Tunisian Journal of Mathematics

an international publication organized by the Tunisian Mathematical Society

Trigonometric series with a given spectrum

Yves Meyer

2020 vol. 2 no. 4

msp

TUNISIAN JOURNAL OF MATHEMATICS
Vol. 2, No. 4, 2020
dx.doi.org/10.2140/tunis.2020.2.881

Trigonometric series with a given spectrum

Yves Meyer

To the memory of Salah Baouendi

Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set. The vector space consisting of all trigonometric sums whose frequencies belong to Λ is denoted by $\mathcal{T}_\Lambda$. Given an exponent $p \in [1, \infty]$ we say that Λ is p-coherent if there exist a compact set $K \subset \mathbb{R}^n$ and a continuous function ω defined on $\mathbb{R}^n$ with values in $[1, \infty)$ such that for every $P \in \mathcal{T}_\Lambda$ and every $y \in \mathbb{R}^n$ one has $\big(\int_{|x-y|\le 1} |P(x)|^p\,dx\big)^{1/p} \le \omega(y)\big(\int_K |P(x)|^p\,dx\big)^{1/p}$. Several properties of p-coherent sets are proved in this essay.

1. Four problems on trigonometric sums

1A. *Summary.* Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set. The vector space consisting of all trigonometric sums whose frequencies belong to Λ is denoted by $\mathcal{T}_\Lambda$. Given an exponent $p \in [1, \infty]$ we say that Λ is p-coherent if there exist a compact set $K \subset \mathbb{R}^n$ and a continuous function ω defined on $\mathbb{R}^n$ with values in $[1, \infty)$ such that for every $P \in \mathcal{T}_\Lambda$ and every $y \in \mathbb{R}^n$ one has $\big(\int_{|x-y|\le 1} |P(x)|^p\,dx\big)^{1/p} \le \omega(y)\big(\int_K |P(x)|^p\,dx\big)^{1/p}$.

A survey of the still incomplete L^2 theory is given in Section 2. A remarkable theorem by S. Jaffard, M. Tucsnak, and E. Zuazua on weighted L^2 estimates is stated and proved in Section 3. Examples of sets which are not p-coherent sets are given in the five following sections. A p-coherent set Λ has a finite Beurling and Malliavin density, as is proved in Section 4. The role of Section 4 is to bridge the gap between the problems raised in Section 1 and growth estimates satisfied by mean periodic functions with a given spectrum. When this growth cannot be controlled by a weight ω we say that Λ is a wild set. In other words a wild set is a set which is not ∞-coherent. We prove (Theorem 5.4) that the digital cone $\Lambda \subset \mathbb{R}^3$ is not p-coherent if $2 < p \le \infty$. However the digital cone is 2-coherent. A more involved example is the Pisot set. The Pisot set Λ_θ is 2-coherent. If θ is a Pisot number, the Pisot set is contained in a quasicrystal. Therefore it is p-coherent for $1 \le p \le \infty$. When θ is not a Pisot number the Pisot set is a wild set. The proof uses

MSC2010: primary 42A32; secondary 42B10.
Keywords: mean periodic functions, almost periodic functions, trigonometric sums.

a famous theorem by Charles Pisot. Unfortunately we do not know whether the Pisot set is p-coherent or not when θ is not a Pisot number and when $2 < p < \infty$. A partial answer is given in Section 8. A third example of a "wild set" is given in Section 6, Theorem 6.1. This wild set has a finite Beurling and Malliavin density but is not p-coherent when $1 \le p \le \infty$. Theorem 6.1 is given another proof in Section 7, where we show in full generality that a p-coherent set has a finite upper uniform density. Finally in Section 8 we provide the reader with sufficient conditions implying that a set Λ is p-coherent and relate these L^p estimates to the spectral properties or to the additive properties of Λ.

1B. ***The wave equation.*** One of the motivations of this essay is control theory [Avdonin 1974; Avdonin and Ivanov 1995; Lions 1984]. To control the vibrations of a surface, one is led to study the wave equation on a bounded domain. Solutions of the wave equation on a compact Riemannian manifold or on a bounded domain are nonperiodic trigonometric series. That is why precise estimates on nonperiodic trigonometric sums are so important. Here are some details of this discussion. Let M be a compact Riemannian manifold and $\Delta : \mathcal{C}^\infty(M) \mapsto \mathcal{C}^\infty(M)$ be the corresponding Laplace–Beltrami operator. The wave equation on M is

$$\partial_t^2 u - \Delta_x u = 0. \tag{1}$$

A solution of (1) is a series

$$u(x,t) = \sum_0^\infty [a_k(x)\exp(i\lambda_k t) + b_k(x)\exp(-i\lambda_k t)], \tag{2}$$

where $-\lambda_k^2 \le 0$ are the eigenvalues of the Laplace–Beltrami operator and the functions a_k and b_k belong to the corresponding eigenspaces. We have $\Delta a_k = -\lambda_k^2 a_k$, $\Delta b_k = -\lambda_k^2 b_k$.

The series (2) is not a periodic function of the time variable in general. Therefore even if $u(x,t)$ is a global continuous solution of (1), its large time behavior can be quite unexpected and surprising. More generally if $T > 0$, the growth as $t \to \infty$ of $I_p(t) = \bigl(\int_M \int_t^{t+T} |u(x,s)|^p \, dx \, ds\bigr)^{1/p}$ can strongly differ if $p \ne 2$ from what happens if $p = 2$. This essay focuses on such problems.

1C. ***Notation.*** Let us fix some notation. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$. The Fourier transform $\mathcal{F}(f) = \hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i x \cdot \xi) \, dx. \tag{3}$$

Throughout, assume $\Lambda \subset \mathbb{R}^n$ is a closed and discrete set. Then Λ can always be ordered as a sequence λ_j, $j \in \mathbb{N}$, with $|\lambda_j|$ tending to infinity. Such a Λ is uniformly discrete if there exists a $\beta > 0$ such that,

$$\text{for all } \lambda \in \Lambda, \text{ for all } \lambda' \in \Lambda, \quad \lambda' \neq \lambda \implies |\lambda' - \lambda| \geq \beta. \tag{4}$$

One writes $P \in \mathcal{T}_\Lambda$ if P is a trigonometric sum whose frequencies belong to Λ:

$$P(x) = \sum_{\lambda \in \Lambda} c(\lambda) \exp(2\pi i \lambda \cdot x). \tag{5}$$

In the case of the wave equation on a compact manifold, x is replaced by the time variable and (5) takes the form

$$u(x_0, t) = \sum_0^\infty [a_k(x_0) \exp(i\lambda_k t) + b_k(x_0) \exp(-i\lambda_k t)]. \tag{6}$$

1D. *Four properties.* We now return to the general case. We are given a closed and discrete set $\Lambda \subset \mathbb{R}^n$. Four properties of Λ are discussed in this essay. The first one is the L^2 theory.

Property 1.1. There exist a compact set $K \subset \mathbb{R}^n$ of positive Lebesgue measure and a constant C such that for every $P \in \mathcal{T}_\Lambda$ one has

$$\left(\sum_{\lambda \in \Lambda} |c(\lambda)|^2 \right)^{1/2} \leq C \left(\int_K |P(x)|^2 \, dx \right)^{1/2}. \tag{7}$$

On the one hand (7) implies that Λ is uniformly discrete. Conversely if Λ is uniformly discrete there exists a positive number $R(\Lambda)$ such that (7) is satisfied when K is a ball of radius larger than $R(\Lambda)$. In dimension $n \geq 2$ we do not know how to compute $R(\Lambda)$ [Kahane 1962].

Definition 1.1. The compact set K in the right-hand side of (7) is minimal if (7) does not hold any more when K is replaced by a compact set $L \subset K$, $|L| < |K|$, the constant C being possibly replaced by a larger constant C'.

In an equivalent formulation of (7) the roles of Λ and K are exchanged. One starts with the Paley–Wiener space $\mathrm{PW}(K) \subset L^2(\mathbb{R}^n)$. It is the Hilbert space consisting of all $f \in L^2(\mathbb{R}^n)$ whose Fourier transform $\hat{f}$ is supported by K.

Definition 1.2. Let $K \subset \mathbb{R}^n$ be a compact set with a positive measure. A uniformly discrete $\Lambda \subset \mathbb{R}^n$ is a set of stable interpolation for the Paley–Wiener space $\mathrm{PW}(K)$ if for every square summable sequence $c(\lambda)$, $\lambda \in \Lambda$, there exists a function $f \in \mathrm{PW}(K)$ such that

$$f(\lambda) = c(\lambda) \quad \text{for all } \lambda \in \Lambda. \tag{8}$$

We then have:

Lemma 1.1. *Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set and $K \subset \mathbb{R}^n$ be a compact set with a positive measure. Then Λ is a set of stable interpolation for* $\mathrm{PW}(K)$ *if and only if* (7) *is satisfied.*

References are [Landau 1967; Meyer 2018b; Olevskii and Ulanovskii 2008].

Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set. If (7) is satisfied for a compact set K it is also satisfied for every compact set L containing K. Given Λ one tries to find a compact set K as small as possible for which (7) is valid. As was already said, K is minimal if there does not exist a compact subset L of K with $L \neq K$ for which (7) is valid. Two examples of minimal sets are given by Theorems 2.1 and 2.2. But K cannot be too small. The Lebesgue measure $|K|$ cannot be smaller than the upper uniform density of Λ. This was proved by H. J. Landau [1967]. We will return to Landau's theorem in Section 2.

Our second problem has the same structure but L^2 norms are replaced by L^∞ norms. This second problem was raised by J.-P. Kahane [1957].

Property 1.2. A uniformly discrete set Λ is a coherent set of frequencies if there exist a compact set K and a constant C such that for every $P \in \mathcal{T}_\Lambda$ one has

$$\|P\|_\infty \leq C \sup_{x \in K} |P(x)|. \tag{9}$$

Coherent sets of frequencies are studied in [Kahane 1957; Meyer 1972]. Property (9) is labeled $Q(\Lambda)$ in Kahane's seminal work. Property $Q(\Lambda)$ also implies that Λ is uniformly discrete but the converse is not true whatever be the size of K. This was observed in [Kahane 1957]. Given a coherent set of frequencies Λ one is interested in finding K as small as possible in (9). Theorem 8.1 gives an answer to this problem. The definition of a minimal compact set K for (9) is the same as the one given for (7). If $\Lambda = \mathbb{Z}$, then $K = [0, 1]$ is minimal for (7) and (9). If $\Lambda = \mathbb{Z} \cup \left\{\frac{1}{2}\right\}$, then $K = [0, 1]$ is still minimal for (9) but (7) does not hold.

In a weaker version of (9), L^∞ norms are replaced by weighted L^∞ norms.

Definition 1.3. A weight is a continuous function ω defined on $\mathbb{R}^n$ with values in $[1, \infty)$. A weight is submultiplicative if $\omega(x + y) \leq \omega(x)\omega(y)$ for all $x, y \in \mathbb{R}^n$.

Property 1.3. There exist a compact set K and a weight ω such that for every $P \in \mathcal{T}_\Lambda$ and every $y \in \mathbb{R}^n$ one has

$$|P(y)| \leq \omega(y) \sup_{x \in K} |P(x)|. \tag{10}$$

This no longer implies that Λ is uniformly discrete. For instance (10) is valid if $\Lambda = \mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}$. We then have $\omega(x) = 1 + |x|$. This easy observation is proved in [Meyer 2018a].

More generally given $p \in [1, \infty]$ the following problem will be studied:

Property 1.4. A closed and discrete set Λ is p-coherent if there exist a compact set K and a weight ω such that for every $P \in \mathcal{T}_\Lambda$ and for every $y \in \mathbb{R}^n$ one has

$$\left(\int_{|x-y| \leq 1} |P(x)|^p \, dx\right)^{1/p} \leq \omega(y) \left(\int_K |P(x)|^p \, dx\right)^{1/p}. \tag{11}$$

If $p=\infty$, this is Property 1.3. If $p=2$ and ω is a constant we are back to Property 1.1. This leads to the following definition:

Definition 1.4. If $1 \le p \le \infty$, if $K \subset \mathbb{R}^n$ is a compact set, and if ω is a weight, $\mathcal{L}(K, \omega, p)$ is the collection of all closed and discrete sets Λ fulfilling (11). Let $\mathcal{L}(p)$ be the union $\bigcup_{K,\omega} \mathcal{L}(K, \omega, p)$. This union is taken over all compact sets K and all weights ω. Finally if $1 \le p \le \infty$, we say that a closed and discrete set Λ is p-wild if it does not belong to $\mathcal{L}(p)$.

Our first task is to find a criterion on Λ implying $\Lambda \in \mathcal{L}(p)$. Our second task is to try to replace (11) by a sharper estimate. This estimate is sharper if the pair (K, ω) is replaced by (K', ω') where K' is "smaller" than K and similarly ω' is smaller than ω. We do not know if $\mathcal{L}(p) \subset \mathcal{L}(q)$ for $2 \le q \le p$. We do not know if $\mathcal{L}(p)$ is stable by finite unions. Lemma 1.2 is the only fact we know.

Lemma 1.2. *Let* $1 \le p \le \infty$. *We have* $\mathcal{L}(\infty) \subset \mathcal{L}(p) \subset \mathcal{L}(2)$.

The proof of the inclusion $\mathcal{L}(p) \subset \mathcal{L}(2),\ 1 \le p \le \infty$, will be given in a forthcoming paper. Let us prove the first assertion of Lemma 1.2. Property 1.3 is equivalent to the following assertion: there exist a compact set K and a constant C such that for every $y \in \mathbb{R}^n$ one can find a Radon measure μ_y with the following properties:

(a) μ_y is supported by K.

(b) $\|\mu_y\| \le \omega(y)$.

(c) $\hat{\mu}_y(\lambda) = \exp(2\pi i \lambda \cdot y)$ for all $\lambda \in \Lambda$.

To prove this remark we consider the linear form L_y on $\mathcal{T}_\Lambda$ defined by $L_y(P) = P(y)$. We consider the Banach space $\mathcal{C}(K)$ of continuous functions on K equipped with the sup-norm. Then (10) implies that the norm of L_y does not exceed $\omega(y)$. Using the Hahn–Banach theorem one extends L_y to $\mathcal{C}(K)$ with the same norm. This provides us with a Radon measure μ_y on K such that $\int_K P\, d\mu_y = P(y)$. Then (a), (b), and (c) are proved.

We now return to the proof of Property 1.4. We observe that (c) implies $P * \mu_y(x) = P(x+y)$ for every $P \in \mathcal{T}_\Lambda$. Therefore

$$\left(\int_{K+y} |P(u)|^p\, du\right)^{1/p} = \left(\int_K |P(x+y)|^p\, dx\right)^{1/p} = \left(\int_K |P * \mu_y|^p(x)\, dx\right)^{1/p}.$$

We now define $Q = P\chi_{(K-K)}$, where χ_E is the indicator function of E. We then have $\left(\int_K |P * \mu_y|(x)^p\, dx\right)^{1/p} \le \left(\int_{\mathbb{R}^n} |Q * \mu_y|(x)^p\, dx\right)^{1/p} \le \|\mu_y\|\, \|Q\|_p$, which ends the proof of Lemma 1.2.

Property 1.4 was studied in [Meyer 1974] on a simpler example. In the one-dimensional case, it was assumed that $\Lambda = \{k + r_k \mid k \in \mathbb{Z}\}$, where $r_k \to 0$ as $|k| \to \infty$, and finally it was assumed that ω has a polynomial growth at infinity. These assumptions are satisfied in the case of a vibrating sphere.

1E. ***The vibrating sphere.*** These problems become trivial when Λ is a lattice and when K is a fundamental domain for the dual lattice Λ^*. The dual lattice Λ^* is defined by $\Lambda^* = \{x \in \mathbb{R}^n \mid \exp(2\pi i x \cdot y) = 1 \text{ for all } y \in \Lambda\}$. A fundamental domain K for a lattice Γ is defined by the following condition: if sets of zero measure are ignored, the translated compacts $\gamma + K$, $\gamma \in \Gamma$, are an exact paving of $\mathbb{R}^n$.

The wave equation on the sphere $\mathbb{S}^2$ which is discussed in [Meyer 1973] provides us with a natural example where property (9) is not satisfied but where (10) holds true.

Theorem 1.1. *There exists a constant C such that for every continuous solution $u(x,t)$ of the wave equation on the sphere $\mathbb{S}^2$, for every $t \geq 2\pi$, and every $x \in \mathbb{S}^2$, one has*

$$|u(x,t)| \leq C\sqrt{t} \sup_{s \in [0,2\pi]} |u(x,s)| \tag{12}$$

and this estimate is optimal.

This follows from [Meyer 1973; 2018a] and explains why there exist continuous solutions of the wave equation on the sphere which are not almost periodic. If the sphere is replaced by the torus, this is no longer true, as will be proved in Theorem 5.4.

2. L^2 estimates

2A. ***Landau's theorem.*** The L^2 theory of nonperiodic trigonometric sums (in the sense given by Property 1.1) was born in the thirties [Ingham 1936; Paley and Wiener 1934]. In the sixties this theory was revitalized by some important applications to control theory [Avdonin 1974; Avdonin and Ivanov 1995; Lions 1984] and to signal processing [Landau 1967]. A main breakthrough was achieved in [Landau 1967]. While he was working at the Bell Labs in Murray Hill, Landau proved that (7) implies $|K| \geq \overline{\text{dens}}\,\Lambda$. The upper uniform density of Λ will be defined below and $|K|$ denotes the Lebesgue measure of K. Can the converse implication be true? Does $|K| \geq \overline{\text{dens}}\,\Lambda$ imply (7)? The simplest counterexample is given by $\Lambda = \mathbb{Z}$ and $K = \left[0, \frac{1}{3}\right] \cup \left[1, 1+\frac{1}{3}\right] \cup \left[2, 2+\frac{1}{3}\right] \cup \left[3, 3+\frac{1}{3}\right]$. The measure of K is $\frac{4}{3}$, which exceeds dens Λ, but (7) is not true since $P \in \mathcal{T}_\Lambda$ is one-periodic and each of the four intervals of K gives the same information on P. We return to the definition of the upper uniform density of Λ. First for every $R > 0$ one computes $N(R) = \sup_{x \in \mathbb{R}^n} \#(\Lambda \cap B(x,R))$, where $B(x,R)$ denotes the ball of radius R centered at x. Then the upper uniform density of Λ is $\limsup_{R \to \infty} N(R)/(c_n R^n)$, where c_n denotes the volume of the unit sphere.

Property 1.1 is well understood if $n = 1$ and if K is an interval: $|K| \geq \overline{\text{dens}}\,\Lambda$ is necessary and $|K| > \overline{\text{dens}}\,\Lambda$ is sufficient. But Property 1.1 is mostly open when $n = 1$ and K is a finite union of intervals, or when $n \geq 2$. Then the arithmetical structure of Λ plays a seminal role, as will be illustrated by Theorem 2.1. For example if $n = 2$, if Λ is a lattice, and if K is a disk, Landau's bound $|K| = \operatorname{dens}\Lambda$ cannot be approached. Indeed we have (7) $\Rightarrow |K| \geq 2\pi/(3\sqrt{3}) \operatorname{dens}\Lambda$ and $2\pi/(3\sqrt{3}) > 1$. This gap comes from the fact that the plane cannot be paved with translated copies of a disk. We conclude that in dimension $n \geq 2$ sharp results are not related to Landau's theorem but depend on a deeper analysis of the structure of Λ.

2B. ***The converse implication in Landau's theorem.*** The fundamental question raised by Landau's theorem is the following: given a discrete and closed set Λ, is it possible that (7) holds for every compact Riemann integrable set K such that $|K| > \overline{\operatorname{dens}}\,\Lambda$? This natural question was only recently solved. As was observed, such a Λ cannot be a lattice. A first solution was given in [Olevskii and Ulanovskii 2008] and then a second one in [Matei and Meyer 2010]. In the latter, we proved (7) when Λ is a simple quasicrystal and K is a compact Riemannian integrable set K such that $|K| > \operatorname{dens}\Lambda$. S. Grepstad and N. Lev [2014] settled the limiting case $|K| = \operatorname{dens}\Lambda$. For the sake of simplicity their result will be stated on an example. Let $\lfloor x \rfloor$ be the integral part of a real number x. Then $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. Let us assume $\alpha > 0$, $\beta > 0$, $\alpha \notin \mathbb{Q}$, $\alpha + \beta^{-1} \notin \mathbb{Q}$. Let $\lambda_k = k + \beta\{\alpha k\}$, $k \in \mathbb{Z}$, and $\Lambda_\alpha = \{\lambda_k \mid k \in \mathbb{Z}\}$. Then Sigrid Grepstad and Nir Lev proved the following theorem.

Theorem 2.1. *Let K be a finite union of disjoint intervals with endpoints in $\alpha\mathbb{Z} + \mathbb{Z}$. Then the exponential functions* $\exp(2\pi i\lambda \cdot x)$, $\lambda \in \Lambda_\alpha$, *are a Riesz basis of $L^2(K)$ if and only if $|K| = 1$.*

Definition 2.1. If H is a Hilbert space, a Riesz basis of H is the image of an orthonormal basis of H by an isomorphism $T : H \mapsto H$.

Let us observe that $|K| = 1$ is Landau's bound. Theorem 2.1 implies that such a K is minimal for Λ_α. Is there an L^p analogue of Theorem 2.1 when $p \neq 2$? We do not know since the proof of Theorem 2.1 given in [Grepstad and Lev 2014] is using Plancherel formula.

Corollary 2.1. *Let K be a finite union of disjoint intervals with endpoints in $\alpha\mathbb{Z} + \mathbb{Z}$. If $|K| = 1$ every $f \in L^2(K)$ is the sum of the Fourier series*

$$f(x) = \sum_{\lambda \in \Lambda_\alpha} c(\lambda) \exp(2\pi i\lambda \cdot x), \tag{13}$$

which converges to f in $L^2(K)$.

Moreover we have $C_1\|f\|_{L^2(K)} \leq \big(\sum_{\lambda\in\Lambda_\alpha} |c(\lambda)|^2\big)^{1/2} \leq C_2\|f\|_{L^2(K)}$ and there exists a dual family $g_\lambda(x) \in L^2(K)$ such that $c(\lambda) = \int_K f(x)\bar{g}_\lambda(x)\,dx$.

2C. ***A second example of a minimal set.*** Another example of a minimal set is given by the following construction. Let $\alpha > 0,\ \beta > 0\ \alpha \notin \mathbb{Q},\ \beta|\sin(\pi\alpha)| \in \left(0, \tfrac{1}{2}\right)$, and $\lambda_k^{(\alpha,\beta)} = k + \beta \sin(2\pi\alpha k),\ k \in \mathbb{Z}$. Let $\Lambda_{\alpha,\beta} = \{\lambda_k^{(\alpha,\beta)} \mid k \in \mathbb{Z}\}$.

Theorem 2.2. *The functions* $\exp(2\pi i\lambda x),\ \lambda \in \Lambda_{\alpha,\beta}$, *are a Riesz basis of* $L^2([0, 1])$.

The proof of Theorem 2.2 mimics what was achieved in [Grepstad and Lev 2014]. The condition $0 < \beta|\sin(\pi\alpha)| < \frac{1}{2}$ implies that $\Lambda_{\alpha,\beta}$ is uniformly discrete. Moreover there exists an integer N such that uniformly in k we have

$$\beta\left|\frac{1}{N}\sum_{k}^{k+N-1} \sin(2\pi\alpha j)\right| \le \theta < \tfrac{1}{5}. \tag{14}$$

Then S. A. Avdonin's theorem [1974], see also [Avdonin and Ivanov 1995], yields the result. (Does an analogue of Grepstad and Lev's theorem hold?) Therefore $[0, 1]$ is a minimal set for $\Lambda_{\alpha,\beta}$. Finally if $\beta|\sin(\pi\alpha)| \ge \frac{1}{2}$, then $\Lambda_{\alpha,\beta}$ is not uniformly discrete and $\exp(2\pi i\lambda x),\ \lambda \in \Lambda_{\alpha,\beta}$, cannot be a basis.

3. Weighted L^2 estimates

If Λ is uniformly discrete then Property 1.1 is satisfied. Given such a Λ, the main issue is to find "small" compact set K for which (7) holds. If $\overline{\text{dens}}\,\Lambda < \infty$ then Λ is the union of at most N uniformly discrete sets $\Lambda_j,\ 1 \le j \le N$. We conjecture that (11) holds with $p = 2$ and $\omega(x) \le C(1 + |x|)^{N-1}$. This was proved by Jaffard, Tucsnak and Zuazua [Jaffard et al. 1997] in a slightly narrower setting. In their theorem Λ is the union of two uniformly discrete sets of real numbers. Their theorem is proved here in a slightly simplified version. In this section 2π is omitted in the definition of the Fourier transform.

Theorem 3.1. *Let* $\lambda_k,\ k \in \mathbb{Z}$, *be an increasing sequence of real numbers such that for a constant* $M \ge 2$ *we have* $1 \le \lambda_{k+2} - \lambda_k \le M,\ k \in \mathbb{Z}$. *Then there exist a* $T > 0$ *and two constants* C_0 *and* C_1 *such that for every sequence* $a_k,\ k \in \mathbb{Z}$,

$$C_0 \int_{-T}^{T} \left|\sum_{k\in\mathbb{Z}} a_k \exp(i\lambda_k t)\right|^2 dt \le \sum_{k\in\mathbb{Z}} [|a_{k+1} + a_k|^2 + |\lambda_{k+1} - \lambda_k|^2(|a_{k+1}|^2 + |a_k|^2)]$$
$$\le C_1 \int_{-T}^{T} \left|\sum_{k\in\mathbb{Z}} a_k \exp(i\lambda_k t)\right|^2 dt.$$

If moreover we have $\lambda_{k+1} - \lambda_k \ge \beta > 0$ for all $k \in \mathbb{Z}$, then Λ is uniformly discrete and Theorem 3.1 coincides with Property 1.1 since $\sum_{k\in\mathbb{Z}} |a_{k+1} + a_k|^2 \le 2\sum_{k\in\mathbb{Z}}(|a_{k+1}|^2 + |a_k|^2)$. Let us prove Theorem 3.1. Let ϕ be a real-valued even function in the Schwartz class $\mathcal{S}(\mathbb{R})$ such that $\phi(t) > 0$ for all $t \in \mathbb{R}$, $\|\phi\|_2 = 1$ and $\hat{\phi} = 0$ outside $\left[-\frac{1}{10}, \frac{1}{10}\right]$. We then have:

Lemma 3.1. *There exist four positive constants C, C', c and c' (depending on the choice of ϕ) such that for every $\epsilon \in [-1/(2\sqrt{C}), 1/(2\sqrt{C})]$ and any coefficients a and b we have*

$$C'\epsilon^2(|a|^2+|b|^2)+(1-C\epsilon^2)|a+b|^2 \leq \int_{-\infty}^{+\infty} |(a\exp(i\epsilon t)+b)\phi(t)|^2\,dt \leq C\epsilon^2(|a|^2+|b|^2)+(1-C'\epsilon^2)|a+b|^2.$$

If $1/(2\sqrt{C}) \leq |\epsilon| \leq 1$ we have

$$c(|a|^2+|b|^2) \leq \int_{-\infty}^{+\infty} |(a\exp(i\epsilon t)+b)\phi(t)|^2\,dt \leq c'(|a|^2+|b|^2).$$

We set $\Phi=\phi^2$; we obviously have $\int_{-\infty}^{+\infty}\Phi(t)t^2\,dt>0$ and Φ is even. Therefore there exist $C>C'>0$ such that

$$1-C\epsilon^2 \leq \widehat{\Phi}(\epsilon) \leq 1-C'\epsilon^2 \quad \text{for all } \epsilon\in[-1,1].$$

On the other hand we have

$$\int_{-\infty}^{+\infty} |(a\exp(i\epsilon t)+b)\phi(t)|^2\,dt = |a|^2+|b|^2+\widehat{\Phi}(\epsilon)(a\bar{b}+b\bar{a}) = \widehat{\Phi}(\epsilon)|a+b|^2+(1-\widehat{\Phi}(\epsilon))(|a|^2+|b|^2).$$

This and the estimates on $\widehat{\Phi}(\epsilon)$ end the proof of the first assertion in Lemma 3.1. When $|\epsilon| \geq 1/(2\sqrt{C})$, we simply use the uniform bound $0 \leq \widehat{\Phi}(\epsilon) \leq \theta < 1$ and conclude as above.

We return to the proof of Theorem 3.1 and estimate $\|P\phi\|_2$ when $P(t)=\sum_{k\in\mathbb{Z}} a_k\exp(i\lambda_k t)$. As in [Jaffard et al. 1997] we denote by A the set of all integers k such that either $\lambda_{k+1}-\lambda_k \leq \frac{1}{5}$ or $\lambda_k-\lambda_{k-1} \leq \frac{1}{5}$. Then we set $B=\mathbb{Z}\setminus A$. If $k\in B$ and $k+1\in A$ we have $\lambda_{k+1}-\lambda_k > \frac{4}{5}$. Similarly if $k\in B$ and $k-1\in A$ we have $\lambda_k-\lambda_{k-1} > \frac{4}{5}$. These estimates hold since $\lambda_{k+2}-\lambda_k \geq 1$. If both k and $k+1$ belong to B we have $\lambda_{k+1}-\lambda_k > \frac{1}{5}$ and similarly if k and $k-1$ belong to B. In all cases we have $\lambda_{k+1}-\lambda_k > \frac{1}{5}$ if $k\in B$. Then if $P(t)=\sum_{k\in\mathbb{Z}} a_k\exp(i\lambda_k t)$, we have $P\phi=f_1+f_2$, where $f_1=\sum_{k\in A} a_k\exp(i\lambda_k t)\phi(t)$ and $f_2=\sum_{k\in B} a_k\exp(i\lambda_k t)\phi(t)$. By the definition of B, the terms in the sum f_2 are pairwise orthogonal since they have disjoint supports in the Fourier domain. Similarly the terms in the sum f_1 appear as pairs $g_k(t)=(a_k\exp(i\lambda_k t)+a_{k+1}\exp(i\lambda_{k+1}t))\phi(t)$, where $\lambda_{k+1}-\lambda_k \leq \frac{1}{5}$. These g_k are pairwise orthogonal and are orthogonal to f_2. Then

$$\|P\phi\|_2^2 = \|f_1\|_2^2+\|f_2\|_2^2 = \sum_{k\in A}\|g_k\|_2^2+\sum_{k\in B}|a_k|^2\,\|\phi\|_2^2.$$

Lemma 3.1 is applied to each term $\|g_k\|_2$. We obtain the right-hand side of Theorem 3.1. Let us observe that in this right-hand side we ignored the sets A and B, which is harmless since $\lambda_{k+1}-\lambda_k > \frac{1}{5}$ if $k\in B$. Then Theorem 3.1 would be proved if the left-hand side was $\|P\phi\|_2^2$. This is not the case but this will be repaired using the following corollary of our calculation.

Corollary 3.1. *We have for every real number* τ

$$I(\tau)=\int_{-\infty}^{+\infty}|P(t+\tau)|^2\phi(t)\,dt\le C(1+\tau^2)\int_{-\infty}^{+\infty}|P(t)|^2\phi(t)\,dt.$$

Indeed we estimate $I(\tau)$ and $I(0)$ by a sum of coefficients given by Theorem 3.1. The only terms that differ in the two sums are

$$S(\tau)=\sum_{k\in\mathbb{Z}}|a_{k+1}\exp(i\tau\lambda_{k+1})+a_k\exp(i\tau\lambda_k)|^2$$

compared to $S(0)=\sum_{k\in\mathbb{Z}}|a_{k+1}+a_k|^2$. Then it suffices to observe that

$$\begin{aligned}|a_{k+1}\exp(i\tau\lambda_{k+1})+a_k\exp(i\tau\lambda_k)| &= |a_{k+1}\exp(i\tau(\lambda_{k+1}-\lambda_k))+a_k|\\ &\le |a_{k+1}\exp(i\tau(\lambda_{k+1}-\lambda_k))-a_{k+1}|+|a_{k+1}+a_k|\\ &\le |\tau|\,|a_{k+1}(\lambda_{k+1}-\lambda_k)|+|a_{k+1}+a_k|.\end{aligned}$$

Finally

$$\begin{aligned}S(\tau)&\le 2S(0)+2|\tau|^2\sum_{k\in\mathbb{Z}}|a_{k+1}(\lambda_{k+1}-\lambda_k)|^2\\ &\le 2(1+\tau^2)\sum_{k\in\mathbb{Z}}[|a_{k+1}+a_k|^2+|\lambda_{k+1}-\lambda_k|^2(|a_{k+1}|^2+|a_k|^2)],\end{aligned}$$

which ends the proof.

To end the proof of Theorem 3.1 we use an obvious trick and prove the equivalence between $\|P\phi\|_2$ and $\left(\int_{-T}^{T}|P(t)|^2\right)^{1/2}$. On the one hand we have

$$\int_{-\infty}^{+\infty}|P(t)\phi(t)|^2\,dt\ge c\int_{-T}^{+T}|P(t)|^2\,dt$$

for a positive constant c. On the other hand

$$\int_{-\infty}^{+\infty}|P(t)\phi(t)|^2\,dt=\sum_{k\in\mathbb{Z}}\int_{(k-1)T}^{(k+1)T}|P(t)\phi(t)|^2\,dt=\sum_{k\in\mathbb{Z}}I_k.$$

Each I_k is estimated from above by Corollary 3.1 when $k\neq 0$. The total contribution does not exceed $\eta(T)\|P\phi\|_2$, while I_0 is estimated from below by $c\|P\phi\|_2$. We chose T large enough to have $c>\eta(T)$. This ends the proof of Theorem 3.1.

4. Mean periodic functions

The goal of this section is to prove that a p-coherent set $\Lambda\subset\mathbb{R}$ has a finite Beurling and Malliavin density. A sharper theorem will be proved in Section 8. Let us begin with the n-dimensional case. Let $\mathcal{C}(\mathbb{R}^n)$ denote the vector space of all continuous functions on $\mathbb{R}^n$, equipped with the topology of *uniform convergence on compact sets*.

Lemma 4.1. *If Λ is a discrete and closed set and if $\mathcal{T}_\Lambda$ is dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets then Λ cannot be a p-coherent set.*

If $\mathcal{T}_\Lambda$ is dense in $\mathcal{C}(\mathbb{R}^n)$ then Property 1.4 cannot be true. Otherwise (11) would still hold for every continuous function, which is clearly impossible.

Lemma 4.2. *The following two properties are equivalent for a closed and discrete set $\Lambda \subset \mathbb{R}^n$:*

(a) *$\mathcal{T}_\Lambda$ is not dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets.*

(b) *There exists a compactly supported Radon measure $\mu \neq 0$ whose Fourier transform vanishes on Λ.*

This is provided by the Hahn–Banach theorem. In the one-dimensional case Beurling and Malliavin [1967] proved that (a) and (b) are equivalent to a remarkable density condition on Λ. It will be proved in Section 6 that there exists a closed and discrete set Λ satisfying conditions (a) and (b) with an infinite upper uniform density. We conclude these remarks by Lemma 4.3.

Lemma 4.3. *If $\Lambda \subset \mathbb{R}$ is a p-coherent set, the Beurling and Malliavin density of Λ is finite.*

As it was observed in [Kahane 1957], Properties 1.2 and 1.3 are strongly motivated by the theory of mean periodic functions.

Definition 4.1. A mean periodic function is a function $f \in \mathcal{C}(\mathbb{R}^n)$ for which there exists a compactly supported Radon measure $\mu \neq 0$ such that $f * \mu = 0$.

Definition 4.2. Let us assume that $\mathcal{T}_\Lambda$ is not dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets. Then the closure of $\mathcal{T}_\Lambda$ in $\mathcal{C}(\mathbb{R}^n)$ will be denoted by $\mathcal{C}_\Lambda$.

These definitions and Lemma 4.2 imply that every function $f \in \mathcal{C}_\Lambda$ is a mean periodic function.

Definition 4.3. The spectrum of a mean periodic function f is the set $S \subset \mathbb{C}^n$ of all $\lambda \in \mathbb{C}^n$ such that $\exp(2\pi i x \cdot \lambda)$ is a limit, for the topology of uniform convergence on compact sets, of linear combinations of translates of f.

If μ is a compactly supported Radon measure such that $\mu * f = 0$ the spectrum of f is contained in the set of zeros of the Fourier–Laplace transform of μ. If $f \in \mathcal{C}_\Lambda$ its spectrum is contained in Λ.

Here is a more constructive definition of a mean periodic function.

Lemma 4.4. *Let us assume that* (a) $f \in \mathcal{C}(\mathbb{R}^n)$ *has polynomial growth at infinity and* (b) $\hat{f} = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda$, *where* δ_a *is the Dirac measure at* a. *Then* $f \in \mathcal{C}_\Lambda$.

Is the converse implication true? What are the sets Λ enjoying the property that every $f \in \mathcal{C}_\Lambda$ has polynomial growth at infinity? This is Problem 1.3 and our essay partially answers this natural question. Problem 1.2 has a simple formulation as the following theorem shows.

Theorem 4.1. *Let* $\Lambda \subset \mathbb{R}^n$ *be a closed and discrete set. Then the following two conditions are equivalent*:

(i) *Every* $f \in \mathcal{C}_\Lambda$ *is an almost periodic function in the sense of H. Bohr.*

(ii) *Property* 1.2 *is satisfied.*

This is proved in [Kahane 1957].

5. Wild sets of frequencies

5A. *Equivalent definitions.*

Definition 5.1. A closed and discrete set $\Lambda \subset \mathbb{R}^n$ is wild if $\Lambda \notin \mathcal{L}(\infty)$.

A wild set Λ is an ∞-wild set. We treat the one-dimensional case and investigate the structure of wild sets. Property 1.3 is given an equivalent version in the following lemma:

Lemma 5.1. *Let* Λ *be a closed and discrete set of real numbers. Then Property* 1.3 *is equivalent to the following*: *there exists a* $T > 0$ *and for each* $R \geq T$ *a finite constant* $C(R)$ *such that for every* $P \in \mathcal{T}_\Lambda$ *one has*

$$\sup_{|x| \leq R} |P(x)| \leq C(R) \sup_{|x| \leq T} |P(x)|. \tag{15}$$

An L^p version of Lemma 5.1 will be given below. One direction is obvious and we can choose $C(R) = \sup_{|x| \leq R} \omega(x)$. The other direction would be obvious if continuity was not imposed on the weight. Here is the argument. We first replace $C(R)$ by

$$\widetilde{C}(R) = \sup_{P \in \mathcal{T}_\Lambda} \frac{\sup_{|x| \leq R} |P(x)|}{\sup_{|x| \leq T} |P(x)|}.$$

Then $\widetilde{C}(R) \leq C(R)$ and $\widetilde{C}(R)$ is obviously a nondecreasing function of R. Then (15) remains true if $C(R)$ is replaced by $\widetilde{C}(R)$. Finally there exists a continuous function $\omega(R)$ such that $\widetilde{C}(R) \leq \omega(R)$. This implies (10).

In summary if Λ is wild, then for every $R > 0$ there exists a $T > R$ and a sequence $P_j \in \mathcal{T}_\Lambda$ such that $\sup_{|x| \leq R} |P_j(x)|$ tends to 0, while $\sup_{|x| \leq T} |P_j(x)| = 1$. We now prove that if Λ is wild, this property is true for every $T > R$, which is a stronger statement.

Theorem 5.1. *Let Λ be a closed and discrete set of real numbers. Then Λ is wild if and only if one of the two following conditions is satisfied*:

(a) *There exists a sequence $P_j \in \mathcal{T}_\Lambda$ such that $P_j(0) = 1$ and such that for every compact set K contained in $(0, \infty)$ we have*

$$\sup_{y \in K} |P_j(y)| \to 0, \quad j \to \infty. \tag{16}$$

(b) *There exists a sequence $Q_j \in \mathcal{T}_\Lambda$ such that $Q_j(0) = 1$ and such that for every compact set K contained in $(-\infty, 0)$ we have $\sup_{y \in K} |Q_j(y)| \to 0,\ j \to \infty$.*

If Property 1.3 holds with $K = [a, b]$, every trigonometric sum $P \in \mathcal{T}_\Lambda$ can be extrapolated outside $[a, b]$ from its knowledge on $[a, b]$. In fact it suffices to extrapolate P on two points, one less than a, the other one larger than b, as the following lemma tells us.

Lemma 5.2. *Let Λ be a closed and discrete set of real numbers. If there exist an interval $I = [a, b]$, a real number $x_0 < a$, and a constant C such that for every $P \in \mathcal{T}_\Lambda$ one has*

$$|P(x_0)| \le C \sup_{y \in I} |P(y)| \tag{17}$$

then there exist a weight $\omega(x)$ and a compact set K such that (10) *holds true for every $x < a$.*

Let $\eta = a - x_0$ and $J = [a, b + \eta]$. Let us fix $u \in [x_0, a]$. Since the space $\mathcal{T}_\Lambda$ is translation invariant, (17) can be applied to $Q(x) = P(x + u - x_0)$. Therefore (17) remains valid when x_0 is replaced by u and I by J. We then proceed inductively from the interval $E_m = [x_0 - m\eta, a - m\eta]$ to E_{m+1}, $m \in \mathbb{N}$. This inductive procedure yields (10) with an exponential weight.

Needless to say Lemma 5.2 is also true if $x_0 < a$ is replaced by $x_1 > b$. The conclusion is the validity of (10) for $x > b$. Finally Lemma 5.2 implies Theorem 5.1.

Here is the argument. Let us assume that Λ is wild. Then one of the two following conditions is satisfied: (1) either for any interval $I = [a, b]$ and any $x_0 < a$ there exists a sequence $P_j \in \mathcal{T}_\Lambda$ such that $P_j(x_0) = 1$ and P_j converges to 0 uniformly on I or (2) for any interval $I = [a, b]$ and any $x_1 > b$ there exists a sequence $P_j \in \mathcal{T}_\Lambda$ such that $P_j(x_1) = 1$ and P_j converges to 0 uniformly on I. Everything being translation invariant, we can assume $x_0 = 0$ if the first case occurs. We set $I_m = [m^{-1}, m]$. For every integer $j \ge 1$ there exists an integer j_m such that $P_{j_m} \in \mathcal{T}_\Lambda,\ P_{j_m}(0) = 1$, and $\sup_{y \in I_m} |P_{j_m}(y)| \le 2^{-j}$. This sequence P_{j_m}, $m \in \mathbb{N}$, is the sequence announced in Theorem 5.1. The second alternative is similar.

Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set and let $K \subset \mathbb{R}^n$ be a compact set. We denote by $\mathcal{C}(K)$ the Banach space of all continuous functions on K and by $\mathcal{C}_\Lambda(K)$ the closure of $\mathcal{T}_\Lambda$ in $\mathcal{C}(K)$.

Corollary 5.1. *Property* 1.3 *is satisfied by a closed and discrete set* Λ *of real numbers if and only if there exist two intervals* $[a, b]$ *and* $[c, d]$ *with* $c < a < b < d$ *such that the restriction operator* $R : \mathcal{C}_\Lambda([c, d]) \mapsto \mathcal{C}_\Lambda([a, b])$ *is an isomorphism.*

It would be interesting to know whether or not this property is valid in a more general setting. Given a closed and discrete set $\Lambda \subset \mathbb{R}^n$ and two compact sets $K, L \subset \mathbb{R}^n$ such that K is contained in the interior of L, we assume that the restriction operator $R : \mathcal{C}_\Lambda(L) \mapsto \mathcal{C}_\Lambda(K)$ is an isomorphism. Does it imply Property 1.3?

5B. ***The Pisot set.*** Here is an illustration of Theorem 5.1. Let $\theta \geq 2$ be a real number and let Λ_θ be the set of all finite sums $\sum_{k\geq 0} \epsilon_k \theta^k$, $\epsilon_k \in \{0, 1\}$. This set Λ_θ is uniformly discrete and will be named the Pisot set.

Theorem 5.2. *Let us assume that* θ *is not a Pisot–Thue–Vijayaraghavan number. Then* Λ_θ *is wild.*

We consider the sequence $P_m(x)$ of finite products

$$\prod_0^{m-1} \left(\frac{1 + \exp(2\pi i \theta^k x)}{2} \right).$$

The spectrum of P_m is contained in Λ. By Pisot's theorem we know that $|P_m(x)| = \prod_0^{m-1} |\cos(\pi \theta^k x)|$ converge uniformly to 0 on every compact set not containing the origin. We have $P_m(0) = 1$, which concludes the proof.

The converse is true. If θ is a Pisot–Thue–Vijayaraghavan number then Λ_θ satisfies Property 1.2, as is proved in [Meyer 1972]. If the sequence θ^k, $k \in \mathbb{N}$, is replaced by a lacunary sequence θ_k such that $\sum_0^\infty (\theta_k/\theta_{k+1})^2 < \infty$, the arithmetical properties of θ_k do not play any role. The set of all finite sums $\sum_{k\geq 0} \epsilon_k \theta_k$, $\epsilon_k \in \{0, 1\}$, satisfies Property 1.2 [Meyer 1972, Theorem IV, Chapter VIII].

5C. ***The wave equation.*** Here is a second example illustrating the definition of wild sets of frequencies. We consider the wave equation on the three-dimensional torus $\mathbb{T}^3$.

Theorem 5.3. *For every* $T_1 > T_0 > 0$ *and every* $\epsilon > 0$ *there exists a solution* $v(x, t)$ *of the wave equation on* $\mathbb{T}^3$ *such that* $v(0, 0) = 1$ *and* $|v(x, t)| \leq \epsilon$ *for all* $t \in [T_0, T_1]$, *for all* $x \in \mathbb{T}^3$.

Corollary 5.2. *The digital cone* $\Lambda \subset \mathbb{R}^4$, *defined by* $\Lambda = \{(k, \pm|k|) \mid k \in \mathbb{Z}^3\}$, *is wild.*

The proof of this simple observation depends on the following remarks. Let $w(x, t)$ be defined on $\mathbb{T}^3 \times \mathbb{R}$ by

$$w(x, t) = t + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sin(2\pi t |k|)}{2\pi |k|} \exp(2\pi i k \cdot x).$$

Then $w(x, t)$ is the solution to the following Cauchy problem (named C-1) for the wave equation on $\mathbb{T}^3 \times \mathbb{R}$:

$$\frac{\partial^2}{\partial t^2} u(x, t) = \Delta u(x, t),$$
$$u(x, 0) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = \delta_0(x).$$

But $w(x, t)$ can also be computed by periodizing the solution of a similar Cauchy problem (named C-2) on $\mathbb{R}^3 \times \mathbb{R}$. This scheme is detailed now. Let σ_t, $t \in \mathbb{R}$, be the normalized surface measure on the sphere $B_t \subset \mathbb{R}^3$ centered at 0 with radius $|t|$ (the total mass of σ_t is 1). Then $u(x, t) = t\sigma_t(x)$ belongs to $\mathcal{C}^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^3))$ and is the solution of the Cauchy problem C-2:

$$\frac{\partial^2}{\partial t^2} u(x, t) = \Delta u(x, t),$$
$$u(x, 0) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = \delta_0(x).$$

Therefore

$$w(x, t) = \sum_{k \in \mathbb{Z}^3} t\sigma_t(x - k) \tag{18}$$

is the solution of the Cauchy problem C-1. Let us consider the distribution $\tau(x, t) = \frac{\partial}{\partial t} w(x, t)$. Let ϕ be a compactly supported smooth function defined on $\mathbb{R}^3$ and for $\epsilon > 0$ let $\phi_\epsilon(x) = g(x/\epsilon)$. If ϵ is small enough, ϕ_ϵ can be viewed as a function defined on $\mathbb{T}^3$. Let $g_\epsilon(x, t)$ be the solution of the wave equation defined by $g_\epsilon = \tau * \phi_\epsilon$, where the convolution product takes place on $\mathbb{T}^3$. Then $g_\epsilon(0, 0) = 1$, while $|g_\epsilon(x, t)| \le C\epsilon$ if $t \in [T_0, T_1]$, as simple estimates show.

The same construction is performed on $\mathbb{T}^2$ instead of $\mathbb{T}^3$. It gives a natural example of a uniformly discrete set Λ for which Property 1.4 fails if $p > 2$. Indeed if $0 < T_0 < T_1$ we have $\int_{T_0}^{T_1} \int_{\mathbb{T}^2} |g_\epsilon|^p \, dt \, dx \le C\epsilon^{p/2+1}$, while for $\eta > 0$ we have $\int_\eta^\eta \int_{\mathbb{T}^2} |g_\epsilon|^p \, dt \, dx \simeq \epsilon^2$. We can conclude:

Theorem 5.4. *The digital cone* $\Lambda \subset \mathbb{R}^3$, *defined by* $\Lambda = \{(k, \pm|k|) \mid k \in \mathbb{Z}^2\}$, *is not p-coherent if* $2 < p \le \infty$.

Let us observe that Λ is uniformly discrete. Therefore Λ is 2-coherent.

6. Unions of lattices

An interesting example of a wild set of frequencies is detailed in this section. It will now be assumed that

$$\Lambda = \bigcup_1^\infty \omega_j \mathbb{Z}, \tag{19}$$

where $1 = \omega_1 < \cdots < \omega_j < \cdots$ and $\sum_1^\infty 1/\omega_j < \infty$. Then we have:

Lemma 6.1. *If* Λ *is defined by* (19) *then* $\mathcal{T}_\Lambda$ *is not dense in* $\mathcal{C}(\mathbb{R}^n)$ *for the topology of uniform convergence on compact sets.*

For proving Lemma 6.1 it suffices to construct a function h with the following properties: h is not identically 0, it is compactly supported, and its Fourier transform vanishes on Λ. Let us begin with $g_j(x),\ j \in \mathbb{N}$, which is defined by $g_j(x) = \pi\omega_j$ on $[-1/(2\pi\omega_j), 1/(2\pi\omega_j)]$ and $g_j(x) = 0$ outside this interval. The convolution products $g_1 * g_2 * \cdots * g_j$ converge to a $\mathcal{C}^\infty$ function g. We have $g \geq 0$, $\int g = 1$ and $\hat{g} = 0$ on $\Lambda \setminus \{0\}$. The function $h = \frac{d}{dx} g$ has the required properties. This ends the proof.

Corollary 6.1. *The Beurling and Malliavin density of the set* Λ *defined by* (19) *is finite.*

Theorem 6.1. *Let us furthermore assume that* $1, 1/\omega_1, 1/\omega_2, \ldots, 1/\omega_j, \ldots$ *are linearly independent over* $\mathbb{Q}$. *Then the upper uniform density of* $\Lambda = \bigcup_1^\infty \omega_j\mathbb{Z}$ *is infinite and* Λ *is wild.*

In the next section it will be proved that the property "infinite upper uniform density" implies "p-wild". Here the two properties will be proved by the same argument. We argue by contradiction and assume that Λ is not wild. Then (10) holds true. The proof of Theorem 6.1 begins with the following definition.

Definition 6.1. Let $\Lambda \subset \mathbb{R}$ be a closed and discrete set and let F be a finite set. We write $F \in \mathcal{F}(\Lambda)$ if there exists a sequence $x_j,\ j \in \mathbb{N}$, of real numbers such that $F + x_j \subset \Lambda + [-1/j, 1/j]$.

If $|x_j| \leq C$, then $F \in \mathcal{F}(\Lambda)$ simply means $F \subset \Lambda + a$ for some a. If $\Lambda = \{k + 2^{-k} \mid k \in \mathbb{N}\}$, then $F \subset \mathbb{Z}$ implies $F \in \mathcal{F}(\Lambda)$. The proof of Theorem 6.1 depends on the following lemma:

Lemma 6.2. *Let* $\Lambda \subset \mathbb{R}$ *be a closed and discrete set. Let us assume that* (10) *holds true for a pair* (Λ, K) *and for a weight* ω. *If* $F \in \mathcal{F}(\Lambda)$ (10) *also holds true for the pair* (F, K) *and the same weight* ω.

Let $P(x) = \sum_{y \in F} c(y) \exp(2\pi i \lambda y)$ be an arbitrary trigonometric sum with frequencies in F. We want to prove that

$$|P(x)| \leq \omega(x) \sup_{y \in K} |P(y)|. \tag{20}$$

For $j \geq 1$ every $y \in F$ can be written as $y = \lambda_{j,y} - x_j + \epsilon_j$, where $|\epsilon_j| \leq 1/j$. We approach $P(x)$ by $P_j(x) = \exp(-2\pi i x_j x) Q_j(x)$, where

$$Q_j(x) = \sum_{y \in F} c(y) \exp(2\pi i \lambda_{y,j} x).$$

But (10) is true for Q_j by assumption. Since $|P_j| = |Q_j|$, (10) is also true for P_j and it suffices to let j tend to infinity to conclude the proof of Lemma 6.2.

Lemma 6.3. *If* Λ *is defined by* (19) *and if the real numbers* $1/\omega_j,\ j \in \mathbb{N}$, *are linearly independent over* $\mathbb{Q}$ *then the finite set* $F_{\epsilon,n} = \{0, \epsilon, 2\epsilon, \dots, (n-1)\epsilon\}$ *belongs to* $\mathcal{F}(\Lambda)$ *for every* $\epsilon > 0$ *and every integer* n.

Lemma 6.3 obviously implies that the upper uniform density of Λ is infinite. This upper uniform density is defined as

$$\limsup_{T\to\infty} T^{-1} \sup_{x\in\mathbb{R}} \#([x, x+T] \cap \Lambda). \tag{21}$$

We return to the proof of Lemma 6.3. The linear independence over $\mathbb{Q}$ of $1, 1/\omega_1, 1/\omega_2, \dots, 1/\omega_n$ implies that the subgroup $(\exp(2\pi i k/\omega_m))_{1\le m\le n},\ k \in \mathbb{Z}$, is dense in $\mathbb{T}^n$. Therefore there exists a sequence k_j of integers and n sequences $l_{j,m}$, $j \in \mathbb{Z},\ 1 \le m \le n$, of integers such that, for $1 \le m \le n$, we have $k_j/\omega_m - l_{j,m} + m\epsilon/\omega_m \to 0,\ j \to \infty$. This convergence takes place on the real line. It yields $k_j - \omega_m l_{j,m} + m\epsilon \to 0,\ j \to \infty$. Returning to Definition 6.1 we have $x_j = k_j$ and $m\epsilon = \lim_{j\to\infty} \omega_m l_{j,m} - x_j$, as announced.

We now disprove the uniform validity of (10) when Λ is replaced by $F_{\epsilon,m}$ and $\epsilon \to 0,\ m \to \infty$. To this end we form $P_{\epsilon,m} = \epsilon^{-m} \sum_0^{m-1} c_k \exp(2\pi i \epsilon k x)$, where $\sum_0^{m-1} c_k k^q = 0,\ 0 \le q \le m-1$. Lemma 6.2 implies that (10) is satisfied by $P_{\epsilon,m}$ uniformly with respect to ϵ and m. But $\lim_{\epsilon\to 0} P_{\epsilon,m} = cx^m$. Therefore (10) is satisfied by x^m uniformly in $m \in \mathbb{N}$, which is impossible. The same argument can be used to disprove (11).

If Λ is replaced by a finite union $\Lambda_N = \bigcup_1^N \omega_j \mathbb{Z}$ then (10) is satisfied with $\omega(x) = (1+|x|)^{N-1}$, as is proved in [Meyer 2018a].

7. Upper uniform densities

Theorem 6.1 is a special instance of a more general fact which is valid for every $p \in [1, \infty]$.

Theorem 7.1. *The upper uniform density of a* p*-coherent set of real numbers is finite.*

The proof begins with the following lemma:

Lemma 7.1. *A closed and discrete set* Λ *is* p*-coherent if and only if there exist a* $T > 0$ *and for every* $R \ge T$ *a constant* $C(R)$ *such that for every* $P \in \mathcal{T}_\Lambda$ *one has*

$$\left(\int_{|x|\le R} |P(x)|^p \, dx\right)^{1/p} \le C(R) \left(\int_{|x|\le T} |P(x)|^p \, dx\right)^{1/p}. \tag{22}$$

The proof is immediate. On one hand if (11) is satisfied, it suffices to set $C(R) = \sup_{|x|\le R} \omega(x)$. On the other hand if (22) is satisfied, we first optimize this estimate. We replace $C(R)$ by the lower bound $\gamma(R) \le C(R)$ of all possible constants for which (22) holds. Then $\gamma(R)$ is an increasing function of R and there exists a continuous increasing $\tilde{\gamma}(R) \geq \gamma(R)$. Finally it suffices to set $\omega(y) = \tilde{\gamma}(|y|)$.

We return to the proof of Theorem 7.1 and fix the notation used in Lemma 7.2. If Λ is p-coherent then $\Lambda + y$ is also p-coherent with the same constants $C(R)$ in (22) and this holds true for every $y \in \mathbb{R}$. Without loss of generality it can be assumed that $K = [-a, a]$. Indeed K can be replaced by a larger compact. We fix $R = a + 1$ in (22). We then write $\gamma = C(R)$.

Lemma 7.2. *Let Λ be a p-coherent set of real numbers. Then for every interval J with length $|J| \geq 1$ we have*

$$\#(J \cap \Lambda) \leq C(a, \gamma)|J|, \tag{23}$$

where $C(a, \gamma)$ only depends on a and γ.

Lemma 7.2 obviously implies Theorem 7.1. To prove Lemma 7.2 for an arbitrary J it suffices to do it when $|J| = [0, 1]$. Indeed the translation invariance of (22) in the Fourier domain will imply Lemma 7.2 for every J with length 1. It suffices to add these estimates to obtain (23) for $|J| \geq 1$. We now prove (23) when $|J| = [0, 1]$. Let q be the conjugate exponent defined by $1/p + 1/q = 1$. We have $K = [-a, a]$ and without loss of generality it can be assumed that $a \geq 1$. Let $I = [a, a+1]$ and let χ_I be the indicator function of I. By (22) and a duality argument there exists a function $g \in L^q(K)$ carried by K such that $\|g\|_q \leq \gamma$ and $\hat{g}(\lambda) = \hat{\chi}_I(\lambda)$ for every $\lambda \in \Lambda$. We now consider the entire function

$$F(z) = \int_{\mathbb{R}} \exp(-i2\pi zt)(\chi_I(t) - g(t))\, dt. \tag{24}$$

Then F vanishes on Λ. Let us compute $F(iy)$ for $y \geq 1$. We have $F(iy) = \int_a^{a+1} \exp(2\pi yt)\, dt - \int_K \exp(2\pi yt) g(t)\, dt$. Therefore

$$\begin{aligned} |F(iy)| &\geq \frac{\exp(2\pi y(a+1)) - \exp(2\pi ya)}{2\pi y} - \|g\|_q \left(\int_K \exp(2\pi pyt)\, dt \right)^{1/p} \\ &\geq C_1 \frac{\exp(2\pi y(a+1))}{y} - C_2 \gamma \frac{\exp(2\pi ya)}{y}. \end{aligned} \tag{25}$$

Here C_1 and C_2 are two absolute constants. Finally there exists a $y_0 \geq 1$, depending only on a and γ, such that $|F(iy_0)| \geq 1$. This y_0 is now fixed. We consider a disc D centered at $z_0 = iy_0$ with radius $R = 2\sqrt{1 + y_0^2}$. Then the disc centered at z_0 with radius $r = R/2$ contains J. The following corollary of Jensen's formula is applied to F and D:

Lemma 7.3. *If F is holomorphic on a neighborhood of a disc D centered at z_0 with radius R, if $|F|$ is bounded by M on D, and if $F(z_0) \neq 0$ then for every $r \in (0, R)$ the number of zeros of F inside the disc $|z - z_0| \leq r$ does not exceed $(\log(R/r))^{-1} \log(M/|F(z_0)|)$.*

We have $z_0 = iy_0$ and $R = 2r = 2\sqrt{1+y_0^2}$. We already know that $|F(z_0)| \geq 1$. Estimating $|F(z)|$ on D is trivial. Indeed if $y \geq 0$ and $x+iy = z \in D$, we have

$$|F(z)| \leq \frac{\exp(2\pi y(a+1)) - \exp(2\pi ya)}{2\pi y} + \gamma\left(\int_K \exp(2\pi pyt)\,dt\right)^{1/p} \leq M(a,\gamma)$$

since $y \leq y_0 + 2\sqrt{1+y_0^2}$ on D. The case $y \leq 0$ is similar. Finally Lemma 7.2 implies (23).

This proof raises a few questions. It seems that we are not using the full strength of the hypothesis since the proof is based on the value $R = a+1$ when $K = [-a,a]$. But this special instance of (22) implies (22) in full generality, as is proved in Lemma 5.1. The second issue is the generalization of Theorem 7.1 to the n-dimensional case. The third problem is the converse statement. If $p = 2$ and if Λ has a finite upper uniform density, does it imply that Λ is 2-coherent? A partial answer is given in [Jaffard et al. 1997].

8. L^p-estimates

8A. *A sufficient condition.* In a particular case the problem raised by Property 1.4 of Section 1 can be answered. The aim of this section is to show how much the L^p theory differs from the L^2 theory when $p \neq 2$. From now on $\Lambda \subset \mathbb{R}^n$ will be a uniformly discrete set and we set $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$.

Definition 8.1. We say that Λ is a gentle set if the distributional Fourier transform $\hat{\sigma}_\Lambda$ of σ_Λ is a Radon measure.

A lattice is a gentle set. A finite union of gentle sets is a gentle set. But the set Λ defined by (19) is not a gentle set. The calculation of the Fourier transform of σ_Λ is an amusing exercise. The set $\Lambda_{\alpha,\beta}$, which was defined in Section 2, is a gentle set, as was proved in [Meyer 2018a]. In this case $\hat{\sigma}_\Lambda$ is an atomic measure. This fact will be used later on.

A gentle set has a finite upper uniform density, which explains why the set defined by (19) is not a gentle set. Indeed let ϕ be a compactly supported continuous function whose Fourier transform $\hat{\phi}$ is nonnegative. Then for every $y \in \mathbb{R}^n$ we have $\int \hat{\phi}(y-x)\,d\sigma_\Lambda(x) = \int \exp(2\pi iy\cdot u)\phi(u)\,d\hat{\sigma}_\Lambda(u) = I(y)$ and $|I(y)| \leq C$ since $\hat{\sigma}_\Lambda$ is a Radon measure.

Let Λ be a gentle set. Then $\mu = \hat{\sigma}_\Lambda$ is a Radon measure. We set $w(y) = \int_{B_y} d|\mu|$, $y \in \mathbb{R}^n$, where B is the ball centered at 0 with radius 1 and $B_y = B + y$. The following theorem was proved in [Meyer 2018a]:

Theorem 8.1. *Let Λ be a gentle set. Let us assume that w has a polynomial growth at infinity and let $\omega \geq 1$ be a continuous and submultiplicative function which is a majorant of w. Then there exists a compact set K such that for every $f \in \mathcal{C}_\Lambda$ we* *have,*

$$\text{for all } y \in \mathbb{R}^n, \quad |f(y)| \le \omega(y) \sup_{u\in K} |f(u)|. \tag{29}$$

This estimate is optimal if $w \simeq \omega$. Indeed we let g be a continuous function supported by the unit ball and normalized by $\|g\|_\infty = 1$. We consider the convolution product $f = \mu * g$. This function belongs to $\mathcal{C}_\Lambda$ and (29) is satisfied. For a given x we have

$$\left| \int g(y-x)\, d\mu(y) \right| \le C'\omega(x), \tag{30}$$

where C' is the total mass of μ on K. We now take the supremum of the left-hand side with respect to g and obtain $w(x) = |\mu|(B_x) \le C'\omega(x)$.

The L^p version of this theorem is given now.

Theorem 8.2. *Let Λ be a gentle set. Let us assume that w has polynomial growth at infinity and let $\omega \ge 1$ be a continuous submultiplicative function such that $\omega \ge w$. Let $1 \le p \le \infty$. Then there exists a compact set K such that for every $f \in \mathcal{C}_\Lambda$ and every $y \in \mathbb{R}^n$ we have*

$$\left(\int_{K+y} |f(x)|^p\, dx \right)^{1/p} \le C\omega(y)^{|2-p|/p} \left(\int_K |f(x)|^p\, dx \right)^{1/p}. \tag{31}$$

Theorem 8.2 is sharp. Indeed the estimate given by (31) is optimal in many instances, as will be proved below. Therefore the L^p theory strongly differs from the L^2 theory. If $p = 2$, Theorem 8.2 is a trivial statement. Indeed Λ is assumed to be uniformly discrete and the L^2 theory is given for free. If $p = \infty$, Theorem 8.2 is identical to Theorem 8.1. The proof of Theorem 8.2 is obtained by interpolating between these two cases. But the Riesz–Thorin interpolation theorem is not true if the operator to which it is applied is restricted to a subspace $V \subset L^2 \cap L^\infty$. That is why we need to build the interpolation scheme on the whole of $L^2 \cap L^\infty$. To prepare the notation for the proof we define $\beta > 0$ by

$$\inf_{\{\lambda \ne \lambda' | \lambda, \lambda' \in \Lambda\}} |\lambda - \lambda'| = \beta > 0.$$

Let $0 < r < r' < \beta/2$, let B_r and B'_r be the balls centered at 0 with radii r and r' respectively. Let ϕ be a function in the Schwartz class $\mathcal{S}$ such that $\hat{\phi} = 1$ on B_r and $\hat{\phi} = 0$ outside B'_r. Let μ_y be the Radon measure μ translated by $-y$ and let $\chi_y(x) = \exp(2\pi i x y)$. Then the Fourier transform of the product $\nu_y = \phi\mu_y$ is the convolution product $\hat{\phi} * \chi_y \sigma_\Lambda$. The following lemma resumes this discussion:

Lemma 8.1. *We have*

$$\widehat{\phi\mu_y}(\xi) = \sum_{\lambda\in\Lambda} \exp(2\pi i \lambda \cdot y) \hat{\phi}(\xi - \lambda). \tag{32}$$

We now estimate the norm of the measure $\phi\mu_y$. We have

$$\|\nu_y\| \le C\omega(y). \tag{33}$$

This estimate results from the definition of w, the rapid decay of ϕ, and the slow growth of w.

The operator norm of the convolution with the measure $\nu_y = \phi\mu_y$ acting on $L^\infty(\mathbb{R}^n)$ does not exceed $\|\nu_y\| \le C\omega(y)$. The same bound is valid on $L^1(\mathbb{R}^n)$. On the other hand this convolution operator acts on $L^2(\mathbb{R}^n)$ with a norm not exceeding C. Indeed (34) shows that $\|\hat{\nu}_y\|_\infty \le C$ uniformly in y. An interpolation between L^2 and L^∞ or L^1 yields the following:

Lemma 8.2. *Let $p \in [1, \infty]$. Then we have, for every $y \in \mathbb{R}^n$ and every $f \in L^p$,*

$$\|\nu_y * f\|_p \le C\omega(y)^{|2/p-1|}\|f\|_p. \tag{34}$$

We now return to the proof of Theorem 8.2. Let g be a positive function in the Schwartz class whose Fourier transform is supported by the ball centered at 0 with radius r. This function g will be used to localize $P \in \mathcal{T}_\Lambda$. Let us set $P_y(x) = P(x+y)$. Then the product gP_y gives access to P around y.

Lemma 8.3. *For every $y \in \mathbb{R}^n$ we have*

$$gP_y = \nu_y * (Pg). \tag{35}$$

It suffices to prove Lemma 9.3 when $P(x) = \exp(2\pi i\lambda \cdot x)$, $\lambda \in \Lambda$. Then the Fourier transform of the left-hand side of (35) is $\exp(2\pi i\lambda \cdot y)\hat{g}(\xi - \lambda)$, while the Fourier transform of the right-hand side is $\exp(2\pi i\lambda \cdot y)\hat{\phi}(\xi - \lambda)\hat{g}(\xi - \lambda)$. But $\hat{\phi} = 1$ on the support of $\hat{g}$, which ends the proof of Lemma 9.3.

We now return to the proof of (31). For simplifying the notation let us set $\omega_p(y) = \omega(y)^{|2/p-1|}$. Then

Lemma 8.4. *For every $P \in \mathcal{T}_\Lambda$, every $y \in \mathbb{R}^n$, and every $R \ge 1$ we have*

$$\left(\int_{|x-y|\le R} |P(x)|^p\, dx\right)^{1/p} \le C_R\omega_p(y)\|Pg\|_p. \tag{36}$$

We have by (34) and (35)

$$\|P_y g\|_p = \|\nu_y * (Pg)\|_p \le C\omega_p(y)\|Pg\|_p. \tag{37}$$

Then (37) implies (36). Indeed it suffices to observe that $g(x) \ge c_R > 0$ on the ball centered at 0 with radius R.

If g were compactly supported, (36) would end the proof of Theorem 8.2. This is not the case but the problem can be easily fixed since g has a rapid decay at infinity. We now give the details of this argument.

Lemma 8.5. *Let Q_T be the cube defined by $|x_1| \le T, \dots, |x_n| \le T$, and let $\mathcal{R}_T = \mathbb{R}^n \setminus Q_T$. For every $\epsilon > 0$ there exists an integer $T \ge 1$ such that for every $P \in \mathcal{T}_\Lambda$ we have*

$$\left(\int_{\mathcal{R}_T} |Pg|^p \, dx\right)^{1/p} \le \epsilon \|Pg\|_p. \tag{38}$$

For proving this estimate we pave $\mathcal{R}_T$ by a disjoint union of cubes $Q^j,\ j \in \mathbb{N}$, of size 1. Then (38) implies $\int_{Q^j} |P|^p \, dx \le C\omega_p^p(x_j)\|Pg\|_p^p$, where x_j is the center of Q^j. Therefore

$$\int_{Q^j} |Pg|^p \, dx \le C\omega_p^p(x_j) \sup_{x \in Q_j} |g(x)|^p \, \|Pg\|_p^p. \tag{39}$$

Adding these estimates yields $\int_{\mathcal{R}_T} |Pg|^p \, dx \le C\sum_j \omega_p^p(x_j) \sup_{Q_j} |g|^p \, \|Pg\|_p^p \le \epsilon^p \|Pg\|_p^p$, which proves Lemma 9.5.

Corollary 8.1. *The three norms $\left(\int_{Q_T} |Pg|^p \, dx\right)^{1/p}$, $\left(\int_{Q_T} |P|^p \, dx\right)^{1/p}$, and $\|Pg\|_p$ are equivalent on $\mathcal{T}_\Lambda$ if T is large enough.*

Finally Corollary 8.1 and (3) imply Theorem 8.2.

8B. *Optimality.* In a special case which is detailed below, these estimates are optimal and this follows from a general result which is given now. Let us assume that $\hat{\sigma}_\Lambda$ is the atomic measure $\mu = \sum_0^\infty a_j \delta_{x_j}$ and, if $1 \le p \le \infty$, let $\omega_{p,R}(x) = \left(\sum_{|x - x_j| \le R} |a_j|^p\right)^{1/p}$. We further assume that there exists a constant C such that $\omega_{p,2R}(x) \le C\omega_{p,R}(x)$ holds true for $R \ge 1$. We write $\omega_{p,1} = \omega_p$.

Theorem 8.3. *Then for every compact set K and for every $y \in \mathbb{R}^n$ there exists a nontrivial $f \in \mathcal{C}_\Lambda$ such that*

$$\left(\int_{|x-y| \le 1} |f(x)|^p \, dx\right)^{1/p} \ge C_R \omega_p(y) \left(\int_K |f(x)|^p \, dx\right)^{1/p}. \tag{40}$$

Let us observe that $\omega_2 \simeq 1$ since Λ is uniformly discrete. It implies $\omega_p(y) \le C$ if $p \ge 2$. Moreover if $1 \le p \le 2$, Hölder's inequality yields

$$\omega_{p,R}(x) = \left(\sum_{|x-x_j| \le R} |a_j|^p\right)^{1/p} \le \left(\sum_{|x-x_j| \le R} |a_j|\right)^{(2-p)/p} \left(\sum_{|x-x_j| \le R} |a_j|^2\right)^{(p-1)/p}.$$

It shows that $\omega_{p,R}(x) \le C\omega(x)^{(2-p)/p}$. Therefore (31) and (40) are compatible. In general there is a gap between the upper bound given by (31) and the lower bound given by (40). But in some exceptional cases $\omega_{p,R}(x) \simeq \omega(x)^{(2-p)/p}$. An example is given below (Theorem 9.4). The proof of (40) mimics the argument used in the proof of Lemma 6.5 in [Matei and Meyer 2010]. Let $\epsilon > 0$ and let ϕ be an even compactly supported smooth function. We define $\phi_{\epsilon,p}(x) = \epsilon^{-n/p}\phi(x/\epsilon)$ and we have $\|\phi_{\epsilon,p}\|_p = \|\phi\|_p$. We consider the convolution product $f_{\epsilon,p} = \mu * \phi_{\epsilon,p}$. This function belongs to $\mathcal{C}_\Lambda$ and is our candidate to prove (40). Local L^p norms of $f_{\epsilon,p}$ are computed as follows:

Lemma 8.6. *Let a_j, $j \in \mathbb{N}$, be a sequence in l^1 and let $x_j \in \mathbb{R}^n$ be a sequence of pairwise disjoint points. Let K be a Riemann integrable compact set whose boundary ∂K does not contain any x_j. Then for $1 \le p \le \infty$ we have*

$$\lim_{\epsilon\to 0}\left(\int_K \left|\sum_j a_j \phi_{\epsilon,p}(x-x_j)\right|^p dx\right)^{1/p} = \left(\sum_{x_j\in K} |a_j|^p\right)^{1/p}. \tag{41}$$

Given $\eta > 0$ one fixes N such that $\sum_{N+1}^\infty |a_j| \le \eta$. The triangle inequality implies

$$\left(\int_K \left|\sum_{N+1}^\infty a_j \phi_{\epsilon,p}(x-x_j)\right|^p dx\right)^{1/p} \le \eta.$$

Next $\epsilon_N > 0$ is fixed such that the supports of $\phi_{\epsilon,p}(x-x_j)$, $x_j \in K$, $0 \le j \le N$, $0 < \epsilon \le \epsilon_N$, are pairwise disjoint. Then

$$\left(\int_K \left|\sum_0^N a_j \phi_{\epsilon,p}(x-x_j)\right|^p dx\right)^{1/p} = \left(\sum_{x_j\in K,\, 0\le j\le N} |a_j|^p\right)^{1/p}. \tag{42}$$

This ends the proof of Lemma 9.6.

If one the condition $x_j \notin \partial K$ is dropped, (41) shall be replaced by

$$\lim_{\epsilon\to 0}\left(\int_K \left|\sum_j a_j \phi_{\epsilon,p}(x-x_j)\right|^p dx\right)^{1/p} \ge \left(\sum_{x_j\in L} |a_j|^p\right)^{1/p}, \tag{43}$$

where L is any compact set contained in the interior of K. Finally (41) and (43) imply Theorem 8.3.

We illustrate these theorems by the one-dimensional example of the set $\Lambda_{\alpha,\beta}$, which was defined in Section 2. In this case the Fourier transform of the measure $\sigma_{\Lambda_{\alpha,\beta}}$ is an explicit atomic measure [Meyer 2018a, Proposition 6.1]. Then Theorems 8.2 and 8.3 and the explicit calculation in [Meyer 2018a] imply the following:

Theorem 8.4. *Let $1 \le p \le \infty$ and $\omega_p(x) = C(1+|x|)^{|1/p-1/2|}$. Then for every $f \in \mathcal{C}_{\Lambda_{\alpha,\beta}}$ and for every y we have*

$$\left(\int_{y-1}^{y+1} |f(x)|^p\, dx\right)^{1/p} \le \omega_p(y)\left(\int_{-1}^{1} |f(x)|^p\, dx\right)^{1/p} \tag{44}$$

and this estimate is optimal if $1 \le p \le 2$.

8C. L^4 *estimates.* In the preceding examples L^p estimates were provided by interpolation between L^∞ and L^2. Here is an example where we do not have L^∞ estimates but a direct access to L^4.

Theorem 8.5. *Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set such that $\Lambda + \Lambda$ is also a uniformly discrete set. Then there exists a compact set $K \subset \mathbb{R}^n$ and a constant C such that for every $P \in \mathcal{T}_\Lambda$ and every $y \in \mathbb{R}^n$ we have*

$$\left(\int_{K+y} |P(x)|^4\,dx\right)^{1/4} \le C\left(\int_K |P(x)|^4\,dx\right)^{1/4}. \tag{45}$$

The proof of Theorem 9.5 will be given after a few remarks. Let us observe that $\Lambda + \Lambda$ is uniformly discrete if and only if $\Lambda - \Lambda$ is uniformly discrete.

But this condition is not necessary for obtaining (45). A one-dimensional counterexample is given by the union $\Lambda = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 = \{2^k \mid k \in \mathbb{N}\}$ and $\Lambda_2 = \{2^k + r_k \mid k \in \mathbb{N}\}$, with $0 < r_1 < \cdots < r_k < r_{k+1} < \cdots < \frac{1}{2}$. The proof of this remark is based on the following lemma:

Lemma 8.7. *Using this notation, there exists a $T > 0$ such that the norms $\left(\int_0^T |P(x)|^2\,dx\right)^{1/2}$ and $\left(\int_0^T |P(x)|^4\,dx\right)^{1/4}$ are equivalent on $\mathcal{T}_\Lambda$.*

If Lemma 9.7 is admitted, (45) follows. Indeed Λ is uniformly discrete. Therefore Property 1.1 is satisfied if $K = [0, T]$ and if T is large enough. Then Lemma 9.7 implies (45).

We now prove Lemma 9.7. Every $P \in \mathcal{T}_\Lambda$ is a sum $P_1 + P_2$, where the spectrum of P_1 is contained in Λ_1 and the spectrum of P_2 in Λ_2. As was already observed, Property 1.1 is satisfied by Λ. Therefore there exists a $T \ge 1$ such that

$$\int_0^T |P(x)|^2\,dx \simeq \int_0^T |P_1(x)|^2\,dx + \int_0^T |P_2(x)|^2\,dx.$$

But

$$\left(\int_0^T |P_1(x)|^2\,dx\right)^{1/2} \simeq \left(\int_0^T |P_1(x)|^4\,dx\right)^{1/4}$$

since Λ_1 is a Sidon set. The same holds true for P_2. Finally

$$\left(\int_0^T |P(x)|^2\,dx\right)^{1/2} \simeq \left(\int_0^T |P(x)|^4\,dx\right)^{1/4}.$$

A Delone set is, by definition, a uniformly discrete set $\Lambda \subset \mathbb{R}^n$ which is relatively dense: for a compact ball B we have $\Lambda + B = \mathbb{R}^n$. We now return to the hypothesis in Theorem 9.5. If Λ and $\Lambda - \Lambda$ are Delone sets, then Λ is a Meyer set and Theorem 9.5 follows from known results on quasicrystals. Finally Theorem 9.5 is not a new fact when Λ is a Delone set.

For proving Theorem 9.5 one observes that the L^4 norm of P is the square root of the L^2 norm of $Q = P^2$. The spectrum of Q is contained in $M = \Lambda + \Lambda$. But M is uniformly discrete. Therefore Property 1.1 is satisfied for the pair (M, K) if K is a large enough ball. More precisely there exists a constant C such that for every $Q \in \mathcal{T}_M$ and every $y \in \mathbb{R}^n$ we have

$$\left(\int_{K+y} |Q(x)|^2\,dx\right)^{1/2} \le C\left(\int_K |Q(x)|^2\,dx\right)^{1/2}, \tag{46}$$

which is exactly (45).

Theorem 9.5 can be applied to the set Λ_θ defined in Section 5. If $\theta \ge 3$ then $\Lambda_\theta + \Lambda_\theta$ is uniformly discrete. Therefore (45) holds true and the arithmetical properties of θ do not play any role. It would be interesting to know if this conclusion remains valid when the assumption $\theta \ge 3$ is replaced by $\theta \ge 2$ and when the exponent 4 is replaced by any $p \in (1, \infty)$. Finally beyond quasicrystals there are many other examples of Delone sets Λ satisfying Property 1.4. These examples cannot be constructed by Theorem 9.5.

Acknowledgements

The anonymous referee was supportive and inspiring. This work was supported by a grant from the Simons Foundation (601950, YM).

References

[Avdonin 1974] S. A. Avdonin, "On the question of Riesz bases of exponential functions in L^2", *Vestnik Leningrad. Univ.* **13** (1974), 5–12. In Russian; translated in *Vestnik Leningrad Univ. Math.* **7** (1979), 203–211. MR Zbl

[Avdonin and Ivanov 1995] S. A. Avdonin and S. A. Ivanov, *Families of exponentials: the method of moments in controllability problems for distributed parameter systems*, Cambridge University Press, 1995. MR Zbl

[Beurling and Malliavin 1967] A. Beurling and P. Malliavin, "On the closure of characters and the zeros of entire functions", *Acta Math.* **118** (1967), 79–93. MR Zbl

[Grepstad and Lev 2014] S. Grepstad and N. Lev, "Universal sampling, quasicrystals and bounded remainder sets", *C. R. Math. Acad. Sci. Paris* **352**:7-8 (2014), 633–638. MR Zbl

[Ingham 1936] A. E. Ingham, "Some trigonometrical inequalities with applications to the theory of series", *Math. Z.* **41**:1 (1936), 367–379. MR Zbl

[Jaffard et al. 1997] S. Jaffard, M. Tucsnak, and E. Zuazua, "On a theorem of Ingham", *J. Fourier Anal. Appl.* **3**:5 (1997), 577–582. MR Zbl

[Kahane 1957] J.-P. Kahane, "Sur les fonctions moyenne-périodiques bornées", *Ann. Inst. Fourier, Grenoble* **7** (1957), 293–314. MR Zbl

[Kahane 1962] J.-P. Kahane, "Pseudo-périodicité et séries de Fourier lacunaires", *Ann. Sci. École Norm. Sup.* (3) **79** (1962), 93–150. MR Zbl

[Landau 1967] H. J. Landau, "Necessary density conditions for sampling and interpolation of certain entire functions", *Acta Math.* **117** (1967), 37–52. MR Zbl

[Lions 1984] J.-L. Lions, "Sur le contrôle ponctuel de systèmes hyperboliques ou du type Petrowski", pp. exposé 20 in *Goulaouic–Meyer–Schwartz seminar, 1983–1984*, École Polytech., Palaiseau, 1984. MR Zbl

[Matei and Meyer 2010] B. Matei and Y. Meyer, "Simple quasicrystals are sets of stable sampling", *Complex Var. Elliptic Equ.* **55**:8-10 (2010), 947–964. MR Zbl

[Meyer 1972] Y. Meyer, *Algebraic numbers and harmonic analysis*, North-Holland Mathematical Library **2**, North-Holland Publishing Co., Amsterdam, 1972. MR Zbl

[Meyer 1973] Y. Meyer, *Trois problèmes sur les sommes trigonométriques*, Astérisque **1**, Société Mathématique de France, Paris, 1973. MR Zbl

[Meyer 1974] Y. Meyer, "Théorie L^p des sommes trigonométriques apériodiques", *Ann. Inst. Fourier* (*Grenoble*) **24**:4 (1974), 189–211. MR Zbl

[Meyer 2018a] Y. Meyer, "Global and local estimates on trigonometric sums", *Skr. K. Nor. Vidensk. Selsk.* **2018** (2018).

[Meyer 2018b] Y. Meyer, "Mean periodic functions and irregular sampling", *Skr. K. Nor. Vidensk. Selsk.* **2018** (2018).

[Olevskii and Ulanovskii 2008] A. Olevskii and A. Ulanovskii, "Universal sampling and interpolation of band-limited signals", *Geom. Funct. Anal.* **18**:3 (2008), 1029–1052. MR Zbl

[Paley and Wiener 1934] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, American Mathematical Society Colloquium Publications **19**, American Mathematical Society, New York, 1934. Zbl

Received 15 Jan 2019. Revised 24 Jul 2019.

YVES MEYER:

yves.meyer@cmla.ens-cachan.fr
CMLA, ENS-Cachan, CNRS, Université Paris-Saclay, France

Tunisian Journal of Mathematics

msp.org/tunis

The Tunisian Journal of Mathematics is an international publication organized by the Tunisian Mathematical Society (http://www.tms.rnu.tn) and published in electronic and print formats by MSP in Berkeley.

See inside back cover or msp.org/tunis for submission instructions.

The subscription price for 2020 is US $320/year for the electronic version, and $380/year (+$20, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Tunisian Journal of Mathematics (ISSN 2576-7666 electronic, 2576-7658 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

TJM peer review and production are managed by EditFlow® from MSP.

Tunisian Journal of Mathematics

2020 vol. 2 no. 4