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TATHAGATA BASAK

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We show that the projectivized complex reflection group Γ of the unique $(1+i)$-modular Hermitian $\mathbb{Z}[i]$-module of signature $(9, 1)$ is a new arithmetic reflection group in $PU(9, 1)$ and we construct an automorphic form on $\mathbb{C}H^9$ with singularities along the mirrors of Γ using Borcherds' singular theta lift. We find 32 complex reflections of order four generating Γ. The mirrors of these 32 reflections form the vertices of a sort of Coxeter–Dynkin diagram D whose edges are determined by the finite geometry of 16 points and 16 affine hyperplanes in $\mathbb{F}_2^4$. The group of automorphisms of D is $2^4 : (2^3 : L_3(2)) : 2$. This group transitively permutes the 32 mirrors of generating reflections and fixes a unique point τ in $\mathbb{C}H^9$. These 32 mirrors are precisely the mirrors closest to τ. These results are strikingly similar to the results satisfied by the complex hyperbolic reflection group at the center of Allcock's monstrous proposal.

1. Introduction

Let $\mathcal{G} = \mathbb{Z}[i]$ be the ring of Gaussian integers. Let $p = (1 + i)$. We study the projectivized complex reflection group $\Gamma = \Gamma_1$ of the unique p-modular[1] Hermitian $\mathcal{G}$–lattice of signature $(9, 1)$. We show that Γ_1 is arithmetic. We find nice generators for Γ_1 and some natural relations among these generators (the notation Γ_1 is only used in the introduction. Afterwards we shall simply write Γ instead of Γ_1). We mention three reasons for our interest in Γ_1:

- There are few known examples of lattices in $PU(n, 1)$ generated by complex reflections when $n > 3$. These are all arithmetic. The three largest values of n for which examples are known are 13, 10 and 9. Sources for these examples

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[1] A $\mathcal{G}$-lattice L is called α-modular for some $\alpha \in \mathcal{G}$, if $\alpha L^\vee = L$ where $L^\vee$ is the dual lattice. Modular lattices were introduced in [Quebbemann 1995] generalizing self-dual or unimodular lattices; also see [Quebbemann 1997; Nebe 2004] for many interesting examples and connections with modular forms. We focus on modular lattices because they happen to have interesting reflection groups. The p-modularity plays a key role in many of our results.

are [Deligne and Mostow 1986; Mostow 1986; 1988; Thurston 1998; Allcock 2000a; Allcock et al. 2011; Looijenga and Swierstra 2008]. The "largest" examples found in [Mostow 1988] and [Thurston 1998] are identical; it is an arithmetic lattice in $PU(9, 1)$. This lattice is denoted by Γ_5 later in this introduction. A single example in dimension 10 was found in [Allcock et al. 2011; Looijenga and Swierstra 2008] and a single example in dimension 13 (Γ_2 in our notation) was found in [Allcock 2000a]. The group Γ_1 studied in this article gives a new example in $PU(9, 1)$.

- Allcock's monstrous proposal conjecture states that the fundamental group of the ball quotient constructed from Γ_2 maps onto $(M \wr 2)$ where M is the monster simple group. The arithmetic lattice $\Gamma_2 \subseteq PU(13, 1)$ plays a central role in the monstrous proposal; see [Allcock 2009; Basak 2007a; Allcock and Basak 2016; 2018]. Our results for Γ_1 have striking similarity with results for Γ_2 obtained in [Basak 2007a].

- The results for Γ_1 and Γ_2 (and a few other lattices in $PU(n, 1)$) tie into a general pattern of phenomena that have an analogy in the theory of Weyl groups; see Theorem 1.1. Understanding this analogy may be useful in finding more interesting examples and in studying complex reflection groups in general.

Before discussing Γ_1 in more detail, we want to describe this general pattern of phenomena. Let $\mathbb{K}$ denote one of the three real division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let V be a $\mathbb{K}$-module with a nondegenerate $\mathbb{K}$-valued Hermitian form that is either positive definite or Lorentzian (i.e., of signature $(n, 1)$). Let G denote the real Lie group of isometries of V. Let Γ be a discrete subgroup of G generated by real, complex or quaternionic reflections. Let X be the projective space over $\mathbb{K}$ if V is positive definite and let X be the hyperbolic space over $\mathbb{K}$ if V is Lorentzian. There is a natural G-invariant metric on the symmetric space X. The discrete group Γ acts properly discontinuously on X by isometries. The set of fixed points of a reflection $s \in \Gamma$ forms a totally geodesic $\mathbb{K}$-hypersurface in X called the *mirror* of s. Nice generators and relations for Γ can often be found as follows: Choose a suitable point τ in X that is a point of symmetry of the mirrors of Γ in an appropriate sense. Let S be the set of reflections in Γ whose mirrors are closest to τ. In many examples, the reflections in S generate Γ and these reflections satisfy nice Coxeter-type relations. First, we give some well known examples:

- Let Γ be a Weyl group of A-D-E type acting on the real vector space V by its natural reflection representation. Let τ be the line in V containing the Weyl vector. Then the mirrors of the simple reflections are exactly the mirrors in $P(V)$ closest to τ. So in this case S is just the set of simple mirrors. In analogy with this classical case, in all the examples described below, the reflections in S will be called *simple reflections* and the corresponding mirrors will be called *simple mirrors*.

- Let Γ be an irreducible finite complex reflection group acting on $V = \mathbb{C}^n$. For most Γ including the infinite family $G(de, e, n)$, there exists a point τ in $P(V)$ such that the mirrors closest to τ generate Γ (see [Basak 2012, Sections 3.10 and 3.11, Remark (5)]). For the infinite family $G(de, e, n)$ one can choose τ such that S is the standard set of generators, given, for example in [Broué et al. 1998]. However, for some exceptional examples, one obtains new sets of generators by choosing τ's that are analogous to Weyl vectors (in a certain sense, explained in [Basak 2012]).

- The reflection group Γ of $\mathrm{II}_{25,1}$ (the unique even self-dual $\mathbb{Z}$-lattice of signature $(25, 1)$) acts on the real hyperbolic space $X = \mathbb{R}H^{25}$. Choose τ to be a "Leech cusp", which means that τ is a line containing a primitive norm zero vector τ_* such that $\tau_*^{\perp}/\mathbb{Z}\tau_*$ is isomorphic to the Leech lattice. Here we are stretching our discussion a little bit since τ is not really a point in $\mathbb{R}H^{25}$ but a point on its boundary. The mirrors closest to τ in horocyclic distance are parametrized by the vectors of the Leech lattice. So they are called Leech mirrors. The reflections in the Leech mirrors generate Γ. These generating reflections obey Coxeter relations governed by the Leech lattice, leading to Conway's [1983] observation that the Leech lattice is the Dynkin diagram of the reflection group of $\mathrm{II}_{25,1}$.

Counting the example studied in this article, we now have at least six examples of complex and quaternionic hyperbolic reflection groups, where similar results hold. To state these results in a uniform manner, we need some notation. Let $\mathcal{G} = \mathbb{Z}[i]$, $\mathcal{E} = \mathbb{Z}[e^{2\pi i/3}]$ and $\mathcal{H} = \mathbb{Z}[i, (1 + i + j + ij)/2]$ denote the ring of Gaussian integers, the ring of Eisenstein integers and the quaternionic ring of Hurwitz integers, respectively. The rings $\mathcal{G}$ and $\mathcal{E}$ are maximal orders in imaginary quadratic number fields while the ring $\mathcal{H}$ is a maximal order in the skew field $\mathbb{Q}[i, j]$ of rational quaternions. Let $\mathcal{O}$ be one of these three rings. Let l be a nonzero prime in $\mathcal{O}$ of smallest possible norm. If $\mathcal{O} = \mathcal{G}$ or $\mathcal{O} = \mathcal{H}$, then we may choose $l = p = (1 + i)$. If $\mathcal{O} = \mathcal{E}$, then we may choose $l = \sqrt{-3}$. Let $\mathcal{O}_{n,1}$ denote an l-modular Hermitian $\mathcal{O}$-lattice of signature $(n, 1)$, if such a lattice exists.[2] In particular, we define $\mathcal{O}_{1,1} = \mathcal{O}e_1 \oplus \mathcal{O}e_2$ where $e_1^2 = e_2^2 = 0$ and $\langle e_1 | e_2 \rangle = \bar{l}$. The lattice $\mathcal{O}_{1,1}$ is the unique l-modular Hermitian $\mathcal{O}$-lattice of signature $(1, 1)$ and we call it the *hyperbolic cell*. We shall consider the following six lattices:

$$L_1 = \mathcal{G}_{9,1}, \quad L_2 = \mathcal{E}_{13,1}, \quad L_3 = \mathcal{H}_{7,1}, \quad L_4 = \mathcal{G}_{5,1}, \quad L_5 = \mathcal{E}_{9,1}, \quad L_6 = \mathcal{H}_{5,1}.$$

In each case, L_j is the unique l-modular Hermitian $\mathcal{O}$-lattice in its given signature. Note that L_{3+j} is a sublattice of L_j. Let $j \in \{1, 2, \ldots, 6\}$. Let $R(L_j)$ be the (complex or quaternionic) reflection group of L_j and let $\Gamma_j = PR(L_j)$ be the image of $R(L_j)$ in $PU(n, 1)$ or $P\mathrm{SP}(n, 1)$. The mirrors of Γ_j are determined by the orthogonal complements of vectors of minimal positive norm in L_j. Since L_j is indefinite, there are infinitely many mirrors. If $\mathcal{O} = \mathcal{E}$, then the reflection group Γ_j

[2] This means $\mathcal{O}_{n,1}$ is a free (right) $\mathcal{O}$–module of rank $(n+1)$ with a $\mathcal{O}$-valued Hermitian form $\langle \, | \, \rangle : \mathcal{O}_{n,1} \times \mathcal{O}_{n,1} \to \mathcal{O}$ of signature $(n, 1)$, and $l^{-1}\mathcal{O}_{n,1}$ is equal to the dual lattice of $\mathcal{O}_{n,1}$.

contains order 3 reflections around the mirrors. If $\mathcal{O} = \mathcal{G}$ or $\mathcal{H}$, then Γ_j contains order 4 and order 2 reflections around the mirrors. The projectivized reflection group Γ_j acts faithfully on the complex or quaternionic hyperbolic space X_j of appropriate dimension. The following results hold:

1.1. Theorem. (a) *Γ_j has finite index in $P\,\mathrm{Aut}(L_j)$. So Γ_j is arithmetic.*

(b) *There is a point τ_j in X_j such that the set of reflections S_j in the mirrors closest to τ_j (i.e., the simple mirrors) generates Γ_j. Further, $P\,\mathrm{Aut}(L_j)$ has a finite subgroup Q_j that acts transitively on the simple mirrors and τ_j is the unique point of X_j fixed by Q_j.*

(c) *The Coxeter relations between the simple reflections S_j are encoded by the edges of a diagram D_j which we call the Dynkin diagram of Γ_j. The vertices of D_j correspond to the simple reflections. The diagram D_2 (resp. D_3) is the incidence graph of $P^2(\mathbb{F}_3)$ (resp. $P^2(\mathbb{F}_2)$). The diagram D_1 has 32 vertices and is defined by the incidence relations of 16 points and 16 hyperplanes in $\mathbb{F}_2^4$. A precise description of D_1 is given later in this introduction. Finally D_{3+j} is a maximal circuit in D_j.*

We should emphasize that the Coxeter relations mentioned in Theorem 1.1(c) are probably not sufficient to obtain a presentation of Γ_j. We do know some extra non-Coxeter relations satisfied by the generators in S_j but at present we do not know if these are enough to give a presentation of Γ_j on the generating set S_j.

In each case, Q_j is more or less the automorphism group of the diagram D_j. Theorem 1.1 is the summary of results from a few articles. For $\Gamma_2, \Gamma_3, \Gamma_5, \Gamma_6$, part (a) is due to Allcock [2000a; 2000b]. For Γ_4, Theorem 1.1 is due to Goertz [2017]. The rest of the results are due to the author. In this paper, we prove Theorem 1.1 for Γ_1. For $\Gamma_2, \Gamma_3, \Gamma_5$, Theorem 1.1 follows from the results in [Basak 2007a], [Basak 2007b] and Section 4.1 of [Basak 2006], respectively. The generators and relations for Γ_2 encoded in the diagram D_2 forms the basis for Allcock's monstrous proposal conjecture [2009]. One of our motivations for studying Γ_1 in detail is the close similarity between Γ_1 and Γ_2 and our interest in Γ_2 stemming from the monstrous proposal conjecture. For Γ_6, the proofs of parts (b) and (c) of Theorem 1.1 have not been written up. However the proofs for Γ_{3+j} are entirely similar to the proofs for Γ_j and easier. A detailed study of the references mentioned and this paper reveals many more similarities between these reflection groups.

For the rest of the introduction, we shall focus on Γ_1 and give some more details. To maintain notational consistency with the references cited above, we shall drop the subscript and write $L = L_1$, $\tau = \tau_1$ and so on. We work over the ring $\mathcal{G} = \mathbb{Z}[i]$. Let $p = (1+i)$. Our objective is to study the complex reflection group of the unique p-modular $\mathcal{G}$-lattice $L = \mathcal{G}_{9,1}$ of signature (9, 1). One has

$$L \simeq 4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1} \simeq \mathrm{BW}_{16}^{\mathcal{G}} \oplus \mathcal{G}_{1,1},$$

where $D_4^{\mathcal{G}}$ and $\mathsf{BW}_{16}^{\mathcal{G}}$ are the Gaussian forms of the D_4 root lattice and the Barnes–Wall lattice respectively. The projective reflection group Γ is a discrete subgroup of $PU(9, 1)$ and acts faithfully by isometries on the complex hyperbolic space

$$\mathbb{B}(L) = \{\mathbb{C}v : v \in L \otimes_{\mathcal{G}} \mathbb{C},\ \langle v|v\rangle < 0\} \simeq \mathbb{C}H^9.$$

Let v be a primitive positive norm vector in L and let ζ be a root of unity. Write $v^2 = \langle v|v\rangle$. The complex ζ-reflection in v takes $x \in L \otimes_{\mathcal{G}} \mathbb{C}$ to

$$x - (1-\zeta)\langle v|x\rangle v/v^2.$$

This complex reflection defines an automorphism of L if and only if $(1-\zeta)\langle v|x\rangle/v^2$ is in $\mathcal{G}$ for all $x \in L$, that is $((1-\bar{\zeta})/v^2)v \in L^\vee = p^{-1}L$. Since v is a primitive vector of L, this forces $p(1-\bar{\zeta})/v^2 \in \mathcal{G}$, which is only possible if $v^2 = 2$ and $\zeta = -1$ or $\pm i$. It follows that the complex reflections in $\mathsf{Aut}(L)$ are the order 4 and 2 complex reflections in the norm 2 vectors of L. By definition, $R(L)$ is the subgroup of $\mathsf{Aut}(L)$ generated by these complex reflections and $\Gamma = PR(L)$ is the image of $R(L)$ in $PU(9, 1)$.

1.2. Theorem. (a) (See Theorem 4.4) *Γ has finite index in $P\mathsf{Aut}(L)$. So Γ is arithmetic.*

(b) (See Theorems 5.11, 5.9) *Γ is generated by thirteen i-reflections satisfying the Coxeter relations of the diagram X_{3333} shown in Figure 1.*

(c) (See Lemma 5.2, Theorem 5.6, Section 5.5) *The above generating set of thirteen i-reflections can be extended to a set of thirty-two i-reflections whose mirrors are equidistant from a point τ in $\mathbb{B}(L)$. These 32 mirrors are precisely the mirrors closest to τ. A subgroup Q of $P\mathsf{Aut}(L)$ isomorphic to $(2^4 : (2^3 : L_3(2))) : 2$ (in the notation of [Conway et al. 1985]) acts transitively on the 32 mirrors and τ is the unique point in $\mathbb{B}(L)$ fixed by Q.*

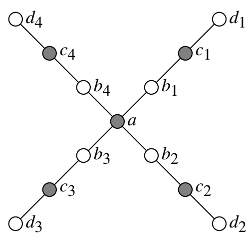
The configuration of the 32 mirrors closest to τ has appealing symmetry related to the geometry of the finite vector space $\mathbb{F}_2^4$. To describe this, fix a point $a \in \mathbb{F}_2^4$. For $u \in \mathbb{F}_2^4$, let $t_u : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ be the translation $t_u(v) = u + v$. Let $\mathcal{K}_0$ be the set of homogeneous hyperplanes in $\mathbb{F}_2^4$ that do not contain a. Let $\mathcal{K}$ be the set of translates of the hyperplanes in $\mathcal{K}_0$. So $\mathcal{K}$ consists of 8 homogeneous and 8 affine hyperplanes in $\mathbb{F}_2^4$. Let $D = \mathbb{F}_2^4 \cup \mathcal{K}$. Note that each t_u permutes $\mathbb{F}_2^4$ and permutes $\mathcal{K}$ and thus defines a permutation of D. The thirty-two i-reflections closest to τ can be labeled by D such that the relations among these i-reflections are dictated by the configuration D. More precisely, let $d, d' \in D$ and let R, R' be the corresponding i-reflections.

- If $\{d, d'\}$ is an incident pair of point and hyperplane, then $RR'R = R'RR'$. This is denoted in the diagram D (Figure 1, right) by a solid edge joining d to d'.

- If $d' = t_a(d)$, then $RR'RR' = R'RR'R$. This is denoted in the diagram D by a dashed edge joining d to d'.
- Otherwise, $RR' = R'R$.

We picture D as a graph with two kinds of edges. The subgroup $2^3 : L_3(2)$ in Q is the stabilizer of a in $L_4(2)$, the 2^4 corresponds to the translation action of $\mathbb{F}_2^4$ on itself, and the extra $\mathbb{Z}/2$ is a symmetry that interchanges the points in $\mathbb{F}_2^4$ and hyperplanes in $\mathcal{K}$ (see Section 5.5 for details). The group Q acts on the set D preserving both kinds of edges. We may think of D as the Dynkin diagram for $R(L)$ and Q as the group of diagram automorphisms.

The proof of arithmeticity of $\Gamma = PR(\mathcal{G}_{9,1})$ is similar to that for $\Gamma_2 = PR(\mathcal{E}_{13,1})$ in [Allcock 2000b] which in turn is adapted from an argument in [Conway 1983]. The statements and proofs in this article often closely parallel those in [Allcock 2000b; Basak 2007a; 2007b]. We shall refrain from mentioning them at every step,



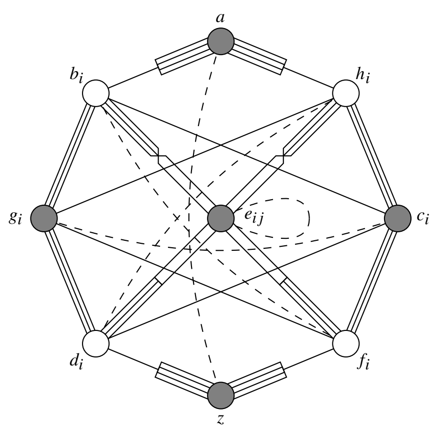


Figure 1. The X_{3333} diagram is on the left. A shorthand drawing of the 32 node diagram D is on the right. In the 32 node diagram, $1 \leq i < j \leq 4$, so the node c_i (resp. e_{ij}) stands for four (resp. six) nodes. A solid (resp. dashed) edge between two vertices x and y indicates the relation $xyx = yxy$ (resp. $xyxy = yxyx$). No edge between x and y means $xy = yx$. The following shorthands are used: The edge between a and b_i means that a and b_i are connected for all i. A single (resp. triple) edge between b_i and c_i (resp. g_i) means that b_i and c_j (resp. g_j) are connected if $i = j$ (resp. $i \neq j$). Notice that there are two kinds of edges out of e_{ij}. The edge between e_{ij} and d_i (resp. b_i) means that e_{ij} is connected to d_k (resp. b_k) if $k \in \{i, j\}$ (resp. $k \notin \{i, j\}$). Finally, the dashed loop joining e_{ij} to itself means that e_{ij} is joined to e_{kl} by a dashed edge if $\{i, j\} \cap \{k, l\} = \varnothing$.

but a couple of remarks comparing $\mathcal{G}_{9,1}$ and $\mathcal{E}_{13,1}$ are worthwhile. Below, $\mathsf{Leech}^{\mathcal{E}}$ denotes the complex Leech lattice (studied in detail in [Wilson 1983]) scaled to have minimal norm 6.

The description $\mathcal{G}_{9,1} \simeq \mathsf{BW}_{16}^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$ is crucial in our proofs, just like the description $\mathcal{E}_{13,1} \simeq \mathsf{Leech}^{\mathcal{E}} \oplus \mathcal{E}_{1,1}$ is crucial in the proofs in [Allcock 2000a; Basak 2007a]. This is because we use two key properties that are shared by the (real) Barnes–Wall and the (real) Leech lattice: they contain no norm 2 vectors and they are very dense, in fact the densest lattices known in the respective dimensions. The absence of a norm 2 vector in $\mathsf{BW}_{16}^{\mathcal{G}}$ provides us with a "Barnes–Wall cusp" at the boundary of $\mathbb{B}(L)$ such that no mirror passes through it. These cusps play a key role in our arguments.

The results on $\mathcal{E}_{13,1}$ use the fact that the covering radius of the Leech lattice is $\sqrt{2}$. In our results on $\mathcal{G}_{9,1}$, the density of the Barnes–Wall lattice is used via Lemma 6.11 of [Allcock 2000b] which is a special case of results in [Borcherds 1990]. This lemma gives a covering of the underlying real vector space of the Barnes–Wall lattice using balls of two sizes. It is curious to note that this lemma does not use the covering radius of the Barnes–Wall lattice. Rather, it uses the covering radius of the Leech lattice and an embedding of the Barnes–Wall lattice in the Leech lattice. Just like in [Allcock 2000b; Basak 2007a; Allcock and Basak 2016], the crucial proofs in this paper would all collapse if the covering radius of the Leech lattice were bigger than $\sqrt{2}$.

In Section 2 we collect the definitions we need regarding complex reflection groups and complex hyperbolic space and the basic facts about the complex lattices we intend to study. In Section 3, we prove some general results about reflection groups of p-modular Gaussian lattices. In Sections 4 and 5, we study the reflection group Γ in detail. Finally, in Section 6, we use Borcherds' singular theta lift to construct a holomorphic automorphic form on $\mathbb{C}H^9$ for the group $\mathrm{Aut}(L)$ with zeros along the mirrors of Γ. This positively answers a question of Borcherds in our example; see Problem 13.1 of [Borcherds 2000b]. The Appendix contains a proof of uniqueness of a p-modular $\mathcal{G}$-lattice of signature $(2d-1,1)$.

2. Preliminaries

2.1. Definition (Gaussian lattices). Let $\mathcal{G} = \mathbb{Z}[i]$ be the ring of Gaussian integers. Let $p = (1+i)$. Let K be a free $\mathcal{G}$-module of finite rank with a $\mathbb{Q}[i]$-valued Hermitian form $\langle\,|\,\rangle : K \times K \to \mathbb{Q}[i]$. Hermitian forms are always assumed to be linear in the second variable. We shall always identify K inside the $\mathbb{Q}[i]$-vector space $K \otimes_{\mathcal{G}} \mathbb{Q}[i]$ and further, inside the complex vector space $K \otimes_{\mathcal{G}} \mathbb{C}$. The Hermitian form linearly extends to these vector spaces. For $v \in K \otimes_{\mathcal{G}} \mathbb{C}$, write $v^2 = \langle v|v\rangle$. We say that v^2 is the *norm* of v. A nonzero vector of norm zero is called a *null vector*.

Let $A \subseteq K \otimes_{\mathcal{G}} \mathbb{C}$ and $m \in \mathbb{R}$. It will be convenient to use the notation

$$A(m) = \{a \in A : a^2 = m\} \quad \text{and} \quad A(\leq m) = \{a \in A : a^2 \leq m\}.$$

Let $A^{\perp} = \{v \in K \otimes_{\mathcal{G}} \mathbb{C} : \langle v|a\rangle = 0 \text{ for all } a \in A\}$. The *radical* of K is defined as $\mathsf{rad}(K) = K^{\perp} \cap K$. The Hermitian form is nonsingular if $\mathsf{rad}(K) = 0$. If $\mathsf{rad}(K) = 0$, then K is called a *$\mathcal{G}$-lattice* or a *Gaussian lattice*. If $\mathsf{rad}(K) \neq 0$, then K is called a *singular $\mathcal{G}$-lattice*. Say that K is *integral* if the Hermitian form takes values in $\mathcal{G}$. Say that K is *Lorentzian* if it has signature $(n, 1)$.

Let K be a $\mathcal{G}$–lattice. Define $K^{\vee} = \{v \in K \otimes_{\mathcal{G}} \mathbb{Q}[i] : \langle v|K\rangle \subseteq \mathcal{G}\}$. Then $K^{\vee}$ is a $\mathcal{G}$-lattice called the *dual lattice* of K. Note that K is integral if and only if $K \subseteq K^{\vee}$. Let $l \in \mathcal{G}$. A $\mathcal{G}$-lattice K is called *l-modular* if $K^{\vee} = l^{-1}K$. Clearly if K_1 and K_2 are l-modular, then so is $K_1 \oplus K_2$. Let $K_{\mathbb{Z}}$ denote the underlying $\mathbb{Z}$-lattice of K. This means that $K_{\mathbb{Z}}$ is the underlying $\mathbb{Z}$-module of K with the bilinear form $\mathsf{Re}\langle\,|\,\rangle$. We say that $K_{\mathbb{Z}}$ is the *real form* of the hermitian lattice K. Real forms of other complex or quaternionic lattices are defined similarly. Note that K is an integral $\mathcal{G}$-lattice if and only if $K_{\mathbb{Z}}$ is an integral $\mathbb{Z}$-lattice and $(K^{\vee})_{\mathbb{Z}} = (K_{\mathbb{Z}})^{\vee}$.

2.2. *The $D_4^{\mathcal{G}}$ lattice.* Let $D_{2n}^{\mathcal{G}}$ be the sublattice of $\mathcal{G}^n$ consisting of all $(x_1, \dots, x_n)$ in $\mathcal{G}^n$ such that $(x_1 + \cdots + x_n) \equiv 0 \bmod p$ with the standard positive definite Hermitian form

$$\langle x|x'\rangle = \bar{x}_1 x_1' + \cdots + \bar{x}_n x_n'.$$

The underlying $\mathbb{Z}$-module of $D_{2n}^{\mathcal{G}}$ with the inner product $\mathsf{Re}\langle x|y\rangle$ is the root lattice D_{2n}, where

$$D_n = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : (x_1 + \cdots + x_n) \equiv 0 \bmod 2\}.$$

Note that $D_4^{\mathcal{G}}$ is p-modular but $D_{2n}^{\mathcal{G}}$ is not p-modular for $n > 2$. A $\mathcal{G}$-basis for $D_4^{\mathcal{G}}$ is $v_1 = (1, 1)$ and $v_2 = (0, \bar{p})$. The discriminant group of $D_4^{\mathcal{G}}$ is $(D_4^{\mathcal{G}})^{\vee}/D_4^{\mathcal{G}} = p^{-1}D_4^{\mathcal{G}}/D_4^{\mathcal{G}} \simeq (\mathbb{Z}/2)^2$. Coset representatives for $p^{-1}D_4^{\mathcal{G}}/D_4^{\mathcal{G}}$ can be chosen to be $\{(0, 0), v_1/\bar{p}, v_2/\bar{p}, (v_2 - v_1)/\bar{p}\}$.

2.3. *The Barnes–Wall lattice over Hurwitz and Gaussian integers.* Express real quaternions in the form $(x + yj)$ where $x, y \in \mathbb{C}$. The ring $\mathcal{H}$ of Hurwitz integers consists of all $(x + yj)$ such that $(x, y) \in \mathcal{G}^2$ or $\left(x + \frac{p}{2}, y + \frac{p}{2}\right) \in \mathcal{G}^2$. Note that $\mathcal{G}$ is a subring of $\mathcal{H}$. The map $(x + yj) \mapsto (x, y)$ defines an isomorphism $\mathcal{H} \simeq p^{-1}D_4^{\mathcal{G}}$ of $\mathcal{G}$-modules. Define

$$\mathsf{BW}_{16}^{\mathcal{G}} = \left\{(x_1, \dots, x_4) : x_j \in p^{-1}D_4^{\mathcal{G}},\ x_j \equiv x_k \bmod D_4^{\mathcal{G}} \text{ for all } j, k,\ \sum_j x_j \in pD_4^{\mathcal{G}}\right\}$$

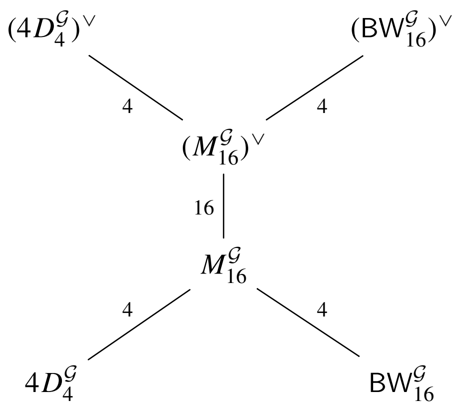
Allcock [2000b] describes a 4-dimensional Hurwitz lattice whose real form (appropriately scaled) is the Barnes–Wall lattice. It is immediate that $\mathsf{BW}_{16}^{\mathcal{G}}$, as defined above, is the Gaussian form of this Hurwitz lattice with the norms scaled by 2. So the real form of $\mathsf{BW}_{16}^{\mathcal{G}}$ is the usual rank 16 Barnes–Wall lattice of minimum norm 4. For more information on the Barnes–Wall lattice, see [Conway and Sloane 1988;

Scharlau and Venkov 1994; Nebe et al. 2002]. The 16-dimensional Barnes–Wall lattice is part of a family of lattices studied widely in the coding theory literature; see [Grigorescu and Peikert 2017; Nebe et al. 2002]. An alternative quick definition of the $\mathcal{G}$-lattice $\mathsf{BW}_{16}^{\mathcal{G}}$ is the $\mathcal{G}$-span of the rows of $\left(\begin{smallmatrix}1&1\\0&p\end{smallmatrix}\right)^{\otimes 4}$ [Grigorescu and Peikert 2017]. It is straightforward to verify the equivalence of the two definitions.

2.4. ***A common over-lattice of $4D_4^{\mathcal{G}}$ and $\mathsf{BW}_{16}^{\mathcal{G}}$.*** Let $M_{16}^{\mathcal{G}}$ be the rank 8 Gaussian lattice

$$M_{16}^{\mathcal{G}} = \{(x_1, x_2, x_3, x_4) : x_j \in p^{-1} D_4^{\mathcal{G}} : x_j \equiv x_k \bmod D_4^{\mathcal{G}} \text{ for all } j, k\}.$$

Then $M_{16}^{\mathcal{G}}$ is an integral Gaussian lattice of minimum norm 2 that contains both $4D_4^{\mathcal{G}}$ and $\mathsf{BW}_{16}^{\mathcal{G}}$. It is easy to verify the following inclusions among lattices with the indices indicated next to the edges:



In particular, $|(\mathsf{BW}_{16}^{\mathcal{G}})^\vee / \mathsf{BW}_{16}^{\mathcal{G}}| = 2^8$. On the other hand, from the definition of $\mathsf{BW}_{16}^{\mathcal{G}}$ one verifies that $(\mathsf{BW}_{16}^{\mathcal{G}})^\vee \supseteq p^{-1}\mathsf{BW}_{16}^{\mathcal{G}}$. So

$$2^8 = |(\mathsf{BW}_{16}^{\mathcal{G}})^\vee / \mathsf{BW}_{16}^{\mathcal{G}}| \geq |p^{-1}\mathsf{BW}_{16}^{\mathcal{G}} / \mathsf{BW}_{16}^{\mathcal{G}}| = |p^{-1}\mathcal{G}^8/\mathcal{G}^8| = 2^8.$$

It follows that equality must hold everywhere and that $(\mathsf{BW}_{16}^{\mathcal{G}})^\vee = p^{-1}\mathsf{BW}_{16}^{\mathcal{G}}$. In particular, all vectors in $\mathsf{BW}_{16}^{\mathcal{G}}$ have even norms.

One crucial property of $\mathsf{BW}_{16}^{\mathcal{G}}$ is that it does not have any norm 2 vector, so it has minimum norm 4. This can be seen quickly from our definition as follows: Note that $p^{-1}D_4^{\mathcal{G}}$ has minimum norm 1. Take $x \in \mathsf{BW}_{16}^{\mathcal{G}}(2)$. Write $x = (x_1, x_2, x_3, x_4)$ with each $x_j \in p^{-1}D_4^{\mathcal{G}}$. Then $\sum_{j=1}^4 x_j^2 = 2$ implies that at least two of the x_j's must be 0. Since the x_j's are all congruent modulo $D_4^{\mathcal{G}}$, it follows that $x_j \in D_4^{\mathcal{G}}$ for all j. Since $D_4^{\mathcal{G}}$ has minimum norm 2, it follows that there exists a $k \in \{1, 2, 3, 4\}$ such that $x_j = 0$ for $j \neq k$. But now $x_k = \sum_j x_j \in pD_4^{\mathcal{G}}$, so $x^2 = x_k^2$ has norm at least 4, which is a contradiction.

The lemma below is taken from [Allcock 2000b] and is a special case of the results in [Borcherds 1990] where it was used to find interesting reflection groups in real hyperbolic space $\mathbb{R}H^{17}$ (see Theorem 3.1 of [Borcherds 1990] and the examples following Theorem 3.3 on page 232). As mentioned in the introduction, the proof depends on the covering radius of the Leech lattice. We quote it below for convenience:

2.5. Lemma [Allcock 2000b, Lemma 6.11]. *The real vector space* $\mathsf{BW}_{16}^{\mathcal{G}} \otimes_{\mathbb{Z}} \mathbb{R}$ *is covered by the closed balls of radius* $\sqrt{2}$ *around the vectors of* $\mathsf{BW}_{16}^{\mathcal{G}}$ *together with closed balls of radius* 1 *around the vectors* $p^{-1}v$ *with* $v \in \mathsf{BW}_{16}^{\mathcal{G}}$ *and* $v^2 \equiv 2 \bmod 4$.

2.6. Definition (roots, reflection groups). Let V be a complex vector space with a Hermitian form $\langle\ |\ \rangle$. Let $v \in V$ be a vector of nonzero norm. A *complex reflection* R in v is a linear automorphism of V of finite order that pointwise fixes $v^{\perp}$. If R has order n, then it follows that $R(v) = \xi v$ where ξ is a primitive n-th root of unity. We shall write $R = R_v^{\xi}$. One has

$$R_v^{\xi}(x) = x - (1-\xi)\frac{\langle v|x\rangle}{v^2}v.$$

Let K be a $\mathcal{G}$-lattice. A *root* of K means a primitive vector v of K of positive norm such that $R_v^{\xi} \in \mathsf{Aut}(K)$ for some nontrivial root of unity ξ. The *reflection group* of K, denoted $R(K)$, is the subgroup of $\mathsf{Aut}(K)$ generated by the reflections in the roots of K. Write $R_v = R_v^{i}$. Let s and t be two norm 2 vectors of K. We say that R_s and R_t *braid* if $R_s R_t R_s = R_t R_s R_t$. One verifies that R_s and R_t commute (resp. braid) if $\langle s|t\rangle = 0$ (resp. $|\langle s|t\rangle| = \sqrt{2}$). Further, if $|\langle s|t\rangle| = 2$, then $i R_s R_t R_s(t) = t$, and hence $R_s R_t R_s R_t = R_t R_s R_t R_s$.

2.7. ***Reflection groups of p-modular lattices.*** Assume K is a p-modular $\mathcal{G}$-lattice. Then the minimal norm of K is at least 2. As explained in the introduction, from the formula for complex reflection we find that $K(2)$ is the set of roots of K. The reflections in $R(K)$ are the order 4 and order 2 reflections in these roots.

2.8. ***The reflection group of*** $\boldsymbol{D_4^{\mathcal{G}}}$**.** The lattice $D_4^{\mathcal{G}}$ has six roots counted up to roots of unity. These are $(p, 0)$, $(0, p)$, $(1, i^r)$, for $r = 0, 1, 2, 3$. If s and t are any two nonproportional and nonorthogonal roots of $D_4^{\mathcal{G}}$, then the finite complex reflection group $R(D_4^{\mathcal{G}})$ is generated by the i-reflections R_s and R_t. These two reflections obey the braiding relation $R_s R_t R_r = R_t R_s R_t$. The relations $R_s^4 = R_t^4 = 1$ and $R_s R_t R_r = R_t R_s R_t$ give a presentation of $R(D_4^{\mathcal{G}})$. This group is called G_8 in the table of finite complex reflection groups given in [Broué et al. 1998].

2.9. ***Complex hyperbolic space.*** Let V be a complex vector space of dimension $(n+1)$ with the standard nondegenerate Hermitian form of signature $(n, 1)$. Let $\mathbb{B}(V)$ be the set of 1-dimensional negative definite subspaces of V. This is an open subset of the projective space $P(V)$. If L is a Lorentzian $\mathcal{G}$-lattice, we write $\mathbb{B}(L) = \mathbb{B}(L \otimes_{\mathcal{G}} \mathbb{C})$.

The group of isometries V is $U(n, 1)$ and $\mathbb{B}(V)$ is a concrete model for the corresponding Hermitian symmetric space, sometimes called the complex hyperbolic space and denoted by $\mathbb{C}H^n$. Up to scaling, there is a unique $PU(n, 1)$-invariant metric on $\mathbb{C}H^n$ called the Bergman metric. We only need a few facts about the associated distance function.

A negative norm vector v in V determines a point $\mathbb{C}v$ in $\mathbb{B}(V)$. A positive norm vector r in V determines a totally geodesic hyperplane $\mathbb{B}(r^\perp)$ in $\mathbb{B}(V)$. For simplicity we shall write v instead of $\mathbb{C}v$ and $r^\perp$ instead of $\mathbb{B}(r^\perp)$ when there is no chance of confusion.

Let u, v be two negative norm vectors in V. The distance between the corresponding points in the complex hyperbolic space is

$$d(u, v) = \cosh^{-1}\sqrt{\frac{|\langle u|v\rangle|^2}{u^2v^2}}.$$

Let r be a negative norm vector in V. Then

$$d(r^\perp, v) = \sinh^{-1}\sqrt{\frac{|\langle r|v\rangle|^2}{-r^2v^2}}.$$

Let r, s be two negative norm vectors in V. If $\mathbb{C}r + \mathbb{C}s$ is positive definite then the hyperplanes $\mathbb{B}(r^\perp)$ and $\mathbb{B}(s^\perp)$ meet in $\mathbb{B}(L)$. Otherwise, one has

$$d(r^\perp, s^\perp) = \cosh^{-1}\sqrt{\frac{|\langle r|s\rangle|^2}{r^2s^2}}.$$

Our distance function differs from the ones in [Goldman 1999] by a factor of 2. This is not an issue because we use these formulas only to compare distances. Let $v \in V$ be a positive norm vector. Let ξ be a primitive k-th root of unity for some $k > 1$. Unless it is necessary, we shall not distinguish between R_v^ξ and its image in $PU(n, 1)$ and refer to either as a ξ-reflection in s. This reflection is an isometry of $\mathbb{B}(V)$ that pointwise fixes the totally geodesic hyperplane $v^\perp$ (called the *mirror* of reflection) and acts as anticlockwise rotation of angle $2\pi/k$ in the normal bundle to the mirror.

2.10. Definition (horocyclic distance). Let V be as in Section 2.9. Let z be a null vector in V. If v is a negative norm vector in V, define

$$d_z(v) = \tfrac{1}{2}\log(|\langle z|v\rangle|^2/(-v^2)).$$

Note that d_z determines a function on the complex hyperbolic space $\mathbb{B}(V)$. We denote this function also by d_z. The null vector z determines a point $\mathbb{C}z$ in the boundary $\partial\mathbb{B}(V)$ of $\mathbb{B}(V)$. As before, we write z instead of $\mathbb{C}z$ if there is no chance of confusion. We say that $d_z(v)$ is the *horocyclic distance* between z and v. This terminology is justified by the lemma below. We include a proof because we could not find a convenient reference.

2.11. Lemma (ideal triangle inequality). (a) *Let x, y be negative norm vectors in V. Then one has $|d_z(x) - d_z(y)| \leq d(x, y)$.*

(b) *The equality $d_z(x) - d_z(y) = d(x, y)$ holds if and only if y lies on the geodesic ray joining x and z.*

Proof. (a) Let $\alpha = \langle z|x\rangle$, $\beta = \langle y|z\rangle$ and $\gamma = \langle x|y\rangle$. By changing x, y by units if necessary, we may assume, without loss, that, $|x|^2 = |y|^2 = -1$. If z, x, y are linearly independent then their span has signature (2, 1), so $\mathsf{det}(\mathsf{gram}(z, x, y)) < 0$, otherwise $\mathsf{det}(\mathsf{gram}(z, x, y)) = 0$. So we have

$$0 \geq \mathsf{det}(\mathsf{gram}(z, x, y)) = \mathsf{det}\begin{pmatrix} 0 & \alpha & \bar{\beta} \\ \bar{\alpha} & -1 & \gamma \\ \beta & \bar{\gamma} & -1 \end{pmatrix} = |\alpha|^2 + |\beta|^2 + 2\,\mathsf{Re}(\alpha\beta\gamma)$$
$$\geq |\alpha|^2 + |\beta|^2 - 2|\alpha\beta\gamma|.$$

It follows that

$$\cosh d(x, y) = |\gamma| \geq \frac{1}{2}\left(\frac{|\alpha|}{|\beta|} + \frac{|\beta|}{|\alpha|}\right) = \frac{1}{2}(e^{d_z(x)-d_z(y)} + e^{d_z(y)-d_z(x)})$$
$$= \cosh|d_z(x) - d_z(y)|.$$

Since $(t \mapsto \cosh t)$ is strictly increasing for $t \in [0, \infty)$, part (a) follows.

(b) Suppose y lies on the geodesic ray joining z and x, Then z, x, y are linearly dependent. So the calculation in part (a) shows that $d(x, y) = |d_z(x) - d_z(y)|$. Now, without loss, assume $\langle z|x\rangle$ is a negative real number and $|x|^2 = -1$. If y is on the geodesic ray joining x and z, then $\mathbb{C}y = \mathbb{C}(x + tz)$ for some $t \geq 0$. So

$$e^{2d_z(y)} = \frac{|\langle x|z\rangle|^2}{-(x+tz)^2} = \frac{|\langle x|z\rangle|^2}{1 - 2t\langle x|z\rangle} < |\langle x|z\rangle|^2 = e^{2d_z(x)}.$$

So $d_z(x) > d_z(y)$ and hence $d(x, y) = d_z(x) - d_z(y)$. The other implication follows from uniqueness of the geodesic which is a consequence of negative curvature. We shall skip the details since we do not need this for our application. □

2.12. Definition (horoballs). Let z be a null vector in V. Let c be a positive real number. A subset $B \subseteq \mathbb{B}(V)$ of the form $B = \{v : d_z(v) < c\}$ is called an open *horoball* around z. Similarly define a closed horoball. The boundary of a horoball around z is called a *horosphere* around z. Pick $v \in \mathbb{B}(V) - B$. Let q be the point where the geodesic ray joining v and z intersects ∂B. Then, one verifies that q is the unique point of the closed horoball $\overline{B}$ that is closest to v, that is,

$$d(q, v) = d(B, v).$$

In other words, q is the *projection* of v on B. Lemma 2.11 implies that

$$d_z(v) = c + d(B, v).$$

Let $\zeta \in \partial\mathbb{B}(V)$ be the point determined by z. We shall say that v_1 is closer to ζ than v_2 in horocyclic distance if and only if $d_z(v_1) < d_z(v_2)$. Note that this notion does not depend on the choice of z. Another way to say this is to note that v_1 is closer to ζ than v_2 if and only if v_1 is closer to B than v_2, where B is any small horoball around ζ that misses v_1 and v_2.

3. Reflection groups of p-modular $\mathcal{G}$-lattices: height reduction

In this section we prove some results about the reflection group of a general p-modular Lorentzian $\mathcal{G}$-lattice L. A null vector $z \in L$ or the point of $\mathbb{B}(L)$ determined by z is called a *cusp* (of $R(L)$). Our first goal is to prove some lemmas that are useful for finding sets of mirrors close to a cusp z such that the reflections in them generate $R(L)$. Formally, these results are of the following sort:

3.1. Lemma. *Let G be a group of isometries of a metric space X. Let $\mathcal{H}$ be a G-stable collection of subsets of X and let $A \subseteq X$ such that $\{d(A, H) : H \in \mathcal{H}\}$ is a discrete subset of* $[0, \infty)$. *Let $d_0 \in [0, \infty)$. Assume that for all $H \in \mathcal{H}$ with $d(A, H) > d_0$, there exists $g \in G$ such that $d(A, gH) < d(A, H)$. Then $\{H \in \mathcal{H} : d(A, H) \le d_0\}$ meets every G-orbit in $\mathcal{H}$.*

In this situation we say that $d(A, H)$ (or some suitable increasing function of it) is the *height* of H (with respect to A). The proof is an obvious induction on height. We call these height reduction arguments. In our application, G will be some subgroup of $R(L)$, $X = \mathbb{B}(L)$, and A will be either a point in $\mathbb{B}(L)$ or a small horoball around some cusp of $R(L)$. The collection $\mathcal{H}$ will be either the set of mirrors of $R(L)$ or a suitable collection of horoballs around the cusps of $R(L)$.

Our second goal of this section is to introduce a discrete Heisenberg group $\mathbb{T}$ sitting in the stabilizer of a cusp in $\mathrm{Aut}(L)$ and show that the reflection group $R(L)$ contains a finite index subgroup of $\mathbb{T}$.

3.2. ***Notation.*** For this section, let Λ denote a p-modular positive definite $\mathcal{G}$-lattice of rank n. Let $\Lambda(r \bmod 4) = \{\lambda \in \Lambda : \lambda^2 \equiv r \bmod 4\}$. Since the underlying $\mathbb{Z}$-lattice of Λ is even, $\lambda \mapsto \frac{1}{2}\lambda^2 \bmod 2$ is a homomorphism $\Lambda \to \mathbb{Z}/2$. The kernel of this homomorphism is $\Lambda(0 \bmod 4)$ and the other coset is $\Lambda(2 \bmod 4)$. If $a \in p\mathcal{G}$, then the homomorphism $\Lambda \to \mathbb{Z}/2$ factors through Λ/a. In other words, all the elements in a coset in Λ/a either have norm $0 \bmod 4$ or have norm $2 \bmod 4$.

3.3. ***p-modular Gaussian Lorentzian lattice.*** Let $L = \Lambda \oplus \mathcal{G}_{1,1}$. Since $\mathcal{G}_{1,1}$ is p-modular, so is L. Note that all vectors of L have even norms. We write a vector of L in the form $(\sigma; m, n)$ where $\sigma \in \Lambda$ and $m, n \in \mathcal{G}$. The Hermitian form on L is given by

$$\langle(\sigma; m, n)|(\sigma'; m', n')\rangle = \langle\sigma|\sigma'\rangle + \bar{m}\bar{p}n' + \bar{n}pm'.$$

In particular,

$$(\sigma; m, n)^2 = \sigma^2 + 2\,\mathsf{Re}(\bar{m}\bar{p}n).$$

Let

$$\rho = (0; 0, 1).$$

This norm zero vector plays a special role throughout. Note that

$$\langle\rho|(\sigma; m, n)\rangle = pm.$$

If $s \in L(N) - \rho^{\perp}$, then one can write s in the form

$$s = \left(\sigma; m, p\bar{m}^{-1}\left(\frac{N - \sigma^2}{4} + \nu\right)\right), \tag{1}$$

where $\nu \in \mathsf{Im}(\mathbb{C})$ is chosen so that the last coordinate of s lies in $\mathcal{G}$. We define the *height* of a primitive lattice vector s (with respect to ρ) to be

$$\mathsf{ht}_\rho(s) = \begin{cases} |\langle s|\rho\rangle|^2/|s^2| & \text{if } s^2 \neq 0, \\ |\langle s|\rho\rangle|^2 & \text{if } s^2 = 0. \end{cases}$$

Given $s \in L(N)$ and $s' \in L(N')$ written in the form (1), we record a formula for their inner product which is verified by direct calculation:

$$\begin{aligned} \mathsf{Re}\left\langle \frac{s}{m} \,\middle|\, \frac{s'}{m'} \right\rangle &= \frac{N}{2|m|^2} + \frac{N'}{2|m'|^2} - \frac{1}{2}\left(\frac{\sigma}{m} - \frac{\sigma'}{m'}\right)^2, \\ \mathsf{Im}\left\langle \frac{s}{m} \,\middle|\, \frac{s'}{m'} \right\rangle &= \mathsf{Im}\left\langle \frac{\sigma}{m} \,\middle|\, \frac{\sigma'}{m'} \right\rangle + \frac{2\nu'}{|m'|^2} - \frac{2\nu}{|m|^2}. \end{aligned} \tag{2}$$

The expression for s in (1) and the inner product formulas in (2) are direct analogs of equations (2-6) and (2-7) of [Allcock and Basak 2018]. They may seem complicated but they are very useful in computation, for example, see the proof of Lemma 3.6.

3.4. ***The roots of L near the cusp ρ.*** The roots of L are the vectors of minimum norm 2. Let $s \in L(2)$ be a root. As in (1), we write

$$s = \left(\sigma; m, p\bar{m}^{-1}\left(\frac{1}{2}\left(1 - \frac{\sigma^2}{2}\right) + \nu\right)\right).$$

Note that $\mathsf{ht}_\rho(s) = |m|^2$. The roots having height $1, 2, 4, \ldots$ are called the roots in the first shell, second shell, third shell and so on. The mirror of a root in the j-th shell is called a j-th shell mirror. Among the mirrors that do not pass through ρ, the first shell mirrors are the mirrors closest to the cusp ρ in horocyclic distance. The second shell mirrors are the next closest and so on. Below we explicitly describe the roots in the first two shells. One verifies easily that the first shell roots of L are of the form

$$i^r\Big(\sigma;\,1,\,p\Big(\frac{1}{2}\Big(1-\frac{\sigma^2}{2}\Big)+\nu\Big)\Big)\quad \text{where } \sigma\in\Lambda,\ \nu\in\frac{i}{2}\mathbb{Z} \text{ and } \frac{2}{i}\nu\equiv\Big(1-\frac{\sigma^2}{2}\Big) \bmod 2.$$

Writing $\nu = ik - \frac{i}{2}\big(1-\frac{\sigma^2}{2}\big)$ we find that the first shell roots are of the form

$$i^r\Big(\sigma;\,1,\,1-\frac{\sigma^2}{2}+ipk\Big),\qquad \text{where } \sigma\in\Lambda \text{ and } k\in\mathbb{Z}.$$

One verifies easily that the second shell roots of L are of the form

$$i^r\Big(\sigma;\,\bar p,\,\frac{1}{2}\Big(1-\frac{\sigma^2}{2}\Big)+\nu\Big)\quad \text{where } \sigma\in\Lambda(2 \bmod 4) \text{ and } \nu\in i\mathbb{Z}.$$

It is useful to note that if s is a first or second shell root written as above and we change ν to $\nu'\in\nu+i\mathbb{Z}$, then we again get a root in the same shell.

Let $l=(\lambda;\,h,\,*)\in L$. Let $s=(\sigma;\,m,\,*)$ be a root of L. Assume $h,m\neq 0$. The lemma below gives us a necessary and sufficient condition for an order 4 reflection in s to decrease the height of l. If l is a root, then this is equivalent to saying that a reflection in s brings the mirror $\mathbb{B}(l^\perp)$ closer to ρ. The condition is conveniently expressed in terms of the quantity

$$y=y(s,l)=|m|^2\Big\langle\frac{s}{m}\;\Big|\;\frac{l}{h}\Big\rangle.$$

3.5. Lemma. *Given a root s as above, an order four reflection in s decreases the height of l if and only if $y=y(s,l)$ belongs to $B(1+i,\sqrt{2})\cup B(1-i,\sqrt{2})$, that is, the union of radius $\sqrt{2}$ open discs in the complex plane centered at* $(1\pm i)$.

Proof. Let $\xi=\pm i$ and let R denote the ξ-reflection in s. We calculate

$$\Big\langle\rho\;\Big|\;R\Big(\frac{l}{h}\Big)\Big\rangle=\Big\langle R^{-1}(\rho)\;\Big|\;\frac{l}{h}\Big\rangle=\Big\langle\rho-\frac{1}{2}(1-\bar\xi)\bar m\bar p s\;\Big|\;\frac{l}{h}\Big\rangle=p-\frac{1}{2}(1-\xi)py.$$

The reflection R decreases the height of l if and only if $\big|\langle\rho\,|\,\frac{l}{h}\rangle\big|>\big|\langle\rho\,|\,R\big(\frac{l}{h}\big)\rangle\big|$, that is,

$$|p|>\big|p-\tfrac{1}{2}(1-\xi)py\big|.$$

Multiplying both sides by $|p^{-1}(1-\bar\xi)|$, the inequality becomes

$$\sqrt{2}>|(1-\bar\xi)-y|$$

which is equivalent to y lying in $B(1+i,\sqrt{2})\cup B(1-i,\sqrt{2})$. □

3.6. Lemma. *Let l be a root of L. As in* (1), *write the root in the form* $l=(\lambda;\,h,\,p\bar m^{-1}((l^2-\lambda^2)/4+\nu_l))$.

(a) *Assume that $|h|>1$ and that there exists $\sigma\in\Lambda$ satisfying $(\sigma-\lambda/h)^2\leq 2$. Then some first shell reflection decreases the height of l.*

(b) *Assume that* $|h| > \sqrt{2}$ *and that there exists* $\sigma \in p^{-1}\Lambda(2 \bmod 4)$ *satisfying* $(\sigma - \lambda/h)^2 \le 1$. *Then some second shell reflection decreases the height of* l.

Proof. In part (a) (resp. (b)) we show that we can choose a root s in the first (resp. second) shell such that $y = y(s, l)$ belongs to the rectangle $(0, 2) \times [-i, i]$. The lemma then follows from Lemma 3.5, since this rectangle is a subset of $B(1+i, \sqrt{2}) \cup B(1-i, \sqrt{2})$.

(a) Choose $\sigma \in \Lambda$ such that $(\sigma - \lambda/h)^2 \le 2$. Consider a first shell root written in the form $s = \big(\sigma; 1, p\big(\frac{1}{2}\big(1 - \frac{1}{2}\sigma^2\big) + \nu\big)\big)$ as in Section 3.4 with ν still to be determined. Using (2), we compute

$$\text{(3)} \quad \mathrm{Re}(y) = 1 + \frac{l^2}{2|h|^2} - \frac{1}{2}\Big(\sigma - \frac{\lambda}{h}\Big)^2 \quad \text{and} \quad \mathrm{Im}(y) = \mathrm{Im}\Big\langle \sigma \;\Big|\; \frac{\lambda}{h}\Big\rangle + \frac{2\nu_l}{|h|^2} - 2\nu.$$

Since $|h| > 1$ and $l^2 = 2$, the choice of σ ensures that $\mathrm{Re}(y) \in (0, 2)$. From Section 3.4, note that we are free to choose $2\nu/i$ either from $2\mathbb{Z}$ or from $2\mathbb{Z}+1$. So we can choose ν to ensure that $\mathrm{Im}(y) \in [-i, i]$.

(b) Choose $\sigma \in \Lambda(2 \bmod 4)$ such that $(\sigma/\bar{p} - \lambda/h)^2 \le 1$. Consider a second shell root written in the form $s = \big(\sigma; \bar{p}, \big(\frac{1}{2}(1 - \frac{1}{2}\sigma^2) + \nu\big)\big)$ as in Section 3.4 with ν still to be determined. Using (2), we compute

$$\text{(4)} \quad \mathrm{Re}(y) = 1 + \frac{l^2}{|h|^2} - \Big(\frac{\sigma}{\bar{p}} - \frac{\lambda}{h}\Big)^2 \quad \text{and} \quad \mathrm{Im}(y) = 2\,\mathrm{Im}\Big\langle \frac{\sigma}{\bar{p}} \;\Big|\; \frac{\lambda}{h}\Big\rangle + \frac{4\nu_l}{|h|^2} - 2\nu.$$

Since $|h| > \sqrt{2}$ and $l^2 = 2$, the choice of σ ensures that $\mathrm{Re}(y) \in (0, 2)$. From Section 3.4, Note that we are free to choose $2\nu/i$ from $2\mathbb{Z}$. So we can choose ν to ensure that $\mathrm{Im}(y) \in [-i, i]$. □

3.7. Lemma. *Let* $l = (\lambda; h, *)$ *be a primitive null vector of* L *with* $h \ne 0$. *Assume that* l *is not orthogonal to any root of* L.

(a) *If there exists* $\sigma \in \Lambda$ *with* $(\sigma - \lambda/h)^2 \le 2$, *then a reflection in a first shell root decreases height of* l.

(b) *If there exists* $\sigma \in p^{-1}\Lambda(2 \bmod 4)$ *with* $(\sigma - \lambda/h)^2 \le 1$, *then a reflection in a second shell root decreases the height of* l.

Proof. The proof is almost identical to the proof of Lemma 3.6. The computation is only slightly different. Equations (3) and (4) still hold but now since $l^2 = 0$, we obtain $y(s, l) \in [0, 1] \times [-i, i]$. This rectangle minus the origin is a subset of $B(1+i, \sqrt{2}) \cup B(1-i, \sqrt{2})$. Since l is assumed to be not orthogonal to any root, we have $y(s, l) \ne 0$. □

3.8. Definition (the Heisenberg group of translations). Let $\mathbb{T}$ be the group of automorphisms of L that fix ρ and act trivially on $\rho^\perp/\rho$. One verifies that

$$\mathbb{T} = \{T_{\lambda,z} : \lambda \in \Lambda, z \in i(\lambda^2/2 + 2\mathbb{Z})\} = \{T_{\lambda, i(\lambda^2/2)+2ik} : \lambda \in \Lambda, k \in \mathbb{Z}\},$$

where $T_{\lambda,z} \in \mathsf{Aut}(L)$ is defined by

$$T_{\lambda,z}(l; a, b) = (l + a\lambda; a, -\bar{p}^{-1}\langle\lambda|l\rangle + a\bar{p}^{-1}(z - \lambda^2/2) + b).$$

Note that for each $\lambda \in \Lambda$, the integer z/i either runs over $2\mathbb{Z}$ or $2\mathbb{Z}+1$. We call $\mathbb{T}$ the group of *translations*. One verifies that the translations form a discrete Heisenberg group whose multiplication law is given by

$$T_{\lambda,z}T_{\lambda',z'} = T_{\lambda+\lambda',z+z'+\mathsf{Im}\langle\lambda'|\lambda\rangle}. \tag{5}$$

Note that the translations of the form $T_{0,z}$ are central, $T_{\lambda,z}^{-1} = T_{-\lambda,-z}$, and

$$T_{\lambda,z}T_{\lambda',z'}T_{\lambda,z}^{-1}T_{\lambda',z'}^{-1} = T_{0,2\,\mathsf{Im}\langle\lambda'|\lambda\rangle}. \tag{6}$$

3.9. Lemma. *Let R_1 and R_2 be the i-reflections in the roots $r_1 = (0^n; 1, 1)$ and $r_2 = (0^n; 1, i)$ respectively. Let $\lambda \in \Lambda$. Choose z such that $T_{\lambda,z} \in \mathbb{T}$. Let G be the group generated by the reflections in $T_{\lambda,z}(r_1),\ T_{\lambda,z}(r_2),\ r_1,\ r_2$. Then $T_{\bar{p}\lambda,i\lambda^2} \in G$.*

Proof. Let β be the automorphism of L that is the identity on Λ and acts on $\mathcal{G}_{1,1}$ as multiplication by $-i$. Then one verifies that $R_1R_2 = \beta T_{0,-4i}$. Since $T_{0,-4i}$ is a central translation, it follows that

$$(R_1R_2)T_{\lambda,z}(R_1R_2)^{-1} = \beta T_{\lambda,z}\beta^{-1} = T_{i\lambda,z}.$$

So G contains $T_{\lambda,z}(R_1R_2)T_{\lambda,z}^{-1}(R_1R_2)^{-1} = T_{\bar{p}\lambda,i\lambda^2}$. □

We finish this section by showing that $R(L)$ contains many translations, specifically, a finite index subgroup of $\mathbb{T}$.

3.10. Corollary. (a) *Fix a $\mathcal{G}$-basis $\lambda_1, \ldots, \lambda_n$ of Λ. For each $j = 1, \ldots, n$, fix $z_j \in \mathsf{Im}(\mathbb{C})$ such that $T_{\lambda_j,z_j} \in \mathbb{T}$. Let G be the subgroup of $R(L)$ generated by reflections in the following set of roots*:

$$\{r_k, T_{\lambda_j,z_j}(r_k), T_{i\lambda_j,z_j}(r_k) : k = 1, 2,\ j = 1, 2, \ldots, n\}.$$

Then G contains a translation of the form $T_{\lambda,*}$ for each $\lambda \in p\Lambda$ and the central translations $T_{0;4im}$ for all $m \in \mathbb{Z}$.

(b) *A full set of coset representatives for $(R(L) \cap \mathbb{T})\backslash\mathbb{T}$ can be chosen from the finite set $\mathbb{T}_* = \{T_{\sigma,i\sigma^2/2}, T_{\sigma,2i+i(\sigma^2/2)} : \sigma \in (\Lambda/p)^{\sim}\}$ where $(\Lambda/p)^{\sim}$ is a full set of coset representatives for Λ/p.*

Proof. (a) Lemma 3.9 implies that G contains a translation of the form $T_{\bar{p}\lambda_j,*}$ and a translation of the form $T_{i\bar{p}\lambda_j,*}$ for each j. From (5), it follows that $R(L)$ contains a translation of the form $T_{\lambda,*}$ for all $\lambda \in p\Lambda$. Since Λ is p-modular, choose $\lambda, \lambda' \in p\Lambda$ with $\langle\lambda'|\lambda\rangle = 2p$. Then (6) implies that G also contains $T_{0,4i}$. So G contains the central translations of the form $T_{0,4im}$ for all $m \in \mathbb{Z}$.

(b) Let $T = T_{\lambda',*} \in \mathbb{T}$. Choose $\sigma \in (\Lambda/p)^{\sim}$ such that $\sigma - \lambda' = p\lambda$ for some $\lambda \in \Lambda$. By part (a) we can choose a translation T_1 in $R(L)$ having the form $T_1 = T_{p\lambda,*}$. Then

$$T_1 T = T_{p\lambda,*} T_{\lambda',*} = T_{p\lambda+\lambda',*} = T_{\sigma,*}.$$

So $T_1 T = T_{\sigma,2ik+i(\sigma^2/2)}$ for some $k \in \mathbb{Z}$. Choose $m \in \mathbb{Z}$ such that $k + 2m \in \{0, 1\}$. Let $T' = T_{0,4mi} T_1$. Then $T' \in R(L)$ and $T'T = T_{\sigma,2i(k+2m)+i(\sigma^2/2)} \in \mathbb{T}_*$. □

4. Reflection group of the p-modular $\mathcal{G}$-lattice of signature (9, 1)

4.1. Lemma. *There is a unique p-modular Gaussian lattice of signature* $(2d-1, 1)$.

Lemma 4.1 follows from the classification of self-dual indefinite $\mathcal{G}$-lattices; see [Allcock 2000b, Theorem 7.1]. The idea of the proof is due to Daniel Allcock and is the same as the proof of Lemma 2.6 of [Basak 2007a], but somewhat more complicated. The only implication of Lemma 4.1 that we need is the isomorphism $4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1} \simeq \mathrm{BW}_{16}^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$, and we shall exhibit an explicit isomorphism in Section 5.10, because we need it for our computations. So strictly speaking, Lemma 4.1 is not needed for our main theorem. However, we feel that the lemma may be useful in other contexts. So we have included a proof in the Appendix.

4.2. *Notation.* In the last section, Λ denoted any positive definite p-modular Gaussian lattice. From here on, unless otherwise stated, we specialize to the case

$$\Lambda = \mathrm{BW}_{16}^{\mathcal{G}} \quad \text{and} \quad L = \mathcal{G}_{9,1} \simeq 4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1} \simeq \Lambda \oplus \mathcal{G}_{1,1}.$$

Both descriptions of L are going to be useful for us. In this section, unless otherwise stated, we identify $L = \Lambda \oplus \mathcal{G}_{1,1}$. In the next section we shall use the identification $L = 4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$. Let $\rho = (0; 0, 1) \in L$. Since Λ has no root, there are no mirrors through the cusp ρ. As before, the mirrors closest to ρ are called the first shell mirrors; the mirrors that are next closest to ρ are called second shell mirrors, and so on. The corresponding roots are called first shell roots, second shell roots, etc.

Our first goal is to show that the projective reflection group $\Gamma = PR(L)$ is an arithmetic lattice in $PU(9, 1)$. The plan of the proof follows Theorem 4.1 of [Allcock 2000a].

4.3. Lemma. *The reflection group $R(L)$ acts transitively on primitive null vectors of L (considered up to 4-th roots of unity) that are not orthogonal to any roots.*

Proof. Let $z = (\zeta; h, *)$ be a primitive null vector of L that is not orthogonal to any roots. Suppose $h \neq 0$. Lemma 2.5 implies that either there exists $\sigma \in \Lambda$ such that $(\sigma - \zeta/h)^2 \leq \sqrt{2}$ or there exists $\sigma \in p^{-1}\Lambda(2 \bmod 4)$ such that $(\sigma - \zeta/h)^2 \leq 1$. Lemma 3.7 then shows that the height of z can be reduced by a reflection in some root (Lemma 3.7 only uses reflections in the first and second shell roots but this is irrelevant for this argument). By induction, it follows that a finite sequence of reflection can bring z to a null vector of the form $z_1 = (\zeta_1\,; 0, *)$. Now, $z_1^2 = 0$ implies $\zeta_1^2 = 0$. Since z_1 is primitive, it follows that z_1 is a unit multiple of ρ. □

4.4. Theorem. $|\Gamma\backslash P\mathrm{Aut}(L)| \le 2^9|\mathrm{Aut}(\Lambda)|$. *In particular*, Γ *is arithmetic.*

Proof. Let $\rho_1 = (0; 1, 0)$. Take $g \in \mathrm{Aut}(L)$. Then $g\rho$ is a primitive null vector that is not orthogonal to any roots. Lemma 4.3 implies that there exists $g_1 \in R(L)$ such that $i^{-m}g_1^{-1}g\rho = \rho$ for some $m \in \mathbb{Z}/4$. So $i^{-m}g_1^{-1}g\rho_1$ is a null vector of the form $(*\,; 1, *)$. One verifies that the group of translations acts transitively on the null vectors of the form $(*\,; 1, *)$. So there exists a translation $T \in \mathrm{Aut}(L)$ such that $T^{-1}i^{-m}g_1^{-1}g$ fixes ρ and ρ_1. Let $\alpha = T^{-1}i^{-m}g_1^{-1}g$. Then $\alpha \in \mathrm{Aut}(\Lambda)$, a finite group. So $g = g_1T\alpha i^m$. By Corollary 3.10(b), there exists a $g_2 \in R(L)$ and $t \in \mathbb{T}_*$ such that $T = g_2t$. So $g = g_1T\alpha i^m = g_1g_2t\alpha i^m \in R(L)t\alpha i^m$. It follows that a full set of coset representatives for $\Gamma\backslash P\mathrm{Aut}(L)$ can be chosen from the finite set $\{t\alpha : t \in \mathbb{T}_*, \alpha \in \mathrm{Aut}(\Lambda)\}$. □

The goal of the rest of the section is to find a finite set of generators for $R(L)$.

4.5. Lemma. *The* i*-reflections in the first and second shell roots generate* $R(L)$.

Proof. The argument is similar to the proof of Lemma 4.3 except that one needs Lemma 3.6 instead of Lemma 3.7. From Lemma 2.5, we know that closed balls of radius $\sqrt{2}$ around vectors in Λ together with closed balls of radius 1 around the vectors in $p^{-1}\Lambda(2 \bmod 4)$ cover the underlying real vector space of Λ. So the lemma follows from parts (a) and (b) of Lemma 3.6, using induction on height of a root. □

Lemma 4.5 give us an infinite set of reflections that generates $R(L)$. The next lemma shows that an explicit finite subset of these reflections is enough to generate $R(L)$. To list the roots of these generating reflections, fix a $\mathcal{G}$-basis $\lambda_1, \dots, \lambda_8$ of Λ. For each $j = 1, \dots, 8$, fix a $z_j \in \mathrm{Im}(\mathbb{C})$ such that $T_{\lambda_j, z_j} \in \mathbb{T}$. Recall that if $m \in \mathcal{G}$, then $(\Lambda/m)^{\sim}$ denotes a full set of coset representatives for Λ/m. Also recall the roots $r_1 = (0^8; 1, 1)$ and $r_2 = (0^8; 1, i)$ introduced in Lemma 3.9. Define

- $S_0 = \{r_k, T_{\lambda_j, z_j}(r_k), T_{i\lambda_j, z_j}(r_k) : j = 1, \dots, 8, k = 1, 2\}$,
- $S_1 = \{(\sigma; 1, 1 - \sigma^2/2 + ipk) : \sigma \in (\Lambda/p)^{\sim}, k = 0, 1\}$.
- $S_2 = \big\{\big(\sigma; \bar{p}, \tfrac{1}{2}(1 - \sigma^2/2) + ik\big) : \sigma \in (\Lambda/2)^{\sim} \cap \Lambda(2 \bmod 4), k = 0, 1, 2, 3\big\}$.

From the discussion in Section 3.2, recall that all the vectors in a coset in $\Lambda/2$ either have norm 0 mod 4 or have norm 2 mod 4. So $(\Lambda/2)^{\sim} \cap \Lambda(2 \bmod 4)$ is a set of representatives of the cosets that consist of vectors of norm 2 mod 4.

4.6. Lemma. *The* i*-reflections in the roots in* $S_0 \cup S_1 \cup S_2$ *generate* $R(L)$.

Proof. Let G be the group generated by the reflections listed. Since G contains the reflections in the roots in S_0, Corollary 3.10(a) implies that G contains a translation of the form $T_{p\lambda,*}$ for all $\lambda \in \Lambda$ and the central translations of the form $T_{0,4in}$ for $n \in \mathbb{Z}$. We make the following claim:

Claim: If s is a root of the form $(*\,;\,1,*)$ (*resp.* $(*\,;\,\bar{p},*)$), *then there exists a translation* $T\in G$ *such that* Ts *belongs to* S_1 (*resp.* S_2).

Let s be a root of the form $s=(\sigma;\,1,*)$. Choose $\sigma_0\in(\Lambda/p)^{\sim}$ such that $\sigma_0-\sigma=p\lambda$ for some $\lambda\in\Lambda$. Choose a translation of the form $T_{p\lambda,*}\in G$ that takes s to a root of the form $s'=(\sigma_0;\,1,*)$. Next, one can choose a central translation in G that takes s' to a root in S_1. This proves the claim for roots of the form $(*\,;\,1,*)$.[3] The argument for roots of the form $(*\,;\,\bar{p},*)$ is similar.

Now let R be an i-reflection in a root in the first or second shell. Then there exists a root s of the form $(*\,;\,1,*)$ or $(*\,;\,\bar{p},*)$ such that $R=R_s$. Choose $T\in G$ such that $Ts\in S_1\cup S_2$. So $R_{Ts}\in G$. It follows that

$$R_s=T^{-1}R_{Ts}T\in G.$$

Thus, G contains the i-reflections in all the roots in the first two shells. Lemma 4.5 completes the proof. □

5. The 32 mirrors closest to a point in $\mathbb{C}H^9$

The goal of this section is to find nice generators and relations for $R(L)$, analogous to the Coxeter generators and relations of Weyl groups. This is only a rough analogy; for example the generators do not form not a minimal set and we do not know if our relations are sufficient to give a presentation of $R(L)$. However, the analogy is reinforced because similar phenomena repeat for other complex hyperbolic reflection groups of interesting lattices, as illustrated by Theorem 1.1. In particular, everything in this section closely parallels the results of [Basak 2007a] and [Basak 2007b].

As for Coxeter groups, our generators and relations can be encoded in a diagram D, a sort of Coxeter-Dynkin diagram of $R(L)$. This diagram can be defined from the intersection pattern of a configuration of points and hyperplanes in the finite vector space $\mathbb{F}_2^4$. We shall define the lattice L starting from D, rather like defining the root lattice from a Cartan matrix. We start by describing these points and hyperplanes and by working out the symmetries of this configuration.

5.1. *A configuration of points and hyperplanes in* $\mathbb{F}_2^4$.

Let

$$g_1=(1,0,0,0),\ldots,g_4=(0,0,0,1)$$

be the standard basis vectors of $\mathbb{F}_2^4$. The 16 points of $\mathbb{F}_2^4$ are named a, c_j, g_j, e_{ij}, z where

$$a=(1,1,1,1),\quad c_i=a+g_i,\quad e_{ij}=a+g_i+g_j,\quad z=(0,0,0,0)$$

[3] $\mathbb{T}$ acts simply transitively on the roots of the form $(*;\,1,*)$. So the argument here is essentially a repeat of the proof of Corollary 3.10(b).

and $1 \le i < j \le 4$. Next, we name 16 hyperplanes in $\mathbb{F}_2^4$. Let

$$d_k = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}_2^4 : x_k = 0\}$$

and

$$f_k = \Big\{(x_1, x_2, x_3, x_4) \in \mathbb{F}_2^4 : x_k = \sum_j x_j\Big\}.$$

So $\mathcal{K}_0 = \{d_1, \dots, d_4, f_1, \dots, f_4\}$ is the set of homogeneous hyperplanes in $\mathbb{F}_2^4$ not containing a. Let

$$b_k = \mathbb{F}_2^4 - f_k \quad \text{and} \quad h_k = \mathbb{F}_2^4 - d_k.$$

Finally let $\mathcal{K} = \{d_k, f_k, b_k, h_k : k = 1, 2, 3, 4\}$ be the set of translates of $\mathcal{K}_0$ and let

$$D = \mathbb{F}_2^4 \cup \mathcal{K}.$$

Let Q_+ be the subgroup of the group of affine transformations of $\mathbb{F}_2^4$ that preserves D. One verifies that $Q_+ \simeq 2^4 : (2^3 : L_3(2))$ where the 2^4 comes from the translation action of $\mathbb{F}_2^4$ on itself and the $2^3 : L_3(2)$ is the stabilizer of a in $L_4(2)$. It will be useful to note that the symmetric group S_4 acting by coordinate permutation on $\mathbb{F}_2^4$ fixes a. So this S_4 is a subgroup of Q_+.

The diagram for our reflection group $R(L)$ is shown in Figure 1. It is a graph with vertex set D and with two kinds of edges as shown in Figure 1. Two vertices u and v are joined by a dashed edge if $v = t_a(u)$ where t_a is the automorphism of D induced by the translation $t_a : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ defined by $t_a(v) = a + v$. If we ignore the dashed edges, then D should be thought of as a directed bipartite graph with a directed edge from v to u if $v \in \mathcal{K}$, $u \in \mathbb{F}_2^4$ and u is incident on v. The automorphism group Q_+ acts on this graph D preserving both kinds of edges.

5.2. Lemma. *Let L° be the Gaussian lattice of rank 32 with a basis $\{s_v^\circ : v \in D\}$ indexed by D and inner product defined by*

$$\langle s_u^\circ | s_v^\circ \rangle = \begin{cases} 2 & \text{if } u = v, \\ p & \text{if } u \in \mathbb{F}_2^4, v \in \mathcal{K} \text{ and } u \in v, \\ \bar{p} & \text{if } u \in \mathcal{K}, v \in \mathbb{F}_2^4 \text{ and } v \in u, \\ -2 & \text{if } v = t_a(u), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Then $L \simeq L^\circ / \operatorname{Rad}(L^\circ)$.

Proof. Identify $L = 4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$. We claim that there are 32 roots $\{s_v : v \in D\}$ in L such that the inner products between them are governed by D as in (7). In other words, there is an inner product preserving linear map $L^\circ \to L$ that sends s_v° to s_v. Recall that there is an obvious S_4 action on D. The symmetric group S_4 also acts on $4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$ by permuting the four copies of $D_4^{\mathcal{G}}$. The map $s_v^\circ \mapsto s_v$ is going to be S_4 equivariant. Define

$$\begin{aligned}
s_a &= [0,0,0,0,0,0,0,0;\,-1,-1],\\
s_{c_1} &= [-1,1,0,0,0,0,0,0;\;\;0,0],\\
s_{e_{12}} &= -[1,1,1,1,0,0,0,0;\;\;i,1],\\
s_{g_1} &= -[0,2,1,1,1,1,1,1;\,2i,2],\\
s_z &= -[1,1,1,1,1,1,1,1;\,-1+2i,1],\\
s_{f_1} &= [0,0,0,p,0,p,0,p;\,ip,p],\\
s_{b_1} &= [0,p,0,0,0,0,0,0;\,-1,0],\\
s_{d_1} &= [-p,0,0,0,0,0,0,0;\;\;0,0],\\
s_{h_1} &= [p,p,0,p,0,p,0,p;\;\;p-3,p].
\end{aligned}$$

The other roots are obtained by using the S_4 symmetry (that is, by permuting the four copies of $D_4^{\mathcal{G}}$). For example $s_{c_2} = [0,0,\ -1,1,\ 0,0,\ 0,0;\ \ 0,0]$. The lemma is proved once one verifies that these roots have the required inner products. □

5.3. Remark. We want to mention how we came up with the 32 roots $\{s_v : v \in D\}$. Write $L_* = 3D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$. The complex reflection group of $D_4^{\mathcal{G}}$ is generated by two braiding i-reflections. In other words, the finite complex reflection group $R(D_4^{\mathcal{G}})$ has Dynkin diagram A_2 with each vertex having order 4. Since $R(3D_4^{\mathcal{G}}) \subseteq R(L_*)$, there are three disjoint copies of A_2 diagrams in $R(L_*)$. One can extend this diagram first to an affine diagram and then to a hyperbolic diagram to obtain a Y_{333} diagram inside $R(L_*)$ (similar computations can be found in [Conway and Sloane 1988, Chapter 30]; also see [Basak 2007a] where the Y_{555} diagram arises in a similar manner starting from $R(3E_8^{\mathcal{E}})$). The graph Y_{333} is a maximal subtree in $\mathrm{Inc}(P^2(\mathbb{F}_2))$, which means the incidence graph of the finite projective plane $P^2(\mathbb{F}_2)$. One can extend the Y_{333} diagram uniquely to an $\mathrm{Inc}(P^2(\mathbb{F}_2))$ diagram in $R(L_*)$. The diagram automorphisms $L_3(2){:}2$ act on $\mathbb{B}(L_*)$ with a unique fixed point τ_* and the 14 mirrors corresponding to the vertices of $\mathrm{Inc}(P^2(\mathbb{F}_2))$ are exactly the mirrors closest to τ. We tried to prove results similar to Theorem 1.1 for the reflection group $R(L_*)$ but could not make the arguments work because the covering radius of the lattice $3D_4^{\mathcal{G}}$ is not small enough. However, these arguments work for $L = 4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$ because of the alternative description $L \simeq \mathrm{BW}_{16}^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$. In the root system of $L = 4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$, we can naturally extend the $\mathrm{Inc}(P^2(\mathbb{F}_2))$ diagram, first by using the obvious S_4 symmetry permuting the four copies of $D_4^{\mathcal{G}}$ and then by looking for a regular graph. This leads to the 32 root diagram D in L.

5.4. *Linear relations among the 32 roots.* One could also prove Lemma 5.2 by working out the 22-dimensional radical of L°. It is useful for us to at least write down enough linear relations among the 32 roots $\{s_v : v \in D\}$, where enough means that the corresponding vectors of $\mathsf{Rad}(L^\circ)$ span $\mathsf{Rad}(L^\circ)\otimes\mathbb{C}$. If both $v, w \in \mathbb{F}_2^4$ or both $v, w \in \mathcal{K}$, then we have the relation

$$s_v + s_{t_a(v)} = s_w + s_{t_a(w)}. \tag{8}$$

We define

$$p_\infty = s_v + s_{t_a(v)} \quad \text{if } v \in \mathbb{F}_2^4,$$

and

$$l_\infty = s_v + s_{t_a(v)} \quad \text{if } v \in \mathcal{K}.$$

Using (7), one verifies that p_∞ and l_∞ are primitive null vectors of L. Explicitly, one computes

$$p_\infty = -(1,1,1,1,1,1,1,1;2i,2) \quad \text{and} \quad l_\infty = (0,p,0,p,0,p,0,p;p-3,p).$$

Further, for each $u \in \mathbb{F}_2^4$ and for each $w \in \mathcal{K}$, we have the relations

$$-2(1+i)s_u + \sum_{v\in\mathcal{K}:u\in v} s_v = 4l_\infty - (1+i)p_\infty, \tag{9}$$

and

$$-2(1-i)s_w + \sum_{v\in\mathbb{F}_2^4:v\in w} s_v = 4p_\infty - (1-i)l_\infty. \tag{10}$$

To verify the relation (8), one just checks, using (7), that both sides of (8) have the same inner product with each vector in $\{s_v : v \in D\}$. The relations in (9) and (10) can be verified similarly, or by direct computation.

5.5. ***The configuration of the 32 mirrors in complex hyperbolic space.*** One verifies that the mirrors $\{s_v^\perp : v \in \mathbb{F}_2^4\}$ meet at the cusp determined by p_∞ and the mirrors $\{s_v^\perp : v \in \mathcal{K}\}$ meet at the cusp determined by l_∞. The group $Q_+ = 2^4 : (2^3 : L_3(2))$ transitively permutes these two sets of 16 mirrors and fixes the $\mathbb{C}H^1$ spanned by p_∞ and l_∞. The set of 32 mirrors $\{s_v^\perp : v \in D\}$ has an extra symmetry that we describe next. Let $\sigma_\mathcal{K} : \mathcal{K} \to \mathbb{F}_2^4$ be the bijection

$$\sigma_\mathcal{K}\left(\left\{(x_1,\dots,x_4) \in \mathbb{F}_2^4 : \sum_j u_j x_j = \epsilon\right\}\right) = (u_1,u_2,u_3,1) + (u_4+\epsilon)(1,1,1,1).$$

(See the Note below if the definition of $\sigma_\mathcal{K}$ seems too ad hoc.) Define $\sigma_D : D \to D$ by $\sigma_D|_\mathcal{K} = \sigma_\mathcal{K}$ and $\sigma_D|_{\mathbb{F}_2^4} = \sigma_\mathcal{K}^{-1}$. One verifies that the involution σ_D preserves the incidence relations between points and hyperplanes and commutes with the action of the translation t_a. It follows that σ_D is an orientation-reversing involution of D. In other words, the involution σ_D acts on D by preserving both kinds of edges and reversing the orientation on the solid edges. Define

$$\sigma_{L^\circ} : L^\circ \to L^\circ \quad \text{by } \sigma_{L^\circ}(s_v^\circ) = \begin{cases} s^\circ_{\sigma_D(v)} & \text{if } v \in \mathcal{K}, \\ -is^\circ_{\sigma_D(v)} & \text{if } v \in \mathbb{F}_2^4. \end{cases}$$

From the definition of the inner product on L°, it follows that σ_{L° is an automorphism of L° whose square is multiplication by $-i$. So σ_{L° descends to define an automorphism $\sigma = \sigma_L$ of L and an involution of $\mathbb{B}(L)$, also denoted by σ. This involution σ interchanges the 16 mirrors meeting at p_∞ with the 16 mirrors meeting at l_∞. One verifies that the group Q generated by σ and Q_+ permutes the 32 mirrors transitively and fixes a unique point in $\mathbb{B}(L)$ which can be represented by the vector

$$\tau = e^{-\pi i/4} l_\infty - p_\infty.$$

The 32 mirrors are all equidistant from τ. Let d_0 be this distance. We compute

$$d_0 = d(\tau, s_v^\perp) \approx 0.4090 \quad \text{for all } v \in D.$$

As already mentioned, we are using the same notation for a vector, say τ (resp. a hyperplane, say $s_j^\perp$) in $L \otimes_{\mathcal{G}} \mathbb{C}$ and the point (resp. hyperplane) in $\mathbb{B}(L)$ it determines. Let B be a small horoball around p_∞ not containing τ and let B' be the image of B under any automorphism of L taking p_∞ to l_∞. Then τ is the point on the real geodesic joining p_∞ and l_∞ that is equidistant from B and B'. We should think of τ as the midpoint between p_∞ and l_∞.

Note. The following discussion may help to clarify the definition of $\sigma_{\mathcal{K}}$. Identify $\mathbb{F}_2^4$ and $\mathcal{K}$ with two affine hyperplanes

$$\mathfrak{F} = \{x \in \mathbb{F}_2^5 : x_5 = 1\} \quad \text{and} \quad \mathfrak{K} = \{x \in \mathbb{F}_2^5 : x_1 + x_2 + x_3 + x_4 = 1\}$$

in $\mathbb{F}_2^5$ respectively via

$$i_{\mathfrak{F}} : v \mapsto \begin{pmatrix} v \\ 1 \end{pmatrix} \quad \text{and} \quad i_{\mathfrak{K}} : \{x \in \mathbb{F}_2^4 : u^T x = \epsilon\} \mapsto \begin{pmatrix} u \\ \epsilon \end{pmatrix}.$$

Note that $v \in \mathbb{F}_2^4$ belongs to the hyperplane $\{x \in \mathbb{F}_2^4 : u \cdot x = \epsilon\}$ if and only if $\binom{v}{1}$ and $\binom{u}{\epsilon}$ are orthogonal with respect to the standard inner product of $\mathbb{F}_2^5$. We need to choose an appropriate $\sigma_{\mathfrak{K}} \in L_5(2)$ taking $\mathfrak{K}$ to $\mathfrak{F}$. For this, let $J_{m,n}$ denote the $m \times n$ matrix whose entries are all equal to 1. Let $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let I_n denote the $n \times n$ identity matrix. Define

$$\sigma_{\mathfrak{K}} = \begin{pmatrix} I_3 & J_{3,2} \\ J_{2,3} & H \end{pmatrix} \in L_5(2).$$

Verify that $\sigma_{\mathfrak{K}}^{-1} : \mathfrak{F} \to \mathfrak{K}$ and $\sigma_{\mathfrak{K}} : \mathfrak{K} \to \mathfrak{F}$ are mutually inverse bijections and $\sigma_{\mathfrak{K}}^2$ is an involution. Let $f \in \mathfrak{F}$ and $k \in \mathfrak{K}$. Since $\sigma_{\mathfrak{K}}$ is self-adjoint, $\sigma_{\mathfrak{K}}^{-1} f$ is orthogonal to $\sigma_{\mathfrak{K}} k$ if and only k is orthogonal to f. Let $\sigma_{\mathcal{K}} = i_{\mathfrak{F}}^{-1} \circ \sigma_{\mathfrak{K}} \circ i_{\mathfrak{K}}$. Then $\sigma_{\mathcal{K}} : \mathcal{K} \to \mathbb{F}_2^4$ is a bijection such that $v \in h$ if and only if $\sigma(h) \in \sigma(v)$ for all $v \in \mathbb{F}_2^4$ and for all $h \in \mathcal{K}$. In other words, $\sigma_{\mathcal{K}}$ and $\sigma_{\mathcal{K}}^{-1}$ interchange the points in $\mathbb{F}_2^4$ with the hyperplanes in $\mathcal{K}$ preserving the incidence relations. For $w \in \mathbb{F}_2^4$, the translation t_w acts on $\mathfrak{F}$ and $\mathfrak{K}$ as the matrices

$$t_w|_{\mathfrak{F}} = \begin{pmatrix} I_4 & w \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t_w|_{\mathfrak{K}} = \begin{pmatrix} I_4 & 0 \\ w^T & 1 \end{pmatrix}.$$

One verifies that $(t_w|_{\mathfrak{F}})\sigma_{\mathfrak{K}} = \sigma_{\mathfrak{K}}\big(t_w|_{\mathfrak{K}}\big)$ if and only if $w = a$ or $w = 0$.

5.6. Theorem. *The* 32 *mirrors* $\{s_v^{\perp} : v \in D\}$ *are precisely the mirrors closest to* τ. *In particular*, τ *does not lie on any mirror.*

Let r be any root of L such that $d(\tau, r^{\perp}) \le d_0$. The lemma below gives us conditions that allow us to restrict the possibilities for r to a finite set.

5.7. Lemma. (a) *Let* w *be a root of* L. *Then*

$$|p^{-1}\langle w|r\rangle|^2 \le 2\cosh^2(d_0 + d(\tau, w^{\perp})).$$

(b) *Let* w *be a primitive null vector of* L. *From* Definition 2.10, *recall that* $d_w(\tau) = \frac{1}{2}\log(|\langle w|\tau\rangle|^2/(-\tau^2))$. *One has*

$$|p^{-1}\langle w|r\rangle|^2 \le e^{2(d_0 + d_w(\tau))}.$$

In each case, *note* $p^{-1}\langle w|r\rangle \in \mathcal{G}$. *So* $|p^{-1}\langle w|r\rangle|^2$ *belongs to* $\{0, 1, 2, 4, 5, 8, 9, \dots\}$. *Thus*, *in each case*, *there are finitely many possibilities for* $\langle w|r\rangle$.

Proof. (a) If the hyperplanes $w^{\perp}$ and $r^{\perp}$ meet in $\mathbb{B}(L) \cup \partial\mathbb{B}(L)$, then $|p^{-1}\langle w|r\rangle|^2 \le 2$, so the required inequality holds trivially. Otherwise, the triangle inequality implies

$$d(r^{\perp}, w^{\perp}) \le d(r^{\perp}, \tau) + d(\tau, w^{\perp}) \le d_0 + d(\tau, w^{\perp}).$$

Part (a) follows using the formula given in Section 2.9 for the distance between two hyperplanes.

(b) Let p_τ be the projection of τ on $r^{\perp}$. Then $d(\tau, p_\tau) = d(\tau, r^{\perp}) \le d_0$. Choose a small horoball B around w that does not meet τ and $r^{\perp}$. Let p_w be the point of $r^{\perp}$ that is nearest B. In other words p_w is the projection of w on $r^{\perp}$. Then p_w is closer to w than p_τ, that is, $d_w(p_w) \le d_w(p_\tau)$. From the ideal triangle inequality (Lemma 2.11), we have

$$\frac{1}{2}\log\left(\frac{|\langle w|p_w\rangle|^2}{-p_w^2}\right) = d_w(p_w) \le d_w(p_\tau) \le d_w(\tau) + d(\tau, p_\tau) \le d_w(\tau) + d_0.$$

So

$$|\langle w|p_w\rangle|^2/(-p_w^2) \le e^{2(d_0 + d_w(\tau))}.$$

The projection p_w of w on $r^\perp$ is represented by the intersection of $\mathbb{C}w+\mathbb{C}r$ and $r^\perp$. So p_w can be represented by the vector $p_w = w - \langle r|w\rangle r/r^2$. One computes that $p_w^2 = -|\langle r|w\rangle|^2/r^2$ and $\langle w|p_w\rangle^2/(-p_w^2) = |\langle r|w\rangle|^2/r^2$. Part (b) follows. □

Lemma 5.7(a) implies, in particular, that $|p^{-1}\langle r|s_v\rangle|^2 \le 2\cosh^2(2d_0) \approx 3.6642$ for all $v \in D$. So

$$p^{-1}\langle r|s_v\rangle \in \mathcal{G}(\le 2) \quad \text{for all } v \in D.$$

(Recall that $\mathcal{G}(\le k)$ denotes the set of elements of $\mathcal{G}$ of norm $\le k$). To obtain further restrictions on r, it will be convenient to use the following basis $v_1, \dots, v_{10}$ for $L \otimes \mathbb{C}$:

$$(v_1, v_2, \dots, v_{10}) = (-s_{d_1}, s_{b_1}, -s_{d_2}, s_{b_2}, -s_{d_3}, s_{b_3}, -s_{d_4}, s_{b_4}, (0^8; 1, 0), l_\infty).$$

The roots s_{d_i}, s_{b_i} were defined in the proof of Lemma 5.2. The inner products between $v_1, \dots, v_{10}$ are described as follows: $v_1, \dots, v_8$ are eight roots that form an orthogonal basis for a maximal positive definite subspace of $L \otimes \mathbb{C}$ whose orthogonal complement has a basis consisting of the two null vectors v_9, v_{10}. Finally, $\langle v_9|v_{10}\rangle = 2$. Write r as a linear combination of $v_1, \dots, v_{10}$ in the form

$$r = p^{-1}(c_1v_1 + c_2v_2 + \cdots + c_{10}v_{10}). \tag{11}$$

The lemma below gives enough conditions on $c_1, \dots, c_{10}$ to allow a computer enumeration of all possible $(c_1, \dots, c_{10})$ and hence, of all possible r.

5.8. Lemma. *One has $c_1, \dots, c_9 \in \mathcal{G}(\le 2)$ and $c_{10} \in \mathcal{G}(\le 9)$. Furthermore,*

$$c_1 + c_2 \equiv c_3 + c_4 \equiv c_5 + c_6 \equiv c_7 + c_8 \equiv c_{10} \bmod p$$

and

$$|c_1|^2 + \cdots + |c_8|^2 = 2 - 2\,\mathrm{Re}(\bar{c}_9 c_{10}) \in \{2, 4, 6, 8, 10\}.$$

Proof. Taking the inner product of r with $v_1, \dots, v_{10}$ we find

$$c_9 = \bar{p}^{-1}\langle v_{10}|r\rangle, \;\; c_{10} = \bar{p}^{-1}\langle v_9|r\rangle, \;\; c_j = \bar{p}^{-1}\langle v_j|r\rangle \quad \text{for } j = 1, \dots, 8.$$

Since L is p-modular, $c_1, \dots, c_{10} \in \mathcal{G}$. Write r in the coordinate system $4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$:

$$r = (c_1, c_2 + c_{10}, c_3, c_4 + c_{10}, c_5, c_6 + c_{10}, c_7, c_8 + c_{10}; r_9, c_{10}),$$

where

$$r_9 = (c_9 + (p-3)c_{10} - c_2 - c_4 - c_6 - c_8)/p. \tag{12}$$

From the definition of $D_4^{\mathcal{G}}$, it follows that $c_{2j-1} + c_{2j} + c_{10} \equiv 0 \bmod p$ for $j = 1, 2, 3, 4$. This implies the congruences on the c_j's.

The bounds on norms of c_j's follow from Lemma 5.7. As already noted, Lemma 5.7(a) implies $p^{-1}\langle r|s_v\rangle \in \mathcal{G}(\leq 2)$ for $v \in D$. In particular, $c_1, \dots, c_8 \in \mathcal{G}(\leq 2)$. Next, Lemma 5.7(b) implies that

$$|p^{-1}\langle v_9|r\rangle|^2 \leq e^{2(d_0+d_{v_9}(\tau))} \approx 9.3379,$$

and

$$|p^{-1}\langle v_{10}|r\rangle|^2 \leq e^{2(d_0+d_{v_{10}}(\tau))} \approx 3.2043.$$

This implies that $c_{10} = \bar{p}^{-1}\langle v_9|r\rangle \in \mathcal{G}(\leq 9)$ and $c_9 = \bar{p}^{-1}\langle v_{10}|r\rangle \in \mathcal{G}(\leq 2)$. Taking norms of the two sides of (11) and rearranging, we obtain

$$|c_1|^2 + \cdots + |c_8|^2 = 2 - 2\,\mathrm{Re}(\bar{c}_9 c_{10}).$$

We already know that there are a small number of possibilities for c_9 and c_{10}. Enumerating these, we find that $\mathrm{Re}(\bar{c}_9 c_{10}) \in [-4, 4] \cap \mathbb{Z}$. Since $|c_1|^2 + \cdots + |c_8|^2 \geq 0$, it follows that $\mathrm{Re}(\bar{c}_9 c_{10}) \in [-4, 1] \cap \mathbb{Z}$. The lemma follows once we argue that $\mathrm{Re}(\bar{c}_9 c_{10}) \neq 1$. If possible, suppose $\mathrm{Re}(\bar{c}_9 c_{10}) = 1$. Then $\sum_{j=1}^{8} |c_j|^2 = 0$, so $c_1 = \cdots = c_8 = 0$. The congruences satisfied by c_j's imply that $c_{10} \equiv 0 \bmod p$. Now the condition $r_9 \in \mathcal{G}$ implies that $c_9 \equiv 0 \bmod p$. But this forces $\mathrm{Re}(\bar{c}_9 c_{10})$ to be in $2\mathbb{Z}$ which is a contradiction. □

Proof of Theorem 5.6. We use a computer program to enumerate all possible tuples $(c_1, \dots, c_{10})$ satisfying the conditions of Lemma 5.8 and subject to the further restriction that $c_9 \in \{0, 1, p\}$. We may assume $c_9 \in \{0, 1, p\}$ since it is enough to enumerate the possible tuples $(c_1, \dots, c_{10})$ up to units. Let $r = \left(\sum_j c_j v_j\right)/p$. Then r is a root of L if and only if $r_9 \in \mathcal{G}$ (see (12)). We run through the possibilities for r and list those for which $r_9 \in \mathcal{G}$ and $d(\tau, r^\perp) \leq d_0$. This produces only the unit multiples of $\{s_v : v \in D\}$. □

5.9. Theorem. *The i-reflections in the* 32 *roots* $\{s_v : v \in D\}$ *generate* $R(L)$*. These generators obey the Coxeter relations dictated by D as stated in the introduction after the statement of Theorem 1.2 and in the caption of Figure 1.*

Proof. Write $S = \{s_v : v \in D\}$. Let G denote the subgroup of $R(L)$ generated by the reflections in S. Lemma 4.6 provides us a finite set of roots $S_0 \cup S_1 \cup S_2$ such that the i-reflections in them generate $R(L)$. We take a root $x_0 \in S_0 \cup S_1 \cup S_2$ and try to find some $s \in S$ and $\xi \in \{i, i^2, i^3\}$ such that $x_1^\perp = R_s^\xi(x_0^\perp)$ is closer to τ than $x_0^\perp$. We repeat this to obtain a sequence of roots $x_0, x_1, x_2, \dots$. If some x_j is a unit multiple of a root in S, then we say that height reduction (with respect to τ) is successful for x_0 and in this case, we obtain $R_{x_0} \in G$.

In a computer calculation, height reduction is successful for most of the 123426 roots in $S_0 \cup S_1 \cup S_2$. For 401 roots (all from S_2) height reduction is not successful. In these cases, we end up with a root $x_j^\perp$ whose distance from τ cannot be decreased by any reflection in S. Let S' be the set of these 401 roots. To deal with these cases, by a little experimentation, we found a root $y \in S_2 - S'$, such that height reduction is successful for all the roots in $\{R_y(x) : x \in S'\}$. This means that $R_y \in G$ and, further, that for each $x \in S'$, one has $R_y R_x R_y^{-1} \in G$; hence $R_x \in G$. This proves that the reflections in S generate $R(L)$. More details on computer calculations are given in Section 5.10. The Coxeter relations between these 32 generators are consequences of the inner products between the roots in S, as given in (7). □

5.10. ***Remarks on computer calculations.*** In the proof of Theorem 5.9 we glossed over one step. The roots $S_0 \cup S_1 \cup S_2$ in Lemma 4.6 are written in the coordinate system $\mathsf{BW}_{16}^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$ while the 32 roots in S are written in the coordinate system $4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$. So we need to find an explicit isomorphism from $\mathsf{BW}_{16}^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$ to $4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$. Let $v_1, \dots, v_{10} \in \mathsf{BW}_{16}^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$ be the rows of the following matrix:

$$\frac{1}{2}\begin{bmatrix} p & p & \bar{p} & -\bar{p} & \bar{p} & -\bar{p} & p & p & 2 & -2 \\ -\bar{p} & -p & -p & -\bar{p} & -p & \bar{p} & \bar{p} & -p & -2 & 2 \\ 0 & 2p & 0 & 0 & 0 & 0 & 0 & 2p & 2 & -2 \\ -2 & -2i & 0 & 0 & -2 & -2i & 0 & 0 & -2 & 2 \\ \bar{p} & p & p & -\bar{p} & p & -\bar{p} & \bar{p} & p & 2 & -2 \\ -2 & -2i & 0 & 0 & 0 & 0 & -2i & -2 & -2 & 2 \\ p & p & p & p & p & p & p & p & 2 & -2 \\ -\bar{p} & -p & p & \bar{p} & -p & \bar{p} & \bar{p} & -p & -2 & 2 \\ 3\bar{p} & 5+i & \bar{p} & p & 3-i & 1+3i & p & 5-i & 4-6i & -4+4i \\ -3\bar{p} & -5-i & -\bar{p} & -p & -3+i & -1-3i & -p & -5+i & -6+6i & 4-6i \end{bmatrix}$$

One verifies that the inner products between $(v_1, \dots, v_{10})$ are the same as the inner products between the ten vectors $(d_1, c_1, d_2, c_2, d_3, c_3, d_4, c_4, (0^8; 1, 0), (0^8; 0, 1))$ that form a basis of $4D_4^{\mathcal{G}} \oplus H$. So sending $v_1, v_2, \dots$ to $d_1, c_1, \dots$ defines an isomorphism from $\mathsf{BW}_{16}^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$ to $4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$. Finding the vectors $v_1, \dots, v_{10}$ required considerable computation using a list of the 4320 short vectors of $\mathsf{BW}_{16}^{\mathcal{G}}$. We shall omit these computational details, since, for the purpose of proving Theorem 5.9, these computations are irrelevant once the vectors $v_1, \dots, v_{10}$ have been found. One has to simply verify that $v_1, v_2, \dots$ has the same gram matrix as $d_1, c_1, \dots$.

The root $y \in 4D_4^{\mathcal{G}} \oplus \mathcal{G}_{1,1}$ used in the proof of Theorem 5.9 to perturb the 401 elements of S' is

$$y = [1 + 2i, 3, 1 + i, 5 + i, 1 + 2i, 4 + i, 1 + 2i, 4 + i, 7, -6i]. \tag{13}$$

All the computer calculations needed in this article were performed using the pari/gp calculator [PARI]. The calculations are contained in the file `bw2.gp`, available on the website math.iastate.edu/~tathagat/codes. The calculations needed for Theorem 5.9 only use exact arithmetic. Running the height reduction algorithm on the 123426 roots to verify Theorem 5.9 took a couple of hours of computation time on a regular laptop computer. However, most of these intensive calculations become redundant after the fact. For each of the 123426 roots we record the finite sequence of simple reflections (or rather the corresponding simple roots) that reduce the height. Once we have this data, Theorem 5.9 can be verified on the computer in a few minutes.

5.11. Theorem. *The* 13 *i-reflections in $s_a, s_{b_k}, s_{c_k}, s_{d_k}$ for $k = 1, 2, 3, 4$ generate $R(L)$.*

Theorem 5.11 follows quickly from Theorem 5.9. Before giving the proof, we recall a definition from [Basak 2012].

5.12. Definition. Let $\{x_j : j \in \mathbb{Z}/k\}$ be the elements in a monoid and let m be a positive integer. Let $\mathcal{C}_m\langle x_0, \dots, x_{k-1}\rangle$ denote the positive homogeneous relation

$$x_0 x_1 \cdots x_{m-1} = x_1 x_2 \cdots x_m.$$

For example, $\mathcal{C}_2\langle x, y\rangle$ (resp. $\mathcal{C}_3\langle x, y\rangle$) denotes the relations $xy = yx$ (resp. $xyx = yxy$). Let $\tilde{A}_{n-1}$ be the affine Dynkin diagram of type A_{n-1} with vertices labeled by $\mathbb{Z}/n$. We call it an n-gon. Let $\{y_j : j \in \mathbb{Z}/n\}$ be the generators for the corresponding Artin group

(14) $\quad y_j y_k y_j = y_k y_j y_j$ if j and k are adjacent $\quad$ and $\quad y_j y_k = y_k y_j$ otherwise.

If $y_j^2 = 1$, then we have a presentation of the affine Weyl group of type A_{n-1}. The relation $\mathcal{C}_{n-1}\langle y_1, \dots, y_n\rangle$ collapses this affine Weyl group to the spherical Weyl group or the symmetric group. Following [Conway and Simons 2001], we call this relation *deflating the n-gon* $(y_1, \dots, y_n)$. One can verify that in the presence of the braiding and commuting relations of (14), the deflation relation $\mathcal{C}_{n-1}\langle y_1, \dots, y_n\rangle$ is equivalent to $\mathcal{C}_{n-1}\langle y_{j+1}, \dots, y_{j+n}\rangle$ for any j (see [Basak 2012], Lemma 4.3(a)).

Proof of Theorem 5.11. Write $r_v = R_{s_v}$. Let G be the subgroup of $R(L)$ generated by the 13 reflections $r_a,\ r_{b_1}, \dots, r_{b_4},\ r_{c_1}, \dots, r_{c_4},\ r_{d_1}, \dots, r_{d_4}$. One verifies that $(d_2, c_2, b_2, a, b_1, c_1, d_1, e_{12})$ is an octagon in the graph D and the deflation relation

$$\mathcal{C}_7\langle r_{d_2}, r_{c_2}, r_{b_2}, r_a, r_{b_1}, r_{c_1}, r_{d_1}, r_{e_{12}}\rangle$$

holds in $R(L)$. This shows that $r_{e_{12}} \in G$. By S_4 symmetry we obtain $r_{e_{jk}} \in G$ for all j, k. Next, observe that one has the octagons

$$(d_1, c_1, b_1, e_{34}, b_2, c_2, d_2, z), \quad (c_2, b_2, a, b_3, e_{24}, d_4, c_4, f_1),$$
$$(c_4, b_4, e_{13}, d_3, z, d_2, c_2, h_1), \quad (d_2, e_{23}, f_3, c_4, h_3, a, b_2, g_1)$$

in D and that deflation relations hold for these four octagons. Applying S_4 symmetry, it successively follows that $r_z, r_{f_k}, r_{h_k}, r_{g_k} \in G$ as well. Now the theorem follows from Theorem 5.9. □

6. An automorphic form on $\mathbb{C}H^9$ related to the reflection group Γ

Borcherds has asked if interesting hyperbolic reflection groups always have reflective modular forms associated to them. In [Borcherds 1998] and [Borcherds 2000b], he finds many examples of this mainly in the $O(1, n)$ and the $O(2, n)$ case. Problem 13.1 of [Borcherds 2000b] asks if this happens in the complex hyperbolic, i.e., the $U(1, n)$ case. Theorem 6.1 provides a positive answer to this question for the complex reflection group Γ of our lattice L. The calculations in this section closely follow [Borcherds 2000a].

6.1. Theorem. *There is a holomorphic automorphic form of weight* 8 *on* $\mathbb{C}H^9$ *for a* 1*-dimensional representation of* $\mathsf{Aut}(L)$ *and its divisor coincides with the set of mirrors of the reflection group* Γ. *It has order* 4 *zeros along each mirror.*

Let M be the underlying $\mathbb{Z}$-lattice of L with the inner product negated. If $x, y \in L$, then their inner product as elements of M is

$$(x|y) = -\operatorname{Re}\langle x|y\rangle.$$

In particular, $(x|x) = -\langle x|x\rangle = -x^2$. The lattice M has signature $(b^+, b^-) = (2, 18)$. The discriminant group $M^\vee/M$ is $(\mathbb{Z}/2)^{10}$ since

$$M^\vee/M \simeq L^\vee/L = p^{-1}L/L \simeq (\mathcal{G}/p\mathcal{G})^{10} \simeq (\mathbb{Z}/2)^{10}.$$

If U is a subset of $L \otimes \mathbb{C}$ (resp. $M \otimes \mathbb{R}$), then we write $\mathbb{B}(U)$ (resp. $G(U)$) for the set of all maximal negative definite complex subspaces (resp. maximal positive definite real subspaces) in the complex (resp. real) span of U. Here $\mathbb{B}$ stands for ball and G stands for Grassmannian. In particular, $\mathbb{C}H^9 \simeq \mathbb{B}(L) \subseteq G(M) \simeq G(2, 18)$. If λ is a positive norm vector in L, then it determines a sub-ball $\mathbb{B}(\lambda^\perp)$ in $\mathbb{B}(L)$ and a subgrassmannian $G(\lambda^\perp)$ in $G(M)$. By slight abuse of notation, one usually denotes both $\mathbb{B}(\lambda^\perp)$ and $G(\lambda^\perp)$ simply by $\lambda^\perp$, but in this section we will try to maintain the distinction. Note that $G(\lambda^\perp) \cap \mathbb{B}(L) = \mathbb{B}(\lambda^\perp)$.

Let $\mathbb{C}[M^\vee/M] = \bigoplus_{\gamma\in M^\vee/M} \mathbb{C}e_\gamma$ be a complex vector space with basis indexed by the elements of $M^\vee/M$. Let $\rho_M : \mathsf{SL}_2(\mathbb{Z}) \to \mathsf{Aut}(\mathbb{C}[M^\vee/M])$ be the Weil representation associated to the discriminant form $M^\vee/M$. It is defined by

$$\rho_M(T)e_\gamma = \mathrm{e}((\gamma|\gamma)/2)e_\gamma \quad \text{and} \quad \rho_M(S)(e_\gamma) = 2^{-5}\sum_\delta \mathrm{e}(-(\gamma|\delta))e_\delta,$$

where $\mathrm{e}(x) = \exp(2\pi i x)$ and $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. To apply the singular theta correspondence, we want to find a vector-valued modular form

$$F(\tau) = \sum_{\gamma\in M^\vee/M} f_\gamma(\tau)e_\gamma$$

of singular weight $1 - b^-/2 = -8$ and of type ρ_M that is holomorphic on the upper half plane with poles at cusps, such that the nonpositive parts of the Fourier expansion of $f_\gamma(\tau)$'s at $i\infty$ have all integral coefficients. In particular, the functions $(f_\gamma)_{\gamma\in M^\vee/M}$ must satisfy

$$f_\gamma(\tau+1)=\mathrm{e}((\gamma|\gamma)/2)f_\gamma(\tau)\quad\text{and}\quad f_\gamma(-\tau^{-1})=2^{-5}\tau^{-8}\sum_\delta f_\delta(\tau)\,\mathrm{e}(-(\delta|\gamma)).$$

Note that if $\gamma\in M^\vee/M$, then $\gamma^2 \bmod 2\mathbb{Z}\in\mathbb{Z}/2\mathbb{Z}$ is well defined. In other words, we have a quadratic form $q: M^\vee/M\to\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ defined by

$$q(\gamma)=\tfrac{1}{2}(\gamma|\gamma) \bmod \mathbb{Z}=\tfrac{1}{2}\gamma^2 \bmod \mathbb{Z}.$$

The bilinear form associated to q is $\partial q:(M^\vee/M)\times(M^\vee/M)\to\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, defined by

$$\partial q(\gamma|\delta)=(\gamma|\delta) \bmod \mathbb{Z}=\mathrm{Re}\langle\gamma|\delta\rangle \bmod \mathbb{Z}.$$

We shall say that a nonzero element γ of $M^\vee/M$ is of *type* 0 (resp. type 1) if $q(\gamma)=0$ $\big($resp. $q(\gamma)=\frac{1}{2}\big)$. We shall also say that the element $0\in M^\vee/M$ has type 00. Some information about the metric group $(M^\vee/M,q)$ is summarized in the following table:

type of u / type of v	00 / 00	00 / 0	00 / 1	0 / 00	0 / 0	0 / 1	1 / 00	1 / 0	1 / 1
$\#\{v:(u\|v)\equiv 0 \bmod \mathbb{Z}\}$	1	527	496	1	271	240	1	255	256
$\#\{v:(u\|v)\equiv \frac{1}{2} \bmod \mathbb{Z}\}$	0	0	0	0	256	256	0	272	240

For example, the entry 272 means that for each $u\in M^\vee/M$ of type 1, there are 272 elements v of type 0 such that $(u|v)\equiv\frac{1}{2} \bmod \mathbb{Z}$. We verify on the computer that this number is independent of the choice of u (and similarly in all the other cases). This suggests that we may search for a modular form F that has only three types of components f_{00}, f_0, f_1, such that

- $f_\gamma=f_{00}$ if $\gamma=0$,
- $f_\gamma=f_0$ for all $\gamma\in(M^\vee/M)-\{0\}$ with $\gamma^2\equiv 0 \bmod 2\mathbb{Z}$,
- $f_\gamma=f_1$ for all $\gamma\in(M^\vee/M)-\{0\}$ with $\gamma^2\equiv 1 \bmod 2\mathbb{Z}$.

Let $f=(f_{00},f_0,f_1)^{\mathrm{tr}}$. Using the information in the table above, we find that the transformation properties of F translate into

$$f(\tau+1)=T_0f(\tau)\quad\text{and}\quad f(-1/\tau)=\tau^{-8}S_0f(\tau), \tag{15}$$

where

$$T_0=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\quad\text{and}\quad S_0=2^{-5}\begin{bmatrix}1&527&496\\1&15&-16\\1&-17&16\end{bmatrix}.$$

We construct a function f which satisfies these transformation properties using eta-quotients. Let $q = \mathrm{e}(\tau)$. The Dedekind eta function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ satisfies the transformation laws

$$\eta(\tau+1) = \mathrm{e}(1/24)\eta(\tau) \quad \text{and} \quad \eta(-1/\tau) = (\tau/i)^{1/2}\eta(\tau). \tag{16}$$

Here $\tau^{1/2} = |\tau|^{1/2}\exp(i\,\mathrm{arg}(\tau)/2)$ where $\mathrm{arg}(\tau) \in (0,\pi)$. Define

$$g(\tau) = \begin{bmatrix} \eta(2\tau)^{16}\eta(\tau)^{-32} \\ \eta(\tau/2)^{16}\eta(\tau)^{-32} \\ \eta((\tau+1)/2)^{16}\eta(\tau+1)^{-32} \end{bmatrix}.$$

The transformation properties of $\eta(\tau)$ in (16) easily imply

$$g(\tau+1) = T_1 g(\tau) \quad \text{and} \quad g(-1/\tau) = \tau^{-8} S_1 g(\tau),$$

where

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad S_1 = \begin{bmatrix} 0 & 2^{-8} & 0 \\ 2^8 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define

$$\begin{bmatrix} f_{00}(\tau) \\ f_0(\tau) \\ f_1(\tau) \end{bmatrix} = f(\tau) = Ag(\tau), \quad \text{where } A = \begin{bmatrix} -120 & 1/2 & 1/2 \\ 8 & 0 & 0 \\ 0 & -1/32 & 1/32 \end{bmatrix}.$$

Then one verifies that $AS_1 = S_0 A$ and $AT_1 = T_0 A$ (Remark 6.2 explains how we choose the matrix A). So f satisfies the transformations in (15). As above, define $F(\tau) = \sum_\gamma f_\gamma(\tau)e_\gamma$ with three types of components $f_{00}(\tau)$, $f_0(\tau)$, $f_1(\tau)$. Then $F(\tau)$ is a vector-valued modular form of singular weight -8 on the upper half plane of type ρ_M. Using $\eta(\tau) = q^{1/24}(1-q-q^2+q^5+\cdots)$, we compute

$$g(\tau) = \begin{bmatrix} 1+O(q) \\ q^{-1} - 16q^{-1/2} + 136 + O(q^{1/2}) \\ q^{-1} + 16q^{-1/2} + 136 + O(q^{1/2}) \end{bmatrix}.$$

From this, we work out the nonpositive parts of the q-expansions of the f_γ's:

$$f_\gamma(\tau) = \begin{cases} q^{-1} + 16 + O(q) & \text{if } \gamma = 0, \\ 8 + O(q) & \text{if } \gamma \in (M^\vee/M) - \{0\} \text{ and } \gamma^2 \equiv 0 \bmod 2\mathbb{Z}, \\ q^{-1/2} + O(q^{1/2}) & \text{if } \gamma \in (M^\vee/M) - \{0\} \text{ and } \gamma^2 \equiv 1 \bmod 2\mathbb{Z}. \end{cases}$$

Using the singular theta correspondence of [Borcherds 1998, Theorem 13.3], one obtains an automorphic form $\Psi = \Psi_M$ on the Grassmannian $G(M) \simeq G(2,18)$. The restriction of Ψ to the complex hyperbolic space $\mathbb{B}(L) \simeq \mathbb{C}H^9$ as in [Allcock 2000a] yields an automorphic form for the complex reflection group Γ.

The weight and zeros and poles of Ψ are determined by the nonpositive parts of Fourier expansions of f_γ's. Write

$$f_\gamma(\tau) = \sum_{n\in\mathbb{Q}} c_\gamma(n) q^n.$$

Then Ψ has weight $c_0(0)/2 = 8$. If $\lambda \in M$ is primitive with $(\lambda \mid \lambda) < 0$, then along the rational quadratic divisor $G(\lambda^\perp)$, the form Ψ has zero (or pole) of order

$$\sum_{0<x\in\mathbb{R}, x\lambda\in M^\vee} c_{x\lambda}\left(\frac{x^2(\lambda|\lambda)}{2}\right) \tag{17}$$

and this describes all the zeros and poles of Ψ.[4]

Let λ be a primitive vector of M. To get a nonzero contribution in the sum in (17), we need $x\lambda \in M^\vee = L^\vee = \frac{1}{p}L \subseteq \frac{1}{2}L = \frac{1}{2}M$. So $2x\lambda \in M$. It follows that $2x = m$ must be an integer. Next, note that the only negative rational numbers n for which $c_\gamma(n) \neq 0$ are $\{-1, -\frac{1}{2}\}$. So, to get a nonzero contribution in the sum in (17), we need $-m^2\lambda^2/8 = -x^2\lambda^2/2 \in \{-1, -\frac{1}{2}\}$. This gives three possible cases and we treat these separately in the following table:

Case	λ^2	m	x	type of $x\lambda$	order of zero (or pole) of Ψ along $G(\lambda^\perp)$
1	2	2	1	00	$c_{x\lambda}(-1) = \text{coefficient of } q^{-1} \text{ in } f_{00}(\tau) = 1$
2	4	1	$\frac{1}{2}$	1	$c_{x\lambda}(-\frac{1}{2}) = \text{coefficient of } q^{-1/2} \text{ in } f_1(\tau) = 1$
3	8	1	$\frac{1}{2}$	0	$c_{x\lambda}(-1) = \text{coefficient of } q^{-1} \text{ in } f_0(\tau) = 0$

Since $\mathsf{rank}(M) > 5$, it follows that Ψ is a holomorphic automorphic form on the Grassmannian $G(M)$. It has a simple zero along $G(\lambda^\perp)$ for each $\lambda \in M$ such that $(\lambda \mid \lambda) = -2$ and for each $\lambda \in M$ such that $(\lambda \mid \lambda) = -4$ and $\lambda/2 \in M^\vee$.

Now consider the restriction of Ψ to $\mathbb{B}(L) \simeq \mathbb{C}H^9$. Note that in Case 2 of the above table, we need $x\lambda = \lambda/2 \in M^\vee = L^\vee = p^{-1}L$, so $p^{-1}\lambda = \mu \in L$. So in Case 2, the vector λ is primitive in M but it is a scalar multiple of the norm 2 vector $\mu \in L$. So when we restrict the divisor $G(\lambda^\perp)$ to $\mathbb{B}(L)$, we find that $G(\lambda^\perp) \cap \mathbb{B}(L) = G(\mu^\perp) \cap \mathbb{B}(L) = \mathbb{B}(\mu^\perp)$. In Case 1, of course λ can be any norm 2 vector of L. It follows that all the zeros of $\Psi|_{\mathbb{B}(L)}$ are along orthogonal complements of norm 2 vectors of L.

Let $\lambda \in L$ with $\lambda^2 = 2$. Then there are four distinct rational quadratic divisors in $G(M)$, namely $G(\lambda^\perp)$, $G((i\lambda)^\perp)$, $G((p\lambda)^\perp)$, $G((ip\lambda)^\perp)$, that restrict to the mirror $\mathbb{B}(\lambda^\perp)$ in $\mathbb{B}(L)$. From Cases 1 and 2 above, we find that the automorphic form Ψ has a simple zero along the four divisors $G(\lambda^\perp)$, $G((i\lambda)^\perp)$, $G((p\lambda)^\perp)$

[4] Warning: note that $(\lambda|\lambda)$ means the norm of λ in M which is the negative of the norm of λ in L. In other words $(\lambda|\lambda) = -\lambda^2$. This minus sign causes an unfortunate clash of notation with [Borcherds 1998].

and $G((ip\lambda)^{\perp})$. So $\Psi|_{\mathbb{B}(L)}$ is a holomorphic automorphic form on the complex hyperbolic space $\mathbb{B}(L)$ for the group $P\,\mathrm{Aut}(L)$ with a zero of order 4 along $\mathbb{B}(\lambda^{\perp})$ for each norm 2 vector of L and no other zeros or poles. This finishes the proof of Theorem 6.1.

6.2. Remark. Up to scalars, there is a one-parameter family of matrices A_x such that $A_x S_1 = S_0 A_x$ and $A_x T_1 = T_0 A_x$. This gives us a one-parameter family of modular forms $\{h^x : x \in \mathbb{C}\}$ that satisfy the right transformation properties (15). From this one-parameter family we chose the modular form f to ensure that the second component of f, namely $f_0(\tau) = 8\eta(2\tau)^{16}\eta(\tau)^{-32}$, is regular at $i\infty$ because this implies that the automorphic form Ψ has no zero or pole along the orthogonal complements of norm 8 vectors of M. We scale f to make all the coefficients of the nonpositive part of its Fourier expansion integral and as small as possible.

Appendix: p-modular $\mathcal{G}$-lattice of signature $(2d-1, 1)$ is unique

The purpose of this appendix is to sketch a proof of Lemma 4.1.

A.1. ***Symmetric bilinear and symplectic forms over*** $\mathbb{F}_2$**.** Let V be a finite-dimensional $\mathbb{F}_2$-vector space with a with a symmetric (or equivalently skew-symmetric) bilinear form $(\ ,\ ) : V \times V \to \mathbb{F}_2$. A subspace L of V is called *isotropic* if $L \subseteq L^{\perp}$. In characteristic 2, the map $q : V \to \mathbb{F}_2$ defined by $q(u) = u^2 = (u, u)$ is a group homomorphism. We say that V (or the bilinear form) is *even* if $\mathrm{ker}(q) = V$, otherwise we say that V is *odd*. If V is odd, then $\mathrm{ker}(q)$ is a codimension 1 subspace in V. We will call the elements of $\mathrm{ker}(q)$ the *even vectors* and the elements of $V - \mathrm{ker}(q)$ the *odd vectors*. For each nonzero even vector z in V, one has a transvection $t_z \in \mathrm{Aut}(V, (\ ,\ ))$ of order 2, defined by

$$t_z(x) = x + (z, x)z.$$

A bilinear form $(\ ,\ )$ on an $\mathbb{F}_2$-vector space V is called *symplectic* if it is nondegenerate and alternating, that is, $(x, x) = 0$ for all $x \in V$. A maximal isotropic subspace of a symplectic vector space is called a *Lagrangian subspace*. If V is symplectic, we write $\mathrm{Sp}(V) = \mathrm{Aut}(V, (\ ,\ ))$. If V is symplectic, then the symplectic group $\mathrm{Sp}(V)$ contains a transvection t_z of order 2 for each $z \in V - \{0\}$. It is known that the symplectic group $\mathrm{Sp}(V)$ is generated by these transvections. It is also known that $\mathrm{Sp}(V)$ acts transitively on the set of Lagrangian subspaces (see [O'Meara 1978]).

A.2. Example. Let $(\ ,\ )$ be the *standard bilinear form* on $\mathbb{F}_2^d$ defined by

$$((x_1, \dots, x_d), (y_1, \dots, y_d)) = x_1 y_1 + \cdots + x_d y_d.$$

The associated quadratic form is

$$q((x_1, \dots, x_d)) = x_1^2 + \cdots + x_d^2 = x_1 + \cdots + x_d.$$

So the even (resp. odd) vectors of $\mathbb{F}_2^d$ are simply the vectors with an even (resp. odd) number of nonzero coordinates. The vector

$$\mathbf{1}_d = (1, 1, \ldots, 1) \in \mathbb{F}_2^d$$

is called the *parity vector* of $\mathbb{F}_2^d$. Note that $\mathbf{1}_d^\perp$ is the set of even vectors of $\mathbb{F}_2^d$ and $(\mathbf{1}_d^\perp)^\perp = \{0, \mathbf{1}_d\}$. We abbreviate $\langle \mathbf{1}_d \rangle := \{0, \mathbf{1}_d\}$. Since any automorphism in $\mathsf{Aut}(\mathbb{F}_2^d, (\ ,\ ))$ preserves the set of even vectors, it follows that any such automorphism must fix the parity vector.

A.3. ***A homomorphism from*** $\mathsf{Aut}(\mathbb{F}_2^d, (\ ,\ ))$ ***to*** $\mathsf{Sp}_{d-2}(\mathbb{F}_2)$ ***for even d.*** Continue with the example above and assume that $d \in 2\mathbb{N}$. Since $\mathbf{1}_d \subseteq \mathbf{1}_d^\perp$, the restriction of the bilinear form $(\ ,\ )$ to $\mathbf{1}_d^\perp$ is singular with a 1-dimensional radical $\langle \mathbf{1}_d \rangle$. So $(\ ,\ )$ induces a symplectic form on $\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle$ and the projection $\mathbf{1}_d^\perp \to \mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle$ is bilinear form preserving. If $g \in \mathsf{Aut}(\mathbb{F}_2^d, (\ ,\ ))$, then g preserves $\mathbf{1}_d$, and hence preserves $\mathbf{1}_d^\perp$. So g induces an automorphism $\bar{g}$ of the symplectic vector space $\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle$ defined by $\bar{g}\bar{v} = \overline{gv}$ for all $v \in \mathbf{1}_d^\perp$. This gives us a group homomorphism $g \mapsto \bar{g}$ from $\mathsf{Aut}(\mathbb{F}_2^d, (\ ,\ ))$ to $\mathsf{Sp}(\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle) \simeq \mathsf{Sp}_{d-2}(\mathbb{F}_2)$. Let $O'(\mathbb{F}_2^d)$ be the subgroup of $\mathsf{Aut}(\mathbb{F}_2^d, (\ ,\ ))$ generated by transvections in the even vectors other than the parity vector.

A.4. Lemma. *Assume $d \in 2\mathbb{N}$.*

(a) *The above homomorphism maps $O'(\mathbb{F}_2^d)$ onto $\mathsf{Sp}(\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle)$.*

(b) *The group $O'(\mathbb{F}_2^d)$ acts transitively on the isotropic subspaces in $\mathbb{F}_2^d$ of dimension $d/2$.*

Proof. (a) Let x be a nonzero element of $\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle$. There exists $v \in \mathbf{1}_d^\perp - \langle \mathbf{1}_d \rangle$ such that $\bar{v} = x$. Then $t_v \in O'(\mathbb{F}_2^d)$ and t_v maps to $t_x \in \mathsf{Sp}(\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle)$. So the image of the homomorphism $O'(\mathbb{F}_2^d) \to \mathsf{Sp}(\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle)$ contains the transvections t_x for all nonzero x in $\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle$. These transvections generate the symplectic group.

(b) Observe that any isotropic subspace of $\mathbb{F}_2^d$ of dimension $d/2$ is necessarily maximal isotropic and hence must contain the parity vector. So if $\mathcal{W}_1$ and $\mathcal{W}_2$ are two such subspaces, then their images $\mathcal{W}_1 / \langle \mathbf{1}_d \rangle$ and $\mathcal{W}_2 / \langle \mathbf{1}_d \rangle$ are two Lagrangian subspaces in $\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle$. So there exists $\gamma \in \mathsf{Sp}(\mathbf{1}_d^\perp / \langle \mathbf{1}_d \rangle)$ that takes $\mathcal{W}_1 / \langle \mathbf{1}_d \rangle$ to $\mathcal{W}_2 / \langle \mathbf{1}_d \rangle$. Using part (a), lift γ to an element $g \in O'(\mathbb{F}_2^d)$. Then $g\mathcal{W}_1 = \mathcal{W}_2$. □

Let $\mathcal{G}^{n,1}$ denote the $\mathcal{G}$-module $\mathcal{G}^{n+1}$ with the hermitian form

$$\langle (x_1, \ldots, x_{n+1}) | (y_1, \ldots, y_{n+1}) \rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n - \bar{x}_{n+1} y_{n+1}.$$

A.5. Lemma. *Write $U = \mathcal{G}^{2d-1,1}$. Denote the projection $U \to U/pU = \mathbb{F}_2^{2d}$ by $r \mapsto \bar{r}$. If $v \in \mathbf{1}_{2d}^\perp - \langle \mathbf{1}_{2d} \rangle$, then there exists a norm 2 vector r in U such that $\bar{r} = v$.*

Proof. First, assume that the last coordinate of v is zero. Then v has the form $v = (1^{2k}, 0^{2d-2k-1}; 0)$ for some $k < d$. There exists $(x_1, x_2, x_3) \in \mathbb{Z}^3$ such that $2k + 2(x_1^2 + x_2^2 - x_3^2) = 2$, since the quadratic form $(x_1^2 + x_2^2 - x_3^2)$ represents all integers. Choose $r \in U$ of the form $r = (1^{2k}, p(x_1 + ix_2), 0^{2d-2k-2}; px_3)$.

Now assume that the last coordinate of v is nonzero. Then v has the form $v = (1^{2k-1}, 0^{2d-2k}; 1)$ for some $k < d$. There exists $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}^5$ such that $2k - 1 + 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - x_5^2 = 2$ since the quadratic form

$$(2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - x_5^2)$$

represents all integers. Note that x_5 is necessarily an odd integer. Choose $r \in U$ of the form $r = (1^{2k-1}, p(x_1 + ix_2), p(x_3 + ix_4), 0^{2d-2k-2}; x_5)$. □

Proof of Lemma 4.1. (existence) Observe that $(d-1)D_4^{\mathcal{G}} \perp \mathcal{G}_{1,1}$ is a p-modular Gaussian lattice of signature $(2d-1, 1)$.

(uniqueness) Let L be any p-modular Gaussian lattice of signature $(2d-1, 1)$. Then vectors of $L^\vee$ have integral norm. So $L^\vee/L$ has the natural $\mathbb{F}_2$-valued quadratic form $q(x) = x^2 \bmod 2\mathbb{Z}$. The bilinear form $\partial q(x, y) := q(x+y) - q(x) - q(y)$ can be computed using

$$\partial q(\bar{x}, \bar{y}) = p\langle x|y\rangle \bmod p\mathcal{G} \in \mathcal{G}/p\mathcal{G} = \mathbb{F}_2$$

for all $x, y \in L^\vee$. One verifies that this bilinear form is nondegenerate since L is p-modular. So $(L^\vee/L, \partial q)$ becomes a symplectic $\mathbb{F}_2$-vector space.

Let W be the preimage in $L^\vee$ of a Lagrangian subspace $\mathcal{W}$ of $L^\vee/L$. Then one verifies that W is a unimodular Gaussian lattice. So [Allcock 2000b, Theorem 7.1] implies that there are at most two possibilities for W; it is either the odd unimodular or the even unimodular $\mathcal{G}$-lattice in the given signature. In either case, W contains a null vector, so L contains a primitive null vector as well. From this, one easily verifies that L splits a p-modular hyperbolic cell $\mathcal{G}_{1,1}$. Since $\mathcal{G}_{1,1} \subseteq L$, the lattice L contains a norm 2 vector v. Note that $p^{-1}v \in L^\vee - L$. Let $\mathcal{U}$ be a Lagrangian subspace of $L^\vee/L$ containing $(p^{-1}v \bmod L)$ and let U be the preimage of $\mathcal{U}$ in $L^\vee$. Then U is a self-dual Gaussian lattice containing L and it is odd since it contains a norm 1 vector $p^{-1}v$. So [Allcock 2000b, Theorem 7.1] implies $U \simeq \mathcal{G}^{2d-1,1}$. So we may assume without loss that $L \subseteq U = \mathcal{G}^{2d-1,1}$ and hence $U = U^\vee \subseteq L^\vee$.

The hermitian form on U induces a nondegenerate symmetric bilinear form on the $\mathbb{F}_2$-vector space $U/pU \simeq \mathbb{F}_2^{2d}$ defined by

$$b(x, y) = \langle x|y\rangle \bmod p\mathcal{G}.$$

One can identify $(U/pU, b) = (\mathbb{F}_2^{2d}, (\ ,\ ))$ where $(\ ,\ )$ is the standard bilinear form on $\mathbb{F}_2^{2d}$ defined in Example A.2. We denote the projection $U \to \mathbb{F}_2^{2d}$ and corresponding map $\mathrm{Aut}(U)$ to $\mathrm{Aut}(\mathbb{F}_2^{2d}, (\ ,\ ))$ by $x \mapsto \bar{x}$. If $v \in \mathbb{F}_2^{2d}$ is an even vector other than the parity vector, then Lemma A.5 implies that there exists a norm 2 vector r in U such that $\bar{r} = v$. The order 2 reflection in r maps to the transvection t_v. So the image of $\mathrm{Aut}(U)$ in $\mathrm{Aut}(\mathbb{F}_2^{2d}, (\ ,\ ))$ contains the subgroup $O'(\mathbb{F}_2^{2d})$ generated by transvections in the even vectors other than the parity vector.

Since $L \subseteq U \subseteq p^{-1}L = L^\vee$, it follows that $pU \subseteq L \subseteq U$. Since $\langle x|y\rangle \in p\mathcal{G}$ for all $x, y \in L$, it follows that $\bar{L} = L/pU$ is an isotropic subspace in $\mathbb{F}_2^{2d}$. Since $[U : L] = [L : pU]$, we find that $\dim_{\mathbb{F}_2}(L/pU) = \frac{1}{2}\dim_{\mathbb{F}_2}(U/pU) = d$. So L determines a d-dimensional maximal isotropic subspace $\bar{L}$ in $\mathbb{F}_2^{2d}$. So if L and L_1 are two p-modular, finite index sublattices of U, then Lemma A.4(b) implies that there exists $\gamma \in O'(\mathbb{F}_2^{2d})$ such that $\gamma\bar{L} = \bar{L}_1$. Since the image of $\mathrm{Aut}(U)$ in $\mathrm{Aut}(\mathbb{F}_2^{2d}, (\ ,\ ))$ contains the subgroup $O'(\mathbb{F}_2^{2d})$, there exists some $g \in \mathrm{Aut}(U)$ such that $\bar{g} = \gamma$. Then $gL = L_1$. So any two p-modular, finite index sublattices of $\mathcal{G}^{2d-1,1}$ are conjugate via $\mathrm{Aut}(\mathcal{G}^{2d-1,1})$. □

A.6. Remark. Two issues, both specific to characteristic 2 and both related to the parity vector, cause some of the extra complications in the argument presented in this appendix. To explain this, assume $d = 5$ in Lemma A.5. This is the example relevant to this article. First, observe that there does not exist any $r \in \mathcal{G}^{9,1}$ such that $r^2 = 2$ and $\bar{r} = \mathbf{1}_{2d} \in \mathbb{F}_2^{10}$ because of obvious congruence obstructions. So the hypothesis $v \neq \mathbf{1}_{2d}$ in Lemma A.5 is necessary. This forces us to deal with the subgroup $O'(\mathbb{F}_2^{10})$. Second, observe that the bilinear form induced on $\mathbb{F}_2^{10} = \mathcal{G}^{9,1}/p\mathcal{G}^{9,1}$ is skew-symmetric but not symplectic and its restriction to the even vectors has a radical spanned by $\mathbf{1}_{2d}$. This forces us to pass to the subquotient $\mathbf{1}_{2d}^\perp/\mathbf{1}_{2d}$ to get a symplectic vector space.

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TATHAGATA BASAK
DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
AMES, IA
UNITED STATES
tathagat@iastate.edu

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