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G-symmetric monoidal categories of modules over equivariant commutative ring spectra

Andrew J. Blumberg and Michael A. Hill

We describe the multiplicative structures that arise on categories of equivariant modules over certain equivariant commutative ring spectra. Building on our previous work on $\mathcal{N}_\infty$ ring spectra, we construct categories of equivariant operadic modules over $\mathcal{N}_\infty$ rings that are structured by equivariant linear isometries operads. These categories of modules are endowed with equivariant symmetric monoidal structures, which amounts to the structure of an "incomplete Mackey functor in homotopical categories". In particular, we construct internal norms which satisfy the double coset formula. One application of the work of this paper is to provide a context in which to describe the behavior of Bousfield localization of equivariant commutative rings. We regard the work of this paper as a first step towards equivariant derived algebraic geometry.

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1. Introduction

Stable homotopy theory has been revolutionized over the last twenty years by the development of symmetric monoidal categories of spectra [Elmendorf et al. 1997; Mandell et al. 2001; Hovey et al. 2000]. Commutative monoids in these categories model E_∞ ring spectra. Arguably the most important consequence of this machinery is the ability to have tractable point-set models for homotopical categories of modules over an E_∞ ring spectrum R. In the equivariant setting, analogous symmetric monoidal categories of G-spectra have been constructed, most notably the category of orthogonal G-spectra [Mandell and May 2002]. Once again commutative monoids model E_∞ ring spectra and so there are good point-set models for homotopical categories of modules over such an equivariant E_∞ ring spectrum.

Modules over a commutative ring orthogonal G-spectrum R form a "G-symmetric monoidal category" [Hill and Hopkins 2016]. Roughly speaking, for each G-set T, we have an internal norm in the category of R-modules; for an orbit G/H, the internal norm is precisely the composite of the R-relative norm ${}_R N_H^G$ and the forgetful functor from R-modules to $\iota_H^* R$-modules. These internal norms are compatible with disjoint unions of G-sets and with restrictions to subgroups, and if the set has a trivial action and cardinality n, then we recover the smash power functors $X \mapsto X^n$.

However, in contrast to the nonequivariant setting, there are many possible notions of E_∞ ring spectra when working over a nontrivial finite group G. The commutative monoids in orthogonal G-spectra are just one end of the spectrum of possible multiplicative structures. We previously described this situation in detail by explaining how such multiplications can be structured by $\mathcal{N}_\infty$ operads [Blumberg and Hill 2015]. Roughly speaking, just as a commutative ring is characterized by compatible multiplication maps $R^{\wedge n} \to R$ as n varies over the natural numbers, a commutative G-ring is characterized by compatible equivariant multiplication maps $R^{\wedge T} \to R$, where here T is a G-set. The $\mathcal{N}_\infty$ operads structure which such equivariant norms exist for a given commutative ring, expressed in terms of compatible families of subgroups of $G \times \Sigma_n$. Specifically, associated to an $\mathcal{N}_\infty$ operad $\mathcal{O}$ there is a coefficient system $\mathcal{C}(\mathcal{O})$ which controls the "admissible" G-sets T for which equivariant multiplications exist. The commutative monoids in orthogonal G-spectra correspond to the "complete" $\mathcal{N}_\infty$ operads which permit all norms.

In this paper, we turn to the study of the equivariant symmetric monoidal structure on categories of operadic modules associated to algebras over particular $\mathcal{N}_\infty$ operads: the linear isometries operads determined by a (possibly incomplete) G-universe U. Here the admissible sets for U will play a second role; for an $\mathcal{O}$-algebra R, the admissible sets determine additional structure on the underlying symmetric monoidal category of R-modules. Specifically, for each admissible G-set T, we have a internal norm in the category of R-modules for an $\mathcal{O}$-algebra R.

In order to describe this structure, it is convenient to instead consider the collection of categories of modules over $\iota_H^* R$, where $H \subseteq G$ is a closed subgroup of G and ι_H^* is the forgetful functor. The extra structure on the category of R-modules then is encoded in functors

$$\iota_H^* : \mathrm{Mod}_{\iota_K^* R} \to \mathrm{Mod}_{\iota_H^* R} \quad \text{and} \quad {}_{(\iota_K^* R)}N_H^K : \mathrm{Mod}_{\iota_H^* R} \to \mathrm{Mod}_{\iota_K^* R}$$

for $H \subset K \subset G$ that assemble into a kind of "incomplete Mackey functor" of homotopical categories. The internal norms arise from the composite functors $N_H^G \iota^* H(-)$, extended to arbitrary G-sets T by decomposing T into a disjoint union of orbits G/H and smashing together the corresponding composites. The compatibility conditions in particular express the double coset formula.

More precisely, we have the following functors:

Theorem 1.1. *Let G be a finite group and U a G-universe. Let R be an algebra in orthogonal G-spectra over $\mathscr{L}_U$, the linear isometries operad structured by U. Then for each $H \subset G$ there exists a symmetric monoidal model category $\mathcal{M}_{\iota_H^* R}$ of $\iota_H^* R$-modules. For each $H \subset K \subset G$ such that K/H is an admissible K-set for U, there exist homotopical functors*

$$_{(\iota_K^* R)}N_{H,\iota_H^* U}^{K,\iota_K^* U} : \mathcal{M}_{\iota_H^* R} \to \mathcal{M}_{\iota_K^* R} \quad \textit{and} \quad \iota_H^* : \mathcal{M}_{\iota_K^* R} \to \mathcal{M}_{\iota_H^* R}.$$

The internal norms now arise from these functors:

Definition 1.2. Let G be a finite group and U be a G-universe. For an H-set T, writing $T = H/K_1 \amalg H/K_2 \amalg \cdots \amalg H/K_m$, define

$$_{(\iota_H^* R)}N^T : \mathcal{M}_{\iota_H^* R} \to \mathcal{M}_{\iota_H^* R}$$

by the formula

$$_{(\iota_H^* R)}N^T X = \left(_{(\iota_H^* R)}N_{K_1}^H \iota_{K_1}^* X\right) \wedge_{\iota_H^* R} \left(_{(\iota_H^* R)}N_{K_2}^H \iota_{K_2}^* X\right) \wedge_{\iota_H^* R} \cdots \wedge_{\iota_H^* R} \left(_{(\iota_H^* R)}N_{K_m}^H \iota_{K_m}^* X\right).$$

More generally, define

$$_R N^T : \mathcal{M}_R \to \mathcal{M}_R$$

by the formula

$$_R N^T X = G_+ \wedge_H \left(_{(\iota_H^* R)}N^T X\right).$$

The equivariant symmetric monoidal structure on $\mathcal{M}_R$ is encoded by the following relations between the internal norms and the forgetful functors:

Theorem 1.3. *Let G be a finite group and U be a G-universe.*

(1) *For $H_1 \subseteq H_2 \subseteq H_3 \subseteq G$, there are natural homeomorphisms*

$$N_{H_2}^{H_3} N_{H_1}^{H_2} \cong N_{H_1}^{H_3} \quad \textit{and} \quad \iota_{H_1}^* \iota_{H_2}^* \cong \iota_{H_1}^*$$

that descend to the derived category,

(2) *For any H-sets T_1 and T_2, there are natural homeomorphisms $N^{T_1\times T_2}X \cong N^{T_1}N^{T_2}X$ for each X that descend to the derived category when T_1 and T_2 are admissible, and*

(3) *For an admissible H-set T, the derived composite $\iota_K^* N^T M$ is naturally equivalent to $N^{\iota_K^* T}\iota_K^* M$.*

The last of these relations is a version of the double coset formula.

When $G = e$, the structure described by Theorem 1.3 is simply the usual symmetric monoidal structure on orthogonal spectra; the functors N^T for a set T are just the smash powers $X^{\wedge|T|}$. When U is the complete universe, this structure is precisely the G-symmetric monoidal structure on R-modules obtained by choosing a model of R that is a commutative monoid in orthogonal G-spectra.

Note that we have avoided trying to precisely formulate the notion of an incomplete Mackey functor of homotopical categories here, choosing instead to explicitly write out the structure and some of the coherences. However, if we are willing to pass to the homotopy category, we can state the following result.

Corollary 1.4. *Let R be an algebra in orthogonal G-spectra over $\mathscr{L}_U$. Let $\mathcal{B}_{G,U}$ denote the bicategory of spans of the admissible sets for $\mathscr{L}_U$. There exists a 2-functor from $\mathcal{B}_{G,U}$ to the 2-category of triangulated categories, exact functors, and natural isomorphisms that takes an admissible set G/H to $\mathcal{M}_{\iota_H^* R,\iota_H^* U}$.*

However, the coherences necessary for the definition of an incomplete Mackey functor at the level of homotopical categories is most easily handled using the formalism of ∞-categories; we expect such a treatment to come from the forthcoming work of [Barwick et al. 2016]. (See also [Bohmann and Osorno 2015] for a treatment of equivariant permutative categories from this kind of perspective. A different approach to equivariant permutative categories is described in [Guillou and May 2017].)

One of the applications of our work is the construction of strict point-set models of N_∞ ring spectra. Specifically, let $\mathbb{S}_G$ be the equivariant sphere spectrum, regarded as an $\mathscr{L}_U$ algebra. Then we have the following straightforward consequence of the proof of Theorem 1.1.

Corollary 1.5. *The category of commutative monoid objects in $\mathcal{M}_{\mathbb{S}_G}$ is equivalent to the category of N_∞ algebras structured by $\mathscr{L}_U$.*

More generally, for an N_∞ algebra R, we obtain a description of N_∞ R-algebras.

Corollary 1.6. *Let R be an N_∞ algebra structured by $\mathscr{L}_U$. The category of commutative monoid objects in $\mathcal{M}_R$ is equivalent to the category of N_∞ R-algebras structured by $\mathscr{L}_U$.*

These corollaries are particularly useful in the context of equivariant Bousfield localization. In their study of the multiplicative properties of equivariant Bousfield localization, the second author and Hopkins showed that localization of an N_∞ ring spectrum can change the universe that structures the multiplication [Hill and Hopkins 2016]. Specifically, [Hill and Hopkins 2016, Theorem 6.3] shows that a Bousfield localization L of orthogonal G-spectra takes $\mathscr{L}_U$ algebras to $\mathscr{L}_U$ algebras precisely when the category of L-acyclics is closed under norms for the indexing system determined by U. Therefore, we obtain the following result.

Theorem 1.7. *Let A be a commutative monoid in $M_{\mathbb{S}^G_U}$, where $\mathbb{S}^G_U$ denotes the sphere spectrum regarded as an $\mathscr{L}_U$ algebra in orthogonal G-spectra. Let L be a Bousfield localization functor with L-acyclics closed under norms specified by the indexing system for a universe U'. Suppose that U'' is a universe with corresponding indexing system contained in the indexing system obtained as the intersection of U and U'. Then LR is a commutative monoid object in $M_{\mathbb{S}^G_{U''}}$.*

In order to explain the restriction to $\mathcal{N}_\infty$ operads that can be modeled as linear isometries operads, we need to explain the strategy of proof for Theorem 1.1. Our approach is to adapt the strategy of EKMM [Elmendorf et al. 1997] to study operadic multiplications on G-spectra. Let $\mathcal{S}p_G$ denote the category of orthogonal G-spectra on a complete universe. Fix a different (possibly incomplete) G-universe U. Then there is a monad $\mathbb{L}_U$ on $\mathcal{S}p_G$, specified by the formula

$$X \mapsto \mathscr{L}(U, U)_+ \wedge X,$$

where $\mathscr{L}(U, U)$ is the G-space of nonequivariant linear isometries from U to U (with G acting by conjugation).

The category $\mathcal{S}p_G[\mathbb{L}_U]$ of $\mathbb{L}_U$-algebras has a model structure that is Quillen equivalent to the standard model structures on $\mathcal{S}p_G$. Moreover, it has a new symmetric monoidal product $\wedge_U$ such that the underlying orthogonal G-spectrum of $X \wedge_U Y$ is equivalent to $X \wedge Y$. But now monoids and commutative monoids for $\wedge_U$ are precisely (non)-symmetric algebras for the G-linear isometries operad for U. Just as in the category of spectra, we can restrict to the unital objects in $\mathcal{S}p_G[\mathbb{L}_U]$ to obtain a symmetric monoidal category $G\mathcal{S}_U$. All of these categories can be equipped with symmetric monoidal model category structures. Using these symmetric monoidal model categories, we construct symmetric monoidal module categories for an $\mathcal{N}_\infty$ ring R structured by the G-linear isometries operad for U.

We expect that Theorem 1.1 is true more generally for any $\mathcal{N}_\infty$ operad, but it is difficult to obtain control on categories of operadic modules over operads other than the linear isometries operad using point-set techniques. In fact, a substantial part of the work of this paper involves verification of delicate point-set facts about the linear isometries operad that are simply not true for an arbitrary N_∞ operad, just as in [Elmendorf et al. 1997]. Unfortunately, as we explain in [Blumberg and Hill 2015, Theorem 4.24], there are equivariant operads which arise from "little disks" constructions that are not equivalent to equivariant linear isometries operads for any universe. Again, we expect that it is more tractable to handle these sorts of homotopical questions in the ∞-categorical setting; specifically, working with equivariant ∞-operads structured over the nerve of distinguished subcategories of the category of finite G-sets.

One benefit of our approach to Theorem 1.1 is that our technical results about the equivariant linear isometries operad validate the multiplicative theory of the equivariant version of EKMM spectra. Although [Elmendorf et al. 1997, 0.1] famously asserts that all of the work of that volume holds mutatis mutandis when assuming that a compact Lie group G acts, verifying such a theorem requires some subtle checks about the behavior of the linear isometries operad (most notably Theorem A.9); and [Elmendorf and May 1997], which amongst other endeavors attempts to justify some of these properties, contains a critical error (in [Elmendorf and May 1997, Theorem 1.2]). As such, our work here supports prior applications of the equivariant category of S-modules, notably [Greenlees and May 1997].

Our interest in Theorem 1.1 comes in large part from examples arising from localization. As explained above, localization of an N_∞ ring spectrum can change the universe that structures the multiplication [Hill and Hopkins 2016]. In Section 6, we discuss a number of examples of this kind that arise in applications. More generally, the multiplicative behavior of localization of equivariant commutative ring spectra has significant consequences for the foundations of equivariant derived algebraic geometry.

Specifically, the possible loss of norms that occurs implies that there is not necessarily a "genuine" affine scheme associated to a commutative ring orthogonal G-spectrum when we work with the Zariski topology. Work of Nakaoka [2012] shows that something similar is true for Tambara functors: there does not exist a sheaf of Tambara functors on the Zariski site of a Tambara functor.

However, by restriction of structure, every equivariant commutative ring spectrum R is also an algebra over $\mathscr{L}_{\mathbb{R}^\infty}$, the linear isometries operad for a trivial universe. Bousfield localization always preserves the property of being an algebra over $\mathscr{L}_{\mathbb{R}^\infty}$, so in particular, we do have a sheaf of such rings in the Zariski topology. Therefore, using the work of this paper we can define equivariant derived affine schemes (and then more general derived schemes by gluing) in this fashion. More generally, Theorem 1.7 explains the situations when we can expect more general affines. We intend to return to the study of equivariant derived schemes in a subsequent paper.

As a concrete example of this circle of ideas, let $\mathcal{X} \to \mathcal{Y}$ be a Galois cover of stacks with Galois group G, and let $\mathcal{Y} \to \mathcal{M}_{\mathrm{Ell}}$ be an étale map to the moduli stack of elliptic curves. We can evaluate the Goerss–Hopkins–Miller sheaf of topological modular forms $\mathcal{O}^{\text{top}}$ on these étale maps, producing commutative ring spectra and maps

$$\text{TMF}(\mathcal{Y}) \to \text{TMF}(\mathcal{X}).$$

The G-action on $\mathcal{X}$ gives a G-action on TMF($\mathcal{X}$), and we can then view this as a genuine commutative equivariant ring spectrum by pushing forward to a complete universe (see [Hill and Meier 2017] for a related discussion). We would like to be able to understand the category of equivariant TMF($\mathcal{X}$)-modules in algebro-geometric terms. The machinery presented in this paper is an essential tool in this endeavor, making it possible to make sense of sheaves of modules on the Zariski site.

2. Review of $\mathcal{N}_\infty$ operads

In this section, we review the framework for describing equivariant commutative ring spectra that we will work with in the paper. We refer the reader to [Blumberg and Hill 2015] for a more detailed discussion.

Let G be a finite group and let $G\mathcal{S}$ denote the category of orthogonal G-spectra structured by a complete universe and with morphisms all (not necessarily equivariant) maps. We will tacitly suppress notation for the "additive" universe implicit in the definition of $G\mathcal{S}$, as we are focused on multiplicative phenomena. Recall that the category $G\mathcal{S}$ is a complete and cocomplete closed symmetric monoidal category under the smash product $\wedge$ with unit the equivariant sphere spectrum $\mathbb{S}_G$. We will write $F(-,-)$ for the internal mapping G-spectrum in $G\mathcal{S}$. The category $G\mathcal{S}$ is enriched over based G-spaces and has tensors and cotensors; for X an object of $G\mathcal{S}$, the tensor with a based G-space A is given by the smash product $A \wedge X$ and the cotensor by the function spectrum $F(A, X)$.

The enrichment of $G\mathcal{S}$ means that we can regard operads in G-spaces as acting on objects of $G\mathcal{S}$ via the addition of a G-fixed disjoint basepoint and the tensor. Given a G-operad in spaces, recall the following definition from [Blumberg and Hill 2015, Definition 3.7].

Definition 2.1. An $\mathcal{N}_\infty$ operad is a G-operad $\mathcal{O}$ such that:

(1) The space $\mathcal{O}_0$ is G-contractible.

(2) The action of Σ_n on $\mathcal{O}_n$ is free.

(3) $\mathcal{O}_n$ is a universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{1\}$.

For any $\mathcal{N}_\infty$ operad $\mathcal{O}$, there is an associated category $\mathcal{O}$-Alg of $\mathcal{O}$-algebras in $G\mathcal{S}$. We will be particularly interested in the algebras associated to the G-linear isometries operads. Fix a possibly incomplete universe U of finite-dimensional G-representations; we adopt the standard convention that U contains a trivial representation and each of its finite-dimensional subrepresentations infinitely often. We do not assume any relationship between U and the "additive" universe that arises in the definition of $G\mathcal{S}$.

Definition 2.2. The G-linear isometries operad $\mathscr{L}_U$ has n-th space

$$\mathcal{L}_U(n) = \mathscr{L}(U^n, U),$$

the G-space of nonequivariant linear isometries $U^n \to U$ equipped with the conjugation action. The distinguished element $1 \in \mathcal{L}_U(1)$ is the identity map and the operad structure maps are induced by composition and direct sum.

Recall from [Blumberg and Hill 2015, Theorem 4.24] that the G-linear isometries operads do not always describe all of the possible $\mathcal{N}_\infty$ operads. Nonetheless, they do capture many examples of interest, in particular including the trivial and complete multiplicative universes.

One of the major themes of our previous study of $\mathcal{N}_\infty$ operads was that the essential structure encoded by an operad $\mathcal{O}$ is the collection of *admissible sets*. We now review the relevant definitions from [Blumberg and Hill 2015, §3].

Definition 2.3. A *symmetric monoidal coefficient system* is a contravariant functor $\underline{\mathcal{C}}$ from the orbit category of G to the category of symmetric monoidal categories and strong symmetric monoidal functors. The *value at H* of a symmetric monoidal coefficient system $\underline{\mathcal{C}}$ is $\underline{\mathcal{C}}(G/H)$, and will often be denoted $\underline{\mathcal{C}}(H)$.

The most important example of a symmetric monoidal coefficient system for us is the coefficient system of finite G-sets.

Definition 2.4. Let $\underline{\mathrm{Set}}$ be the symmetric monoidal coefficient system of finite sets. The value at H is Set^H, the category of finite H-sets and H-maps. The symmetric monoidal operation is disjoint union.

We will associate to every $\mathcal{N}_\infty$ operad a subcoefficient system of $\underline{\mathrm{Set}}$. The operadic structure gives rise to additional structure on the coefficient system.

Definition 2.5. We say that a full subsymmetric monoidal coefficient system $\mathcal{F}$ of $\underline{\mathrm{Set}}$ is *closed under self-induction* if whenever $H/K \in \mathcal{F}(H)$ and $T \in \mathcal{F}(K)$, $H \times_K T \in \mathcal{F}(H)$.

Definition 2.6. Let $\mathcal{C} \subset \mathcal{D}$ be a full subcategory. We say that $\mathcal{C}$ is a *truncation subcategory* of $\mathcal{D}$ if whenever $X \to Y$ is monic in $\mathcal{D}$ and Y is in $\mathcal{C}$, then X is also in $\mathcal{C}$. A truncation subcoefficient system of a symmetric monoidal coefficient system $\underline{\mathcal{D}}$ is a subcoefficient system that is levelwise a truncation subcategory.

In particular, for finite G-sets, truncation subcategories are subcategories that are closed under passage to subobjects and which are closed under isomorphism.

Definition 2.7. An *indexing system* is a truncation subsymmetric monoidal coefficient system $\underline{\mathcal{F}}$ of $\underline{\mathrm{Set}}$ that contains all trivial sets and is closed under self induction and Cartesian product.

One of the main structural theorems about $\mathcal{N}_\infty$ operads [Blumberg and Hill 2015, Theorem 4.17] is that an $\mathcal{N}_\infty$ operad $\mathcal{O}$ determines an indexing system of admissible sets. This connection arises from the standard observation that subgroups Γ of $G \times \Sigma_n$ such that $\Gamma \cap (\{1\} \times \Sigma_n) = \{1\}$ arise as the graphs of homomorphisms $H \to \Sigma_n$, for some $H \subseteq G$.

3. Point-set categories of modules over an $\mathcal{N}_\infty$ algebra

In this section, we describe an approach to constructing categories of modules over $\mathcal{N}_\infty$ algebras that proceeds via a rigidification argument. Of course, for any given $\mathcal{N}_\infty$ operad $\mathcal{O}$ and an $\mathcal{O}$-algebra R in orthogonal G-spectra, we can construct a category of operadic modules over R. However, experience in the nonequivariant case teaches us that for practical work it is extremely convenient to have rigid models of such categories that are equipped with a symmetric monoidal smash product.

Specifically, for each linear isometries operad $\mathcal{O} = \mathcal{L}(U)$, we will construct a symmetric monoidal structure on a category Quillen equivalent to orthogonal G-spectra such that monoids and commutative monoids correspond to $\mathcal{O}$-algebras. We describe how to produce such a structure by adapting the techniques pioneered in the development of the EKMM category of S-modules. See also [Blumberg 2006; Blumberg et al. 2010; Kříž and May 1995; Spitzweck 2001] for other categories in which this kind of approach has been developed. We can then define modules over an $\mathcal{O}$-algebra R in the evident fashion.

We are not able to rigidify algebras and modules over $\mathcal{N}_\infty$ operads which are not equivalent to equivariant linear isometries operads. Although in these cases we can give a homotopical construction of the tensor product of operadic $\mathcal{O}$-modules in terms of the bar construction, we do not have good point-set control.

3.1. ***The point-set theory of $\mathbb{L}_U$-algebras in orthogonal spectra.*** We begin by discussing the point-set details of the category of algebras for a monad obtained from the first part of the equivariant linear isometries operad. Recall that $G\mathcal{S}$ denotes the category of orthogonal G-spectra structured by a complete G-universe. Since the universe implicit in the definition of $G\mathcal{S}$ does not play an essential role in what follows (once the weak equivalences are fixed), we continue to suppress this choice from the notation.

Remark 3.1. It is possible to carry out the work of this paper in the context of an incomplete additive universe on $G\mathcal{S}$; the simplest case arises when the additive and multiplicative universes are the same. We leave this elaboration (and its attendant complications) to the interested reader.

Fix a (possibly incomplete) universe U. Let

$$\mathcal{L}_U(1) = \mathcal{L}(U, U)$$

denote the G-space of linear isometries $U \to U$; i.e., the space of nonequivariant linear isometries $U \to U$ equipped with the conjugation action. More generally, we write $\mathcal{L}_U(n)$ to denote $\mathcal{L}(U^n, U)$, the n-th space of the equivariant linear isometries operad.

Since $G\mathcal{S}$ is tensored over based G-spaces, the formula

$$X \mapsto \mathcal{L}_U(1)_+ \wedge X$$

specifies a monad $\mathbb{L}_U : G\mathcal{S} \to G\mathcal{S}$. The monadic structure maps are induced by the identity element $\mathrm{id}_U \in \mathcal{L}_U(1)$ and the composition

$$\mathcal{L}(U, U) \times \mathcal{L}(U, U) \to \mathcal{L}(U, U).$$

Definition 3.2. Let $G\mathcal{S}[\mathbb{L}_U]$ denote the category of $\mathbb{L}_U$-algebras in $G\mathcal{S}$.

Since the monad $\mathbb{L}_U$ has a right adjoint $F(\mathcal{L}_U(1)_+, -)$, the observation of [Elmendorf et al. 1997, I.4.3] implies that this right adjoint determines a comonad $\mathbb{L}_U^\sharp$ such that the category of coalgebras over $\mathbb{L}_U^\sharp$ is equivalent to $G\mathcal{S}[\mathbb{L}_U]$. As a consequence we conclude the following result about the existence of limits and colimits.

Lemma 3.3. *The category $G\mathcal{S}[\mathbb{L}_U]$ is complete and cocomplete, with limits and colimits created in $G\mathcal{S}$. Similarly, $G\mathcal{S}[\mathbb{L}_U]$ has tensors and cotensors with based G-spaces; the indexed colimits and limits are created in $G\mathcal{S}$.*

The category $G\mathcal{S}[\mathbb{L}_U]$ is equipped with mapping G-spectra $F_{G\mathcal{S}[\mathbb{L}_U]}(-, -)$ defined by the equalizer

$$F(X, Y) \rightrightarrows F(\mathbb{L}_U X, Y),$$

where the maps are induced by the action $\mathbb{L}_U X \to X$ and the adjoint of the composite

$$(\mathbb{L}_U X) \wedge F(X, Y) \cong \mathbb{L}_U(X \wedge F(X, Y)) \to \mathbb{L}_U Y \to Y.$$

Next, we note that any orthogonal G-spectrum can be given a trivial $G\mathcal{S}[\mathbb{L}_U]$ structure. Specifically, in addition to the free $\mathbb{L}_U$-algebra functor

$$\mathcal{L}_U(1)_+ \wedge (-) : G\mathcal{S} \to G\mathcal{S}[\mathbb{L}_U],$$

there is another functor $p^*: G\mathcal{S} \to G\mathcal{S}[\mathbb{L}_U]$ determined by the unique projection map $p: \mathscr{L}_U(1) \to *$; i.e., we can equip any orthogonal G-spectrum X with the trivial structure map

$$\mathscr{L}_U(1)_+ \wedge X \to (*)_+ \wedge X \cong X.$$

We will be most interested in the sphere spectrum $\mathbb{S}_G$ regarded as an $\mathbb{L}_U$-algebra in this fashion. The pullback functor is the right adjoint of a functor $Q: G\mathcal{S}[\mathbb{L}_U] \to G\mathcal{S}$ specified by the formula $QX = \mathbb{S}_G \wedge_{\Sigma^\infty_+ \mathscr{L}_U(1)} X$.

We now define a closed weak symmetric monoidal structure on $G\mathcal{S}[\mathbb{L}_U]$ with unit $\mathbb{S}_G$. (Recall that a weak symmetric monoidal category has a product and a unit satisfying all of the axioms of a symmetric monoidal category except that the unit map is not required to be an isomorphism [Elmendorf et al. 1997, II.7.1].)

Definition 3.4. Let X, Y be objects of $G\mathcal{S}[\mathbb{L}_U]$. We define the smash product $\wedge_U$ to be the coequalizer of the diagram

$$\big(\mathscr{L}_U(2) \times \mathscr{L}_U(1) \times \mathscr{L}_U(1)\big)_+ \wedge (X \wedge Y) \rightrightarrows \mathscr{L}_U(2)_+ \wedge (X \wedge Y) \to X \wedge_U Y,$$

where the maps are specified by the actions of $\mathscr{L}_U(1)_+$ on X and Y and the right action of $\mathscr{L}_U(1) \times \mathscr{L}_U(1)$ on $\mathscr{L}_U(2)$ via block sum and precomposition.

We will sometimes write this coequalizer using the notation

$$\mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} (X \wedge Y).$$

Here the left action of $\mathscr{L}_U(1)$ on $\mathscr{L}_U(2)$ induces a left action of $\mathscr{L}_U(1)$ on $X \wedge_U Y$ which endows it with the structure of an $\mathbb{L}_U$ algebra. As an example, when $X = \mathbb{L}_U A$ and $Y = \mathbb{L}_U B$ are free $\mathbb{L}_U$-algebras,

$$X \wedge_U Y \cong \mathscr{L}_U(2)_+ \wedge (A \wedge B). \tag{3.5}$$

Analogously, we have an internal function object in $G\mathcal{S}[\mathbb{L}_U]$ that satisfies the usual adjunction.

Definition 3.6. Let X, Y be objects of $G\mathcal{S}[\mathbb{L}_U]$. We define the mapping $\mathbb{L}_U$-spectrum $F_{\mathscr{L}_U}(X, Y)$ to be the equalizer of the diagram

$$F_{G\mathcal{S}[\mathbb{L}_U]}(\mathscr{L}_U(2)_+ \wedge X, Y) \rightrightarrows F_{G\mathcal{S}[\mathbb{L}_U]}\big(\big(\mathscr{L}_U(2) \times \mathscr{L}_U(1) \times \mathscr{L}_U(1)\big)_+ \wedge X, Y\big),$$

where the maps are induced by the action of $\mathscr{L}_U(1) \times \mathscr{L}_U(1)$ on $\mathscr{L}_U(2)$ by block sum and via the adjunction homeomorphism

$$\begin{aligned} F_{G\mathcal{S}[\mathbb{L}_U]}&\big(\big(\mathscr{L}_U(2) \times \mathscr{L}_U(1) \times \mathscr{L}_U(1)\big)_+ \wedge X, Y\big) \\ &\cong F_{G\mathcal{S}[\mathbb{L}_U]}\big((\mathscr{L}_U(2) \times \mathscr{L}_U(1))_+ \wedge X, F_{G\mathcal{S}[\mathbb{L}_U]}(\mathscr{L}_U(1)_+ \wedge \mathbb{S}_G, Y)\big) \end{aligned}$$

along with the action $\mathscr{L}_U(1)_+ \wedge X \to X$ as well as the coaction

$$Y \to F_{G\mathcal{S}[\mathbb{L}_U]}(\mathscr{L}_U(1)_+ \wedge \mathbb{S}_G, Y).$$

In what follows, we will repeatedly make use of the fact that for any $n > 0$ and admissible set T, we can choose a G-equivariant homeomorphism $\mathbb{R}\{T\} \otimes U \to U$ (see Lemma A.1 in Appendix A). We now establish the basic properties of $\wedge_U$.

Theorem 3.7. *Let X, Y, and Z be objects of $G\mathcal{S}[\mathscr{L}_U]$. There is a natural commutativity homeomorphism*

$$\tau : X \wedge_U Y \to Y \wedge_U X$$

and a natural associativity homeomorphism

$$(X \wedge_U Y) \wedge_U Z \cong X \wedge_U (Y \wedge_U Z).$$

More generally, there is a canonical natural homeomorphism

$$X_1 \wedge_U \ldots \wedge_U X_k \cong \mathscr{L}_U(k) \times_{\underbrace{(\mathscr{L}_U(1) \times \cdots \times \mathscr{L}_U(1))}_{k}} (X_1 \wedge \cdots \wedge X_k),$$

where the left-hand side is associated in any order and the right-hand side denotes the evident coequalizer generalizing the definition of $\wedge_U$.

Proof. Commutativity is essentially immediate (see [Elmendorf et al. 1997, I.5.2]) and associativity is a consequence of the equivariant analogue of [Elmendorf et al. 1997, I.5.4], that is, the homeomorphism

$$\mathscr{L}_U(i+j) \cong \mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} \mathscr{L}_U(i) \times \mathscr{L}_U(j),$$

which we prove as Lemma A.4. Associativity and the last formula now follow from the arguments for [Elmendorf et al. 1997, I.5.6]. Specifically, we have natural homeomorphisms $\mathscr{L}_U(1) \times_{\mathscr{L}_U(1)} X \cong X$ for all X in $G\mathcal{S}[\mathbb{L}_U]$, and therefore there are natural homeomorphisms

$$\begin{aligned} &X \wedge_U Y \wedge_U Z \\ &\cong \mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} \big(\mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} (X \wedge Y)\big) \wedge (\mathscr{L}_U(1) \times_{\mathscr{L}_U(1)} Z) \\ &\cong \big(\mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} \mathscr{L}_U(2) \times \mathscr{L}_U(1)\big) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)\times\mathscr{L}_U(1)} (X \wedge Y \wedge Z) \\ &\cong \mathscr{L}_U(3) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)\times\mathscr{L}_U(1)} (X \wedge Y \wedge Z). \end{aligned}$$

□

Next, we construct the unit map, which is a consequence of the equivariant analogue of a basic point-set property of spaces of linear isometries; see Lemma A.2.

Corollary 3.8. *There is a natural homeomorphism of $\mathbb{L}_U$-spectra*

$$\lambda : \mathbb{S}_G \wedge_U \mathbb{S}_G \cong \mathbb{S}_G$$

such that $\lambda\tau = \lambda$.

The argument for [Elmendorf et al. 1997, I.8.3] now generalizes without change to the equivariant setting:

Theorem 3.9. *Let X be an object of $G\mathcal{S}[\mathbb{L}_U]$. Then there exists a natural map*

$$\psi : \mathbb{S}_G \wedge_U X \to X$$

which is compatible with the commutativity and associativity homeomorphisms.

Combining Theorems 3.7 and 3.9, we have proved the following result.

Theorem 3.10. *The category $G\mathcal{S}[\mathbb{L}_U]$ is a closed weak symmetric monoidal category with product $\wedge_U$, unit $\mathbb{S}_G$, and function object $F_{\mathscr{L}_U}(-,-)$.*

Just as in the setting of spaces [Blumberg 2006; Blumberg et al. 2010] and spectra [Elmendorf et al. 1997], we can actually work with the closed symmetric monoidal category obtained by restricting to the unital objects.

Definition 3.11. Let $G\mathcal{S}_U$ denote the full subcategory of $G\mathcal{S}[\mathbb{L}_U]$ consisting of those objects for which $\psi : \mathbb{S}_G \wedge_U X \to X$ is a homeomorphism. For X, Y objects in $G\mathcal{S}_U$, let $F_U(X,Y)$ denote $\mathbb{S}_G \wedge_U F_{\mathscr{L}_U}(X,Y)$ and abusively denote by $X \wedge_U Y$ the coequalizer regarded as an object of $G\mathcal{S}_U$.

Corollary 3.8 implies that there is a functor $\mathbb{S}_G \wedge_U (-) : G\mathcal{S}[\mathbb{L}_U] \to G\mathcal{S}_U$ which is the left adjoint to the functor $F_{\mathscr{L}_U}(\mathbb{S}_G, -)$ and the right adjoint to the forgetful functor. As a consequence, we can deduce the following result.

Proposition 3.12. *The category $G\mathcal{S}_U$ is complete and cocomplete. Colimits are created in $G\mathcal{S}[\mathbb{L}_U]$ (and hence in $G\mathcal{S}$). Limits are formed by applying $\mathbb{S}_G \wedge_U (-)$ to the limit in $G\mathcal{S}[\mathbb{L}_U]$. Similarly, $G\mathcal{S}_U$ has tensors and cotensors with based G-spaces. Tensors are created in $G\mathcal{S}[\mathbb{L}_U]$ (and hence in $G\mathcal{S}$). Cotensors are formed by applying $\mathbb{S}_G \wedge_U (-)$ to the cotensor in $G\mathcal{S}[\mathbb{L}_U]$.*

It is now straightforward to conclude the following result.

Theorem 3.13. *The category $G\mathcal{S}_U$ is a closed symmetric monoidal category with unit $\mathbb{S}_G$, product $\wedge_U$, and function object $F_U(-,-)$.*

3.2. ***Point-set multiplicative change of universe functors.*** A counterintuitive but useful fact about the category of orthogonal G-spectra is that the point-set change of universe functors are always symmetric monoidal equivalences of categories, although they are not always Quillen equivalences. In particular, for any universe U, there is an equivalence of categories between G-objects in the category of (nonequivariant) orthogonal spectra and orthogonal G-spectra on U. In this section, we explain the corresponding result in the context of *multiplicative* change of universe functors for the categories $G\mathcal{S}[\mathbb{L}_U]$ as U varies; the underlying (complete) additive universe that structures $G\mathcal{S}$ remains constant.

Let U and U' be G-universes, and denote by $\mathscr{L}(U, U')$ the G-space of nonequivariant linear isometries $U \to U'$, where G acts by conjugation. When $U = U'$, note that $\mathscr{L}(U, U) = \mathscr{L}_U(1)$.

Definition 3.14. Let U and U' be G-universes. We define the functor

$$\mathscr{L}\mathscr{I}_U^{U'} : G\mathcal{S}[\mathbb{L}_U] \to G\mathcal{S}[\mathbb{L}_{U'}]$$

by setting $\mathscr{L}\mathscr{I}_U^{U'} X$ to be the coequalizer of the diagram

$$\mathscr{L}(U, U')_+ \wedge \mathscr{L}(U, U)_+ \wedge X \rightrightarrows \mathscr{L}(U, U')_+ \wedge X,$$

where the maps are determined by the action of $\mathscr{L}_U(1)$ on X and the composition $\mathscr{L}(U, U') \times \mathscr{L}(U, U) \to \mathscr{L}(U, U')$. The action of $\mathscr{L}_{U'}(1)$ on $\mathscr{I}_U^{U'} X$ is also induced by the composition map $\mathscr{L}(U', U') \times \mathscr{L}(U, U') \to \mathscr{L}(U, U')$.

As explained in [Elmendorf and May 1997, Corollaries 1.3, 1.4], we have the following point-set result about the behavior of these functors. We include the proof here in order to make this paper more self-contained.

Theorem 3.15. *Let U and U' be G-universes. The functors $\mathscr{L}\mathscr{I}_U^{U'}$ and $\mathscr{L}\mathscr{I}_{U'}^{U}$ are inverse equivalences of categories between $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}[\mathbb{L}_{U'}]$. Both functors are strong symmetric monoidal. As a consequence, the change of universe functors descend to the categories $G\mathcal{S}_U$ and $G\mathcal{S}_{U'}$.*

Proof. This result follows from the identification of the coequalizer

$$\mathscr{L}(U', U'') \times \mathscr{L}(U', U') \times \mathscr{L}(U, U') \rightrightarrows \mathscr{L}(U', U'') \times \mathscr{L}(U, U')$$

as $\mathscr{L}(U, U'')$, for any universes U, U', and U'' [Elmendorf and May 1997, Lemma 2.2] and where the maps are all induced by the composition γ. Since coequalizers in G-spaces are computed using the forgetful functor to spaces, it suffices to show that this is a coequalizer diagram of nonequivariant spaces. But in this setting, the diagram is a split coequalizer. The splitting is constructed as follows. Choose an isomorphism $s : U \to U'$ and define

$$h : \mathscr{L}(U, U'') \to \mathscr{L}(U', U'') \times \mathscr{L}(U, U')$$

and

$$k : \mathscr{L}(U', U'') \times \mathscr{L}(U, U') \to \mathscr{L}(U', U'') \times \mathscr{L}(U', U') \times \mathscr{L}(U, U')$$

via the formulas $h(f) = (f \circ s^{-1}, s)$ and $k(g', g) = (g', g \circ s^{-1}, s)$. Then $\gamma \circ h = \mathrm{id}$, $(\mathrm{id} \times \gamma) \circ k = \mathrm{id}$, and $(\gamma \times \mathrm{id}) \circ k = h \circ \gamma$. □

In particular, we have the following surprising corollary.

Corollary 3.16. *Let U be any G-universe. The categories $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$ are equivalent to the categories $G\mathcal{S}[\mathbb{R}^\infty]$ and $G\mathcal{S}_{\mathbb{R}^\infty}$ respectively.*

In our work in this paper, we will make critical use of this equivalence to establish some point-set properties of our categories $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$, notably about the multiplicative norm and the fixed-point functors. In fact, as pointed out by an anonymous referee, we could simplify some of the work of the previous section by using the fact that $G\mathcal{S}_{\mathbb{R}^\infty}$ can be described as a diagram category; the construction of the smash product, colimits, and limits is then immediate from general results about diagram categories and Corollary 3.16.

Remark 3.17. The additive version of this phenomenon was originally discovered in the context of equivariant Γ-spaces by Shimakawa [1991] and was proved for orthogonal G-spectra in [Mandell and May 2002, §V.1]. In the multiplicative setting, the use of these formulas to simplify the point-set theory for the equivariant stable category is sketched in [May 1996, §XXIII.4], in the context of the equivariant version of EKMM spectra [Elmendorf et al. 1997]; this exposition followed [Elmendorf and May 1997].

Although these facts were known to experts for a long time, the observation has become prominent after its use in the definition of the Hill–Hopkins–Ravenel multiplicative norm [Hill et al. 2016]; it is vastly simpler to define the norm directly on G-objects and use the universe only to study the homotopy theory. Another important recent application of these ideas comes from global equivariant homotopy theory; this technique is essentially required to make the point-set approach to global equivariant homotopy theory tractable [Schwede 2018]. Likely motivated by this fact, Schwede [2019] has advocated for developing the foundations of equivariant stable homotopy theory from this perspective (although on the other hand see [Mandell and May 2002, Remark V.1.9] for a contrary view).

Nonetheless, we believe that despite Corollary 3.16, it is conceptually clarifying in our work to keep track of the multiplicative universe at the point-set level. The issue is simply that we have two universes in play, the universe structuring the additive theory and the universe structuring the multiplicative theory. We believe that the approach outlined in [May 1996, §XXIII.4] works best when there is only a single universe, i.e., when the additive and multiplicative universe coincide. Moreover, when doing homotopical work, there is of course no way to avoid incorporating the universe explicitly when writing down formulas for fibrant replacement and (right) derived functors.

3.3. ***Rings and modules in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$.*** We now turn to the characterization of multiplicative objects in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$. The key observation about $\wedge_U$ is that (in direct analogy with the nonequivariant case), monoids for $\wedge_U$ are algebras over the non-Σ linear isometries operad $\mathscr{L}_U$ and commutative monoids for $\wedge_U$ are algebras over the linear isometries operad $\mathscr{L}_U$. More precisely, let $\mathbb{T}$ and $\mathbb{P}$ denote the monads structuring associative and commutative monoid objects in $G\mathcal{S}[\mathbb{L}_U]$ respectively. Concretely, for X an object of $G\mathcal{S}[\mathbb{L}_U]$,

$$\mathbb{T}X = \bigvee_{k\geq 0} \underbrace{X \wedge_U \cdots \wedge_U X}_{k} \quad \text{and} \quad \mathbb{P}X = \bigvee_{k\geq 0} \big(\underbrace{X \wedge_U \cdots \wedge_U X}_{k}\big)/\Sigma^k,$$

where X^0 is defined to be $\mathbb{S}_G$.

Monadic algebras over the analogous monads $\mathbb{T}$ and $\mathbb{P}$ in $G\mathcal{S}_U$ are simply algebras in $G\mathcal{S}[\mathbb{L}_U]$ that are unital; this is clear for $\mathbb{T}$, and follows for $\mathbb{P}$ from the fact that colimits in $G\mathcal{S}_U$ are created in $G\mathcal{S}[\mathbb{L}_U]$. Moreover, there are functors

$$\mathbb{S}_G \wedge_U (-) : (G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}] \to G\mathcal{S}_U[\mathbb{T}] \quad \text{and} \quad \mathbb{S}_G \wedge_U (-) : (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] \to G\mathcal{S}_U[\mathbb{P}].$$

The next result connects the categories $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$ and $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$ of monadic algebras to categories of operadic $\mathcal{N}_\infty$ algebras [Blumberg and Hill 2015].

Theorem 3.18. *The category $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$ is isomorphic to the category of non-Σ $\mathscr{L}_U$-algebras in $G\mathcal{S}$. The category $(G\mathcal{S}[\mathbb{L}_U][\mathbb{P}])$ is isomorphic to the category of $\mathscr{L}_U$-algebras in $G\mathcal{S}$.*

Proof. The argument is the same as the proof of [Elmendorf et al. 1997, II.4.6], using the homeomorphism of Equation (3.5) levelwise. □

In light of the previous theorem, we will refer to monoids and commutative monoids in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}$ as associative and commutative $\mathcal{N}_\infty$ ring orthogonal G-spectra, respectively.

Next, the arguments of [Elmendorf et al. 1997, II.7] extend to prove the following:

Theorem 3.19. *The categories $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$, $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$, $G\mathcal{S}_U[\mathbb{T}]$, and $G\mathcal{S}_U[\mathbb{P}]$ are complete and cocomplete, with limits created in $G\mathcal{S}$. The categories $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$ and $G\mathcal{S}_U[\mathbb{T}]$ are tensored and cotensored over based G-spaces, with cotensors created in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$ respectively. The categories $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$ and $G\mathcal{S}_U[\mathbb{P}]$ are tensored and cotensored over unbased G-spaces, with cotensors created in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$ respectively (regarding these categories as cotensored over unbased spaces via the functor that adjoins a disjoint G-fixed basepoint).*

As an aside, we note the following standard observation, which follows as usual simply by checking the universal property.

Lemma 3.20. *The symmetric monoidal product $\wedge_U$ is the coproduct on $G\mathcal{S}_U[\mathbb{P}]$.*

Finally, for any monoid or commutative monoid R, there are associated categories of (left) R-modules in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$. Since the theory is cleanest in the case of $G\mathcal{S}_U$, we focus on the unital setting in the following discussion. The multiplication and unit maps for R give the functor $R \wedge_U (-)$ the structure of a monad on $G\mathcal{S}_U$.

Definition 3.21. Let R be an object in $G\mathcal{S}_U[\mathbb{T}]$ or $G\mathcal{S}_U[\mathbb{P}]$. The category $\mathcal{M}_{R,U}$ of R-modules in $G\mathcal{S}_U[\mathbb{P}]$ is the category of algebras for the monad $R \wedge_U (-)$ in $G\mathcal{S}_U$.

Such categories of R-modules are complete and cocomplete, with limits and colimits created in $G\mathcal{S}_U$. When R is commutative, the category of R-modules is closed symmetric monoidal with unit R and product $X \wedge_{R,U} Y$ defined as the coequalizer of the diagram

$$X \wedge_U R \wedge_U Y \rightrightarrows X \wedge_U Y,$$

where the maps are induced by the right action of R on X via the symmetry homeomorphism and the left action of R on Y. The function object is defined as the equalizer of the diagram

$$F_U(X, Y) \rightrightarrows F_U(R \wedge_U X, Y),$$

where the maps are induced by the action of R on X and the adjoint of the composite

$$R \wedge_U X \wedge_U F_U(X, Y) \to R \wedge_U Y \to Y.$$

There are also the evident categories of R-algebras and commutative R-algebras.

Definition 3.22. Let R be an object in $G\mathcal{S}_U[\mathbb{P}]$. Abusively denote by $\mathbb{T}$ and $\mathbb{P}$ the monads in $\mathcal{M}_{R,U}$ that structure monoids and commutative monoids. We refer to the categories $\mathcal{M}_{R,U}[\mathbb{T}]$ and $\mathcal{M}_{R,U}[\mathbb{P}]$ as the categories of R-algebras and commutative R-algebras respectively.

3.4. ***Change of group and fixed-point functors.*** In this section, we study change-of-group and fixed-point functors in the context of the categories $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$. If we are content to ignore the monoidal structure, the point-set theory of the change of group and fixed-point functors is the same as for $G\mathcal{S}$. The interaction of these functors with the action of $\mathscr{L}_U(1)$ is more subtle. Our discussion relies on observations from [Mandell and May 2002, §VI.1].

Let $\iota_H : H \to G$ be the inclusion of a subgroup. Denote by WH the quotient NH/H, where NH is the normalizer of H in G. For X an object of $G\mathcal{S}$, there is a homeomorphism

$$\iota_H^* \mathbb{L}_U X \cong \mathbb{L}_{(\iota_H^* U)}(\iota_H^* X).$$

This homeomorphism is easily seen to be compatible with the monad structure, and so we obtain a functor

$$\iota_H^* : G\mathcal{S}[\mathbb{L}_U] \to H\mathcal{S}[\mathbb{L}_{(\iota_H^* U)}],$$

where the additive universe on $H\mathcal{S}$ here is ι^* applied to the complete universe structuring $G\mathcal{S}$. Analogously, for Y an object of $H\mathcal{S}$, we have a homeomorphism

$$G_+ \wedge_H \mathbb{L}_{(\iota_H^* U)} Y \cong \mathbb{L}_U(G_+ \wedge_H Y)$$

that is compatible with the monad structure, producing a functor

$$G_+ \wedge_H (-) : H\mathcal{S}[\mathbb{L}_{(\iota_H^* U)}] \to G\mathcal{S}[\mathbb{L}_U]$$

that is the left adjoint to ι_H^*. Finally, there is also a homeomorphism

$$F_H(G, \mathbb{L}^\sharp_{(\iota_H^* U)} Y) \cong \mathbb{L}^\sharp F_H(G, Y)$$

(here recall that the comonad $\mathbb{L}^\sharp$ is described just prior to the proof of Lemma 3.3) that is compatible with the comonad structure and thus produces the right adjoint

$$F_H(G, -) : H\mathcal{S}[\mathbb{L}_{(\iota_H^* U)}] \to G\mathcal{S}[\mathbb{L}_U]$$

to ι_H^*.

Furthermore, all of these functors are compatible with the functors creating the unital objects, and so descend to functors $\iota_H^* : G\mathcal{S}_U \to H\mathcal{S}_{(\iota_H^* U)}$ and the attendant left and right adjoints.

Finally, it is evident that ι_H^* is symmetric monoidal and so it restricts to categories of monoids and commutative monoids.

Proposition 3.23. *Let H be a subgroup of G. Then there are forgetful functors*

$$\iota_H^* : (G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}] \to (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{T}] \quad \textit{and} \quad \iota_H^* : G\mathcal{S}_U[\mathbb{T}] \to G\mathcal{S}_{\iota_H^* U}[\mathbb{T}]$$

and

$$\iota_H^* : (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] \to (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \quad \textit{and} \quad \iota_H^* : G\mathcal{S}_U[\mathbb{P}] \to G\mathcal{S}_{\iota_H^* U}[\mathbb{P}].$$

Next, we turn to the question of the categorical fixed points. Our definition is built from the categorical fixed point functor $(-)^H$ on $G\mathcal{S}$ [Mandell and May 2002, Definition V.3.9].

Theorem 3.24. *Let H be a subgroup of G. Then the categorical H-fixed point functor on $G\mathcal{S}$ induces a lax monoidal categorical H-fixed point functor*

$$(-)^H : G\mathcal{S}[\mathbb{L}_U] \to WH\mathcal{S}[\mathbb{L}_{U^H}]$$

specified (*in mild abuse of notation*) *by the formula*

$$X^H = (\mathscr{L}\mathscr{I}_U^{U^H} X)^H,$$

where the $(-)^H$ on the right-hand side denotes the categorical fixed points in $G\mathcal{S}$. The fixed point functor has an op-lax symmetric monoidal left adjoint

$$\epsilon_H^* : WH\mathcal{S}[\mathbb{L}_{U^H}] \to G\mathcal{S}[\mathbb{L}_U],$$

which assigns to a WH-spectrum X the G-spectrum obtained by pulling back along the quotient $NH \to WH$, changing (*additive*) *universe, and inducing up to G, and changing multiplicative universe. When H is normal, the left adjoint is strong symmetric monoidal.*

Proof. Since

$$(\mathscr{L}(U^H, U^H)_+ \wedge X)^H \cong \mathscr{L}(U^H, U^H)_+ \wedge X^H,$$

the categorical H-fixed point functor restricts to a functor

$$G\mathcal{S}[\mathbb{L}_{U^H}] \to WH\mathcal{S}[\mathbb{L}_{U^H}].$$

Analogously,

$$\epsilon_H^*(\mathscr{L}(U, U)_+ \wedge Y) \cong \mathscr{L}(U, U)_+ \wedge \epsilon_H^* Y,$$

which implies that ϵ_H^* restricts to a functor from $WH\mathcal{S}[\mathbb{L}_{U^H}]$ to $G\mathcal{S}[\mathbb{L}_U]$.

Next, we consider the interaction of $(-)^H$ with the monoidal structure. Since the action of H on $\mathscr{L}(U^H, U^H)$ is trivial and $(-)^H$ is lax monoidal on $G\mathcal{S}$ [Mandell and May 2002, Proposition V.3.8], for X and Y in $G\mathcal{S}[\mathbb{L}_U]$ and $H \subseteq G$ we have a natural map

$$(\mathscr{L}_{U^H}(2) \times_{\mathscr{L}_{U^H}(1) \times \mathscr{L}_{U^H}(1)})_+ \wedge X^H \wedge Y^H \to (\mathscr{L}_{U^H}(2) \times_{\mathscr{L}_{U^H}(1) \times \mathscr{L}_{U^H}(1)})_+ \wedge (X \wedge Y)^H$$

which lands in the fixed-points

$$\big((\mathscr{L}_{U^H}(2) \times_{\mathscr{L}_{U^H}(1) \times \mathscr{L}_{U^H}(1)})_+ \wedge (X \wedge Y)\big)^H,$$

and so we deduce that $(-)^H$ is a lax symmetric monoidal functor

$$G\mathcal{S}[\mathbb{L}_U] \to WH\mathcal{S}[\mathbb{L}_{U^H}].$$

Finally, when H is normal, the left adjoint is strong symmetric monoidal since the pullback and additive change of universe are. □

The situation for the geometric fixed point functor is analogous; again, we construct Φ^H on $G\mathcal{S}[\mathbb{L}_U]$ by considering the composite $\Phi^H(\mathscr{L}\mathscr{I}_U^{U^H} X)$.

Theorem 3.25. *Let H be a subgroup of G. Then there is a lax symmetric monoidal geometric H-fixed point functor*

$$\Phi^H : G\mathcal{S}[\mathbb{L}_U] \to WH\mathcal{S}[\mathbb{L}_{U^H}].$$

Proof. The compatibility of the geometric fixed points functor with $\mathscr{L}_U(1)$ action is clear. Next, once again the fact that the actions of H on $\mathscr{L}_{U^H}(2)$ and $\mathscr{L}_{U^H}(1)$ are trivial and the fact that Φ^H is lax symmetric monoidal on $G\mathcal{S}$ implies that it is lax symmetric monoidal on $G\mathcal{S}[\mathbb{L}_U]$. □

3.5. ***The point-set theory of the norm.*** In this subsection, we construct multiplicative norm functors in the sense of [Hill et al. 2016] on the categories $G\mathcal{S}[\mathbb{L}_U]$, $G\mathcal{S}_U$, and $\mathcal{M}_{R,U}$ for R a commutative algebra in $G\mathcal{S}_U$. Fix a subgroup $H \subseteq G$ and let

$\widehat{U}$ denote an H-universe. The norm functor $N_H^G : H\mathcal{S} \to G\mathcal{S}$ is strong symmetric monoidal and so there is a natural homeomorphism

$$N_H^G(\mathscr{L}_{\widehat{U}}(1)_+ \wedge X) \cong F_H(G, \mathscr{L}_{\widehat{U}}(1))_+ \wedge N_H^G X.$$

This leads to the following definition, which can be viewed as a form of Theorem 3.7 where we have allowed the group to act on the cartesian factors.

Definition 3.26. We define the functor

$$N_{H,\widehat{U}}^{G,U} : H\mathcal{S}[\mathbb{L}_{\widehat{U}}] \to G\mathcal{S}[\mathbb{L}_U]$$

on objects X via the coequalizer of the diagram

$$\mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U)_+ \wedge F_H(G, \mathscr{L}_{\widehat{U}}(1))_+ \wedge N_H^G X \rightrightarrows \mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U)_+ \wedge N_H^G X,$$

where the left action of $\mathscr{L}_U(1)$ on $\mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U)$ provides the structure of an $\mathbb{L}_U$ algebra.

In the coequalizer, the other map is specified by the action of $F_H(G, \mathscr{L}_{\widehat{U}}(1))$ on $\mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U)$ via the map of monoids

$$I_{(-)} : F_H(G, \mathscr{L}_{\widehat{U}}(1)) \to \mathscr{L}_{\operatorname{Ind}_H^G \widehat{U}}(1)$$

given by

$$f \mapsto I_f = \big(g \otimes u \mapsto g \otimes f(g)(u)\big),$$

the target of which is underlain by the orthogonal sum of isometries and hence is an isometry.

There is an alternate characterization of $N_{H,\iota_H^* U}^{G,U}$ which can be given using the multiplicative change of universe functors.

Lemma 3.27. *There is a natural homeomorphism*

$$N_{H,\iota_H^* U}^{G,U} X \cong \mathscr{L}\mathscr{I}_{\mathbb{R}^\infty}^{U}\big(N_{H,\mathbb{R}^\infty}^{G,\mathbb{R}^\infty}(\mathscr{L}\mathscr{I}_{\iota_H^* U}^{\mathbb{R}^\infty} X)\big)$$

Proof. To establish the identification, we expand the right-hand side, writing $\mathbb{R}$ in place of $\mathbb{R}^\infty$ for concision:

$$\begin{aligned}
&\mathscr{L}(\mathbb{R}, U) \times_{\mathscr{L}(\mathbb{R},\mathbb{R})} N_{H,\mathbb{R}}^{G,\mathbb{R}} \mathscr{L}(\iota_H^* U, \mathbb{R}) \times_{\mathscr{L}(\iota_H^* U, \iota_H^* U)} X \\
&\quad \cong \mathscr{L}(\mathbb{R}, U) \times_{\mathscr{L}_{\mathbb{R}}(1)} \big(\mathscr{L}(\operatorname{Ind}_H^G \mathbb{R}, \mathbb{R}) \times_{F_H(G, \mathscr{L}_{\mathbb{R}}(1))} N_H^G\big(\mathscr{L}(\iota_H^* U, \mathbb{R}) \times_{\mathscr{L}_{\iota_H^* U}(1)} X\big)\big) \\
&\quad \cong \mathscr{L}(\operatorname{Ind}_H^G \mathbb{R}, U) \times_{F_H(G, \mathscr{L}(\mathbb{R},\mathbb{R}))} N_H^G \mathscr{L}(\iota_H^* U, \mathbb{R}) \times_{\mathscr{L}(\iota_H^* U, \iota_H^* U)} X \\
&\quad \cong \mathscr{L}(\operatorname{Ind}_H^G \mathbb{R}, U) \times_{F_H(G, \mathscr{L}(\mathbb{R},\mathbb{R}))} F_H(G, \mathscr{L}(\iota_H^* U, \mathbb{R})) \times_{F_H(G, \mathscr{L}(\iota_H^* U, \iota_H^* U))} N_H^G X \\
&\quad \cong \mathscr{L}(\operatorname{Ind}_H^G \iota_H^* U, U) \times_{F_H(G, \mathscr{L}(\iota_H^* U, \iota_H^* U))} N_H^G X \\
&\quad \cong N_{H,\iota_H^*}^{G,U} X.
\end{aligned}$$

In these expansions, note that we use the fact that the norm preserves reflexive coequalizers [Hill et al. 2016, Remark A.54]. □

We now show that norm is strong symmetric monoidal. This can be done using Lemma 3.27, but it is convenient in the homotopical analysis to give a slightly more expansive proof that involves a bit more work with the linear isometries operad, also given in Appendix A. (In contrast, compare the proof of Theorem 3.31 below.)

Theorem 3.28. *The functor $N_{H,\widehat{U}}^{G,U}$ is strong symmetric monoidal.*

Proof. By Lemma A.3, we see that $N_{H,\widehat{U}}^{G,U}$ preserves the unit. We now compare $N_{H,\widehat{U}}^{G,U}(X \wedge_{\widehat{U}} Y)$ and $(N_{H,\widehat{U}}^{G,U} X) \wedge_U (N_{H,\widehat{U}}^{G,U} Y)$ by direct computation. By definition, we have

$$N_{H,\widehat{U}}^{G,U}(X \wedge_{\widehat{U}} Y) = \mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))} \big(N_H^G(\mathscr{L}_{\widehat{U}}(2) \times_{\mathscr{L}_{\widehat{U}}(1)\times\mathscr{L}_{\widehat{U}}(1)} (X \wedge Y)\big).$$

Since the norm functor commutes with reflexive coequalizers and is symmetric monoidal as a functor on orthogonal H-spectra, this is isomorphic to

$$\mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))} \big(F_H(G, \mathscr{L}_{\widehat{U}}(2)) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))\times F_H(G,\mathscr{L}_{\widehat{U}}(1))} (N_H^G X \wedge N_H^G Y)\big).$$

Applying Corollary A.7, we rewrite this as

$$\mathscr{L}(\operatorname{Ind}_H^G \widehat{U} \oplus \operatorname{Ind}_H^G \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))\times F_H(G,\mathscr{L}_{\widehat{U}}(1))} (N_H^G X \wedge N_H^G Y).$$

On the other hand, writing out $(N_{H,\widehat{U}}^{G,U} X) \wedge_U (N_{H,\widehat{U}}^{G,U} Y)$ we have

$$\mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} \Big(\big(\mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))} N_H^G X\big) \wedge \big(\mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))} N_H^G Y\big)\Big).$$

Applying Corollary A.5, we can rewrite this as

$$\mathscr{L}(\operatorname{Ind}_H^G \widehat{U} \oplus \operatorname{Ind}_H^G \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))\times F_H(G,\mathscr{L}_{\widehat{U}}(1))} (N_H^G X \wedge N_H^G Y).$$

Finally, the naturality of the homeomorphisms above make it clear that the pentagon identities hold. □

As a consequence of Theorem 3.28, we have the following corollary.

Corollary 3.29. *The functor $N_{H,\widehat{U}}^{G,U}$ restricts to a functor*

$$N_{H,\widehat{U}}^{G,U} : H\mathcal{S}_{\widehat{U}} \to G\mathcal{S}_U$$

which we abusively refer to with the same notation.

Remark 3.30. Lemma 3.27 can now be interpreted as the statement that the norm can be described as the indexed product on $H\mathcal{S}_{\mathbb{R}^\infty}$; this makes it clear that the norm is functorial in both the group and the input spectrum.

We now turn to establish the following adjunction on commutative ring objects. We will be predominantly interested in the case where $\widehat{U} = \iota_H^* U$, as this is relevant for describing the equivariant symmetric monoidal structures on the categories $\mathcal{M}_{R,U}$.

Theorem 3.31. *There are adjoint pairs with left adjoints*

$$N_{H,\iota_H^* U}^{G,U} : (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \to (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] \quad \textit{and} \quad N_{H,\iota_H^* U}^{G,U} : (H\mathcal{S}_{\iota_H^* U})[\mathbb{P}] \to G\mathcal{S}_U[\mathbb{P}]$$

and right adjoints

$$\iota_H^* : (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] \to (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \quad \textit{and} \quad \iota_H^* : G\mathcal{S}_U[\mathbb{P}] \to H\mathcal{S}_{\iota_H^* U}[\mathbb{P}]$$

respectively.

Proof. First, observe that the conclusion of the theorem follows immediately when $U = \mathbb{R}^\infty$: since for any G the category $G\mathcal{S}[\mathbb{L}_{\mathbb{R}^\infty}]$ is equivalent to the category of G-objects in $\mathcal{S}[\mathbb{L}_{\mathbb{R}^\infty}]$, we can apply [Hill et al. 2016, Corollary A.56]. We now use the alternate characterization of the norm from Lemma 3.27 and the fact that by Theorem 3.15 the change of universe functors are symmetric monoidal equivalences of categories. □

An immediate corollary of Theorem 3.31 is that commutative ring objects have an "internal norm" map arising from the counit of the adjunction.

Corollary 3.32. *Let R be an object in $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$ or $G\mathcal{S}_U[\mathbb{P}]$. Then there is a natural map*

$$N_{H,\iota_H U}^{G,U} \iota_H^* R \to R.$$

Using the counit of the adjunction of Theorem 3.31 and the absolute norm functor described in Definition 3.26, we can express the R-relative norm for a commutative ring object R in $G\mathcal{S}_U$ using base-change:

Definition 3.33. Let R be an object in $G\mathcal{S}_U[\mathbb{P}]$. We define the functor

$${}_R N_{H,\iota_H^* U}^{G,U} : \mathcal{M}_{\iota_H^* R, \iota_H^* U} \to \mathcal{M}_{R,U}$$

via the formula

$$X \mapsto R \wedge_{N_{H,\iota_H^* U}^{G,U} R} N_{H,\iota_H^* U}^{G,U} X,$$

where the coequalizer is over the counit map and the map induced by the action of R on X.

It is clear from the definition and Theorem 3.28 that the R-relative norm is also a strong symmetric monoidal functor.

4. Homotopical categories of modules over an N_∞ algebra

In this section, we describe model structures on the categories $G\mathcal{S}[\mathbb{L}_U]$, $G\mathcal{S}_U$, and categories of algebras and modules over an algebra. The main goal of our efforts is to describe the derived functors of the norm and forgetful functors as a prelude to the construction of the equivariant symmetric monoidal structure.

4.1. ***The homotopical theory of $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$.*** We begin by quickly reviewing some of the less commonly used terminology from the theory of model categories that we will employ in the statements of results below. Recall from [Mandell et al. 2001, Definition 5.9] that a cofibrantly generated topological model structure is compactly generated if the domains of the generating cofibrations and acyclic cofibrations are compact and satisfy the "cofibration hypothesis" [Mandell et al. 2001, Cofibration Hypothesis 5.3]. Let $\mathcal{C}$ be a complete and cocomplete topologically enriched category. An h-cofibration in $\mathcal{C}$ is a map that is the analogue of a Hurewicz cofibration; i.e., a map $X \to Y$ such that the induced map $Y \cup_X (X \otimes I) \to Y \otimes I$ has a retraction. The cofibration hypothesis for a set of maps I in a model category $\mathcal{A}$ equipped with a forgetful functor $\mathcal{A} \to \mathcal{C}$ specifies that the following two conditions are satisfied.

(1) For a coproduct $A \to B$ of maps in I, in any pushout

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

in $\mathcal{A}$, the cobase change $X \to Y$ is an h-cofibration in $\mathcal{C}$.

(2) Given a sequential colimit in $\mathcal{A}$ along maps that are h-cofibrations in $\mathcal{C}$, the colimit in $\mathcal{A}$ is equal to the colimit in $\mathcal{C}$.

In order to be able to apply Bousfield localization, it is convenient to add the requirements that:

(1) The domains of the generating acyclic cofibrations are small with respect to the generating cofibrations, and

(2) the cofibrations are effective monomorphisms.

A compactly generated model category that satisfies these additional conditions is cellular [Hirschhorn 2003, Definition 12.1.1], and so admits left Bousfield localizations very generally. In mild abuse of terminology, we will use the term compactly generated to refer to a compactly generated model category that is cellular in this paper.

A model category is G-topological if it is enriched over G-spaces and satisfies the analogue of Quillen's SM7 [Mandell and May 2002, Definition III.1.14]. There is an evident G-equivariant version of the cofibration hypothesis. The building block for our work in this section is the complete model structure on $G\mathcal{S}$ [Hill et al. 2016, Proposition B.63]. (Note that although the cited reference refers to the positive complete model structure, the existence of the complete model structure is clear.) Recall that the complete model structure has generating cofibrations given by the set of maps

$$\{G_+ \wedge_H S^{-V} \wedge S^{n-1}_+ \to G_+ \wedge_H S^{-V} \wedge D^n_+\},$$

where V varies over the (additive) universe, $n \geq 0$, and $H \subseteq G$.

Lemma 4.1. *The complete model structure is a compactly generated G-topological model structure.*

Proof. The discussion proving [Hill et al. 2016, Proposition B.63] establishes that the complete model structure is cofibrantly generated. Since the generating cofibrations in the complete model structure are h-cofibrations [Hill et al. 2016, Remark B.64], it is straightforward to see that the complete model structure satisfies the cofibration hypothesis. Finally, the cofibrations are effective monomorphisms since $S^{n-1}_+ \to D^n_+$ is for all $n \geq 0$, and the compactness criterion for the domain of the generating acyclics is clearly satisfied. □

Theorem 4.2. *The category $G\mathcal{S}[\mathbb{L}_U]$ is a compactly generated weak symmetric monoidal proper G-topological model category in which the weak equivalences and fibrations are detected by the forgetful functor* $\mathrm{u} : G\mathcal{S}[\mathbb{L}_U] \to G\mathcal{S}$.

Proof. The monad $\mathbb{L}_U$ evidently satisfies the hypotheses of (the equivariant analogue of) [Mandell et al. 2001, Proposition 5.13], and so we can conclude that there is a compactly generated G-topological model structure on $G\mathcal{S}[\mathbb{L}_U]$. The proof of the unit axiom follows from the equivariant analogue of from [Elmendorf et al. 1997, XI.3.1], which holds by the same proof as in the nonequivariant case. To check the monoid axiom, observe that it suffices to check on the generating (acyclic) cofibrations, and since these are obtained by $\mathscr{L}_U(1) \wedge_+ (-)$ applied to generating (acyclic) cofibrations of $G\mathcal{S}$, the result holds since it does in $G\mathcal{S}$. Finally, it is clear that $G\mathcal{S}[\mathbb{L}_U]$ is proper. □

By construction, the adjoint pair $(\mathbb{L}_U, \mathrm{u})$ is a Quillen adjunction. Since $\mathscr{L}_U(1)$ is G-contractible, we can conclude that this pair induces a Quillen equivalence between $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}$.

Proposition 4.3. *The adjoint pair* $(\mathbb{L}_U, \mathrm{u})$ *forms a Quillen equivalence between* $G\mathcal{S}[\mathbb{L}_U]$ *and* $G\mathcal{S}$.

Although $\mathbb{L}_U$ is not strong symmetric monoidal, it is close: as a consequence of Lemma A.1, there is a homeomorphism

$$\mathbb{L}_U X \wedge_U \mathbb{L}_U Y \cong \mathbb{L}_U (X \wedge Y)$$

and more generally homeomorphisms

$$\mathbb{L}_U X_1 \wedge_U \cdots \wedge_U \mathbb{L}_U X_k \cong \mathbb{L}_U (X_1 \wedge \cdots \wedge X_k).$$

(The failure of $\mathbb{L}_U$ to be strong symmetric monoidal is a consequence of the fact that these homeomorphisms ultimately depend on choices of homeomorphisms $U^k \to U$.) On the other hand, the functor $Q(-) = \mathbb{S}_G \wedge_{\Sigma^\infty_+ \mathscr{L}_U(1)} (-)$ is strong symmetric monoidal [Blumberg et al. 2010, Theorem 4.14]. As a consequence, we have the following comparison result (where here recall that p^* denotes the right adjoint to Q which gives an object of $G\mathcal{S}$ the trivial $\mathscr{L}_U(1)$-action).

Proposition 4.4. *The adjoint pair (Q, p^*) is a weak symmetric monoidal Quillen equivalence.*

Proof. Since p^* preserves fibrations and weak equivalences, this is clearly a Quillen adjunction. Taking $\mathbb{L}\mathbb{S}_G$ as a cofibrant replacement of the unit in $G\mathcal{S}$, we compute that $Q\mathbb{L}\mathbb{S}_G \cong \mathbb{S}_G$ and so the adjunction is monoidal. Finally, evaluation of Q on the generating cofibrations makes it clear that the natural map $QX \to \mathrm{u}X$ is a weak equivalence for cofibrant X, and so the adjunction is a Quillen equivalence. □

In order to retain homotopical control over $G\mathcal{S}_U$, we need to prove the equivariant analogue of [Elmendorf et al. 1997, I.8.4, XI.2.2], i.e., that the canonical unit map $\lambda : \mathbb{S}_G \wedge_U X \to X$ is always a weak equivalence. The proof of the required result follows the outline of [Elmendorf et al. 1997, I.8.5], using Theorem A.9.

Theorem 4.5. *For any X in $G\mathcal{S}[\mathbb{L}_U]$, the unit map*

$$\lambda : \mathbb{S}_G \wedge_U X \to X$$

is a weak equivalence.

Theorem 4.5 now allows us to prove the following theorem.

Theorem 4.6. *The category $G\mathcal{S}_U$ is a compactly generated symmetric monoidal proper G-topological model category in which the weak equivalences are detected by the forgetful functor and the fibrations are detected by the functor $F_U(\mathbb{S}_G, -)$.*

Proof. Although $\mathbb{S}_G \wedge_U (-)$ is not a monad, the argument for [Mandell et al. 2001, Proposition 5.13] again applies. As in the corresponding proof in [Elmendorf et al. 1997, VI.4.6], consideration of the category of counital objects in $G\mathcal{S}[\mathbb{L}_U]$ is illuminating. □

Remark 4.7. By adjunction, a map $\mathbb{S}_G \wedge_U \mathbb{L}_U S^n \to X$ in $G\mathcal{S}_U$ is the same as a map $\mathbb{S}^n_G \to X$ in $G\mathcal{S}$. As a consequence, the "internal" homotopy groups in $G\mathcal{S}_U$ determined by the free objects on spheres coincide with the homotopy groups on the underlying orthogonal G-spectrum.

The functor $\mathbb{S}_G \wedge_U (-) : G\mathcal{S}[\mathbb{L}_U] \to G\mathcal{S}_U$ is a Quillen left adjoint and is a symmetric monoidal functor. In fact, the following proposition is straightforward to verify.

Proposition 4.8. *The adjoint pair* $(\mathbb{S}_G \wedge_U (-), F_U(\mathbb{S}_G, -))$ *forms a weak symmetric monoidal Quillen equivalence between* $G\mathcal{S}[\mathbb{L}_U]$ *and* $G\mathcal{S}_U$.

As a consequence of these results, we have the following comparison result.

Lemma 4.9. *For cofibrant* $X, Y \in G\mathcal{S}_U$ *there is a natural equivalence*

$$X \wedge_U Y \to X \wedge Y$$

and more generally for cofibrant $\{X_1, X_2, \ldots, X_n\} \in G\mathcal{S}_U$ *there are natural equivalences*

$$X_1 \wedge_U X_2 \wedge_U \cdots \wedge_U X_n \to X_1 \wedge X_2 \wedge \cdots \wedge X_n.$$

We now turn to the study of the multiplicative structure on $G\mathcal{S}_U$. The following result explains the equivariant homotopical content of the operadic smash product $\wedge_U$.

Theorem 4.10. *Let* X *be a cofibrant object of* $G\mathcal{S}_U$. *Then there is a natural weak equivalences of* $G \times \Sigma_n$ *spectra*

$$(E_{\mathcal{F}_U}\Sigma_n)_+ \wedge X^{\wedge n} \simeq X^{\wedge_U n},$$

and a natural weak equivalence of G*-spectra*

$$(E_{\mathcal{F}_U}\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \simeq X^{\wedge_U n}/\Sigma_n,$$

where here $\mathcal{F}_U$ *denotes the family of* $G \times \Sigma_n$ *specified by* U.

Proof. When X is free as an object of $G\mathcal{S}_U$ (i.e., $X = \mathbb{S}_G \wedge_U \mathbb{L}_U Y$), then the result follows immediately from Theorem 3.7 and the fact that $\mathscr{L}(U^n, U) \simeq E_{\mathcal{F}_U}\Sigma_n$. The general result now follows by inductively reducing to the free case using the filtration argument of [Hill et al. 2016, (B.117)]. □

In particular, Theorem 4.10 makes clear the way in which $G\mathcal{S}_U$ depends on the choice of U. Specifically, the $G\times\Sigma_n$-equivariant homotopy type of the n-fold $\wedge_U$ power of X is controlled by U, and is precisely the universal space for the family associated to $\mathscr{L}_U(n)$.

Corollary 4.11. *Let* $X \to X'$ *be an acyclic cofibration in* $G\mathcal{S}_U$. *Then the induced maps*

$$\mathbb{T}X \to \mathbb{T}X' \quad \text{and} \quad \mathbb{P}X \to \mathbb{P}X'$$

are weak equivalences.

Corollary 4.11 provides the essential technical input for the next theorem, which is again proved using the standard outline (e.g., see [Mandell et al. 2001, Proposition 5.13] or [Hill et al. 2016, (B.130)]).

Theorem 4.12. *The categories $G\mathcal{S}_U[\mathbb{T}]$ and $G\mathcal{S}_U[\mathbb{P}]$ are compactly generated proper G-topological model categories with weak equivalences and fibrations determined by the forgetful functor to $G\mathcal{S}_U$.*

For a fixed ring object R, we have the following relative version of the preceding theorem.

Theorem 4.13. *For an object R in $G\mathcal{S}_U[\mathbb{T}]$ or $G\mathcal{S}_U[\mathbb{P}]$, the category of R-modules in $G\mathcal{S}_U$ is a compactly generated proper G-topological model category with weak equivalences and fibrations determined by the forgetful functor to $G\mathcal{S}_U$. When R is commutative (i.e., an object in $G\mathcal{S}_U[\mathbb{P}]$), then*

(1) *the category of R-modules in $G\mathcal{S}_U$ is a compactly generated proper G-topological symmetric monoidal model category, and*

(2) *the category of R-algebras is a compactly generated proper G-topological model category.*

4.2. *The homotopical theory of change of group and fixed-point functors.* In this section, we describe how to compute the derived functors of the change-of-group and fixed-point functors described in Section 3.4. Our analysis bootstraps from the analogous theory in the setting of $G\mathcal{S}$; the following two lemmas establish that the homotopical theory for $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$ can be understood in terms of the homotopical theory for $G\mathcal{S}$.

Lemma 4.14. *Let X be an object of $G\mathcal{S}[\mathbb{L}_U]$ or $G\mathcal{S}_U$. If X is cofibrant, then the underlying orthogonal G-spectrum associated to X has the homotopy type of a cofibrant object. The analogous results hold for $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$ and $G\mathcal{S}_U[\mathbb{T}]$.*

Proof. This follows from inspection of the generating cells and the "cofibration hypothesis" in this context. We can assume without loss of generality that X is a cellular object. Then $X = \operatorname{colim}_n X_n$, where the colimit is sequential and along h-cofibrations. The cofibration hypothesis then implies that we can compute the colimit in the underlying category, and so it suffices to consider each X_n. Since each X_n is formed from X_{n-1} by attaching cells, the cofibration hypothesis again allows us to inductively reduce this to consideration of the generating cells, where the result is clear. □

Lemma 4.15. *Let X be a fibrant object in $G\mathcal{S}[\mathbb{L}_U]$, $G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$, or $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$. Then X is fibrant in $G\mathcal{S}$. Analogously, if X is fibrant in $G\mathcal{S}_U$, $G\mathcal{S}_U[\mathbb{T}]$, or $G\mathcal{S}_U[\mathbb{P}]$, then $F_U(\mathbb{S}_G, X)$ is fibrant in $G\mathcal{S}$.*

Proof. The statements about modules imply the statements about monoids and commutative monoids, as fibrations in the model structures on the categories of algebras are determined by the forgetful functors to $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$ respectively. The first assertion is clear for $G\mathcal{S}[\mathbb{L}_U]$ since the fibrations are created by the forgetful functor to $G\mathcal{S}$. For $G\mathcal{S}_U$, the result follows from Proposition 4.8; the functor $F_U(\mathbb{S}_G, -) : G\mathcal{S}_U \to G\mathcal{S}[\mathbb{L}_U]$ is a Quillen right adjoint. □

In order to understand the behavior of the fixed point functors on $G\mathcal{S}_U$, we need to describe the homotopical behavior of the point-set multiplicative change of universe functors. In contrast to the situation for the additive functors in orthogonal spectra, these always induce Quillen equivalences.

Proposition 4.16. *Let U and U' be G-universes. The multiplicative change of universe functors $\mathscr{L}\mathscr{I}_U^{U'}$ are left (and right) Quillen functors that preserve weak equivalences between cofibrant objects and therefore induce Quillen equivalences between $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}[\mathbb{L}_{U'}]$ and $G\mathcal{S}_U$ and $G\mathcal{S}'_U$, respectively.*

Warning 4.17. What is not preserved by $\mathscr{L}\mathscr{I}_U^{U'}$ is not the underlying additive homotopy theory but the multiplicative norms. Specifically, the derived functor of $\mathscr{L}\mathscr{I}_U^{U'}$ preserves only those multiplicative norms corresponding to G-sets that are admissible in both U and U'. Put another way, these functors do not preserve the homotopical equivariant symmetric monoidal structure.

We now turn to the fixed points. The forgetful functors ι_H^* preserve all weak equivalences, and so are already derived. Their left and right adjoints can be derived by cofibrant or fibrant approximation, as a consequence of the preceding lemmas. Similarly, Proposition 4.16 implies that the (right) derived functors of the categorical fixed points can be computed by fibrant replacement and the (left) derived functors of geometric fixed points by cofibrant replacement. We summarize the situation in the following result.

Proposition 4.18. (1) *The forgetful functors ι_H^* preserve all weak equivalences on $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$.*

(2) *The left adjoint $G_+ \wedge_H (-)$ to ι_H^* preserves weak equivalences between cofibrant objects on $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$. The right adjoint $F_H(G, -)$ to ι_H^* preserves weak equivalences between fibrant objects on $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$.*

(3) *The categorical fixed point functor $(-)^H$ preserves weak equivalences between fibrant objects in $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$.*

(4) *The geometric fixed point functor Φ^H preserves weak equivalences between cofibrant objects in $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$.*

Finally, we have the following result which shows that the geometric fixed-point functor is strong monoidal in the homotopical sense.

Proposition 4.19. *Let X and Y be cofibrant objects in $G\mathcal{S}[\mathbb{L}_U]$ or $G\mathcal{S}_U$. Then the natural map*

$$\Phi^H X \wedge_U \Phi^H Y \to \Phi^H (X \wedge_U Y)$$

is a weak equivalence.

Proof. First consider the case of $G\mathcal{S}[\mathbb{L}_U]$. The result follows from the result for Φ^H on $G\mathcal{S}$ [Mandell and May 2002, Proposition V.4.7] when X and Y are generating cells, since

$$\mathbb{L}_{U^H} X' \wedge_{U^H} \mathbb{L}_{U^H} Y' \cong \mathscr{L}_{U^H}(2)_+ \wedge (X' \wedge Y')$$

for any X' and Y' and WH acts trivially on $\mathscr{L}U^H(2)_+$. Since $(-) \wedge_{U^H} (-)$ preserves colimits in either variable and preserves weak equivalences between cofibrant objects, we can conclude the general statement. The case of $G\mathcal{S}_U$ follows from analogous considerations. □

4.3. *The homotopical theory of the norm.* In this section, we show that the norm $N^{G,U}_{H,\iota^*_H U}$ is a homotopical functor and participates in a Quillen adjunction when restricted to commutative ring objects.

Theorem 4.20. *Let X be a cofibrant object in $G\mathcal{S}[\mathbb{L}_U]$ or $G\mathcal{S}_U$. The natural map*

$$N^{G,U}_{H,\iota^*_H U} X \to N^G_H X$$

is a weak equivalence when G/H is admissible for U.

Proof. By induction over the cellular filtration, it suffices to consider the case when X is free. In this case, we're looking at the map

$$\mathscr{L}(\mathrm{Ind}^G_H \iota^*_H U, U)_+ \wedge N^G_H X \to N^G_H X$$

given by the collapse map $\mathscr{L}(\mathrm{Ind}^G_H \iota^*_H U, U)_+ \to S^0$. Since the collapse is a G-equivalence when G/H is admissible, the result follows. □

Remark 4.21. When G/H is not admissible for U, it is not clear in general what the homotopy type of $N^{G,U}_{H,\iota^*_H U}$ is. For free objects, the homotopy type is controlled by $\mathscr{L}(\mathrm{Ind}^G_H \iota^*_H U, U)$, which has no G-fixed points.

Corollary 4.22. *The functor $N^{G,U}_{H,\iota^*_H U}$ preserves weak equivalences between cofibrant objects in $H\mathcal{S}_{\iota^*_H U}$ and $H\mathcal{S}[\mathbb{L}_{\iota^*_H U}]$ when G/H is admissible for U.*

The next lemma provides homotopical control on the output of the norm functor.

Lemma 4.23. *Let X be a cofibrant object in $H\mathcal{S}[\mathbb{L}_{\widehat{U}}]$ or $H\mathcal{S}_{\widehat{U}}$. Then $N^{G,U}_{H,\iota^*_H U} X$ is cofibrant in $G\mathcal{S}[\mathbb{L}_U]$.*

Proof. Using the filtration of [Hill et al. 2016, §A.3.4], we can inductively reduce to the case when X is of the form $\mathscr{L}_{\widehat{U}}(1)_+ \wedge Y$. In this case,

$$N^{G,U}_{H,\iota^*_H U}(\mathscr{L}_{\widehat{U}}(1)_+ \wedge Y) \cong \mathscr{L}(\mathrm{Ind}^G_H \widehat{U}, U)_+ \wedge N^G_H Y.$$

Since $\mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U) \cong \mathscr{L}_1(U)$ by Lemma A.1, the result follows. □

As a consequence, we have the following result about the composition of the norm functor.

Proposition 4.24. *Fix $H_1 \subseteq H_2 \subseteq G$. Let X be a cofibrant object in $H_1\mathcal{S}[\mathbb{L}_{\iota_{H_1}^* U}]$ or $H_1\mathcal{S}_{\iota_{H_1}^* U}$. Then there is a natural weak equivalence*

$$N_{H_1,\iota_{H_1}^* U}^{G,U} X \simeq N_{H_2,\iota_{H_2}^* U}^{G,U} N_{H_1,\iota_{H_1}^* U}^{H_2,\iota_{H_2}^* U} X.$$

Proof. Expanding using the definition, we have

$$N_{H_2,\iota_{H_2}^* U}^{G,U} N_{H_1,\iota_{H_1}^* U}^{H_2,\iota_{H_2}^* U} X = \mathscr{L}(\mathrm{Ind}_{H_2}^G \iota_{H_2}^* U, U) \times_{F_{H_2}(G,\mathscr{L}_{\iota_{H_2}^* U}(1))} N_{H_2}^G (N_{H_1,\iota_{H_1}^* U}^{H_2,\iota_{H_2}^* U} X)$$

and

$$N_{H_1,\iota_{H_1}^* U}^{H_2,\iota_{H_2}^* U} X = \big(\mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U) \times_{F_{H_1}(H_2,\mathscr{L}_{\iota_{H_1}^* U}(1))} N_{H_1}^{H_2} X\big)$$

which implies that

$$N_{H_2}^G (N_{H_1,\iota_{H_1}^* U}^{H_2,\iota_{H_2}^* U} X) \cong \big(F_{H_2}(G, \mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U)) \times_{F_{H_2}(G,\mathscr{L}_{\iota_{H_1}^* U}(1))} N_{H_1}^G X\big).$$

Next, we show that

$$\mathscr{L}(\mathrm{Ind}_{H_2}^G \iota_{H_2}^* U, U) \times_{F_{H_2}(G,\mathscr{L}_{\iota_{H_2}^* U}(1))} F_{H_2}\big(G, \mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U)\big)$$

is isomorphic to $\mathscr{L}(\mathrm{Ind}_{H_1}^G \iota_{H_1}^* U, U)$. There is an equivariant map

$$\mathscr{L}(\mathrm{Ind}_{H_2}^G \iota_{H_2}^* U, U) \times F_{H_2}\big(G, \mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U)\big) \to \mathscr{L}(\mathrm{Ind}_{H_1}^G \iota_{H_1}^* U, U)$$

induced by composition and the natural map

$$F_{H_2}\big(G, \mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U)\big) \to \mathscr{L}(\mathrm{Ind}_{H_1}^G \iota_{H_1}^* U, \mathrm{Ind}_{H_2}^G \iota_{H_2}^* U)$$

induced by the direct sum. This map is compatible with the maps determining the coequalizer, and so it suffices to check that the underlying nonequivariant diagram is a reflexive coequalizer. This now follows from Lemma A.6. The theorem is now a consequence of the preceding homeomorphism and Lemma 4.23. □

In the case of commutative monoid objects, it is straightforward to check that the adjunction involving the norm and the forgetful functor is homotopical; it is clear that ι_H^* preserves fibrations and weak equivalences.

Theorem 4.25. *The adjoint pairs*

$$N_{H,\iota_H^* U}^{G,U} : (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \leftrightarrows (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] : \iota_H^*$$

and

$$N_{H,\iota_H^* U}^{G,U} : H\mathcal{S}_{\iota_H^* U}[\mathbb{P}] \leftrightarrows G\mathcal{S}_U[\mathbb{P}] : \iota_H^*$$

are Quillen adjunction.

Note however that the derived functor of the norm $N^{G,U}_{H,\iota^*_H U}$ on commutative rings only agrees with the derived functor of the module norm when G/H is admissible for U; the following result is a consequence of the fact that the derived functor of the norm on commutative rings in orthogonal H-spectra agrees with the underlying norm [Hill et al. 2016, Remark B.148].

Proposition 4.26. *Let X be a cofibrant object in $G\mathcal{S}[\mathbb{L}_U][\mathbb{P}]$. The natural map*

$$N^{G,U}_{H,\iota^*_H U} X \to N^G_H X$$

is a weak equivalence when G/H is admissible for U.

We now turn to the relative norm construction.

Theorem 4.27. *The functor ${}_R N^{G,U}_{H,\iota^*_H U}$ preserves weak equivalence between cofibrant objects in $\mathcal{M}_{\iota^*_H R,\iota^*_H U}$ when G/H is admissible for U.*

Proof. Since the R-relative norm is strong symmetric monoidal, it suffices to show that when X is cofibrant in $\mathcal{M}_{\iota^*_H R,\iota^*_H U}$, $N^{G,U}_{H,\iota^*_H U} X$ is cofibrant as an $N^{G,U}_{H,\iota^*_H U} R$ module. Once again, it suffices to check this on free objects, where it is straightforward. □

5. G-symmetric monoidal categories of modules over an N_∞ algebra

In this section, we describe the homotopical G-symmetric monoidal structure on $\mathcal{M}_{R,U}$. More precisely, we have a $\mathscr{L}_U$-symmetric monoidal structure, where we mean an equivariant symmetric monoidal structure specified by the coefficient system of admissible sets for $\mathscr{L}_U$ [Hill and Hopkins 2016, Definition 4.4]. We characterize this structure in terms of a homotopical exponential functor

$${}_R N^T : \mathcal{M}_{R,U} \to \mathcal{M}_{R,U}$$

for any admissible G-set T. We explain how this "internal norm" arises from structure on the collection of norms and forgetful functors on the categories $\mathcal{M}_{\iota^*_H R,\iota^*_H U}$ as H varies over the closed subgroups of G; these functors assemble into an incomplete Mackey functor in homotopical categories. We also explain the resulting structure on commutative monoid objects, recovering the characterizations of [Blumberg and Hill 2015, Theorem 6.11].

5.1. ***The G-symmetric monoidal structure on $G\mathcal{S}$ and $\mathcal{M}_R$.*** In this subsection, we review the canonical G-symmetric monoidal structure on $G\mathcal{S}$ and $\mathcal{M}_R$ for R a commutative ring orthogonal G-spectrum. We begin by recalling from [Blumberg and Hill 2015, §6] the definition of the internal norm in orthogonal spectra.

Definition 5.1. Let $H \subset G$ be a subgroup. The internal norm of an orthogonal G-spectrum X is specified by the formula

$$N^{G/H} X = N_H^G \iota_H^* X.$$

For an arbitrary G-set T, we define the internal norm by decomposing T into a disjoint union of orbits $\coprod_i G/H_i$ and defining

$$N^T X = \bigwedge_i N^{G/H_i} X.$$

For example, when T is a trivial G-set, $N^T M$ is simply the smash-power of $|T|$ copies of M. Note that this definition extends in the evident way to categories of modules over a commutative ring orthogonal G-spectrum.

There is another equivalent description for this which will make the properties of the norm (summarized in Theorem 5.5 below) more transparent. If T is a finite G-set, then let $B_T G$ denote the translation category of T. This has object set T itself and the morphism set is $T \times G$ with structure maps the projection onto T and the action. Given a G-spectrum X, we have a $B_G T$-shaped diagram X^T described by $t \mapsto X$ and (t, g) acts as multiplication by g on X.

A map of finite G-sets $f : T \to S$ produces a covering category $B_T G \to B_S G$ as in [Hill et al. 2016, Definition A.24], and therefore we have an associated indexed monoidal product $f_*^{\otimes}$.

Proposition 5.2. *Let $p : T \to *$ be the terminal map. There is a canonical homeomorphism*

$$p_*^{\otimes} X^T \cong N^T(X).$$

Proof. Since both sides take disjoint unions to smash products (the left by construction and the right by definition), it suffices to construct the canonical homeomorphism when $T = G/H$. In this case, each side is then an indexed product.

Using the additive change of universe equivalence, we can work in the category of orthogonal spectra with a G-action and prove the desired equality there. In this case, both sides are the indexed product associated to $p : G/H \to *$, so it will suffice to show that the resulting diagrams are isomorphic. For the left-hand side, the diagram is the constant diagram $X^{G/H}$. For the right-hand side, the diagram is determined by choosing coset representatives and sending a coset gH to the gHg^{-1}-spectrum $g \cdot i_H^* X$ (where here $g \cdot Y$ for an H-spectrum Y is just the restriction along the isomorphism $gHg^{-1} \cong H$). However, we then have an equivariant isomorphism of diagrams

$$X^{G/H} \cong (gH \mapsto g \cdot i_H^* X)$$

which at a coset gH is simply multiplication by g. The indexed products are therefore isomorphic. □

Remark 5.3. The key step in the argument is the same as the one showing that we have canonical homeomorphisms

$$F_H(G_+, i_H^* X) \cong F(G/H_+, X) \quad \text{and} \quad G_+ \wedge_H i_H^* X \cong G/H_+ \wedge X.$$

In each case, we have the same two diagrams as the one given above and then we compare the associated indexed monoidal products.

Because N_H^G, $- \wedge -$, and ι_H^* preserve weak equivalences and cofibrant objects, the internal norm is a homotopical functor.

Lemma 5.4. *For any G-set T, the internal norm*

$$N^T : G\mathcal{S} \to G\mathcal{S}$$

preserves acyclic cofibrations.

We can now recall the basic theorem establishing the G-symmetric monoidal structure on G-spectra. All of this follows easily from Appendix A of [Hill et al. 2016]; for convenience, we include details here.

Theorem 5.5. (1) *For $H_1 \subseteq H_2 \subseteq G$, there is a natural homeomorphism*

$$\iota_{H_1}^* \cong \iota_{H_1}^* \iota_{H_2}^*.$$

(2) *For G-sets T_1 and T_2, there is a natural homeomorphism*

$$N^{T_1 \times T_2} X \simeq N^{T_1} N^{T_2} X.$$

(3) *For $K \subset H$, there is a natural homeomorphism*

$$\iota_K^* N^T X \simeq N^{\iota_K^* T} \iota_K^* X.$$

Proof. The first part is obvious. For the second and third parts, we use the alternative description of $N^T X$ given by Proposition 5.2.

For the second, observe that $B_{T_1 \times T_2} G \cong B_{T_1} G \times B_{T_2} G$, and the composite of the norms is the composites of the indexed products

$$T_1 \times T_2 \to T_1 \to *.$$

The composite of the indexed products is the indexed product of the composites [Hill et al. 2016, Proposition A.29].

The third is the variant of the double coset formula here. If T is a finite G-set, then we have a pullback diagram of categories

$$\begin{array}{ccc} B_{G/H\times T}G & \longrightarrow & B_T G \\ \downarrow & & \downarrow \\ B_{G/H}G & \longrightarrow & BG. \end{array}$$

Since $G/H \times T \cong G \times_H i_H^* T$, the left-hand side of this diagram is equivalent to $B_{i_H^* T} H \to BH$. Since the map on spectra induced by pulling back along $B_{G/H}G \to BG$ is i_H^*, we conclude by [Hill et al. 2016, Proposition A.31] that

$$i_H^* N^T X \cong N^{i_H^* T} i_H^{*X}. \qquad \square$$

The analogue of Theorem 5.5 for modules over a commutative ring orthogonal G-spectrum R follows from the characterization of the R-relative norm via the formula

$${}_R N_H^G X \cong R \wedge_{N_H^G R} N_H^G X$$

and the fact that the norm N_H^G is the left adjoint to the restriction functor ι_H^* on commutative rings. We explain in detail the argument below in the proof of Theorem 5.10.

5.2. ***The $\mathscr{L}_U$-symmetric monoidal structure on $G\mathcal{S}_U$ and $\mathcal{M}_{R,U}$.*** We now provide the analogous definitions in our context.

Definition 5.6. Given a subgroup $H \subset G$, we define the internal norm

$${}_R N_U^{G/H} M : \mathcal{M}_{R,U} \to \mathcal{M}_{R,U}$$

as the composite

$${}_R N_U^{G/H}(-) := {}_R N_{H,\iota_H^*}^{G,U} \iota_H^*(-).$$

We extend the internal norm to an arbitrary G-set T by decomposing T into a disjoint union of orbits $\coprod_i G/H_i$ and specifying that

$${}_R N_U^T M = \bigwedge_i {}_R N^{G/H_i} M.$$

We now describe the homotopical properties of the internal norm. We begin by considering the absolute case where $R = S$.

Lemma 5.7. *Let T be an admissible G-set. The functor*

$$N_U^T : G\mathcal{S}_U \to G\mathcal{S}_U$$

preserves weak equivalences between cofibrant objects.

Proof. This is a consequence of the fact that ι_H^* preserves cofibrant orthogonal G-spectra [Mandell and May 2002, Lemma V.2.2], colimits, and the identification

$$\iota_H^* \mathscr{L}_U(1)_+ \wedge X \cong \mathscr{L}_{\iota_H^* U}(1)_+ \wedge (\iota_H^* X). \qquad \square$$

Furthermore, we can also identify the interaction of N^T with the cartesian product.

Lemma 5.8. *When T_1 and T_2 are admissible G-sets and M is a cofibrant object in $G\mathcal{S}[\mathbb{L}_U]$, there is a natural weak equivalence*

$$N_U^{T_1 \times T_2} M \simeq N_U^{T_1}(N_U^{T_2} M).$$

Proof. This follows from Lemma 5.7, Lemma 5.4, and Theorem 4.20. $\square$

Proposition 4.24 shows that the norm functors compose as expected, and it is clear that for $H_1 \subseteq H_2 \subseteq G$, $\iota_{H_1}^* \cong \iota_{H_1}^* \iota_{H_2}^*$.

Theorem 5.9. *Fix $K \subseteq H$, let T be an admissible H-set, and let M be a cofibrant object in $G\mathcal{S}[\mathbb{L}_U]$. The composite $\iota_K^* N_U^T M$ is naturally equivalent to $N_U^{\iota_K^* T} \iota_K^* M$.*

Proof. This again follows from Theorem 4.20 and the fact that the desired equivalence holds for the norm in orthogonal spectra. $\square$

When R is no longer necessarily the sphere, we have corresponding analogues of the preceding results; we summarize the situation in the following theorem.

Theorem 5.10. *Let R be a cofibrant object in $G\mathcal{S}_U[\mathbb{P}]$.*

(1) *The functor ${}_R N_U^T$ preserves weak equivalences between cofibrant objects.*

(2) *When T_1 and T_2 are admissible G-sets and M is a cofibrant object in $\mathcal{M}_{R,U}$, there is a natural equivalence*

$${}_R N_U^{T_1 \times T_2} M \simeq {}_R N_U^{T_1}({}_R N_U^{T_2} M).$$

(3) *For $H_1 \subseteq H_2 \subseteq G$, there is a natural homeomorphism*

$$\iota_{H_1}^* \cong \iota_{H_1}^* \iota_{H_2}^*.$$

(4) *For $K \subset H$ and T an admissible G-set, there is a natural equivalence*

$$\iota_K^* {}_R N_U^T M \simeq {}_R N_U^{\iota_K^* T} \iota_K^* M$$

when M is a cofibrant object in $\mathcal{M}_{R,U}$.

Proof. The first of these follows from Theorem 4.27. The second is a consequence of Theorem 4.20; the proof is analogous to the proof of Lemma 5.8, along with the observation that the smash product defining the relative norm computes the derived smash product under our hypotheses. The third is immediate. For the fourth, we can leverage the absolute result as follows.

Since ι_K^* is a strong symmetric monoidal functor, we have the homeomorphisms

$$\iota_K^*{}_R N_U^T M \cong \iota_K^*(N_U^T M \wedge_{N_U^T R} R) \cong (\iota_K^* N_U^T M) \wedge_{\iota_K^* N_U^T R} \iota_K^* R.$$

By Theorem 5.9, we know that

$$\iota_K^* N_U^T M \simeq N^{\iota_K^* T} \iota_K^* M.$$

Moreover, since N_U^T is a left adjoint on commutative rings, we have an homeomorphism

$$\iota_K^* N_U^T R \cong N_{\iota_H^* U}^{\iota_H^* T} \iota_H^* R,$$

which is compatible with the counit $N_U^T R \to R$ used in the formation of the relative smash product. Since the hypotheses guarantee we are computing the derived smash product, we end up with a natural weak equivalence

$$\iota_K^*{}_R N_U^T M \simeq N^{\iota_K^* T} \iota_K^* M \wedge_{N_{\iota_H^* U}^{\iota_H^* T} \iota_H^* R} \iota_H^* R \cong {}_R N_U^{\iota_K^* T} \iota_K^* M. \qquad \square$$

5.3. ***The multiplicative structure on N_∞ algebras.*** In this subsection, we explain how the $\mathscr{L}_U$-symmetric monoidal structure on $G\mathcal{S}_U$ induces additional multiplicative structure on objects of $G\mathcal{S}_U[\mathbb{P}]$. Of course, Theorem 3.18 implies that an object of $G\mathcal{S}_U[\mathbb{P}]$ is an N_∞ algebra structured by the equivariant linear isometries operad determined by U, and [Blumberg and Hill 2015, Theorem 6.11] explains the extra structure this gives. Our purpose here is to demonstrate that this structure is essentially an immediate consequence of Theorem 5.10.

Let R be a cofibrant object of $G\mathcal{S}_U[\mathbb{P}]$. The adjunction of Theorem 4.25 yields homotopical counit maps

$$N_U^{G/H} = N_{H,\iota_H^*}^G \iota_H^* R \to R$$

for admissible G/H, which clearly induce natural maps

$$N_U^T R \to R \quad \text{and} \quad G_+ \wedge_H N_U^S \iota_K^* R \to R,$$

for admissible G-sets T and admissible $K \subseteq G$ sets S. The argument of [Blumberg and Hill 2015, Theorem 6.8] extends without change to produce a map

$$N_U^T R \to N_U^S R$$

given any G-map $f : S \to T$.

It is clear from the definition of N_U^T that the diagram

$$\begin{array}{ccc} N_U^{S \amalg T} R \cong N_U^S R \wedge N_U^T R & \longrightarrow & R \wedge R \\ \downarrow & \swarrow & \\ R & & \end{array}$$

commutes. Assertion (2) of Theorem 5.10 implies that the diagram

$$\begin{array}{ccc} N_U^{S\times T}R \cong N_U^S N_U^T R & \longrightarrow & N_U^T R \\ \downarrow & \swarrow & \\ R & & \end{array}$$

commutes. Finally, assertion (4) of Theorem 5.10 implies that for any admissible sets S and T such that for some $K \subseteq G$ we have $\iota_K^* S \cong \iota_K^* T$, the diagram

$$\begin{array}{ccccc} \iota_K^* N_U^S R \cong N_{\iota_K^* U}^{\iota_K^* S} \iota_K^* R & & \xrightarrow{\quad\cong\quad} & & N_{\iota_K^* U}^{\iota_K^* T} \iota_K^* R \cong \iota_K^* N_U^T R \\ & \searrow & & \swarrow & \\ & & R & & \end{array}$$

commutes. Thus, we precisely recover the characterizations of [Blumberg and Hill 2015, Theorem 6.11].

6. Examples and applications

We close with several examples in which the technology in this paper can be used to construct symmetric monoidal structures on categories of equivariant modules. The most basic example comes from algebras over the nonequivariant E_∞ operad regarded as a G-operad with trivial action. Algebras over this operad have no multiplicative norms, and so their modules cannot be described in terms of the usual symmetric monoidal model structure on G-spectra. In particular, we do not get a G-symmetric monoidal category of modules. But since this operad can be modeled by the (nonequivariant) linear isometries operad, Theorem 1.1 implies that we can produce a symmetric monoidal category of modules.

More generally, there are many examples that arise when studying smashing Bousfield localization in the equivariant setting. The examples in the first family we study in Sections 6.1 and 6.2 below (generalizing the trivial E_∞ operad) are necessary ingredients in the work of Greenlees and Shipley [2018; 2014] on monoidal equivalences between various models for rational G-spectra. The second class of examples, studied in Sections 6.3 and 6.4 below, is relevant to understanding chromatic localizations in the equivariant setting.

The phenomenon generating all of these results is the following theorem of Hill and Hopkins [2016].

Theorem 6.1. *Let $\mathcal{O}$ be an $\mathcal{N}_\infty$ operad, and let $\underline{\mathcal{C}}_{\mathcal{O}}$ denote the associated indexing system. Let L be a Bousfield localization on the category $G\mathcal{S}$ and let $\underline{\mathcal{Z}}$ denote the coefficient system of acyclics for L (i.e., the value at G/H is the subcategory of the homotopy category of $H\mathcal{S}$ consisting of those H-spectra which are acyclic for the* *restriction of L). Then if $\underline{\mathcal{Z}}$ is closed under the (derived) norms specified by $\underline{\mathcal{C}}_{\mathcal{O}}$, L preserves $\mathcal{O}$-algebras.*

In particular, this theorem reduces questions about what structure a localization preserves to determining categorical structure on the categories of acyclics.

6.1. ***Isotropic localization.*** As was first observed by McClure [1996], the localization which nullifies anything induced does not preserve genuine equivariant commutative rings (e.g., algebras over the linear isometries operad for a complete universe U). In particular, $\Sigma^\infty \tilde{E}\mathcal{P}$ cannot be made into a genuine equivariant commutative ring spectrum: since the restriction to any proper subgroup of $\Sigma^\infty \tilde{E}\mathcal{P}$ is contractible, then the putative counit map determined by the commutative ring structure

$$N_H^G i_H^* \Sigma^\infty \tilde{E}\mathcal{P} \to \Sigma^\infty \tilde{E}\mathcal{P}$$

cannot be unital.

More generally, we can apply Theorem 6.1 to produce immediate strengthenings of this observation. Let $\mathcal{F}$ be a family of subgroups of G. For any $\mathcal{F}$ there exists a smashing localization $L_{\mathcal{F}}$ which nullifies any G-spectrum with isotropy in $\mathcal{F}$. The canonical localization sequence is then precisely the isotropy separation sequence:

$$E\mathcal{F}_+ \wedge X \to X \to \tilde{E}\mathcal{F} \wedge X.$$

Proposition 6.2. *Let $\mathcal{F}$ be a family of subgroups of G which is not the trivial family. Then $\tilde{E}\mathcal{F}$ is not a genuine equivariant commutative ring spectrum. (It is however always a naïve E_∞ ring spectrum.)*

Proof. If $\mathcal{F}$ is nontrivial, then $i_e^* \Sigma^\infty \tilde{E}\mathcal{F}$ is contractible. The argument above now shows that if $\Sigma^\infty \tilde{E}\mathcal{F}$ had a genuine equivariant commutative ring structure, then the absolute norm would factor through the zero ring; we arrive at the same contradiction as above. The final observation is always satisfied by Bousfield localizations. □

Proposition 6.2 shows that the category of local objects (equivalently, the category of modules over $\Sigma^\infty \tilde{E}\mathcal{F}$) cannot be given a symmetric monoidal structure when working with the symmetric monoidal category of orthogonal G-spectra. In contrast, using Theorem 1.1 above, we can obtain a symmetric monoidal category of modules.

Corollary 6.3. *For any family $\mathcal{F}$ of subgroups of G, the category of local spectra for $L_{\mathcal{F}}$ can always be modeled by a symmetric monoidal category.*

More interestingly, we can describe localizations that result in richer equivariant structures on categories of local objects (i.e., modules).

Theorem 6.4. *Let $\mathcal{F}$ be a family of subgroups of G. Let $\mathscr{L}_U$ be such that for all admissible H/K and for H' in the family, the isotropy of*

$$N_K^H i_K^*(\Sigma_+^\infty H/H') = \Sigma_+^\infty \operatorname{Map}_K(H, H/H')$$

is in $\mathcal{F}$. Then $\Sigma^\infty \tilde{E}\mathcal{F}$ is a $\mathscr{L}_U$-algebra and its category of modules is a $\mathscr{L}_U$-symmetric monoidal category.

This provides a very satisfying sanity check. If N is a normal subgroup of G and if $\mathcal{F}_N$ is the family of subgroups which do not contain N, then there is a composite Quillen equivalence

$$\Sigma^\infty \tilde{E}\mathcal{F}_N\text{-Mod} \leftrightarrows (G/N)\mathcal{S},$$

where the right adjoint is essentially just the N-fixed points (e.g., see [Greenlees and Shipley 2014, Propositions 3.2, 3.3]). The target is a $\underline{\mathrm{Set}}^{G/N}$-monoidal category as recalled in Section 5.1 above. Our work can be used to promote this Quillen equivalence to a structured equivalence via the following result.

Corollary 6.5. *Let N be a normal subgroup of G, and let $\mathcal{F}_N$ denote the family of subgroups of G which do not contain N. Then the category of $\Sigma^\infty \tilde{E}\mathcal{F}_N$-modules can be modeled as a $\underline{\mathrm{Set}}^{G/N}$-symmetric monoidal category.*

6.2. ***Idempotent splittings of the sphere spectrum.*** Dress studied idempotent elements in the Burnside ring and established a decomposition of the sphere spectrum $\mathbb{S}$ as the product of localizations $\mathbb{S}[e_L^{-1}]$, where e_L is a primitive idempotent corresponding to a perfect subgroup $L \subset G$. A natural problem is to describe the N_∞ structures on each term in the product. In his thesis, Böhme [2019] solves this problem and establishes that $\mathbb{S}[e_L^{-1}]$ is an N_∞ algebra structured by an operad corresponding to an explicitly described indexing system $\mathcal{O}_L$ determined by L. When $\mathcal{O}_L$ corresponds to a linear isometries operad, the main results in this paper now permit the construction of module categories over these localizations equipped with $\mathcal{O}_L$-monoidal structures.

An interesting specialization of these results shows that the idempotent splitting for the rational equivariant sphere $\mathbb{S}_\mathbb{Q}$ consists of terms which do not possess any norms, i.e., algebras over the nonequivariant E_∞ operad regarded as a G-operad. This situation is closely related to the failure of the algebraic model for rational G-spectra obtained in [Kędziorek 2017] to capture multiplicative norms, as explained in [Barnes et al. 2019].

6.3. ***A nonflat arithmetic localization.*** Even various kinds of arithmetic localizations of equivariant commutative rings can have counterintuitive properties. This shows that care must be taken with the ways one might consider Zariski localizations of commutative rings. We include a basic, somewhat surprising, example here.

If $\underline{R}$ is a Green functor for a finite group G, then given any collection of elements

$$\{a_i \in \underline{R}(G/H_i) \mid i \in \mathcal{I}\},$$

we can form a new Green functor $\underline{R}[a_i^{-1}]$ which is initial amongst all Green functors under $\underline{R}$ in which all of the a_i become units [Blumberg and Hill 2018]. There are several ways to form this, but the simplest is by mirroring a classical algebra construction. Recall that the forgetful functor

$$\mathrm{u} : \mathcal{G}reen^G \to \mathcal{M}ackey^G$$

has a left-adjoint, the symmetric algebra functor Sym. Recall also that the covariant functors

$$\underline{M} \mapsto \underline{M}(T)$$

are representable for any finite G-set T, with representing object $\underline{A}_T$.

Definition 6.6 [Blumberg and Hill 2018, Definition 5.4]. For any finite G-set T, let

$$\underline{A}[x_T] = \mathrm{Sym}(\underline{A}_T).$$

The Green functors $\underline{A}[x_T]$ represent the functors

$$\underline{R} \mapsto \underline{R}(T),$$

and hence act like polynomial rings.

We restrict attention now to $G = C_2$, inverting the element 2 in the underlying ring. Here, we can take advantage of the explicit descriptions of the free Green functors from [Blumberg and Hill 2017, Lemma 3.2].

Definition 6.7. For $G = C_2$, let $\underline{A}\left[\frac{1}{2}_e\right]$ be given by the pushout in commutative Green functors

$$\begin{array}{ccc} \underline{A}[x_{C_2}] & \xrightarrow{2x} & \underline{A}[x_{C_2}] \\ \Big\downarrow{\scriptstyle 1_e} & & \Big\downarrow \\ \underline{A} & \longrightarrow & \underline{A}\left[\frac{1}{2}_e\right]. \end{array}$$

Here the map labeled $2x$ is the map adjoint to the element

$$2x \in \underline{A}[x_{C_2}](C_2) = \mathbb{Z}[x, \bar{x}],$$

where $\bar{x}$ is the Weyl conjugate of x. The map labeled 1_e is the map adjoint to the element

$$1 \in \underline{A}(C_2) = \mathbb{Z}.$$

Proposition 6.8. *The Green functor* $\underline{A}\left[\frac{1}{2_e}\right]$ *splits*:

$$\underline{A}\left[\tfrac{1}{2_e}\right] \cong \underline{I} \times \underline{\mathbb{Z}\left[\tfrac{1}{2}\right]},$$

where $\underline{I}$ *is the augmentation ideal of the Burnside Mackey functor and where* $\underline{\mathbb{Z}\left[\frac{1}{2}\right]}$ *is the constant Mackey functor with value* $\mathbb{Z}\left[\frac{1}{2}\right]$.

Proof. The underlying ring for $\underline{A}\left[\frac{1}{2_e}\right]$ is just $\mathbb{Z}\left[\frac{1}{2}\right]$, by construction. The fixed points are more interesting, however (and are non-Noetherian!). These are the subrings of

$$\mathbb{Z}\left[\tfrac{1}{2}\right][t]/t^2-2t$$

which give an integer when evaluated at $t=0$. The restriction sends t to 2 and the transfer sends 1 to t. In the fixed ring, there are two orthogonal idempotents:

$$e=\tfrac{1}{2}t, \quad 1-e,$$

and these split the fixed points into the product of rings

$$\underline{A}\left[\tfrac{1}{2_e}\right](C_2/C_2) \cong \mathbb{Z} \times \mathbb{Z}\left[\tfrac{1}{2}\right].$$

The projection onto $\mathbb{Z}$ is given by multiplication by $1-e$, and hence this factor restricts to zero. The projection onto $\mathbb{Z}\left[\frac{1}{2}\right]$ is given by multiplication by e, and then restricts isomorphically onto $\mathbb{Z}\left[\frac{1}{2}\right]$. Similarly, the transfer of the element 1 is the element $t=2e$, and hence lands in the factor $\mathbb{Z}\left[\frac{1}{2}\right]$. This gives our splitting in Green functors. □

Remark 6.9. We can use these two idempotents to split C_2-Mackey functors provided 2 is inverted in the underlying ring. This is a weaker condition than 2 being inverted in the fixed points, and so can be viewed as a more general form of the rational splittings above.

Proposition 6.10. *There is no Tambara functor structure on* $\underline{A}\left[\frac{1}{2_e}\right]$ *such that the unit map* $\underline{A} \to \underline{A}\left[\frac{1}{2_e}\right]$ *is a map of Tambara functors.*

Proof. The element $2 \in \mathbb{Z} \cong \underline{A}(C_2)$ has norm

$$N_e^{C_2}(2) = 2+t.$$

If the unit is a map of Tambara functors, then this maps to the element $2+t$, which in our splitting of rings is the pair

$$(2,4) \in \mathbb{Z} \times \mathbb{Z}\left[\tfrac{1}{2}\right].$$

Since the norm is a map of multiplicative monoids, if 2 is inverted, then this must be a unit, and we have reached a contradiction. □

We can mirror all of this in spectra. The role of Sym is just the free E_∞ ring spectrum $\mathbb{P}$, and the representables $\underline{A}_T$ are just $\Sigma^\infty_+ T$. This lets us easily describe the result of inverting an element in the underlying homotopy.

Definition 6.11. Let $S^0\big[\frac{1}{2}_e\big]$ be the pushout in E_∞-ring spectra

$$\begin{array}{ccc} \mathbb{P}(C_{2+}) & \xrightarrow{2x_e} & \mathbb{P}(C_{2+}) \\ \downarrow{\scriptstyle 1_e} & & \downarrow \\ S^0 & \longrightarrow & S^0\big[\tfrac{1}{2}_e\big] \end{array}$$

Proposition 6.12. *The E_∞-ring spectrum $S^0\big[\frac{1}{2}_e\big]$ cannot be made into a commutative ring spectrum.*

Proof. Since the sphere spectrum and $\mathbb{P}(C_2)$ are both (-1)-connected, the zeroth homotopy Green functor of $S^0\big[\frac{1}{2}_e\big]$ is just the pushout of corresponding diagram after applying $\underline{\pi}_0$ levelwise. This is the diagram in Definition 6.7. Proposition 6.10 shows that this has no Tambara functor structure, and so by work of Brun [2007], this shows that $S^0\big[\frac{1}{2}_e\big]$ cannot be a commutative ring spectrum. □

However, Theorem 1.1 guarantees that we again have a good, symmetric monoidal category of modules for $S^0\big[\frac{1}{2}_e\big]$.

6.4. ***Chromatic localization.*** The localization in the previous section can be thought of as a localization which nullifies the spectrum $C_{2+} \wedge M(\mathbb{Z}/2)$. In other words, it is a kind of chromatic localization. Work of Balmer and Sanders [2017] and of Barthel, Hausmann, Naumann, Nikolaus, Noel, and Stapleton [Barthel et al. 2019] has (up to a small ambiguity) classified the triangulated subcategories of $G\mathcal{S}$. These are determined by the topology on the spectrum (in the sense of Balmer [2005]) of $G\mathcal{S}$: triangulated subcategories of $G\mathcal{S}$ are in bijective correspondence with Thomason subsets of the spectrum, i.e., the subsets which are a union of closed subsets with quasicompact complement. Balmer and Sanders showed that the prime ideals are exactly the inverse images under various geometric fixed points functors of the classical Devinatz–Hopkins–Smith type n-spectra.

Given a Thomason subset V, let L_V denote the associated localization nullifying the triangulated subcategory associated to V. Theorem 6.1 above specifies when L_V preserves equivariant multiplicative structures (and a complete classification of such localizations is forthcoming), so we single out a particular case of interest.

Fix a prime p such that $p \mid |G|$ and let $(G\mathcal{S})_p$ denote the category $G\mathcal{S}$ localized at p. Let $V_{n,G}$ denote the triangulated subcategory of $(G\mathcal{S})_p$ generated by $G_+ \wedge M(n)$, where $M(n)$ is any type n-spectrum.

Proposition 6.13. *The localization $L_{V_{n,G}}$ does not preserve genuine equivariant commutative ring spectra.*

Proof. Everything in the triangulated category $V_{n,G}$ has the property that the geometric fixed points are contractible. However, the diagonal map provides an isomorphism in the derived category

$$E \cong \Phi^G N_e^G E$$

for any spectrum E. In particular, taking

$$E = i_e^* G_+ \wedge M(n) \simeq \bigvee_{|G|} M(n)$$

shows that the geometric fixed points of the norm of the generator of the acyclics is not acyclic. □

In particular, there is little hope for any of the equivariant chromatic categories to be G-symmetric monoidal categories. Once again, Theorem 1.1 above guarantees that we can construct models that are symmetric monoidal categories, however, work of the second author builds on this in several other examples [Hill 2018].

Appendix A: The equivariant linear isometries operad

In this section, we collect some technical results about the behavior of the equivariant linear isometries operad.

Lemma A.1. *Let U be any G-universe. If T is a nonempty admissible set for $\mathscr{L}(U)$, then there is a G-equivariant homeomorphism*

$$\mathbb{R}\{T\} \otimes U \to U.$$

Proof. By definition of admissibility, for the linear isometries operad we have an equivariant embedding

$$\mathbb{R}\{T\} \otimes U \to U.$$

This implies that every isomorphism class of representations in $\mathbb{R}\{T\} \otimes U$ is contained in U. The inclusion of a trivial summand in $\mathbb{R}\{T\}$ (which exists since T is nonempty) guarantees that every irreducible representation of U is also in $\mathbb{R}\{T\} \otimes U$. □

Lemma A.2. *The orbit space $\mathscr{L}_U(2)/(\mathscr{L}_U(1) \times \mathscr{L}_U(1))$ consists of a single point. More generally, the orbit space $\mathscr{L}_U(n)/\mathscr{L}_U(1)^{\times n}$ consists of a single point.*

Proof. The right action map $\mathscr{L}_U(2) \times \mathscr{L}_U(1) \times \mathscr{L}_U(1) \to \mathscr{L}_U(2)$ is clearly a map of G-spaces. As a consequence, we can compute the orbit space as the colimit of underlying spaces, and so in this case the result follows from the nonequivariant identification of the orbit space [Elmendorf et al. 1997, I.8.1].

We deduce the general case by induction: We can use Theorem 3.7 to write

$$\mathscr{L}_U(n)/\mathscr{L}_U(1)^{\times n} \cong \big(\mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} (\mathscr{L}_U(1) \times \mathscr{L}_U(n-1))\big)/\mathscr{L}_U(1)^{\times n}.$$

Since coequalizers commute, the result for n now follows from the base case $n = 2$ and the induction hypothesis. □

More generally, we have the following result.

Lemma A.3. *The orbit space* $\mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U)/F_H(G, \mathscr{L}_{\widehat{U}}(1))$ *consists of a single point.*

Proof. As in the proof of Lemma A.2, since the action map

$$\mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U) \times F_H(G, \mathscr{L}_{\widehat{U}}(1)) \to \mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U)$$

is a map of G-spaces, it suffices to compute the orbit space in terms of the colimit of the underlying spaces. In this case, we can deduce the result from Lemma A.2. □

We also have a series of generalizations of [Elmendorf et al. 1997, I.5.4].

Lemma A.4. *Let T and T' be nonempty admissible sets for U. There are natural isomorphisms*

$$\mathscr{L}(\mathbb{R}\{T\}\otimes\widehat{U} \oplus \mathbb{R}\{T'\}\otimes\widehat{U}, U) \cong \mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} \big(\mathscr{L}(\mathbb{R}\{T\}\otimes\widehat{U}, U) \times \mathscr{L}(\mathbb{R}\{T'\}\otimes\widehat{U}, U)\big).$$

Proof. First, observe that it suffices to show that nonequivariantly this isomorphism arises from a reflexive coequalizer diagram. Now using Lemma A.1 to choose isomorphisms $\widehat{U}\otimes\mathbb{R}\{T'\}\cong\widehat{U}$, the required nonequivariant splittings arise just as in the proof of [Elmendorf et al. 1997, I.5.4]. □

A particularly useful corollary of Lemma A.4 is the following:

Corollary A.5. *There is a natural isomorphism*

$$\mathscr{L}(\operatorname{Ind}_H^G \widehat{U} \oplus \operatorname{Ind}_H^G \widehat{U}, U) \cong \mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} \big(\mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U) \times \mathscr{L}(\operatorname{Ind}_H^G \widehat{U}, U)\big).$$

We also have another kind of analogue of [Elmendorf et al. 1997, I.5.4].

Lemma A.6. *Let T be a nonempty admissible set for U. Then there is a natural isomorphism*

$$\mathscr{L}\big((\mathbb{R}\{T\}\otimes\widehat{U}) \oplus (\mathbb{R}\{T\}\otimes\widehat{U}), U\big) \cong \mathscr{L}(\mathbb{R}\{T\}\otimes\widehat{U}, U) \times_{\mathscr{L}_{\widehat{U}}(1)^T} \mathscr{L}_{\widehat{U}}(2)^T.$$

Proof. Again, the result follows by producing a reflexive coequalizer after forgetting the G-action. Specifically, we need to show that the diagram

$$\begin{array}{c}\mathscr{L}(\mathbb{R}\{T\}\otimes\widehat{U},U)\times\mathscr{L}_{\widehat{U}}(1)^T\times\mathscr{L}_{\widehat{U}}(2)^T\\ \downarrow\downarrow\downarrow\\ \mathscr{L}(\mathbb{R}\{T\}\otimes\widehat{U},U)\times\mathscr{L}_{\widehat{U}}(2)^T\\ \downarrow\\ \mathscr{L}((\mathbb{R}\{T\}\otimes\widehat{U})\oplus(\mathbb{R}\{T\}\otimes\widehat{U}),U)\end{array}$$

is a reflexive coequalizer. Choosing $|T|$ isomorphisms $h_i:\widehat{U}^2\cong\widehat{U}$ such that the sum assembles to an isomorphism $h:\mathbb{R}\{T\}\otimes\widehat{U}\oplus\mathbb{R}\{T\}\otimes\widehat{U}\cong\mathbb{R}\{T\}\otimes\widehat{U}$, we can define the splitting map

$$\mathscr{L}\big((\mathbb{R}\{T\}\otimes\widehat{U})\oplus(\mathbb{R}\{T\}\otimes\widehat{U}),U\big)\to\mathscr{L}(\mathbb{R}\{T\}\otimes\widehat{U},U)\times\mathscr{L}_{\widehat{U}}(2)^T$$

via $f\mapsto(f\circ h,h_1,h_2,\ldots,h_{|T|})$. The argument now proceeds exactly as in [Elmendorf et al. 1997, I.5.4]. □

This has the following corollary.

Corollary A.7. *For $H\subseteq G$, there is a natural isomorphism*

$$\mathscr{L}(\mathrm{Ind}_H^G\widehat{U}\oplus\mathrm{Ind}_H^G\widehat{U},U)\cong\mathscr{L}(\mathrm{Ind}_H^G\widehat{U},U)\times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))}F_H(G,\mathscr{L}_{\widehat{U}}(2)).$$

Finally, we turn to the main technical theorem about the equivariant linear isometries operad that justifies the use of the unital objects. In the proof, we make use of the following standard technical lemma:

Lemma A.8. *Let X have a left H-action and right G-action which are compatible* (*i.e.*, *X is an $H\times G$-space*). *Then the coequalizer*

$$(-)\times_H X$$

specifies a functor from the category of $G'\times H$-spaces and equivariant maps to $G'\times G$-spaces and equivariant maps.

Proof. Let Y be a $G'\times H$-space. It is clear that $Y\times_H X$ has a $G'\times G$ action inherited from the G'-action on Y and the G-action on X. Let $f:Y\to Y'$ be a map of $G'\times H$-spaces. Then there is an induced map of spaces

$$\theta_f:Y\times_H X\to Y'\times_H X$$

defined by $(y,x)\mapsto(f(y),x)$. This is a left G'-map since f is a $G'\times H$-map; $\theta_f((g'y,x))=(f(g'y),x)=(g'f(y),x)=g'\theta_f((y,x))$. Similarly, it is a right G-map. □

Theorem A.9. *For each $k > 0$, the map*

$$\gamma_k : \hat{\mathscr{L}}_U(k) = \mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} (\mathscr{L}_U(0) \times \mathscr{L}_U(k)) \to \mathscr{L}_U(k)$$

induced by the operadic structure map

$$\mathscr{L}_U(2) \times \mathscr{L}_U(0) \times \mathscr{L}_U(k) \to \mathscr{L}_U(k)$$

is a homotopy equivalence of $G\times\Sigma_k$-spaces.

Proof. First, consider the case where $k = 1$. In this case, we are considering the map

$$\gamma_1 : \mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} (\mathscr{L}_U(0) \times \mathscr{L}_U(1)) \to \mathscr{L}_U(1).$$

The proof of [Elmendorf et al. 1997, XI.2.2] goes through in the equivariant context to show that γ_1 is a homotopy equivalence of G-spaces. It is helpful to decompose γ_1 as follows [Mandell and May 2002, §VI.6]:

$$\mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} (\mathscr{L}_U(0) \times \mathscr{L}_U(1)) \xrightarrow{\theta_1} \mathscr{L}_U(2)/\mathscr{L}_U(1) \xrightarrow{\theta_2} \mathscr{L}_U(1),$$

where $\mathscr{L}_U(2)/\mathscr{L}_U(1)$ is the orbit space for the right action of $\mathscr{L}_U(1)$ on $\mathscr{L}_U(2)$ given by $(f, h) \mapsto f\circ(h\oplus\mathrm{id})$ and equipped with the right action of $\mathscr{L}_U(1)$ specified by $([f], h) \mapsto [f \circ (\mathrm{id}\oplus h)]$, θ_2 is the restriction to the second summand, and θ_1 is specified by $(g, 0, f) \mapsto g \circ (\mathrm{id}\oplus f)$. Both maps are $G\times\mathscr{L}_U(1)$-maps, and θ_1 is a homeomorphism.

Now take $k > 1$. Then γ_k factors as the composite

$$\hat{\mathscr{L}}_U(k) \cong (\mathscr{L}_U(2)/\mathscr{L}_U(1)) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k) \to \mathscr{L}_U(1) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k) \cong \mathscr{L}_U(k),$$

induced by γ_1, where we are using the homeomorphism

$$\hat{\mathscr{L}}_U(k) \cong (\mathscr{L}_U(2) \times_{\mathscr{L}_U(1)\times\mathscr{L}_U(1)} (\mathscr{L}_U(0) \times \mathscr{L}_U(1))) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k).$$

To see this, observe that $\gamma_k((g, 0, f)) = g \circ (f \oplus 0)$. On the other hand, the composite above first takes $(g, 0, f)$ to $((g\circ\mathrm{id}), f)$, then $((g\circ\mathrm{id}), f)$ to $(\theta_2(g), f)$, and finally $(\theta_2(g), f)$ to $\theta_2(g)\circ f = g\circ(f \oplus 0)$.

Since $\mathscr{L}_U(k)$ is a universal space for the family of subgroups of $G \times \Sigma_k$ prescribed by U, it suffices to show that $(\mathscr{L}_U(2)/\mathscr{L}_U(1)) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k)$ is also a universal space for the same family. To do this, we will unpack part of the proof of [Elmendorf et al. 1997, XI.2.2].

Write $U \cong U_1 \oplus U_2$ as G-spaces, where U_1 and U_2 are G-universes such that $U_1 \cong U$ and $U_2 \cong U$; we can do this by Lemma A.1. Define $\mathscr{K}(2) \subset \mathscr{L}_U(2)$ to be $\{f \mid f(\{0\}\oplus U) \subset U_2\}$, equipped with the conjugation G-action. Next, we define

$$\hat{\mathscr{K}_1} = \mathscr{K}(2)/\mathscr{L}_U(1)$$

and we let $\mathscr{K}_1 \subset \mathscr{L}_U(1)$ be $\{f \mid f(U) \subseteq U_2\}$ with the conjugation G-action. The map θ_2 restricts to give a G-map $\hat{\mathscr{K}_1} \to \mathscr{K}_1$ which is compatible with the action of $\mathscr{L}_U(1)$ and so by Lemma A.8 we have an induced $G\times\Sigma_k$-map

$$\hat{\mathscr{K}_1} \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k) \to \mathscr{K}_1 \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k).$$

The nonequivariant argument in [Elmendorf et al. 1997, XI.2.2] extends to the equivariant case to show that the map $\hat{\mathscr{K}_1} \to \mathscr{K}_1$ is a homeomorphism of G-spaces. On the other hand, we have a homeomorphism of G-spaces $\mathscr{K}_1 \cong \mathscr{L}(U, U_2) \cong \mathscr{L}_U(1)$ which is compatible with the action of $\mathscr{L}_U(1)$, and so Lemma A.8 implies that there is a composite $G\times\Sigma_k$-map

$$\mathscr{K}_1 \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k) \to \mathscr{L}_U(1) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k) \cong \mathscr{L}_U(k)$$

which is a homeomorphism. Putting these together, we have a $G\times\Sigma_k$-map

$$\hat{\mathscr{K}_1} \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k) \cong \mathscr{L}_U(k)$$

that is a homeomorphism.

To finish the argument, observe that the proof in [Elmendorf et al. 1997, XI.2.2] extends to the equivariant context to show that the inclusion

$$\mathscr{K}(2) \to \mathscr{L}_U(2)$$

is a G-homotopy equivalence of right $\mathscr{L}_U(1)\times\mathscr{L}_U(1)$-spaces and therefore

$$\hat{\mathscr{K}_1} \to (\mathscr{L}_U(2)/\mathscr{L}_U(1))$$

is a G-homotopy equivalence of right $\mathscr{L}_U(1)$-spaces. As a consequence, the induced map

$$\hat{\mathscr{K}_1} \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k) \to (\mathscr{L}_U(2)/\mathscr{L}_U(1)) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k)$$

is a $G\times\Sigma_k$-homotopy equivalence. □

Appendix B: Compact Lie groups

In this appendix, we quickly outline what aspects of our work in this paper continue to hold when G is an infinite compact Lie group. Basically, all of the foundational material in this paper goes through except the results on multiplicative norms; when G is an infinite compact Lie group, norms exist only for subgroups H of finite index and hence we can only work with admissible finite sets. With this modification, the theorems of the paper remain true.

To be more precise, the work of the paper depends on various results about the linear isometries operad, mostly collected in Appendix A. Lemma A.1 holds with the same proof for finite G-sets; however, in all of our applications of Lemma A.1, this case suffices. Lemmas A.2 and A.3 hold with the same proofs; these arguments do not rely on the finiteness of G. Lemmas A.4 and A.6 again require finite G-sets, but this suffices to conclude Lemmas A.5 and A.7, respectively. In the body of the paper, Theorem 3.15 goes through with the same proof, as does the essential Theorem A.9.

As a consequence, the work of the remainder of the paper goes through without modification in the arguments except for the material on the norm in Sections 3.5, 4.3, and 5. Here, the results on N_H^G require that G/H be a finite G-set, i.e., that the subgroups have finite index.

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