in \Omega_F^n \wedge dx$ for all i. Expanding $\eta = \sum_{i=0}^{p^r-1} \alpha^i \eta_i$ and $\theta = \sum_{i=0}^{p^r-1} \alpha^i \theta_i + \sum_{i=0}^{p^r-1} \alpha^i \theta_i' \wedge \alpha^{-1} d\alpha$ for $\eta_i \in \Omega_F^{n+1}$, $\theta_i \in \Omega_F^n$ and $\theta_i' \in \Omega_F^{n-1}$ gives the $2p^r$ equations, one for each summand in $\Omega_L^n = \big(\bigoplus_{i=0}^{p^r-1} \alpha^i F \cdot \Omega_F^n\big) \oplus \big(\bigoplus_{i=0}^{p^r-1} \alpha^i F \cdot \Omega_F^{n-1} \wedge \alpha^{-1} d\alpha\big)$:

$$0 = \wp(\eta_0) + \sum_{k=1}^{p-1} x^k \Phi(\eta_{kp^{r-1}}) + d\theta_0,$$

$$0 = \alpha^{ip^{r-1}}\Big(-\eta_{ip^{r-1}} + d\theta_{ip^{r-1}} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{ip^{r-2}+kp^{r-1}})\Big), \quad 1 \le i \le p-1,$$

$$\vdots$$

$$(*_1) \qquad 0 = \alpha^{q'p^2}\Big(-\eta_{q'p^2} + d\theta_{q'p^2} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{q'p+kp^{r-1}})\Big), \quad 1 \le q' \le p^{r-2}-1, \; (q',p)=1,$$

$$0 = \alpha^{qp}\Big(-\eta_{qp} + d\theta_{qp} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{q+kp^{r-1}})\Big), \quad 1 \le q \le p^{r-1}-1, \; (q,p)=1,$$

$$0 = \alpha^j(-\eta_j + d\theta_j), \quad 1 \le j \le p^r-1, \; (j,p)=1,$$

and

$$(*_2)\qquad \begin{aligned} &\Big(\wp(\omega_0) - \sum_{i=1,\,(i,p)=1}^{p^r-1} x^i\,\Phi^r(\omega_i) + \Psi_0\Big)\wedge \frac{d\alpha}{\alpha} = d\theta_0'\wedge\frac{d\alpha}{\alpha},\\ &\alpha^{ip^{r-1}}\Psi_{ip^{r-1}}\wedge\frac{d\alpha}{\alpha} = \alpha^{ip^{r-1}}d\theta'_{ip^{r-1}}\wedge\frac{d\alpha}{\alpha},\quad 1\le i\le p-1,\\ &\qquad\vdots\\ &\alpha^{qp}\Psi_{qp}\wedge\frac{d\alpha}{\alpha} = \alpha^{qp}d\theta'_{qp}\wedge\frac{d\alpha}{\alpha},\quad 1\le q\le p^{r-1}-1,\ (q,p)=1,\\ &\alpha^j(\omega_j+\Psi_j)\wedge\frac{d\alpha}{\alpha} = \alpha^j(d\theta_j'+j\theta_j)\wedge\frac{d\alpha}{\alpha},\quad 1\le j\le p^r-1,\ (j,p)=1. \end{aligned}$$

We note that in using each equation we keep in mind that $\ker\big(i_{L/F}:\Omega_F^n\to\Omega_L^n\big)=\Omega_F^{n-1}\wedge dx$.

The bottom equations from $(*_1)$ show that $\eta_j = d\theta_j+\chi_j'\wedge dx$ for some χ_j'. As $d\Psi_j\in\Omega_F^n\wedge dx$, by the bottom equations of $(*_2)$ we have

$$d\theta_j = j^{-1}(d\omega_j+d\Psi_j)+\chi_j''\wedge dx = j^{-1}d\omega_j+\chi_j'''\wedge dx\quad\text{for some }\ \chi_j'',\chi_j'''.$$

These give $\eta_j = j^{-1}d\omega_j+\chi_j\wedge dx$ for some χ_j whenever $1\le j\le p^r-1$ and $(j,p)=1$.

If $1\le q\le p^{r-1}-1$ and $(q,p)=1$ the second to last equations of $(*_1)$ show for some $\chi_{qp}\in\Omega_F^{n-1}$ that

$$\eta_{qp} = d\theta_{qp}+\sum_{k=0}^{p-1}x^k\Phi(\eta_{q+kp^{r-1}})+\chi_{qp}\wedge dx.$$

If $1\le q'\le p^{r-2}-1$ and $(q',p)=1$ the third to last equations of $(*_1)$ show for some $\chi_{q'p^2},\chi'_{q'p^2}\in\Omega_F^{n-1}$ and $\Theta_{q'p^2}\in\sum_{k=0}^{p-1}x^kB_2\Omega_F^{n+1}$, that

$$\begin{aligned} \eta_{q'p^2} &= d\theta_{q'p^2}+\sum_{k'=0}^{p-1}x^{k'}\Phi(\eta_{q'p+k'p^{r-1}})+\chi'_{q'p^2}\wedge dx\\ &= d\theta_{q'p^2}+\sum_{k'=0}^{p-1}x^{k'}\Phi\Big(d\theta_{q'p+k'p^{r-1}}+\sum_{k=0}^{p-1}x^k\Phi(\eta_{q'+k'p^{r-2}+kp^{r-1}})+\chi_{q'p+k'p^{r-1}}\wedge dx\Big)\\ &\qquad\qquad + \chi'_{q'p^2}\wedge dx\\ &= \Theta_{q'p^2}+\sum_{k'=0}^{p-1}\sum_{k=0}^{p-1}x^{k'+kp}\Phi^2(\eta_{q'+k'p^{r-2}+kp^{r-1}})+\chi_{q'p^2}\wedge dx\\ &= \Theta_{q'p^2}+\sum_{k=0}^{p^2-1}x^k\Phi^2(\eta_{q'+kp^{r-2}})+\chi_{q'p^2}\wedge dx. \end{aligned}$$

Repeating this process, substituting iteratively into the preceding equations in $(*_1)$, when the second set of equations is reached we have for $1 \le i \le p-1$,

$$\eta_{ip^{r-1}} = \Theta_{ip^{r-1}} + \sum_{k=0}^{p^{r-1}-1} x^k \Phi^{r-1}(\eta_{i+kp}) + \chi_{ip^{r-1}} \wedge dx,$$

where $\Theta_{ip^{r-1}} \in \sum_{i=0}^{p^{r-2}} x^i B_{r-1}\Omega_F^{n+1}$. At last, we find when the first equation is reached we substitute these expressions for the $\eta_{i\,p^{r-1}}$ where $1 \le i \le p-1$ and find that

$$\sum_{i=1,\,(i,p)=1}^{p^r-1} x^i \Phi^r(\eta_i) \in \wp(\Omega_F^{n+1}) + d\Omega_F^n + \Omega_F^n \wedge dx + \sum_{k=0}^{p^{r-1}-1} x^k B_r \Omega_F^{n+1}.$$

Using $\eta_i = i^{-1} d\omega_i + \chi_i \wedge dx$ gives that

$$\sum_{i=1,\,(i,p)=1}^{p^r-1} i^{-1} x^i \Phi^r(d\omega_i) \in \wp(\Omega_F^{n+1}) + d\Omega_F^n + \Omega_F^n \wedge dx + \sum_{k=0}^{p^{r-1}-1} x^k B_r \Omega_F^{n+1},$$

as required. □

In the applications the set up is slightly different than they are in the hypotheses of Theorem 2.2. The following corollary takes care of this for us.

Corollary 2.3. *Let $x \in F - F^p$ and $r, e \ge 1$. Suppose*

$$\sum_{i=1,\,(i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) \in \Gamma_F^n + \sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n + \sum_{i=1,\,(i,p)=1}^{p^{r+1}-1} x^i \Phi^{r+1}(\Omega_F^n),$$

where $\omega_i \in \Omega_F^n$. Then

$$\sum_{i=1,\,(i,p)=1}^{p^{r-1}-1} i^{-1} x^i \Phi^r(d\omega_i) \in \Gamma_F^{n+1} + \sum_{i=1}^{p^{r-1}-1} x^i B_r \Omega_F^{n+1}.$$

Proof. We express $\sum_{i=1,\,(i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) = \gamma + \sum_{i=1}^{p^{r+1}-1} x^i \Phi^{r+1}(\theta_i)$, where $\gamma \in \Gamma_F^n + \sum_{i=1}^{p^{e-1}-1} x^i B_e \Omega_F^n$ and $\theta_i \in \Omega_F^n$. If $\tilde{\omega}_i = \omega_i - \sum_{k=0}^{p-1} \Phi(\theta_{i+kp^r})$ we then have $\sum_{i=1}^{p^{r-1}-1} x^i \Phi^r(\tilde{\omega}_i) \in \Gamma_F^n + \sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n$. But we have $d\tilde{\omega}_i = d\omega_i$, so the result follows by Theorem 2.2. □

3. Background on the de Rham Witt complex

As noted in the introduction, the groups $H_{p^m}^{n+1} F$ for $m \ge 1$ require for their definition, either the groups $Q^n(F, m)$ as defined by Izhboldin, or the de Rham Witt groups $W_m \Omega_F^n$. Aside from the proof of Theorem 3.13 below, we work with the latter. In this section we collect results about $W_m\Omega^n_F$ needed to prove our main theorem. As references we cite [Illusie 1979] and Section 2 of [Aravire et al. 2018].

The module $W_m\Omega^n_F$ is a quotient of $\Omega^n_{W_mF}$ where W_mF is the ring of length m Witt vectors over F. As such it is generated by elements of the form $a\cdot db_1\wedge\cdots\wedge db_n$, where $a, b_1, \dots, b_n \in W_mF$. For $a \in F$ we define $[a]_m := (a, 0, \dots, 0) \in W_mF$ and note that these "representatives" are multiplicative but not additive. There are important interactions between these generators and the shift $V : W_m\Omega^n_F \to W_{m+1}\Omega^n_F$, a few of which seem unnatural until one is used to them. We do have $V(a)\cdot dV(b) = V(a\cdot db)$, as the first defining relations for $W_m\Omega^n_F$ ensure this. But one needs to be careful, for V and d do not commute. The relation just given shows that $V(dy) = V([1]_m\cdot dy) = V([1]_m)\cdot dV(y) = p\cdot dV(y)$, for although $[1]_m = 1 \in W_mF$ its shift $V([1]_m) = p \in W_mF$. The generalization of this is given in Lemma 3.2(i).

The second defining relation, $V(a)d[x]_m = V(a[x]^{p-1}_{m-1})dV[x]_{m-1}$, may appear odd at first but is critical to the theory. Using the Witt vector relation, $[b]_mV(a) = V([b^p]_{m-1}a)$, this relation shows that V commutes with the logarithmic derivative of multiplicative elements as follows: $V(a)[x]_m{}^{-1}d[x]_m = V(a[x]^{-p}_{m-1})\,d[x]_m = V(a[x]^{-p}_{m-1}[x]^{p-1}_{m-1})\,dV([x]_{m-1}) = V(a[x]_{m-1}{}^{-1}d[x]_{m-1})$. The logarithmic derivative also links Milnor K-theory to the de Rham Witt groups as follows. For a generator $\tau = \ell(a_1)\otimes\cdots\otimes\ell(a_n)$ of K^M_nF we define

$$\mathrm{dlog}_m(\tau) := \frac{d[a_1]_m}{[a_1]_m}\wedge\cdots\wedge\frac{d[a_n]_m}{[a_n]_m} \in W_m\Omega^n_F.$$

By [Illusie 1979, 3.23, p. 580] when $n = 1$ the map sends products to sums, and then the defining relations for $W_m\Omega^n_F$ ensure one obtains a well-defined homomorphism $\mathrm{dlog}_m : K^M_nF \to W_m\Omega^n_F$.

The next result was mentioned in the introduction. It appears as [Aravire et al. 2018, Lemma 2.2] and its proof is by induction using the second defining relation.

Lemma 3.1. *For all $x \in W_{m-s}F$ with $m > s \geq 1$ and $a \in F$ we have*

$$V^s(x)\,d[a]_m = V^s(x[a]^{p^s-1}_{m-s})\,dV^s[a]_{m-s} \in W_m\Omega^1_F.$$

The Frobenius $\Phi : W_mF \to W_mF$ given by $\Phi(a_1, \dots, a_m) = (a_1^p, \dots, a_m^p)$ extends with a level shift to $\Phi : W_m\Omega^m_F \to W_{m-1}\Omega^m_F$ (see [Illusie 1979]). In analogy with the $m = 1$ case where $\Phi : \Omega^n_F \to \Omega^n_F/d\Omega^{n-1}_F$, there is a lift to $\Phi : W_m\Omega^m_F \to W_m\Omega^m_F/dV^{m-1}\Omega^n_F$. We will use the same notation for both maps as the one in use will be clear from context. Since $p\cdot(a_1, \dots, a_m) = (0, a_1^p, \dots, a_{m-1}^p) \in W_mF$ we have $p\cdot = V\circ\Phi = \Phi\circ V$ on W_mF. The definitions on $W_m\Omega^n_F$ are set so that this rule generalizes as given in Lemma 3.2(iii) below. In addition to the s-fold composite of V, $V^s : W_{m-s}\Omega^n_F \to W_m\Omega^n_F$, there are restriction maps $R^s : W_m\Omega^n_F \to W_s\Omega^n_F$ which generalize the same map on Witt vectors given by $R^s(a_1, \dots, a_m) = (a_1, \dots, a_s)$. This map also behaves properly with respect to logarithmic derivatives where one has $R^s(a[x]_m{}^{-1}d[x]_m) = R^s(a)[x]_s{}^{-1}d[x]_s$. Finally we note that Φ behaves well with respect to logarithmic derivatives of multiplicative elements. According to [Illusie 1979, Proposition 2.18.5, p. 564] we have $\Phi(d[x]_m) = [x]_{m-1}^{p-1}d[x]_{m-1}$, so interpreted according to the lift gives $\Phi([x]_m{}^{-1}d[x]_m) = [x]_m^{p-1}d[x]_m^p/[x]_m^p = [x]_m{}^{-1}d[x]_m$. Since Φ is multiplicative we have $\Phi(\omega\wedge[x]_m{}^{-1}d[x]_m) = \Phi(\omega)\wedge[x]_m{}^{-1}d[x]_m$ for $\omega\in W_m\Omega_F^{n-1}$. There are other basic interactions between, d, V and Φ which we summarize for reference in this next lemma (see [Aravire et al. 2018, Lemma 2.9] or [Illusie 1979] for more details).

Lemma 3.2. *Suppose* $\operatorname{char}(F) = p$. *Then*:

(i) $Vd = pdV$ *as functions* $W_{m-1}\Omega_F^{n-1}\to W_m\Omega_F^n$.

(ii) $\Phi dV = d$ *as functions* $W_m\Omega_F^{n-1}\to W_m\Omega_F^n$.

(iii) $p\cdot = \Phi\circ V = V\circ\Phi$ *as functions* $W_m\Omega_F^n\to W_m\Omega_F^n$.

(iv) $d\,\Phi = p\,\Phi d$ *as functions* $W_m\Omega^n F\to W_{m-1}\Omega_F^{n+1}$.

(v) *If* $\omega_1\in W_m\Omega_F^r$ *and* $\omega_2\in W_{m-1}\Omega_F^s$ *then* $\omega_1V(\omega_2) = V(R^{m-1}\circ\Phi(\omega_1)\omega_2)\in W_m\Omega_F^{r+s}$.

For computations involving $\Phi : W_m\Omega_F^n\to W_m\Omega_F^n$, we will use logarithmic derivatives like what we used in the previous sections for the case of $m = 1$. Since we are working with the lift of Φ these representative values are well-defined (mod $dV^{m-1}(\Omega_F^{n-1})$), which is absorbed into the calculations when used. By the above we can use

$$\Phi\left(a\frac{d[b_1]_m}{[b_1]_m}\wedge\cdots\wedge\frac{d[b_n]_m}{[b_n]_m}\right) = \Phi(a)\frac{d[b_1]_m}{[b_1]_m}\wedge\cdots\wedge\frac{d[b_n]_m}{[b_n]_m}\quad\text{for } a, b_1,\ldots,b_n\in F.$$

Since Φ and V commute we can then use

$$\Phi\left(V^s\left(a\frac{d[b_1]_{m-s}}{[b_1]_{m-s}}\wedge\cdots\wedge\frac{d[b_n]_{m-s}}{[b_n]_{m-s}}\right)\right) = V^s\left(\Phi(a)\frac{d[b_1]_{m-s}}{[b_1]_{m-s}}\wedge\cdots\wedge\frac{d[b_n]_{m-s}}{[b_n]_{m-s}}\right).$$

These two formulae suffice for our computations in this and the next section.

The shift map on the ring of Witt vectors gives an exact sequence

$$0\to W_{m-s}F\xrightarrow{V^s}W_mF\xrightarrow{R^s}W_sF\to 0.$$

The analogue for the de Rham Witt complex is the following result from [Illusie 1979, Proposition 3.2, p. 568], upon which the induction used to prove the main theorem will be based.

Theorem 3.3. *Let* $R^s : W_m\Omega_F^n\to W_s\Omega_F^n$ *be the restriction. Then the following sequence is exact*,

$$0\to\left(V^sW_{m-s}\Omega_F^n + dV^sW_{m-s}\Omega_F^{n-1}\right)\hookrightarrow W_m\Omega_F^n\xrightarrow{R^s}W_s\Omega_F^n\to 0,$$

where $\left(V^s W_{m-s}\Omega_F^n + dV^s W_{m-s}\Omega_F^{n-1}\right)$ *is viewed as its image inside* $W_m\Omega_F^n$.

We remark that it is *not* a consequence of Theorem 3.3 that V^s in injective — indeed it is not and its kernel plays a special role in the sequel. In the theorem only its image in $W_m\Omega_F^n$ is considered which also takes into account relations between $V^s W_{m-s}\Omega_F^n$ and $dV^s W_{m-s}\Omega_F^{n-1}$ in $W_m\Omega_F^n$.

We next generalize the definition of the groups $B_r\Omega_F^n$ to the de Rham Witt complex. These are discussed in detail in [Aravire et al. 2018] and are important in relating the $Q^n(F,m)$ groups of Izhboldin to the de Rham Witt complex. (We also remark that Izhboldin must have understood such details, but decided to spell out his work in a different way.)

Definition 3.4. For $r, m, n \geq 1,\ B_r W_m\Omega_F^n \subseteq W_m\Omega_F^n$ is defined to be

$$B_r W_m\Omega_F^n := \ker\left(V^r : W_m\Omega_F^n \to W_{m+r}\Omega_F^n\right).$$

For convenience, $B_0 W_m\Omega_F^n = \{0\}$.

According to [Illusie 1979, 3.21.1.4, p. 579], for all $r, m, n \geq 1$ we have

$$\ker\left(V^r : W_m\Omega_F^n \to W_{m+r}\Omega_F^n\right) = \Phi^r(dV^m W_r\Omega_F^{n-1}).$$

This formula uses the shifted Φ and Φ^r is the r-fold composite of these shifted Φ. We note that one consequence of the relation $\Phi dV = d$ is that $\Phi(dV^s(W_{m-s}\Omega_F^n)) = dV^{s-1}(W_{m-s}\Omega_F^n)$. So for $r \leq m$ the definitions give $B_r W_m\Omega_F^n = dV^{m-r}(W_r\Omega_F^n)$, so in particular $B_1 W_m\Omega_F^n = dV^{m-1}\Omega_F^{n-1}$ and $B_m W_m\Omega_F^n = dW_m\Omega_F^{n-1}$. When $r \leq m$ this also gives that $B_r W_m\Omega_F^n = \Phi(B_{r-1}W_m\Omega_F^n)$.

This next lemma collects for reference the basic properties of $B_r W_m\Omega_F^n$. The Φ here are the unshifted Φ which is not a problem because $dV^{m-1}\Omega_F^{n-1} \subseteq B_r W_m\Omega_F^n$ for all $r \geq 1$.

Lemma 3.5. (i) $B_r W_m\Omega_F^n = \Phi^r(dV^m W_r\Omega_F^{n-1})$.

(ii) $B_1 W_m\Omega_F^n = dV^{m-1}\Omega_F^{n-1}$.

(iii) *For* $r < m$, $B_r W_m\Omega_F^n = dV^{m-r} W_r\Omega_F^{n-1}$.

(iv) $B_m W_m\Omega_F^n = dW_m\Omega_F^{n-1}$.

(v) *For* $r > m$, $B_r W_m\Omega_F^n = \Phi(B_{r-1}W_m\Omega_F^n) = \Phi^{r-m}(dW_m\Omega_F^{n-1})$.

Proof. (i) is the result from [Illusie 1979] just cited. (ii) is (i) with $r = 1$ together with Lemma 3.2(ii). (iii) follows from (i) and (ii) as Lemma 3.2(ii) gives $\Phi\, dV = d$. (iv) is (iii) with $r = m$, and (v) follows from (iv) using (i). □

It is important to understand how the $B_r W_m\Omega_F^n$ interact with V. This is given next.

Lemma 3.6. *For all* r, m, n *we have* $V(B_r W_m\Omega_F^n) \subseteq B_{r-1}W_{m+1}\Omega_F^n$.

Proof. We have the commutative diagram

$$\begin{array}{ccccc} B_r W_m \Omega^n_F & \longrightarrow & W_m \Omega^n_F & \xrightarrow{V^r} & W_{m+r} \Omega^n_F \\ {\scriptstyle V}\downarrow & & {\scriptstyle V}\downarrow & & {\scriptstyle =}\downarrow \\ \ker(V^{r-1}) & \longrightarrow & W_{m+1}\Omega^n_F & \xrightarrow{V^{r-1}} & W_{m+r}\Omega^n_F \end{array}$$

and since $B_{r-1} W_{m+1} \Omega^n_F = \ker\big(V^{r-1} : W_{m+1}\Omega^n_F \to W_{m+r}\Omega^n_F\big)$ the result follows by definition. □

We remark that the shift of indices in Lemma 3.6 will play an important role in the sequel. The next result generalizes well-known properties of $B_r \Omega^n_F$ to the more general case.

Lemma 3.7. *For all* r, m, n:

(i) *If* $a \in F$ *then* $[a]_m^{p^r} B_r W_m \Omega^n_F \subseteq B_r W_m \Omega^n_F$.

(ii) *If* $x \in F$ *and* $r \geq m$ *then* $[x]_m^{p^{r-m}} B_r W_m \Omega^n_F \subseteq B_r W_m \Omega^n_F + W_m \Omega^{n-1}_F \wedge d[x]_m$.

(iii) *For all* i, $[x]^i_m B_{m-1} W_m \Omega^n_F \subseteq B_{m-1} W_m \Omega^n_F + W_m \Omega^{n-1}_F \wedge d[x]_m$.

Proof. For (i) we proceed by induction on r. We have $B_1 W_m \Omega^n_F = dV^{m-1}\Omega^{n-1}_F$ by definition. Let $\omega \in \Omega^{n-1}_F$ and then

$$d([a]^p_m V^{m-1}(\omega)) = p[a]^{p-1}_m V^{m-1}(\omega) d[a]_m + [a]^p dV^{m-1}(\omega)$$

so as $pV^{m-1}(\omega) = 0$ we find $[a]^p dV^{m-1}(\omega) \in dV^{m-1} W_m \Omega^{n-1}_F$ giving the result when $r = 1$.

Next, assuming $[a]_m^{p^r} B_r W_m \Omega^n_F \subseteq B_r W_m \Omega^n_F$, we use Lemma 3.5(v) and the inductive hypothesis to see that

$$\begin{aligned} [a]_m^{p^{r+1}} B_{r+1} W_m \Omega^n_F = [a]_m^{p^{r+1}} \Phi(B_r W_m \Omega^n_F) &= \Phi([a]_m^{p^r} B_r W_m \Omega^n_F) \\ &\subseteq \Phi(B_r W_m \Omega^n_F) = B_{r+1} W_m \Omega^n_F. \end{aligned}$$

This gives (i).

For (ii) we note if $r = m$ and $\omega \in W_m \Omega^{n-1}_F$, by Lemma 3.5(iv) $B_m W_m \Omega^n_F = d W_m \Omega^{n-1}_F$ so we have $[x]_m d\omega = d([x]_m \omega) - (-1)^n \omega \wedge d[x]_m$ and the result follows in that case. The same argument applies when $r = m - 1$ as $B_{r-1} W_m \Omega^n_F = V W_{m-1} \Omega^n_F$ and $[x]^i_m V(\omega) = V([x]^{ip}_{m-1}\omega)$, giving (iii). To finish (ii), by induction, assuming the result for $r \geq m$ we note that for a generator $\Phi^{r+1-m}(d\omega)$ of $B_{r+1} W_m \Omega^n_F$ (mod $B_r W_m \Omega^n_F$) we have

$$\begin{aligned} [x]_m^{p^{r+1-m}} \Phi^{r+1-m}(d\omega) = \Phi^{r+1-m}([x]_m d\omega) &= \Phi^{r+1-m}\big(d([x]_m \omega) - \omega \wedge d[x]_m\big) \\ &\in B_{r+1} W_m \Omega^n_F + W_m \Omega^{n-1}_F \wedge d[x]_m. \end{aligned}$$

This gives (ii). □

According to Lemma 3.5(i) we have

$$\ker\big(V^r : W_m\Omega^n_F \to W_{m+r}\Omega^n_F\big) = \Phi^r(dV^m W_r\Omega^{n-1}_F).$$

This characterization gives a set of generators for the $B_rW_m\Omega^n_F$ which is spelled out next.

Theorem 3.8. *For $r \geq m$, $B_rW_m\Omega^n_F$ is generated by*

$$B_mW_m\Omega^n_F \quad \textit{and} \quad [s]^{p^t}_m \frac{d[s]_m}{[s]_m} \wedge \operatorname{dlog}_m(\tau),$$

$$\textit{where} \quad t \leq r-m,\ s \in F \textit{ and } \tau \in K^M_{n-1}F \quad \text{(Milnor K-theory)}.$$

In particular, $B_rW_m\Omega^n_F = B_rW_m\Omega^1_F \wedge \operatorname{dlog}_m(K^M_{n-1}F) + B_mW_m\Omega^n_F$.

Proof. We proceed by induction on $r \geq m$, the case of $r = m$ being trivial. Applying d to Theorem 3.3 when $s = 1$ we find that $B_mW_m\Omega^n_F = dW_m\Omega^{n-1}_F$ is generated by elements $d[s]_m \wedge \operatorname{dlog}_m(\tau)$ for $s \in F$ and $\tau \in K^M_{n-1}F$ over $B_{m-1}\Omega^n_F = dVW_{m-1}\Omega^{n-1}_F$. As $\Phi(d[s]_m \wedge \operatorname{dlog}_m(\tau)) = [s]^p_m[s]_m{}^{-1}d[s]_m \wedge \operatorname{dlog}_m(\tau)$ we are done for $r = m+1$ as $B_{m+1}W_m\Omega^n_F = \Phi(B_mW_m\Omega^n_F)$ and these will give generators over $\Phi(B_{m-1}W_m\Omega^n_F) = B_mW_m\Omega^n_F$. For $r > m+1$, as $B_rW_m\Omega^n_F = \Phi(B_{r-1}W_m\Omega^n_F)$ it suffices to note by the properties of Φ that

$$\Phi\left([s]^{p^t}_m \frac{d[s]_m}{[s]_m} \wedge \operatorname{dlog}_m(\tau)\right) = [s]^{p^{t+1}}_m \frac{d[s]_m}{[s]_m} \wedge \operatorname{dlog}_m(\tau).$$

This gives the first statement. The second statement follows from the first, proving the result. □

We now recall the machinery that is used to define the groups $H^{n+1}_{p^m}F$. These results are discussed in detail in [Aravire et al. 2018, Section 2]. To begin we recall the definition of Izhboldin's groups.

Definition 3.9 [Izhboldin 1991, p. 139]. The group $Q^n(F,m)$ is the quotient $A^n_m/\mathcal{J}^n_m$, where we define $A^n_m := W_mF^+ \otimes F^* \otimes \cdots \otimes F^*$ and $\mathcal{J}^n_m$ is the subgroup generated by the tensors

(i) $a \otimes t_1 \otimes \cdots \otimes t_n$, where $a \in W_mF$ and $t_i = t_j$ for some $i \neq j$, and

(ii) $V^s[a^\ell]_{m-s} \otimes a \otimes t_2 \otimes \cdots \otimes t_n$, where $a, t_2, \ldots t_n \in F^*$, $0 \leq s < m$, and $\ell \geq 1 \in \mathbb{N}$.

The next result shows how the Izhboldin groups are linked to the de Rham Witt complex. For this, one checks (see [Aravire et al. 2018, Section 2]), that $q^n_m : W_m\Omega^n_F \to Q^n(F,m)$ defined on generators by

$$q^n_m\left(a\frac{d[t_1]_m}{[t_1]_m} \wedge \cdots \wedge \frac{d[t_n]_m}{[t_n]_m}\right) = [a \otimes t_1 \otimes \cdots \otimes t_m] \in Q^n(F,m)$$

gives a well-defined homomorphism. Further, Izhboldin showed for all m and n that there is a well-defined homomorphism $\delta^n_m : K^M_nF \to Q^n(F,m)$ defined on generators by $\delta_m^n(\ell(t_1)\otimes\cdots\otimes\ell(t_n)) = [1\otimes t_1\otimes\cdots\otimes t_n]$. It is clear by the definitions that $\delta_m^n(\tau) = q_m^n \circ \mathrm{dlog}_m(\tau)$ for all $\tau \in K_n^M F$. We define $B_\infty W_m\Omega_F^n := \bigcup_{r\geq 1} B_r W_m\Omega_F^n$. By [Aravire et al. 2018, Theorem 2.27] we have the following.

Theorem 3.10. *The homomorphism $q_m^n : W_m\Omega_F^n \to Q^n(F,m)$ induces an isomorphism $Q^n(F,m) \cong W_m\Omega_F^n / B_\infty W_m\Omega_F^n$.*

Following [Aravire et al. 2018], using $\Phi(B_{r-1}W_m\Omega_F^n) \subseteq B_r W_m\Omega_F^n$, for $r \geq m$ we have a well defined map

$$\wp := (\Phi - 1) : \frac{W_m\Omega_F^n}{B_{r-1}W_m\Omega_F^n} \to \frac{W_m\Omega_F^n}{B_r W_m\Omega_F^n}.$$

whose kernel will temporarily be labeled $\mathrm{ker}_{m,r}^n$ and whose cokernel will temporarily be labeled $\mathrm{coker}_{m,r}^n$. With this notation we have the following result [Aravire et al. 2018, Lemma 2.30].

Lemma 3.11. *For all n, m and $r \geq m$ the following diagram is commutative with down-arrows induced by the inclusions $B_r W_m\Omega_F^n \subseteq B_{r+1} W_m\Omega_F^n \subseteq B_{r+2} W_m\Omega_F^n$ and induced isomorphisms as indicated.*

$$\begin{array}{ccccccc}
\mathrm{ker}_{m,r}^n & \longrightarrow & \dfrac{W_m\Omega_F^n}{B_{r-1}W_m\Omega_F^n} & \xrightarrow{\wp} & \dfrac{W_m\Omega_F^n}{B_r W_m\Omega_F^n} & \longrightarrow & \mathrm{coker}_{m,r}^n \\
\cong\downarrow & & \downarrow & & \downarrow & & \cong\downarrow \\
\mathrm{ker}_{m,r+1}^n & \longrightarrow & \dfrac{W_m\Omega_F^n}{B_r W_m\Omega_F^n} & \xrightarrow{\wp} & \dfrac{W_m\Omega_F^n}{B_{r+1}W_m\Omega_F^n} & \longrightarrow & \mathrm{coker}_{m,r+1}^n
\end{array}$$

Combining Lemma 3.11 with Theorem 3.10 and passing to direct limits we obtain the following result [Aravire et al. 2018, Theorem 2.31]. The first exact sequence provides the basis for our computations and the second is due to Izhboldin.

Theorem 3.12. *For all r with $m \leq r \leq \infty$ we have $\mathrm{ker}_{m,r}^n \cong K_n^M F / p^m K_m^M F$ and each $\mathrm{coker}_{m,r}^n \cong H_{p^m}^{n+1} F$. In particular we have exact sequences*

$$0 \longrightarrow K_n F/p^m K_n F \longrightarrow \frac{W_m\Omega_F^n}{B_{m-1}W_m\Omega_F^n} \xrightarrow{\wp} \frac{W_m\Omega_F^n}{B_m W_m\Omega_F^n} \longrightarrow H_{p^m}^{n+1} F \longrightarrow 0,$$

and

$$0 \longrightarrow K_n F/p^m K_n F \longrightarrow Q^n(F,m) \xrightarrow{\wp} Q^n(F,m) \longrightarrow H_{p^m}^{n+1} F \longrightarrow 0.$$

As $B_m W_m\Omega_F^n = dW_m\Omega_F^{n-1}$, the characterization of $H_{p^m}^{n+1}F$ given in the introduction

$$W_m\Omega_F^n \xrightarrow{\wp} \frac{W_m\Omega_F^n}{dW_m\Omega_F^{n-1}} \to H_{p^m}^{n+1} F \to 0$$

is a consequence of the first exact sequence of Theorem 3.12. Whenever $\omega \in W_m\Omega_F^n$ we use $\bar{\omega}$ to denote the class it determines in $W_m\Omega_F^n/\big(\wp(W_m\Omega_F^n) + dW_m\Omega_F^{n-1}\big) \cong H_{p^m}^{n+1}F$.

Although the shift-restriction short exact sequence for Witt Vectors doesn't generalize completely to the de Rham Witt complex, it does generalize nicely in cohomology. Although this was known by Izhboldin, we include a proof to illustrate the tools. The injectivity of V is critical in the sequel.

Theorem 3.13. *For all $m \geq 2$ we have exact sequences*

$$0 \to H_{p^{m-1}}^{n+1}F \xrightarrow{V} H_{p^m}^{n+1}F \xrightarrow{R} H_p^{n+1}F \to 0.$$

In particular, for $1 \leq \ell < m$ the maps $V^\ell : H_{p^{m-\ell}}^{n+1}F \to H_{p^m}^{n+1}F$ are injective.

Proof. By Izhboldin's work [1991, Proposition 6.3], the following diagram has exact rows and columns,

$$\begin{array}{ccccccccc}
 & & & & K_n^M F & \xrightarrow{1} & K_n^M F & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \delta_m^n} & & \downarrow{\scriptstyle \delta_1^n} & & \\
0 & \longrightarrow & Q^n(F, m-1) & \xrightarrow{V} & Q^n(F, m) & \xrightarrow{R} & Q^n(F, 1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \wp} & & \downarrow{\scriptstyle \wp} & & \downarrow{\scriptstyle \wp} & & \\
0 & \longrightarrow & Q^n(F, m-1) & \xrightarrow{V} & Q^n(F, m) & \xrightarrow{R} & Q^n(F, 1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \wp} & & \\
 & & H_{p^{m-1}}^{n+1}(F) & \longrightarrow & H_{p^m}^{n+1}(F) & \longrightarrow & H_p^{n+1}(F) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The standard chase gives the required exact sequence. □

To characterize cohomological kernels we need the next definition.

Definition 3.14. For $x \in F - F^p$ we set

$$\mathcal{L}_{m,e}^{n+1} := \sum_{r=1}^{e} \sum_{i=1,\,(i,p)=1}^{p^r-1} \overline{[x]_m^i B_r W_m \Omega_F^n} \subseteq H_{p^m}^{n+1}F,$$

$$\text{and} \quad K_{m,e}^{n+1} := \ker\big(i_{L/F} : H_{p^m}^{n+1}F \to H_{p^m}^{n+1}F(\sqrt[p^e]{x})\big).$$

We next describe the $V^\ell(xB_rW_{m-\ell}\Omega_F^n)$ images in $H_{p^m}^{n+1}F$.

Lemma 3.15. *Suppose $m > \ell \geq t$. Then for $x, s \in F$ we have*

$$V^\ell\big([x]_{m-\ell}[s]_{m-\ell}^{p^t-1} d[s]_{m-\ell}\big) \equiv p^{\ell-t} \cdot [xs^{p^t}]_m \frac{d[x]_m}{[x]_m} \pmod{\wp(W_m\Omega_F^n) + B_m W_m \Omega_F^n}.$$

In particular, whenever $r \leq m$,

$$\overline{V^\ell([x]_{m-\ell} B_r W_{m-\ell}\Omega_F^n)} \subseteq p^{m-r} \cdot \overline{W_m \Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1} F$$

and for all $m \geq e$, $\overline{\mathcal{L}_{m,e}^{n+1}} \subseteq p^{m-e} \cdot \overline{W_m \Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1} F$.

Proof. First we note that as $\ell \geq t$,

$$\begin{aligned} V^\ell\big([x]_{m-\ell}\Phi^t(d[s]_{m-\ell})\big) &= V^\ell\big([x]_{m-\ell}[s]_{m-\ell}^{p^t-1} d[s]_{m-\ell}\big) \\ &= V^\ell([xs^{p^t-1}]_{m-\ell}) V^\ell(d[s]_{m-\ell}) \\ &= V^\ell\big([xs^{p^t-1}]_{m-\ell}[s]_{m-\ell}^{1-p^\ell}\big)\, d[s]_m \\ &= V^\ell\big([xs^{p^t-1}s^{1-p^\ell}]_{m-\ell}[s]_{m-\ell}^{p^\ell}\big) \frac{d[s]_m}{[s]_m} \\ &= V^\ell([xs^{p^t}]_{m-\ell}) \frac{d[s]_m}{[s]_m}, \end{aligned}$$

where the second line follows using Lemma 3.1. Next we have

$$\begin{aligned} V^\ell([xs^{p^t}]_{m-\ell}) \frac{d[s]_m}{[s]_m} &\equiv V^\ell([xs^{p^t}]_{m-\ell}^{p^\ell}) \frac{d[s]_m}{[s]_m} \pmod{\wp(W_m\Omega_F^n) + B_m W_m\Omega_F^n} \\ &= p^\ell \cdot [xs^{p^t}]_m \frac{d[s]_m}{[s]_m} = p^{\ell-t} \cdot [xs^{p^t}]_m \frac{d[s^{p^t}]_m}{[s^{p^t}]_m} \\ &\equiv p^{\ell-t} \cdot [xs^{p^t}]_m \frac{d[x]_m}{[x]_m} \pmod{\wp(W_m\Omega_F^n) + B_m W_m\Omega_F^n}, \end{aligned}$$

where we used $\wp = \Phi - 1$ in the first line and recall that $B_m W_m \Omega_F^n = dW_m\Omega_F^{n-1}$ in the third. These show that if $m > \ell \geq t$ then

$$\overline{V^\ell([x]_{m-\ell}\Phi^t(d[s]_{m-\ell}))} \in p^{\ell-t} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^2 F$$

and from this, $\overline{V^\ell([x]_{m-\ell} B_{t+(m-\ell)} W_{m-\ell}\Omega_F^n)} \subseteq p^{\ell-t} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1} F$ using the second statement of Theorem 3.8. As $m + (t-\ell) \leq m$, setting $r = t+m-\ell$ so $\ell - t = m - r$ this can be reformulated as

$$\overline{V^\ell([x]_{m-\ell} B_r W_{m-\ell}\Omega_F^n)} \subseteq p^{m-r} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1} F$$

whenever $r \leq m$. Using the fact that $r \leq e$ in the definition of $\mathcal{L}_{m,e}^{n+1}$, the result just given shows when $m \geq e$ that $\overline{\mathcal{L}_{m,e}^{n+1}} \subseteq p^{m-e} \cdot \overline{W_e\Omega_F^{n-1} \wedge d[x]_e} \subseteq H_{p^e}^{n+1} F$. This proves the lemma. □

This final lemma considers what happens when we pass to $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$. We remark that part (iii) in this lemma describes the case where special computations will be required to carry out the induction in the proof of the main theorem in the next section.

Lemma 3.16. *Suppose $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$ and $e \geq m > \ell$.*

(1) *Then $V^\ell(\mathcal{L}^{n+1}_{m-\ell,e}) \subseteq K^{n+1}_{m,e}$.*

(2) *Suppose $t \leq e + \ell - m$. Then*

$$i_{L/F}\left(\overline{V^\ell\left([x^i s^{p^t}]_{m-\ell}\frac{d[s]_{m-\ell}}{[s]_{m-\ell}}\right)}\right) = 0 \in H^2_{p^m} L.$$

(3) *When $t = e + \ell - m + 1$ we have*

$$i_{L/F}\left(\overline{V^\ell\left([x^i s^{p^t}]_{m-\ell}\frac{d[s]_{m-\ell}}{[s]_{m-\ell}}\right)}\right) = -\overline{V^{m-1}\left(i x^i s^{p^t}\frac{d\alpha}{\alpha}\right)} \in H^2_{p^m} L.$$

Proof. For (i) we have to show that $i_{L/F}\big(\overline{V^\ell([x]^i_{m-\ell} B_r W_{m-\ell}\Omega^n_F)}\big) = \{0\} \subseteq H^{n+1}_{p^m} L$ for $1 \leq r \leq e$. For $r \leq m$, $B_r W_m \Omega^n_F = dV^{m-r} W_r \Omega^{n-1}_F$ and therefore $[x]^i_m B_r W_m \Omega^n_F \subseteq dW_m\Omega^{n-1}_F + W_m\Omega^{n-1}_F \wedge d[x]_m$ which as $x \in L^{p^e}$ gives the result in this case. For $r \geq m$, by Theorem 3.8 the generators of $V^\ell([x]^i_{m-\ell} B_r W_{m-\ell}\Omega^n_F)$ are of the form

$$V^\ell\left([x^i]_{m-\ell}[s]^{p^t}_{m-\ell}\frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \wedge \operatorname{dlog}_{m-\ell}(\tau)\right),$$

where $t \leq r - (m - \ell)$, $s \in F$ and $\tau \in K^M_{n-1} F$. From this, $\ell + e - t \geq \ell + e - (r - (m - \ell)) = e - r + m$, so as $e - r \geq 0$ we find $\ell + e - t \geq m$. We have the following in $H^2_{p^m} L$ whenever $t < r \leq e$:

$$\begin{aligned} i_{L/F}\left(\overline{V^\ell\left([x^i s^{p^t}]_{m-\ell}\frac{d[s]_{m-\ell}}{[s]_{m-\ell}}\right)}\right) &= \overline{V^\ell\left([\alpha^{ip^e} s^{p^t}]_{m-\ell}\frac{d[s]_{m-\ell}}{[s]_{m-\ell}}\right)} \\ &= \overline{V^\ell\left([\alpha^{ip^{e-t}} s]^{p^t}_{m-\ell}\frac{d[s]_{m-\ell}}{[s]_{m-\ell}}\right)} \\ &= \overline{V^\ell\left([\alpha^{ip^{e-t}} s]_{m-\ell}\frac{d[s]_{m-\ell}}{[s]_{m-\ell}}\right)} \\ &= -ip^{e-t}\overline{V^\ell\left([\alpha^{ip^{e-t}} s]_{m-\ell}\frac{d[\alpha]_{m-\ell}}{[\alpha]_{m-\ell}}\right)}. \end{aligned}$$

This latter element is 0 as long as $\ell + e - t \geq m$. This gives the desired result when $r \geq m$. Part (ii) is a consequence of the latter computation.

For part (iii), when $t = e + \ell - m + 1$, as $m - \ell - 1 \geq 0$ we have by (ii)

$$\begin{aligned} i_{L/F}\left(\overline{V^\ell\left([x^i s^{p^t}]_{m-\ell}\frac{d[s]_{m-\ell}}{[s]_{m-\ell}}\right)}\right) &= -ip^{e-t}\,\overline{V^\ell\left([\alpha^{ip^{e-t}}s]_{m-\ell}\frac{d[\alpha]_{m-\ell}}{[\alpha]_{m-\ell}}\right)} \\ &= -ip^{m-\ell-1}\,\overline{V^\ell\left([\alpha^{ip^{m-\ell-1}}s]_{m-\ell}\frac{d[\alpha]_{m-\ell}}{[\alpha]_{m-\ell}}\right)} \\ &= -i\,\overline{V^{m-1}\left(\alpha^{ip^{2(m-\ell-1)}}s^{p^{m-\ell-1}}\frac{d\alpha}{\alpha}\right)} \\ &= -\overline{V^{m-1}\left(i\alpha^{ip^e}s^{p^{e+\ell-m+1}}\frac{d\alpha}{\alpha}\right)} \\ &= -\overline{V^{m-1}\left(ix^i s^{p^t}\frac{d\alpha}{\alpha}\right)}. \end{aligned}$$

This gives (iii). □

4. The main result

We begin by restating Theorem 1.8 in the case of a simple extension. Although our notation is slightly different, this is the result from [Sobiech 2018, Theorem 3.16] and when $p = 2$ it is [Aravire et al. 2019, Theorem 3.1]. This formulation is convenient for the proof of Theorem 4.2.

Theorem 4.1 (Aravire–Laghribi–O'Ryan–Sobiech). *Suppose $e \geq 1$ and $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$. Suppose $\omega \in \Omega_F^n$ with $\bar\omega \in \ker : H_p^{n+1}F \to H_p^{n+1}L$. Then*

$$\omega \in \sum_{r=1}^{e}\sum_{\substack{i=1\\(i,p)=1}}^{p^{r-1}-1} x^i B_r\Omega_F^n + \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1}\wedge dx.$$

We now give the main theorem. Set $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$ and $e \geq 1$. We recall

$$K_{m,e}^{n+1} := \ker\bigl(i_{L/F} : H_{p^m}^{n+1}F \to H_{p^m}^{n+1}L\bigr)$$

$$\text{and}\quad \mathcal{L}_{m,e}^{n+1} := \sum_{r=1}^{e}\sum_{i=1,\,(i,p)=1}^{p^r-1}[x]_m^i B_r W_m\Omega_F^n \subseteq W_m\Omega^n F.$$

The result is that $K_{m,e}^{n+1} = \sum_{\ell=0}^{m-1}\overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})}$. By Lemma 3.16, $\overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})} \subseteq K_{m,e}^{n+1}$, so our task is to show that $K_{m,e}^{n+1} \subseteq \sum_{\ell=0}^{m-1}\overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})}$. The proof is inductive and the theorem is stated that way for the reader's convenience. The idea behind the induction is to combine Theorem 4.1 with the exact sequence in Theorem 3.3. The subtlety in the argument is that one could proceed with the induction in a simplistic fashion, but then the value of the e in $V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})$ would grow by 1 at each increment in m. This would yield a result, but not what is desired; in particular part (iv), the key to the result described in the introduction, would not be obtained.

Theorem 4.2. *Suppose $\omega \in W_m\Omega_F^n$ where $m, n \geq 1$. Set $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$ and $e \geq 1$. Set $K_{m,e}^{n+1} := \ker\big(i_{L/F} : H_{p^m}^{n+1}F \to H_{p^m}^{n+1}L\big)$.*

(i) *If $\bar{\omega} \in K_{m,e}^{n+1}$ then $\bar{\omega} \in \overline{\mathcal{L}_{m,e}^{n+1}} + V(K_{m-1,e}^{n+1})$.*

(ii) *If $\bar{\omega} \in \overline{\mathcal{L}_{m,e}^{n+1}} + V(K_{m-1,e}^{n+1})$ then $\bar{\omega} \in \sum_{\ell=0}^{m-1} \overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})}$.*

(iii) *For all $m \geq 1$, $K_{m,e}^{n+1} = \sum_{\ell=0}^{m-1} \overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})}$.*

(iv) *When $m \geq e$ we have $K_{m,e}^{n+1} = p^{m-e} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1}F$.*

(v) *For all $m, n \geq 1$,*

$$\ker(H_{p^m}^{n+1}F \to H_{p^m}^{n+1}L) = \ker(H_{p^m}^2F \to H_{p^m}^2L) \wedge \overline{\mathrm{dlog}_m(K_{n-1}^MF)}.$$

Proof. The proof proceeds by induction on m. When $m = 1$, since there are no V terms parts (i), (ii) and (iii) follow by Theorem 4.1. When $m = 1$ part (iv) is Theorem 1.10. For $m > 1$ we start with part (i) and assume the theorem for $m - 1$. Suppose $\bar{\omega} \in K_{m,e}^{n+1}$. As $\overline{R^{m-1}(\omega)} = 0 \in H_{p^{m-1}}^{n+1}F$, by the $m - 1$ case part (iii) we know $R^{m-1}(\omega) \in \sum_{\ell=0}^{m-2} V^\ell(\mathcal{L}_{m-\ell-1,e}^{n+1}) + \wp(W_{m-1}\Omega_F^n) + dW_{m-1}\Omega_F^{n-1}$. By Theorem 3.8, for $r \geq m - \ell - 1$, if $t \leq r - (m - \ell - 1)$ and $s \in F$ we have $[s^{p^t}]_{m-\ell-1}[s]_{m-\ell-1}{}^{-1}d[s]_{m-\ell-1} \in B_rW_{m-\ell-1}\Omega_F^1$ and are a set of generators over dW_mF. We can lift such a generator to $[s^{p^t}]_{m-\ell}[s]_{m-\ell}{}^{-1}d[s]_{m-\ell}$ and note that as $t \leq r + 1 - (m - \ell)$ it is a generator for $B_{r+1}W_{m-\ell}\Omega_F^1$. So these lifts to $W_{m-\ell}\Omega_F^n$ lie in $\sum_{r=1}^{e} \sum_{i=1,\,(i,p)=1}^{p^r-1} [x]_{m-\ell}^i B_{r+1}W_{m-\ell}\Omega_F^n \subseteq \mathcal{L}_{m-\ell,e+1}^{n+1}$. We note the shift from e to $e + 1$ occurs because of the increase of r to $r + 1$ in the subscript of B_{r+1} (even though in the sum we have $r \leq e$.) The technical details of this proof involve dealing with this shift from e to $e + 1$ in the subscript of $\mathcal{L}_{m-\ell,e+1}^{n+1}$ that are a result of this lifting of the inductive hypothesis.

For the proof we will carefully select $\omega_0 \in W_m\Omega_F^n$ with $R^{m-1}(\omega_0) = R^{m-1}(\omega)$. For such ω_0, by Theorem 3.3, there exist $\hat{\omega} \in \Omega_F^n$ and $\hat{\omega}' \in \Omega_F^{n-1}$ with

$$\omega = \omega_0 + V^{m-1}(\hat{\omega}) + dV^{m-1}(\hat{\omega}') \in W_m\Omega_F^n.$$

As $\overline{dV^{m-1}(\hat{\omega}')} = 0 \in H_{p^m}^{n+1}F$ we have

$$\bar{\omega} = \overline{\omega_0 + V^{m-1}(\hat{\omega}) + dV^{m-1}(\hat{\omega}')} = \overline{\omega_0 + V^{m-1}(\hat{\omega})} \in K_{m,e}^{n+1}.$$

By the previous paragraph

$$R^{m-1}(\omega_0) = R^{m-1}(\omega) \in \sum_{\ell=0}^{m-2} V^\ell(\mathcal{L}_{m-\ell-1,e}^{n+1}) + \wp(W_{m-1}\Omega_F^n) + dW_{m-1}\Omega_F^{n-1}$$

and we can choose the lifts of $\mathcal{L}^{n+1}_{m-\ell-1,e}$ elements to lie in

$$\sum_{r=1}^{e} \sum_{i=1,\,(i,p)=1}^{p^r-1} [x]^i_{m-\ell} B_{r+1} W_{m-\ell}\Omega^n_F \subseteq \mathcal{L}^{n+1}_{m-\ell,e+1}.$$

Pulling out the $r = e$ summand we decompose

$$\begin{aligned}\sum_{r=1}^{e} \sum_{i=1,\,(i,p)=1}^{p^r-1} & [x]^i_{m-\ell} B_{r+1} W_{m-\ell}\Omega^n_F \\ &= \sum_{i=1,\,(i,p)=1}^{p^e-1} [x]^i_{m-\ell} B_{e+1} W_{m-\ell}\Omega^n_F + \sum_{r=1}^{e-1} \sum_{i=1,\,(i,p)=1}^{p^r-1} [x]^i_{m-\ell} B_{r+1} W_{m-\ell}\Omega^n_F,\end{aligned}$$

where the second sum lies in $\mathcal{L}^{n+1}_{m-\ell,e}$. So we can choose ω_0 with

$$\omega_0 \in \sum_{\ell=0}^{m-2} V^\ell(\mathcal{L}^{n+1}_{m-\ell,e+1}) + \wp(W_m\Omega^n_F) + dW_m\Omega^{n-1}_F,$$

where we can further decompose $\omega_0 = \omega_0' + \omega_0''$ with $\omega_0' = \sum_{\ell=0}^{m-2} V^\ell(\omega'_{0\ell})$ with carefully chosen $\omega'_{0\ell} \in \sum_{i=1,\,(i,p)=1}^{p^e-1}[x]^i_{m-\ell} B_{e+1} W_{m-\ell}\Omega^n_F$ (specified below) and where $\omega_0'' \in \sum_{\ell=0}^{m-2} V^\ell(\mathcal{L}^{n+1}_{m-\ell,e}) + \wp(W_m\Omega^n_F) + dW_m\Omega^{n-1}_F$.

We claim that if further ω_0' is chosen in such a way that $\overline{\omega'_{00}} \in \overline{\mathcal{L}^{n+1}_{m,e}} + \overline{V(W_{m-1}\Omega^n_F)}$ part (i) will be proved. For as $\omega_0' = \sum_{\ell=0}^{m-2} V^\ell(\omega'_{0\ell})$ we would then have $\overline{\omega_0'} \in \overline{\mathcal{L}^{n+1}_{m,e}} + \overline{V(W_{m-1}\Omega^n_F)}$. By the set up,

$$\omega = \omega_0 + V^{m-1}(\hat\omega) + dV^{m-1}(\hat\omega') = \omega_0' + \omega_0'' + V^{m-1}(\hat\omega) + dV^{m-1}(\hat\omega'),$$

where $\omega_0'' \in \mathcal{L}^{n+1}_{m,e} + V(W_{m-1}\Omega^n_F)$. So given the claim we would have

$$\bar\omega \in \overline{\mathcal{L}^{n+1}_{m,e}} + \overline{V(W_{m-1}\Omega^n_F)}.$$

We express $\bar\omega = \overline{\omega'} + V(\overline{\omega''})$, where $\omega' \in \mathcal{L}^{n+1}_{m,e}$ and $\omega'' \in W_{m-1}\Omega^n_F$. By Lemma 3.16 we know $\overline{\omega'} \in K^{n+1}_{m,e}$, so as $\bar\omega \in K^{n+1}_{m,e}$ we also have $V(\overline{\omega''}) \in K^{n+1}_{m,e}$. By Theorem 3.13, the injectivity of $V : H^{n+1}_{p^{m-1}}L \to H^{n+1}_{p^m}L$ shows that $\overline{\omega''} \in K^{n+1}_{m-1,e}$. By induction we find $\overline{\omega''} \in \overline{\mathcal{L}^{n+1}_{m-1,e}} + V(K^{n+1}_{m-2,e}) \subseteq K^{n+1}_{m-1,e}$. From this, $\bar\omega = \overline{\omega'} + V(\overline{\omega''}) \in \overline{\mathcal{L}^{n+1}_{m,e}} + V(K^{n+1}_{m-1,e})$ and (i) follows.

We next carefully choose ω_0 and ω_0' to meet the requirements. According to Theorem 3.8, since we are working with $B_{e+1}W_{m-\ell}\Omega^n_F$ taken (mod $B_e W_{m-\ell}\Omega^n_F$), we know $\omega_0' = \sum_{\ell=0}^{m-2} V^\ell(\omega'_{0\ell})$ where $\omega'_{0\ell} \in \sum_{i=1,\,(i,p)=1}^{p^e-1}[x]^i_{m-\ell} B_{e+1} W_{m-\ell}\Omega^n_F$ can be expanded as a sum

$$\omega'_{0\ell} := \sum_{i=1,\,(i,p)=1}^{p^e-1} [x]^i_{m-\ell}\Bigl(\sum_j [s_{\ell ij}]_{m-\ell}^{p^{e+1-(m-\ell)}} \frac{d[s_{\ell ij}]_{m-\ell}}{[s_{\ell ij}]_{m-\ell}} \wedge \operatorname{dlog}_{m-\ell}(\tau_{\ell ij})\Bigr)$$

for $s_{\ell ij} \in F$ and $\tau_{\ell ij} \in K^M_{n-1}F$, where the j depend upon the ℓ, i and their number may vary. For reference below, we note using the $\ell = 0$ terms, these definitions give

$$R^1(\omega_0') = R^1(\omega_{00}') = \sum_{i=1,\,(i,p)=1}^{p^e-1} x^i \left(\sum_j s_{0ij}^{p^{e+1-m}} \frac{ds_{0ij}}{s_{0ij}} \wedge \mathrm{dlog}_1(\tau_{0ij}) \right).$$

Our goal is to show that $R^1(\omega_0') \in \wp(\Omega^n_F) + d\Omega^{n-1}_F + \Omega^{n-1}_F \wedge dx + \sum_{i=1}^{p^e-1} x^i B_e \Omega^n_F$, for then it will follow we can modify ω_{00}' so that $\overline{\omega_{00}'} \in \mathcal{L}^{n+1}_{m,e} + \overline{V(W_{m-1}\Omega^n_F)}$ and we would have what we need. To do this we have to analyze

$$i_{L/F}(\overline{\omega_0'}) = \sum_{\ell=0}^{m-2} i_{L/F}(\overline{V^\ell(\omega_{0\ell}')}) \in H^{n+1}_{p^m} L$$

more closely, in particular each summand $i_{L/F}(\overline{V^\ell(\omega_{0\ell}')})$ including those with $\ell > 0$. Applying Lemma 3.16(iii) when $t = e + \ell - m + 1$ we have

$$\overline{i_{L/F}\left(V^\ell \left([x^i s^{p^t}]_{m-\ell} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \right) \right)} = -\overline{V^{m-1}\left(i x^i s^{p^t} \frac{d\alpha}{\alpha} \right)} \in H^{n+1}_{p^m} F.$$

From this we find

$$\begin{aligned}
\overline{i_{L/F}(\omega_0')} &= \overline{\sum_{\ell=0}^{m-2} i_{L/F}\left(V^\ell \left(\sum_{\substack{i=1\\(i,p)=1}}^{p^e-1} [x]^i_{m-\ell} \left(\sum_j [s_{\ell ij}]_{m-\ell}^{p^{e+1-(m-\ell)}} \frac{d[s_{\ell ij}]_{m-\ell}}{[s_{\ell ij}]_{m-\ell}} \wedge \mathrm{dlog}_{m-\ell}(\tau_{\ell ij}) \right) \right) \right)} \\
&= -\overline{V^{m-1}\left(\sum_{\ell=0}^{m-2} \sum_{\substack{i=1\\(i,p)=1}}^{p^e-1} i x^i \sum_j \left(s_{\ell ij}^{p^{e+1-(m-\ell)}} \frac{d\alpha}{\alpha} \wedge \mathrm{dlog}_1(\tau_{\ell ij}) \right) \right)} \\
&= \overline{V^{m-1}\left(\left(\sum_{\ell=0}^{m-2} \hat{\omega}_\ell \right) \wedge \frac{d\alpha}{\alpha} \right)},
\end{aligned}$$

where using $\alpha^{-1} d\alpha \wedge \mathrm{dlog}_1(\tau_{\ell ij}) = (-1)^{n-1} \mathrm{dlog}_1(\tau_{\ell ij}) \wedge \alpha^{-1} d\alpha$ we define

$$\begin{aligned}
\hat{\omega}_\ell &:= (-1)^n \sum_{i=1,\,(i,p)=1}^{p^e-1} i x^i \left(\sum_j \left(s_{\ell ij}^{p^{e+1-(m-\ell)}} \wedge \mathrm{dlog}_1(\tau_{\ell ij}) \right) \right) \\
&= (-1)^n \sum_{i=1,\,(i,p)=1}^{p^e-1} i x^i \Phi^{p^{e+1-(m-\ell)}}(\omega_{\ell i}).
\end{aligned}$$

and we define $\omega_{\ell i} := \sum_j (s_{\ell ij} \wedge \mathrm{dlog}_1(\tau_{\ell ij}))$. Altogether this means we have

$$\sum_{\ell=0}^{m-2} \hat{\omega}_\ell = (-1)^n \sum_{\ell=0}^{m-2} \sum_{\substack{i=1\\(i,p)=1}}^{p^e-1} i x^i \Phi^{p^{e+1-(m-\ell)}}(\omega_{\ell i}).$$

We have

$$0=\overline{i_{L/F}(\omega)}=\overline{i_{L/F}(\omega_0)}+\overline{i_{L/F}(V^{m-1}(\hat{\omega}))}=\overline{i_{L/F}(\omega_0')}+\overline{i_{L/F}(\omega_0'')}+\overline{i_{L/F}(V^{m-1}(\hat{\omega}))}.$$

By Lemma 3.16(ii) we have $\overline{i_{L/F}(\omega_0'')}=0$ so we find $\overline{i_{L/F}(\omega_0')}+\overline{i_{L/F}(V^{m-1}(\hat{\omega}))}=0$. Combining this with the previous shows

$$V^{m-1}\left(\overline{i_{L/F}\left(\sum_{\ell=0}^{m-2}\hat{\omega}_\ell\right)\wedge\frac{d\alpha}{\alpha}}\right)+V^{m-1}(\overline{i_{L/F}(\hat{\omega})})=0.$$

The injectivity of $V^{m-1}:H_p^{n+1}L\to H_{p^m}^{n+1}L$ gives

$$\overline{i_{L/F}(\hat{\omega})+i_{L/F}\left(\sum_{\ell=0}^{m-2}\hat{\omega}_\ell\right)\wedge\frac{d\alpha}{\alpha}}=0\in H_p^{n+1}L.$$

From this we have

$$i_{L/F}(\hat{\omega})+i_{L/F}\left(\sum_{\ell=0}^{m-2}\hat{\omega}_\ell\right)\wedge\frac{d\alpha}{\alpha}\in\wp(\Omega_L^n)+d\Omega_L^{n-1}.$$

We know

$$L=F(\alpha)=\bigoplus_{i=0}^{p^e-1}\alpha^iF\quad\text{and}\quad\Omega_L^n=L\cdot\Omega_F^n\oplus L\cdot\Omega_F^{n-1}\wedge\frac{d\alpha}{\alpha}.$$

Expanding $i_{L/F}(\hat{\omega})+i_{L/F}\big(\sum_{\ell=0}^{m-2}\hat{\omega}_\ell\big)\wedge\alpha^{-1}d\alpha=\wp(\eta)+d\theta\in\Omega_L^n$, where $\eta=\sum_{i=0}^{p^e-1}\alpha^i\eta_i+\sum_{i=0}^{p^e-1}\alpha^i\eta_i'\wedge\alpha^{-1}d\alpha$ and $\theta=\sum_{i=0}^{p^e-1}\alpha^i\theta_i+\sum_{i=0}^{p^e-1}\alpha^i\theta_i'\wedge\alpha^{-1}d\alpha$ for $\eta_i\in\Omega_F^n$, $\theta_i,\eta_i'\in\Omega_F^{n-1}$ and $\theta_i'\in\Omega_F^{n-2}$ gives the $2p^e$ equations, one for each summand in $\Omega_L^n=\big(\bigoplus_{i=0}^{p^e-1}\alpha^iF\cdot\Omega_F^n\big)\oplus\big(\bigoplus_{i=0}^{p^e-1}\alpha^iF\cdot\Omega_F^{n-1}\wedge\alpha^{-1}d\alpha\big)$:

$$i_{L/F}(\hat{\omega})=\wp(\eta_0)+\sum_{k=1}^{p-1}x^k\Phi(\eta_{kp^{e-1}})+d\theta_0,$$

$$0=\alpha^{ip^{e-1}}\left(-\eta_{ip^{e-1}}+d\theta_{ip^{e-1}}+\sum_{k=0}^{p-1}x^k\Phi(\eta_{ip^{e-2}+kp^{e-1}})\right),\quad 1\le i\le p-1,$$

$(*_{\mathrm{i}})$

$$\vdots$$

$$0=\alpha^{qp}\left(-\eta_{qp}+d\theta_{qp}+\sum_{k=0}^{p-1}x^k\Phi(\eta_{q+kp^{e-1}})\right),\quad 1\le q\le p^{e-1}-1,\quad (q,p)=1,$$

$$0=\alpha^j(-\eta_j+d\theta_j),\quad 1\le j\le p^e-1,\quad (j,p)=1,$$

and

$$i_{L/F}\left(\sum_{\ell=0}^{m-2}\hat{\omega}_\ell\right)\wedge\frac{d\alpha}{\alpha}=\left(\wp(\eta'_0)+\sum_{k=1}^{p-1}x^k\Phi(\eta'_{kp^{e-1}})+d\theta'_0\right)\wedge\frac{d\alpha}{\alpha},$$

$$0=\alpha^{ip^{e-1}}\left(-\eta'_{ip^{e-1}}+d\theta'_{ip^{e-1}}+\sum_{k=0}^{p-1}x^k\Phi(\eta'_{ipe-2+kp^{e-1}})\right)\wedge\frac{d\alpha}{\alpha},\quad 1\le i\le p-1,$$

$(*_{\text{ii}})$

$$\vdots$$

$$0=\alpha^{qp}\left(-\eta'_{qp}+d\theta'_{qp}+\sum_{k=0}^{p-1}x^k\Phi(\eta'_{q+kp})\right)\wedge\frac{d\alpha}{\alpha},\quad 1\le q\le p^{e-1}-1,\ (q,p)=1,$$

$$0=\alpha^j\left(-\eta'_j+d\theta'_j+j\theta_j\right)\wedge\frac{d\alpha}{\alpha},\quad 1\le j\le p^e-1,\ (j,p)=1.$$

We note that in using each equation we keep in mind that $\ker(i_{L/F}:\Omega^n_F\to\Omega^n_L)=\Omega^{n-1}_F\wedge dx$.

Using the equations in $(*_{\text{ii}})$ collectively, starting from the bottom and iteratively solving for each η'_j in terms of the $d\theta'_i$ and the θ_j for $(j,p)=1$, and substituting to find $\sum_{\ell=0}^{m-2}\hat{\omega}_\ell$, gives

$$\sum_{\ell=0}^{m-2}\hat{\omega}_\ell\in\wp(\Omega^{n-1}_F)+d\Omega^{n-2}_F+\Omega^{n-2}_F\wedge dx+\sum_{i=1}^{p^e-1}x^iB_e\Omega^{n-1}_F+\sum_{\substack{j=1\\(j,p)=1}}^{p^{e-1}-1}x^j\Phi^e(\Omega^{n-1}_F).$$

By the above,

$$\sum_{\ell=0}^{m-2}\hat{\omega}_\ell=(-1)^n\sum_{\ell=0}^{m-2}\sum_{\substack{i=1\\(i,p)=1}}^{p^e-1}ix^i\Phi^{p^{e+1-(m-\ell)}}(\omega_{\ell i}).$$

By Corollary 2.3 we have for the minimum $e+1-(m-\ell)$-value of $e+1-m<e$ when $\ell=0$,

$$\sum_{i=1,\,(i,p)=1}^{p^e-1}i^{-1}ix^i\Phi^{p^{e+1-m}}(d\omega_{0i})\in\wp(\Omega^n_F)+d\Omega^{n-1}_F+\Omega^{n-1}_F\wedge dx+\sum_{i=1}^{p^e-1}x^iB_e\Omega^n_F.$$

We note that $\Phi^{p^{e+1-m}}(d\omega_{0i})=\sum_j s_{rij}^{p^{e+1-m}}s_{0ij}{}^{-1}ds_{0ij}\wedge\operatorname{dlog}_1(\tau_{0ij})$. This gives

$$\begin{aligned}R^1(\omega'_{00})&=\sum_{i=1,\,(i,p)=1}^{p^e-1}x^i\left(\sum_j s_{0ij}^{p^{e+1-m}}\frac{ds_{0ij}}{s_{0ij}}\wedge\operatorname{dlog}_1(\tau_{0ij})\right)\\&=\sum_{\substack{i=1\\(i,p)=1}}^{p^e-1}x^i\Phi^{p^{e+1-m}}(d\omega_{0i})\in\wp(\Omega^n_F)+d\Omega^{n-1}_F+\Omega^{n-1}_F\wedge dx+\sum_{i=1}^{p^e-1}x^iB_e\Omega^n_F.\end{aligned}$$

As $\sum_{i=1}^{p^e-1} x^i B_e\Omega_F^n \subseteq R^1(\mathcal{L}_{m,e}^{n+1})$ we can express $\overline{\omega_{00}'} = \overline{\chi} + \overline{V(\hat{\omega}_{00}')} \in H_{p^{m-\ell}}^{n+1} F$, where $\overline{\chi} \in \overline{\mathcal{L}_{m,e}^{n+1}}$. From this we can select

$$\overline{\omega_0'} = \sum_{\ell=0}^{m-2} \overline{V^\ell(\omega_{0\ell}')} = \overline{\chi} + \overline{V(\hat{\omega}_{00}')} + \sum_{\ell=1}^{m-2} V^\ell(\overline{\hat{\omega}_{0\ell}'}) \in \overline{\mathcal{L}_{m,e}^{n+1}} + \overline{V(W_{m-1}\Omega_F^n)},$$

giving the above claim required to prove (i).

For (ii) we have $\omega \in W_m\Omega_F^n$ with $\bar{\omega} \in K_{m,e}^{n+1}$ and $\bar{\omega} \in \overline{\mathcal{L}_{m,e}^{n+1}} + \overline{V(K_{m-1,e}^{n+1})}$. By induction using part (iii) for $m-1$ we have $K_{m-1,e}^{n+1} = \sum_{\ell=0}^{m-2} \overline{V^\ell(\mathcal{L}_{m-1-\ell,e}^{n+1})}$. Part (ii) follows combining these. Part (iii) is immediate by parts (i) and (ii). For (iv), when $m \geq e$, by Lemma 3.15 we have $\mathcal{L}_{m,e}^{n+1} \subseteq p^{m-e} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1} F$. Part (iv) of the theorem now follows from part (iii).

For (v), when $r \leq m$ we have

$$[x]_m^i B_r W_m\Omega_F^n = [x]_m^i dV^{m-r}(W_r\Omega_F^{n-1}) \subseteq W_m\Omega^{n-1}F \wedge d[x]_m + dW_m\Omega_F^{n-1},$$

and when $r > m$ by Theorem 3.8 we have

$$B_r W_m\Omega_F^n = B_r W_m\Omega_F^1 \wedge \mathrm{dlog}_m(K_{n-1}^M F) + B_m W_m\Omega_F^n.$$

As $\mathcal{L}_{m,e}^{n+1} := \sum_{r=1}^{e} \sum_{i=1,\,(i,p)=1}^{p^r-1} [x]_m^i B_r W_m\Omega_F^n \subseteq W_m\Omega^n F$, the result follows by part (iii). □

As noted in the introduction, using Theorem 4.2 one obtains a characterization of the kernel of purely inseparable modular extensions when $n+1=2$. A direct generalization of this result is not possible for $n+1 \geq 3$ (see Remark 4.4 below).

Corollary 4.3. *Suppose $x_1, x_2, \ldots, x_s \in F$ are p-independent and set*

$$L = F(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \textit{where } \alpha_i^{p^{r_i}} = x_i.$$

Set $\widetilde{m} := \max\{r_i \mid 1 \leq i \leq s\}$ and define $L_i := F(\alpha_i)$. If $m \geq \widetilde{m}$ then

$$\ker\bigl(H_{p^m}^2 F \to H_{p^m}^2 L\bigr) = \sum_{i=1}^{s} p^{m-r_i} \cdot \overline{W_m F \cdot d[x_i]_m} = \sum_{i=1}^{s} \ker\bigl(H_{p^m}^2 F \to H_{p^m}^2 L_i\bigr).$$

Proof. The proof proceeds by induction on s, where the case $s=1$ follows from Theorem 4.2. By the $s-1$ case,

$$\ker\bigl(H_{p^m}^2 L_1 \to H_{p^m}^2 L\bigr) = \sum_{i=2}^{s} p^{m-r_i} \cdot \overline{W_m L_1 \wedge d[x_i]_m}.$$

So if $\omega \in W_m\Omega_F^1$ and $i_{L/F}(\bar{\omega}) = 0$ then we can express

$$i_{L_1/F}(\bar{\omega}) = \sum_{i=2}^{s} p^{m-r_i} \cdot \overline{\psi_i \wedge \frac{d[x_i]_m}{[x_i]_m}}, \quad \text{where } \psi_i \in W_m L_1.$$

Since $L_1 = F(\alpha_1)$ and $\alpha_1^{p^{r_1}} \in F$ we know that $\Phi^{r_1}(\psi_i) \in W_m F$. As

$$\overline{\psi_i \wedge \frac{d[x_i]_m}{[x_i]_m}} = \overline{\Phi^{r_1}(\psi_i) \wedge \frac{d[x_i]_m}{[x_i]_m}} \in H^2_{p^m} L_1,$$

we find

$$\overline{\omega_1} := \bar{\omega} - \overline{\sum_{i=2}^{s} p^{m-r_i} \cdot \Phi^{r_1}(\psi_i) \wedge \frac{d[x_i]_m}{[x_i]_m}} \in \ker\bigl(H^2_{p^m} F \to H^2_{p^m} L_1\bigr).$$

By the $s = 1$ case we have $\overline{\omega_1} \in p^{m-r_1} \cdot \overline{W_m \Omega_F^{n-1} \wedge d[x_1]_m}$ and the first equation follows. The second equation is an immediate consequence of the first in view of the $s = 1$ case. □

Remark 4.4. (i) The result in Corollary 4.3(i) coincides with the cohomological interpretation of Albert's work on p-algebras, see for example [Albert 1939, Theorem 28, p. 108] or [Jacobson 1996, 4.2.16, p. 162].

(ii) Although Theorem 4.2 is valid for all m and n, it is not possible to generalize Corollary 4.3 directly when $m \geq 2$ and $n \geq 3$. For example,

$$\overline{W_2 F \cdot d[x]_2 \wedge d[y]_2} \subseteq \ker\bigl(H^3_{p^2} F \to H^3_{p^2} F(\sqrt[p]{x}, \sqrt[p]{y})\bigr)$$

and it is possible for an element in $\overline{W_2 F \cdot d[x]_2 \wedge d[y]_2}$ to have order p^2. However, all elements in

$$\ker\bigl(H^3_{p^2} F \to H^3_{p^2} F(\sqrt[p]{x})\bigr) + \ker\bigl(H^3_{p^2} F \to H^3_{p^2} F(\sqrt[p]{y})\bigr)$$

have order p. Hence the result in Corollary 4.3(i) cannot be generalized without taking into account additional elements of this type. There are other complexities as well. The characterization of these kernels is the subject of ongoing work.

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References

[Albert 1939] A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloq. Publ. **24**, Amer. Math. Soc., New York, 1939. MR Zbl

[Amitsur et al. 1979] S. A. Amitsur, L. H. Rowen, and J.-P. Tignol, "Division algebras of degree 4 and 8 with involution", *Israel J. Math.* **33**:2 (1979), 133–148. MR Zbl

[Arason et al. 2007] J. K. Arason, R. Aravire, and R. Baeza, "On some invariants of fields of characteristic $p > 0$", *J. Algebra* **311**:2 (2007), 714–735. MR Zbl

[Aravire and Baeza 2003] R. Aravire and R. Baeza, "The behavior of quadratic and differential forms under function field extensions in characteristic two", *J. Algebra* **259**:2 (2003), 361–414. MR Zbl

[Aravire and Laghribi 2013] R. Aravire and A. Laghribi, "Results on Witt kernels of quadratic forms for multi-quadratic extensions", *Proc. Amer. Math. Soc.* **141**:12 (2013), 4191–4197. MR Zbl

[Aravire et al. 2018] R. Aravire, B. Jacob, and M. O'Ryan, "The de Rham Witt complex, cohomological kernels and p^m-extensions in characteristic p", *J. Pure Appl. Algebra* **222**:12 (2018), 3891–3945. MR Zbl

[Aravire et al. 2019] R. Aravire, A. Laghribi, and M. O'Ryan, "The behavior of differential forms under purely inseparable extensions", *J. Pure Appl. Algebra* **223**:1 (2019), 439–457. MR Zbl

[Elman et al. 1980] R. Elman, T. Y. Lam, and A. R. Wadsworth, "Quadratic forms under multiquadratic extensions", *Indag. Math.* **42**:2 (1980), 131–145. MR Zbl

[Gille and Szamuely 2006] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Stud. Adv. Math. **101**, Cambridge Univ. Press, 2006. MR Zbl

[Illusie 1979] L. Illusie, "Complexe de de Rham–Witt et cohomologie cristalline", *Ann. Sci. École Norm. Sup.* (4) **12**:4 (1979), 501–661. MR Zbl

[Izhboldin 1991] O. Izhboldin, "On p-torsion in K_*^M for fields of characteristic p", pp. 129–144 in *Algebraic K-theory*, edited by A. A. Suslin, Adv. Soviet Math. **4**, Amer. Math. Soc., Providence, RI, 1991. MR Zbl

[Jacob 1991] B. Jacob, "Indecomposable division algebras of prime exponent", *J. Reine Angew. Math.* **413** (1991), 181–197. MR Zbl

[Jacobson 1996] N. Jacobson, *Finite-dimensional division algebras over fields*, Springer, 1996. MR Zbl

[Kato 1982] K. Kato, "Galois cohomology of complete discrete valuation fields", pp. 215–238 in *Algebraic K-theory, II* (Oberwolfach, 1980), edited by R. K. Dennis, Lecture Notes in Math. **967**, Springer, 1982. MR Zbl

[Sobiech 2018] M. Sobiech, "The behavior of differential, quadratic and bilinear forms under purely inseparable field extensions", *J. Algebra* **499** (2018), 151–182. MR Zbl

[Tignol 1987] J.-P. Tignol, "Algèbres indécomposables d'exposant premier", *Adv. Math.* **65**:3 (1987), 205–228. MR Zbl

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