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SPECTRUM OF THE LAPLACIAN AND THE JACOBI OPERATOR ON ROTATIONAL CMC HYPERSURFACES OF SPHERES

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Let $M \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be a compact CMC rotational hypersurface of the $(n+1)$-dimensional Euclidean unit sphere. Denote by $|A|^2$ the square of the norm of the second fundamental form and $J(f) = -\Delta f - nf - |A|^2 f$ the stability or Jacobi operator. In this paper we compute the spectra of their Laplace and Jacobi operators in terms of eigenvalues of second order Hill's equations. For the minimal rotational examples, we prove that the stability index — the numbers of negative eigenvalues of the Jacobi operator counted with multiplicity — is greater than $3n+4$ and we also prove that there are at least 2 positive eigenvalues of the Laplacian of M smaller than n. When H is not zero, we have that every nonflat CMC rotational immersion is generated by rotating a planar profile curve along a geodesic called the axis of rotation. We assume that the coordinates of this plane has been set up so that the axis of rotation goes through the origin. The planar profile curve is made up of m copies, each one of them is a is rigid motion of a single curve that we will call the fundamental piece. For this reason every nonflat rotational CMC hypersurface has Z_m in its group of isometries. If θ denotes the change of the angle of the fundamental piece when written in polar coordinates, then $l = \frac{m\theta}{2\pi}$ is a nonnegative integer. For unduloids (a subfamily of the rotational CMC hypersurfaces that include all the known embedded examples), we show that the number of negative eigenvalues of the operator J counted with multiplicity is at least $(2l-1)n + (2m-1)$.

1. Introduction

Rotational constant mean curvature hypersurfaces of spheres provide a variety of examples that can be used to understand the nature of CMC hypersurfaces in general. In this paper we derive a formula for the Laplacian on these hypersurfaces that allows us to compute the spectra of their Jacobi and Laplace operators to any desired degree of accuracy. Results on the stability index are relevant in both cases, the case $H = 0$ and the case $H \neq 0$. Let us denote by $\mathcal{M}^n$ any minimal, not necessarily

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rotational, compact n-dimensional minimal hypersurface of the sphere. The spectra of the Laplacian and the stability operators on compact minimal hypersurfaces $\mathcal{M}^n \subset \mathbb{S}^{n+1}$ have been one of the central topics in differential geometry. Let us denote by $\lambda_1(\mathcal{M}^n)$, the first nonzero eigenvalue of the Laplace operator of $\mathcal{M}$. For minimal hypersurfaces of spheres, the spectrum of the Laplacian is known only when $\mathcal{M}^n$ is an Euclidean sphere, $\mathcal{M}^n$ is the product of Euclidean spheres, or $\mathcal{M}^n$ is a cubic isoparametric hypersurface, [Solomon 1990a, 1990b]. It is known that for any minimal hypersurface of the sphere, n is one of the eigenvalues of the Laplacian. Yau [1982] has conjectured that if $\mathcal{M}^n$ is compact and embedded, then $\lambda_1(\mathcal{M}^n) = n$. A positive partial result of this conjecture was given by Choi and Wang [1983]. They showed that if $\mathcal{M}^n$ is compact and embedded, then $\lambda_1(\mathcal{M}^n) \geq \frac{n}{2}$. In this paper we prove that if $\mathcal{M}^n$ is rotational, then $\lambda_1(\mathcal{M}^n) < n$. Moreover we show that there are at least two positive eigenvalues of the Laplace operator smaller n.

Regarding the spectrum of the Jacobi operator, we have that the number of negative eigenvalues of the operator J counted with multiplicity is known as the stability index of $\mathcal{M}^n$ and it is denoted as $\mathrm{ind}(\mathcal{M}^n)$. It has been conjectured that if $\mathcal{M}^n$ is not an Euclidean sphere, then $\mathrm{ind}(\mathcal{M}^n) = n+3$ implies that $\mathcal{M}^n$ is a product of Euclidean spheres. This conjecture was proven by Urbano [1990], when $n = 2$. For general n only partial results are known. In [Perdomo 2004a], the author shows that if $\mathrm{ind}(\mathcal{M}^n) = n+3$, then $\int_M |A|^2 \leq \int_M (n-1)$ with equality only if $\mathcal{M}^n$ is a product of Euclidean spheres. On the other hand in [Perdomo 2004b], the author shows that if $\mathcal{M}^n$ is rotational, then, $\int_M |A|^2 \leq \int_M (n-1)$. Therefore, it may seem that the rotational minimal hypersurfaces are good candidates for a counterexample of the conjecture. This is not the case. Due to their symmetries, their stability index must be greater than $n+3$. See [Perdomo 2001].

In this paper we show that there is a big jump in the stability index among the minimal rotational examples. We prove that if $\mathcal{M}^n$ is rotational and minimal and it is not a product of spheres then $\mathrm{ind}(\mathcal{M}^n) \geq 3n+5$. Some other partial results on this conjecture are found in [Simons 1968; Perdomo 2002; Savo 2010]. One of the most important applications of this type of estimate on the stability index was given by Marques and Neves [2014], where, among other tools, they used Urbano's result to prove Willmore's conjecture.

To motivate the study of the stability index when H is a nonzero constant, we would like to point out that any time we have a one-parametric family of CMC hypersurfaces, a jump in the stability index as a function of the parameter indicates a potential bifurcation point, a hypersurface that is part of another one-parametric family of CMC hypersurfaces. As an example of this fact, the families of embedded rotational CMC hypersurfaces on the sphere explained by the author in [Perdomo 2010] can be viewed as families emerging from bifurcation points of the family $S^1(r) \times S^{n-1}(\sqrt{1-r^2}) \subset \mathbb{S}^n$. See [Alías and Piccione 2013].

To illustrate the way we can use our results to estimate the eigenvalues, we pick a 3-dimensional rotational minimal hypersurface in $\mathbb{S}^4$ and prove that the first three eigenvalues of the Laplace operator are: 0, a number near 0.4404 with multiplicity 2, and 3 with multiplicity 5. We also show that the negative eigenvalues of the Jacobi operator are: a number near -8.6534 with multiplicity 1, a number near -8.52078 with multiplicity 2, -3 with multiplicity 5, a number near -2.5596 with multiplicity 6, and a number near -1.17496 with multiplicity 1. The stability index of this hypersurface is thus 15.

When H is not zero, we prove (see Theorem 3.7) a lower bound for the number of negative eigenvalues of the Jacobi operator that generalizes to any dimension the result on the stability index of constant mean curvature tori of revolution in the 3-sphere proven by Rossman and Sultana [2007].

We will be using the *oscillation theorem* for the periodic problem on the Hill's equation (a proof can be found in [Magnus and Winkler 1966]).

Theorem 1.1. *Consider the differential equation*

$$z''(t) + (\lambda + Q(t))z(t) = 0 \tag{1-1}$$

where Q is a smooth T-periodic function. For any λ let us define

$$\delta(\lambda) := z_1(T, \lambda) + z_2'(T, \lambda),$$

where $z_1(t, \lambda)$ and $z_2(t, \lambda)$ are solutions of (1-1) *such that $z_1(0, \lambda) = 1$, $z_1'(0, \lambda) = 0$ and $z_2(0, \lambda) = 0$, $z_2'(0, \lambda) = 1$. There exists an increasing infinite sequence of real numbers $\lambda_1, \lambda_2 \dots$ such the differential equation* (1-1) *has a T-periodic solution if and only if $\lambda = \lambda_j$. Moreover the λ_j are the roots of the equation $\delta(\lambda) = 2$. The function δ is called the discriminant function of the operator $K[z] = z''(t) + Q(t)z(t)$.*

We will also be using the following theorem proven by Haupt [1914] from the Hill's equation theory. The next presentation of the Haupt theorem can be found in [Magnus and Winkler 1966].

Theorem 1.2 [Haupt 1914]. *Let us denote by $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \cdots$ the sequence of eigenvalues of the Hill's equation presented in Theorem 1.1. If $z(t)$ is a nonzero T-periodic solution of* (1-1) *with $\lambda = \lambda_i$ then, the number of zeros of $z(t)$ in the interval* $[0, T)$ *is* $2\lfloor \frac{i}{2} \rfloor$.

We would like to point out that Beekmann and Lökes [1990] have used the Hill equation to find bounds on the eigenvalues of the Laplacian on toroidal surfaces.

2. Describing rotational CMC hypersufaces of spheres

Any compact CMC rotational hypersurface of $\mathbb{S}^{n+1}$ is given by an immersion $\phi : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{S}^{n+1}$ where,

$$\phi(y, t) = \big(r(t)\, y, \sqrt{1 - r(t)^2}\cos(\theta(t)), \sqrt{1 - r(t)^2}\sin(\theta(t))\big) \tag{2-1}$$

and $r(t)$ is a positive T-periodic function that satisfies the conditions

$$(r')^2 + r^2(1 + \lambda^2) = 1, \tag{2-2}$$

with

$$\lambda = H + c^{-n/2} r^{-n}, \quad \theta(t) = \int_0^t \frac{r(\tau)\lambda(\tau)}{1 - r^2(\tau)}\, d\tau, \tag{2-3}$$

and c is a positive real number that satisfies

$$\theta(T) = 2\pi \frac{l}{m} \quad \text{where } l \text{ and } m \text{ are relative prime integers.} \tag{2-4}$$

The condition on c in (2-4) guarantees that the immersion ϕ satisfies

$$\phi(y, t + mT) = \phi(y, t)$$

and makes M compact. Recall that the function $r(t)$ depends on c since $\lambda(t)$ depends on c. Also, since the function $\theta(t)$ depends on $r(t)$ and $\lambda(t)$, it follows that $\theta(T)$ depends on c as well.

Remark 2.1. When M is minimal, Otsuki [1972; 1993], showed that the expression

$$\theta(T) = K(c) = \int_0^{T/2} \frac{c^{-1/n} r^{1-n}(\tau)}{1 - r^2(\tau)}\, d\tau$$

lies between lies π and $\sqrt{2}\pi$. As a consequence the equation $\theta(T) = 2\pi\frac{l}{m}$ cannot be solved with $l = 1$ and therefore no minimal rotational hypersurface is embedded.

The principal curvatures of M are λ with multiplicity $(n - 1)$ and

$$\mu = H - (n - 1)c^{-n/2} r^{-n}$$

with multiplicity one. Differentiating (2-2) we obtain

$$r'' + r + r\lambda\mu = 0. \tag{2-5}$$

The next expression explicitly provides the Gauss map ν of the immersion (2-1),

$$\nu(y, t) = \left(-r\lambda y, \frac{r^2\lambda\cos\theta - r'\sin\theta}{\sqrt{1 - r(t)^2}}, \frac{r^2\lambda\sin\theta + r'\cos\theta}{\sqrt{1 - r(t)^2}}\right) \tag{2-6}$$

All the details of the construction of these hypersurfaces can be found in [Perdomo 2010].

3. Main theorems

Before stating the following theorem, we recall that m is the integer given in (2-4), T is the period of the function $r(t)$, and mT is the period of the immersion ϕ.

Theorem 3.1. *Let M be the rotational symmetric hypersurface defined in* (2-1). *For any function $\bar{f} : \mathbb{S}^{n-1} \to \mathbb{R}$ we define $f : M \to \mathbb{R}$ as $f(\phi(t, y)) = \bar{f}(y)$. Likewise, for any mT-periodic function $\bar{g} : \mathbb{R} \to \mathbb{R}$ we define $g : M \to \mathbb{R}$ as $g(\phi(t, y)) = \bar{g}(t)$. We will denote by $\bar{\Delta}$ the Laplacian operator on $\mathbb{S}^{n-1}$. With this notation we have*

$$\Delta(fg) = f\Big(\bar{g}'' + (n-1)\frac{r'}{r}\bar{g}'\Big) + \frac{\bar{\Delta}(\bar{f})}{r^2}\, g\,. \tag{3-1}$$

Proof. The proof is a direct computation using the fact that $\nabla(fg) = \frac{g}{r}\bar{\nabla}\bar{f} + f\bar{g}'\frac{\partial}{\partial t}$ and that $\operatorname{div}\big(\frac{\partial}{\partial t}\big) = (n-1)\frac{r'}{r}$. □

Remark 3.2. We would like to point out that we are using the fact that the ambient space is $\mathbb{R}^{n+2}$ in the argument above. In particular we are using the natural identification of all tangent spaces $T_x\mathbb{R}^{n+2}$ with $\mathbb{R}^{n+2}$. If we decide to work intrinsically we will notice that the formula that compares the gradients will have an additional factor of r. From the point of view of intrinsic differential geometry, the formula in Theorem 3.1 can be generalized to warped products. See [Marrocos and Gomes 2019].

Theorem 3.3. *Let $\alpha_1 = 0, \alpha_2 = (n-1), \dots, \alpha_k = (k-1)(n+k-3), \dots$ denote the spectrum of $\mathbb{S}^{n-1}$. The spectrum of the Laplace operator Δ on M is given by $\bigcup_{k=1}^{\infty} \Gamma_k$, where,*

$$\Gamma_k = \{\lambda(k, 1), \lambda(k, 2), \dots\}$$

is the ordered spectrum of the operator (*defined on the set of mT periodic functions*)

$$K_{\Delta,k}[z] = z'' + (n-1)\frac{r'}{r}z' - \frac{\alpha_k}{r^2}z$$

The spectrum of the Jacobi operator J on M is given by $\bigcup_{k=1}^{\infty} \mathbb{F}_k$, where

$$\mathbb{F}_k = \{\tilde{\lambda}(k, 1), \tilde{\lambda}(k, 2), \dots\}$$

is the ordered spectrum of the operator (*defined on the set of mT periodic functions*)

$$K_{J,k}[z] = z'' + (n-1)\frac{r'}{r}z' + \left(n + nH^2 + n(n-1)c^{-n}r^{-2n} - \frac{\alpha_k}{r^2}\right)z$$

Proof. Let us prove the case of the Laplacian. By Theorem 3.1 we have that if $\bar{f}_k$ is an eigenfunction of the Laplacian on $\mathbb{S}^{n-1}$ with eigenvalue α_k and $\bar{g}_l(t)$ is an eigenfunction of the operator $K_{\Delta,k}$ with eigenvalue $\lambda(k,l)$, then $h_{k,l}: M \to \mathbb{R}$ given by $h_{kl} = f_k g_l$ is an eigenfunction of the Laplacian with eigenvalue $\lambda(k,l)$. Therefore $\bigcup_{k=1}^{\infty} \Gamma_k$ is contained in the spectrum of Δ. To prove the reverse inclusion, we only need to point out that every smooth function $h : M \to \mathbb{R}$ can be written as a series of eigenfunctions of the form h_{kl}. There exists a basis

$$\bar{f}_{1,1},\ \bar{f}_{2,1},\ \bar{f}_{2,2},\ \dots,\ \bar{f}_{2,n},\ \bar{f}_{3,1},\ \bar{f}_{3,2},\ \dots,\ \bar{f}_{3,2n},\ \bar{f}_{4,1},\ \dots$$

for $L^2(\mathbb{S}^{n-1})$ with $\bar{\Delta}\bar{f}_{k,j} + \alpha_k \bar{f}_{k,j} = 0$. So any $h : M \to \mathbb{R}$ can be written as a sum of the form

$$a_{1,1}(t)\bar{f}_{1,1} + a_{2,1}(t)\bar{f}_{2,1} + \cdots + a_{2,n}(t)\bar{f}_{2,n} + a_{3,1}(t)\bar{f}_{3,1} + \cdots$$

We obtain the desired expression for the function $h : M \to \mathbb{R}$, by noticing that each $a_{k,l}(t)$ can now be expanded in eigenfunctions of the operator $K_{\Delta,k}$. The proof for the Jacobi operator is similar and uses the following expression for $|A|^2$:

$$|A|^2 = n\big(H^2 + (n-1)c^{-n}r^{-2n}\big). \qquad \square$$

The following lemma allows us to use the oscillation theorem to compute the eigenvalues for the second order differential equations on Theorem 3.3.

Lemma 3.4. *Let us denote by $\lambda = H + c^{-n/2}r^{-n}$ and $\mu = H - (n-1)c^{-n/2}r^{-n}$. The change of variables $u = r^{(n-1)/2}z$ gives us,*

$$\begin{aligned} K_{\Delta,k} &= z'' + (n-1)\frac{r'}{r}z' - \frac{\alpha_k}{r^2}z \\ &= \frac{1}{r^{(n-1)/2}}\Bigg(u'' + \bigg(\tfrac14\lambda^2((n-1)(n-3)) + \tfrac12\lambda\mu(n-1) - \frac{4\alpha_k + (n-3)(n-1)}{4r^2} \\ &\qquad\qquad + \tfrac14(n-1)^2\bigg)u\Bigg) \end{aligned}$$

and

$$\begin{aligned} K_{J,k} &= z'' + (n-1)\frac{r'}{r}z' + \left(n + nH^2 + n(n-1)c^{-n}r^{-2n} - \frac{\alpha_k}{r^2}\right)z \\ &= \frac{1}{r^{(n-1)/2}}\Bigg(u'' + \tfrac14\bigg(\lambda^2(n^2-1) + 2\lambda\mu(n-1) + 4\mu^2 + (n+1)^2 \\ &\qquad\qquad - \frac{4\alpha_k + (n-3)(n-1)}{r^2}\bigg)u\Bigg) \end{aligned}$$

Theorem 3.5. *If $\mathcal{M} \subset \mathbb{S}^{n+1}$ is a rotational minimal compact hypersurface, then* $\operatorname{ind}(\mathcal{M}^n) \geq 3n+5$ *and there are at least two positive eigenvalues of the Laplace operator smaller than n. More precisely, if m and l are the relative prime integers in* (2-4), *then*

- $l \geq 2,\ m \geq 3,$

- $\operatorname{ind}(\mathcal{M}^n) \geq (2l-1)n + (2m-1)$ *and,*
- *there are exactly* $2l-2$ *positive eigenvalues of the Laplacian smaller than* n.

Proof. As pointed out in Remark 2.1, the positive integer l that satisfies the equation $\theta(T) = 2\pi\frac{l}{m}$ must be greater than 1. We also have that $m \geq 3$. Recall that Otsuki has shown that, for any c, the integral that defines $\theta(T)$ is a number between π and $\sqrt{2}\pi$. In this proof we will be using the notation for $\lambda(k, j)$ and $\tilde{\lambda}(k, j)$ introduced in Theorem 3.3. More precisely, the eigenvalues of the operator $K_{\Delta,k}$ are

$$\lambda(k, 1) < \lambda(k, 2) \leq \lambda(k, 3) \leq \cdots$$

and the eigenvalues of the operator $K_{J,k}$ are

$$\tilde{\lambda}(k, 1) < \tilde{\lambda}(k, 2) \leq \tilde{\lambda}(k, 3) \leq \cdots$$

A direct verification shows that the functions $f_1(t) = \sqrt{1-r^2(t)}\cos(\theta)$ and $f_2(t) = \sqrt{1-r^2(t)}\sin(\theta)$ satisfy the equation $K_{\Delta,1}(f_i) + nf_i = 0$, for $i = 1, 2$. Since the functions f_1 and f_2 have $2l$ zeroes in the interval $[0, mT)$ then, by Theorem 1.2 we conclude that $n = \lambda(1, 2l) = \lambda(1, 2l+1)$. Therefore, the eigenvalues of $K_{\Delta,1}$ are

$$\lambda(1, 1) = 0 < \lambda(1, 2) \leq \lambda(1, 3) < \cdots < \lambda(1, 2l) = \lambda(1, 2l+1) = n < \cdots .$$

This shows the existence of $2l-2$ positive eigenvalues (counted with multiplicity) of the Laplace operator smaller than n. A direct verification shows that $r(t)$ satisfies the equation $K_{\Delta,2}(r) + nr = 0$. Since $r(t)$ is positive, then $\lambda(2, 1) = n$. Therefore all the eigenvalues of the Laplace operator smaller than n comes from the operators $K_{\Delta,1}$. We conclude that there are exactly $2l-2$ positive eigenvalues of the Laplacian smaller than n. Let us show the inequality for the stability index. A direct verification shows that the function r' satisfy the equation $K_{J,1}(r') = 0$. The numbers of zeroes of r' in the interval $[0, mT)$ is $2m$. By Theorem 1.2 we conclude that either $0 = \tilde{\lambda}(1, 2m)$ or $0 = \tilde{\lambda}(1, 2m+1)$ Therefore the operator $K_{J,1}$ has at least $2m-1$ negative eigenvalues. Let us analyze the spectrum of the operator $K_{J,2}$. A direct verification shows that the functions

$$f_3(t) = \frac{-c^{-n/2}r^{1-n}\cos(\theta) + rr'\sin(\theta)}{\sqrt{1-r^2(t)}} \quad \text{and} \quad f_4(t) = \frac{c^{-n/2}r^{1-n}\sin(\theta) + rr'\cos(\theta)}{\sqrt{1-r^2(t)}}$$

satisfy the equation $K_{J,2}(f_i) = 0$ for $i = 3, 4$. The numbers of zeroes f_3 and f_4 in the interval $[0, mT)$ is at least $2l$. Let us check this property for the function $f_3(t)$. We have that $\theta(t)$ changes from 0 to $2\pi l$ when t moves from 0 mT. Let $t_1, t_2, \ldots, t_{2l-1}$ be increasing values of t such that $\theta(t_i) = i\pi$. Recall that $\theta(0) = 0$ and $\theta(mT) = 2\pi l$.

Since

$$f_3(0) = -\frac{c^{-n/2} r(0)^{1-n}}{\sqrt{1-r(0)^2}} < 0 \quad \text{and} \quad f_3(t_1) = \frac{c^{-n/2} r(0)^{1-n}}{\sqrt{1-r(0)^2}} > 0,$$

there is at least one zero of f_3 between 0 and t_1. The same argument shows that there must be at least one zero of $f_3(t)$ between t_1 and t_2. By continuing with this argument we conclude that f_3 must have at least $2l$ zeroes. A similar argument shows that f_4 has also at least $2l$ zeroes. By Theorem 1.2 we conclude that $0 = \tilde{\lambda}(2, 2k) = \tilde{\lambda}(2, 2k+1)$, where $2k$ is the number of zeroes of f_4 and f_5 on the interval $[0, mT)$. Recall that $k \geq l$. Therefore the operator $K_{J,2}$ has at least $2l-1$ negative eigenvalues. Since the multiplicity of the eigenvalue $\alpha_2 = (n-1)$ of the Laplace operator on S^{n-1} is n, we conclude that $\operatorname{ind}(\mathcal{M}) \geq (2l-1)n + (2m-1)$. □

Remark 3.6. There is a geometrical reason for the functions r' and f_i, $i = 1, 2, 3, 4$ to satisfy the respective Hill's equation mentioned in the proof of the previous theorem. If we write the immersion ϕ in (2-1) as $(\phi_1, \ldots, \phi_{n+2})$ and the Gauss map ν in (2-6) as $(\nu_1, \ldots, \nu_{n+2})$ then, the functions f_1 and f_2 are the last two coordinates of the immersion ϕ and, in general, all the coordinates of an immersion of a minimal hypersurface of $\mathbb{S}^{n+1}$ are eigenfunctions of the Laplacian associated with the eigenvalue n. The function r' is the simplification of the expression $\phi_{n+2}\nu_{n+1} - \phi_{n+1}\nu_{n+2}$ and in general all the functions of the form $\nu_i\phi_j - \nu_j\phi_i$ are either the zero function or they are eigenfunctions of the stability operator associated with the eigenvalue 0. Finally, we have that f_4 and f_5 are factors of the expression $\phi_1\nu_{n+1} - \phi_{n+1}\nu_1$ and $\phi_1\nu_{n+2} - \phi_{n+2}\nu_1$.

Theorem 3.7. *Let $M \subset \mathbb{S}^{n+1}$ be a rotational CMC compact hypersurface described in* (2-1) *with $H \geq 0$. If m and l are the relative prime integers in* (2-4) *then, the number of negative eigenvalues of the Jacobi operator counted with multiplicity is at least* $(2l-1)n + (2m-1)$

Proof. The proof is similar to the one presented for the minimal case. A direct verification shows that $J_{J,0}[r']$ vanishes and the numbers of zeroes of r' in the interval $[0, mT)$ is $2m$. By Theorem 1.2 we conclude that either $0 = \tilde{\lambda}(1, 2m)$ or $0 = \tilde{\lambda}(1, 2m+1)$. Therefore the operator $K_{J,1}$ has at least $2m-1$ negative eigenvalues. Let us analyze the spectrum of the operator $K_{J,2}$. A direct verification shows that the functions

$$f_1(t) = \frac{c^{-n/2} r(t)\big(c^{n/2}\big(H\cos(\theta(t)) - r'(t)\sin(\theta(t))\big) + r(t)^{-n}\cos(\theta(t))\big)}{\sqrt{1-r(t)^2}}$$

and

$$f_2(t) = \frac{c^{-n/2} r(t)\big(c^{n/2}\big(r'(t)\cos(\theta(t)) + H\sin(\theta(t))\big) + r(t)^{-n}\sin(\theta(t))\big)}{\sqrt{1-r(t)^2}}$$

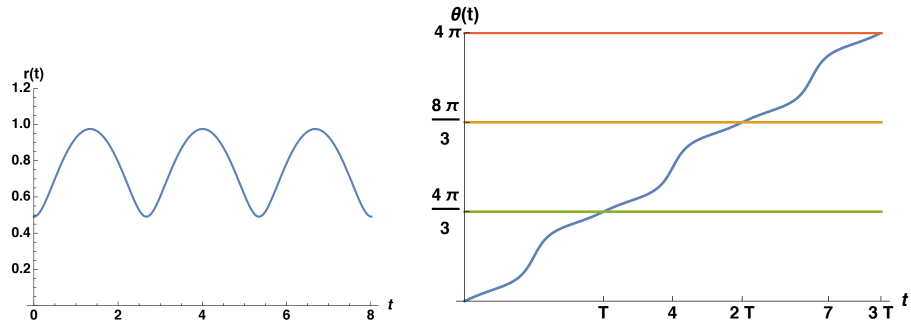

Figure 1. On the left we have the graph of the solution $r(t)$ for the value $ac = 2.8284247911397589$. On the right we have the graph of the function $\theta(t)$ defined in (2-3).

satisfy the equation $K_{J,2}(f_i) = 0$ for $i = 1, 2$. The numbers of zeroes f_1 and f_2 in the interval $[0, mT)$ is at least $2l$. By Theorem 1.2 we conclude that $0 = \tilde{\lambda}(2, 2k) = \tilde{\lambda}(2, 2k+1)$, where $2k$ is the number of zeroes of f_1 and f_2 on the interval $[0, mT)$. Recall that $k \geq l$. Therefore the operator $K_{J,2}$ has at least $2l-1$ negative eigenvalues. Since the multiplicity of the eigenvalue $\alpha_2 = (n-1)$ of the Laplace operator on S^{n-1} is n, we conclude that the number of negative eigenvalue of the operator J is greater than $(2l-1)n + (2m-2)$. $\square$

4. An example that illustrates the method

In this section we pick an explicit rotational 3-dimensional minimal hypersurface $M \subset \mathbb{S}^4$ and we compute the first three eigenvalues of the Laplacian and its stability index.

Construction of the particular example. By the intermediate value theorem, it is easy to see that there is a value of c near $ac = 2.82842479911$ such that the function $r(t)$ has period T near $aT = 2.6722005616$ and $\theta(T) = \frac{4\pi}{3}$. See Equation (2-4). In this case $l = 2$, $m = 3$ and our manifold M is defined by this choice of c. Recall that the immersion $\phi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^4$ is given by

$$\phi(y,t) = \Big(r(t)y, \sqrt{1-r(t)^2}\cos(\theta(t)), \sqrt{1-r(t)^2}\sin(\theta(t))\Big). \tag{4-1}$$

Figure 1 shows the solution $r(t)$ that produces the manifold M, and Figure 2 shows the profile curve of the rotational manifold M.

Computing the first three eigenvalues of the Laplacian. In order to use the oscillation theorem (Theorem 1.1) we notice that making $u = rz$ we obtain (see also Lemma 3.4)

$$K_{\Delta,k}[z] = z'' + 2\frac{r'}{r}z' - \frac{\alpha_k}{r^2}z = \frac{1}{r}\left(u'' + \left(1 - \frac{2}{c^3 r^6} - \frac{\alpha_k}{r^2}\right)u\right).$$

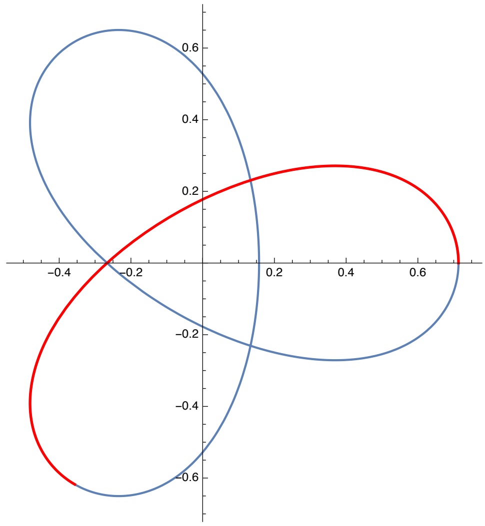

Figure 2. Profile curve of the M. This curve is parametrized by $\left(\sqrt{1-r(t)^2}\cos(\theta(t)), \sqrt{1-r(t)^2}\sin(\theta(t))\right)$. The red piece represent the curve when t moves from 0 to T. Recall that the period of the immersion is $3T$.

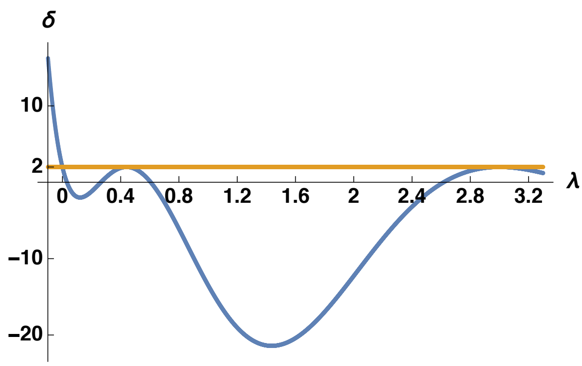

Figure 3. Graph of the function $\delta(\lambda)$. The roots of the equation $\delta(\lambda) = 2$ give us eigenvalues of the Laplacian of the form $\lambda(1, j)$ defined on Theorem 3.3.

Therefore $\lambda(k, i)$ is an eigenvalue of the operator $K_{\Delta,k}$ if and only if $\lambda(k, i)$ is an eigenvalue of the operator

$$\bar{K}_{\Delta,k}[u] = u'' + \left(1 - \frac{2}{c^3 r^6} - \frac{\alpha_k}{r^2}\right)u.$$

For $\alpha_1 = 0$, Figure 3 shows the discriminant function $\delta(\lambda)$ for the operator $\bar{K}_{\Delta,1}$.

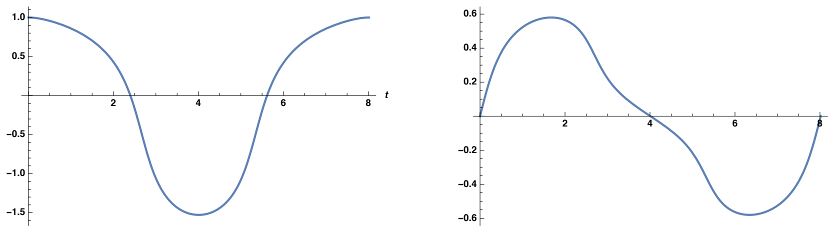

Figure 4. Graph of two solutions ξ_1 and ξ_2 of the equation $K_{\Delta,1}[z]+0.4404z=0$.

Figure 3 was made by taking 3400 values of λ between -0.1 and 3.3, one every 0.001. For each value of λ we solve two differential equations to find the functions $z_1(t,\lambda)$ and $z_2(t,\lambda)$ defined in Theorem 1.1. Once we have $z_1(t,\lambda)$ and $z_2(t,\lambda)$ we computed $\delta(\lambda)$. The crossing of the graph of $\delta(\lambda)$ with the horizontal line $y=2$ at $\lambda(1,1)=0$ was expected since $z(t)=1$ is an eigenfunction and the crossing at $\lambda=3$ with multiplicity 2 was expected because the last two coordinates of the immersion, the functions $\sqrt{1-r^2}\cos(\theta)$ and $\sqrt{1-r^2}\sin(\theta)$, are eigenfunctions of the Laplacian of M, see [Simons 1968]. Regarding the crossing near 0.44 we can check that $|\delta(0.4404)-2|$ is smaller than 10^{-6}. Figure 4 shows two linearly independent solutions ξ_1 and ξ_2 of the equation $K_{\Delta,1}[z]+0.4404z=0$. All together we have 3 eigenvalues of $K_{\Delta,1}$ smaller than 3, which agrees with Theorem 3.5.

We will move now to study the operator $K_{\Delta,2}$. Since the coordinates of the immersion ϕ are eigenfunction of the Laplacian we have that the function $r(t)$ satisfies the equation $K_{\Delta,2}(r(t))=-3r(t)$. The previous equation also follows from (2-5). Since $r(t)$ is positive, $\lambda(2,1)=3$ is the first eigenvalue of $K_{\Delta,2}$ and it has multiplicity 1.

Remark 4.1. Since the sequence α_k is increasing, the sequence $\lambda(k,1)$ is also increasing.

From the previous remark we deduce that all other eigenvalues of the Laplacian of M are greater than 3.

Computing the negative eigenvalues of the Jacobi operator. Once again we use the oscillation theorem (Theorem 1.1). The change of variables $u=rz$ gives us (see also Lemma 3.4)

$$K_{J,k}[z]=z''+2\frac{r'}{r}z'+\left(\frac{6}{c^3r^6}+3-\frac{\alpha_k}{r^2}\right)z=\frac{1}{r}\left(u''+\left(4+\frac{4}{c^3r^6}-\frac{\alpha_k}{r^2}\right)u\right).$$

Similar to the case of the Laplacian operator, we can compute the eigenvalues of the Jacobi operator by computing the eigenvalues of the operator

$$\overline{K}_{J,k}[u]=u''+\left(4+\frac{4}{c^3r^6}-\frac{\alpha_k}{r^2}\right)u.$$

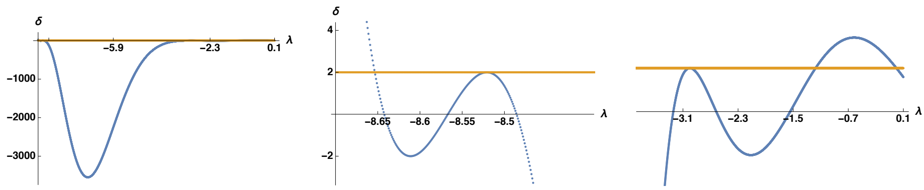

Figure 5. Graph of the function $\delta(\lambda)$ for the operator $\overline{K}_{J,1}$. The roots of the equation $\delta(\lambda) = 2$ give us eigenvalues of the Jacobi operator of the form $\lambda(1, j)$. The graph on the center and on the right shows the function δ on smaller intervals

Figure 5 shows the discriminant δ for the operator $\overline{K}_{J,1}$. A closer look at the function tell us that the negative solutions of the equation $\delta(\lambda) = 2$ are on the intervals $[-8.7, -8.5]$ and $[-3.1, 0]$.

For the first eigenvalue of the Jacobi operator, it is easy to use the intermediate value theorem to show that $\tilde{\lambda}(1, 1) = -8.6534$ within an error of 10^{-4}. This eigenvalue has multiplicity one and Figure 6 shows a nonzero periodic eigenfunction of the operator $K_{J,1}$. For the next value we have that $|\delta(-8.53078) - 2| < 10^{-5}$. For this value of λ the two fundamental solutions of the equation $K_{J,1}[z] + \lambda z = 0$ are shown in Figure 7. The next eigenvalue is -3 with multiplicity 2, this eigenvalue was expected due to the fact that the coordinate functions of the Gauss map are eigenfunctions of the Jacobi operator. For the next eigenvalue we have that $|\delta(-1.1749673) - 2| < 10^{-5}$. The existence of an eigenvalue near -1.17496 with multiplicity one is given by the Intermediate Value Theorem, see Figure 5. We know that this is the last negative eigenvalue because 0 is known to be an eigenvalue of $K_{J,1}$.

We now study the operator $K_{J,2}$. Figure 8 shows the discriminant δ for the operator $\overline{K}_{J,2}$. We can directly check that $K_{J,2}(r^{-2}) = 3r^{-2}$. Since $r(t)$ is positive, then we have that -3 is the first eigenvalue of $K_{J,2}$ with multiplicity one. Since we have that $|\delta(-2.5596) - 2| < 10^{-5}$, then there is an eigenvalue of $K_{J,2}$ with multiplicity 2 near -2.5596. We can check that the first eigenvalue of the operator $K_{J,3}$ is close to 4.3484453. Therefore we have gotten all negative eigenvalues of the Jacobi operator, in summary we have

Remark 4.2. Since the first two eigenvalues of $\mathbb{S}^2$ are 0 with multiplicity 1 and 2 with multiplicity 3, then the stability index of M is 15. Counting with multiplicity we have that the first eigenvalue of the stability operator J on M is near -8.6534 and has multiplicity one. We have two eigenvalues near -8.52078, it could be only one with multiplicity 2. We have -3 with multiplicity 5. Even though this was known, it is interesting to point out that the multiplicity is 5 because -3 is an eigenvalue with multiplicity 2 of $K_{J,1}$ and -3 is an eigenvalue of multiplicity 1 of

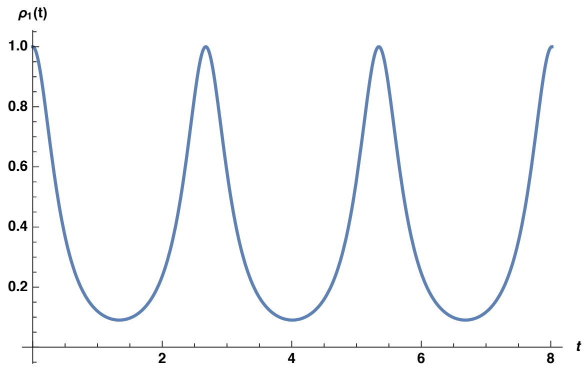

Figure 6. Graph of an eigenfunction associated with $\tilde{\lambda}(1, 1) = -8.65\ldots$. This function also represents the first eigenfunction of the stability operator.

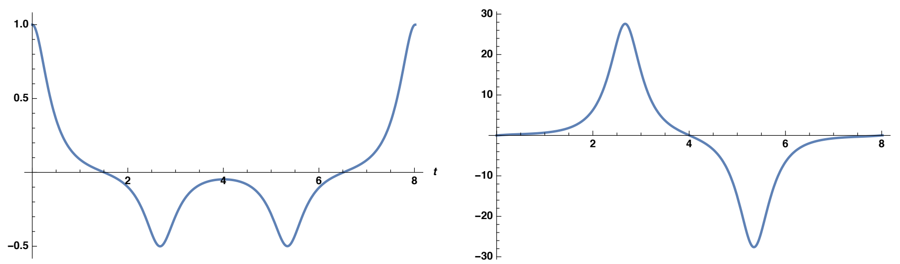

Figure 7. Two solutions of the equation $K_{J,1}[z] - 8.53078z = 0$.

$K_{J,2}$, this multiplicity one needs to be multiply by 3 because the eigenvalue $\alpha_2 = 2$ of the Laplace operator on $\mathbb{S}^2$ has multiplicity 3. The next eigenvalues are six near -2.5596, they are either two with multiplicity 3 or one with multiplicity 6. The last negative eigenvalue is one near -1.17496.

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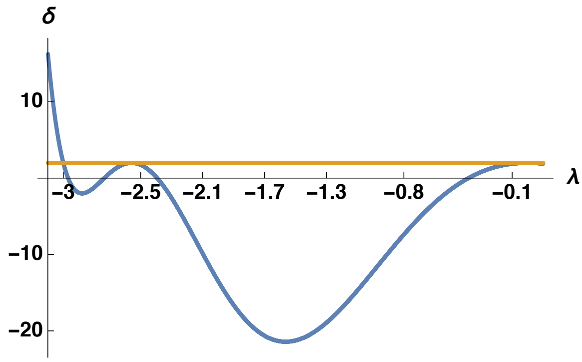

Figure 8. Graph of the discriminant of the operator $K_{J,2}$.

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