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Almost $\mathbb{C}_p$ Galois representations and vector bundles

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Let K be a finite extension of $\mathbb{Q}_p$ and G_K the absolute Galois group. Then G_K acts on the fundamental curve X of p-adic Hodge theory and we may consider the abelian category $\mathcal{M}(G_K)$ of coherent $\mathcal{O}_X$-modules equipped with a continuous and semilinear action of G_K.

An *almost $\mathbb{C}_p$-representation of G_K* is a p-adic Banach space V equipped with a linear and continuous action of G_K such that there exists $d \in \mathbb{N}$, two G_K-stable finite dimensional sub-$\mathbb{Q}_p$-vector spaces U_+ of V, U_- of $\mathbb{C}_p^d$, and a G_K-equivariant isomorphism

$$V/U_+ \to \mathbb{C}_p^d/U_-.$$

These representations form an abelian category $\mathcal{C}(G_K)$. The main purpose of this paper is to prove that $\mathcal{C}(G_K)$ can be recovered from $\mathcal{M}(G_K)$ by a simple construction (and vice-versa) inducing, in particular, an equivalence of triangulated categories

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)).$$

1. Introduction

1A. We fix a prime number p, an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and a finite extension K of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. We set $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ and $\mathbb{C}_p$ the p-adic completion of $\overline{\mathbb{Q}}_p$ on which G_K acts by continuity.

The fundamental curve $X_{\mathbb{Q}_p,\mathbb{C}_p^\flat}$ of p-adic Hodge theory, denoted by X below, was introduced in [Fargues and Fontaine 2018]. It is a separated noetherian regular scheme of dimension 1 defined over $\mathbb{Q}_p$; i.e., $H^0(X, \mathcal{O}_X) = \mathbb{Q}_p$. The structural sheaf is naturally equipped with a topology (Section 3D): if U is any open subset of X, then $\mathcal{O}_X(U)$ is a locally convex $\mathbb{Q}_p$-algebra. There is a natural action of

Jean-Marc Fontaine passed away on 29 January 2019. I saw him last in late November 2018, when he mentioned to me that he wanted to submit this paper to Tunisian Journal of Mathematics after making some small changes, and asked me if I could take care of the paper in case he could not do it himself; to which I, of course, agreed. Contributing to Fontaine's program has been one of the joys of my mathematical career and this paper puts the final touch to the geometrization of this program via the Fargues–Fontaine curve. – Pierre Colmez, 4 August 2019.

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G_K on X which is continuous. We may consider the abelian category $\mathcal{M}(G_K)$ of *G_K-equivariant coherent $\mathcal{O}_X$-modules*, that is of coherent $\mathcal{O}_X$-modules equipped with a semilinear and continuous action of G_K.

Any nonzero $\mathcal{F} \in \mathrm{Ob}(\mathcal{M}(G_K))$ has a degree $\deg(\mathcal{F}) \in \mathbb{Z}$ and a rank $\mathrm{rk}(\mathcal{F}) \in \mathbb{N}$, hence also a slope $s(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rk}(\mathcal{F}) \in \mathbb{Q} \cup \{+\infty\}$ (with the convention that $s(\mathcal{F}) = +\infty$ if $\mathcal{F}$ is a torsion $\mathcal{O}_X$-module). As in the classical case, one says that a coherent $\mathcal{O}_X[G_K]$-module $\mathcal{F}$ is *semistable* if $\mathcal{F} \neq 0$ and if $s(\mathcal{F}') \leq s(\mathcal{F})$ for any nonzero subobject $\mathcal{F}'$ of $\mathcal{F}$.

We may consider the full subcategory $\mathcal{M}^0(G_K)$ of $\mathcal{M}(G_K)$ whose objects are semistable of slope 0. One of the main results of [Fargues and Fontaine 2018] is that, if $\mathcal{F}$ is any object of $\mathcal{M}^0(G_K)$, then $\mathcal{F}(X) = H^0(X, \mathcal{F})$ is a finite-dimensional $\mathbb{Q}_p$-vector space, hence is an object of the abelian category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of p-adic representations of G_K (that is of finite-dimensional $\mathbb{Q}_p$-vector spaces equipped with a linear and continuous action of G_K) and that the functor

$$\mathcal{M}^0(G_K) \to \mathrm{Rep}_{\mathbb{Q}_p}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of categories (with $V \mapsto \mathcal{O}_X \otimes_{\mathbb{Q}_p} V$ as a quasi-inverse).

The main purpose of this paper is to discuss the following question: Is there an extension of this result enabling us to give an analogous Galois description of all objects of $\mathcal{M}(G_K)$?

1B. In [Fontaine 2003], I introduced the category of *almost $\mathbb{C}_p$-representations of G_K*: A *Banach representation of G_K* is a p-adic Banach space (i.e., a topological $\mathbb{Q}_p$-vector space whose topology can be defined by a norm and which is complete) equipped with a linear and continuous action of G_K. With an obvious definition of morphisms, Banach representations of G_K form an additive category $\mathcal{B}(G_K)$ containing the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ as a full subcategory. By continuity, G_K acts on the p-adic completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$ has a natural structure of a Banach representation. The category $\mathcal{C}(G_K)$ of *almost $\mathbb{C}_p$-representations of G_K* is the full subcategory of $\mathcal{B}(G_K)$ whose objects are those V's for which one can find $d \in \mathbb{N}$, two G_K-stable finite-dimensional sub-$\mathbb{Q}_p$-vector spaces U_+ of V and U_- of $\mathbb{C}_p^d$ and an isomorphism $V/U_+ \to \mathbb{C}_p^d/U_-$ in $\mathcal{B}(G_K)$. This category turns out to be abelian (*loc. cit.*).

The curve X has only one closed point ∞ which is G_K-stable and the orbit under G_K of any other closed point is infinite. This implies that a torsion object of $\mathcal{M}(G_K)$ is supported at ∞. As the completion of $\mathcal{O}_{X,\infty}$ is the ring B_{dR}^+ of p-adic periods, the category $\mathcal{M}^\infty(G_K)$ of torsion objects of $\mathcal{M}(G_K)$ ($\iff$ semistable objects of slope ∞) can be identified with the category $\mathrm{Rep}^{\mathrm{tor}}_{B_{dR}^+}(G_K)$ of B_{dR}^+-modules of finite length equipped with a semilinear and continuous action of G_K. The topology of any B_{dR}^+-module of finite length is the topology of a p-adic Banach space and we

may consider the forgetful functor

$$\mathrm{Rep}^{\mathrm{tor}}_{B^+_{dR}}(G_K) \to \mathcal{B}(G_K).$$

We proved in *loc. cit.* that this functor is fully faithful and that the essential image $\mathcal{C}^\infty(G_K)$ is contained in $\mathcal{C}(G_K)$. Hence, setting $\mathcal{C}^0(G_K) = \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, we see that for $s \in \{0, \infty\}$, the functor $\mathcal{F} \mapsto \mathcal{F}(X)$ induces an equivalence of categories

$$\mathcal{M}^s(G_K) \to \mathcal{C}^s(G_K).$$

Similarly as for a smooth projective curve over a field, we defined in [Fargues and Fontaine 2018] the Harder–Narasimhan filtration of any $\mathcal{F} \in \mathcal{M}(G_K)$: this is the unique filtration

$$0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^{r-1} \subset \mathcal{F}^r = \mathcal{F}$$

such that all the $\mathcal{F}^i/\mathcal{F}^{i-1}$ are semistable and that $s(\mathcal{F}^i/\mathcal{F}^{i-1}) > s(\mathcal{F}^{i+1}/\mathcal{F}^i)$ for $0 < i < r$. We call the $s(\mathcal{F}^i/\mathcal{F}^{i-1})$, for $1 \le i \le r$, the *HN-slopes of* $\mathcal{F}$.

Let $\mathcal{M}^{\ge 0}(G_K)$ the full subcategory of $\mathcal{M}(G_K)$ whose objects are *effective*, i.e., such that all their HN-slopes are ≥ 0.

Similarly let $\mathcal{C}^{\ge 0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ whose objects are *effective*, i.e., those V's which are isomorphic to a subobject (in $\mathcal{C}(G_K)$) of an object of $\mathcal{C}^\infty(G_K)$.

If $\mathcal{F}$ is any coherent $\mathcal{O}_X[G_K]$-module, then $\mathcal{F}(X)$ is a topological $\mathbb{Q}_p$-vector space equipped with a linear and continuous action of G_K. Our main result is this:

Theorem 5.9. *If $\mathcal{F}$ is any coherent $\mathcal{O}_X[G_K]$-module, $\mathcal{F}(X)$ is an effective almost $\mathbb{C}_p$-representation of G_K. By restriction to $\mathcal{M}^{\ge 0}(G_K)$ the functor $\mathcal{F} \mapsto \mathcal{F}(X)$ induces an equivalence of categories*

$$\mathcal{M}^{\ge 0}(G_K) \to \mathcal{C}^{\ge 0}(G_K).$$

This equivalence doesn't extend to an equivalence between $\mathcal{M}(G_K)$ and $\mathcal{C}(G_K)$. Nevertheless each of these two categories can be reconstructed from the other: The above functor induces an equivalence of triangulated categories

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K))$$

and each of them can be reconstructed as the heart of a t-structure. More precisely:

• Denote by $\mathcal{M}^{<0}(G_K)$ the full subcategory of $\mathcal{M}(G_K)$ whose objects are those for which all HN-slopes are < 0. Then $t = (\mathcal{M}^{\ge 0}(G_K), \mathcal{M}^{<0}(G_K))$ is what is called a torsion pair on $\mathcal{M}(G_K)$. From this torsion pair, we can construct an other abelian category $\mathrm{f}(\mathcal{M}(G_K))^t$ which is the full subcategory of $D^b(\mathcal{M}(G_K))$ whose objects are those $\mathcal{F}^\bullet$ such that $\mathcal{F}^i = 0$ for $i \notin \{0, 1\}$, while

$$H^0(\mathcal{F}^\bullet) \text{ is an object of } \mathcal{M}^{<0}(G_K) \quad \text{and} \quad H^1(\mathcal{F}^\bullet) \text{ is an object of } \mathcal{M}^{\ge 0}(G_K).$$

There is a natural equivalence $(\mathcal{M}(G_K))^t \to \mathcal{C}(G_K)$.

• Similarly, denote by $\mathcal{C}^{<0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ whose objects are those V's for which $\mathrm{Hom}(V, W) = 0$ for all W in $\mathcal{C}^{\infty}(G_K)$. Then

$$t' = (\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$$

is a torsion pair on $\mathcal{C}(G_K)$ which can be used to define the abelian subcategory $(\mathcal{C}(G_K))^{t'}$ which is the full subcategory of $D^b(\mathcal{C}(G_K)$ whose objects are those $V^\bullet$ such that $V^i = 0$ for $i \notin \{0, 1\}$, while

$$H^0(V^\bullet) \text{ is an object of } \mathcal{C}^{\geq 0}(G_K) \quad \text{and} \quad H^1(V^\bullet) \text{ is an object of } \mathcal{C}^{<0}(G_K).$$

There is a natural equivalence $(\mathcal{C}(G_K))^{t'} \to \mathcal{M}(G_K)$.

A description à la Beauville–Lazlo of vector bundles on X gives an equivalence of categories between G_K-equivariant vector bundles on X and Berger's B-pairs [Berger 2008]. Specializing the above results to the subcategory $\mathrm{Bund}_X(G_K)$ of $\mathcal{M}(G_K)$ of vector bundles recovers (via this equivalence of categories) some results of Berger [2009].

1C. ***Contents.*** In Section 2, we recall and slightly extend the results of [Fontaine 2003] on almost $\mathbb{C}_p$-representations. We first recall (Section 2A) some basic facts about locally convex spaces over a nonarchimedean field. We introduce (Section 2B) the category of (p-adic) ind-Fréchet representations (of G_K). Then (Section 2C), we recall some basic facts about the ring of periods B_{dR}^+ and B_{dR} that we equip with a locally convex topology. In Section 2D, we discuss some properties of B_{dR}^+-representations and B_{dR}-representations (of G_K).

We describe (Section 2E) the main properties of the category $\mathcal{C}(G_K)$ of almost $\mathbb{C}_p$-representations and of its full subcategories $\mathcal{C}^0(G_K)$ of finite-dimensional p-adic representations and $\mathcal{C}^{\infty}(G_K)$ of B_{dR}^+-representations of finite length. In Section 2E, we also introduce the category $\widehat{\mathcal{C}}(G_K)$ of representations of G_K which are suitable limits (in the category of locally convex p-adic representations of G_K) of almost $\mathbb{C}_p$-representations. In Section 2F, we recall the notion of almost split exact sequence of $\mathcal{B}(G_K)$ and the fact that an extension in $\mathcal{B}(G_K)$ of two almost $\mathbb{C}_p$-representations is an almost $\mathbb{C}_p$-representation if and only if the associated short exact sequence almost splits.

In Section 3, we study the category $\mathrm{Rep}_{B_e}(G_K)$ of *B_e-representations of G_K* (several of the results we obtain are already in [Berger 2008; 2009]). We also recall and make more precise some of the results of [Fargues and Fontaine 2018] on *coherent $\mathcal{O}_X[G_K]$-modules*. We first recall (Section 3A) some basic facts about the sub-$\mathbb{Q}_p$-algebras B_{cris}^+ and B_e of B_{dR} which are stable under the action of G_K and equipped with a natural topology of locally convex algebras. Then we introduce (Section 3B) $\mathrm{Rep}_{B_e}(G_K)$ and show that this is a $\mathbb{Q}_p$-linear abelian category.

We recall (Section 3C) the definition of the fundamental curve $X = X_{\mathbb{Q}_p, \mathbb{C}_p^\flat}$ of p-adic Hodge theory introduced in [Fargues and Fontaine 2018] on which G_K acts and give a description of the category $\mathrm{Coh}(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules. We discuss (Section 3D) the topology on the structural sheaf $\mathcal{O}_X$ and give a description of the category $\mathcal{M}(G_K)$ of coherent $\mathcal{O}_X[G_K]$-modules (Section 3E). We describe (Section 3F) the Harder–Narasimhan filtration on any $\mathcal{F} \in \mathcal{M}(G_K)$.

We consider two full subcategories of $\mathcal{M}(G_K)$:

– the category $\mathcal{M}^0(G_K)$ of the semistable objects of slope 0,

– the category $\mathcal{M}^\infty(G_K)$ of objects whose underlying $\mathcal{O}_X$-module is torsion.

We show (Section 3G) that the global sections functor induces equivalence of categories

$$\mathcal{M}^0(G_K) \to \mathcal{C}^0(G_K) \quad \text{and} \quad \mathcal{M}^\infty(G_K) \to \mathcal{C}^\infty(G_K).$$

In Section 3H, we introduce two kinds of twists of the objects of $\mathcal{M}(G_K)$, the *Tate twists* and the *Harder–Narasimhan twists*.

Say that a B_e-representation Λ is *trivialisable* if there exists $U \in \mathcal{C}^0(G_K)$ and an isomorphism $B_e \otimes_{\mathbb{Q}_p} U \to \Lambda$. In Section 3I, we show that $\mathrm{Rep}_{B_e}(G_K)$ is the smallest subcategory of itself containing trivialisable B_e-representations and stable under taking extensions and direct summands.

In Section 3A0, we show that, if Λ is a B_e-representation of G_K, then the underlying topological $\mathbb{Q}_p$-vector space equipped with its action of G_K is an object of $\widehat{\mathcal{C}}(G_K)$ and that the forgetful functor

$$\mathrm{Rep}_{B_e}(G_K) \to \widehat{\mathcal{C}}(G_K)$$

is exact and fully faithful. (This was already known to Berger [2009, théorème B].)

We conclude this section by discussing the cohomology of coherent $\mathcal{O}_X$-modules (Section 3A1) and of coherent $\mathcal{O}_X[G_K]$-modules (Section 3A2). We show that, taking the global sections, we get a functor

$$\mathcal{M}(G_K) \to \mathcal{C}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X) = H^0(X, \mathcal{O}_X)$$

whose essential image is contained in $\mathcal{C}^{\geq 0}(G_K)$.

The aim of Section 4 is to construct a left adjoint

$$\mathcal{C}(G_K) \to \mathcal{M}(G_K), \quad V \mapsto \mathcal{F}_V$$

of the functor $\mathcal{F} \mapsto \mathcal{F}(X)$.

We show (Section 4C) that any almost $\mathbb{C}_p$-representation V has a *B_e-hull*, i.e., there is a pair $V_e = (V_e, \iota_e^V)$ with V_e a B_e-representation (of G_K) and $\iota_e^V : V \to V_e$ a morphism in $\widehat{\mathcal{C}}(G_K)$ such that, for all $\Lambda \in \mathrm{Rep}_{B_e}(G_K)$, the map

$$\mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(V_e, \Lambda) \to \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, \Lambda)$$

induced by ι_e^V is bijective.

Similarly with obvious definitions, we show that V has a B_{dR}^+-hull V_{dR}^+ and a B_{dR}-hull V_{dR}.

Using the existence of these hulls and the relations between them and knowing the description of $\mathcal{M}(G_K)$ given in Section 3E, the construction of the functor $V \mapsto \mathcal{F}_V$ is quite simple.

The proof of the existence of these hulls relies heavily on the description of all extensions in $\mathcal{C}(G_K)$ of an object of $\mathcal{C}^\infty(G_K)$ by an object of $\mathcal{C}^0(G_K)$, which is given in Section 4B.

The aim of Section 5 is to prove our main result (Theorem 5.9).

We show in Section 5A (resp. 5B) that $\mathcal{M}^{\geq 0}(G_K)$ (resp. $\mathcal{C}^{\geq 0}(G_K)$) is the smallest full subcategory of $\mathcal{M}(G_K)$ (resp. $\mathcal{C}(G_K)$) containing $\mathcal{M}^0(G_K)$ and $\mathcal{M}^\infty(G_K)$ (resp. $\mathcal{C}^0(G_K)$ and $\mathcal{C}^\infty(G_K)$) and stable under extensions and direct summands.

In Section 5C we prove by dévissage that the functor

$$\mathcal{M}^{\geq 0}(G_K) \to \mathcal{C}^{\geq 0}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of exact categories (see Section 1E), the functor $V \mapsto \mathcal{F}_V$ being a quasi-inverse.

The purpose of Section 6 is to extend the main result to the categories $\mathcal{M}(G_K)$ and $\mathcal{C}(G_K)$.

After some general nonsense on derived categories of exact subcategories of abelian categories (Section 6A), we first extend the main result to an equivalence of of triangulated categories (Section 6B),

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)).$$

To go further, we need to introduce the full subcategories $\mathcal{M}^{<0}(G_K)$ of $\mathcal{M}(G_K)$ and $\mathcal{C}^{<0}(G_K)$ of $\mathcal{C}(G_K)$ of coeffective objects. The main theorem said that, if $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$, then $H^0(X, \mathcal{F})$ has a natural structure of an object of $\mathcal{C}^{\geq 0}(G_K)$ and this structure determines $\mathcal{F}$. We prove in Section 6C that, if $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$, then $H^1(X, \mathcal{F})$ has a natural structure of an object of $\mathcal{C}^{<0}(G_K)$ and this structure determines $\mathcal{F}$.

Using this result, we can build $\mathcal{C}(G_K)$ from $\mathcal{M}(G_K)$ and conversely. We give two different recipes (with independent proofs) for that. In Section 6D we describe explicitly the heart of the t-structure on $D^b(\mathcal{M}(G_K))$ corresponding to $\mathcal{C}(G_K)$ and of the t-structure on $D^b(\mathcal{C}(G_K))$ corresponding to $\mathcal{M}(G_K))$. In Section 6E, we explain that $(\mathcal{M}^{\geq 0}(G_K), \mathcal{M}^{<0}(G_K))$ is a torsion pair on $\mathcal{M}(G_K)$. One can use it to construct a new abelian category equipped with a torsion pair. Up to equivalence, it is $\mathcal{C}(G_K)$ equipped with the torsion pair $(\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$.

1D. ***A remark on possible generalisations.*** The results of this paper are obviously a special case of a much more general result where K is replaced by any reasonable rigid analytic, Berkovich or adic space. Let's sketch a description of the case where K is now any field complete with respect to a nonarchimedean nontrivial absolute value with perfect residue field of characteristic p.

• We can define the abelian category $\mathrm{Coh}(\mathcal{O}_{X_K})$ of *coherent modules on the curve* X_K. When K is a perfectoid field, X_K is the curve $X_{\mathbb{Q}_p,K^\flat}$ constructed in [Fargues and Fontaine 2018]. If K is not perfectoid, then X_K doesn't exist but one can define the category of coherent modules over this virtual curve. When K is a finite extension of $\mathbb{Q}_p$, there is a natural equivalence of categories

$$\mathrm{Coh}(\mathcal{O}_{X_K}) \to \mathcal{M}(G_K).$$

• We still have the Harder–Narasimhan filtration on $\mathrm{Coh}(\mathcal{O}_{X_K})$ and may consider its exact subcategories $\mathrm{Coh}^{\geq 0}(\mathcal{O}_{X_K})$ and $\mathrm{Coh}^{<0}(\mathcal{O}_{X_K})$ which form a torsion pair t on $\mathrm{Coh}(\mathcal{O}_{X_K})$.

• The construction of the curve X_K is functorial in K. If C is the completion of a separable closure K^s of K, for any coherent $\mathcal{O}_{X_K}$-module $\mathcal{F}$, we may consider the pull-back $f^*\mathcal{F}$ of $\mathcal{F}$ via $f : X_C \to X_K$.

If $G_K = \mathrm{Gal}(K^s/K)$, we may consider the exact category $\mathcal{B}(G_K)$ of p-adic Banach representations of G_K and we have exact and faithful functors

$$\mathrm{Coh}^{\geq 0}(\mathcal{O}_{X_K}) \to \mathcal{B}(G_K), \quad \mathcal{F} \mapsto H^0(X_C, f^*\mathcal{F}),$$
$$\mathrm{Coh}^{<0}(\mathcal{O}_{X_K}) \to \mathcal{B}(G_K), \quad \mathcal{F} \mapsto H^1(X_C, f^*\mathcal{F}).$$

But, in general, these functors are not fully faithful. Working with $\mathcal{B}(G_K)$ amounts to work over the small pro-étale site of K and we need to work with a bigger site. A possibility is to use the big pro-étale site $K_{\text{proét}}$ of K as defined in [Scholze 2017, §8][1] and to replace $\mathcal{B}(G_K)$ with the category $\mathrm{Vect}_{\mathbb{Q}_p}(K)$ of $\mathbb{Q}_p$-sheaves over $K_{\text{proét}}$, and $\mathcal{C}(G_K)$ with the category of *pseudo-geometric* $\mathbb{Q}_p$*-sheaves*, an abelian full subcategory of $\mathrm{Vect}_{\mathbb{Q}_p}(K)$ defined by imitating the definition of $\mathcal{C}(G_K)$ as a full subcategory of $\mathcal{B}(G_K)$.

The correspondence $K \mapsto X_K$ can be extended to a functor

$$U \mapsto X_U$$

[1]More precisely, we fix an uncountable cardinal κ satisfying the properties of [Scholze 2017, Lemma 4.1]. The underlying category is the category of perfectoid spaces over K which are κ-small [loc. cit., Definition 4.3] and coverings are as defined in [loc. cit., Definition 8.1] (the only difference with the big pro-étale site of Scholze is that we restrict ourself to perfectoid spaces lying over the given nonarchimedean field K).

from the category of perfectoid spaces to the category of $\mathbb{Q}_p$-schemes. We also have exact and faithful functors

$$\mathrm{Coh}^{\geq 0}(\mathcal{O}_{X_K}) \to \mathrm{Vect}_{\mathbb{Q}_p}(K), \quad \mathcal{F} \mapsto \big(U \mapsto H^0(X_U, f_U^*\mathcal{F})\big),$$
$$\mathrm{Coh}^{<0}(\mathcal{O}_{X_K}) \to \mathrm{Vect}_{\mathbb{Q}_p}(K), \quad \mathcal{F} \mapsto \big(U \mapsto H^1(X_U, f_U^*\mathcal{F})\big),$$

where $f_U : X_U \to X_K$ is the structural morphism.

It seems likely (and not so hard to prove) that these functors are fully faithful and that one can describe their essential images $\mathrm{Vect}_{\mathbb{Q}_p}^{pg,\geq 0}(K)$ and $\mathrm{Vect}_{\mathbb{Q}_p}^{pg,<0}(K)$. These two functors seem to induce an equivalence of categories

$$(\mathrm{Coh}(\mathcal{O}_{X_K}))^t \to \mathrm{Vect}_{\mathbb{Q}_p}^{pg}(K)$$

the induced torsion pair on $\mathrm{Vect}_{\mathbb{Q}_p}^{pg}(K)$ being $t' = (\mathrm{Vect}_{\mathbb{Q}_p}^{<0}(K), \mathrm{Vect}_{\mathbb{Q}_p}^{\geq 0}(K))$.

In the case where K is the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, this result has been proved by Le Bras [2018]. We hope to come back soon to this generalisation.

1E. ***Conventions and notations.*** If $\mathcal{C}$ is a category, we often write $C \in \mathcal{C}$ for $C \in \mathrm{Ob}(\mathcal{C})$.

An *exact subcategory of an abelian category* $\mathcal{A}$ is a strictly full subcategory of $\mathcal{A}$ containing 0 and stable under extensions.

If $\mathcal{B}$ is an exact subcategory of $\mathcal{A}$, we say that a sequence of morphisms of $\mathcal{A}$ is *exact* if it is exact as a sequence of morphisms in $\mathcal{A}$. In particular, we have the obvious notion of a *short exact sequence*. It is easy to see that, equipped with this class of short exact sequences, $\mathcal{B}$ is an exact category in the sense of Quillen (cf. [Quillen 1973], see also [Laumon 1983]). Actually, any exact category $\mathcal{B}$ in the sense of Quillen can be viewed as an exact subcategory of an abelian category (cf. [Quillen 1973, §2]).

As usual $\mathbb{Z}_p(1)$ is the Tate module of the multiplicative group, and, for all $n \in \mathbb{N}$,

$$\mathbb{Z}_p(n) = \mathrm{Sym}^n_{\mathbb{Z}_p}\mathbb{Z}_p(1), \quad \mathbb{Z}_p(-n) = \mathcal{L}_{\mathbb{Z}_p}(\mathbb{Z}_p(n), \mathbb{Z}_p).$$

If M is any $\mathbb{Z}_p$-module equipped with a linear action of G_K, for all $n \in \mathbb{Z}$,

$$M(n) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n).$$

2. Representations of G_K

In this paper, each time we say “representation", we mean “representation of G_K". In this section, we introduce a few categories of such representations and describe some of their properties. Most of them are already known (see in particular [Fontaine 2003]) or easy consequences of known properties.

2A. ***Banach, Fréchet, ind-Banach and ind-Fréchet.*** We refer to [Emerton 2017] and [Schneider 2002] for basic facts about p-adic functional analysis. All results of this paragraph are either contained or easy consequences of results contained in at least one of these two memoirs.

We fix a nonarchimedean field E, i.e., a field complete with respect to a nontrivial nonarchimedean absolute value, and denote by $\mathcal{O}_E$ its valuation ring. In the applications in this paper, E will be $\mathbb{Q}_p$.

- A *locally convex E-vector space* is a topological E vector space V such that the open sub-$\mathcal{O}_E$-modules of V form a fundamental system of neighbourhood of 0.
- A *Fréchet E-vector space* or an *E-Fréchet* is a locally convex E-vector space which is metrisable and complete.
- A *Banach E-vector space* or an *E-Banach* is a Fréchet vector space whose topology can be defined by a norm.
- An *ind-Fréchet* (resp. *ind-Banach*) E-vector space or an *ind-E-Fréchet* (resp. *ind-E-Banach*) is a locally convex E-vector space V, such that one can find an increasing sequence $(V_n)_{n\in\mathbb{N}}$ of closed sub-E-vector spaces such that
  (i) $V = \bigcup_{n\in\mathbb{N}} V_n$,
  (ii) each V_n, with the induced topology, is an E-Fréchet (resp. an E-Banach),
  (iii) the topology of V is the coarsest locally convex topology with these properties.

Condition (iii) is equivalent to the fact that a sub-$\mathcal{O}_E$-module L of V is open if and only if $L \cap V_n$ is open in V_n for all $n \in \mathbb{N}$.

If V is a topological E-vector space, V is an E-Fréchet if and only if V is complete and its topology can be defined by a countable family $(q_n)_{n\in\mathbb{N}}$ of seminorms.

In this situation, replacing each q_n by $q_n' = \sup_{0\le i\le n} q_i$, we may assume that $q_n \le q_{n+1}$ for all n. Then, if $\overline{V}_n$ is the Hausdorff completion of V, with respect to q_n, this is an E-Banach and we have an homeomorphism

$$V \mapsto \varprojlim_{n\in\mathbb{N}} \overline{V}_n$$

(with the inverse limit topology on the RHS). Conversely, any inverse limit, indexed by $\mathbb{N}$, of E-Banach is an E-Fréchet.

Let V be a topological E-vector space. We say that a decreasing filtration $(F^nV)_{n\in\mathbb{Z}}$ by closed sub-E-vector spaces of V is *admissible* if

(i) $\bigcup_{n\in\mathbb{Z}} F^nV = V$ and $\bigcap_{n\in\mathbb{Z}} F^nV = 0$,
(ii) if $m \in \mathbb{Z}$ and $r \in \mathbb{N}$, then $F^mV/F^{m+r}V$, equipped with the induced topology, is an E-Banach,

(iii) if $m \in \mathbb{Z}$, the natural map

$$F^m V \to \varprojlim_{r\in\mathbb{N}} F^m V/F^{m+r}V$$

is an homeomorphism (with the inverse limit topology on the RHS),

(iv) a sub-$\mathcal{O}_E$-module L of V is open if and only if $L \cap F^n V$ is open in $F^n V$ for all n.

The following result is obvious:

Proposition 2.1. *Let V be a topological E-vector space.*

(i) *V is an ind-E-Fréchet if and only if it has an admissible filtration.*

(ii) *V is an E-Banach (resp. an E-Fréchet, resp. an ind-E-Banach) if and only if has an admissible filtration $(F^n V)_{n\in\mathbb{Z}}$ such that $F^0V = V$ and $F^1V = 0$ (resp. $F^0V = V$, resp. $F^1V = 0$).*

Proposition 2.2. *Let V_1 and V_2 two ind-E-Fréchet, $(F^n V_1)_{n\in\mathbb{Z}}$ an admissible filtration of V_1 and $(F^n V_2)_{n\in\mathbb{Z}}$ an admissible filtration of V_2. Let $u : V_1 \to V_2$ an E-linear map. The following are equivalent*:

(i) *The map u is continuous. For all $m \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $u(F^m V_1) \subset F^n V_2$ and the induced map*

$$F^m V_1 \to F^n V_2$$

is continuous.

Proof. (ii)$\Longrightarrow$(i): It is enough to show that, if L is an open lattice in V_2, then $f^{-1}(L)$ is open in V_1 which means that if $m \in \mathbb{Z}$, then $f^{-1}(L) \cap F^m V_1$ is open in $F^m V_1$ which is indeed true as, if n is such that $f(F^m V_1) \subset F^n V_2$, this is the inverse image of the continuous map $F^m V_1 \to F^n V_2$ which is induced by f.

(i)$\Longrightarrow$(ii): All the $F^n V_2$ are E-Fréchet. For each fixed m, so is $F^m V_1$ and the existence of such an n is explained in [Schneider 2002, Corollary 8.9]. □

Corollary 2.3. *Let V be an ind-E-Fréchet and $(F^n V)_{n\in\mathbb{Z}}$ an admissible filtration. Then V is an E- Banach (resp. an E-Fréchet, resp. an ind-E-Banach) if and only if there exists $m \leq n$ such that $F^m V = V$ and $F^n V = 0$ (resp. m such that $F^m V = V$, resp. n such that $F^n V = 0$).*

Corollary 2.4. *Let V be an ind-E-Fréchet and $(F_1^n V)_{n\in\mathbb{Z}}$ and $(F_2^n V)_{n\in\mathbb{Z}}$ two admissible filtrations. For all $m \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $F_1^m V \subset F_2^n V$.*

An *ind Fréchet E-algebra* is a topological E-algebra B which has a *multiplicative admissible filtration*, i.e., an admissible filtration $(F^n B)_{n\in\mathbb{Z}}$ of the underlying topological E-vector space such that, if $m, n \in \mathbb{Z}$, and, if $b \in F^m B$, $b' \in F^n B$, then $bb' \in F^{m+n} B$.

A *Banach* (resp. *Fréchet*, resp. *ind-Banach*) E-algebra is an ind Fréchet E-algebra B which has a multiplicative admissible filtration $(F^n B)_{n\in\mathbb{Z}}$ such that $F^0 B = B$ and $F^1 B = 0$ (resp. $F^0 B = B$, resp. $F^1 B = 0$).

2B. ***Ind-Fréchet representations.*** From now on E will be $\mathbb{Q}_p$. We will say *Banach*, *Fréchet*, *ind-Banach*, *ind-Fréchet* instead of $\mathbb{Q}_p$-Banach, $\mathbb{Q}_p$-Fréchet, ind-$\mathbb{Q}_p$-Banach, ind-$\mathbb{Q}_p$-Fréchet. We will say *Banach algebra*, *Fréchet algebra*, and so on, instead of $\mathbb{Q}_p$-Banach algebra, $\mathbb{Q}_p$-Fréchet algebra.

The category $\mathcal{IF}(G_K)$ of *ind-Fréchet representations* (*of* G_K) is the category whose objects are ind-Fréchet equipped with a $\mathbb{Q}_p$-linear and continuous action of G_K, and whose morphisms are G_K-equivariant continuous $\mathbb{Q}_p$-linear map.

The category $\mathcal{IF}(G_K)$ is an additive $\mathbb{Q}_p$-linear category and any morphism

$$f : V_1 \to V_2$$

has a kernel and a cokernel: the kernel is the G_K-stable closed sub-$\mathbb{Q}_p$-vector space which is the kernel of the underlying $\mathbb{Q}_p$-linear map. The cokernel is the quotient of V_2 by the G_K-stable closed sub-$\mathbb{Q}_p$-vector space which is the closure of $f(V_1)$.

We say that a morphism f is *strict* if the map

$$\mathrm{Coim}(f) \to \mathrm{Im}(f)$$

is an homeomorphism.

Similarly one can define in an obvious way the categories $\mathcal{B}(G_K)$, $\mathcal{IB}(G_K)$ and $\mathcal{F}(G_K)$ of *Banach, ind-Banach, Fréchet representations (of* G_K*)*. This is consistent with the definition of $\mathcal{B}(G_K)$ already given in the introduction.

2C. ***The rings*** B_{dR}^+ ***and*** B_{dR} ***and their topologies.*** We denote by B_{dR} the usual field of p-adic periods. Recall (from [Fontaine 1994, §1.5], for instance) that this is the fraction field of a discrete valuation ring B_{dR}^+, that G_K acts naturally on these two $\mathbb{Q}_p$-algebras and that $\mathbb{Z}_p(1)$ is naturally a G_K-stable sub-$\mathbb{Z}_p$-module of B_{dR}^+. We choose a generator t of $\mathbb{Z}_p(1)$. This is also a generator of the maximal ideal of B_{dR}^+. Therefore, for all $d \in \mathbb{Z}$, the d-th power of this ideal is

$$\mathrm{Fil}^d B_{dR} = B_{dR}^+ . t^d = B_{dR}^+(d)$$

and is stable under G_K. For each $d \geq 0$, we set

$$B_d = B_{dR}^+ / \mathrm{Fil}^d B_{dR}.$$

Recall [Fontaine 1994, §1.5.3] that B_d has a natural structure of a Banach algebra on which the action of G_K is continuous, that, in particular, $B_1 = \mathbb{C}_p$, and that, for each $d \in \mathbb{N}$, the projection $B_{d+1} \to B_d$ is also continuous. Equipped with the

topology of the inverse limit, B_{dR}^+ becomes a Fréchet algebra on which G_K acts continuously.

For all $n \in \mathbb{Z}$, multiplication by t^n defines a bijection $B_{dR}^+ \to \mathrm{Fil}^n B_{dR}$ and we equip $\mathrm{Fil}^n B_{dR}$ with the induced topology (for which the action of G_K is continuous); note that multiplication by t^n does not commute with the action of G_K.

If $n \in \mathbb{Z}$, then $\mathrm{Fil}^{n+1} B_{dR}$ is closed in $\mathrm{Fil}^n B_{dR}$ and we equip B_{dR} with its natural locally convex topology. (A sub-$\mathbb{Z}_p$-module L of B_{dR} is open if and only if, for all $n \in \mathbb{Z}$, the $\mathbb{Z}_p$-module $L \cap \mathrm{Fil}^n B_{dR}$ is open in $\mathrm{Fil}^n B_{dR}$.)

We see that B_{dR} is an ind-Fréchet K-algebra, with $(\mathrm{Fil}^n B_{dR})_{n\in\mathbb{Z}}$ as a G_K-equivariant multiplicative admissible filtration. In particular B_{dR} has a natural structure of an ind-Fréchet K-representation of G_K.

2D. B_{dR}^+ and B_{dR}-representations. Any B_{dR}^+-module of finite type has a natural structure of a K-Fréchet and any finite-dimensional B_{dR}-vector space has a natural structure of an ind-Fréchet K-vector space.

A B_{dR}^+-représentation (resp. a *B_{dR}-representation) (of G_K)* is a B_{dR}^+-module of finite type (resp. a finite -dimensional B_{dR}-vector space) equipped with a semilinear and continuous action of G_K. With the G_K-equivariant B_{dR}^+-linear maps as morphisms, these representations form a category that we denote by $\mathrm{Rep}_{B_{dR}^+}(G_K)$ (resp. $\mathrm{Rep}_{B_{dR}}(G_K)$).

The category $\mathrm{Rep}^{\mathrm{tor}}_{B_{dR}^+}(G_K) = \mathcal{C}^\infty(G_K)$ of *torsion B_{dR}^+-representations (of G_K)* defined in the introduction (Section 1B) is the full subcategory of $\mathrm{Rep}_{B_{dR}^+}(G_K)$ whose objects are such that the underlying B_{dR}^+-module is torsion ($\Longleftrightarrow$ of finite length).

Recall (from [Stacks, 02MN], for instance) that a *Serre subcategory* $\mathcal{C}$ of an abelian category $\mathcal{A}$ is a strictly full subcategory of $\mathcal{A}$ containing 0 which is stable under subobjects, quotients and extensions. In particular, this is an abelian category. Given $\mathcal{A}$ and $\mathcal{C}$, one can define the quotient category $\mathcal{A}/\mathcal{C}$ which is an abelian category, solution of the obvious universal problem.

Proposition 2.5. *The category $\mathcal{C}^\infty(G_K)$ is a Serre subcategory of* $\mathrm{Rep}_{B_{dR}^+}(G_K)$. *The functor*

$$\mathrm{Rep}_{B_{dR}^+}(G_K) \to \mathrm{Rep}_{B_{dR}}(G_K), \quad W \mapsto B_{dR} \otimes_{B_{dR}^+} W$$

is essentially surjective and induces an equivalence

$$\mathrm{Rep}_{B_{dR}^+}(G_K)/\mathcal{C}^\infty(G_K) \xrightarrow{\simeq} \mathrm{Rep}_{B_{dR}}(G_K).$$

Proof. The essential surjectivity comes from the fact that, for any B_{dR}-representation W, there is a G_K-stable lattice B_{dR}^+-lattice W^+. This result itself comes from the fact that if W_0^+ is a B_{dR}^+-lattice of W, then W_0 is an ind-Fréchet K-vector space

with $(t^n W_0^+)_{n\in\mathbb{Z}}$ forming an admissible filtration. For each $w \in W$, the $g(w)$'s for $g \in G_K$ form a compact subset of W, hence it is bounded which implies (by [Schneider 2002, Proposition 5.6]) that it is contained in $t^{-n}W_0^+$ for $n \gg 0$. Hence, if $e_1, e_2, \ldots, e_d$ is a basis of W over B_{dR}, there exists $n \in \mathbb{N}$ such that $g(e_i) \in t^{-n}W_0^+$ for $1 \le i \le d$ and $g \in G_K$. Therefore the sub-B_{dR}^+-module W^+ of W generated by all these $g(e_i)$'s is also contained in $t^{-n}W_0^+$ and is a G_K-stable B_{dR}^+-lattice of W. The continuity of the action of G_K on W implies the continuity of the action on W^+ which is an object of $\mathrm{Rep}_{B_{dR}^+}(G_K)$. We have an obvious identification of $B_{dR} \otimes_{B_{dR}^+} W^+$ to W and the functor is essentially surjective.

The rest of the proof is straightforward. □

If W is any object of $\mathcal{C}^\infty(G_K)$, there is an integer d such that the underlying B_{dR}^+-module is a B_d-module of finite type. As B_d is a Banach $\mathbb{Q}_p$-algebra, the underlying topological $\mathbb{Q}_p$-vector space is a Banach and W has a natural structure of a p-adic Banach representation.

Proposition 2.6 [Fontaine 2003, théorème 3.1]. *The forgetful functor*

$$\mathcal{C}^\infty(G_K) \to \mathcal{B}(G_K)$$

is fully faithful.

In other words, given a p-adic Banach representation W of G_K, there is at most one structure of B_{dR}^+-module of finite length on W extending the action of $\mathbb{Q}_p$ such that W becomes a torsion B_{dR}^+-representation.

We use this result to identify $\mathcal{C}^\infty(G_K)$ to a full subcategory of $\mathcal{B}(G_K)$.

We denote by

$$\widehat{\mathcal{C}}^\infty(G_K)$$

the full subcategory of $\mathcal{IF}(G_K)$ whose objects are those W's which admit a G_K-equivariant admissible filtration $(F^n W)_{n\in\mathbb{Z}}$ such that $F^m W/F^n W \in \mathcal{C}^\infty(G_K)$ for all $m \le n$ in $\mathbb{Z}$. By passing to the limit, the previous proposition implies that, on such a W, there is a unique structure of B_{dR}^+-module such that the action of G_K is semilinear and each $F^m W$ is a sub-B_{dR}^+-module (and this structure is independent of the choice of $(F^n W)_{n\in\mathbb{Z}}$). We also see that $\widehat{\mathcal{C}}^\infty(G_K)$ is an abelian category and that any morphism of $\widehat{\mathcal{C}}^\infty(G_K)$ is B_{dR}^+-linear.

Moreover $\mathrm{Rep}_{B_{dR}^+}(G_K)$ can be identified with a full subcategory of $\widehat{\mathcal{C}}^\infty(G_K)$. Proposition 2.5 implies that this is also true for $\mathrm{Rep}_{B_{dR}}(G_K)$.

Proposition 2.7. *Let $d \in \mathbb{N}$.*

(i) *Let W_1 be an object of $\mathcal{C}^\infty(G_K)$ such that* $\mathrm{length}_{B_{dR}^+} W_1 \ge d$. *There exists a finite extension K' of K contained in $\overline{\mathbb{Q}}_p$ and a $G_{K'}$-stable sub-B_{dR}^+-module W_1' of W_1 of length d.*

(ii) *Let* W_2 *be an object of* $\mathrm{Rep}_{B^+_{dR}}(G_K)$ *with* $\mathrm{length}_{B^+_{dR}} W_2 \geq d$. *There exists a finite extension* K' *of* K *contained in* $\overline{\mathbb{Q}}_p$ *and a* $G_{K'}$*-stable sub-*B^+_{dR}*-module* W_2' *of* W_2 *such that* $\mathrm{length}_{B^+_{dR}} W_2/W_2' = d$.

Proof. (i) Via an obvious induction, we see that it is enough to check it for $d = 1$. Replacing W_1 by the kernel of the multiplication by t in W_1, we may assume that W_1 is a $\mathbb{C}_p$-representation.

Recall some basic facts of Sen's theory [1980/81]:

Let $\chi : G_K \to \mathbb{Z}_p^*$ be the cyclotomic character, H_K the kernel of χ and $L = (\mathbb{C}_p)^{H_K}$ which is also the completion of $K_\infty = \overline{\mathbb{Q}}_p^{H_K}$. We set $\Gamma_K = G_K/H_K = \mathrm{Gal}(K_\infty/K)$. The character χ factors through a character $\Gamma_K \to \mathbb{Z}_p^*$ that we still denote by χ.

For any $\mathbb{C}_p$-representation W (of G_K), denote by W_K^f the union of the finite-dimensional sub-K-vector spaces of W^{H_K} stable under the action of G_K (acting through Γ_K). This is a finite dimensional K_∞-vector space equipped with a semi-linear action of Γ_K. With obvious notations, we have:

- The functor
$$\mathrm{Rep}_{\mathbb{C}_p}(G_K) \to \mathrm{Rep}_{K_\infty}(\Gamma_K), \quad W \mapsto W_K^f$$
is exact and fully faithful.
- For any $W \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$, the obvious map
$$\mathbb{C}_p \otimes_{K_\infty} W_K^f \to W$$
is an isomorphism.
- For all $W \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$, there exists a unique endomorphism $\alpha_{W,K}$ of the K_∞-vector space W_K^f such that

  for all $w \in W_K^f$, there is an open subgroup Γ_w of Γ_K such that, if $\gamma \in \Gamma_w$, then
  $$\gamma(w) = \exp(\log(\chi(\gamma)).\alpha_{W,K})(w).$$
  (The series $\exp(\lambda\alpha_{W,K})$ converges to an endomorphism of W_K^f for all small enough $\lambda \in \mathbb{Z}_p$.)

It is easy to see that, if K_1 is a finite extension of K contained in $\overline{\mathbb{Q}}_p$, then $W_{K_1}^f$ can be identified with $(K_1)_\infty \otimes_{K_\infty} W_K^f$ and that α_{W,K_1} is the $(K_1)_\infty$-endomorphism of $W_{K_1}^f$ deduced from $\alpha_{W,K}$ by scalar extension.

Choose such a K_1 containing an eigenvalue λ of $\alpha_{W,K}$, hence also of α_{W,K_1} and choose a nonzero eigenvector $\omega_0 \in W_{K_1}^f$ for α_{W,K_1}^f. There is a finite extension K' of K_1 contained in $\overline{\mathbb{Q}}_p$ such that, for all $\gamma \in \Gamma_{K'}$, we have
$$\gamma(w_0) = \exp(\log(\chi(\gamma)).\lambda).w.$$

We can view w_0 as a nonzero element of $W_{K'}^f$ and we see that for all $b \in K'$ and all $\gamma \in \Gamma_{K'}$, we have

$$\gamma(bw_0) = \gamma(b).\exp(\log(\chi(\gamma)).\lambda).w,$$

hence the K'-line of $W_{K'}^f$ generated by w_0 is stable under the action of $\Gamma_{K'}$. Therefore the $\mathbb{C}_p$-line W_1' of W_1 generated by w_0 is stable under the action of $G_{K'}$.

(ii) Replacing W_2 by $W_2/t^r W_2$ with r big enough, we may assume that W_2 is an object of $\mathcal{C}^\infty(G_K)$. The result follows by duality from the assertion (i) applied to the Pontryagin dual $W = \mathcal{L}_{B_{dR}^+}(W_2, B_{dR}/B_{dR}^+)$ of W_2. □

2E. ***Almost $\mathbb{C}_p$-representations.*** If V_1 and V_2 are two objects of $\mathcal{IF}(G_K)$, an *almost isomorphism*

$$f : V_1 \rightsquigarrow V_2, \quad \text{also denoted by} \quad \tilde{f} : V_1/U_1 \to V_2/U_2,$$

is a triple $f = (U_1, U_2, \tilde{f})$ where U_1 is a finite-dimensional G_K-stable sub-$\mathbb{Q}_p$-vector space of V_1, U_2 is a finite dimensional G_K-stable sub-$\mathbb{Q}_p$-vector space of V_2 and

$$\tilde{f} : V_1/U_1 \to V_2/U_2$$

is an isomorphism of ind-Fréchet representations.

We say that two objects V_1 and V_2 of $\mathcal{IF}(G_K)$ are *almost isomorphic* if there exists an almost isomorphism

$$f : V_1 \rightsquigarrow V_2.$$

Proposition 2.8 [Fontaine 2003, théorème 5.3]. *Let V be an object of $\mathcal{B}(G_K)$. The following are equivalent*:

(i) *V is almost isomorphic to a torsion B_{dR}^+-representation.*

(ii) *V is almost isomorphic to a $\mathbb{C}_p$-representation.*

(iii) *There is $d \in \mathbb{N}$ such that V is almost isomorphic to $\mathbb{C}_p^d$ (equipped with the natural action of G_K).*

We denote by $\mathcal{C}(G_K)$ the category of *almost $\mathbb{C}_p$-representations (of G_K)*, that is the full subcategory of $\mathcal{B}(G_K)$ whose objects satisfy the equivalent conditions of the previous proposition. This is coherent with the definition given in the introduction (Section 1B).

The category $\mathcal{C}(G_K)$ contains $\mathcal{C}^\infty(G_K) = \mathrm{Rep}^{\mathrm{tor}}_{B_{dR}^+}(G_K)$ and $\mathcal{C}^0(G_K) = \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ as full subcategories.

A *weak Serre subcategory* $\mathcal{B}$ of an abelian category $\mathcal{A}$ is a strictly full subcategory which is abelian, such that the inclusion functor is exact and which is closed under taking extensions.

The following results are essentially contained in [Fontaine 2003]:

Theorem 2.9. *The category $\mathcal{C}(G_K)$ is abelian and any morphism of $\mathcal{C}(G_K)$ is strict as a morphism of $\mathcal{B}(G_K)$. A sequence of morphisms of $\mathcal{C}(G_K)$ is exact if and only if the underlying sequence of $\mathbb{Q}_p$-vector spaces is exact. The category $\mathcal{C}^0(G_K)$ is a Serre subcategory of $\mathcal{C}(G_K)$and $\mathcal{C}^\infty(G_K)$ is a weak Serre subcategory of $\mathcal{C}(G_K)$.*

Furthermore:

(i) *If $U \in \mathcal{C}^0(G_K)$ and $W \in \mathcal{C}^\infty(G_K)$, then* $\mathrm{Hom}_{\mathcal{C}(G_K)}(W, U) = 0$.

(ii) *There exists additive functions*

$$d : \mathrm{Ob}\, \mathcal{C}(G_K) \to \mathbb{N} \quad \text{and} \quad h : \mathrm{Ob}\, \mathcal{C}(G_K) \to \mathbb{Z},$$

uniquely determined respectively by $d(U) = 0$ if $U \in \mathcal{C}^0(G_K)$ and $d(\mathbb{C}_p) = 1$ (resp. $h(U) = \dim_{\mathbb{Q}_p}(U)$ if $U \in \mathcal{C}^0(G_K)$ and $h(\mathbb{C}_p) = 0$); moreover, if $W \in \mathcal{C}^\infty(G_K)$, then $d(W) = \mathrm{length}_{B_{dR}^+}(W)$ and $h(W) = 0$.

Proof. This is [Fontaine 2003, théorème 5.1] with some extras:

• The fact that $\mathcal{C}^0(G_K)$ is a Serre subcategory of $\mathcal{C}(G_K)$, which is a triviality.

• The fact that $\mathcal{C}^\infty(G_K)$ is a weak Serre subcategory of $\mathcal{C}(G_K)$. The only thing which is not obvious is the stability under extensions of $\mathcal{C}^\infty(G_K)$ inside of $\mathcal{C}(G_K)$, which is contained in [loc. cit., proposition 6.3].

• The fact that if $U \in \mathcal{C}^0(G_K)$ and $W \in \mathcal{C}^\infty(G_K)$, then $\mathrm{Hom}_{\mathcal{C}(G_K)}(W, U) = 0$, which is the corollary [loc. cit., théorème 5.1]. □

For instance, we see that, if U is a G_K-stable finite dimensional sub-$\mathbb{Q}_p$-vector space of $\mathbb{C}_p$, then $d(\mathbb{C}_p/U) = 1$ and $h(\mathbb{C}_p/U) = -\dim_{\mathbb{Q}_p} U$.

If $V \in \mathcal{C}(G_K)$, $W \in \mathcal{C}^\infty(G_K)$ and $\tilde{f} : V/U_+ \to W/U_-$ is an almost isomorphism, from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_+ & \longrightarrow & V & \longrightarrow & V/U_+ & \longrightarrow & 0 \\ & & & & & & \downarrow \simeq & & \\ 0 & \longrightarrow & U_- & \longrightarrow & W & \longrightarrow & W/U_- & \longrightarrow & 0 \end{array}$$

whose lines are exact, we deduce that

$$d(V) = d(W), \quad h(V) = h(U_+) - h(U_-) = \dim_{\mathbb{Q}_p}(U_+) - \dim_{\mathbb{Q}_p}(U_-).$$

Corollary 2.10. (i) *For any $V \in \mathcal{C}(G_K)$, we have $V \in \mathcal{C}^0(G_K) \iff d(V) = 0$ (in which case $h(V) = \dim_{\mathbb{Q}_p} V \geq 0$).*

(ii) *If $g : V \to W$ is a monomorphism of $\mathcal{C}(G_K)$ with $W \in \mathcal{C}^\infty(G_K)$ such that $d(V) = d(W)$, then g is an isomorphism.*

Proof. Looking at an almost isomorphism as above, the first assertion is immediate.

For the second, let U be the cokernel of g. We have $d(U) = 0$, hence $U \in \mathcal{C}^0(G_K)$, hence $U = 0$, as there is no nontrivial morphism from W to an object of $\mathcal{C}^0(G_K)$. □

Remark 2.11. As $\mathcal{C}^0(G_K)$ is a Serre subcategory of $\mathcal{C}(G_K)$, we may consider the quotient

$$\widetilde{\mathcal{C}}(G_K) = \mathcal{C}(G_K)/\mathcal{C}^0(G_K)$$

It is known [Fontaine 2003, proposition 7.1] that this abelian category is semisimple with exactly one isomorphism class of simple objects which is the class of $\mathbb{C}_p$ viewed as an object of this category. Hence $\widetilde{\mathcal{C}}(G_K)$ is completely determined, up to equivalence, by the somewhat mysterious huge skew field $\mathcal{D}_K$ of the endomorphisms of $\mathbb{C}_p$ in this category [loc. cit., proposition 7.2].

We denote by

$$\widehat{\mathcal{C}}(G_K)$$

the full subcategory of $\mathcal{IF}(G_K)$ whose objects are those V's which admit a G_K-equivariant admissible filtration $(F^n V)_{n\in\mathbb{Z}}$ such that $F^m V/F^n V \in \mathcal{C}(G_K)$ for all $m \le n$ in $\mathbb{Z}$.

By passing to the limit, we see that the previous theorem implies:

Proposition 2.12. *Any morphism of $\widehat{\mathcal{C}}(G_K)$ is strict (as a morphism of $\mathcal{IF}(G_K)$) and this category is abelian. A sequence of morphisms of $\widehat{\mathcal{C}}(G_K)$ is exact if and only if the underlying sequence of $\mathbb{Q}_p$-vector spaces is exact. The category $\mathcal{C}(G_K)$ is a Serre subcategory of $\widehat{\mathcal{C}}(G_K)$ of which $\widehat{\mathcal{C}}^\infty(G_K)$ is a weak Serre subcategory.*

Remark 2.13. As $\mathrm{Rep}_{B_{dR}^+}(G_K)$ and $\mathrm{Rep}_{B_{dR}}(G_K)$ are Serre subcategories of $\widehat{\mathcal{C}}^\infty(G_K)$, these two categories are also weak Serre subcategories of $\widehat{\mathcal{C}}(G_K)$.

2F. ***Almost split exact sequences.*** We say that a sequence of morphisms of $\mathcal{IF}(G_K)$ is *exact* if the underlying sequence of $\mathbb{Q}_p$-vector spaces is exact.

An *almost splitting* of a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

in $\mathcal{IF}(G_K)$ is a G_K-stable closed sub-$\mathbb{Q}_p$-vector space S of V such that

(i) the compositum $S \subset V \to V''$ is onto,

(ii) the $\mathbb{Q}_p$-vector space $S \cap V'$ is finite-dimensional.

We say that such an exact sequence *almost splits* if there exists such an almost splitting. This is equivalent to saying that there exists a G_K-stable finite-dimensional sub-$\mathbb{Q}_p$-vector space U of V' such that the sequence

$$0 \to V'/U \to V/U \to V'' \to 0$$

splits.

We observe that any almost splitting S of a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

defines, in an obvious way, almost isomorphisms

$$V \rightsquigarrow V' \oplus V'' \rightsquigarrow S \oplus V''.$$

Proposition 2.14 [Fontaine 2003, théorème 5.2]. *Let*

$$0 \to W' \to W \to W'' \to 0$$

be a short exact sequence in $\mathcal{IF}(G_K)$ with W' and W'' in $\mathcal{C}^\infty(G_K)$. Then W is in $\mathcal{C}^\infty(G_K)$ if and only if the sequence almost splits.

Proposition 2.15 [Fontaine 2003, proposition 5.2]. *Let*

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence in $\mathcal{IF}(G_K)$ with V' and V'' in $\mathcal{C}(G_K)$. Then V is in $\mathcal{C}(G_K)$ if and only if the sequence almost splits.

Corollary 2.16. *Among the strictly full subcategories of $\mathcal{B}(G_K)$ which are abelian, containing $\mathbb{C}_p$ and $\mathcal{C}^0(G_K)$ and stable under almost split extensions, there is a smallest one. This is $\mathcal{C}(G_K)$.*

Proof. Clear! □

3. B_e-representations and coherent $\mathcal{O}_X[G_K]$-modules

3A. ***The topological $\mathbb{Q}_p$-algebras B^+_{cris} and B_e.*** Recall (from, e.g., [Fontaine 1994, §2.3 and §4.1]) that B^+_{cris} is a Banach algebra equipped with a continuous endomorphism φ and a continuous action of G_K commuting with φ. There is a natural G_K-equivariant continuous injective homomorphism of topological $\mathbb{Q}_p$-algebras

$$B^+_{\mathrm{cris}} \to B^+_{dR}$$

that we use to identify B^+_{cris} to a subring of B^+_{dR} containing t.

For each $d \in \mathbb{N}$, we set

$$P^d = \{b \in B^+_{\mathrm{cris}} \mid \varphi(b) = p^d b\}.$$

This is a G_K-stable closed sub-$\mathbb{Q}_p$-vector space of B^+_{cris} as well as of B^+_{dR} (e.g. [Kisin 2003, Lemma 3.3]). Moreover B^+_{cris} and B^+_{dR} induce the same topology on P^d which can be viewed as a Banach representation of G_K. We have a canonical short exact sequence (see [Colmez and Fontaine 2000, proposition 1.3], for instance)

$$0 \to \mathbb{Q}_p(d) \to P^d \to B_d \to 0$$

where $\mathbb{Q}_p(d) = \mathbb{Q}_p t^d$ and $P^d \to B_d$ is the compositum $P^d \subset B^+_{\mathrm{cris}} \subset B^+_{dR} \overset{\mathrm{proj}}{\to} B_d$. In particular we see that P^d is an almost $\mathbb{C}_p$-representation with $d(P^d) = d$ and $h(P^d) = 1$.

As usual, we set $B_{\mathrm{cris}} = B_{\mathrm{cris}}^{+}[1/t]$, which we can view as a G_K-stable subring of B_{dR}.

We have $\varphi(t) = pt$ and φ extends uniquely to B_{cris}. Moreover the natural map $B_{\mathrm{cris}} \to B_{dR} = B_{dR}^{+}[1/t]$ is still injective and we use it to identify B_{cris} to a G_K-stable sub-$\mathbb{Q}_p$-algebra of B_{dR}.

Recall that

$$B_e = \{b \in B_{\mathrm{cris}} \mid \varphi(b) = b\}$$

is also a G_K-stable sub-$\mathbb{Q}_p$-algebra of B_{dR}. We endow it with the topology induced by the (locally convex) topology of B_{dR}.

Then, we have

$$B_e = \varinjlim_{d\in\mathbb{N}} \mathrm{Fil}^{-d} B_e = \bigcup_{d\in\mathbb{N}} \mathrm{Fil}^{-d} B_e$$

where, for all $d \in \mathbb{N}$,

$$\mathrm{Fil}^{-d} B_e = B_e \cap B_{dR}^{+} t^{-d} = P^d.t^{-d} = P^d(-d)$$

is an almost $\mathbb{C}_p$-representation (with $d(P^d(-d)) = d$ and $h(P^d(-d)) = 1$) homeomorphic to P^d as a Banach. Setting $P^d = P^d(-d) = 0$ for $d > 0$, we see that B_e is an ind-Banach algebra with $(P^{-n}(n))_{n\in\mathbb{Z}}$ a G_K-stable multiplicative admissible filtration.

3B. ***B_e-representations.*** The topology of B_e induces on each B_e-module of finite type a natural topology for which it is an ind-Fréchet (actually an ind-Banach). A *B_e-representation (of G_K)* is a B_e-module of finite type equipped with a semilinear and continuous action of G_K. With the G_K-equivariant B_e-linear maps as morphisms, B_e-representations form a category that we denote by $\mathrm{Rep}_{B_e}(G_K)$.

Proposition 3.1. *The B_e-module underlying any B_e-representation is free of finite rank. The category* $\mathrm{Rep}_{B_e}(G_K)$ *is a $\mathbb{Q}_p$-linear abelian category.*

Proof. Recall that B_e is a principal ideal domain [Fargues and Fontaine 2018, théorème 6.5.2]. In particular it is a noetherian ring and the fact that $\mathrm{Rep}_{B_e}(G_K)$ is a $\mathbb{Q}_p$-linear abelian category is obvious.

Moreover [loc. cit., proposition 10.1.1], for any maximal ideal $\mathfrak{p}$ of B_e, the orbit of $\mathfrak{p}$ under the action of G_K is infinite. This implies that there is no nontrivial G_K-equivariant ideal of B_e. If Λ is any nonzero B_e-representation of G_K, the annihilator of its torsion sub-module is a proper G_K-equivariant ideal and must be 0. Therefore the B_e-module underlying Λ is torsion free, hence free of finite rank. □

Remark 3.2. Let C_e be the fraction field of B_e. This is the union of the fractional ideals of B_e. For each such ideal $\mathfrak{a}$, the choice of a generator a defines a bijection

$$B_e \to \mathfrak{a}, \quad b \mapsto ba,$$

and we put on $\mathfrak{a}$ the topology defined by transport de structure, which is independent of the choice of the generator. Hence each $\mathfrak{a}$ is naturally an ind-Banach ($\mathbb{Q}_p$-vector space). If $\mathfrak{a} \subset \mathfrak{b}$ are two fractional ideals, this inclusion is continuous and $\mathfrak{a}$ is a closed sub-$\mathbb{Q}_p$-vector space of $\mathfrak{b}$. Hence we may endow C_e with the coarsest locally convex topology such that, for all fractional ideal $\mathfrak{a}$, the map $\mathfrak{a} \to C_e$ is continuous (a lattice $\mathcal{L}$ in C_e is open if and only if $\mathcal{L} \cap \mathfrak{a}$ is open in $\mathfrak{a}$ for all $\mathfrak{a}$).

The action of G_K on C_e is continuous for this topology (but C_e doesn't seem to be an object of $\mathcal{IF}(G_K)$) and we may consider the category $\mathrm{Rep}_{C_e}(G_K)$ of C_e-representations (of G_K), that is of finite-dimensional C_e-vector spaces equipped with a semilinear and continuous action of G_K. This is obviously a $\mathbb{Q}_p$-linear abelian category.

We have an obvious exact $\mathbb{Q}_p$-linear functor

$$\mathrm{Rep}_{B_e}(G_K) \to \mathrm{Rep}_{C_e}(G_K), \quad \Lambda \mapsto C_e \otimes_{B_e} \Lambda.$$

This functor is fully faithful: if $M \in \mathrm{Rep}_{C_e}(G_K)$ is a C_e-representation of dimension d, there is at most one G_K-equivariant sub-B_e-module of rank d because if Λ_1 and Λ_2 are two of them, so are $\Lambda_1 + \Lambda_2$ and $(\Lambda_1 + \Lambda_2)/\Lambda_1$ is torsion, hence 0.

Remark 3.3. If Λ is any B_e-representation of G_K, the underlying $\mathbb{Q}_p$-vector space is locally convex and Λ inherits a natural structure of an object of $\mathcal{IF}(G_K)$. We will see later that the forgetful functor

$$\mathrm{Rep}_{B_e}(G_K) \to \mathcal{IF}(G_K)$$

is fully faithful (Proposition 3.11) and that its essential image is contained in $\widehat{\mathcal{C}}(G_K)$ (Proposition 3.12).

Proposition 3.4. *Let $W \in \mathcal{C}^\infty(G_K)$ and $\Lambda \in \mathrm{Rep}_{B_e}(G_K)$. Then*

$$\mathrm{Hom}_{\mathcal{IF}(G_K)}(W, \Lambda) = 0.$$

Proof. Let $f : W \to \Lambda$ such a morphism. We see that $B_{dR} \otimes_{B_e} \Lambda$ is a B_{dR}-representation of G_K and that

$$g : \Lambda \to B_{dR} \otimes_{B_e} \Lambda, \quad \lambda \mapsto 1 \otimes \lambda$$

is a morphism of $\mathcal{IF}(G_K)$. But $gf : W \to B_{dR} \otimes_{B_e} \Lambda$ must be B_{dR}^+-linear (Section 2D). As the B_{dR}^+-module W is torsion, and $B_{dR} \otimes \Lambda$ is torsion free, we have $gf = 0$, hence also $f = 0$ as g is injective. □

3C. *Coherent $\mathcal{O}_X$-modules.* We know that B_e is a PID and we may consider the "open curve"

$$X_e = \mathrm{Spec}\ B_e,$$

a noetherian regular affine scheme of dimension 1 whose function field is the fraction field C_e of B_e that we can see as a subfield of B_{dR}. For each closed point x of X, the local ring $\mathcal{O}_{X,x}$ is a DVR and we denote by v_x the corresponding valuation on C_e normalised by $v_x(C_e^*) = \mathbb{Z}$.

Recall (cf. [Fargues and Fontaine 2018, §6.5.1]) that the curve $X = X_{\mathbb{Q}_p, \mathbb{C}_p^\flat}$ can be defined as the compactification at ∞ of X_e. More precisely, as B_{dR} is the fraction field of the discrete valuation ring B_{dR}^+, it is naturally equipped with a valuation v_{dR}: if $b \in B_{dR}$ is $\neq 0$, then $v_{dR}(b)$ is the largest $n \in \mathbb{Z}$ such that $b \in \mathrm{Fil}^n B_{dR}$. We denote by v_∞ the restriction of v_{dR} to C_e. The topological space underlying X is obtained from the topological space underlying X_e by adding the closed point ∞ defined by v_∞. Hence, the function field of X is C_e and, if U is any nonempty open subspace of X, we have

$$\mathcal{O}_X(U) = \{b \in C_e \mid v_x(b) \geq 0,\ \forall x \in U\}.$$

We have $X \setminus \{\infty\} = X_e$, the ring B_{dR}^+ is the completion of $\mathcal{O}_{X,\infty}$ and B_{dR} is the completion of C_e for the topology defined by v_∞.

Consider the following category $\mathrm{Coh}(\mathcal{O}_X)$:

- An object of $\mathrm{Coh}(\mathcal{O}_X)$ is a triple $(\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F})$ with $\mathcal{F}_e$ a B_e-module of finite type, $\mathcal{F}_{dR}^+$ a B_{dR}^+-module of finite type and
$$\iota_\mathcal{F} : \mathcal{F}_{dR}^+ \to B_{dR} \otimes_{B_e} \mathcal{F}_e$$
a B_{dR}^+-linear map inducing an isomorphism of B_{dR}-vector spaces
$$B_{dR} \otimes_{B_{dR}^+} \mathcal{F}_{dR}^+ \to B_{dR} \otimes_{B_e} \mathcal{F}_e.$$
- A morphism $(\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F}) \to (\mathcal{G}_e, \mathcal{G}_{dR}^+, \iota_\mathcal{G})$ is a pair (f_e, f_{dR}^+) with $f_e : \mathcal{F}_e \to \mathcal{G}_e$ a B_e-linear map and $f_{dR}^+ : \mathcal{F}_{dR}^+ \to \mathcal{G}_{dR}^+$ a B_{dR}^+-linear map such that the obvious diagram commutes.

To any coherent $\mathcal{O}_X$-module $\mathcal{F}$, we can associate an object $(\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F})$ of this category:

- $\mathcal{F}_e = \mathcal{F}(X_e)$,
- $\mathcal{F}_{dR}^+ = B_{dR}^+ \otimes_{\mathcal{O}_{X,\infty}} \mathcal{F}_\infty$, the completion of the fiber of $\mathcal{F}$ at ∞,
- the completion at ∞ of the general fiber is $B_{dR} \otimes_{B_{dR}^+} \mathcal{F}_{dR}^+$ as well as $B_{dR} \otimes_{B_e} \mathcal{F}_e$ and $\iota_\mathcal{F} : \mathcal{F}_{dR}^+ \to B_{dR} \otimes_{B_e} \mathcal{F}_e$ is the natural map.

This correspondence is obviously functorial and it is immediate to see that it gives an equivalence of categories. We use it to identify the category of coherent $\mathcal{O}_X$-modules to $\mathrm{Coh}(\mathcal{O}_X)$. In this equivalence we see that the category $\mathrm{Bund}(X)$ of vector bundles over X, i.e., of torsion free coherent $\mathcal{O}_X$-modules, can be identified

with the full subcategory of $\mathrm{Coh}(\mathcal{O}_X)$ whose objects are triples $(\mathcal{F}_e, \mathcal{F}^+_{dR}, \iota_{\mathcal{F}})$ such that the B_e-module $\mathcal{F}_e$ and the B^+_{dR}-module $\mathcal{F}^+_{dR}$ are torsion free ($\Longleftrightarrow$ free).

3D. ***The topology on $\mathcal{O}_X$.*** The curve X can be also described ([FF], §6.5.1) as

$$X = \mathrm{Proj} \bigoplus_{d\in\mathbb{N}} P^d$$

and there is (*loc. cit.*, théorème 6.5.2) a one to one correspondence between the closed points of X and the $\mathbb{Q}_p$-lines in P^1 (the map associating to such a line the prime ideal of $P = \bigoplus_{d\in\mathbb{N}} P^d$ that it generates is a bijection between the set of these lines and the set of nonzero homogeneous prime ideals of P different from $\bigoplus_{d>0} P^d$). In this correspondence ∞ corresponds to the line generated by t.

Moreover, if $x_1, x_2, \ldots, x_r$ are closed points of X and if, for $1 \leq i \leq r$, we choose a generator t_i of the $\mathbb{Q}_p$-line associated to x_i, we see that the $\mathbb{Q}_p$-algebra $\mathcal{O}_X(X \setminus \{x_1, x_2, \ldots, x_r\})$ has a natural topology: If we set $u = t_1 t_2 \ldots t_r$, we have

$$\mathcal{O}_X(X \setminus \{x_1, x_2, \ldots, x_r\}) = \bigcup_{n\in\mathbb{N}} P^{nr} u^{-n}$$

and we see that it is an ind-Banach algebra with $(P^{nr}u^{-n})_{n\in\mathbb{N}}$ a multiplicative admissible Banach filtration. Thus we may consider $\mathcal{O}_X$ as a sheaf of ind-Banach algebras (the restriction maps are obviously continuous).

3E. ***The category $\mathcal{M}(G_K)$.*** The group G_K acts continuously on X and it makes sense to speak of the category $\mathcal{M}(G_K)$ of coherent $\mathcal{O}_X[G_K]$-modules, that is of coherent $\mathcal{O}_X$-modules equipped with a semilinear and continuous action of G_K.

We see that:

– the open subset $X_e = \mathrm{Spec}\ B_e$ is stable under G_K and G_K acts continuously on the ind-Banach algebra B_e,

– the point ∞ is fixed by G_K and the action of G_K on the Fréchet algebra B^+_{dR} (resp. on the ind-Fréchet algebra B_{dR}), completion at ∞ of $\mathcal{O}_{X,\infty}$ (resp. of the function field C_e of X) is continuous.

From the description of coherent $\mathcal{O}_X$-modules of the previous paragraph, we see that we can identify $\mathcal{M}(G_K)$ to the following category:

- An object is a triple $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}^+_{dR}, \iota_{\mathcal{F}})$, where $\mathcal{F}_e$ is a B_e-representation, $\mathcal{F}^+_{dR}$ is a B^+_{dR}-representation and

  $$\iota_{\mathcal{F}} : \mathcal{F}^+_{dR} \to B_{dR} \otimes_{B_e} \mathcal{F}_e$$

  is a G_K-equivariant homorphism of B^+_{dR}-modules such that the induced B_{dR}-linear map

  $$B_{dR} \otimes_{B^+_{dR}} \mathcal{F}^+_{dR} \to B_{dR} \otimes_{B_e} \mathcal{F}_e$$

  is bijective.

- A morphism
$$f : (\mathcal{F}_e, \mathcal{F}^+_{dR}, \iota_{\mathcal{F}}) \to (\mathcal{G}_e, \mathcal{G}^+_{dR}, \iota_{\mathcal{G}})$$
is a pair (f_e, f^+_{dR}) with $f_e : \mathcal{F}_e \to \mathcal{G}_e$ (resp. $f^+_{dR} : \mathcal{F}^+_{dR} \to \mathcal{G}^+_{dR}$) a morphism of B_e-representations (resp. B^+_{dR}-representations) such that the obvious diagram commutes.

  When there is no ambiguity about the map $\iota_{\mathcal{F}}$, we write abusively
$$\mathcal{F} = (\mathcal{F}_e, \mathcal{F}^+_{dR})$$
  We also denote by
$$\mathcal{F}_{dR} = B_{dR} \otimes_{B_e} \mathcal{F}_e = B_{dR} \otimes_{C_e} (C_e \otimes_{B_e} \mathcal{F}_e)$$
the completion at ∞ of the generic fiber $\mathcal{F}_\eta = C_e \otimes_{B_e} \mathcal{F}_e$ of $\mathcal{F}$.

The category $\mathrm{Bund}_X(G_K)$ of *G_K-equivariant vector bundles over X* is the full subcategory of $\mathcal{M}(G_K)$ whose objects are those for which the underlying $\mathcal{O}_X$-module is torsion free. From the fact that any B_e-representation is torsion free, we see that, if $\mathcal{F}$ is any coherent $\mathcal{O}_X[G_K]$-module, there is no torsion away from ∞. Therefore $\mathrm{Bund}_X(G_K)$ is the full subcategory of $\mathcal{M}(G_K)$ whose objects are those $\mathcal{F}$ such that the B^+_{dR}-module $\mathcal{F}^+_{dR}$ is free ($\Longleftrightarrow$ torsion free), i.e., the B-pairs of [Berger 2008].

3F. ***The Harder–Narasimhan filtration.*** The abelian category $\mathrm{Coh}(\mathcal{O}_X)$ is equipped with two additive functions, the *rank* and the *degree* [Fargues and Fontaine 2018, chapitre 5]:
$$\mathrm{rk} : \mathrm{Coh}(\mathcal{O}_X) \to \mathbb{N}, \quad \deg : \mathrm{Coh}(\mathcal{O}_X) \to \mathbb{Z}$$
The rank of $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}^+_{dR}, \iota_{\mathcal{F}})$ is the rank of the B_e-module $\mathcal{F}_e$. It is 0 if and only if $\mathcal{F}$ is torsion. It is more difficult to compute the degree. But this additive function is characterised by the following facts:

- if D is a divisor, then
$$\deg(\mathcal{L}(D)) = \deg(D) = \sum_{\substack{\text{closed} \\ \text{points of } X}} n_x \quad \text{if } D = \sum n_x[x],$$
- if $\mathcal{F}$ is a vector bundle of rank r, then
$$\deg(\mathcal{F}) = \deg(\wedge^r \mathcal{F}),$$
- if $\mathcal{F}$ is a torsion $\mathcal{O}_X$-module, then
$$\deg(\mathcal{F}) = \sum_{\substack{\text{closed} \\ \text{points of } X}} \mathrm{length}_{\mathcal{O}_{X,x}} \mathcal{F}_x.$$

The slope of a nonzero coherent O_X-module $\mathcal{F}$ is

$$\mathrm{slope}(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rank}(\mathcal{F}) \in \mathbb{Q} \cup \{+\infty\}$$

(with the convention that the slope of a nonzero torsion coherent $\mathcal{O}_X$-module is $+\infty$).

The following statements are similar to the classical case:

- A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *semistable* if it is nonzero and if $\mathrm{slope}(\mathcal{F}') \leq \mathrm{slope}(\mathcal{F})$ for any nonzero coherent sub-$\mathcal{O}_X$-module of $\mathcal{F}$.
- The Harder–Narasimhan filtration of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is the unique increasing filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{m-1} \subset \mathcal{F}_m = \mathcal{F}$$

by coherent sub-$\mathcal{O}_X$-modules such that each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable with

$$\mathrm{slope}(\mathcal{F}_1/\mathcal{F}_0) > \mathrm{slope}(\mathcal{F}_2/\mathcal{F}_1) > \cdots > \mathrm{slope}(\mathcal{F}_{m-1}/\mathcal{F}_{m-2}) > \mathrm{slope}(\mathcal{F}_m/\mathcal{F}_{m-1}).$$

The Harder–Narasimhan filtration splits continuously but not canonically. The slopes of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ for $1 \leq i \leq m$ are called *the HN-slopes of* $\mathcal{F}$.

If $\mathcal{F}$ is an object of $\mathcal{M}(G_K)$, the unicity of the Harder–Narasimhan filtration implies that this filtration is by subobjects in $\mathcal{M}(G_K)$. In general, there is no G_K-equivariant splitting of this filtration.

3G. ***The equivalences $\mathcal{M}^0(G_K) \to \mathcal{C}^0(G_K)$ and $\mathcal{M}^\infty(G_K) \to \mathcal{C}^\infty(G_K)$.*** For all $s \in \mathbb{Q} \cup \{+\infty\}$, we denote by $\mathcal{M}^s(G_K)$ the full subcategory of $\mathcal{M}(G_K)$ whose objects are semistable of slope s. We also write $\mathcal{M}^\infty(G_K) = \mathcal{M}^{+\infty}(G_K)$.

We have $H^0(X, \mathcal{O}_X) = \mathbb{Q}_p$. A central result of [Fargues and Fontaine 2018] (théorème 8.2.10) is that a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is semistable of slope 0 if and only if it is isomorphic to $\mathcal{O}_X^r$ for some positive integer r. From that we deduce:

Proposition 3.5. *If $\mathcal{F} \in \mathcal{M}^0(G_K)$, then $\mathcal{F}(X) \in \mathcal{C}^0(G_K)$ and* $\mathrm{rank}(\mathcal{F}) = \dim_{\mathbb{Q}_p} \mathcal{F}(X)$. *The functor*

$$\mathcal{M}^0(G_K) \to \mathcal{C}^0(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of categories. The functor

$$\mathcal{C}^0(G_K) \to \mathcal{M}^0(G_K), \quad U \mapsto \mathcal{O}_X \otimes U = (B_e \otimes_{\mathbb{Q}_p} U, B_{dR}^+ \otimes_{\mathbb{Q}_p} U)$$

is a quasi-inverse.

If $\mathcal{F} \in \mathcal{M}(G_K)$, as there is no torsion away from ∞, we have $\mathcal{F} \in \mathcal{M}^\infty(G_K)$ if and only if $\mathcal{F}_e = 0$. From that, we deduce:

Proposition 3.6. *If $\mathcal{F} \in \mathcal{M}^\infty(G_K)$, then $\mathcal{F}(X) = \mathcal{F}_{dR}^+$ and belongs to $\mathcal{C}^\infty(G_K)$. Moreover*

$$\deg(\mathcal{F}) = \operatorname{length}_{B_{dR}^+} \mathcal{F}(X).$$

The functor

$$\mathcal{M}^\infty(G_K) \to \mathcal{C}^\infty(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of categories. The functor

$$\mathcal{C}^\infty(G_K) \to \mathcal{M}^\infty(G_K), \quad W \mapsto \underline{W} = (0, W)$$

is a quasi-inverse.

For any $s \in \mathbb{Q}$, we denote by $\mathcal{M}^{\geq s}(G_K)$ (resp. $\mathcal{M}^{<s}(G_K)$) the full subcategory of $\mathcal{M}(G_K)$ whose objects are those which have all their HN-slopes $\geq s$ (resp. $< s$).

For any $\mathcal{F} \in \mathcal{M}(G_K)$, we denote by $\mathcal{F}^{\geq 0}$ the largest term of the Harder–Narasimhan filtration which belongs to $\mathcal{M}^{\geq 0}(G_K)$ and $\mathcal{F}^{<0} = \mathcal{F}/\mathcal{F}^{\geq 0}$. We have a short exact sequence

$$0 \to \mathcal{F}^{\geq 0} \to \mathcal{F} \to \mathcal{F}^{<0} \to 0$$

with $\mathcal{F}^{\geq 0} \in \mathcal{M}^{\geq 0}(G_K)$ and $\mathcal{F}^{<0} \in \mathcal{M}^{<0}(G_K)$.

The category $\mathcal{M}(G_K)$ is equipped with a tensor product. From the classification of vector bundles over X [Fargues and Fontaine 2018, théorème 8.2.10], we get the fact that if $s, t \in \mathbb{Q} \cup \{+\infty\}$, if $\mathcal{F} \in \mathcal{M}^s(G_K)$ and if $\mathcal{G} \in \mathcal{M}^t(G_K)$, then $\mathcal{F} \otimes \mathcal{G} \in \mathcal{M}^{s+t}(G_K)$ (with the convention that $s + t = +\infty$ if s or t is $+\infty$).

The additive category $\mathrm{Bund}_X(G_K)$ has an internal hom

$$(\mathcal{F}, \mathcal{G}) \mapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

We see that $(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_e = \mathcal{L}_{B_e}(\mathcal{F}_e, \mathcal{G}_e)$ is the B_e-module of the B_e-linear maps $\mathcal{F}_e \to \mathcal{G}_e$, and $(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_{dR}^+ = \mathcal{L}_{B_{dR}^+}(\mathcal{F}_{dR}^+, \mathcal{G}_{dR}^+)$ is the B_{dR}^+-module of the B_{dR}^+-linear maps $\mathcal{F}_{dR}^+ \to \mathcal{G}_{dR}^+$.

In $\mathrm{Bund}_X(G_K)$, there is also a duality: The dual of $\mathcal{F}$ is $\mathcal{F}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If $\mathcal{F}, \mathcal{G} \in \mathrm{Bund}_X(G_K)$, then $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathcal{F}^\vee \otimes \mathcal{G}$. If $\mathcal{F}$ is semistable of slope s, then $\mathcal{F}^\vee$ is semistable of slope $-s$.

3H. ***Tate and Harder–Narasimhan twists.*** Recall that, for any p-adic vector space V equipped with a linear action of G_K and $n \in \mathbb{Z}$, the n-th Tate's twist of V is

$$V(n) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$$

where $\mathbb{Q}_p(n) = \mathbb{Q}_p t^n \subset B_{dR}$. This construction is functorial.

For any $n \in \mathbb{Z}$, we denote by

$$\mathcal{O}_X(n)_T = \mathcal{O}_X \otimes \mathbb{Q}_p(n) = (B_e(n), B_{dR}^+(n)) = (B_e.t^n, B_{dR}^+.t^n)$$

(where $B_e.t^n$ (resp. $B_{dR}^+.t^n$) is the sub-B_e-module (resp. B_{dR}^+-module) of B_{dR} generated by t^n) the G_K-equivariant line bundle of slope 0 associated to $\mathbb{Q}_p(n)$.

For $\mathcal{F} \in \mathcal{M}(G_K)$ and $n \in \mathbb{Z}$, the *n-th Tate twist* of $\mathcal{F}$ is

$$\mathcal{F}(n)_T = \mathcal{F} \otimes \mathcal{O}_X(n)_T = (\mathcal{F}_e(n), \mathcal{F}_{dR}^+(n), \iota_{\mathcal{F}}(n)).$$

It has the same degree, the same rank and the same slope as $\mathcal{F}$.

For any $n \in \mathbb{Z}$, we consider the G_K-equivariant line bundle

$$\mathcal{O}_X(n)_{HN} = (B_e, B_{dR}^+(-n)) = (B_e, B_{dR}^+.t^{-n}).$$

There is an obvious short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(n)_{HN} \to (0, B_n(-n)) \to 0 \quad \text{if } n \geq 0,$$
$$0 \to \mathcal{O}_X(n)_{HN} \to \mathcal{O}_X \to (0, B_{-n}) \to 0 \quad \text{if } n < 0,$$

In particular, $\mathcal{O}_X(n)_{HN}$ is a modification of $\mathcal{O}_X$ and is of degree n. It is semistable of slope n.

For $\mathcal{F} \in \mathcal{M}(G_K)$ and $n \in \mathbb{Z}$, we define the *n-th Harder–Narasimhan twist* of $\mathcal{F}$ as

$$\mathcal{F}(n)_{HN} = \mathcal{F} \otimes \mathcal{O}_X(n)_{HN} = (\mathcal{F}_e, \mathcal{F}_{dR}^+(-n), \iota_{\mathcal{F}}(-n)) = (\mathcal{F}_e, t^{-n}.\mathcal{F}_{dR}^+, \iota_{\mathcal{F}}(-n)).$$

It has the same rank as $\mathcal{F}$. If $\mathcal{F}$ is semistable of slope s, then $\mathcal{F}(n)_{HN}$ is semistable of slope $s+n$.

These two constructions are obviously functorial and commute with Harder–Narasimhan filtration. In particular:

- If $\mathcal{F}$ is semistable of slope s, then $\mathcal{F}(n)_T$ is semistable of slope s, and $\mathcal{F}(n)_{HN}$ is semistable of slope $s+n$.
- The HN-slopes of $\mathcal{F}(n)_T$ are the same as the HN-slopes of $\mathcal{F}$, and the HN-slopes of $\mathcal{F}(n)_{HN}$ are the $s+n$ for s running through the HN-slopes of $\mathcal{F}$.

These constructions commute: for $m, n \in \mathbb{Z}$, we have

$$\mathcal{F}(m)_T(n)_{HN} = \mathcal{F}(n)_{HN}(m)_T.$$

Remark 3.7. In [Fargues and Fontaine 2018, définition 8.2.1] the G_K-equivariant line bundle $\mathcal{O}_X(n)_{HN}(n)_T$ is denoted $\mathcal{O}_X(n)$. We have to avoid confusion between the three G_K-equivariant line bundles $\mathcal{O}_X(n)_T$, $\mathcal{O}_X(n)_{HN}$ and

$$\mathcal{O}_X(n) = (B_e(n), B_{dR}^+) = (B_e.t^n, B_{dR}^+).$$

3I. ***Potentially trivialisable B_e-representations.*** Let Λ be a B_e-representation of G_K and K' a finite extension of K contained in $\overline{\mathbb{Q}}_p$. We say that Λ is $G_{K'}$-*trivialisable* if there is $U \in \mathcal{C}^0(G_{K'})$ and a $G_{K'}$-equivariant isomorphism of B_e-modules

$$B_e \otimes_{\mathbb{Q}_p} U \simeq \Lambda.$$

We say that Λ is *trivialisable* if it is G_K-trivialisable and *potentially trivialisable* if there is a finite extension K' of K contained in $\overline{\mathbb{Q}}_p$ such that Λ is $G_{K'}$-trivialisable.

Proposition 3.8. *Any absolutely irreducible B_e-representation of G_K is potentially trivialisable.*

Proof. Let Λ be such a B_e-representation. Then $\Lambda_{dR} = B_{dR} \otimes_{B_e} \Lambda$ is a B_{dR}-representation. Let $\mathcal{L}$ be the set of G_K-stable B_{dR}^+-lattices of Λ_{dR}. We know (Proposition 2.5) that $\mathcal{L}$ is not empty. For each $L \in \mathcal{L}$, we may consider the G_K-equivariant vector bundle over X

$$\mathcal{F}_L = (\Lambda, L).$$

Such an $\mathcal{F}_L$ is semistable (otherwise the Harder–Narasimhan filtration would be nontrivial and would induce a nontrivial filtration of the B_e-representation $(\mathcal{F}_L)_e = \Lambda$ which is not possible as Λ is irreducible).

Chose such an $\mathcal{F}_L$. Replacing $\mathcal{F}_L$ with $\mathcal{F}_L(n)_{HN}$ with $n \in \mathbb{N}$ big enough, we may assume that the degree d of $\mathcal{F}_L$ is ≥ 0. By Proposition 2.7, we can find a finite extension K' of K contained in $\overline{\mathbb{Q}}_p$ and a $G_{K'}$-stable sub-B_{dR}^+-lattice $L_0 \subset L$ such that $\text{length}_{B_{dR}^+}(L/L_0) = d$. Then $\mathcal{F}_{L_0} = (\Lambda, L_0)$ is a $G_{K'}$-equivariant vector bundle over X of degree $d - d = 0$. As the B_e-representation Λ is absolutely irreducible, it is irreducible as a B_e-representation of $G_{K'}$. Hence, $\mathcal{F}_{L_0}$ is semistable of slope 0. By Proposition 3.5, there is a $\mathbb{Q}_p$-representation U of $G_{K'}$ such that

$$\mathcal{F}_{L_0} \simeq \mathcal{O}_X \otimes U.$$

Therefore Λ, as a B_e-representation of $G_{K'}$, is isomorphic to $B_e \otimes_{\mathbb{Q}_p} U$. □

Corollary 3.9. *The category* $\text{Rep}_{B_e}(G_K)$ *is the smallest full subcategory of itself containing potentially trivialisable B_e-representations and stable under taking extensions. This is also the smallest full subcategory of itself containing trivialisable B_e-representations and stable under taking extensions and direct summands.*

Proof. For any B_e-representation Λ of G_K, one can find a finite extension K_1 of K contained in $\overline{\mathbb{Q}}_p$ such that Λ, viewed as a B_e-representation of G_{K_1}, can be viewed as a successive extension of absolutely irreducible B_e-representations of G_{K_1} and the first assumption results from the previous proposition. Hence we may find a finite extension K' of K contained in $\overline{\mathbb{Q}}_p$ such that Λ, as a B_e-representation of $G_{K'}$, is a successive extension of $G_{K'}$-trivialisable B_e-representations. Therefore the induced B_e-representation of G_K

$$\Lambda' = B_e[G_K] \otimes_{B_e[G_{K'}]} \Lambda = \mathbb{Q}[G_K] \otimes_{\mathbb{Q}[G_{K'}]} \Lambda$$

is a successive extension of trivialisable B_e-representations of G_K. But the obvious G_K-equivariant projection $\Lambda' \to \Lambda$ splits (as, if $\Lambda^\vee$ denotes the B_e-dual of Λ and

if $H = \mathrm{Gal}(K'/K)$, we have a short exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(\Lambda, \Lambda') \to \mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_{K'})}(\Lambda, \Lambda) \to H^1(H, \Lambda^\vee \otimes_{B_e} \Lambda')$$

and, as B_e is of characteristic 0, we have $H^1(H, \Lambda^\vee \otimes_{B_e} \Lambda') = 0$). Therefore, Λ is a direct summand of a successive extension of trivialisable B_e-representations. □

Remark 3.10. The results of this paragraph can also be deduced from the work of Berger ([Berger 2008] and [Berger 2009]) relating (φ, Γ)-modules on the Robba ring and B_e-pairs.

3J. *The forgetful functor* $\mathbf{Rep}_{B_e}(G_K) \to \widehat{\mathcal{C}}(G_K)$.

Proposition 3.11. *The forgetful functor*

$$\mathrm{Rep}_{B_e}(G_K) \to \mathcal{IF}(G_K)$$

is fully faithful.

Proof. Let Λ and Λ' two B_e-representations. We want to prove that any G_K-equivariant continuous map

$$\Lambda \xrightarrow{\alpha} \Lambda'$$

is B_e-linear.

Let K' be a finite Galois extension of K contained in $\overline{\mathbb{Q}}_p$ such that Λ and Λ' are successive extensions of trivialisable B_e-representations of $G_{K'}$. If $H = \mathrm{Gal}(K'/K)$, we have

$$\mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(\Lambda, \Lambda') = (\mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_{K'})}(\Lambda, \Lambda'))^H,$$
$$\mathrm{Hom}_{\mathcal{IF}(G_K)}(\Lambda, \Lambda') = (\mathrm{Hom}_{\mathcal{IF}(G_{K'})}(\Lambda, \Lambda'))^H.$$

Therefore, replacing K by K' we may assume again that there is $r \in \mathbb{N}$ and a filtration of Λ by sub-B_e-representations

$$0 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{r-1} \subset \Lambda_r = \Lambda$$

such that each Λ_i/Λ_{i-1} is trivialisable.

We proceed by induction on r, the case $r = 0$ being trivial. Assume $r \geq 1$ and that $\Lambda_r/\Lambda_{r-1} = B_e \otimes_{\mathbb{Q}_p} U$ for some $U \in \mathcal{C}^0(G_K)$. Chose a B_e-linear section $s : B_e \otimes U \to \Lambda$ of the projection $\Lambda \to B_e \otimes U$. We have a decomposition of Λ as a B_e-module into a direct sum

$$\Lambda = \Lambda_{r-1} \oplus s(B_e \otimes U) = \Lambda_{r-1} \oplus (B_e \otimes s(U)).$$

By induction, the restriction of α to Λ_{r-1} is B_e-linear. Hence there is a unique B_e-linear map

$$\alpha_0 : \Lambda \to \Lambda'$$

such that $\alpha_0(\lambda) = \alpha(\lambda)$ if $\lambda \in \Lambda_{r-1}$ and $\alpha_0(s(u)) = \alpha(s(u))$ for all $u \in U$. It is easy to check that α_0 is continuous and G_K-equivariant. The maps

$$\alpha, \alpha_0 : \Lambda \to \Lambda'$$

coincide on $\Lambda_{r-1} \oplus s(U)$ and the map $\alpha - \alpha_0$ induces, by going to the quotient, a morphism in $\mathcal{IF}(G_K)$

$$\beta : \Lambda/(\Lambda_{r-1} \oplus s(U)) \to \Lambda'.$$

Recall (cf. eg [Colmez and Fontaine 2000], proposition 1.3) that $B_{dR} = B_e + B_{dR}^+$, and $B_e \cap B_{dR}^+ = \mathbb{Q}_p$. Hence, if we set $\widetilde{B}_{dR} = B_{dR}/B_{dR}^+$, we can identify $B_e/\mathbb{Q}_p$ to $\widetilde{B}_{dR}$.

Therefore we have

$$\Lambda/(\Lambda_{r-1} \oplus s(U)) = (\Lambda_r/\Lambda_{r-1})/U = B_e \otimes U/U = \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U.$$

and $\beta \in \mathrm{Hom}_{\mathcal{IF}(G_K)}(\widetilde{B}_{dR} \otimes U, \Lambda')$.

We see that $\widetilde{B}_{dR}$ is the direct limit of the $B_d(-d)$, for $d \in \mathbb{N}$, hence

$$\widetilde{B}_{dR} \otimes U = \varinjlim_{d \in \mathbb{N}} B_d(-d) \otimes_{\mathbb{Q}_p} U.$$

Each $B_d(-d) \otimes U$ is an object of $\mathcal{C}^\infty(G_K)$. Hence, Proposition 3.4, implies that

$$\mathrm{Hom}_{\mathcal{IF}(G_K)}(B_d(-d) \otimes U, \Lambda') = 0.$$

Therefore $\beta = 0$ and $\alpha = \alpha_0$ is B_e-linear. □

We use this result to identify $\mathrm{Rep}_{B_e}(G_K)$ to a full subcategory of $\mathcal{IF}(G_K)$.

Proposition 3.12. *We have*

$$\mathrm{Rep}_{B_e}(G_K) \subset \widehat{\mathcal{C}}(G_K).$$

More precisely, for any B_e-representation Λ of G_K, there is a G_K-equivariant admissible filtration $(F^n\Lambda)_{n\in\mathbb{Z}}$ with $F^1\Lambda = 0$ and $F^n\Lambda \in \mathcal{C}(G_K)$ for all n. Moreover, we may choose this filtration so that, if $b \in \mathrm{Fil}^{-d} B_e$ and $\lambda \in F^n\Lambda$ (with $d \in \mathbb{N}$, $n \in \mathbb{Z}$), then $b\lambda \in F^{n-d}\Lambda$.

Proof. Assume first that Λ is a successive extension of trivialisable B_e-representations, i.e., that there is $r \in \mathbb{N}$ and a filtration by sub-B_e-representations

$$0 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{r-1} \subset \Lambda_r = \Lambda$$

such that each Λ_i/Λ_{i-1} is trivialisable. We proceed by induction on r, the case $r = 0$ being trivial. Assume $r \geq 1$. Setting $\Lambda_{r-1} = \Lambda'$ and choosing $U \in \mathcal{C}^0(G_K)$

such that $\Lambda_r/\Lambda_{r-1} \simeq B_e \otimes_{\mathbb{Q}_p} U$, we may assume that we have a short exact sequence of B_e-representations

$$0 \to \Lambda' \to \Lambda \to B_e \otimes U \to 0$$

and, using induction hypothesis, that we have an admissible filtration $(F^n\Lambda')_{n\in\mathbb{Z}}$ of Λ' satisfying the required properties. Let $s : B_e \otimes U \to \Lambda$ a B_e-linear section of the projection $\Lambda \to B_e \otimes U$, so that we have a decomposition of the B_e-module Λ into a direct sum

$$\Lambda = \Lambda' \oplus s(B_e \otimes U) = \Lambda' \oplus (B_e \otimes s(U)).$$

The map

$$\rho : G_K \times U \to \Lambda', \quad (g, u) \mapsto g(s(u)) - s(g(u))$$

is continuous. Therefore, if T is a G_K-stable lattice of U, then $\rho(G_K \times T)$ is compact, hence bounded which implies (by [Schneider 2002, proposition 5.6]) that there exists $m \in \mathbb{Z}$ such that $\rho(G_K \times T)$, hence also $\rho(G_K \times U)$ is contained in $F^m\Lambda'$.

If, for $n \in \mathbb{Z}$, we set

$$F^n\Lambda = \begin{cases} F^n\Lambda' \oplus (F^{n-m}B_e \otimes U) & \text{if } n \le m, \\ 0 & \text{if } n > m, \end{cases}$$

we see that $(F^n\Lambda)_{n\in\mathbb{N}}$ is an admissible filtration satisfying the required properties.

– In the general case, we choose a finite extension K' of K such that Λ is a successive extension of trivialisable B_e-representation of $G_{K'}$. Therefore we can find a $G_{K'}$-equivariant decreasing admissible filtration

$$(F^n_{K'}\Lambda)_{n\in\mathbb{Z}}$$

such that, if $n \in \mathbb{Z}$, then $F^n_{K'}\Lambda \in \mathcal{C}(G_{K'})$ and that, if $b \in \mathrm{Fil}^{-d}B_e$, for some $d \in \mathbb{N}$ and $\lambda \in F^n_{K'}\Lambda$, then $b\lambda \in F^{n-d}_{K'}\Lambda$.

For each $n \in \mathbb{Z}$, denote by $F^n\Lambda$ the smallest sub-$\mathbb{Q}_p$-vector space of Λ containing $F^n_{K'}\Lambda$ and stable under G_K. This is also the image of the obvious map

$$\mathbb{Q}_p[G_K] \otimes_{\mathbb{Q}_p[G_{K'}]} F^n_{K'}\Lambda \to \Lambda.$$

If $h_1, h_2, \ldots, h_m$ is a system of representatives of $G_K/G_{K'}$ in G_K, this is also $\sum_{i=1}^m h_i(F^n\Lambda_{K'}) \subset \Lambda$ which is still bounded and it is clear that the $(F^n\Lambda)_{n\in\mathbb{Z}}$ satisfy the required properties. □

Remark 3.13. We see immediately that $\mathrm{Rep}_{B_e}(G_K)$ is a weak Serre subcategory of $\widehat{\mathcal{C}}(G_K)$.

3K. ***Cohomology of coherent $\mathcal{O}_X$-modules.*** We denote by $\overline{B}_{dR}$ the B_e-module B_{dR}/B_e. It is not of finite type but, as the cokernel of the inclusion $B_e \to B_{dR}$ which is a morphism of $\widehat{\mathcal{C}}(G_K)$, it can be viewed as an object of this category. The equalities $B_{dR} = B_e + B_{dR}^+$ and $\mathbb{Q}_p = B_e \cap B_{dR}^+$ imply that $\overline{B}_{dR}$, as an object of $\widehat{\mathcal{C}}(G_K)$, can also be identified with $B_{dR}^+/\mathbb{Q}_p$.

If $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}}) \in \mathrm{Coh}(\mathcal{O}_X)$. The map

$$\mathcal{F}_e \to \mathcal{F}_{dR} = B_{dR} \otimes_{B_e} \mathcal{F}_e, \quad x \mapsto 1 \otimes x$$

is injective, we use it to identify $\mathcal{F}_e$ to a sub-B_e-module of $\mathcal{F}_{dR}$ and we denote by $\overline{\mathcal{F}}_{dR}$ the quotient $\mathcal{F}_{dR}/\mathcal{F}_e$.

Proposition 3.14 [Fargues and Fontaine 2018, proposition 8.2.3]. *For any $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_X)$, we have $H^i(X, \mathcal{F}) = 0$ for $i \notin \{0, 1\}$ and*

$$\begin{aligned} \mathcal{F}(X) = H^0(X, \mathcal{F}) \neq 0 &\iff \mathcal{F}^{\geq 0} \neq 0, \\ H^1(X, \mathcal{F}) \neq 0 &\iff \mathcal{F}^{<0} \neq 0. \end{aligned}$$

Moreover, there is a canonical exact sequence of $\mathbb{Q}_p$-vector spaces

$$0 \to H^0(X, \mathcal{F}) \to \mathcal{F}_e \oplus \mathcal{F}_{dR}^+ \xrightarrow{d_{\mathcal{F}}} \mathcal{F}_{dR} \to H^1(X, \mathcal{F}) \to 0 \tag{1}$$

(where $d_{\mathcal{F}}(x, y) = \iota_{\mathcal{F}}(y) - x$) which is functorial in $\mathcal{F}$.

We have a commutative diagram of $\mathbb{Q}_p$-vector spaces

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{F}_e & = & \mathcal{F}_e & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & \mathcal{F}_e \oplus \mathcal{F}_{dR}^+ & \longrightarrow & \mathcal{F}_{dR} & H^1(X, \mathcal{F}) & \longrightarrow 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{F}_{dR}^+ & \longrightarrow & \overline{\mathcal{F}}_{dR} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

whose columns and the two first lines are exact. Hence we have also an exact sequence

$$0 \to H^0(X, \mathcal{F}) \to \mathcal{F}_{dR}^+ \xrightarrow{\overline{d}_{\mathcal{F}}} \overline{\mathcal{F}}_{dR} \to H^1(X, \mathcal{F}) \to 0 \tag{2}$$

where $\overline{d}_{\mathcal{F}}(y)$ is the image of $\iota_{\mathcal{F}}(y)$ in $\overline{\mathcal{F}}_{dR}$.

3L. ***Cohomology of coherent $\mathcal{O}_X[G_K]$-modules.*** We say that an almost $\mathbb{C}_p$-representation is *effective* if this object of $\mathcal{C}(G_K)$ is isomorphic to a sub-object of $\mathcal{C}^\infty(G_K)$. We denote by $\mathcal{C}^{\geq 0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ whose objects are those which are effective.

Proposition 3.15. *Let $f : W \to V$ a morphism of $\mathcal{C}(G_K)$ with $W \in \mathcal{C}^\infty(G_K)$ and $V \in \mathcal{C}^{\geq 0}(G_K)$. Then the kernel of f belongs to $\mathcal{C}^\infty(G_K)$.*

Proof. By assumption, there exists a monomorphism $g : V \to W'$ in $\mathcal{C}(G_K)$ with $W' \in \mathcal{C}^\infty(G_K)$. The kernel of f is the same as the kernel of $gf : W \to W'$. As W and W' are in $\mathcal{C}^\infty(G_K)$, so is this kernel. □

Proposition 3.16. *Let $\mathcal{F} \in \mathcal{M}(G_K)$. Then $H^0(X, \mathcal{F}) \in \mathcal{C}^{\geq 0}(G_K)$.*

Proof. We see that $\mathcal{F}_e$, $\mathcal{F}_{dR}^+$ and $\mathcal{F}_{dR}$ can be viewed as objects of the abelian category $\widehat{\mathcal{C}}(G_K)$. The inclusion $\mathcal{F}_e \hookrightarrow \mathcal{F}_{dR}$ is a morphism of this category, hence $\overline{\mathcal{F}}_{dR}$ can also viewed as an object of $\widehat{\mathcal{C}}(G_K)$. The map $\overline{d}_{\mathcal{F}}$ of the exact sequence (2) is obviously a morphism of this category, hence

$$H^0(X, \mathcal{F}) = \ker \overline{d}_{\mathcal{F}} \quad \text{and} \quad H^1(X, \mathcal{F}) = \operatorname{coker} \overline{d}_{\mathcal{F}}$$

are objects of $\widehat{\mathcal{C}}(G_K)$.

For $m \in \mathbb{N}$, big enough, $\mathcal{F}(-m)_{HN}$ has all its HN-slopes strictly negative and $H^0(X, \mathcal{F}(-m)_{HN}) = 0$. But this is the kernel of the map

$$\mathcal{F}_{dR}^+(m) \to \overline{\mathcal{F}}_{dR}, \quad b \otimes t^m \mapsto t^m b \ (\text{mod } \mathcal{F}_e).$$

Hence we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{F}_{dR}^+(m) & \longrightarrow & \overline{\mathcal{F}}_{dR} \\ & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & \mathcal{F}_{dR}^+ & \longrightarrow & \overline{\mathcal{F}}_{dR} \end{array}$$

(the first nonzero vertical arrow sends $b \otimes t^m$ to $t^m b$) whose lines are exact. Therefore, the compositum $H^0(X, \mathcal{F}) \to \mathcal{F}_{dR}^+ \to \mathcal{F}_{dR}^+/t^m \mathcal{F}_{dR}^+$ is injective and $H^0(X, \mathcal{F})$, subobject in $\widehat{\mathcal{C}}(G_K)$ of $\mathcal{F}_{dR}^+/t^m.\mathcal{F}_{dR}^+ \in \mathcal{C}^\infty(G_K)$ is in $\mathcal{C}^{\geq 0}(G_K)$. □

4. Hulls and construction of the functor $V \mapsto \mathcal{F}_V$

4A. ***Generalities.*** In what follows, $B_?$ is either B_e, B_{dR}^+ or B_{dR}.

We know (Remarks 3.13 and 2.13) that $\mathrm{Rep}_{B_?}(G_K)$ can be identified with a weak Serre subcategory of $\widehat{\mathcal{C}}(G_K)$. We have "inclusions" of weak Serre subcategories

$$\begin{array}{ccccc} \mathrm{Rep}_{B_{dR}^+}(G_K) & & & & \\ & \searrow & & & \\ & & \mathrm{Rep}_{B_{dR}}(G_K) & \longrightarrow & \widehat{\mathcal{C}}(G_K) \\ & \nearrow & & & \\ \mathrm{Rep}_{B_e}(G_K) & & & & \end{array}$$

Let V be an almost $\mathbb{C}_p$-representation. We say that V *has a* $B_?$*-hull* if the functor

$$\mathrm{Rep}_{B_?}(G_K) \to \mathbb{Q}_p\text{-vector spaces}, \quad W \mapsto \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, W)$$

is representable, i.e., if there is a (necessarily unique up to unique isomorphism) pair $(V_?, \iota_?^V)$, with $V_?$ a $B_?$-representation and $\iota_?^V : V \to V_?$ a G_K-equivariant continuous $\mathbb{Q}_p$-linear map, such that, for all $B_?$-representation W, the map

$$\mathrm{Hom}_{\mathrm{Rep}_{B_?}(G_K)}(V_?, W) \to \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, W),$$

induced by $\iota_?^V$, is bijective.

When it is the case, we call $(V_?, \iota_?^V)$, or abusively $V_?$, the $B_?$*-hull of* V.

Our purpose is to show that such an hull always exists and to use these hulls to construct a functor

$$\mathcal{C}(G_K) \to \mathcal{M}(G_K), \quad V \mapsto \mathcal{F}_V.$$

Remark 4.1. Let V be an almost $\mathbb{C}_p$-representation and let I_V the class of morphisms

$$\iota : V \to W_\iota$$

of $\widehat{\mathcal{C}}(G_K)$ whose source is V and target a $B_?$-representation. With suitable conventions and abuses, to say that V has a $B_?$-hull means that the directed inverse limit

$$V_? = \varprojlim_{\iota \in I_V} W_\iota$$

exists and that the $B_?$-module underlying this "pro-$B_?$-representation of G_K" is of finite type.

Restricted to the full subcategory of $\mathcal{C}(G_K)$ of almost $\mathbb{C}_p$-representations admitting a $B_?$-hull, the correspondence $V \mapsto V_?$ is obviously functorial.

Let $V \in \mathcal{C}(G_K)$ such that, with obvious notations, $(V_{dR}^+, \iota_{dR}^{V,+})$ exists, let $M \in \mathrm{Rep}_{B_{dR}}(G_K)$ and $f : V \to M$ a morphism in $\widehat{\mathcal{C}}(G_K)$. We see that the sub B_{dR}^+-module W of M generated by $f(V)$ is an object of $\mathcal{C}^\infty(G_K)$, hence there is a unique morphism (in $\widehat{\mathcal{C}}(G_K)$ or, in this case, equivalently in $\mathcal{C}^\infty(G_K)$)

$$g : V_{dR}^+ \to W \subset M$$

such that $f = g \circ \iota_{dR}^{V,+}$ and we have

$$\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, M) = \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V_{dR}^+, M) =$$
$$\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(B_{dR} \otimes_{B_{dR}^+} V_{dR}^+, M) = \mathrm{Hom}_{\mathrm{Rep}_{B_{dR}(G_K)}}(B_{dR} \otimes_{B_{dR}^+} V_{dR}^+, M).$$

Therefore V_{dR} exists and can be identified with $B_{dR} \otimes_{B_{dR}^+} V_{dR}^+$.

The same argument applies to the case where (V_e, ι_e^V) exists. Hence we have:

Proposition 4.2. *Let $V \in \mathcal{C}(G_K)$.*

(i) *If V_{dR}^+ exists, V_{dR} exists and is, canonically and functorially, $B_{dR} \otimes_{B_{dR}^+} V_{dR}^+$.*

(ii) *If V_e exists, V_{dR} exists and is, canonically and functorially, $B_{dR} \otimes_{B_e} V_e$.*

Proposition 4.3. *Let $B_?$ as above and let V be an almost $\mathbb{C}_p$-representation of G_K which has a $B_?$-hull $(V_?, \iota_?^V)$.*

(i) *The image of $\iota_?^V$ generates $V_?$ as a $B_?$-module.*

(ii) *If moreover*

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence in $\mathcal{C}(G_K)$, then V'' has a $B_?$-hull which is the quotient of $V_?$ by the sub-$B_?$-module of $V_?$ generated by the image of V'.

(iii) *In this situation, if V' has a $B_?$-hull, then the sequence*

$$V_?' \to V_? \to V_?'' \to 0$$

is exact.

Proof. (i) Let W_0 be the sub-$B_?$-module of $V_?$ generated by the image of V. As $B_?$ is noetherian, this is a $B_?$-module of finite type. By the universal property of $V_?$, there is a unique morphism $\nu : V_? \to W_0$ such that the map $V \to W_0$ is $\nu \circ \iota_?^V$ and we see that $V_? = W_0 \oplus \ker \nu$. The fact that $\mathrm{id}_{V_?}$ is the unique endomorphism of $V_?$ such that $\nu \circ \iota_?^V = \iota_?^V$ forces $\ker \nu$ to be 0.

(ii) If W is any $B_?$-representation, we have

$$\begin{aligned}\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V'', W) &= \left\{ f \in \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, W) \mid f(V') = 0 \right\} \\ &= \left\{ f \in \mathrm{Hom}_{\mathrm{Rep}_{B_?}(G_K)}(V_?, W) \mid f(\iota_?^V(V')) = 0 \right\} \\ &= \mathrm{Hom}_{\mathrm{Rep}_{B_?}(G_K)}(V_?/B_?\iota_?^V(V'), W).\end{aligned}$$

(iii) Let N be the kernel of the projection $V_? \to V_?''$. The image of $V_?'$ in $V_?$ is clearly contained in N. As N is the sub-$B_?$-module generated by the image of V', the map $V_?' \to N$ is surjective and

$$V_?' \to V_? \to V_?'' \to 0$$

is exact. □

4B. ***Construction of trivialisable almost $\mathbb{C}_p$-representations.*** A *trivialisation* of an almost $\mathbb{C}_p$-representation V is a short exact sequence

$$0 \to U \to V \to W \to 0$$

in $\mathcal{C}(G_K)$ with $U \in \mathcal{C}^0(G_K)$ and $W \in \mathcal{C}^\infty(G_K)$.

An almost $\mathbb{C}_p$-representation is *trivialisable* if it admits a trivialisation.

If $V \in \mathcal{C}(G_K)$, if $\tilde{f} : V/U_+ \to W/U_-$ is an almost isomorphism with $W \in \mathcal{C}(G_K)$ and if $\widehat{V} = W \times_{W/U_-} V$, we have, in $\mathcal{C}(G_K)$, a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U_+ & = & U_+ & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & U_- & \to & \widehat{V} & \to & V & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & U_- & \to & W & \to & V/U_+ & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and V is a quotient of $\widehat{V}$ which is trivialisable as it is an extension of W by $U_+ \in \mathcal{C}^0(G_K)$.

Given $U \in \mathcal{C}^0(G_K)$ and $W \in \mathcal{C}^\infty(G_K)$, it is easy to construct all almost $\mathbb{C}_p$-representations which are extensions of W by U:

Recall that

$$B_{dR} = B_e + B_{dR}^+ \quad \text{and} \quad B_e \cap B_{dR}^+ = \mathbb{Q}_p$$

and that we set

$$\widetilde{B}_{dR} = B_{dR}/B_{dR}^+ = B_e/\mathbb{Q}_p.$$

Let U be an object of $\mathcal{C}^0(G_K)$ and W an object of $\mathcal{C}^\infty(G_K)$. Tensoring the exact sequence

$$0 \to \mathbb{Q}_p \to B_e \to \widetilde{B}_{dR} \to 0$$

by U we get a short exact sequence in $\widehat{\mathcal{C}}(G_K)$

$$0 \to U \to B_e \otimes_{\mathbb{Q}_p} U \to \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U \to 0$$

inducing a map

$$\begin{array}{ccc}
\delta_{U,W} : \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(W, \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U) & \longrightarrow & \mathrm{Ext}^1_{\widehat{\mathcal{C}}(G_K)}(W, U) \\
\| & & \| \\
\mathrm{Hom}_{\widehat{\mathcal{C}}^\infty(G_K)}(W, \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U) & & \mathrm{Ext}^1_{\mathcal{C}(G_K)}(W, U)
\end{array}$$

Proposition 4.4 [Fontaine 2003, proposition 3.7]. *Let $U \in \mathcal{C}^0(G_K)$ and $W \in \mathcal{C}^\infty(G_K)$. The map*

$$\delta_{U,W} : \mathrm{Hom}_{\widehat{\mathcal{C}}^\infty(G_K)}(W, \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U) \to \mathrm{Ext}^1_{\mathcal{C}(G_K)}(W, U)$$

is an isomorphism.

Hence if V is a trivialisable almost $\mathbb{C}_p$-representation and if

$$(T) \qquad 0 \to U \to V \to W_0 \to 0$$

is a trivialisation of V, there is a unique

$$\rho_T \in \mathrm{Hom}_{\widehat{\mathcal{C}}^\infty(G_K)}(W_0, \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U)$$

such that the square

$$\begin{array}{ccc} V & \longrightarrow & W_0 \\ \downarrow & & \downarrow \rho_T \\ B_e \otimes_{\mathbb{Q}_p} U & \longrightarrow & \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U \end{array}$$

is cartesian.

4C. *Construction of the hulls.*

Proposition 4.5. *Any almost $\mathbb{C}_p$-representation V has a B_e-hull V_e, a B^+_{dR}-hull V^+_{dR} and a B_{dR}-hull V_{dR}. We have*

$$V_{dR} = B_{dR} \otimes_{B_e} V_e = B_{dR} \otimes_{B^+_{dR}} V^+_{dR},$$
$$\mathrm{rank}_{B^+_{dR}} V^+_{dR} = \mathrm{rank}_{B_e} V_e = \dim_{B_{dR}} V_{dR} \geq h(V)$$

and equality holds when V is trivialisable.

Moreover:

(i) *If $U \in \mathcal{C}^0(G_K)$, then $U_e = B_e \otimes_{\mathbb{Q}_p} U$ and $U^+_{dR} = B^+_{dR} \otimes_{\mathbb{Q}_p} U$,*

(ii) *If $W \in \mathcal{C}^\infty(G_K)$, then $W_e = 0$ and $W^+_{dR} = W$,*

(iii) *If*

$$(T) \qquad 0 \to U \to V \to W_0 \to 0$$

is a trivialisation of an almost $\mathbb{C}_p$-representation V, then

(a) *the map $U_e = B_e \otimes_{\mathbb{Q}_p} U \to V_e$ is an isomorphism, and*

(b) *we have a short exact sequence*

$$0 \to B^+_{dR} \otimes_{\mathbb{Q}_p} U \to V^+_{dR} \to W_0 \to 0$$

More precisely, V^+_{dR} is the fiber product $(B_{dR} \otimes_{\mathbb{Q}_p} U) \times_{\widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U} W_0$ (where $W_0 \to \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U$ is the map ρ_T).

Proof. From Proposition 4.2, we see that the existence of V_e and V_{dR}^+ implies the existence of V_{dR} and the equalities:

$$V_{dR} = B_{dR} \otimes_{B_e} V_e = B_{dR} \otimes_{B_{dR}^+} V_{dR}^+,$$
$$\mathrm{rank}_{\mathrm{B_{dR}^+}} \mathrm{V_{dR}^+} = \mathrm{rank}_{\mathrm{B_e}} \mathrm{V_e} = \dim_{\mathrm{B_{dR}}} \mathrm{V_{dR}}.$$

(i) Let $U \in \mathcal{C}^0(G_K)$. By adjunction, for any B_e-representation Λ, we have

$$\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(U, \Lambda) = \mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda)$$

hence U_e exists and is $B_e \otimes_{\mathbb{Q}_p} U$. Similarly, for any object $W_0 \in \mathcal{C}^\infty(G_K)$, we have

$$\mathrm{Hom}_{\mathcal{C}(G_K)}(U, W_0) = \mathrm{Hom}_{\widehat{\mathcal{C}}^\infty(G_K)}(B_{dR}^+ \otimes_{\mathbb{Q}_p} U, W_0)$$

hence U_{dR}^+ exists and is $B_{dR}^+ \otimes_{\mathbb{Q}_p} U$. In particular, $\dim_{B_{dR}} U_{dR} = h(U)$.

(ii) Let $W \in \mathcal{C}^\infty(G_K)$. For all B_e-representation Λ, we have $\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(W, \Lambda) = 0$ (Proposition 3.4). Therefore W_e exists and is $= 0$. For any $W_0 \in \mathcal{C}^\infty(G_K)$, we have $\mathrm{Hom}_{\mathcal{C}(G_K)}(W, W_0) = \mathrm{Hom}_{\mathcal{C}^\infty(G_K)}(W, W_0)$ (Proposition 2.6) hence W_{dR}^+ exists and is W. In particular $\dim_{B_{dR}} W_{dR} = 0 = h(W)$.

(iii) Let V a trivialisable almost $\mathbb{C}_p$-representation and

$$(T) \qquad 0 \to U \to V \to W_0 \to 0$$

a trivialisation.

(a) Let Λ be a B_e-representation. The inclusion $U \to V$ induces a map

$$\alpha : \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, \Lambda) \to \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(U, \Lambda)$$
$$\xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda) = \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda)$$

(Propositions 3.11 and 3.12). We have a cartesian square (Section 4B)

$$(S) \qquad \begin{array}{ccc} V & \longrightarrow & W_0 \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho_T} \\ B_e \otimes_{\mathbb{Q}_p} U & \longrightarrow & \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U \end{array}$$

and we may use ρ to get a map

$$\beta : \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda) \to \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, \Lambda)$$

Let $f \in \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda)$ and $f' = \alpha(\beta(f))$. If $\sum b_i \otimes u_i \in B_e \otimes_{\mathbb{Q}_p} U$, we have

$$f'\Big(\sum b_i \otimes u_i\Big) = \sum b_i(\beta(f)(u_i)) = \sum b_i f(u_i) = f\Big(\sum b_i \otimes u_i\Big)$$

as f is B_e-linear, hence $f' = f$.

Let $g \in \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, \Lambda)$ and $g' = \alpha(\beta(g))$. If $u \in U$, as $\rho(u) = u$, we have

$$g'(u) = \beta(\alpha(g))(u) = \alpha(g)(u) = g(u)$$

Hence $g' - g$ factors through a morphism in $\widehat{\mathcal{C}}(G_K)$

$$W_0 \to \Lambda$$

which is necessarily 0 (Theorem 2.9), hence $g' = g$. Therefore we see that α is an isomorphism. It is implies that V_e exists and is equal to $U_e = B_e \otimes_{\mathbb{Q}_p} U$.

(b) We want to show that V_{dR}^+ exists and is equal to

$$W_1 = (B_{dR} \otimes_{\mathbb{Q}_p} U) \times_{\widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U} W_0.$$

Using the cartesian square (S) and the inclusion $B_e \otimes_{\mathbb{Q}_p} U \subset B_{dR} \otimes_{\mathbb{Q}_p} U$, we get a morphism of $\widehat{\mathcal{C}}(G_K)$

$$V \to W_1$$

and we have a commutative diagram in $\widehat{\mathcal{C}}(G_K)$

$$(*) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & U_{dR}^+ & \longrightarrow & W_1 & \longrightarrow & W_0 & \longrightarrow & 0 \end{array}$$

whose lines are exact.

If W is any B_{dR}^+-representation, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(W_0, W) & \longrightarrow & \mathrm{Hom}(V, W) & \longrightarrow & \mathrm{Hom}(U, W) & \longrightarrow & \mathrm{Ext}^1(W_0, W) \\ & & \| & & \downarrow & & \downarrow \simeq & & \| \\ 0 & \longrightarrow & \mathrm{Hom}(W_0, W) & \longrightarrow & \mathrm{Hom}(W_1, W) & \longrightarrow & \mathrm{Hom}(U_{dR}^+, W) & \longrightarrow & \mathrm{Ext}^1(W_0, W) \end{array}$$

(where all the Hom and Ext^1 are computed in $\widehat{\mathcal{C}}(G_K)$) which implies that

$$\mathrm{Hom}_{\mathcal{C}(G_K)}(V, W) \to \mathrm{Hom}_{\mathcal{C}(G_K)}(W_1, W) = \mathrm{Hom}_{\mathcal{C}^\infty(G_K)}(W_1, W)$$

is an isomorphism. Hence V_{dR}^+ exists and is equal to W_1.

Finally, let V be any object of $\mathcal{C}(G_K)$. We can find an exact sequence

$$0 \to U \to \widehat{V} \to V \to 0$$

with $\widehat{V}$ trivialisable. The existence of $\widehat{V}_e$ and $\widehat{V}_{dR}^+$ implies (Proposition 4.3) the existence of V_e and V_{dR}^+. The exactness of the sequence

$$U_{dR} \to \widehat{V}_{dR} \to V_{dR} \to 0$$

implies that

$$\dim_{B_{dR}} V_{dR} \geq \dim_{B_{dR}} \widehat{V}_{dR} - \dim_{B_{dR}} U_{dR} = h(\widehat{V}) - h(U) = h(V). \qquad \square$$

4D. ***The functor $V \mapsto \mathcal{F}_V$.*** For any almost $\mathbb{C}_p$-representation V, denote

$$\iota_V : V_{dR}^+ \to V_{dR} = B_{dR} \otimes_{B_e} V_e$$

the natural map. It induces an isomorphism $B_{dR} \otimes_{B_{dR}^+} V_{dR}^+ \to V_{dR}$. Therefore

$$\mathcal{F}_V = (V_{dR}^+, V_e, \iota_V)$$

is a coherent $\mathcal{O}_X[G_K]$-module. This construction is clearly functorial and we get an additive functor

$$\mathcal{C}(G_K) \to \mathcal{M}(G_K), \quad V \mapsto \mathcal{F}_V.$$

From the universal properties of the functor $V \mapsto V_{dR}^+$ and $V \mapsto V_e$, we deduce the fact that $V \mapsto \mathcal{F}_V$ is left adjoint to $\mathcal{F} \mapsto \mathcal{F}(X)$.

5. The equivalence $\mathcal{M}^{\geq 0}(G_K) \to \mathcal{C}^{\geq 0}(G_K)$

5A. ***A characterisation of effective coherent $\mathcal{O}_X[G_K]$-modules.***

Theorem 5.1. *The category $\mathcal{M}^{\geq 0}(G_K)$ is the smallest strictly full subcategory of $\mathcal{M}(G_K)$ containing $\mathcal{M}^0(G_K)$ and $\mathcal{M}^\infty(G_K)$ and stable under taking extensions and direct summands.*

Lemma 5.2. *Let s be a positive rational number. There exists $\mathcal{G}_s \in \mathcal{M}^s(G_K)$ which is an extension of an object of $\mathcal{M}^\infty(G_K)$ by an object of $\mathcal{M}^0(G_K)$.*

Proof of the theorem given the lemma. As a subcategory of $\mathcal{M}(G_K)$, the category $\mathcal{M}^{\geq 0}(G_K)$ is obviously stable under taking extensions and direct summands. Hence, it suffices to show that any $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$ can be written as a direct summand of successive extensions of direct summands of objects which are extensions of an object of $\mathcal{M}^\infty(G_K)$ by an object of $\mathcal{M}^0(G_K)$. Using the Harder–Narasimhan filtration, it is enough to show that, if $\mathcal{F}$ is semistable of slope $s \geq 0$, then $\mathcal{F}$ is such a direct summand.

If $s = 0$, then $\mathcal{F} \in \mathcal{M}^0(G_K)$ and, if $s = +\infty$, then $\mathcal{F} \in \mathcal{M}^\infty(G_K)$ and we may assume that s is a positive rational number.

Let $\mathcal{G}_s$ as in the lemma, so that we have a short exact sequence

$$0 \to \mathcal{G}_s^0 \to \mathcal{G}_s \to \mathcal{G}_s^\infty \to 0$$

with $\mathcal{G}_s^0 \in \mathcal{M}^0(G_K)$ ans $\mathcal{G}_s^\infty \in \mathcal{M}^\infty(G_K)$. As $\mathcal{G}_s$ is a vector bundle (it has no torsion), its dual $\mathcal{G}_s^\vee$ is well defined and semistable of slope $-s$. Therefore

$$\mathcal{F}_0 = \mathcal{F} \otimes \mathcal{G}_s^\vee$$

is semistable of slope 0. We have a short exact sequence

$$0 \to \mathcal{F}_0 \otimes \mathcal{G}_s^0 \to \mathcal{F}_0 \otimes \mathcal{G}_s \to \mathcal{F}_0 \otimes \mathcal{G}_s^\infty \to 0$$

and $\mathcal{F}_0 \otimes \mathcal{G}_s$ is an extension of $\mathcal{F}^0 \otimes \mathcal{G}_s^\infty \in \mathcal{M}^\infty(G_K)$ by $\mathcal{F}^0 \otimes \mathcal{G}_s^0 \in \mathcal{M}^0(G_K)$.

But, with obvious notations,

$$\mathcal{F}_0 \otimes \mathcal{G}_s = \mathcal{F} \otimes \mathcal{G}_s^\vee \otimes \mathcal{G}_s = \mathcal{F} \otimes \mathrm{End}(\mathcal{G}_s).$$

If $\mathrm{End}^0(\mathcal{G}_s)$ is the subsheaf of elements of trace 0 in $\mathrm{End}(\mathcal{G}_s)$, we have

$$\mathrm{End}(\mathcal{G}_s) = \mathcal{O}_X \oplus \mathrm{End}^0(\mathcal{G}_s)$$

hence

$$\mathcal{F}_0 \otimes \mathcal{G}_s = \mathcal{F} \otimes (\mathcal{O}_X \oplus \mathrm{End}^0(\mathcal{G}_s)) = \mathcal{F} \oplus (\mathcal{F} \otimes \mathrm{End}^0(G_s))$$

and $\mathcal{F}$ is a direct summand of $\mathcal{F}_0 \otimes \mathcal{G}_s$. □

Proof of the lemma. We may assume $K = \mathbb{Q}_p$. Recall the following facts ([Fargues and Fontaine 2018, proposition 10.5.3]; see also [Colmez and Fontaine 2000, §5]):

- A *filtered φ-module over $\mathbb{Q}_p$* is a pair (D, Fil) consisting of
  - (a) a *φ-module over $\mathbb{Q}_p$*, i.e., a finite-dimensional $\mathbb{Q}_p$-vector space D equipped with an automorphism $\varphi : D \to D$,
  - (b) an exhausted and separated decreasing filtration $(\mathrm{Fil}^n D)_{n\in\mathbb{Z}}$.
  - (i) There is a fully faithful additive functor

$$(D, \mathrm{Fil}) \mapsto \mathcal{F}_{D,\mathrm{Fil}}$$

from the category of filtered φ-modules over $\mathbb{Q}_p$ to the category of $G_{\mathbb{Q}_p}$-equivariant vector bundles over X (the essential image consists of those equivariant vector bundles which are *crystalline*, i.e., those $\mathcal{F}$'s such that the natural map

$$B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \left(B_{\mathrm{cris}} \otimes_{B_e} \mathcal{F}_e\right)^{G_K} \to B_{\mathrm{cris}} \otimes_{B_e} \mathcal{F}_e$$

is bijective): we have $\mathcal{F}_{D,\mathrm{Fil}} = (\mathcal{F}_{D,\mathrm{Fil},e}, \mathcal{F}^+_{D,\mathrm{Fil},dR})$ where

– $\mathcal{F}_{D,\mathrm{Fil},e}$ is the B_e-module $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} D)_{\varphi=1}$ which implies that

$$\mathcal{F}_{D,\mathrm{Fil},dR} = B_{dR} \otimes_{B_e} \mathcal{F}_{D,e} = B_{dR} \otimes_{\mathbb{Q}_p} D,$$

– $\mathcal{F}^+_{D,\mathrm{Fil},dR} = \mathrm{Fil}^0(B_{dR} \otimes_{\mathbb{Q}_p} D) = \sum_{n\in\mathbb{Z}} \mathrm{Fil}^{-n} B_{dR} \otimes \mathrm{Fil}^n D.$

Set $s = d/h$ with d, h positive integers, prime together.

Consider the φ-module D over $\mathbb{Q}_p$ whose underlying $\mathbb{Q}_p$-vector space is of dimension h, with $(e_r)_{r\in\mathbb{Z}/h\mathbb{Z}}$ as a basis and

$$\varphi(e_r) = \begin{cases} e_{r+1} & \text{if } r+1 \neq 0, \\ p^{-d} e_0 & \text{if } r+1 = 0. \end{cases}$$

We equip D with two distinct filtrations Fil and Fil_0:

$$\mathrm{Fil}^n D = \begin{cases} D & \text{if } n \le 0, \\ 0 & \text{if } n > 0, \end{cases} \qquad \mathrm{Fil}_0^n D = \begin{cases} D & \text{if } n \le -d, \\ \bigoplus_{r \ne 0} \mathbb{Q}_p e_r & \text{if } -d < n \le 0, \\ 0 & \text{if } n > d. \end{cases}$$

Set $\mathcal{G}_s = \mathcal{F}_{D,\mathrm{Fil}}$ and $\mathcal{G}_s^0 = \mathcal{F}_{D,\mathrm{Fil}_0}$. Both are coherent $\mathcal{O}_X[G_K]$-module of rank h. As the polynomial $X^h - p^{-d}$ is irreducible over $\mathbb{Q}_p$, the $\mathbb{Q}_p[\varphi]$-module D is irreducible which implies that $\mathcal{G}_s$ and $\mathcal{G}_s^0$ are stable, hence semistable. An easy computation shows that $\deg(\mathcal{G}_s) = d$ and $\deg(\mathcal{G}_s^0) = 0$, hence $\mathcal{G}_s$ is semistable of slope $d/h = s$ and $\mathcal{G}_s^0$ is semistable of slope 0, hence belongs to $\mathcal{M}^0(G_K)$. We see that $\mathcal{G}_{s,e}^0 = \mathcal{G}_{s,e}$ and that $(\mathcal{G}_s^0)_{dR}^+ \subset (\mathcal{G}_s)_{dR}$. Therefore $\mathcal{G}_s^0$ is a subobject of $\mathcal{G}_s$ and the cokernel $\mathcal{G}_s^\infty$ is torsion, and so belongs to $\mathcal{M}^\infty(G_K)$. □

5B. ***Some properties of effective almost $\mathbb{C}_p$-representations.*** Recall (Section 1E) that an exact subcategory of an abelian category is a strictly full subcategory containing 0 and stable under extensions. For instance the previous theorem shows that $\mathcal{M}^{\ge 0}(G_K)$ is an exact subcategory of $\mathcal{M}(G_K)$.

Theorem 5.3. *Let $V \in \mathcal{C}(G_K)$. The following conditions are equivalent*:

(i) *V is effective (i.e., $V \in \mathcal{C}^{\ge 0}(G_K)$).*

(ii) *There is a finite extension K' of K contained in $\overline{Q}_p$ such that V, as an object of $\mathcal{C}(G_{K'})$ is a successive extension of objects belonging either to $\mathcal{C}^0(G_{K'})$ or to $\mathcal{C}^\infty(G_{K'})$.*

(iii) *V belongs to the smallest strictly full subcategory of $\mathcal{C}(G_K)$ containing $\mathcal{C}^0(G_K)$ and $\mathcal{C}^\infty(G_K)$ and stable under taking extensions and direct summands.*

Moreover $\mathcal{C}^{\ge 0}(G_K)$ is an exact subcategory of $\mathcal{C}(G_K)$.

Before proving this theorem, let's state an other result. Recall (Section 4D) that, to any $V \in \mathcal{C}(G_K)$, we associated the coherent $\mathcal{O}_X[G_K]$-module

$$\mathcal{F}_V = (V_{dR}^+, V_e, \iota_V).$$

We have

$$(\mathcal{F}_V)_{dR}^+ = V_{dR}^+, \quad (\mathcal{F}_V)_e = V_e, \quad \iota_{\mathcal{F}_V} = \iota_V.$$

Therefore, if we set $\overline{V}_{dR} = \overline{\mathcal{F}_V}_{dR} = V_{dR}/V_e$, we have (cf. Section 3L) an exact sequence

$$(C) \qquad 0 \to H^0(X, \mathcal{F}_V) \to V_{dR}^+ \xrightarrow{\bar{\iota}_V} \overline{V}_{dR} \to H^1(X, \mathcal{F}_V) \to 0$$

(where $\bar{\iota}_V = \bar{\iota}_{\mathcal{F}_V}$ is the compositum of ι_V with the projection $V_{dR} \to V_{dR}/V_e$) and, as $V \subset V_e$ is injective, the image of V in V_{dR}^+ is contained in $\mathcal{F}_V(X) = H^0(X, \mathcal{F}_V)$.

Proposition 5.4. *Let $V \in \mathcal{C}^{\geq 0}(G_K)$.*

(i) *We have $h(V) \geq 0$ and $\dim_{B_{dR}} V_{dR} = h(V)$.*

(ii) *We have $V \in \mathcal{C}^{\infty}(G_K) \Longleftrightarrow h(V) = 0$,*

(iii) *The sequence*

$$0 \to V \to V_{dR}^{+} \xrightarrow{\bar{\iota}_V} \overline{V}_{dR} \to 0$$

is exact.

(iv) *the map $V \to H^0(X, \mathcal{F}_V)$ is bijective and $\mathcal{F}_V \in \mathcal{M}^{\geq 0}(G_K)$.*

Moreover, the restriction to $\mathcal{C}^{\geq 0}(G_K)$ of the four functors $\mathcal{C}(G_K) \to \widehat{\mathcal{C}}(G_K)$

$$V \mapsto V_{dR}^{+}, \quad V \mapsto V_e, \quad V \mapsto V_{dR} \quad V \mapsto \overline{V}_{dR}$$

and of the functor

$$\mathcal{C}(G_K) \to \mathcal{M}(G_K), \quad V \mapsto \mathcal{F}_V$$

are exact.

Proof of the theorem and beginning of the proof of the proposition. For any $V \in \mathcal{C}^{\geq 0}(G_K)$, we denote by d_V the infimum of the $d(W)$'s for all $W \in \mathcal{C}^{\infty}(G_K)$ such that V is isomorphic to a subobject of W (note that $d(V) \leq d_V$).

Denote by $\mathcal{K}$ the set of finite extensions L of K contained in $\overline{\mathbb{Q}}_p$. For any $L \in \mathcal{K}$, let $\mathcal{C}^{?}(G_L)$ the full subcategory of $\mathcal{C}(G_L)$ whose objects can be written as a successive extension of objects belonging either to $\mathcal{C}^0(G_L)$ or to $\mathcal{C}^{\infty}(G_L)$.

We now show assertion (i) of the proposition and the implication (i)$\Longrightarrow$(ii) of the theorem, i.e., that, if $V \in \mathcal{C}^{\geq 0}(G_K)$, then

$\dim_{B_{dR}} V_{dR} = h(V)$ (so $h(V) \geq 0$) and there exists $K' \in \mathcal{K}$ such that $V \in \mathcal{C}^{?}(G_{K'})$.

We proceed by induction on d_V, the case $d_V = 0$ being trivial.

Let $V \subset W$ an embedding of V into an object $W \in \mathcal{C}^{\infty}(G_K)$ satisfying $d(W) = d_V > 0$. We can find (cf. Proposition 2.7) $K_1 \in \mathcal{K}$ and a G_{K_1}-stable sub-B_{dR}^{+}-module W' of W of length 1. Setting $W'' = W/W'$, $V' = V \cap W'$ and denoting V'' the image of V in W'', we get a commutative diagram in $\mathcal{C}(G_{K_1})$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \end{array}$$

whose rows are exact and vertical arrows are injective which implies that V' and V'' belong to $\mathcal{C}^{\geq 0}(G_{K_1})$. We have $d(V') \leq d(W') = 1$. From Corollary 2.10, we get that either $d(V') = 1$ in which case $V' = W'$ or $d(V') = 0$ which implies that $V' \in \mathcal{C}^0(G_{K_1})$.

– If $V' = W'$, we have $h(V') = 0$ and $(V')_{dR}^{+} = W'$ hence $V'_{dR} = 0$.

– If $V' \in \mathcal{C}^0(G_{K_1})$, we have $h(V') = \dim_{\mathbb{Q}_p} V'$ and $V'_{dR} = B_{dR} \otimes_{\mathbb{Q}_p} V'$.

In both cases, we have $\dim_{B_{dR}} V'_{dR} = h(V')$. By induction, we have $\dim_{B_{dR}} V''_{dR} = h(V'')$. The exactness of the sequence

$$V'_{dR} \to V_{dR} \to V''_{dR} \to 0$$

implies that

$$\dim_{B_{dR}} V_{dR} \leq \dim_{B_{dR}} V'_{dR} + \dim_{B_{dR}} V''_{dR} = h(V') + h(V'') = h(V),$$

hence, as $\dim_{B_{dR}} V_{dR} \geq h(V)$ (Proposition 4.5), we get $\dim_{B_{dR}} V_{dR} = h(V)$, i.e the assertion (i) of the proposition.

Also by induction, as V'' belongs to $\mathcal{C}^{\geq 0}(G_{K_1})$, there is $K' \in \mathcal{K}$ containing K_1 such that $V'' \in \mathcal{C}^?(G_{K'})$. Then V, as a representation of $G_{K'}$, is an extension of V'' by either an object of $\mathcal{C}^\infty(G_{K'})$ (if $d(V') = 1$) or by an object of $\mathcal{C}^0(G_{K'})$ (if $d(V') = 0$). In both cases, V belongs to $\mathcal{C}^?(G_{K'})$.

Therefore, given $V \in \mathcal{C}^{\geq 0}(G_K)$, there is $K' \in \mathcal{K}$ and a filtration of V by subobjects in $\mathcal{C}^+(G_{K'})$

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = V$$

such that, if $i = 1, 2, \ldots, r$, then V_i / V_{i-1} belongs either to $\mathcal{C}^0(G_{K'})$ or to $\mathcal{C}^\infty(G_{K'})$. This proves the implication (i)$\Longrightarrow$(ii) of the theorem.

In particular, we have $h(V) = \sum_{i=1}^r h(V_i / V_{i-1})$ which is > 0 unless $h(V_i / V_{i-1})$ vanishes for all i, which means that V_i / V_{i-1} belongs to $\mathcal{C}^\infty(G_K)$. As $\mathcal{C}^\infty(G_K)$ is stable under taking extensions, we get the equivalence

$$h(V) = 0 \Longleftrightarrow V \in \mathcal{C}^\infty(G_K)$$

which is the assertion (ii) of the proposition.

The implication (ii)$\Longrightarrow$(iii) of the theorem is obvious: If V satisfies (ii), the induced representation $\mathbb{Q}_p[G_K] \otimes_{\mathbb{Q}_p[G_{K'}]} V$ belongs to $\mathcal{C}^?(G_K)$ and V is a direct summand of this representation.

As a full subcategory of $\mathcal{C}(G_K)$, the category $\mathcal{C}^{\geq 0}(G_K)$ is obviously stable under taking direct summands. Hence, we see that the implication (iii)$\Longrightarrow$(i) of the theorem and the fact that $\mathcal{C}^{\geq 0}(G_K)$ is an exact subcategory of $\mathcal{C}(G_K)$ result from the following:

Lemma 5.5. *Assume we have a short exact sequence in $\mathcal{C}(G_K)$*

$$0 \to V_0 \to V_1 \to V_2 \to 0 \tag{1}$$

with $V_2 \in \mathcal{C}^{\geq 0}(G_K)$ and V_0 belonging either to $\mathcal{C}^0(G_K)$ or to $\mathcal{C}^\infty(G_K)$. Then $V_1 \in \mathcal{C}^{\geq 0}(G_K)$ and the sequence

$$0 \to V^+_{0,dR} \to V^+_{1,dR} \to V^+_{2,dR} \to 0$$

is exact.

Proof of the lemma. Assume first that V_0 belongs to $\mathcal{C}^0(G_K)$: we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & V_{0,dR}^+ & \longrightarrow & V_{1,dR}^+ & \longrightarrow & V_{2,dR}^+ & \longrightarrow & 0 \end{array}$$

whose rows are exact, the maps $V_0 \to V_{0,dR}^+$ and $V_2 \to V_{2,dR}^+$ being injective. I will show that the map $V_{0,dR}^+ \to V_{1,dR}^+$ is injective. As $V_{0,dR}^+ = B_{dR}^+ \otimes_{\mathbb{Q}_p} V_0$ is a torsion free B_{dR}^+-module, it is enough to check that $V_{0,dR} \to V_{1,dR}$ is injective. If it were not true, we would have

$$\dim_{B_{dR}} V_{1,dR} < \dim_{B_{dR}} V_{0,dR} + \dim_{B_{dR}} V_{2,dR} = h(V_0) + h(V_2) = h(V_1).$$

As we have (Proposition 4.5) $\dim_{B_{dR}} V_{1,dR} \geq h(V_1)$, this can't happen. This forces $V_1 \to V_{1,dR}^+$ to be also injective, hence $V_1 \in \mathcal{C}^{\geq 0}(G_K)$.

Now assume instead that V_0 belongs to $\mathcal{C}^\infty(G_K)$. As the sequence (1) almost splits (Proposition 2.15), we can find an extension S in $\mathcal{C}^0(G_K)$ of V_2 by some $U \in \mathcal{C}^0(G_K)$ such that $V_1 = V_0 \oplus_U S$. By what we just saw, $S \in \mathcal{C}^{\geq 0}(G_K)$ and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & S & \longrightarrow & V_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_{dR}^+ & \longrightarrow & S_{dR}^+ & \longrightarrow & V_{2,dR}^+ & \longrightarrow & 0 \end{array}$$

whose line are exacts and vertical arrows are injective.

We also have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & W \oplus S & \longrightarrow & V_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & U_{dR}^+ & \longrightarrow & W \oplus S_{dR}^+ & \longrightarrow & V_{1,dR}^+ & \longrightarrow & 0 \end{array}$$

(the map $U \to W \oplus S$ send u to $(u, -u)$) whose rows are exact and the two first vertical arrows are injective.

The injectivity of $U_{dR}^+ \to S_{dR}^+$ implies the injectivity of $U_{dR}^+ \to W \oplus S_{dR}^+$. To finish the proof we only need to show that the map $V_1 \to V_{1,dR}^+$ is injective or, with obvious identifications, that inside of $W \oplus S_{dR}^+$, we have

$$U_{dR}^+ \cap (W \oplus S) = U.$$

Assume $(w, s) \in W \oplus S$ belongs to U_{dR}^+. This implies that $s \in S \cap U_{dR}^+$ which is U as the map $V_2 \to V_{2,dR}^+$ is injective. We then need $w = -s$ and (w, s) is the image of $-s \in U$. □

Proof of the exactness of the functors $V \mapsto V_{dR}$, $V \mapsto V_e$ and $V \mapsto \overline{V}_{dR}$. If

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence in $\mathcal{C}^{\geq 0}(G_K)$, we know that the sequences

$$V'_e \to V_e \to V''_e \to 0, \quad V'_{dR} \to V_{dR} \to V''_{dR} \to 0$$

are exact. As

$$\dim_{B_{dR}} V_{dR} = h(V) = h(V') + h(V'') = \dim_{B_{dR}} V'_{dR} = \dim_{B_{dR}} V''_{dR}$$

the map $V'_{dR} \to V_{dR}$ must be injective and the functor $V \mapsto V_{dR}$ is exact.

As the B_e-modules V'_e, V_e and V''_e are torsion free and as

$$\begin{aligned}
\operatorname{rank}_{B_e}(V'_e) &= \dim_{B_{dR}} V'_{dR}, \\
\operatorname{rank}_{B_e}(V_e) &= \dim_{B_{dR}} V_{dR}, \\
\operatorname{rank}_{B_e}(V''_e) &= \dim_{B_{dR}} V''_{dR}.
\end{aligned}$$

the same argument shows the exactness of $V \mapsto V_e$.

We then have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V'_e & \longrightarrow & V_e & \longrightarrow & V''_e & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V'_{dR} & \longrightarrow & V_{dR} & \longrightarrow & V''_{dR} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{V}'_{dR} & \longrightarrow & \overline{V}_{dR} & \longrightarrow & \overline{V}''_{dR} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose three columns and the two first rows are exact. This implies the exactness of the last row.

Lemma 5.6. *Let*

$$0 \to V' \to V \to V'' \to 0$$

a short exact sequence in $\mathcal{C}^{\geq 0}(G_K)$. Assume the sequences

$$0 \to V' \to (V')^+_{dR} \to \overline{V}'_{dR} \to 0$$

and

$$0 \to V'' \to (V'')^+_{dR} \to \overline{V}''_{dR} \to 0$$

are exact. Then the sequences

$$0 \to V \to V_{dR}^+ \to \overline{V}_{dR} \to 0$$

and

$$0 \to (V')_{dR}^+ \to V_{dR}^+ \to (V'')_{dR}^+ \to 0$$

are exact.

Proof of the lemma: We have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (V')_{dR}^+ & \longrightarrow & V_{dR}^+ & \longrightarrow & (V'')_{dR}^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{V}'_{dR} & \longrightarrow & \overline{V}_{dR} & \longrightarrow & \overline{V}''_{dR} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose first and third rows are exact. By assumption, the first and the third columns are also exact. We also know that, except may be in $(V')_{dR}^+$, the second line is exact and, as $V \in \mathcal{C}^{\geq 0}(G_K)$, that the map $V \to V_{dR}^+$ is injective. By diagram chasing, we get the fact that the second line and the second column are also exact. □

We resume the proof of the proposition.

We first prove (iii), i.e., for all $V \in \mathcal{C}^{\geq 0}(G_K)$, the exactness of the sequence

$$0 \to V \to V_{dR}^+ \xrightarrow{\bar{\iota}_V} \overline{V}_{dR} \to 0.$$

(a) If $V \in \mathcal{C}^{\infty}(G_K)$, as $V_{dR}^+ = V$ and $V_{dR} = \overline{V}_{dR} = 0$, exactness is obvious.

(b) If $V \in \mathcal{C}^0(G_K)$, this sequence can be rewritten

$$0 \to V \to B_{dR}^+ \otimes_{\mathbb{Q}_p} V \to \overline{B}_{dR} \otimes_{\mathbb{Q}_p} V \to 0$$

and exactness is deduced by tensoring with V from the exactness of

$$0 \to \mathbb{Q}_p \to B_{dR}^+ \to \overline{B}_{dR} \to 0$$

(recall that $B_{dR} = B_e + B_{dR}^+$, that $\overline{B}_{dR} = B_{dR}/B_e$ and that $B_e \cap B_{dR}^+ = \mathbb{Q}_p$).

(c) In general, we proceed by induction on the smallest integer r_V such that there is $K' \in \mathcal{K}$ with the property that V is a successive extension of r_V objects belonging either to $\mathcal{C}^0(G_{K'})$ or to $\mathcal{C}^{\infty}(G_{K'})$. Replacing K by K' if necessary, we may assume

$K' = K$. We just proved it is OK if $r_V = 1$. Assume $r_V \geq 2$, so that we can find a short exact sequence in $\mathcal{C}^{\geq 0}(G_K)$

$$0 \to V' \to V \to V'' \to 0$$

with $r_{V'}$ and $r_{V''} < r_V$. Then, by induction, the sequences

$$0 \to V' \to (V')^+_{dR} \to \overline{V'}_{dR} \to 0$$
$$0 \to V'' \to (V'')^+_{dR} \to (\overline{V''})_{dR} \to 0$$

are exact and the result follows from the two assertions of the previous lemma.

From the exact sequence (C), we see that $V = H^0(X, \mathcal{F}_V)$ and that $H^1(X, \mathcal{F}_V) = 0$ hence that $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$, which proves (iv).

We are left to prove the exactness of the functors $V \mapsto V^+_{dR}$ and $V \mapsto \mathcal{F}_V$, i.e., that, if

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence in $\mathcal{C}^{\geq 0}(G_K)$, then the sequences

$$0 \to (V')^+_{dR} \to V^+_{dR} \to (V'')^+_{dR} \to 0$$

and

$$0 \to \mathcal{F}_{V'} \to \mathcal{F}_V \to \mathcal{F}_{V''} \to 0$$

are exact. As we now know the assertion (iii) of the proposition, the exactness of the first sequence is a consequence of the previous lemma. Finally, we see that exactness of the second is equivalent to the exactness of

$$0 \to (V')^+_{dR} \to V^+_{dR} \to (V'')^+_{dR} \to 0$$

and of

$$0 \to V'_e \to V_e \to V''_e \to 0$$

and we are done. □

Proposition 5.7. *Let $V \in \mathcal{C}(G_K)$. Any decreasing sequence of subobjects of V*

$$V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots$$

is stationary.

Proof. Chose $\widehat{V} \in \mathcal{C}^{\geq 0}(G_K)$ such that V is a quotient of $\widehat{V}$. For all $n \in \mathbb{N}$, set

$$\widehat{V}_n = \widehat{V} \times_V V_n.$$

The $\widehat{V}_n$ form a decreasing sequence of subobject of $\widehat{V}$ and, for all $n \in \mathbb{N}$, we have a canonical isomorphism $\widehat{V}_n / \widehat{V}_{n+1} \simeq V_n / V_{n+1}$. In particular

$$V_{n+1} = V_n \iff \widehat{V}_{n+1} = \widehat{V}_n.$$

Replacing V by $\widehat{V}$ and the V_n's by the $\widehat{V}_n$'s if necessary we ay assume that V, therefore also the V_n's are in $\mathcal{C}^{\geq 0}$.

As $d(V_{n+1}) \leq d(V_n)$ and $d(V_n) \geq 0$, there is an integer m such that $d(V_n) = d(V_{n+1})$ for $n \geq m$.

For $n \geq m$, we have $d(V_n/V_{n+1}) = 0$, hence $V_n/V_{n+1} \in \mathcal{C}^0(G_K)$ and, if we set $h_n = \dim_{\mathbb{Q}_p}(V_n/V_{n+1})$ $(\in \mathbb{N})$, we have $h(V_{n+1}) = h(V_n) - h_n$. As $V_{n+1} \in \mathcal{C}^{\geq 0}(G_K)$, we have $h(V_{n+1}) \geq 0$. Therefore, there is an integer $m' \geq m$ such that $h_n = 0$ if $n \geq m'$. This implies that $V_{n+1} = V_n$. □

Remark 5.8. On the other hand, there are objects of $\mathcal{C}(G_K)$ which admit nonstationary increasing sequences of subobjects. For instance, it is easy to see that $\mathbb{C}_p$ contains infinitely many subobjects belonging to $\mathcal{C}^0(G_K)$. From that, one can constructs nonstationary increasing sequences

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset V_{n+1} \subset \cdots$$

of subobjects of $\mathbb{C}_p$ belonging to $\mathcal{C}^0(G_K)$.

5C. ***The main result.*** We may consider the functors

$$\mathcal{M}^{\geq 0}(G_K) \to \mathcal{C}^{\geq 0}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

and

$$\mathcal{C}^{\geq 0}(G_K) \to \mathcal{M}^{\geq 0}(G_K), \quad V \mapsto \mathcal{F}_V.$$

Theorem 5.9. *The functor*

$$\mathcal{M}^{\geq 0}(G_K) \to \mathcal{C}^{\geq 0}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of exact categories and

$$\mathcal{C}^{\geq 0}(G_K) \to \mathcal{M}^{\geq 0}(G_K), \quad V \mapsto \mathcal{F}_V$$

is a quasi-inverse.

Proof. As the functor $V \mapsto \mathcal{F}_V$ is left adjoint to $\mathcal{F} \mapsto \mathcal{F}(X)$ (Section 4D), we are reduced to checking the following claims:

(i) If $V \in \mathcal{C}^{\geq 0}(G_K)$, the map $V \to \mathcal{F}_V(X)$ coming from adjunction is an isomorphism,

(ii) If $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$, the map $\mathcal{F}_{\mathcal{F}_V(X)} \to \mathcal{F}$ coming from adjunction is an isomorphism.

(iii) If

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence of $\mathcal{C}^{\geq 0}(G_K)$, the sequence

$$0 \to \mathcal{F}_{V'} \to \mathcal{F}_V \to \mathcal{F}_{V''} \to 0$$

is exact.

(iv) If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of $\mathcal{M}^{\geq 0}(G_K)$, the sequence

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0$$

is exact.

(1) and (3) have already been proved (Proposition 5.4) and (4) results from the fact that, if $\mathcal{F}' \in \mathcal{M}^{\geq 0}(G_K)$, then $H^1(X, \mathcal{F}') = 0$ (Proposition 3.14).

Let's prove (2): Let $\mathcal{M}$ the full subcategory of $\mathcal{M}^{\geq 0}(G_K)$ whose objects are those $\mathcal{F}$'s for which $\mathcal{F}_{\mathcal{F}_V(X)} \to \mathcal{F}$ is an isomorphism. It is obviously stable under taking direct summands. By exactness of the functors $\mathcal{F} \to \mathcal{F}(X)$ and $V \mapsto \mathcal{F}_V$, it is stable under extensions. It contains $\mathcal{M}^0(G_K)$ and $\mathcal{M}^\infty(G_K)$. Then Theorem 5.1 implies that $\mathcal{M} = \mathcal{M}^{\geq 0}(G_K)$. □

6. From $\mathcal{M}(G_K)$ to $\mathcal{C}(G_K)$ and conversely

6A. ***Some general nonsense.*** Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be an exact subcategory of $\mathcal{A}$. Recall (cf., e.g., [Laumon 1983, §1.1]) that one can define the derived category of bounded complexes of $\mathcal{B}$ that we denote $D^b_{\mathcal{A}}(\mathcal{B})$: in the triangulated category $\mathcal{K}^b(\mathcal{B})$ of bounded complexes of $\mathcal{B}$ up to homotopies, the full subcategory $\mathcal{N}$ of bounded acyclic complexes (in $\mathcal{B}$) form a null system and we set

$$\mathcal{D}^b_{\mathcal{A}}(\mathcal{B}) = \mathcal{K}^b(\mathcal{B})/\mathcal{N}.$$

Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ an exact subcategory of $\mathcal{A}$ and $\mathcal{D}$ a strictly full subcategory of $\mathcal{B}$ which is a Serre subcategory of $\mathcal{A}$ (hence $\mathcal{D}$ is abelian).

- We say that *the exact embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ is left big with respect to $\mathcal{D}$* if,

(i) any quotient in $\mathcal{A}$ of an object of $\mathcal{B}$ belongs to $\mathcal{B}$,

(ii) for any object A of $\mathcal{A}$, one can find a short exact sequence

$$0 \to A \to B \to D \to 0$$

of $\mathcal{A}$ with B an object of $\mathcal{B}$ and D an object of $\mathcal{D}$.

- We say that *the exact embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ is right big with respect to $\mathcal{D}$* if $\mathcal{B}^{\mathrm{op}} \hookrightarrow \mathcal{A}^{\mathrm{op}}$ is left big with respect to $\mathcal{D}^{\mathrm{op}}$ which amounts to requiring that

(i) any subobject in $\mathcal{A}$ of an object of $\mathcal{B}$ belongs to $\mathcal{B}$,

(ii) for any object A of $\mathcal{A}$, one can find a short exact sequence

$$0 \to D \to B \to A \to 0$$

of $\mathcal{A}$ with B an object of $\mathcal{B}$ and D an object of $\mathcal{D}$.

We say that an exact embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ is *left big* (resp. *right big*) if one can find a Serre subcategory $\mathcal{D}$ of $\mathcal{A}$ contained in $\mathcal{B}$ such that $\mathcal{B} \hookrightarrow \mathcal{A}$ is left big (resp. right big) with respect to $\mathcal{D}$.

Proposition 6.1. *Let $\mathcal{B} \hookrightarrow \mathcal{A}$ an exact embedding which is either left big or right big. Then the natural functor*

$$D^b_{\mathcal{A}}(\mathcal{B}) \to D^b(\mathcal{A})$$

is an equivalence of triangulated categories.

It is a formal consequence of the more precise following statement:

Proposition 6.2. *Let $\mathcal{B} \hookrightarrow \mathcal{A}$ be an exact embedding and $\mathcal{D}$ a Serre subcategory of $\mathcal{A}$ contained in $\mathcal{B}$ such that $\mathcal{B} \hookrightarrow \mathcal{A}$ is left big (resp. right big) with respect to $\mathcal{D}$ and let $A^\bullet$ a bounded complex of $\mathcal{A}$.*

(i) *There exists a short exact sequence of bounded complexes of $\mathcal{A}$*

$$0 \to A^\bullet \to B^\bullet \to D^\bullet \to 0 \ \ (\textit{resp.}\ \ 0 \to D^\bullet \to B^\bullet \to A^\bullet \to 0\ )$$

with $B^\bullet$ a bounded complex of $\mathcal{B}$ and $D^\bullet$ an acyclic complex of $\mathcal{D}$.

(ii) *If*

$$0 \to A^\bullet \to B'^\bullet \to D'^\bullet \to 0 \ (\textit{resp.}\ \ 0 \to D'^\bullet \to B'^\bullet \to A^\bullet \to 0\ )$$

isan other short exact sequence of the same kind, there exists a a third short exact sequence of the same kind

$$0 \to A^\bullet \to B''^\bullet \to D''^\bullet \to 0 \ (\textit{resp.}\ \ 0 \to D''^\bullet \to B''^\bullet \to A^\bullet \to 0\ )$$

together with morphisms of complexes

$$B^\bullet \to B''^\bullet \textit{ and } B'^\bullet \to B''^\bullet \ (\textit{resp. } B''^\bullet \to B^\bullet \textit{ and } B''^\bullet \to B'^\bullet)$$

such that the diagram

$$\begin{array}{ccccc} & & A^\bullet & & \\ & \swarrow & \downarrow & \searrow & \\ B'^\bullet & \longrightarrow & B''^\bullet & \longleftarrow & B^\bullet \end{array} \qquad \left(\textit{resp.} \quad \begin{array}{ccccc} B'^\bullet & \longrightarrow & B''^\bullet & \longleftarrow & B^\bullet \\ & \searrow & \downarrow & \swarrow & \\ & & A^\bullet & & \end{array} \right)$$

is commutative.

Proof. It is enough to treat the case were the strict embedding is right big. Assume this is the case. To prove (i), by induction, we are reduced to proving this:

Lemma 6.3. *Let $r \in \mathbb{Z}$ and let*

$$0 \to D_r^\bullet \to B_r^\bullet \to A^\bullet \to 0$$

a short exact sequence of bounded complexes of $\mathcal{A}$. Assume that $D_r^\bullet$ is an acyclic complex of $\mathcal{D}$, that $D_r^n = 0$ for $n \geq r$ and that B_r^n is an object of $\mathcal{B}$ for all $n < r$. Then, there exists a short exact sequence of bounded complexes of $\mathcal{A}$

$$0 \to D_{r+1}^\bullet \to B_{r+1}^\bullet \to A^\bullet \to 0$$

where $D_{r+1}^\bullet$ is an acyclic complex of $\mathcal{D}$ with $D_{r+1}^n = 0$ for $n \geq r+1$ and B_{r+1}^n an object of $\mathcal{B}$ for all $n < r+1$.

Proof of the lemma. We can identify B_r^n to A^n for $n \geq r$. Granted to right bigness of $\mathcal{B} \hookrightarrow \mathcal{A}$, we can find a short exact sequence

$$0 \to D \to B \to A^r \to 0$$

with B an object of $\mathcal{B}$ and D an object of $\mathcal{D}$. Set

$$B_{r+1}^n = \begin{cases} B_r^n & \text{for } n < r-1, \\ B_r^{r-1} \times_{A_r} B & \text{for } n = r-1, \\ B & \text{for } n = r, \\ B_r^n = A^n & \text{for } n > r. \end{cases}$$

We have a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & D & = & D & & \\ & & & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & B_{r+1}^{r-2} & \longrightarrow & B_{r+1}^{r-1} & \longrightarrow & B_{r+1}^{r} & \longrightarrow & B_{r+1}^{r+1} \longrightarrow \cdots \\ & & \| & & \downarrow & & \downarrow & & \| \\ \cdots & \longrightarrow & B_{r}^{r-2} & \longrightarrow & B_{r}^{r-1} & \longrightarrow & B_{r}^{r} & \longrightarrow & B_{r}^{r+1} \longrightarrow \cdots \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

whose rows are complexes and columns are exact. Hence we have a quasi-isomorphism $B_{r+1}^\bullet \to B_r^\bullet$. Moreover B_{r+1}^n is an object of $\mathcal{B}$ for all $n < r+1$ (for $n = r-1$, this is due to the fact that B_{r+1}^{r-1} is a subobject of $B_r^{r-1} \oplus B$ which belongs to $\mathcal{B}$).

The compositum

$$B_{r+1}^\bullet \to B_r^\bullet \to A^\bullet$$

is a surjective morphism of complexes which is a quasi-isomorphism. Then the kernel $D^{\bullet}_{r+1}$ is acyclic. As it is the complex

$$\cdots \to D_r^{r-3} \to D_r^{r-2} \to D_{r+1}^{r-1} \to D \to 0 \to \cdots \to 0 \to \cdots$$

we see that $D^n_{r+1} = 0$ for $n \geq r+1$ and that all the D^n_{r+1} belong to $\mathcal{D}$ (for $n = r-1$, this is due to the fact that we have a short exact sequence

$$0 \to D'' \to D_{r+1}^{r-1} \to D \to 0$$

with $D'' = \operatorname{coker}(D_r^{r-3} \to D_r^{r-2}) \in \mathcal{D}$, hence, as D_{r+1}^{r-1} is an extension in $\mathcal{A}$ of $D \in \mathcal{D}$ by $D'' \in \mathcal{D}$, it belongs to $\mathcal{D}$). □

To prove part (ii) of the proposition we take, for each $n \in \mathbb{Z}$, the fiber product

$$B''^n = B'^n \times_{A^n} B^n.$$

For each n, we have an exact sequence

$$0 \to D''^n \to B''^n \to A^n \to 0$$

with $D''^n = D'^n \oplus D^n$ and all the required properties are obviously fulfilled. □

6B. ***The equivalence of triangulated categories.***

Theorem 6.4. *The equivalence of categories of Theorem 5.9 extends uniquely to an equivalence of triangulated categories*

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)).$$

Proof. Uniqueness is obvious.

Recall (Section 5C) that $\mathcal{M}^{\geq 0}(G_K)$ is an exact subcategory of $\mathcal{M}(G_K)$ and $\mathcal{C}^{\geq 0}(G_K)$ is an exact subcategory of $\mathcal{C}(G_K)$.

• The category $\mathcal{M}^{\infty}(G_K)$ is a Serre subcategory of $\mathcal{M}(G_K)$ contained in $\mathcal{M}^{\geq 0}(G_K)$ and any quotient $\mathcal{F}''$ in $\mathcal{M}(G_K)$ of an object $\mathcal{F}$ of $\mathcal{M}^{\geq 0}(G_K)$ is in $\mathcal{M}^{\geq 0}(G_K)$

(as $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K) \iff H^1(X, \mathcal{F}) = 0 \Longrightarrow H^1(X, \mathcal{F}'') = 0 \iff \mathcal{F}'' \in \mathcal{M}^{\geq 0}(G_K)$).

If $\mathcal{F} \in \mathcal{M}(G_K)$, for all $n \in \mathbb{N}$, as, for all $n \in \mathbb{N}$, the HN-slopes of $\mathcal{F}(n)_{HN}$ are the $s+n$ for s describing the HN-slopes of $\mathcal{F}$ (cf. Section 3H), for $n \gg 0$, we have $\mathcal{F}(n)_{HN} \in \mathcal{M}^{\geq 0}(G_K)$.

Tensoring with $\mathcal{F}$ the short exact sequence (Section 3H)

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(n)_{HN} \to (0, B_n(-n)) \to 0$$

we get a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}(n)_{HN} \to (0, \mathcal{F}^+_{dR} \otimes_{B^+_{dR}} B_n(-n)) \to 0.$$

As $\mathcal{F}(n)_{HN}$ belongs to $\mathcal{M}^{\geq 0}(G_K)$ and $(0, \mathcal{F}^+_{dR} \otimes_{B^+_{dR}} B_n(-n))$ belongs to $\mathcal{M}^{\infty}(G_K)$, it shows that the exact embedding $\mathcal{M}^{\geq 0}(G_K) \to \mathcal{M}(G_K)$ is left big with respect to $\mathcal{M}^{\infty}(G_K)$. Therefore (Proposition 6.1) the natural functor

$$D^b_{\mathcal{M}(G_K)}(\mathcal{M}^{\geq 0}(G_K)) \to D^b(\mathcal{M}(G_K))$$

is an equivalence of triangulated categories.

• Similarly, the category $\mathcal{C}^0(G_K)$ is a Serre subcategory of $\mathcal{C}(G_K)$ contained in $\mathcal{C}^{\geq 0}(G_K)$ and any subobject in $\mathcal{C}(G_K)$ of an object of $\mathcal{C}^{\geq 0}(G_K)$ belongs to $\mathcal{C}^{\geq 0}(G_K)$.

Let $V \in \mathcal{C}(G_K)$ and choose an almost isomorphism $V/U_+ \simeq W/U_-$ with $W \in \mathcal{C}^{\infty}(G_K)$ (cf. Section 2E). Set

$$\hat{V} = V \times_{W/U_-} W$$

(where the map $V \to W/U_-$ is the compositum of the projection $V \to V/U_+$ with the isomorphism $V/U_+ \to W/U_-$).

We have a diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & U_+ & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & U_- & \longrightarrow & \widehat{V} & \longrightarrow & V & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & W & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

whose line and column are exacts. The column shows that $\widehat{V} \in \mathcal{C}^{\geq 0}(G_K)$ and, therefore, the line shows that V is a quotient of an object of $\mathcal{C}^{\geq 0}(G_K)$ by an object of $\mathcal{C}^0(G_K)$. In other words, the exact embedding $\mathcal{C}^{\geq 0}(G_K) \to \mathcal{C}(G_K)$ is right big with respect to $\mathcal{C}^0(G_K)$. Hence (Proposition 6.1) the natural functor

$$D^b_{\mathcal{C}(G_K)}(\mathcal{C}^{\geq 0}(G_K)) \to D^b(\mathcal{C}(G_K))$$

is an equivalence of triangulated categories.

As the equivalence $\mathcal{M}^{\geq 0}(G_K) \xrightarrow{\sim} \mathcal{C}^{\geq 0}(G_K)$ is an equivalence of exact categories, it extends uniquely to an equivalence of triangulated categories

$$D^b_{\mathcal{M}(G_K)}(\mathcal{M}^{\geq 0}(G_K)) \to D^b_{\mathcal{C}(G_K)}(\mathcal{C}^{\geq 0}(G_K)).$$

• It is now clear that there is a unique equivalence of triangulated categories

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K))$$

such that the square

$$\begin{array}{ccc} D^b_{\mathcal{M}(G_K)}(\mathcal{M}^{\geq 0}(G_K)) & \longrightarrow & D^b_{\mathcal{C}(G_K)}(\mathcal{C}^{\geq 0}(G_K)) \\ \downarrow & & \downarrow \\ D^b(\mathcal{M}(G_K)) & \longrightarrow & D^b(\mathcal{C}(G_K)) \end{array}$$

is commutative and that this equivalence extends that of Theorem 5.9. □

6C. ***The equivalence $\mathcal{M}^{<0}(G_K) \to \mathcal{C}^{<0}(G_K)$.*** We say that a coherent $\mathcal{O}_X[G_K]$-module is *co-effective* if all its HN slopes are < 0. We saw (Proposition 3.14) that $\mathcal{F} \in \mathcal{M}(G_K)$ is co-effective if and only if $H^0(X, \mathcal{F}) = 0$. The full subcategory of $\mathcal{M}(G_K)$ whose objects are co-effective is $\mathcal{M}^{<0}(G_K)$ and is stable under taking subobjects and extensions.

Any $\mathcal{F} \in \mathcal{M}(G_K)$ as a biggest quotient $\mathcal{F}^{<0}$ belonging to $\mathcal{M}^{<0}(G_K)$and the sequence

$$0 \to \mathcal{F}^{\geq 0} \to \mathcal{F} \to \mathcal{F}^{<0} \to 0$$

is exact.

We say that an almost $\mathbb{C}_p$-representation V is *co-effective* if, for all $W \in \mathcal{C}^\infty(G_K)$, we have $\mathcal{H}om_{\mathcal{C}(G_K)}(V, W) = 0$. We denote $\mathcal{C}^{<0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ whose objects are co-effective. It is obviously stable undertaking quotients and extensions.

Proposition 6.5. *Let V be an almost $\mathbb{C}_p$-representation. The following conditions are equivalent*:

(i) *V is co-effective.*

(ii) $V^+_{dR} = 0$.

(iii) $\mathcal{F}_V = 0$.

These conditions also imply

$$V_e = V_{dR} = \overline{V}_{dR} = 0.$$

Proof. The equivalence (i)$\Longleftrightarrow$(ii) results from the universal property of V^+_{dR} and (ii)$\Longleftrightarrow$(iii) is trivial. If $V^+_{dR} = 0$, we have $V_{dR} = B_{dR} \otimes_{B^+_{dR}} V^+_{dR} = 0$, hence also $V_e = 0$ as the map $V_e \to V_{dR}$ is injective and therefore $\overline{V}_{dR} = V_{dR}/V_e = 0$. □

Proposition 6.6. *Let $V \in \mathcal{C}(G_K)$. The set of subobjects of V in $\mathcal{C}(G_K)$ belonging to $\mathcal{C}^{<0}(G_K)$ has a biggest element $V^{<0}$ and the set of quotients of V in $\mathcal{C}(G_K)$ belonging to $\mathcal{C}^{\geq 0}(G_K)$ as a biggest element $V^{\geq 0}$. Moreover $V^{<0}$ (resp. $V^{\geq 0}$) is the kernel (resp. the image) of the natural map $V \to V^+_{dR}$. The sequence*

$$0 \to V^{<0} \to V \to V^{\geq 0} \to 0$$

is exact.

Proof. If V' and V'' are subobjects of V belonging to $\mathcal{C}^{<0}(G_K)$, we see that $V'+V''$ also. Hence to show the existence of $V^{<0}$ it is enough to show that any increasing sequence

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset V_{n+1} \subset \cdots$$

of subobjects of V belonging to $\mathcal{C}^{<0}(G_K)$ is stationary. As the sequence of the integers $d(V_n)$ is increasing and bounded by $d(V)$, there exists $m \in \mathbb{N}$ such that $d(V_n) = d(V_m)$ for all $n \geq m$. For such an n, we have $d(V_{n+1}/V_n) = 0$, hence $V_{n+1}/V_n \in \mathcal{C}^0(G_K)$. This implies $V_{n+1} = V_n$ as, otherwise, the compositum of the projection of V_{n+1} onto V_n with the injective map

$$V_{n+1}/V_n \to B_{dR}^+ \otimes_{\mathbb{Q}_p} (V_{n+1}/V_n), \quad v \mapsto 1 \otimes v$$

would be a nonzero morphism from V_{n+1} to an object of $\mathcal{C}^\infty(G_K)$.

If $\overline{V}'$ and $\overline{V}''$ are quotients of V belonging to $\mathcal{C}^{\geq 0}(G_K)$, then the image of $V \to \overline{V}' \oplus \overline{V}''$ also (as it is a subobject of $\overline{V}' \oplus \overline{V}'' \in \mathcal{C}^{\geq 0}(G_K)$). Hence to show the existence of $V^{\geq 0}$ it suffices to show that any sequence

$$\cdots \to \underline{V}_{n+1} \to \underline{V}_n \to \cdots \to \underline{V}_1 \subset \underline{V}_0$$

of quotients of V (belonging to $\mathcal{C}^{<0}(G_K)$) is stationary. If $\widetilde{V}_n$ is the kernel of the projection $V \to \overline{V}_n$, the sequence $(\widetilde{V}_n)_{n\in\mathbb{N}}$ is a decreasing sequence of objects of $\mathcal{C}(G_K)$, hence is stationary (Proposition 5.7), therefore the sequence of the $\overline{V}_n$'s also.

Set $V_0 = \ker(V \to V_{dR}^+)$. We obviously have $V^{<0} \subset V_0$ and to show the equality it is enough to show that $V_0 \in \mathcal{C}^{<0}(G_K)$. Otherwise, we could find a nonzero morphism $f : V^0 \to W$ with $W \in \mathcal{C}^\infty(G_K)$. Let $V_1 = \ker f$ and consider the short exact sequence

$$0 \to V_0/V_1 \to V/V_1 \to V/V_0 \to 0.$$

As V_0/V_1 injects into W, it belongs to $\mathcal{C}^{\geq 0}(G_K)$. As V/V_0 injects into V_{dR}^+, it also belongs to $\mathcal{C}^{\geq 0}(G_K)$. Therefore, as $\mathcal{C}^{\geq 0}(G_K)$ is stable under extensions, $V/V_1 \in \mathcal{C}^{\geq 0}(G_K)$. Hence the sequence

$$0 \to (V_0/V_1)_{dR}^+ \to (V/V_1)_{dR}^+ \to (V/V_0)_{dR}^+ \to 0$$

is exact. As obviously $(V/V_1)_{dR}^+ = (V/V_0)_{dR}^+ = V_{dR}^+$, it contradicts the fact that, as V_0/V_1 is a nonzero object of $\mathcal{C}^{\geq 0}(G_K)$, we have $(V_0/V_1)_{dR}^+ \neq 0$.

Let $V_2 = \mathrm{im}(V \to V_{dR}^+)$. As the map $V_2 \to V_{dR}^+$ is injective, V_2 belongs to $\mathcal{C}^{\geq 0}(G_K)$ and is, therefore a quotient of $V^{\geq 0}$. The kernel V_3 of the projection

$V^{\geq 0} \to V_2$ belongs also to $\mathcal{C}^{\geq 0}(G_K)$ (as this category is stable under taking subobjects) and we have an exact sequence in $\mathcal{C}^{\geq 0}(G_K)$

$$0 \to V_3 \to V^{\geq 0} \to V_2 \to 0$$

Therefore the sequence

$$0 \to V_{3,dR}^{+} \to V_{dR}^{\geq 0,+} \to V_{2,dR}^{+} \to 0$$

is also exact.

As $V^{\geq 0}$ is a quotient of V, we see that $V_{dR}^{\geq 0,+}$ is a quotient of V_{dR}^{+}. But clearly $V_{2,dR}^{+} = V_{dR}^{+}$. Therefore $V_{dR}^{\geq 0,+} = V_{dR}^{+}$ and $V_{3,dR}^{+} = 0$. As $V_3 \in \mathcal{C}^{\geq 0}(G_K)$, this implies $V_3 = 0$, hence $V^{\geq 0} = V_2$.

The exactness of

$$0 \to V^{<0} \to V \to V^{\geq 0} \to 0$$

is now clear. □

Remarks 6.7. (i) To any $V \in \mathcal{C}(G_K)$, we just associated the canonical short exact sequence

$$0 \to V^{<0} \to V \to V^{\geq 0} \to 0$$

It is worth comparing with the canonical short exact sequence

$$0 \to \mathcal{F}^{\geq 0} \to \mathcal{F} \to \mathcal{F}^{<0} \to 0$$

associated to any $\mathcal{F} \in \mathcal{M}(G_K)$.

(ii) We know that, for any $\mathcal{F} \in \mathcal{M}(G_K)$, the natural map $\mathcal{F}^{\geq 0}(X) \to \mathcal{F}(X)$ is an isomorphism.The two previous propositions together imply that, for any $V \in \mathcal{C}(G_K)$, the natural map $\mathcal{F}_V \mapsto \mathcal{F}_{V^{\geq 0}}$ is an isomorphism. In particular, $\mathcal{F}_V$ always belongs to $\mathcal{M}^{\geq 0}(G_K)$.

It is clear that $\mathcal{M}^{<0}(G_K)$ is an exact subcategory of $\mathcal{M}(G_K)$, and $\mathcal{C}^{<0}(G_K)$ is an exact subcategory of $\mathcal{C}(G_K)$.

Proposition 6.8. *If $\mathcal{F} \in \mathcal{M}(G_K)$, then $H^1(X, \mathcal{F}) \in \mathcal{C}^{<0}(G_K)$ and the map*

$$H^1(X, \mathcal{F}) \mapsto H^1(X, \mathcal{F}^{<0})$$

is an isomorphism.

Moreover, the functor

$$\mathcal{M}^{<0}(G_K) \to \mathcal{C}^{<0}(G_K), \quad \mathcal{F} \mapsto H^1(X, \mathcal{F})$$

is an equivalence of exact categories.

Proof. If $\mathcal{F} \in \mathcal{M}(G_K)$, we may find a short exact sequence in $\mathcal{M}(G_K)$

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to 0$$

with $\mathcal{F}^0 \in \mathcal{M}^{\geq 0}(G_K)$. As $H^1(X, \mathcal{F}^0) = 0$, we see that $H^1(X, \mathcal{F})$ is the cokernel of $H^0(X, \mathcal{F}^0) \to H^0(X, \mathcal{F}^1)$, hence belongs to $\mathcal{C}(G_K)$.

We know that $H^1(X, \mathcal{F})$ is a quotient of $\overline{\mathcal{F}}_{dR}$ therefore also of $\mathcal{F}_{dR}$. If $f : H^1(X, \mathcal{F}) \to W$ were a nonzero morphism of $\mathcal{C}(G_K)$ with $W \in \mathcal{C}^\infty(G_K)$, the compositum $\mathcal{F}_{dR} \to H^1(X, \mathcal{F}) \to W$ would be a nonzero morphism in $\widehat{\mathcal{C}}^\infty(G_K)$ and, therefore, would be B_{dR}^+-linear. As multiplication by t is invertible in $\mathcal{F}_{dR}$ and nilpotent in W, the map must be 0 which implies that $H^1(X, \mathcal{F}) \in \mathcal{C}^{<0}(G_K)$.

If A is an object of an abelian category and $d \in \mathbb{Z}$, we denote $A[d]$ the bounded complex in $\mathcal{A}$ which is A in degree $-d$ and 0 elsewhere.

Denote by

$$\Gamma : D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)) \ \ (\text{resp. } \Delta : D^b(\mathcal{C}(G_K)) \to D^b(\mathcal{M}(G_K)) \ )$$

the functor extending $\mathcal{F} \mapsto \mathcal{F}(X)$ (resp. $V \mapsto \mathcal{F}_V$). If $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$ and if

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to 0$$

is as above (observe that $\mathcal{F}^0 \in \mathcal{M}^{\geq 0}(G_K) \Longrightarrow \mathcal{F}^1 \in \mathcal{M}^{\geq 0}(G_K)$), we see that (with obvious conventions)

$$\Gamma(\mathcal{F}[0]) = \Gamma(\mathcal{F}^0 \to \mathcal{F}^1) = (H^0(X, \mathcal{F}^0) \to H^0(X, \mathcal{F}^1)) = H^1(X, \mathcal{F})[-1]$$

(as $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$ and $\mathcal{F}^0 \in \mathcal{M}^{\geq 0}(G_K)$, the sequence

$$0 \to H^0(X, \mathcal{F}^0) \to H^0(X, \mathcal{F}^1) \to H^1(X, \mathcal{F}) \to 0$$

is exact).

Let $V \in \mathcal{C}^{<0}(G_K)$. We can find a short exact sequence in $\mathcal{C}(G_K)$

$$0 \to V^0 \to V^1 \to V \to 0$$

with $V^1 \in \mathcal{C}^{\geq 0}(G_K)$ which implies $V^0 \in \mathcal{C}^{\geq 0}(G_K)$. With obvious conventions, we have

$$\Delta(V[-1]) = \Delta(V^0 \to V^1) = (\mathcal{F}_{V^0} \to \mathcal{F}_{V^1}) = \mathcal{F}[0]$$

with $\mathcal{F}$ the kernel of $\mathcal{F}_{V^0} \to \mathcal{F}_{V^1}$ (as $V \in \mathcal{C}^{<0}(G_K)$, we have $V_{dR}^+ = V_e = 0$ which implies that

$$\mathcal{F}_{V^0} = (V_{dR}^{0+}, V_e^0, \iota_{V^0}) \to \mathcal{F}_{V^1} = (V_{dR}^{1+}, V_e^1, \iota_{V^1})$$

is an epimorphism).

We have a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(X,\mathcal{F}) & \longrightarrow & V^0 & \longrightarrow & V^1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}^+_{dR} & \longrightarrow & V^{0+}_{dR} & \to & V^{1+}_{dR} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}_e & \longrightarrow & V^0_e & \longrightarrow & V^1_e
\end{array}$$

whose rows and columns are exact. The injectivity of $V^0 \to V^1$ implies that $H^0(X,\mathcal{F}) = 0$, i.e., that $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$.

Finally, we see that, if we view

- $\mathcal{M}^{<0}(G_K)$ as the full subcategory of $D^b(\mathcal{M}(G_K))$ whose objects are of the form $\mathcal{F}[0]$ with $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$,
- $\mathcal{C}^{<0}(G_K)$ as the full subcategory of $D^b(\mathcal{C}(G_K))$ whose objects are of the form $V[-1]$ with $V \in \mathcal{C}^{<0}(G_K)$,

then Γ induces the required equivalence of categories. □

6D. *t-Structures and hearts.* The functors

$$\Gamma : D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)) \text{ and } \Delta : D^b(\mathcal{C}(G_K)) \to D^b(\mathcal{M}(G_K))$$

are as in the proof of the previous proposition.

Let $(D^{\le 0}_{\mathcal{M}}, D^{\ge 0}_{\mathcal{M}})$ be the canonical t-structure on $D^b(\mathcal{M}(G_K))$: we see that $D^{\le 0}_{\mathcal{M}}$ (resp. $D^{\ge 0}_{\mathcal{M}}$) is the full subcategory of $D^b(\mathcal{M}(G_K))$ whose objects are those $\mathcal{F}^\bullet$ such that $H^i(\mathcal{F}^\bullet) = 0$ for $i > 0$ (resp. $i < 0$). Therefore if we denote by $\Gamma(D^{\le 0}_{\mathcal{M}})$ (resp. $\Gamma(D^{\ge 0}_{\mathcal{M}})$) the essential image under Γ of $D^{\le 0}_{\mathcal{M}}$ (resp. $D^{\ge 0}_{\mathcal{M}}$), we see that $(\Gamma(D^{\le 0}_{\mathcal{M}}), \Gamma(D^{\ge 0}_{\mathcal{M}}))$ is a t-structure on $D^b(\mathcal{C}(G_K))$ whose heart $\Gamma(D^{\le 0}_{\mathcal{M}}) \cap \Gamma(D^{\ge 0}_{\mathcal{M}})$ is an abelian category equivalent via Δ to $\mathcal{M}(G_K)$.

Similarly, let $(D^{\le 0}_{\mathcal{C}}, D^{\ge 0}_{\mathcal{C}})$ the canonical t-structure on $D^b(\mathcal{C}(G_K))$: hence $D^{\le 0}_{\mathcal{C}}$ (resp . $D^{\ge 0}_{\mathcal{C}}$) is the full subcategory of $D^b(\mathcal{C}G_K)$ whose objects are those $V^\bullet$ such that $H^i(V^\bullet) = 0$ for $i > 0$ (resp. $i < 0$). Therefore if we denote by $\Delta(D^{\le 0}_{\mathcal{C}})$ (resp. $\Delta(D^{\ge 0}_{\mathcal{C}})$) the essential image under Δ of $D^{\le 0}_{\mathcal{C}}$ (resp. $D^{\ge 0}_{\mathcal{C}}$), we see that $(\Delta(D^{\le 0}_{\mathcal{C}}), \Delta(D^{\ge 0}_{\mathcal{C}}))$ is a t-structure on $D^b(\mathcal{M}(G_K))$ whose heart $\Delta(D^{\le 0}_{\mathcal{C}}) \cap \Delta(D^{\ge 0}_{\mathcal{C}})$ is an abelian category equivalent via Γ to $\mathcal{C}(G_K)$.

Proposition 6.9. (i) $\Gamma(D^{\ge 0}_{\mathcal{M}})$ *(resp.* $\Gamma(D^{\le 0}_{\mathcal{M}})$*) is the full subcategory of* $D^b(\mathcal{C}(G_K))$ *whose objects are those* $V^\bullet$*'s such that* $H^r(V^\bullet) = 0$ *for* $r < 0$ *and* $H^0(V^\bullet) \in \mathcal{C}^{\ge 0}(G_K)$ *(resp.* $H^r(V^\bullet) = 0$ *for* $r > 1$ *and* $H^1(V^\bullet) \in \mathcal{C}^{<0}(G_K)$*).*

(ii) $\Delta(D_{\mathcal{C}}^{\geq 0})$ (*resp.* $\Delta(D_{\mathcal{C}}^{\leq 0})$) *is the full subcategory of* $D^b(\mathcal{M}(G_K))$ *whose objects are those* $\mathcal{F}^\bullet$*'s such that* $H^r(\mathcal{F}^\bullet) = 0$ *for* $r < -1$ *and* $H^{-1}(\mathcal{F}^\bullet) \in \mathcal{M}^{<0}(G_K)$ (*resp.* $H^r(\mathcal{F}^\bullet) = 0$ *for* $r > 0$ *and* $H^0(\mathcal{F}^\bullet) \in \mathcal{M}^{\geq 0}(G_K)$).

Proof. Let's prove that the description of $\Gamma(D_{\mathcal{M}}^{\geq 0})$ is correct (the proof of the three other statements are similar):

Any object $\underline{\mathcal{F}}$ of $D_{\mathcal{M}}^{\geq 0}$ can be represented by a bounded complex $\mathcal{F}^\bullet$ such that $\mathcal{F}^i = 0$ for $i < 0$. From the fact that, for any $\mathcal{F} \in \mathcal{M}(G_K)$, one can find a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0$$

with $\mathcal{F}_0, \mathcal{F}_1 \in \mathcal{M}^{\geq 0}(G_K)$ and the fact that any quotient, in $\mathcal{M}(G_K)$, of an object of $\mathcal{M}^{\geq 0}(G_K)$ still belongs to $\mathcal{M}^{\geq 0}(G_K)$, one easily deduces that the complex $\mathcal{F}^\bullet$ is quasi-isomorphic to a bounded complex $\mathcal{F}_0^\bullet$ with $\mathcal{F}_0^r = 0$ for $r < 0$ and $\mathcal{F}_0^r \in \mathcal{M}^{\geq 0}(G_K)$ for all $r \in \mathbb{N}$. Therefore $\Gamma(\underline{\mathcal{F}})$ is represented by the bounded complex

$$\cdots \to 0 \to \cdots \to 0 \to \mathcal{F}_0^0(X) \to \mathcal{F}_0^1(X) \to \cdots \to \mathcal{F}_0^r(X) \to \mathcal{F}_0^{r+1}(X) \to \cdots$$

all of whose terms belong to $\mathcal{C}^{\geq 0}(G_K)$. In particular, as $\mathcal{C}^{\geq 0}(G_K)$ is stable under taking subobjects in $\mathcal{C}(G_K)$, we see that $\Gamma(\underline{\mathcal{F}})$ belongs to the full subcategory $D_{\mathcal{C},\mathcal{M}}^{\geq 0}$ of $D^b(\mathcal{C}(G_K))$ whose objects are those $\underline{V}$'s such that $H^r(\underline{V}) = 0$ for $r < 0$ and $H^0(\underline{V}) \in \mathcal{C}^{\geq 0}(G_K)$.

Conversely, any object $\underline{V}$ of $D_{\mathcal{C},\mathcal{M}}^{\geq 0}(G_K)$ can be represented by a complex $V_0^\bullet$ such that $V_0^r = 0$ for $r < 0$ and that the kernel of $V^0 \to V^1$ belongs to $\mathcal{C}^{\geq 0}(G_K)$. Using the fact that, for any $V \in \mathcal{C}(G_K)$ one can find a short exact sequence in $\mathcal{C}(G_K)$

$$0 \to V_1 \to V_0 \to V \to 0$$

with $V_1, V_0 \in \mathcal{C}^{\geq 0}(G_K)$, one easily deduces that the complex $V_0^\bullet$ is quasi-isomorphic to a bounded complex $V^\bullet$ with $V^r = 0$ for $r < 0$ and $V^r \in \mathcal{C}^{\geq 0}(G_K)$ for $r > 0$.

We have a short exact sequence (with $d : V^0 \to V^1$ the differential in the complex $V^\bullet$)

$$0 \to (V_{d=0}^0) \to V^0 \to dV^0 \to 0$$

The inclusion $dV^0 \subset V^1$ implies that $dV^0 \in \mathcal{C}^{\geq 0}(G_K)$. As $V_{d=0}^0 = H^0(V^\bullet)$, we have $(V^0)_{d=0} \in \mathcal{C}^{\geq 0}(G_K)$. We know that $\mathcal{C}^{\geq 0}(G_K)$, as a full subcategory of $\mathcal{C}(G_K)$, is stable under extension. Therefore $V^0 \in \mathcal{C}^{\geq 0}(G_K)$.

As all the V^r's belong to $\mathcal{C}^{\geq 0}(G_K)$, we see that $\Delta(\underline{V})$ is represented by the bounded complex

$$\cdots \to 0 \to \cdots \to 0 \to \mathcal{F}_{V^0} \to \mathcal{F}_{V^1} \to \cdots \to \mathcal{F}_{V^r} \to \mathcal{F}_{V^{r+1}} \to \cdots$$

hence belong to $D_{\mathcal{M}}^{\geq 0}$. □

6E. *Torsion pairs in $\mathcal{M}(G_K)$ and in $\mathcal{C}(G_K)$.* The langage of torsion pairs (see [Happel et al. 1996, Chapter 1]) is very convenient to give an explicite description of the way to go from $\mathcal{M}(G_K)$ to $\mathcal{C}(G_K)$ and conversely. The results of this subsection and of the next one are independent of those of the previous one and give an other proof of the description of the heart of the t-structures we considered (Proposition 6.9).

Recall (*loc. cit.*) that a *torsion pair* in an abelian category $\mathcal{A}$ is a pair $t = (\mathcal{A}^+, \mathcal{A}^-)$ of full subcategories of $\mathcal{A}$ containing 0 such that:

(i) If B is an object of $\mathcal{A}^+$ and C is an object of $\mathcal{A}^-$, then $\mathrm{Hom}_{\mathcal{A}}(B, C) = 0$,

(ii) for any object A of $\mathcal{A}$, there is a short exact sequence in $\mathcal{A}$

$$0 \to A^+ \to A \to A^- \to 0$$

with $A^+ \in \mathrm{Ob}(\mathcal{A}^+)$ and $A^- \in \mathrm{Ob}(\mathcal{A}^-)$.

Condition (1) implies that the exact sequence of (2) is unique up to a unique isomorphism and that the correspondences $A \mapsto A^+$ and $A \mapsto A^-$ are functorial.

We define the *heart* $\mathcal{A}^t$ of t as the full subcateogry of the derived category $D^b(\mathcal{A})$ whose objects are those $A^\bullet$ such that

$$H^{-1}(A^\bullet) \in \mathrm{Ob}(\mathcal{A}^-), \quad H^0(A^\bullet) \in \mathrm{Ob}(\mathcal{A}^+), \quad H^n(A^\bullet) = 0 \text{ if } n \notin \{-1, 0\}.$$

Proposition 6.10. *Let $t = (\mathcal{A}^+, \mathcal{A}^-)$ be a torsion pair in an abelian category $\mathcal{A}$. Consider the full subcategories $D^{\leq 0} = D_t^{\leq 0}(\mathcal{A})$ and $D^{\geq 0} = D_t^{\geq 0}(\mathcal{A})$ of $D = D^b(\mathcal{A})$ defined by*

(i) $\mathrm{Ob}(D^{\leq 0}) = \{A^\bullet \in \mathrm{Ob}(D^b(\mathcal{A})) \mid H^1(A^\bullet) \in \mathrm{Ob}(\mathcal{A}^+)$ *and* $H^n(A^\bullet) = 0, \forall n > 1\}$,

(ii) $\mathrm{Ob}(D^{\geq 0}) = \{A^\bullet \in \mathrm{Ob}(D^b(\mathcal{A})) \mid H^0(A^\bullet) \in \mathrm{Ob}(\mathcal{A}^-)$ *and* $H^n(A^\bullet) = 0, \forall n < 0\}$.

Then $(D^{\leq 0}, D^{\geq 0})$ is a t-structure on D whose heart is $\mathcal{A}^t$.

Proof. To show that $(D^{\geq 0}, D^{\leq 0})$ is a t-structure, we have to check (cf. [Kashiwara and Schapira 1990, Definition 10.1.1]) that (with standard notations)

(i) $D^{\leq -1} \subset D^{\leq 0}$ and $D^{\geq 1} \subset D^{\geq 0}$,

(ii) $\mathrm{Hom}_D(X, Y) = 0$ for $X \in \mathrm{Ob}(D^{\leq 0})$ and $Y \in \mathrm{Ob}(D^{\geq 1})$,

(iii) For any $X \in \mathrm{Ob}(D)$, there exists a distinguished triangle $X_0 \to X \to X_1 \xrightarrow{+1}$ in D with $X_0 \in \mathrm{Ob}(D^{\geq 0})$ and $X_1 \in \mathrm{Ob}(D^{\geq 1})$.

(1) is obvious. (2) is clear as, if $f : X \to Y$ with $X \in \mathrm{Ob}(D^{\leq 0})$ and $Y \in \mathrm{Ob}(D^{\geq 1})$, we have $H^n(f) = 0$ for $n \leq 0$ (as $H^n(Y) = 0$), for $n > 1$ (as $H^n(X) = 0$) and for $n = 1$ (as $H^1(X) \in \mathrm{Ob}(\mathcal{A}^+)$ and $H^1(Y) \in \mathrm{Ob}(\mathcal{A}^-)$). Let's check (3): we have

$H^1(X) = X^1_{d=0}/dX^0$. Let $U = (\widehat{H^1(X)})^+$ where $\widehat{H^1(X)}$ is the inverse image of $H^1(X)$ in $X^1_{d=0}$. We have a short exact sequence of complexes

$$0 \to X_0 \to X \to X_1 \to 0$$

where

$$X_0^n = \begin{cases} X^n & \text{if } n < 1, \\ U & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \qquad X_1^n = \begin{cases} 0 & \text{if } n < 1, \\ X^1/U & \text{if } n = 1, \\ X^n & \text{if } n > 1, \end{cases}$$

which gives the desired distinguished triangle.

We have $\mathcal{A}^t = D^{\leq 0} \cap D^{\geq 0}$ and the last assertion is obvious. □

In particular, $\mathcal{A}^t$ is an abelian category [Kashiwara and Schapira 1990, proposition 10.1.11].

Denote by $\mathcal{A}^t_0$ the full subcategory of $\mathcal{A}^t$ whose objects are those $A^\bullet$ such that $A^n = 0$ for $n \notin \{0, 1\}$. To give an object $A^\bullet$ of $\mathcal{A}^t_0$ amounts to give a morphism

$$d_A = d^0_{A^\bullet} : A^0 \to A^1$$

of $\mathcal{A}$ such that $\ker(d_A)$ is an object of $\mathcal{A}^-$ and $\operatorname{coker}(d_A)$ an object of $\mathcal{A}^+$.

The inclusion functor $\mathcal{A}^t_0 \to \mathcal{A}^t$ is obviously an equivalence of categories: there is even a canonical quasi-inverse

$$\mathcal{A}^t \to \mathcal{A}^t_0,$$

which sends $A^\bullet$ to $A^{-1}/dA^{-2} \to (A^0)_{d=0}$.

We have an obvious functor

$$\iota^+_t : \mathcal{A}^+ \to \mathcal{A}^t_0, \quad A \mapsto (0 \to A).$$

It is easy to check that this functor is fully faithful and we denote $\mathcal{A}^{t,-}_0$ its essential image.

Similarly, it is easy to check that the functor

$$\iota^-_t : \mathcal{A}^- \to \mathcal{A}^t_0 \ : \ A \mapsto (A \to 0)$$

is fully faithful and we denote by $\mathcal{A}^{t,+}_0$ its essential image.

It is also easy to check that $\widetilde{t} = (\mathcal{A}^{t,+}_0, \mathcal{A}^{t,-}_0)$ is a torsion pair in $\mathcal{A}^t_0$.

Proposition 6.11. (i) *$t = (\mathcal{M}^{\geq 0}(G_K), \mathcal{M}^{<0}(G_K))$ is a torsion pair in $\mathcal{M}(G_K)$.*

(ii) *$t' = (\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$ is a torsion pair in $\mathcal{C}(G_K)$.*

Proof. (i) We already know (Section 6C) that, for any object $\mathcal{F}$ of $\mathcal{M}(G_K)$, we have a canonical exact sequence

$$0 \to \mathcal{F}^{\geq 0} \to \mathcal{F} \to \mathcal{F}^{<0} \to 0$$

with $\mathcal{F}^{\geq 0} \in \mathcal{M}^{\geq 0}(G_K)$ and $\mathcal{F}^{<0} \in \mathcal{M}^{<0}(G_K)$.

If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{M}(G_K)$, it sends $\mathcal{F}^{\geq 0}$ to $\mathcal{G}^{\geq 0}$. Therefore if $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$ ($\iff \mathcal{F}^{\geq 0} = \mathcal{F}$) and if $\mathcal{G} \in \mathcal{M}^{<0}(G_K)$ ($\iff \mathcal{G}^{\geq 0} = 0$), we have $f = 0$.

(ii) We already know (Proposition 6.6) that, for any object V of $\mathcal{C}(G_K)$, we have a canonical exact sequence

$$0 \to V^{<0} \to V \to V^{\geq 0} \to 0$$

with $V^{<0} \in \mathcal{C}^{<0}(G_K)$ and $V^{\geq 0} \in \mathcal{C}^{\geq 0}(G_K)$. Let $f : V_1 \to V_2$ be a morphism of $\mathcal{C}(G_K)$ with $V_1 \in \mathcal{C}^{<0}(G_K)$ and $V_2 \in \mathcal{C}^{\geq 0}(G_K)$. We can find a monomorphism $V_2 \to W$ with $W \in \mathcal{C}^{\infty}(G_K)$. As any morphism from V_1 to W is 0, the compositum $V_1 \to V_2 \to W$ is 0, hence $f = 0$. □

Denote by $\mathrm{Ar}^t(\mathcal{M}(G_K))$ the full subcategory of the categories of arrows of $\mathcal{M}^{\geq 0}(G_K)$ whose objects are those $d_{\mathcal{F}} : \mathcal{F}^0 \to \mathcal{F}^1$ such that $\ker d_{\mathcal{F}} \in \mathcal{M}^{<0}(G_K)$. Denote $(\mathcal{M}(G_K))^t_{00}$ the full subcategory of $(\mathcal{M}(G_K))^t_0$ whose objects are of the form

$$d_{\mathcal{F}} : \mathcal{F}^0 \to \mathcal{F}^1$$

with $\mathcal{F}^0$ and $\mathcal{F}^1$ objects of $\mathcal{M}^{\geq 0}(G_K)$.

As $\mathcal{M}^{\geq 0}(G_K)$ is stable by taking quotients, $(\mathcal{M}(G_K))^t_{00}$ and $\mathrm{Ar}^t(\mathcal{M}(G_K))$ have the same objects. With obvious conventions, $(\mathcal{M}(G_K))^t_{00}$ is the category deduced from $\mathrm{Ar}^t(\mathcal{M}(G_K))$ by working up to homotopies and inverting quasi-isomorphisms.

Proposition 6.12. *The inclusion functor*

$$(\mathcal{M}(G_K))^t_{00} \to (\mathcal{M}(G_K))^t_0$$

is an equivalence of categories.

Proof. It means that any object $d_{\mathcal{F}} : \mathcal{F}^0 \to \mathcal{F}^1$ of $(\mathcal{M}(G_K))^t_0$ is quasi-isomorphic to an object of $(\mathcal{M}(G_K))^t_{00}$. Indeed, we may find a monomorphism $\mathcal{F}^0 \to \mathcal{G}^0$ of $\mathcal{M}(G_K)$ with $\mathcal{G}^0 \in \mathcal{M}^{\geq 0}(G_K)$. Set

$$\mathcal{G}^1 = \mathcal{G}^0 \oplus_{\mathcal{F}^0} \mathcal{F}^1.$$

We have a short exact sequence

$$0 \to \overline{\mathcal{G}}^0 \to \mathcal{G}^1 \to \mathrm{coker}\, d_{\mathcal{F}} \to 0$$

where $\overline{\mathcal{G}}^0$ is a quotient of $\mathcal{G}^0$. Then $\mathrm{coker}\, d_{\mathcal{F}} \in \mathcal{M}^{\geq 0}(G_K)$ by assumption and $\overline{\mathcal{G}}^0$ also because $\mathcal{M}^{\geq 0}(G_K)$ is stable under taking quotients. As it is also stable under extensions, $\mathcal{G}^1$ also belongs to $\mathcal{M}^{\geq 0}(G_K)$. Hence, $\mathcal{G}^0 \to \mathcal{G}^1$ is an object of $(\mathcal{M}(G_K))^t_{00}$ which is quasi-isomorphic to $\mathcal{F}^0 \to \mathcal{F}^1$. □

Similarly, denote by $\mathrm{Ar}^{t'}(\mathcal{C}(G_K))$ the full subcategory of the categories of arrows of $\mathcal{C}^{\geq 0}(G_K)$ whose objects are those $d_V : V^0 \to V^1$ such that $\operatorname{coker} d_V \in \mathcal{C}^{<0}(G_K)$. Denote $(\mathcal{C}(G_K))^{t'}_{00}$ the full subcategory of $(\mathcal{C}(G_K))^{t'}_0$ whose objects are of the form

$$d_V : V^0 \to V^1$$

with V^0 and V^1 objects of $\mathcal{C}^{\geq 0}(G_K)$.

As $\mathcal{C}^{\geq 0}(G_K)$ is stable by taking subobjects, $(\mathcal{C}(G_K))^{t'}_{00}$ and $\mathrm{Ar}^{t'}(\mathcal{C}(G_K))$ have the same objects. With obvious conventions, $(\mathcal{C}(G_K))^{t'}_{00}$ is the category deduced from $\mathrm{Ar}^{t'}(\mathcal{C}(G_K))$ by working up to homotopies and inverting quasi-isomorphisms.

Proposition 6.13. *The inclusion functor*

$$(\mathcal{C}(G_K))^{t'}_{00} \to (\mathcal{C}(G_K))^{t'}_0$$

is an equivalence of categories.

Proof. The proof is entirely similar to the proof of the previous proposition: It means that any object $d_V : V^0 \to V^1$ of $\mathcal{C}(G_K)^{t'}_0$ is quasi-isomorphic to an object of $(\mathcal{C}(G_K))^{t'}_{00}$. Indeed, we may find an epimorphism $W^1 \to V^1$ of $\mathcal{C}(G^K)$ with $V^1 \in \mathcal{C}^{\geq 0}(G_K)$. Set

$$W^0 = V_0 \times_{V^1} W^1$$

We have a short exact sequence

$$0 \to \ker d_V \to W^0 \to W' \to 0$$

where W' is a subobject of $\mathcal{G}^0$. Then $\ker d_V \in \mathcal{C}^{\geq 0}(G_K)$ by assumption and W' also because $\mathcal{C}^{\geq 0}(G_K)$ is stable under taking subobjects. As it is also stable under extensions, W^0 also belongs to $\mathcal{C}^{\geq 0}(G_K)$. Hence, $V^0 \to V^1$ is an object of $(\mathcal{C}(G_K))^{t'}_{00}$ which is quasi-isomorphic to $V^0 \to V^1$. □

Theorem 6.14. (i) *The functor*

$$\widehat{\Gamma} : \mathrm{Ar}^t(\mathcal{M}(G_K)) \to \mathcal{C}(G_K), \quad (d_{\mathcal{F}} : \mathcal{F}^0 \to \mathcal{F}^1) \mapsto \operatorname{coker}(\mathcal{F}^0(X) \to \mathcal{F}^1(X))$$

factors uniquely through a functor

$$\Gamma : \mathcal{M}(G_K)^t_{00} \to \mathcal{C}(G_K)$$

and Γ *is an equivalence of categories.*

(ii) *The functor*

$$\widehat{\Delta} : \mathrm{Ar}^{t'}(\mathcal{C}(G_K)) \to \mathcal{M}(G_K), \quad (d_V : V^0 \to V^1) \mapsto \ker(\mathcal{F}_{V^0} \to \mathcal{F}_{V^1})$$

factors uniquely through a functor

$$\Delta : \mathcal{C}(G_K)^{t'}_{00} \to \mathcal{M}(G_K)$$

and Δ is an equivalence of categories.

Proof. Let's prove (i). Set $\widehat{\mathcal{M}} = \mathrm{Ar}^t(\mathcal{M}(G_K))$ and $\mathcal{M} = \mathcal{M}(G_K)^t_{00}$. If $d_{\mathcal{F}} = \mathcal{F}^0 \to \mathcal{F}^1$ is an object of one of these categories we denote it also $d_{\mathcal{F}}$ or $\mathcal{F}^0 \to \mathcal{F}^1$.

We see that $\mathcal{M}$ has an obvious structure of an exact category and that the natural functor $\widehat{\mathcal{M}} \to \mathcal{M}$ is exact.

• Let $\widehat{\mathcal{M}}^+$ (resp. $\mathcal{M}^+$) the full subcategory of $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) whose objects are those $d_{\mathcal{F}}$'s such that $\mathrm{coker}\, d_{\mathcal{F}} = 0$. For such an object, as $\ker d_{\mathcal{F}} \in \mathcal{M}^{<0}(G_K)$, and $\mathcal{F}^0$ and $\mathcal{F}^1$ belong to $\mathcal{M}^{\geq 0}(G_K)$, the long exact sequence of coherent cohomology associated to the exact sequence of $\mathcal{M}(G_K)$

$$0 \to \ker d_{\mathcal{F}} \to \mathcal{F}^0 \to \mathcal{F}^1 \to 0$$

is reduced to

$$0 \to \mathcal{F}^0(X) \to \mathcal{F}^1(X) \to H^1(X, \ker d_{\mathcal{F}}) \to 0.$$

Granted Proposition 6.8, this shows that the restriction of $\widehat{\Gamma}$ to $\widehat{\mathcal{M}}^+$ factors through a functor

$$\Gamma^+ : \mathcal{M}^+ \to \mathcal{C}^{<0}(G_K)$$

which is an equivalence of categories.

• Let $\widehat{\mathcal{M}}^-$ (resp. $\mathcal{M}^-$) the full subcategory of $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) whose objects are those $d_{\mathcal{F}}$ such that $\mathcal{F}^0 = 0$. The natural functor $\widehat{\mathcal{M}}^- \to \mathcal{M}^-$ is an equivalence of categories and, granted Theorem 5.9, the restriction of $\widehat{\Gamma}$ to $\widehat{\mathcal{M}}^+$ factors through an equivalence of categories

$$\Gamma^- : \mathcal{M}^- \to \mathcal{C}^{\geq 0}(G_K).$$

• For any $d_{\mathcal{F}} \in \widehat{M}$, we have a canonical short exact sequence

$$0 \to d_{\mathcal{F}_+} \to d_{\mathcal{F}} \to d_{\mathcal{F}_-} \to 0$$

with $d_{\mathcal{F}_+} = (\mathcal{F}^0 \to \mathrm{im}\, d_{\mathcal{F}}) \in \widehat{\mathcal{M}}^+$ and $d_{\mathcal{F}_-} = (0 \to \mathcal{F}^1) \in \widehat{\mathcal{M}}^-$ and this construction is functorial. Moreover, we see that the sequence

$$0 \to \widehat{\Gamma}(d_{\mathcal{F}_+}) \to \widehat{\Gamma}(d_{\mathcal{F}}) \to \widehat{\Gamma}(d_{\mathcal{F}_-}) \to 0$$

is exact.

From these facts, we see that $\widehat{\Gamma}$ factors through a functor $\Gamma : \mathcal{M} \to \mathcal{C}(G_K)$ and that this functor is faithful. It is also straightforward to check that it is exact.

We are left to check the essential surjectivity: Let $V \in \mathcal{C}(G_K)$. We can find a short exact sequence in $\mathcal{C}(G_K)$

$$0 \to U \to \widehat{V} \to V \to 0$$

with $U \in \mathcal{C}^0(G_K)$ and $\widehat{V} \in \mathcal{C}^{\geq 0}(G_K)$. Let $\mathcal{F}^-$ be the kernel of the morphism $\mathcal{F}_U \to \mathcal{F}_{\widehat{V}}$ of $\mathcal{M}(G_K)$. As the functor global section is left exact, we have an exact sequence

$$0 \to \mathcal{F}^-(X) \to \mathcal{F}_U(X) \to \mathcal{F}_{\widehat{V}}(X).$$

But $\mathcal{F}_U(X) = U$, $\mathcal{F}_{\widehat{V}}(X) = \widehat{V}$ and the map $U \to \widehat{V}$ is the given map which is injective. Therefore $\mathcal{F}^-(X) = 0$ which means that $\mathcal{F}^- \in \mathcal{M}^{<0}(G_K)$ and

$$d_{\mathcal{F}} = (\mathcal{F}_U \to \mathcal{F}_{\widehat{V}})$$

is an object of $\mathcal{M}$. Clearly $\Gamma(d_{\mathcal{F}}) = V$, i.e., Γ is essentially surjective.

The proof of (ii) is entirely similar and we leave it to the reader. □

Remark 6.15. The category $\mathcal{M}(G_K)^t_{00}$ is a full subcategory of $D^b(\mathcal{M}(G_K))$ and $\mathcal{C}(G_K)$ is a full subcategory of $D^b(\mathcal{C}(G_K))$. The functor Γ of the previous theorem is the restriction to $\mathcal{M}(G_K)^t_{00}$ of the functor $\Gamma : D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K))$ considered in Section 6D. Similarly, the functor Δ of the previous theorem is the restriction to $\mathcal{C}(G_K)^{t'}_{00}$ of the functor $\Delta : D^b(\mathcal{C}(G_K)) \to D^b(\mathcal{M}(G_K))$ considered in Section 6D.

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