CONJUGACY CLASSES OF p-ELEMENTS AND NORMAL p-COMPLEMENTS

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We study the structure of finite groups with a large number of conjugacy classes of *p*-elements for some prime *p*. As a consequence, we obtain some new criteria for the existence of normal *p*-complements in finite groups.

1. Introduction

Let p be a prime. Let G be a finite group and let P be a Sylow p-subgroup of G. Denote by $k(G)$ and $k_p(G)$ the number of conjugacy classes of G and the number of conjugacy classes of p-elements of G, respectively. By Sylow's theorem, we can choose a complete set Γ of representatives for the conjugacy classes of p-elements of G in such a way that $\Gamma \subseteq P$. This yields that $k_p(G) \le k(P)$. Also $k_p(G) \ge 2$ unless G is a p'-group. Hence if p divides $|G|$, then $2 \le k_p(G) \le k(P) \le |P|$. In [Külshammer et al. 2014], the authors study finite groups G with $k_p(G) = 2$. They show that the Sylow p-subgroup P of such a group G must be either elementary abelian or extra-special of order p^3. In this paper, we will look at the case when $k_p(G)$ is large in comparison to $|P|$.

Recall that a subgroup N of a finite group G is called a normal p-complement of G if N is a normal subgroup of G whose order is relatively prime to p and whose index is a power of p. A finite group G is said to be p-nilpotent if it has a normal p-complement. A classical result in group theory states that a finite group G is p-nilpotent if and only if P controls its own fusion in G. (See [Isaacs 2008, Theorem 5.25] and the definitions in Section 2). The latter condition is equivalent to $x^G \cap P = x^P$ for every $x \in P$, which is equivalent to the condition $k_p(G) = k(P)$. Thus G is p-nilpotent if and only if $k_p(G) = k(P)$. If we assume that $k_p(G) = |P|$, then $k_p(G) = k(P) = |P|$; hence G is p-nilpotent and has an abelian Sylow p-subgroup. So, we may ask whether G is still p-nilpotent, if $k_p(G)/|P|$ is close to 1.

It turns out that the fraction $k_p(G)/|P|$ is related to the commuting probability $d(G)$ of a finite group G, which is defined to be the probability that two randomly chosen elements of G commute. Gustafson [1973] shows that $d(G) = k(G)/|G|$. The invariant $d(G)$ is also called the commutativity degree of G.

Here is our first result for the prime $p = 2$.

Theorem A. *Let G be a finite group and let P be a Sylow 2-subgroup of G. Then $k_2(G) > |P|/2$ if and only if G has a normal 2-complement and $k(P) > |P|/2$.*

Clearly, any finite group with a normal 2-complement is solvable by the Feit–Thompson theorem. Also, the Sylow 2-subgroup P in Theorem A is nilpotent of class at most 2. (See Lemma 2.7.) Theorem A does not hold if we allow equality. For example, if $G = A_4$ and $P \in \mathrm{Syl}_2(G)$, then $k_2(G) = 2$ and $|P| = 4$, so $k_2(G) = |P|/2$ but G is not 2-nilpotent. Also, we cannot replace 2 by an odd prime. Indeed, if $G = A_5$ and $P \in \mathrm{Syl}_3(G)$, then $k_3(G) = 2$ and $|P| = 3$; hence $k_3(G) = \frac{2}{3}|P| > \frac{1}{2}|P|$ but G is not 3-nilpotent.

In view of Lemma 2.8, to investigate the structure of finite groups G with $k_p(G)/|P|$ a specified constant, we may assume that $\boldsymbol{O}_{p'}(G) = 1$.

Theorem B. *Let G be a finite group and let $P \in \mathrm{Syl}_2(G)$. Suppose that $\boldsymbol{O}_{2'}(G) = 1$ and $k_2(G) = |P|/2$. Then*

(1) $G/\boldsymbol{Z}(G) \cong A_4$ *or* S_4; *or*

(2) $G/\boldsymbol{Z}(G)$ *is an almost simple group with a nonabelian simple socle isomorphic to* $\mathrm{PSL}_2(q)$ *with* $3 < q \equiv 3, 5 \pmod 8$.

Let π be a set of primes. Let $k_\pi(G)$ be the number of conjugacy classes of π-elements of G. Let $|G|_\pi$ be the π-part of the order $|G|$ of G. Define $d_\pi(G)$ to be $k_\pi(G)/|G|_\pi$. If $\pi = \{p\}$, then we write $d_p(G)$ and $k_p(G)$ instead of $d_{\{p\}}(G)$ or $k_{\{p\}}(G)$. We now investigate the structure of finite groups G with $d_\pi(G) > \frac{1}{2}$, where π is a set of primes containing 2.

Theorem C. *Let G be a finite group and let π be a set of primes with $2 \in \pi$. Let $\sigma = \pi \setminus \{2\}$. Suppose that $d_\pi(G) > \frac{1}{2}$. Then G has a normal π-complement and an abelian Hall σ-subgroup.*

We should point out that our proofs of Theorems A–C do not depend on the classification of finite simple groups.

For odd primes p, we obtain the following result. Here our proof depends on the odd version of Glauberman's Z^*-theorem and thus depends on the classification of finite simple groups.

Theorem D. *Let G be a finite group and let p be an odd prime. Then $d_p(G) > (p+1)/2p$ if and only if G has a normal p-complement and an abelian Sylow p-subgroup.*

This bound cannot be improved since $d_p(D_{2p}) = (p+1)/2p$ but D_{2p} is not p-nilpotent, where p is an odd prime. For nonsolvable examples, let $f \geq 2$ be an integer and p be a prime such that $4^f - 1$ is divisible by p but not by p^2. Then $d_p(\mathrm{PSL}_2(2^f)) = (p+1)/2p$.

Theorem E. *Let G be a finite group and let π be a set of odd primes. Let p be the smallest prime in π. Suppose that $d_\pi(G) > (p+1)/2p$. Then G has a normal π-complement and an abelian Hall π-subgroup.*

In [Maróti and Nguyen 2014], the authors show that if $d_\pi(G) > \frac{5}{8}$, then $d_\pi(G) = 1$ or $\frac{2}{3}$. They also study the structure of finite groups G such that $\boldsymbol{O}_{3'}(G) = 1$ and $d_3(G) = \frac{2}{3}$. Thus if $p = 3$ in Theorem D, then our results follow immediately from their results. However, if $p \geq 5$, then $(p+1)/2p < \frac{5}{8}$. Hence our Theorems C and E above improve their Theorem 1 and finally our last result includes Theorem 2 in [Maróti and Nguyen 2014].

Theorem F. *Let G be a finite group and let p be an odd prime. Let P be a Sylow p-subgroup of G. Suppose that $\boldsymbol{O}_{p'}(G) = 1$ and $d_p(G) = (p+1)/2p$. Then P is abelian, $N_G(P)/C_G(P)$ has order 2, $[P, N_G(P)]$ has order p and $G \cong A \times B$, where B is an abelian p-group and A is either a dihedral group of order $2p$ or an almost simple group with a Sylow p-subgroup of order p contained in the socle of A.*

The paper is organized as follows. We collect some results needed for the proofs of the main theorems in Section 2. We prove Theorems A–C in Section 3 and prove Theorems D–F in Section 4.

2. Control of fusion and Glauberman's Z*-theorem

Let G be a finite group and let $K \leq H \leq G$ be subgroups of G. We say that H *controls G-fusion in K* if and only if every pair of G-conjugate elements of K are H-conjugate; that is, if $x, x^g \in K$ for some $g \in G$, then $x^g = x^h$ for some $h \in H$. Let p be a prime and let H be a subgroup of G. We say that H *controls p-fusion in G* if H contains a Sylow p-subgroup P of G and H controls G-fusion in P. We first recall some classical results on the existence of normal p-complements as well as the control of fusion in finite groups.

Lemma 2.1. *Let G be a finite group and let P be a Sylow p-subgroup of G for some prime p.*

(1) *$N_G(P)$ controls G-fusion in $C_G(P)$.*

(2) *If $P \subseteq \boldsymbol{Z}(N_G(P))$, then G has a normal p-complement.*

(3) *G has a normal p-complement if and only if P controls its own fusion in G.*

Proof. These are well known results; for proofs, see Lemma 5.12 and Theorems 5.13 and 5.25 in [Isaacs 2008]. $\square$

Parts (1) and (2) above are known as Burnside's lemma and Burnside's normal p-complement theorem, respectively. Here are some obvious consequences of the lemma.

Corollary 2.2. *Let G be a finite group and let P be a Sylow p-subgroup of G for some prime p.*

(1) $k_p(G) \le k(P)$, *and equality holds if and only if G has a normal p-complement.*

(2) $k_p(G) = |P|$, *or equivalently $d_p(G) = 1$, if and only if G has a normal p-complement and an abelian Sylow p-subgroup.*

Note that part (1) of the corollary is equivalent to the statement that $d_p(G) \le d(P)$ and equality holds if and only if G has a normal p-complement. The following result is a consequence of the definitions above and Sylow's theorem.

Lemma 2.3. *Let G be a finite group and let P be a Sylow p-subgroup of G for some prime p. Let $x \in P$. Then $x^G \cap P = \{x\}$ if and only if $C_G(x)$ controls p-fusion in G.*

Proof. Let $x \in P$. Assume that $x^G \cap P = \{x\}$. We claim that $C_G(x)$ controls p-fusion in G. Since $x^P \subseteq x^G \cap P = \{x\}$, we see that $x \in Z(P)$ and thus $P \le C_G(x)$. Now assume that $y, y^g \in P$ for some $g \in G$. We need to show that $y^g = y^h$ for some $h \in C_G(x)$. We have that $\{y, y^g\} \subseteq P \subseteq C_G(x)$, which implies that $\{x, x^{g^{-1}}\} \subseteq C_G(y)$. Let U be a Sylow p-subgroup of $C_G(y)$ containing x. By Sylow's theorem, $U \le P^t$ for some $t \in G$. It follows that $x^{t^{-1}} \in P \cap x^G = \{x\}$; hence $x^{t^{-1}} = x$, so $t \in C_G(x)$. Now $x^{g^{-1}} \in U^c$ for some $c \in C_G(y)$ as $x^{g^{-1}} \in C_G(y)$ is a p-element. We now have that $x^{g^{-1}c^{-1}t^{-1}} \in P$ and thus $x^{g^{-1}c^{-1}t^{-1}} = x$, which implies that $g^{-1}c^{-1} \in C_G(x)$. Therefore $cg = h \in C_G(x)$. Now $y^g = y^{cg} = y^h$ as wanted.

For the converse, let $P_1 \in \mathrm{Syl}_p(G)$ and assume that $P_1 \subseteq C_G(x)$ and that $C_G(x)$ controls G-fusion in P_1. It follows that $x \in P_1$. By Sylow's theorem, $P = P_1^t$ for some $t \in G$. Since $x \in P$, $x^{t^{-1}} \in P_1 \le C_G(x)$. As $C_G(x)$ controls G-fusion in P_1, it follows that $x^{t^{-1}} = x^h$ for some $h \in C_G(x)$. Hence $x^{t^{-1}} = x^h = x$ and so $t \in C_G(x)$. In particular, $P = P_1^t \subseteq C_G(x)$. Finally, if $x^g \in P$ for some $g \in G$, then $x^g = x^h$ for some $h \in C_G(x)$ and so $x^g = x^h = x$. Therefore $x^G \cap P = \{x\}$.   □

For a finite group G and a prime p, we define $Z_p^*(G)$ to be the normal subgroup of G such that $Z_p^*(G)/O_{p'}(G) = Z(G/O_{p'}(G))$.

We first state the original form of Glauberman's Z^*-theorem, whose proof does not depend on the classification of finite simple groups.

Lemma 2.4 (Glauberman's Z^*-theorem). *Let G be a finite group and let P be a Sylow 2-subgroup of G. If $x \in P$ and $x^G \cap P = \{x\}$, then $x \in Z_2^*(G)$.*

Proof. This is a restatement of Theorem 3 in [Glauberman 1966].   □

The odd version of Glauberman's Z^*-theorem, which is called Glauberman's Z_p^*-theorem, says that if $x \in P$ is an element of order p and $x^G \cap P = \{x\}$, then $x \in Z_p^*(G)$. The proof of this theorem depends on the classification (for a sketch proof, see [Guralnick and Robinson 1993, Theorem 4.1]). By Sylow's theorem, it

is easy to see that if $x^G \cap P = \{x\}$ then x does not commute with any G-conjugate $x^g \neq x$ of x. Finally, we note that the conclusion of Glauberman's Z_p^*-theorem can be written as $G = \boldsymbol{C}_G(x)\boldsymbol{O}_{p'}(G)$.

To use Glauberman's Z_p^*-theorem for an arbitrary p-element $x \in P$ which is not of prime order satisfying $x^G \cap P = \{x\}$, we need the following lemma.

Lemma 2.5. *Let G be a finite group and let P be a Sylow p-subgroup of G for some prime p. Let $x \in P$. If $x^G \cap P = \{x\}$, then $y^G \cap P = \{y\}$ for every $y \in \langle x \rangle$.*

Proof. Suppose that $x^G \cap P = \{x\}$ and $y \in \langle x \rangle$. Then $P \leq \boldsymbol{C}_G(x) \leq \boldsymbol{C}_G(y)$. By Lemma 2.3, we need to show that $\boldsymbol{C}_G(y)$ controls G-fusion in P. Let $z, z^g \in P$ for some $g \in G$. By Lemma 2.3, $\boldsymbol{C}_G(x)$ controls p-fusion in G so $z^g = z^t$ for some $t \in \boldsymbol{C}_G(x)$. As $\boldsymbol{C}_G(x) \leq \boldsymbol{C}_G(y)$, we have $t \in \boldsymbol{C}_G(y)$ and the claim follows.   □

We will need the following results.

Lemma 2.6. *Let G be a finite group and let π be a nonempty set of primes.*

(1) *If $\mu \subseteq \pi$ is a nonempty subset, then $d_\pi(G) \leq d_\mu(G) \leq 1$.*

(2) *If $N \trianglelefteq G$, then $d_\pi(G) \leq d_\pi(G/N)d_\pi(N)$.*

(3) *If G is a nonabelian p-group for some prime p, then $d(G) < (p+1)/p^2$.*

(4) *If G does not have a normal Sylow p-subgroup for some prime p, then $d(G) \leq 1/p$.*

Proof. Part (1) can be found in [Maróti and Nguyen 2014, Proposition 5] and Part (2) is Lemma 2.3 in [Fulman and Guralnick 2012]. Finally, the last two parts can be found in Lemma 2 in [Guralnick and Robinson 2006].   □

Finite groups G with $d(G) \geq \frac{1}{2}$ were classified by Lescot [1995; 2001]. To state the result, we need the following notation. For any integer $m \geq 1$, denote by G_m the group defined by

$$G_m = \langle a, b : a^3 = b^{2^m} = 1, a^b = a^{-1} \rangle.$$

Note that $G_1 \cong S_3$. We have that $|G_m| = 3 \cdot 2^m$, $\boldsymbol{Z}(G_m) = \langle b^2 \rangle$, $G'_m = \langle a \rangle$, and $G_m/\boldsymbol{Z}(G_m) \cong S_3$.

Lemma 2.7. *Let G be a finite group. Then $d(G) \geq \frac{1}{2}$ if and only if one of the following holds*:

(i) *G is abelian and $d(G) = 1$.*

(ii) *$G \cong P \times A$, where A is abelian of odd order and P is a Sylow 2-subgroup of G with $|G'| = |P'| = 2$ and $d(G) = d(P) = (1 + 4^{-m})/2$ and $G/\boldsymbol{Z}(G)$ is elementary abelian of order 4^m for some integer $m \geq 1$. Further, $\frac{1}{2} < d(G) \leq \frac{5}{8}$.*

(iii) *$G \cong G_m \times A$ and $d(G) = \frac{1}{2}$, where A is abelian and $m \geq 1$.*

Proof. This is a combination of Theorem 3.1 in [Lescot 2001] and Corollary 3.2 in [Lescot 1995]. □

It follows from Lemma 2.7 that there is no 2-group G with $d(G) = \frac{1}{2}$. Also, if G is of odd order with $d(G) \geq \frac{1}{2}$, then G is abelian and $d(G) = 1$.

Lemma 2.8. *Let G be a finite group and let p be a prime. If $N \trianglelefteq G$ is a p'-subgroup, then $k_p(G) = k_p(G/N)$ and so $d_p(G) = d_p(G/N)$.*

Proof. Let N be a normal p'-subgroup of G and let $P \in \text{Syl}_p(G)$. Write $\overline{G} = G/N$. Since $p \nmid |N|$, the Sylow p-subgroups of G and $\overline{G}$ have the same order, thus it suffices to show that $k_p(G) = k_p(\overline{G})$. By Lemma 2.6(2), we have $d_p(G) \leq d_p(\overline{G})$ since $d_p(N) \leq 1$. It follows that $k_p(G) \leq k_p(\overline{G})$. The reverse inequality is obvious. □

3. Conjugacy classes of 2-elements

We will prove Theorems A, B and C in this section. Recall that a p-group P is said to be extra-special if $P' = \Phi(P) = \mathbf{Z}(P)$ and $|\mathbf{Z}(P)| = p$. The following lemma is key to our proofs.

Lemma 3.1. *Let G be a finite group and let P be a Sylow 2-subgroup of G. Suppose that $\mathbf{O}_{2'}(G) = \mathbf{Z}(G) = 1$ and $|P| > 1$. Then $k_2(G) \leq |P|/2$ and $d_2(G) \leq \frac{1}{2}$. Moreover, if $k_2(G) = |P|/2$, then one of the following holds:*

(1) *$k_2(G) = 2$ and P is elementary abelian of order 4.*

(2) *$k_2(G) > 2$ and P is an extra-special group of order 2^{1+2m} for some integer $m \geq 1$ with $\mathbf{Z}(P) = \langle z \rangle$ a cyclic group of order 2. Moreover, $|z^G \cap P| = 3$ and for any $1 \neq y \in P$ with $y \notin z^G$, $|y^G \cap P| = 2$.*

Proof. The hypothesis of the lemma implies that $\mathbf{Z}_2^*(G) = 1$. Let P be a Sylow 2-subgroup of G. By Lemma 2.4, if $1 \neq x \in P$, then $|x^G \cap P| \geq 2$. Let $k = k_2(G) - 1$ be the number of nontrivial conjugacy classes of 2-elements in G. Clearly, we can choose a complete set $\Gamma = \{x_i\}_{i=1}^k$ of representatives for all nontrivial conjugacy classes of 2-elements in G such that $\Gamma \subseteq P \setminus \{1\}$. Notice that $k \geq 1$ as otherwise P is trivial.

Observe that $|x_i^G \cap P| \geq 2$ for all i with $1 \leq i \leq k$, and $P \setminus \{1\} = \bigcup_{i=1}^k x_i^G \cap P$. We have that

$$|P \setminus \{1\}| = \sum_{i=1}^k |x_i^G \cap P| \geq \sum_{i=1}^k 2 = 2k.$$

Hence $|P| - 1 \geq 2k$ and so $2k \leq |P| - 2$ since $|P| - 1$ is odd. Therefore $k_2(G) = k + 1 \leq |P|/2$ and $d_2(G) \leq \frac{1}{2}$ as wanted.

Next, assume that $k_2(G) = |P|/2$. We know that $|x_i^G \cap P| \geq 2$ for all $i = 1, 2, \ldots, k$. If $k = 1$, then $k_2(G) = 2$ and $|P| = 4$, so $|x_1^G \cap P| = 3$ and P is elementary abelian of order 4. Thus part (1) holds. Assume that $k \geq 2$. Since $|x_i^G \cap P| \geq 2$

for every i and $k = |P|/2 - 1$, we obtain that $|x_j^G \cap P| = 3$ for a unique index j and $|x_i^G \cap P| = 2$ for all $1 \le i \ne j \le k$. So we may assume that $|x_1^G \cap P| = 3$ and $|x_i^G \cap P| = 2$ for $2 \le i \le k$.

By Corollary 2.2, we have $\frac{1}{2} = d_2(G) \le d(P)$ and thus either P is abelian or $|P'| = 2$, $d(P) = (1 + 4^{-m})/2$ and $P/\mathbf{Z}(P)$ is elementary abelian of order 4^m by Lemma 2.7. We claim that P is nonabelian. By way of contradiction, assume that P is abelian. Let $H = \mathbf{N}_G(P)$. By Lemma 2.1(1), H controls G-fusion in P (since P is abelian) and so $x_k^G \cap P = x_k^H$. Moreover, as P is abelian, $P \le \mathbf{C}_H(x_k) \le H$ and hence $|x_k^H| \ge 1$ is odd. However $|x_k^H| = |x_k^G \cap P| = 2$ by the result in the previous paragraph, which is a contradiction.

Next, we claim that P is extra-special. It suffices to show that $\mathbf{Z}(P) = P'$. Write $P' = \langle z \rangle$. Since $|P'| = 2$, we have $P' \le \mathbf{Z}(H) \cap \mathbf{Z}(P)$. Let $1 \ne u \in \mathbf{Z}(P)$. We claim that $|u^G \cap P| = 3$. Assume by contradiction that $u^G \cap P = \{u, v\}$, where $u \ne v \in P$. If $v \in \mathbf{Z}(P)$, then $v = u^h$ for some $h \in H$ by Lemma 2.1(1). It follows that $|u^H| = 2$, which is impossible as $P \le \mathbf{C}_H(u) \le H$. Thus $v \notin \mathbf{Z}(P)$. Now $u^G \cap P = v^G \cap P = \{u, v\}$. Since $v \in P \setminus \mathbf{Z}(P)$, we have $|v^P| > 1$ whence $v^P = \{u, v\}$. In particular $u = v^t$ for some $t \in P$. Hence $v = u^{t^{-1}} = u$ as $u \in \mathbf{Z}(P)$. This contradiction shows that $|u^G \cap P| = 3$ for every $1 \ne u \in \mathbf{Z}(P)$. In particular, $|z^G \cap P| = 3$. Now if $\mathbf{Z}(P) \ne P'$, then we can choose $u \in \mathbf{Z}(P) \setminus P'$ and by our previous claim, $|u^G \cap P| = 3$. It follows that u and z are G-conjugate as there is only one class of 2-elements satisfying the previous condition. Again this is a contradiction by using Lemma 2.1(1) and the fact that $z \in \mathbf{Z}(H)$. The proof is now complete. $\square$

We are now ready to prove our first theorem.

Proof of Theorem A. Let G be a finite group. Assume first that G has a normal 2-complement and $d(P) > \frac{1}{2}$ for some Sylow 2-subgroup P of G. By Corollary 2.2(1), $d_2(G) = d(P) > \frac{1}{2}$. Conversely, assume that $d_2(G) > \frac{1}{2}$. Let P be a Sylow 2-subgroup of G. If G has a normal 2-complement, then $d(P) = d_2(G) > \frac{1}{2}$. Thus we only need to show that G has a normal 2-complement. We proceed by induction on $|G|$. Observe that if N is a proper nontrivial normal subgroup of G, then $d_2(N)$ and $d_2(G/N)$ are strictly larger than $\frac{1}{2}$ by Lemma 2.6(2). By induction, both N and G/N have normal 2-complements. Hence if N is of odd order or G/N is a 2-group then G has a normal 2-complement and we are done. Therefore, we may assume that $\mathbf{O}_{2'}(G) = 1$ and $G = \mathbf{O}^2(G)$.

Suppose that $\mathbf{Z}(G)$ is nontrivial. As $\mathbf{O}_{2'}(G) = 1$, $\mathbf{Z}(G)$ must be a 2-group. Now $G/\mathbf{Z}(G)$ has a normal 2-complement, say $K/\mathbf{Z}(G) \trianglelefteq G/\mathbf{Z}(G)$, for some normal subgroup K of G with $\mathbf{Z}(G) \le K$. Hence $\mathbf{Z}(G) \trianglelefteq K \trianglelefteq G$ and G/K is a 2-group. Since $G = \mathbf{O}^2(G)$, we obtain that $G = K$. We now see that $\mathbf{Z}(G)$ is a normal Sylow 2-subgroup of G and thus G has a normal 2-complement by Lemma 2.1(2). So we may assume that $\mathbf{Z}(G) = 1$. Now Lemma 3.1 yields a contradiction. $\square$

We now study the structure of finite groups G with $d_2(G) = \frac{1}{2}$. We first consider the solvable case.

Lemma 3.2. *Let G be a finite solvable group. Suppose that $\boldsymbol{O}_{2'}(G) = 1$ and $G = \boldsymbol{O}^2(G)$. Then $d_2(G) = \frac{1}{2}$ if and only if $G \cong \boldsymbol{A}_4$.*

Proof. If $G \cong \boldsymbol{A}_4$, then $d_2(G) = \frac{1}{2}$ as $k_2(G) = 2$ and $|P| = 4$. Conversely, assume that $d_2(G) = \frac{1}{2}$. We proceed by induction on $|G|$. By Corollary 2.2, we have $\frac{1}{2} = d_2(G) \leq d(P)$. Lemma 2.7 yields that either P is abelian or $|P'| = 2$ and $P/\boldsymbol{Z}(P)$ is elementary abelian of order 4^m. In both cases, $d(P) > \frac{1}{2}$. It follows that G is not a 2-group and so P is noncyclic by Corollary 5.14 in [Isaacs 2008]. As $\boldsymbol{O}_{2'}(G) = 1$, $\boldsymbol{C}_G(\boldsymbol{O}_2(G)) \subseteq \boldsymbol{O}_2(G)$ by [Isaacs 2008, Theorem 3.21], hence $\boldsymbol{Z}(G) \leq P$.

We claim that G/Z satisfies the hypothesis of the lemma for any central subgroup $Z \leq \boldsymbol{Z}(G)$. Clearly, $\boldsymbol{O}^2(G/Z) = G/Z$ since $G = \boldsymbol{O}^2(G)$. Next, assume that $K/Z = \boldsymbol{O}_{2'}(G/Z)$, where $Z \leq K \trianglelefteq G$. Then K has a central Sylow 2-subgroup Z and so by Lemma 2.1(2), K has a normal 2-complement $\boldsymbol{O}_{2'}(K)$. Since $K \trianglelefteq G$, $\boldsymbol{O}_{2'}(K) \leq \boldsymbol{O}_{2'}(G) = 1$. Hence $K = Z$ and so $\boldsymbol{O}_{2'}(G/Z) = 1$.

By Lemma 2.6(2), we have $\frac{1}{2} = d_2(G) \leq d_2(G/Z)d_2(Z) = d_2(G/Z)$. If $d_2(G/Z) > \frac{1}{2}$, then G/Z has a normal 2-complement by Theorem A but this would imply that G/Z is a 2-group and so G is a 2-group, a contradiction. Hence $d_2(G/Z) = \frac{1}{2}$. Therefore, by using induction on $|G|$, if Z is nontrivial, then $G/Z \cong \boldsymbol{A}_4$. We now consider two cases separately, according to whether P is abelian or not.

Case 1: P is abelian. As $\boldsymbol{C}_G(\boldsymbol{O}_2(G)) \subseteq \boldsymbol{O}_2(G)$, we have $P = \boldsymbol{O}_2(G)$ and so $P = \boldsymbol{C}_G(P) \trianglelefteq G$. Clearly, $P \neq \boldsymbol{Z}(G)$, as otherwise G is a 2-group by applying Lemma 2.1(2). Thus $|P : \boldsymbol{Z}(G)| \geq 2$.

Assume first that $\boldsymbol{Z}(G) = 1$. By Lemma 3.1, P is elementary abelian of order 4 and $k_2(G) = 2$. As $P = \boldsymbol{C}_G(P) \trianglelefteq G$, G/P embeds into $\mathrm{GL}_2(2) \cong \boldsymbol{S}_3$. Since G/P is of odd order and nontrivial, $G/P \cong \boldsymbol{C}_3$. It is not hard to see that $G \cong \boldsymbol{A}_4$.

Next, assume that $\boldsymbol{Z}(G)$ is nontrivial. Then $G/\boldsymbol{Z}(G) \cong \boldsymbol{A}_4$. By [Isaacs 2008, Theorem 5.18], $G' \cap \boldsymbol{Z}(G) = G' \cap P \cap \boldsymbol{Z}(G) = 1$. Let R be a Sylow 3-subgroup of G. Then $G = PR$, $|R| = 3$ and R acts nontrivially and coprimely on P; hence $\boldsymbol{Z}(G) = \boldsymbol{C}_P(R)$ and $[P, R] = G' \leq P$. Since R acts coprimely on P, we have $P = [P, R] \times \boldsymbol{C}_P(R) = G' \times \boldsymbol{Z}(G)$. Moreover, $G'R \trianglelefteq G$ and $|G/G'R|$ is a 2-power, so $G = G'R$ forcing $\boldsymbol{Z}(G) = 1$, a contradiction.

Case 2: P is nonabelian. We have $P' \leq \boldsymbol{Z}(P) \leq \boldsymbol{C}_G(\boldsymbol{O}_2(G)) \leq \boldsymbol{O}_2(G)$. Observe that $G/\boldsymbol{O}_{2,2'}(G)$ has an abelian Sylow 2-subgroup, so $G/\boldsymbol{O}_{2,2'}(G)$ has a normal Sylow 2-subgroup by using Hall–Higman Lemma 1.2.3 ([Isaacs 2008, Theorem 3.21]); hence $G = \boldsymbol{O}_{2,2',2,2'}(G)$. (For the definitions of $\boldsymbol{O}_{2,2'}(G)$ and $\boldsymbol{O}_{2,2',2,2'}(G)$, see [Gorenstein 1968, Section 6.3].)

Assume first that $G = \boldsymbol{O}_{2,2'}(G)$. Then $\boldsymbol{C}_G(P) \leq P \trianglelefteq G$. It follows that $P' \leq \boldsymbol{Z}(G)$ as $|P'| = 2$. Thus $G/P' \cong A_4$ and $|G| = 24$. It is easy to check that $G \cong 2 \cdot A_4 \cong$ SL$_2(3)$ as $G = \boldsymbol{O}^2(G)$. However, $d_2(\text{SL}_2(3)) = \frac{3}{8} < \frac{1}{2}$.

Assume that $G/\boldsymbol{O}_{2,2'}(G)$ is nontrivial. Let $L = \boldsymbol{O}_{2,2'}(G) \trianglelefteq G$. Since $\boldsymbol{O}_{2'}(L) = 1$, by using Theorem A and Lemma 2.6(2), we see that $d_2(L) = \frac{1}{2}$. But then this forces $d_2(G/L) = 1$. By Corollary 2.2(2), G/L has a normal 2-complement and since $G = \boldsymbol{O}^2(G)$, we deduce that G/L is a $2'$-group, forcing $G = L$, which is a contradiction. This completes our proof. $\qquad\square$

Lemma 3.3. *Let G be a finite solvable group. Suppose that $\boldsymbol{O}_{2'}(G) = 1$. If $d_2(G) = \frac{1}{2}$, then $G/\boldsymbol{Z}(G) \cong A_4$ or S_4.*

Proof. Suppose that G is a finite solvable group with $d_2(G) = \frac{1}{2}$ and $\boldsymbol{O}_{2'}(G) = 1$. Let $L = \boldsymbol{O}^2(G)$. Then $\boldsymbol{O}_{2'}(L) = 1$ and $L = \boldsymbol{O}^2(L)$. By Lemma 2.6(2), $d_2(L) \geq \frac{1}{2}$. If $d_2(L) > \frac{1}{2}$, then L has a normal 2-complement by Theorem A. However, as $\boldsymbol{O}_{2'}(L) = 1$, L must be a 2-group and hence G is a 2-group with $d_2(G) = d(G) = \frac{1}{2}$, which is impossible by Lemma 2.7. Therefore, $d_2(L) = \frac{1}{2}$. So $L \cong A_4$ by Lemma 3.2.

Let $C = \boldsymbol{C}_G(L) \trianglelefteq G$. As $\boldsymbol{Z}(L) = 1$, we have $C \cap L = 1$. Then $A_4 \cong LC/C \trianglelefteq G/C \leq$ Aut$(A_4) = S_4$. Hence $G/C \cong A_4$ or S_4. It remains to show that $C = \boldsymbol{Z}(G)$. Let P be a Sylow 2-subgroup of G.

As $C \times L = CL \trianglelefteq G$, we have $\frac{1}{2} = d_2(G) \leq d_2(CL) \leq d_2(L)d_2(C) = d_2(C)/2$ and so $d_2(C) = 1$. Thus C has a normal 2-complement and an abelian Sylow 2-subgroup. However, as $\boldsymbol{O}_{2'}(C) \leq \boldsymbol{O}_{2'}(G) = 1$, C must be an abelian 2-group and $C \leq P$. We also have that $\frac{1}{2} = d_2(G) \leq d_2(G/L)d_2(L) = d_2(G/L)/2$, so $d_2(G/L) = 1$ where G/L is a 2-group. It follows that G/L is an abelian 2-group. In particular, $G' \leq L$. Thus $[P, C] \subseteq G' \cap C \subseteq L \cap C = 1$, so $[P, C] = 1$. As $[L, C] = 1$ and $G = PL$, we have $C \leq \boldsymbol{Z}(G)$. Since $\boldsymbol{Z}(G/C)$ is trivial, we must have $C = \boldsymbol{Z}(G)$ as wanted. $\quad\square$

We next classify all finite nonabelian simple groups S such that $d_2(S) = \frac{1}{2}$.

Lemma 3.4. *Let S be a finite nonabelian simple group. Then $d_2(S) = \frac{1}{2}$ if and only if $S \cong \text{PSL}_2(q)$ with $3 < q \equiv 3, 5 \pmod 8$, where q is a prime power.*

Proof. Let S be a finite nonabelian simple with a Sylow 2-subgroup P. If $S \cong$ PSL$_2(q)$ with $3 < q \equiv 3, 5 \pmod 8$, then P is elementary abelian of order 4 and S has only one class of involutions so $k_2(S) = 2$ and thus $d_2(S) = \frac{1}{2}$. Conversely, assume that S is a finite nonabelian simple group with $d_2(S) = \frac{1}{2}$. By Lemma 3.1, either P is elementary abelian of order 4 with $k_2(P) = 2$ or P is extra-special of order 2^{1+2m} for some integer $m \geq 1$.

Assume first that P is elementary abelian of order 4. It follows from [Walter 1969, Theorem I] that S is isomorphic to PSL$_2(2^f)$, where $f \geq 2$, PSL$_2(q)$ with $3 < q \equiv 3, 5 \pmod 8$ ^{2}G$_2(3^{2n+1})$, where $n \geq 1$, or the first Janko group J$_1$. Since $|P| = 4$, we deduce that $S \cong$ PSL$_2(q)$ with $3 < q \equiv 3, 5 \pmod 8$. Note PSL$_2(4) \cong$ PSL$_2(5)$.

Assume now that P is extra-special of order 2^{1+2m}, $m \geq 1$. In this case, P is nilpotent of class 2. It follows from the Main Theorem of [Gilman and Gorenstein 1975a; 1975b] that S is isomorphic to one of the groups $\mathrm{PSL}_2(q)$ with $q \equiv 7, 9 \pmod{16}$, A_7, $\mathrm{Sz}(2^n)$, $\mathrm{PSU}_3(2^n)$, $\mathrm{PSL}_3(2^n)$ or $\mathrm{PSp}_4(2^n)$ with $n \geq 2$. However, except for the first two groups, the centers of the Sylow 2-subgroups of the remaining simple groups have order at least 4. For A_7, we can check that $k_2(A_7) = 3$ so $d_2(A_7) = \frac{3}{8}$ as a Sylow 2-subgroup of A_7 is isomorphic to D_8. Similarly, the Sylow 2-subgroup of $S = \mathrm{PSL}_2(q)$ with $q \equiv 7, 9 \pmod{16}$ is also isomorphic to D_8. Again, except for the identity, S has two nontrivial classes of 2-elements, one consisting of all involutions in S and another consisting of elements of order 4. Thus these cases cannot occur. $\square$

For a finite group G, we denote by $\mathrm{Sol}(G)$ the solvable radical of G, that is, the largest solvable normal subgroup of G.

Lemma 3.5. *Let G be a finite group. Suppose that $\mathrm{Sol}(G) = 1$ and $d_2(G) = \frac{1}{2}$. Then G is a finite almost simple group.*

Proof. Let M be a minimal normal subgroup of G. As G has a trivial solvable radical, $M \cong S^k$, where S is a nonabelian simple group and $k \geq 1$ is an integer. By Lemma 2.6(2), we have $\frac{1}{2} = d_2(G) \leq d_2(G/M)d_2(M) \leq d_2(M)$. By applying this lemma repeatedly, we have $\frac{1}{2} \leq d_2(M) \leq d_2(S)^k$. By Lemma 3.1, $d_2(S) \leq \frac{1}{2}$; so $\frac{1}{2} \leq d_2(S)^k \leq (\frac{1}{2})^k$, forcing $k = 1$ and $d_2(S) = \frac{1}{2}$.

Let $C = C_G(M)$. Then $C \trianglelefteq G$ and $CM = C \times M \trianglelefteq G$. By Lemma 2.6(2),

$$\tfrac{1}{2} = d_2(G) \leq d_2(G/MC)d_2(MC) \leq d_2(MC) \leq d_2(M)d_2(C) = d_2(C)/2.$$

Hence $d_2(C) = 1$ and so C is solvable by Corollary 2.2(2) and the Feit–Thompson theorem. Since $\mathrm{Sol}(G) = 1$ and C is a solvable normal subgroup of G, we must have $C = 1$ so G is almost simple with simple socle M. $\square$

Lemma 3.6. *Let G be a finite perfect group. Suppose that $O_{2'}(G) = 1$ and $d_2(G) = \frac{1}{2}$. Then $G \cong \mathrm{PSL}_2(q)$ with $3 < q \equiv 3, 5 \pmod{8}$, where q is a prime power.*

Proof. Let U be the solvable radical of G. Then G/U is nonsolvable. Since $\frac{1}{2} = d_2(G) \leq d_2(U)d_2(G/U)$ by Lemma 2.6, both $d_2(U)$ and $d_2(G/U)$ are at least $\frac{1}{2}$. By Theorem A, $d_2(G/U) = \frac{1}{2}$ as otherwise G/U is solvable. By Lemmas 3.4 and 3.5 and the fact that G is perfect, $G/U \cong \mathrm{PSL}_2(q)$, where $q \equiv 3, 5 \pmod{8}$. We have $d_2(U) = 1$ and since $O_{2'}(G) = 1$, U is an abelian 2-group. We will show that G is nonabelian simple by induction on $|G|$.

If $U = 1$, then G is simple and we are done. Assume that U is nontrivial. Assume first that $Z := Z(G)$ is nontrivial. Then Z must be a 2-group. Consider the quotient group G/Z. Observe that G/Z is perfect, $d_2(G/Z) = \frac{1}{2}$ and $O_{2'}(G/Z) = 1$. Since $|G/Z| < |G|$, the inductive hypothesis implies that G/Z is nonabelian simple and

thus $Z = U$. It follows that $G \cong \mathrm{SL}_2(q)$, the only Schur cover of $\mathrm{PSL}_2(q)$ with $q \equiv 3, 5 \pmod{8}$. However, it is easy to see that $\mathrm{SL}_2(q)$ has only two classes of nontrivial 2-elements and the Sylow 2-subgroup of $\mathrm{SL}_2(q)$ with $q \equiv 3, 5 \pmod 8$ has order 8, so $d_2(G) = \frac{3}{8} < \frac{1}{2}$, which is a contradiction. Hence we may assume that $\mathbf{Z}(G) = 1$. Since U is a normal abelian subgroup of G, we have $U \leq \mathbf{C}_G(U) \trianglelefteq G$. Since U is not central in G and G/U is nonabelian simple, we must have that $U = \mathbf{C}_G(U)$.

Let $P \in \mathrm{Syl}_2(G)$. Note that the hypothesis of Lemma 3.1 holds for G, that is, $\mathbf{O}_{2'}(G) = \mathbf{Z}(G) = 1$, $|P| > 1$ and that $d_2(G) = \frac{1}{2}$. We claim that $k_2(G) > 2$. Assume by contradiction that $k_2(G) = 2$. Then P is elementary abelian of order 4 by Lemma 3.1. However the Sylow 2-subgroup of $\mathrm{PSL}_2(q)$ with $q \equiv 3, 5 \pmod 8$ has order 4. So $U = 1$, which is a contradiction. Therefore $k_2(G) > 2$ and so part (2) of Lemma 3.1 holds. Obviously $|U| \geq 4$ and $P' = \langle z \rangle = \mathbf{Z}(P) < U$. If $z^G = U \setminus \{1\}$, then U is elementary abelian of order 4 and thus G/U embeds into $\mathrm{GL}_2(2)$ which is impossible. Thus there exists $1 \neq y \in U \setminus z^G$ and so $|y^G \cap P| = 2$. Since $y \in U \trianglelefteq G$, we have $y^G \subseteq U \leq P$, so $|y^G \cap P| = |y^G| = 2$ which implies that $U \leq \mathbf{C}_G(y) < G$ and $|G : \mathbf{C}_G(y)| = 2$. Therefore, G/U has a subgroup of index 2 which is impossible as G/U is nonabelian simple. $\square$

Proof of Theorem B. Let G be a finite group and assume that $d_2(G) = \frac{1}{2}$ and $\mathbf{O}_{2'}(G) = 1$. If G is solvable, then $G/\mathbf{Z}(G) \cong A_4$ or S_4 by Lemma 3.3. So part (1) of the theorem holds. Assume that G is nonsolvable. Let L be the last term of the derived series of G. By Theorem A and Lemma 2.6(2), $d_2(L) = \frac{1}{2}$. Moreover L is perfect and $\mathbf{O}_{2'}(L) = 1$. By Lemma 3.6, $L \cong S$ where $S = \mathrm{PSL}_2(q)$ $q \equiv 3, 5 \pmod 8$. Write $q = p^f$, where p is a prime and $f \geq 1$ is an integer. We see that f must be odd and thus $\mathrm{Out}(S) = C_2 \times C_f$.

Let $C = \mathbf{C}_G(L)$. Then $C \trianglelefteq G$, $C \cap L = 1$ and G/C is an almost simple group with socle isomorphic to S. Since $d_2(L) = \frac{1}{2}$, we see that $d_2(C) = 1$ and since $\mathbf{O}_{2'}(G) = 1$, C is a normal abelian 2-subgroup of G. We also have that

$$\tfrac{1}{2} = d_2(G) \leq d_2(G/L) d_2(L) = d_2(G/L)/2$$

so $d_2(G/L) = 1$ and so G/L has a normal 2-complement W/L and an abelian Sylow 2-subgroup PL/L by Corollary 2.2, where P is any Sylow 2-subgroup of G containing C. Since CL/L and W/L are normal subgroups of G/L and have coprime orders, we deduce that $[C, W] \leq L$. As $C \trianglelefteq G$, we have $[C, W] \leq L \cap C = 1$. Thus $[C, W] = 1$. On the other hand, PL/L is abelian, thus $[C, P] \leq L$. With the same reasoning, we have $[C, P] \leq L \cap C = 1$. Since $G = PW$, we obtain that $[C, G] = 1$. In particular, $C \leq \mathbf{Z}(G)$ and since G/C is almost simple, we must have that $C = \mathbf{Z}(G)$. Therefore, we have shown that $G/\mathbf{Z}(G)$ is almost simple with socle S as required. $\square$

We will need the following result for our proof of Theorem C.

Lemma 3.7. *Let G be a finite group of odd order and let σ be a nonempty set of primes. If $d_\sigma(G) \geq \frac{1}{2}$, then G has a normal σ-complement and an abelian Hall σ-subgroup.*

Proof. By the Feit–Thompson theorem, we know G is solvable. By Lemma 2.6(2), if $N \unlhd G$, then $d_\sigma(N) \geq \frac{1}{2}$ and $d_\sigma(G/N) \geq \frac{1}{2}$. Assume that G has a normal σ-complement K. Let H be a Hall σ-subgroup of G. We claim that H is abelian. As $G/K \cong H$, we have $\frac{1}{2} \leq d_\sigma(H) = d(H)$, where the last equality holds as H is a σ-group. Thus $d(H) \geq \frac{1}{2}$ where H is a group of odd order. By Lemma 2.7, H must be abelian as wanted. Therefore, it suffices to show that G has a normal σ-complement. We will prove this claim by induction on $|G|$.

Let N be a minimal normal subgroup of G. Then N is an elementary abelian p-subgroup for some odd prime p. As $d_\sigma(G/N) \geq \frac{1}{2}$, by induction on $|G|$, G/N has a normal σ-complement, say M/N. If $p \notin \sigma$, then M is also a normal σ-complement of G, and we are done. Thus we may assume that $O_{\sigma'}(G) = 1$ and $p \in \sigma$. We have $M \unlhd G$ and $d_\sigma(M) \geq \frac{1}{2}$. Therefore, by induction again, M has a normal σ-complement whenever $M < G$; but then this would imply that M is a σ-subgroup since $O_{\sigma'}(M) \subseteq O_{\sigma'}(G) = 1$ and hence G is a σ-group. So, we can assume $M = G$, hence G/N is a σ'-group.

Since G/N is solvable, let T/N be a maximal normal subgroup of G/N of prime index $r \notin \sigma$. Since $T \unlhd G$, we have $d_\sigma(T) \geq \frac{1}{2}$ and again by induction, T has a normal σ-complement which implies that $T = N$. Thus N is a maximal normal subgroup of G and $|G/N| = r$ is a prime different from p.

If $C_G(x) = G$ for some $1 \neq x \in N$, then $\langle x \rangle = N \leq Z(G)$ and $G \cong C_p \times C_r$ by Lemma 2.1(2), which is a contradiction as $O_{\sigma'}(G) = 1$. So, we may assume that $C_G(x) < G$ for all $1 \neq x \in N$. Since N is maximal in G, $C_G(x) = N$ for every $1 \neq x \in N$. Thus G is a Frobenius group with Frobenius kernel N and Frobenius complement isomorphic to C_r. Set $|N| = p^k$ for some integer $k \geq 1$. We can see that $\sigma = \{p\}$ and that $k_p(G) = (p^k - 1)/r + 1$ and so $d_p(G) = 1/r + (r-1)/(rp^k) \geq \frac{1}{2}$. Notice that $r \neq p \geq 3$ and $r \mid p^k - 1$. We consider the following cases:

(1) $r = 3$ and $p \geq 5$. In this case, we have

$$d_p(G) \leq \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{5} = \frac{7}{15} < \frac{1}{2}.$$

(2) $r \geq 5$ and $p = 3$. Since $r > p$, $k \geq 2$. We have

$$d_p(G) \leq \frac{1}{5} + \frac{1}{9} = \frac{14}{45} < \frac{1}{2}.$$

(3) $r \geq 5$ and $p \geq 5$. Clearly, we have

$$d_p(G) < \frac{1}{5} + \frac{1}{5} = \frac{2}{5} < \frac{1}{2}.$$

Thus we have shown that G has a normal σ-complement as wanted. $\square$

We are now ready to prove Theorem C.

Proof of Theorem C. Let G be a finite group and let π be a set of primes containing 2 and let $\sigma = \pi \setminus \{2\}$. Suppose $d_\pi(G) > \frac{1}{2}$. By Lemma 2.6(1), $\frac{1}{2} < d_\pi(G) \le d_2(G)$ and thus by Theorem A, G has a normal 2-complement K and by Lemma 2.6(2), we have

$$\tfrac{1}{2} < d_\pi(G) \le d_\pi(K) d_\pi(G/K) \le d_\pi(K) = d_\sigma(K).$$

By Lemma 3.7 , K has a normal σ-complement, say N, and an abelian Hall σ-subgroup T. It follows that $G = PTN$, where N is also a normal π-complement of G. □

4. Conjugacy classes of p-elements with p odd

We now consider odd primes. We start with the following easy result.

Lemma 4.1. *Let p be an odd prime. Let G be a finite group and let P be a Sylow p-subgroup of G. If $d_p(G) \ge (p+1)/2p$, then P is abelian.*

Proof. Let G be a finite group such that $d_p(G) \ge (p+1)/2p$. By Corollary 2.2, we have $d_p(G) \le d(P)$ which implies that $d(P) \ge (p+1)/2p$. If P is abelian, then we are done. So, assume that P is nonabelian. By Lemma 2.6(3), we have $d(P) < (p+1)/p^2$. Since p is odd, we can check that $(p+1)/2p > (p+1)/p^2$ and so $d(P) < (p+1)/p^2 < (p+1)/2p \le d(P)$, which is a contradiction. □

Proof of Theorem D. Let p be an odd prime. Let G be a finite group. Assume that G has a normal p-complement and an abelian Sylow p-subgroup P. By Corollary 2.2(2), we have $d_p(G) = d(P) = 1 > (p+1)/2p$. Conversely, assume that $d_p(G) > (p+1)/2p$. By Lemma 4.1, P is abelian. It remains to show that G has a normal p-complement. We proceed by using induction on $|G|$.

We first claim that $\mathbf{O}_{p'}(G) = 1$. Assume by contradiction that $\mathbf{O}_{p'}(G)$ is nontrivial. By Lemma 2.6(2), $d_p(G) \le d_p(G/\mathbf{O}_{p'}(G)) d_p(\mathbf{O}_{p'}(G)) \le d_p(G/\mathbf{O}_{p'}(G))$, so by induction $G/\mathbf{O}_{p'}(G)$ has a normal p-complement; hence G will have a normal p-complement. Thus we may assume that $\mathbf{O}_{p'}(G) = 1$.

We next claim that $G = \mathbf{O}^p(G)$. Indeed, if $N = \mathbf{O}^p(G)$ is a proper subgroup of G, then $(p+1)/2p < d_p(G) \le d_p(N)$; thus by induction again, N has a normal p-complement $\mathbf{O}_{p'}(N)$. Clearly, this is also a normal p-complement of G.

We now show that G is p-solvable. In fact, suppose that G is not p-solvable and let M/N be a nonabelian chief factor of G with p dividing $|M/N|$. There exists a nonabelian simple group S and an integer $k \ge 1$ such that $M/N \cong S^k$. By applying Lemma 2.6(2) repeatedly, we have $(p+1)/2p < d_p(S)^k \le d_p(S)$. (Note that p divides $|S|$.) Let $T \in \mathrm{Syl}_p(S)$ and let $H = \mathbf{N}_S(T)$. Clearly T is abelian, so by Lemma 2.1(1), H controls S-fusion in T. Thus $x^S \cap T = x^H \subseteq T$ for every $x \in T$. Since S is nonabelian simple, $\mathbf{Z}_p^*(S) = 1$. Now Lemmas 2.3 and 2.5 together with

Glauberman's Z_p^*-theorem imply that $|x^S \cap T| \geq 2$ for all $1 \neq x \in T$. It follows that $|T| - 1 \geq 2(k_p(S) - 1)$. This implies that $k_p(S) \leq (|T| + 1)/2$ and hence

$$(p + 1)/2p < d_p(S) \leq (|T| + 1)/(2|T|) \leq (p + 1)/2p$$

as $|T| \geq p$. This contradiction shows that G is p-solvable.

By Hall–Higman Lemma 1.2.3 ([Isaacs 2008, Lemma 3.21]) and the fact that P is abelian, we have $P \leq \boldsymbol{C}_G(\boldsymbol{O}_p(G)) \leq \boldsymbol{O}_p(G)$, so

$$P = \boldsymbol{O}_p(G) \trianglelefteq G.$$

Let P/N be a chief factor of G. Assume that N is nontrivial. Then

$$(p + 1)/2p < d_p(G) \leq d_p(G/N)$$

and so by induction G/N has a normal p-complement K/N. However, as $G = \boldsymbol{O}^p(G)$, $G = K$ which is impossible. So, we can assume that P is an elementary abelian minimal normal p-subgroup of G.

If $|x^G| = 1$ for some $1 \neq x \in P$, then $x \in \boldsymbol{Z}(G) \cap P$ which forces $P = \langle x \rangle \subseteq \boldsymbol{Z}(G)$. In this case, G has a normal p-complement by Lemma 2.1(2). Hence we can also assume that $|x^G| \geq 2$ for all $1 \neq x \in P$ whence $k_p(G) \leq (|P| + 1)/2$. Since $d_p(G) > (p + 1)/2p$,

$$|P|(p + 1)/2p < (|P| + 1)/2.$$

However, this inequality cannot occur as $|P| \geq p$. $\hspace{2cm}\square$

Proof of Theorem E. Let π be a nonempty set of odd primes and let p be the smallest member in π. Let G be a finite group with $d_\pi(G) > (p + 1)/2p$. For every $r \in \pi$, we see that

$$(r + 1)/2r \leq (p + 1)/2p < d_\pi(G) \leq d_r(G)$$

by Lemma 2.6(1), so $d_r(G) > (r + 1)/2r$. By Theorem D, G has a normal r-complement and an abelian Sylow r-subgroup. It follows that G has a normal π-complement $N = \boldsymbol{O}_{\pi'}(G) \trianglelefteq G$ and G is π-solvable. By [Isaacs 2008, Theorem 3.20], G has a Hall π-subgroup H. Clearly $G = HN$ and $G/N \cong H$. Since

$$(p + 1)/2p < d_\pi(G) \leq d_\pi(H)d_\pi(N) \leq d_\pi(H) = d(H),$$

we deduce that

$$d(H) > (p + 1)/2p > 1/p \geq 1/r$$

and so by Lemma 2.6(4), H has a normal Sylow r-subgroup. It follows that H is nilpotent and thus H is abelian. $\hspace{2cm}\square$

Proof of Theorem F. Let G be a finite group with a Sylow p-subgroup P, where p is an odd prime. Suppose that $d_p(G) = (p + 1)/2p$ and $\boldsymbol{O}_{p'}(G) = 1$. By Lemma 4.1,

P is abelian and thus by Lemma 2.1(1), $d_p(N_G(P)) = d_p(G) = (p+1)/2p$. By Theorem 7.4.4 in [Gorenstein 1968], we have $P = P \cap N' \times P \cap Z(N)$, where $N = N_G(P)$. Set $Z = P \cap Z(N)$ and $U = P \cap N'$. We have that $k_p(N) \le (|P| + |Z|)/2$ as for any $x \in P \setminus Z$, $|x^N| \ge 2$. It follows that

$$(p+1)/2p \le (|P| + |Z|)/2|P|$$

and thus $|P : Z| \le p$. Clearly, $|P : Z| > 1$ as otherwise $P \subseteq Z(N)$ and thus G has a normal p-complement which forces $G = P$ since $O_{p'}(G) = 1$. But then $d_p(G) = 1$, a contradiction. Thus $|P : Z| = p$; hence $|U| = p$ and $|Z| = |P|/p$. Moreover $|x^N| = 2$ for every $x \in P \setminus Z$. Set $U = \langle y \rangle$. Then $C_G(P) = C_N(y) \trianglelefteq N$. As $|y^N| = 2$, we have $|N : C_G(P)| = 2$. Since $U = P \cap N'$ has order p, we see that $U = [P, N]$. Furthermore, by Theorem 7.4.4 in [Gorenstein 1968], $P \cap G' = P \cap N' = U$ is cyclic of order p and by Theorem 5.18 in [Isaacs 2008], $G' \cap Z = 1$.

Let $F^*(G)$ be the generalized Fitting subgroup of G. Then $F^*(G) = F(G)E(G)$ is the central product of the Fitting subgroup $F(G)$ and the layer $E(G)$ of G, which is the product of all components of G, that is, subnormal quasisimple subgroups of G. Bender's theorem ([Isaacs 2008, Theorem 9.8]) says that $C_G(F^*(G)) \subseteq F^*(G)$.

Assume first that $E(G) = 1$. Then $F^*(G) = F(G)$ is a p-group since $O_{p'}(G) = 1$. As $C_G(F(G)) \subseteq F(G)$ and P is abelian, $P = F(G)$ and thus $P = C_G(P) \trianglelefteq G$. It follows that $|G : P| = 2$ and $G' = [G, P] = U$ is cyclic of order p; moreover $Z = Z(G) \cap P = Z(G)$ as P is self-centralizing. Now G/G' is an abelian group of order $2|Z|$. Hence G/G' has a normal Sylow 2-subgroup A/G' and a normal Sylow p-subgroup $P/G' = ZG'/G'$, so $G/G' = A/G' \times ZG'/G'$ which implies that $G = Z \times A$, where A is a nonabelian group of order $2p$ and it has a normal cyclic Sylow p-subgroup of order p. It is easy to see that $A \cong D_{2p}$, the dihedral group of order $2p$.

Assume now that $E := E(G)$ is nontrivial. Since $G' \cap P$ is cyclic of order p, the center of E is either trivial or cyclic of order p. If $|Z(E)| = p$, then $E/Z(E)$ is a p'-group which is impossible by Corollary 5.4 in [Isaacs 2008]. Thus $Z(E) = 1$. Hence E has a Sylow p-subgroup of order p which forces E to be a nonabelian simple group. Now we have that $F^*(G) = E \times F$ where $F = F(G) \le P$ is a p-subgroup. Since $E \cap P \le G' \cap P = U$ is of order p, we have $U = E \cap P = G' \cap P$. Therefore, $F \cap G' \le F \cap G' \cap P = E \cap P \cap F = 1$, so $F \cap G' = 1$, whence $F \le Z(G)$. Hence $C_G(E) = F = Z(G)$ and so G/F is an almost simple group with socle isomorphic to E. Since E has a cyclic Sylow p-subgroup of order p, we deduce from Lemma 2.3 in [Tong-Viet 2018] that $|\mathrm{Out}(E)|$ is prime to p. In particular, G/EF is a solvable p'-group. Thus G/E has a central Sylow p-subgroup $EF/E \cong F$. By Lemma 2.1(2), G/E has a normal p-complement A/E and $G/E = EF/E \times A/E$. Since $E \cap F = 1$, we have $G = A \times F$ and so A is almost simple with socle E. $\square$

Acknowledgments

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