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STATIONARY COMPRESSIBLE NAVIER–STOKES EQUATIONS WITH INFLOW CONDITION IN DOMAINS WITH PIECEWISE ANALYTICAL BOUNDARIES

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We show the existence of strong solutions in Sobolev–Slobodetskii spaces to the stationary compressible Navier–Stokes equations with inflow boundary condition. Our result holds provided a certain condition on the shape of the boundary around the points where characteristics of the continuity equation are tangent to the boundary, which holds in particular for piecewise analytical boundaries. The mentioned situation creates a singularity which limits regularity at such points. We show the existence and uniqueness of regular solutions in a vicinity of given laminar solutions under the assumption that the pressure is a linear function of the density. The proofs require the language of suitable fractional Sobolev spaces. In other words our result is an example where the application of fractional spaces is irreplaceable, although the subject is a classical system.

1. Introduction

We investigate the existence of regular solutions to stationary barotropic compressible Navier–Stokes equations in a two-dimensional bounded domain Ω with nonzero inflow/outflow through the boundary. The complete system reads

$$\rho v \cdot \nabla v - \mu \Delta v - (\mu + \nu)\nabla \operatorname{div} v + \nabla \pi(\rho) = 0 \quad \text{in } \Omega, \tag{1a}$$

$$\operatorname{div}(\rho v) = 0 \quad \text{in } \Omega, \tag{1b}$$

$$n \cdot 2\mu \boldsymbol{D}(v) \cdot \tau + f v \cdot \tau = b, \quad \text{on } \Gamma, \tag{1c}$$

$$n \cdot v = d \quad \text{on } \Gamma, \tag{1d}$$

$$\rho = \rho_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \tag{1e}$$

where the velocity field of the fluid v and the density ρ are the unknown functions describing the flow. We distinguish the parts of the boundary

$$\begin{aligned} \Gamma_{\text{in}} &= \{x \in \Gamma : d < 0\}, \quad \Gamma_{\text{out}} = \{x \in \Gamma : d > 0\}, \\ \Gamma_0 &= \{x \in \Gamma : d = 0\}, \quad \Gamma_* = \bar{\Gamma}_0 \cap \overline{\Gamma_{\text{in}} \cup \Gamma_{\text{out}}}. \end{aligned} \tag{2}$$

We show the existence of a solution in fractional Sobolev spaces $u \in W^{1+s}_p(\Omega)$, $\rho \in W^s_p(\Omega)$, where u is the velocity field of the fluid and ρ is the density. Our choice of functional spaces allows us to overcome the problem of singularity in the continuity equation and obtain boundedness of the density. Before we formulate the problem more precisely we give a brief overview of recent developments in this topic,

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focusing on the scope of interest of this paper, that is, on regular stationary solutions, mentioning also the most important results concerning global weak stationary solutions. For more complete overview of known results in the mathematical theory of compressible flows we refer to [Novotný and Straškraba 2004; Plotnikov and Sokołowski 2012].

The mathematical theory of stationary solutions to the Navier–Stokes equations describing compressible flows started to develop in early 80s with certain results on the existence of regular solutions, first in Hilbert spaces [Valli 1983] and later in L_p framework [Beirão da Veiga 1987]. However, all of these results required certain smallness assumptions on the data and concerned mostly homogeneous boundary conditions with vanishing normal component of the velocity.

The famous result of [Lions 1998] on the existence of weak solutions for homogeneous Dirichlet boundary conditions triggered the development of the global existence theory of weak solutions. The result was improved by Novo and Novotný [2002], who adopted the nonsteady approach of [Feireisl et al. 2001], and then it was extended to the case of slip boundary conditions in the barotropic case in [Mucha and Pokorný 2006; Pokorný and Mucha 2008] and for the full system, including thermal effects in [Mucha and Pokorný 2009]; see also the result for a system involving radiation effects in [Kreml et al. 2013]. Further improvements in the theory of regular solutions were made in [Novotný and Padula 1994; Novotný and Pileckas 1998] but mostly for homogeneous boundary data.

It should be emphasized that all above-mentioned global results concern the case of the normal component of the velocity vanishing on the boundary. If the normal component of the velocity does not vanish, substantial mathematical difficulties arise in the analysis of the continuity equation, which can be reduced to a stationary transport equation. Namely, the hyperbolicity of the continuity equation makes it necessary to prescribe the density on the part of the boundary where the fluid enters the domain, called briefly the inflow part. Solvability of either a time-dependent or stationary transport equation is of utmost importance in the mathematical analysis of the Navier–Stokes equations for compressible flow. For a recent application of the theory of the transport equation in the context of the existence of weak solutions to the compressible Navier–Stokes equation we can refer to [Bresch and Jabin 2018]. Important developments in the theory of the transport equation, strengthening the classical results of [DiPerna and Lions 1989], were made in [Crippa and De Lellis 2008; Mucha 2010].

The mathematical investigation of inflow/outflow problems began with Valli and Zajączkowski [1986], who investigated the time-dependent problem obtaining also an existence result in the stationary case. Then the development of existence theory for inhomogeneous boundary data was hindered by both mathematical difficulties and a shift in interest toward global existence of weak solutions, until [Kweon and Kellogg 1997]. More recently, the existence theory has been developed, motivated by applications in shape optimization, by Plotnikov, Ruban and Sokolowski [Plotnikov et al. 2008; 2009]; see also [Plotnikov and Sokołowski 2012]. All above results require certain smallness assumptions. Concerning large data problems, there are only few particular results on the global existence of weak solutions for nonstationary problems; see [Girinon 2011]. In the stationary case, due to nontrivial boundary terms it has been for a long time impossible to get basic a priori estimates and further problems are encountered with the issue of existence and uniqueness for the continuity equation. The first global existence result was obtained very recently by Feireisl and Novotný [2018], under the assumption that the pressure is a nondecreasing C^1 function of the density satisfying $\lim_{\rho\to\bar\rho} p(\rho) = +\infty$ for some positive constant $\bar\rho$. The proof is based on appropriate regularization of both continuity and momentum equations. The key estimates for the approximate systems are obtained using a suitable extension of the boundary velocity which is constructed in such a way that it satisfies a certain smallness condition even though the data can be arbitrarily large.

At first glance a natural functional space for regular solutions is W^1_p for the density and W^2_p for the velocity. A regular solution is then understood as a function with weak derivatives satisfying the equations almost everywhere. However, except for some special classes of domains, we are not able to obtain the solutions in the above class for arbitrarily large p; see [Kweon and Kellogg 1997]. The reason is a singularity arising in the solution of the steady transport equation around the points where characteristics of this hyperbolic equation become tangent to the boundary; we refer to these points as singularity points.

On the other hand, the range $p > n$ is important since it gives boundedness of the density due to the imbedding theorem. The results from [Kweon and Kellogg 1997] cover a part of this range, namely $2 < p < 3$. However, a further increase of p is impossible even under relaxation of the boundary singularity. Further investigation of this singularity is therefore an interesting question in view of the development of the theory of regular solutions.

One possible way to obtain existence for $n < p < \infty$ is to investigate some special domains, such as a cylindrical domain in [Mucha and Piasecki 2014; Piasecki 2009; 2010; Guo et al. 2015] for the barotropic case and [Piasecki and Pokorný 2014] for a system with thermal effects or an unbounded domain contained between two parallel planes in [Kweon and Kellogg 1998]. A possible way to overcome the singularity problem described above in a general domain is an appropriate choice of functional spaces. In [Plotnikov et al. 2008; 2009] the existence and uniqueness of solutions in fractional Sobolev spaces (velocity in W^{1+s}_p and density in W^s_p) is shown under certain assumption relating the inflow velocity and shape of the boundary around the singularity points. However, this result requires an additional assumption that the gradient of the density and the second gradient of the velocity are in L_2.

In this paper we show existence of solutions in fractional Sobolev spaces as above. However we do not require the existence of $\nabla\rho$ and $\nabla^2 u$. Our analysis shows that we are able to show existence of the solutions for $sp > n$, which gives boundedness of the density. We need to impose a certain limitation on the boundary around the singularity points; however this assumption is weaker than in [Kweon and Kellogg 1997; Plotnikov et al. 2008; 2009] and turns out to be quite natural; in particular it is satisfied by analytical boundaries.

The only result giving uniqueness of solutions to the compressible Navier–Stokes equations for large data without information on $\nabla\rho$ is [Hoff 2006] for a time-dependent problem. The key idea there is to show uniqueness in quite low regularity, namely L_2 for the velocity and H^{-1} for the density. Then we have to estimate the H^{-1}-norm of the pressure with H^{-1}-norm of the density and for this purpose it is required that the pressure is a linear function of the density (or satisfies slightly more general constraint — see (1.16) in [Hoff 2006]).

Our result, which is to our knowledge the first one giving uniqueness of solutions without any information on the gradient of the density in the stationary case, combines the ideas from [Hoff 2006] in the context of uniqueness with the approach used to obtain existence of regular solutions in the series of papers [Piasecki 2010; Mucha and Piasecki 2014; Piasecki and Pokorný 2014]. It shows that the choice of fractional Sobolev spaces is in a sense natural for the considered problem and therefore is not only of purely mathematical interest. In particular it may indicate a possible direction for the development of the theory of global existence.

1.1. ***Functional spaces.*** In order to formulate more precisely the problem and main result we shall first recall the definitions of the functional spaces in which we work. We use standard Sobolev spaces W_p^k with natural k, which consist of functions with weak derivatives up to order k in $L_p(\Omega)$; for the definition we refer for example to [Adams and Fournier 2003]. However, most important for our result are Sobolev–Slobodetskii spaces W_p^s with fractional s. For the sake of completeness we recall the definition here; see [Triebel 1978, Section 4.4]. By $W_p^s(\Omega)$ we denote the space of functions for which the norm

$$\|f\|_{W_p^s(\Omega)} = \left(\int_\Omega |f|^p\,dx\right)^{1/p} + \left(\iint_{\Omega^2} \frac{|f(x)-f(y)|^p}{|x-y|^{2+sp}}\,dx\,dy\right)^{1/p} \tag{3}$$

is finite and $s \in (0,1)$. Furthermore, $W_p^{1+s}(\Omega)$ is a space of functions with first weak derivatives in $W_p^s(\Omega)$ with the norm

$$\|f\|_{W_p^{1+s}(\Omega)} = \|f\|_{W_p^s(\Omega)} + \|\nabla f\|_{W_p^s(\Omega)}. \tag{4}$$

Let us recall two important features of Sobolev–Slobodetskii spaces. We formulate them in a simplified way convenient for our applications.

Fact. Assume $\Omega \subset \mathbb{R}^2$ be bounded with $\partial\Omega \in C^2$, Let $u \in W_p^s$ with $sp > n$. Then for all $q \le \infty$, $\delta > 0$

$$\|u\|_{L_q} \le C(q,\Omega)\|u\|_{W_p^s}, \tag{5}$$

$$\|u\|_{L_q} \le \delta\|u\|_{W_p^s} + C(\delta)\|u\|_{L_2}. \tag{6}$$

The proofs of more general versions of the above facts can be found in [Triebel 1978]. Furthermore, for given $\epsilon > 0$ let us define

$$\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \epsilon\}.$$

Then by $B^r_{p,\infty}(\Omega)$ we denote a space of functions for which (see [Triebel 1978, Section 4.4] or [DeVore and Sharpley 1993, Section 2])

$$\|u\|_{B^r_{p,\infty}(\Omega)} = \sup_{0<|h|\le h_0} |h|^{-r}\|u(\cdot+h)-u(\cdot)\|_{L_p(\Omega_{|h|})} < \infty, \tag{7}$$

where $h_0 = \sup\{|h| : |\Omega_h| > 0\}$. Then for Ω bounded with $\partial\Omega \in C^2$ we have (see [Triebel 1978, Section 4.6])

$$W_p^r(\Omega) \subset B^r_{p,\infty}(\Omega). \tag{8}$$

Finally, let us define

$$V = \{v \in W_2^1(\Omega) : v \cdot n|_\Gamma = 0\}. \tag{9}$$

1.2. ***Problem formulation.*** Let us move to a precise statement of the problem under consideration. We investigate stationary flow of a barotropic fluid in a two-dimensional, bounded domain described by the system (1). The system is supplied with inhomogeneous slip boundary conditions on the velocity. In particular, the normal component of the velocity does not vanish and, as explained above, we have to prescribe the density on the part of the boundary where the flow enters the domain.

Let us have a closer look at the definition of different parts of the boundary (2). We see that Γ_* consists of points where the characteristics of the continuity equation (1b) become tangent to the boundary; we will call it the set of singularity points. Moreover, let Γ_{in}^s be a certain boundary neighborhood of the set of singularity points. Formally we define it as

$$\Gamma_{\text{in}}^s = \{x \in \Gamma_{\text{in}} : \operatorname{dist}(x, \Gamma_*) < 2\eta\}, \tag{10}$$

where $\eta > 0$ is a small number to be made precise later.

Our goal is to show the existence of a solution $(u, \rho) \in W_p^{1+s} \times W_p^s$ to the system (1), where $s > n/p$, which is close to the constant flow

$$(\bar{v}, \bar{\rho}) \equiv ([v^*, 0], 1), \tag{11}$$

where v^* is a positive constant. Our method works for a wider class of solutions in which x_1 is in a sense a dominating direction. Our motivation for the choice of this fractional-order space for the density has been explained above; we want to solve the problem of singularity in the solution of the continuity equation around the singularity points. On the other hand, by the imbedding theorem we have $W_p^s(\Omega) \in L_\infty(\Omega)$. Then the choice of the space W_p^{1+s} for the velocity follows naturally from the structure of (1). Obviously such a solution no longer satisfies the equations almost everywhere. Therefore in order to define the solutions we need a weak formulation of the problem (1). A natural way is to multiply (1a) by $\psi \in V$ and (1b) by a smooth function ϕ such that $\phi|_{\Gamma \setminus \Gamma_{\text{in}}=0}$, integrate by parts and apply boundary conditions (1c), (1d). However, because of inhomogeneous condition (1d) we would obtain boundary terms with derivatives of v. Therefore we first remove the inhomogeneity and introduce a weak formulation of the perturbed problem (30). We obtain the following definition:

Definition 1. A strong solution to the system (1) is a pair $(v, \rho) \in W_p^{1+s} \times W_p^s$ such that $v = \bar{v} + u + u_0$ and $\rho = \bar{\rho} + w$, where (u, w) is a solution to the system (30) in the sense of Definition 7 and u_0 is defined in (29).

In order to formulate our main result let us introduce the following quantity to measure the distance of the data of the problem (1) from $(\bar{v}, \bar{\rho})$:

$$D_0 = \|b - f\tau_1\|_{W_p^{s-1/p}(\Gamma)} + \|d - n_1\|_{W_p^{1+s-1/p}(\Gamma)} + \|\rho_{\text{in}} - 1\|_{W_p^s(\Gamma_{\text{in}})}. \tag{12}$$

As the formulation our main result involves certain properties of the boundary, it is useful to describe first the domain, introducing the necessary assumptions.

1.3. ***The domain*****:** ***representative case.*** Conducting the proof for a general domain with multiple singularity points would lead to unnecessary complications which would likely hide the main ideas. For clarity

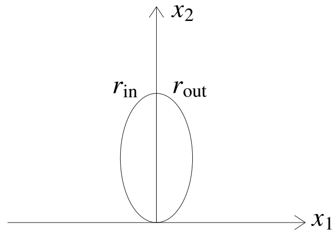

Figure 1. The domain: simple representative case with two singularity points.

of the proof we consider a simple domain with inflow and outflow parts of the boundary given by

$$\begin{aligned} \Gamma_{\text{in}} &= \{(\underline{x}_1(x_2), x_2) : x_2 \in (0, b)\}, \\ \Gamma_{\text{out}} &= \{(\bar{x}_1(x_2), x_2) : x_2 \in (0, b)\}, \end{aligned} \tag{13}$$

with

$$\lim_{x_2 \to 0^+} \underline{x}_1(x_2) = \lim_{x_2 \to b^-} \underline{x}_1(x_2) = \lim_{x_2 \to 0^+} \bar{x}_1(x_2) = \lim_{x_2 \to b^-} \bar{x}_1(x_2) = 0 \tag{14}$$

and

$$\begin{aligned} \lim_{x_2 \to 0^+} \underline{x}_1'(x_2) = \lim_{x_2 \to b^-} \bar{x}_1'(x_2) = -\infty, \\ \lim_{x_2 \to 0^+} \bar{x}_1'(x_2) = \lim_{x_2 \to b^-} \underline{x}_1'(x_2) = +\infty. \end{aligned} \tag{15}$$

Then we have two singularity points:

$$\Gamma_* = \Gamma_0 = \{(0, 0), (0, b)\}.$$

We assume further that these are the only singularity points; that is, the are no singularity points "inside" Γ_{in} and Γ_{out}. An example of such domain is shown in Figure 1. It is well known and was already mentioned in the introduction that existence of regular solutions to the inflow problem (1) requires certain assumptions on the shape of the boundary around the singularity points. We also need an assumption of this kind; in order to formulate it notice that around each singularity point the boundary is given as a function $x_2(x_1)$. We assume it satisfies the condition

$$\begin{aligned} &\text{there exists } N \in \mathbb{N} \text{ such that } |x_2(x_1) - x_2(y_1)| \geq C|x_1 - y_1|^N \\ &\qquad \text{for all } x_1, y_1 \text{ such that } (x_1, x_2(x_1)), (y_1, x_2(y_1)) \in \Gamma_{\text{in}}^s. \end{aligned} \tag{16}$$

We emphasize that the above condition is required only on the inflow part, therefore it can be rewritten as

$$\text{there exists } \delta > 0 \text{ such that } |\underline{x}_1(x_2) - \underline{x}_1(y_2)| \leq C|x_2 - y_2|^\delta, \quad \text{with } \delta = \frac{1}{N}. \tag{17}$$

Remark 2. Notice that if (16) is satisfied by some N^* then it is also satisfied for $N > N^*$ for small $|x_2 - y_2|$. This condition means that the boundary around a singularity point must be less flat than some polynomial. Moreover we shall emphasize that (16) is a necessary but not sufficient condition; we also require sufficient global regularity of the boundary in order to solve an auxiliary elliptic problem. In order to avoid additional technicalities we assume that Ω is a C^3 domain. Such regularity is obviously not assured by (16); in particular a Lipschitz boundary may satisfy (16) for any $N \geq 1$.

Remark 3. A similar domain is considered in [Kweon and Kellogg 1997] and it is worth comparing to the condition (17) with a similar constraint there with $\delta \geq \frac{1}{2}$. Therefore our condition is clearly weaker and means that the boundary around the singularity points is less flat than some polynomial.

Another interesting observation concerning the condition (16) is that a polynomial behavior of the boundary near the singularity points turns out to be important in a completely different context of singularity of boundary layers in the stationary Munk equation; see [Dalibard and Saint-Raymond 2018].

Although condition (17) seems quite technical in the above formulation, it is in fact satisfied by a wide class of functions, in particular by piecewise analytical boundaries, which is shown in the following lemma:

Lemma 4. *Assume that x_2 is an analytic function of x_1 around the singularity points. Then* (16) *holds.*

Proof. By (15) we have $x_2'(x_1) = 0$ at the singularity points. Therefore it is enough to show that if $f : \mathbb{R} \to \mathbb{R}$ is analytic in some $[-r, r]$, $f(0) = f'(0) = 0$, and $f \neq 0$ then

$$|f(x)| \geq C|x|^N \quad \text{for } x \in (-l, l) \tag{18}$$

for some $C > 0$, $N \geq 2$, and $l < r$ sufficiently small. Since $f \neq 0$ and f is analytic, we must have $f^{(n)}(0) \neq 0$ for some $n \geq 2$. Let $f^{(k)}(0)$ be the first derivative not vanishing in 0. Then we have

$$f(x) = \frac{f^{(k)}(0)}{k!} x^k + R^{k+1}(x),$$

where $|R^{k+1}(x)| \leq M|x|^{k+1}$ for $x \in (-r, r)$. Hence

$$|f(x)| \geq \left| \frac{f^k(0)}{k!} \right| |x|^k - M|x|^{k+1} = \left(\frac{f^k(0)}{k!} - M|x| \right) |x|^k. \qquad \square$$

1.4. *The domain: general case.* Our result holds for a wide class of domains where the inflow and outflow parts are defined in a natural way. A general setting is to consider inflow and outflow described as

$$\begin{aligned} \Gamma_{\text{in}} &= \{(\underline{x}_1(x_2), x_2) : x_2 \in (a, b)\}, \\ \Gamma_{\text{out}} &= \{(\bar{x}_1(x_2), x_2) : x_2 \in (a, b)\}, \end{aligned} \tag{19}$$

with singularity points given by $\{(\underline{x}_1(x_2), x_2) : x_2 = k_{\text{in}}^i\}$ and $\{(\bar{x}_1(x_2), x_2) : x_2 = k_{\text{out}}^j\}$, where

$$\lim_{x_2 \to k_{\text{in}}^{i-}} |\underline{x}_1'(x_2)| = \lim_{x_2 \to k_{\text{in}}^{i+}} |\underline{x}_1'(x_2)| = \infty,$$
$$\lim_{x_2 \to k_{\text{out}}^{j-}} |\bar{x}_1'(x_2)| = \lim_{x_2 \to k_{\text{out}}^{j+}} |\bar{x}_1'(x_2)| = \infty.$$

Furthermore we assume

$$\begin{aligned} \lim_{x_2 \to a^+} \underline{x}_1(x_2) &= c_1, & \lim_{x_2 \to b^-} \underline{x}_1(x_2) &= c_2, \\ \lim_{x_2 \to a^+} \bar{x}_1(x_2) &= d_1, & \lim_{x_2 \to b^-} \bar{x}_1(x_2) &= d_2, \end{aligned} \tag{20}$$

with $c_i \leq d_i$. Then we have

$$\Gamma_0 = (c_1, d_1) \times \{a\} \cup (c_2, d_2) \times \{b\}. \tag{21}$$

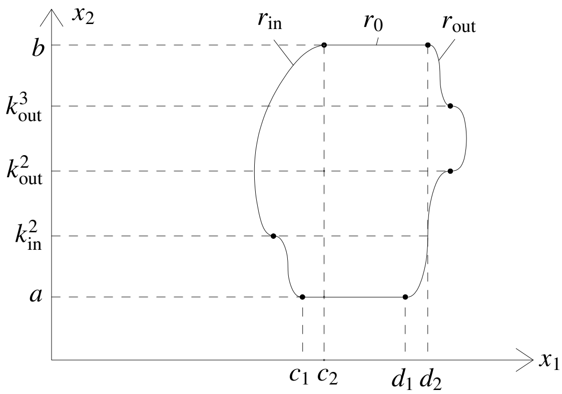

Figure 2. The domain: general case.

An example of an above-described general domain is shown in Figure 2. Again, around singularity points x_2 is given as a function $x_2^i(x_1)$ and we assume these functions satisfy (16).

1.5. *Main result.*

We are now in a position to formulate our main result.

Theorem 5. *Assume that Ω satisfies the conditions introduced above, in particular (17). Assume the following regularity of the boundary data:*

$$b \in W_p^{s-1/p}(\Gamma), \quad d \in W_p^{1+s-1/p}(\Gamma), \tag{22}$$

$$\rho_{\text{in}} \in W_p^s(\Gamma_{\text{in}}) \cap W_p^r(\Gamma_{\text{in}}^s), \tag{23}$$

where

$$s < \delta, \quad r > \frac{s}{\delta}, \quad sp > n. \tag{24}$$

Let the pressure be in the form

$$\pi(\rho) = K\rho \tag{25}$$

for some positive constant K. Let the viscosity μ be sufficiently large compared to $|\Omega|$, $\|\bar{v}\|_{L_\infty}$ and K. Assume further that the boundary data satisfy the following additional assumption:

$$|(d - n_1)|\sqrt{1 + |\underline{x}_1'(x_2)|^2} \le \kappa < 1. \tag{A1}$$

Assume that D_0 defined in (12) is small enough and let f be large enough on Γ_{in}. Then there exists a solution $(v, \rho) \in W_p^{1+s} \times W_p^s$ to the system (1) such that

$$\|v - \bar{v}\|_{W_p^{1+s}} + \|\rho - \bar{\rho}\|_{W_p^s} \le E(D_0), \tag{26}$$

where $E(\cdot)$ is a Lipschitz function, $E(0) = 0$. Moreover, this solution is unique in the class of solutions satisfying (26).

Remark 6. Condition (23) is required to obtain the estimate for the density near the singularity points; the details are shown in the proofs of Lemmas 15 and 17. Therefore, for δ given by the geometry of the domain in (17) we can choose an arbitrarily small r provided we choose a sufficiently small s, but we have to compensate for it with sufficiently large p. Note that in particular we can assume

$$\rho_{\text{in}} \in W^s_p(\Gamma_{\text{in}}) \cap W^1_p(\Gamma^s_{\text{in}}). \tag{27}$$

The rest of the paper is organized as follows. In the remainder of the present section we reformulate the problem (1) introducing perturbations as new unknowns, obtaining system (30). Then we recall basic properties of the functional spaces we use. In Section 2 we introduce the linearization of (30) and show a priori estimates. First we show the energy estimate. Then in Section 2.2 we move to the estimate in W^s_p for the steady transport equation, which is the main difficulty in the proof. This result makes it possible to conclude the estimate in $W^{1+s}_p \times W^s_p$ in Sections 2.3 and 2.4. In Section 3 we prove Theorem 5 with an iterative scheme using our estimates for the linear system to show the convergence of the sequence of approximations. Finally we finish with a short concluding section. Without loss of generality we assume in the proofs $v^* = 1$ in (11), except in the proof of Proposition 23 where we need to track the dependence of the viscosity on this constant.

To remove inhomogeneity from the boundary condition (30d), we construct $u_0 \in W^{1+s}_p$ such that

$$u_0 \cdot n|_\Gamma = d - n_1, \quad \|u_0\|_{W^{1+s}_p} \le C\|d - n_1\|_{W^{1+s-1/p}_p(\Gamma)}. \tag{28}$$

For our purpose we find u_0 as a solution to the problem

$$-\mu\Delta u_0 - (\mu+\nu)\nabla \operatorname{div} u_0 = 0, \quad u_0 \cdot n = d - n_1. \tag{29}$$

In order to construct u_0 we supply (29) with the condition $u_0 \cdot \tau = 0$ and define $u_0 = u^1_0 + u^2_0$, where $u^1_0 \in W^{1+s}_p$ is any extension of the boundary data $d - n_1$ and u^2_0 solves

$$-\mu\Delta u^2_0 - (\mu+\nu)\nabla \operatorname{div} u^2_0 = \mu\Delta u^1_0 + (\mu+\nu)\nabla \operatorname{div} u^1_0, \quad u^2_0|_\Gamma = 0.$$

Introducing the perturbations

$$u = v - \bar{v} - u_0 \quad \text{and} \quad w = \rho - \bar{\rho},$$

we obtain the system

$$\begin{aligned}
\partial_{x_1} u - \mu\Delta u - (\nu+\mu)\nabla \operatorname{div}\, u + K\,\nabla w &= F(u,w) && \text{in } \Omega, && (30a)\\
\operatorname{div}\, u + \partial_{x_1} w + (u+u_0)\cdot\nabla w &= G(u,w) && \text{in } \Omega, && (30b)\\
n \cdot 2\mu \boldsymbol{D}(u)\cdot\tau + f\, u\cdot\tau &= B && \text{on } \Gamma, && (30c)\\
n\cdot u &= 0 && \text{on } \Gamma, && (30d)\\
w &= w_{\text{in}} && \text{on } \Gamma_{\text{in}}, && (30e)
\end{aligned}$$

where

$$\begin{aligned}
F(u,w) &= -w\,(u+\bar{v}+u_0)\cdot\nabla(u+u_0) - \partial_{x_1} u_0 - (u+u_0)\cdot\nabla(u+u_0),\\
G(u,w) &= -(w+1)\ \operatorname{div} u_0 - w\ \operatorname{div} u,\\
B &= b - f\tau^{(1)} - 2\mu\, n\cdot \boldsymbol{D}(u_0)\cdot\tau.
\end{aligned} \tag{31}$$

Notice that in general $F(u, w)$ contains a term $\mu\Delta u_0 + (\nu+\mu)\nabla \operatorname{div} u_0$, which vanishes due to our definition of u_0. It can be seen easily that

$$\begin{aligned}\|F(u,w)\|_{L_p} &\le C[\|w\|_{W_p^s}\|u\|_{W_p^{1+s}} + E(\|\nabla u, u, w\|_{L_p}+1)],\\ \|G(u,w)\|_{W_p^s} &\le C[\|w\|_{W_p^s}\|u\|_{W_p^{1+s}} + E(\|w\|_{W_p^s}+1)],\\ \|B\|_{W_p^{s-1/p}} &\le C[\|b-f\tau\|_{W_p^{s-1/p}} + \|d-n_1\|_{W_p^{1+s-1/p}(\Gamma)}],\end{aligned} \tag{32}$$

where E is a small constant dependent on $\|u_0\|_{W_p^{1+s}}$ and V^* is a dual space to V defined in (9).

From now on we focus on the system (30). Our goal is to show the existence of a solution $(u, w) \in W_p^{1+s} \times W_p^s$ for given small function $u_0 \in W_p^{1+s}$. Recall in particular that the solution to the system (30) is used in the definition of the solution to the original problem (1). In order to define the solution to (30) we need its weak formulation. For this purpose we apply the identity

$$\begin{aligned}&\int_\Omega (-\mu\Delta u - (\nu+\mu)\nabla \operatorname{div} u)\cdot v\,dx\\ &\quad= \int_\Omega 2\mu \boldsymbol{D}(u) : \nabla v + \nu \operatorname{div} u \operatorname{div} v\,dx - \int_\Gamma n\cdot[2\mu\boldsymbol{D}(u)]\cdot v\,d\sigma - \int_\Gamma n\cdot[\nu(\operatorname{div} u)\mathbf{Id}]\cdot v\,d\sigma.\end{aligned} \tag{33}$$

Then a natural definition is the following:

Definition 7. A regular W_p^s solution to the problem (30) is a pair $(u, w) \in W_p^{1+s} \times W_p^s$ such that

$$\begin{aligned}&\int_\Omega \psi\,\partial_{x_1} u\,dx + \int_\Omega (2\mu\boldsymbol{D}(u):\nabla\psi + \nu \operatorname{div} u \operatorname{div}\psi)\,dx\\ &\quad+ \int_\Gamma [f(u\cdot\tau)(\psi\cdot\tau) - b(\psi\cdot\tau)]\,d\sigma - \int_\Omega w \operatorname{div}\psi\,dx = \int_\Omega F(u,w)\cdot\psi\,dx \quad \text{for all } \psi\in V\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\int_\Omega \phi \operatorname{div} u\,dx - \int_\Omega w((\bar v + u + u_0)\cdot\nabla\phi + \phi \operatorname{div}(u+u_0))\,dx + \int_{\Gamma_{\text{in}}} w_{\text{in}}\phi\,d\sigma = \int_\Omega G(u,w)\phi\,dx\\ &\qquad\qquad \text{for all } \phi\in C^1(\Omega) \text{ such that } \phi|_{\Gamma\setminus\Gamma_{\text{in}}} = 0.\end{aligned} \tag{35}$$

2. Linearization and a priori bounds

In this section we derive a priori estimates for the following linearization of system (30):

$$\partial_{x_1} u - \mu\Delta u - (\nu+\mu)\nabla \operatorname{div} u + K\,\nabla w = F \quad \text{in } \Omega, \tag{36a}$$

$$\operatorname{div} u + \partial_{x_1} w + U\cdot\nabla w = G \quad \text{in } \Omega, \tag{36b}$$

$$n\cdot 2\mu\boldsymbol{D}(u)\cdot\tau + f\,u\cdot\tau = B \quad \text{on } \Gamma, \tag{36c}$$

$$n\cdot u = 0 \quad \text{on } \Gamma, \tag{36d}$$

$$w = w_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \tag{36e}$$

where $U \in W_p^{1+s}$ is small and satisfies $U\cdot n|_\Gamma = d - n_1$ and on the right-hand side we have

$$F \in L_p(\Omega), \quad G \in W_p^s(\Omega), \quad \text{and} \quad B \in W_p^{s-1/p}(\Gamma). \tag{37}$$

We start with the energy estimate, then we deal with the steady transport equation, which is the main difficulty of our proof, and finally we show the estimate in the solution space.

2.1. ***Energy estimate.*** In this section we show an energy estimate for the solutions of (36).

Lemma 8. *Let (u, w) be a sufficiently smooth solution to system (36) with given functions $(F, G, B) \in (V^* \times L_2 \times L_2(\Gamma))$. Then*

$$\|u\|_{W_2^1} + \|w\|_{L_2} \le C[\|F\|_{V^*} + \|G\|_{L_2} + \|B\|_{L^2(\partial\Omega)}], \tag{38}$$

where C is independent from the boundary data.

Proof. Multiplying (36a) by u and integrating over Ω we get, using the boundary condition (36c) and the identity (33),

$$\int_\Omega 2\mu \boldsymbol{D}^2(u) + \nu \operatorname{div}^2 u\, dx + \int_{\partial\Omega} \Big(f + \frac{n_1}{2}\Big) u^2\, d\sigma + \int_\Omega K \nabla w u\, dx \\ = -\int_\Omega u\, \partial_{x_1} u\, dx + \int_\Omega F u\, dx + \int_{\partial\Omega} B(u\cdot\tau)\, d\sigma. \tag{39}$$

The boundary term on the left-hand side will be positive for $f \ge 0$ on Γ_{out} and $f \ge n_1/2$ on Γ_{in}. Next we integrate by parts the last term of the left-hand side of (39). Using (36b) we obtain

$$K\int_\Omega u\nabla w\, dx = -K\int_\Omega w \operatorname{div} u\, dx = K\int_\Omega (w\partial_{x_1} w + U\cdot w\nabla w - Gw)\, dx \\ = K\Big[\frac{1}{2}\int_\Gamma w^2 n_1\, d\sigma - \frac{1}{2}\int_\Omega w^2 \operatorname{div}\, U - \int Gw\, dx\Big]. \tag{40}$$

We will also use the Korn inequality

$$\int_\Omega 2\mu \mathbf{D}^2(u) + \nu \operatorname{div}^2 u\, dx \ge C_K \|u\|^2_{W_2^1(\Omega)}, \tag{41}$$

where $C_K = C_K(\Omega)$. Using (39), (40) and (41) we get

$$\|u\|^2_{W_2^1(\Omega)} + \int_{\Gamma_{\text{out}}} w^2 n_1\, d\sigma \le C\Big[\int_\Omega w^2 \operatorname{div}\, U\, dx + \gamma\int_\Omega Gw\, dx + \int_\Omega Fu\, dx + \int_\Gamma B(u\cdot\tau)\, d\sigma + \int_{\Gamma_{\text{in}}} w_{\text{in}}^2 n_1\, d\sigma\Big].$$

Next, using Hölder and Young inequalities, the fact that $w^2 n_1 > 0$ on Γ_{out}, and the trace theorem on the boundary term, we get for any $\delta > 0$

$$\|u\|^2_{W_2^1(\Omega)} \le (\delta + \|U\|_{W_p^{1+s}(\Omega)})\|w\|^2_{L_2(\Omega)} + C(\delta)\|G\|^2_{L_2(\Omega)} + C[(\|F\|_{V^*} + \|B\|_{L_2(\Gamma)})\|u\|_{W_2^1(\Omega)}],$$

which yields

$$\|u\|_{W_2^1(\Omega)} \le (\delta + \|U\|_{W_p^{1+s}(\Omega)})\|w\|_{L_2(\Omega)} + C(\delta)\|G\|_{L_2(\Omega)} + C(\|F\|_{V^*} + \|B\|_{L_2(\Gamma)}). \tag{42}$$

To estimate the first term of the right-hand side we find a bound on $\|w\|_{L_2}$. From (36b) we have

$$\partial_{x_1} w + U\cdot\nabla w = G - \operatorname{div}\, u.$$

In order to estimate $\|w\|_{L_2}$, for $x \in \Omega$ let us denote by γ_x a characteristic of the operator $\partial_{x_1} + U \cdot \nabla$ connecting x with Γ_{in}, i.e., a solution to

$$\begin{aligned} \dot{\gamma}_x(t) &= -(1 - U_1(\gamma_x(t)), U_2(\gamma_x(t))), \\ \gamma_x(0) &= x. \end{aligned} \tag{43}$$

Notice that we take the tangential vector with opposite sign since we consider a backwards characteristic starting at a given point inside Ω. Denote by $x^{\mathrm{in}}(x)$ the intersection of γ_x with Γ_{in}. Due to regularity and the smallness of U, γ_x is close to a straight line $\{x_2 = c\}$. Now we can write

$$w(x) = w_{\mathrm{in}}(x^{\mathrm{in}}(x)) + \int_{\gamma_x} (G - \operatorname{div} u)\, dl_{\gamma_x}. \tag{44}$$

By Jensen's inequality we have

$$\left(\int_{\gamma} (G - \operatorname{div} u)\, dl_{\gamma_x} \right)^2 \leq |\gamma_x| \int_{\gamma_x} |G|^2 + |\operatorname{div} u|^2\, dl_{\gamma_x}.$$

Hence applying (44) we get

$$\|w\|_{L_2(\Omega)} \leq C(\Omega)(\|w_{\mathrm{in}}\|_{L_2(\Gamma_{\mathrm{in}})} + \|G\|_{L_2(\Omega)} + \|u\|_{W_2^1(\Omega)}). \tag{45}$$

Now we combine (42) and (45). By the smallness of U we can fix δ in (42) small enough to put the term $\delta + \|U\|_{W_p^{1+s}}$ on the left, obtaining (38), which completes the proof of the lemma. □

2.2. *Steady transport equation.* In this section we show the estimate in W_p^s for the steady transport equation with inflow condition, which is a crucial step in showing an a priori estimate for the linear problem (36).

Proposition 9. *Let Ω be a set defined at the end of Section 1, satisfying (17). Let w solve*

$$\bar{K} w + w_{x_1} + U \cdot \nabla w = H, \quad w|_{\Gamma_{\mathrm{in}}} = w_{\mathrm{in}}, \tag{46}$$

where $\bar{K}$ is a positive constant, $U \in W_\infty^1(\Omega)$ is small and satisfies $U \cdot n|_\Gamma = d - n_1$, where d and n_1 satisfy (A1). Assume $H \in W_p^s(\Omega)$ and let w_{in}, s, and p satisfy the assumptions of Theorem 5. Then

$$\|w\|_{W_p^s(\Omega)} \leq C[\|H\|_{W_p^s(\Omega)} + \|w_{\mathrm{in}}\|_{W_p^s(\Gamma_{\mathrm{in}}) \cap W_p^r(\Gamma_s)}], \tag{47}$$

where $C = C(s, p, \Omega)$ and r is from (24).

Remark 10. Notice that the assumption $U \in W_\infty^1(\Omega)$ is weaker than $U \in W_p^{1+s}(\Omega)$ for $sp > 2$ due to the imbedding theorem.

Remark 11. Proposition 9 generalizes the result from [Piasecki 2018], where it is assumed $U = [u^1, 0]$, which enables a much simpler proof than the one presented below and no additional regularity of the density around the singularity points is required.

For simplicity we set $\bar{K} = 1$, which is allowed as we assume anyway sufficient smallness of the data. The proof of Proposition 9 is quite technical since we have to treat carefully the boundary terms. For the reader's convenience we divide it into several lemmas.

Lemma 12. *Let the assumptions of Proposition 9 hold. Then*

$$\|w\|_{W^s_p(\Omega)} + \frac{1}{p}\iint_{\Omega^2} \frac{\partial_{x_1}|w(x)-w(y)|^p + \partial_{y_1}|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\,dx\,dy$$
$$+\frac{1}{p}\iint_{\Omega^2} \frac{U(x)\cdot\nabla_x|w(x)-w(y)|^p + U(y)\cdot\nabla_y|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\,dx\,dy \le \|H\|_{W^s_p(\Omega)}\|w\|^{p-1}_{W^s_p}. \tag{48}$$

Proof. Recalling the definition of Sobolev–Slobodetskii norm we write (46) in x and y. Using identities of the type $\nabla_x w(y)=0$ we can write

$$w(x)+\partial_{x_1}[w(x)-w(y)]+U(x)\cdot\nabla_x[w(x)-w(y)] = H(x),$$
$$w(y)+\partial_{y_1}[w(y)-w(x)]+U(y)\cdot\nabla_y[w(y)-w(x)] = H(y).$$

We multiply the first equation by

$$\frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{\phi_\epsilon(x,y)}$$

and the second by

$$\frac{|w(x)-w(y)|^{p-2}(w(y)-w(x))}{\phi_\epsilon(x,y)},$$

where

$$\phi_\epsilon(x,y)=\epsilon+|x-y|^{2+sp}.$$

Then we add the equations and integrate twice over Ω, with respect to x and y. Since

$$w(x)(w(x)-w(y))+w(y)(w(y)-w(x)) = [w(x)-w(y)]^2$$

and

$$H(x)(w(x)-w(y))+H(y)(w(y)-w(x)) = (H(x)-H(y))(w(x)-w(y)),$$

we obtain on the left-hand side

$$\iint_{\Omega^2}\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\,dx\,dy \to_{\epsilon\to 0} \|w\|^p_{W^s_p}. \tag{49}$$

On the right-hand side we have using the Hölder inequality

$$\iint_{\Omega^2}\frac{(H(x)-H(y))|w(x)-w(y)|^{p-2}(w(x)-w(y))}{\phi_\epsilon(x,y)}$$
$$\le \left(\iint_{\Omega^2}\frac{|H(x)-H(y)|^p}{\phi_\epsilon(x,y)}\right)^{1/p}\left(\iint_{\Omega^2}\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\right)^{1-1/p} \to_{\epsilon\to 0} \|H\|_{W^s_p}\|w\|^{p-1}_{W^s_p}, \tag{50}$$

which completes the proof of (48). □

The rest of the proof of Proposition 9 consists in dealing with the integral terms in (48). This is where all the difficulties are hidden and our assumptions on the boundary and boundary data will come into play. We start with observing that under the assumptions of Proposition 9 we have

$$n_1 + U\cdot n \ge 0 \quad \text{on } \Gamma_{\text{out}}. \tag{51}$$

Indeed, we have

$$n(x)|_{\Gamma_{\rm out}} = \frac{[1, -\bar{x}_1'(x_2)]}{\sqrt{1+[\bar{x}_1'(x_2)]^2}}, \quad n(x)|_{\Gamma_{\rm in}} = \frac{[-1, \underline{x}_1'(x_2)]}{\sqrt{1+[\underline{x}_1'(x_2)]^2}}; \tag{52}$$

therefore by (A1)

$$\left|\frac{U\cdot n}{n_1}\right| = \sqrt{1+[\bar{x}_1'(x_2)]^2}\,|U\cdot n| \le \kappa,$$

and so (51) holds.

We now proceed with transforming (48). Since we want to have a W^s_p-norm, we have to get rid of the derivatives of w and for this purpose we integrate by parts. This step is presented in the following lemma:

Lemma 13. *Let the assumptions of Proposition 9 be satisfied. Then*

$$\|w\|^p_{W^s_p(\Omega)} + \frac{2}{p}\int_\Omega\int_{\Gamma_{\rm in}} \frac{|w(x)-w(y)|^p(n_1+U\cdot n)(x)}{\phi_\epsilon}\,dS(x)\,dy \le \|H\|_{W^s_p(\Omega)}\|w\|^{p-1}_{W^s_p(\Omega)} + C\|\nabla U\|_{L_\infty(\Omega)}\|w\|^p_{W^s_p(\Omega)}. \tag{53}$$

Proof. We will obtain (53) from (48). Let us start with the first integral term on the left-hand side of (48). We have

$$\frac{\partial_{x_1}|w(x)-w(y)|^p}{\phi_\epsilon(x,y)} = \partial_{x_1}\left(\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\right) - |w(x)-w(y)|^p\partial_{x_1}\left(\frac{1}{\phi_\epsilon(x,y)}\right). \tag{54}$$

By the definition of $\phi_\epsilon(x,y)$ we have

$$\nabla_x\phi_\epsilon(x,y) = -\nabla_y\phi_\epsilon(x,y); \tag{55}$$

therefore using (54) we get

$$\begin{aligned}\iint_{\Omega^2}&\left\{\frac{\partial_{x_1}|w(x)-w(y)|^p}{\phi_\epsilon(x,y)} + \frac{\partial_{y_1}|w(y)-w(x)|^p}{\phi_\epsilon(x,y)}\right\}dx\,dy\\ &= \iint_{\Omega^2}\left\{\partial_{x_1}\left(\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\right) + \partial_{y_1}\left(\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\right)\right\}dx\,dy\\ &\quad - \iint_{\Omega^2}\left\{|w(x)-w(y)|^p\partial_{x_1}\left(\frac{1}{\phi_\epsilon(x,y)}\right) + |w(x)-w(y)|^p\partial_{y_1}\left(\frac{1}{\phi_\epsilon(x,y)}\right)\right\}dx\,dy.\end{aligned} \tag{56}$$

Taking into account (55), the integrand in the second integral on the right-hand side of (56) vanishes identically. Combining (55) with the identities

$$\nabla_x|w(x)-w(y)|^p = p|w(x)-w(y)|^{p-2}(w(x)-w(y))\nabla_x w(x), \tag{57}$$

$$\nabla_y|w(x)-w(y)|^p = -p|w(x)-w(y)|^{p-2}(w(x)-w(y))\nabla_y w(y), \tag{58}$$

we see that the first integral on the right-hand side of (56) adds up to

$$\frac{2}{p}\iint_{\Omega^2}\partial_{x_1}\left(\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\right)dx\,dy = \frac{2}{p}\int_\Omega\int_\Gamma \frac{|w(x)-w(y)|^p n_1}{\phi_\epsilon(x,y)}\,dS(x)\,dy. \tag{59}$$

Now consider the second integral on the left-hand side of (48). We have

$$\iint_{\Omega^2} U(x)\frac{\nabla_x|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\,dx\,dy = \iint_{\Omega^2} U(x)\nabla_x\left(\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\right)dx\,dy - \iint_{\Omega^2} U(x)|w(x)-w(y)|^p\nabla_x\left(\frac{1}{\phi_\epsilon(x,y)}\right)dx\,dy \tag{60}$$

and

$$\iint_{\Omega^2} U(y)\frac{\nabla_y|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\,dx\,dy = \iint_{\Omega^2} U(y)\nabla_y\left(\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\right)dx\,dy - \iint_{\Omega^2} U(y)|w(x)-w(y)|^p\nabla_y\left(\frac{1}{\phi_\epsilon(x,y)}\right)dx\,dy. \tag{61}$$

Recalling (55) we see that the terms with $\nabla(1/\phi)$ cancel. But this time the integrand does not vanish identically and we have to check whether the sum of these integrals makes sense when $\epsilon \to 0$. This sum equals

$$\left|\iint_{\Omega^2}[U(x)-U(y)]|w(x)-w(y)|^p\frac{(2+sp)|x-y|^{sp}(x-y)}{(\epsilon+|x-y|^{2+sp})^2}\,dx\,dy\right| \longrightarrow_{\epsilon\to 0} (2+sp)\left|\iint_{\Omega^2}\frac{U(x)-U(y)}{x-y}\frac{|w(x)-w(y)|^p}{|x-y|^{2+sp}}\,dx\,dy\right| \le C\|\nabla U\|_{L_\infty}\|w\|^p_{W^s_p}.$$

Now consider the terms with $\nabla|w(x)-w(y)|^p/\phi_\epsilon$ on the right-hand side of (60) and (61). Combining (57) and (58) with (55) we see that these terms add up to

$$\frac{2}{p}\iint_{\Omega^2} U(x)\nabla_x\left(\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\right)dx\,dy = -\int_\Omega dy\int_\Omega\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\operatorname{div}_x U(x)\,dx + \int_\Omega dy\int_\Gamma\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}U(x)\cdot n\,dS(x). \tag{62}$$

The first integral is straightforward:

$$\left|\int_\Omega dy\int_\Omega\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x,y)}\operatorname{div}_x U(x)\,dx\right| \le C(\Omega)\|\nabla U\|_{L_\infty(\Omega)}\|w\|^p_{W^s_p(\Omega)}. \tag{63}$$

Combining (48), (59), (62) and (63) we get (53). □

In order to complete the proof of Proposition 9 it remains to find an estimate on the integral term in (53). Notice that it is sufficient to consider the integral over Γ_{in} since the outflow part is nonnegative due to (51) and therefore we already skipped it in (53). However, the inflow term is negative because of n_1 and therefore must be estimated. The treatment of this term is quite technical and in fact constitutes the core of the proof of Proposition 9. Let us start with introducing some further notation. In the estimate which will follow, $y = (y_1, y_2)$ will denote a point inside Ω, while $\underline{x} = (\underline{x}_1, \underline{x}_2)$ is a point on Γ_{in}. In several places we will replace an integral with respect to the boundary measure on Γ_{in} by an integral with respect to x_2. Then we will write $\underline{x}(x_2) = (\underline{x}_1(x_2), x_2)$. At this point we should also fix η in the definition (10). For this purpose observe that (A1) implies

$$|(d-n_1)|\sqrt{1+|\underline{x}_1'(x_2)|^2} \le \frac{\tilde{\kappa}}{\sqrt{1+\lambda^2}} \tag{64}$$

for some $\kappa < \tilde{\kappa} < 1$ and $\lambda > 0$. Now in (10) we choose η such that

$$|x_2'(x_1)| < \lambda \quad \text{on } \Gamma_{\text{in}}^s. \tag{65}$$

Notice that such η exists due to assumed regularity of the boundary. Next we define

$$\begin{aligned}
&\Gamma_{\text{in}}^r := \{x \in \Gamma_{\text{in}} :\ \eta < x_2 < b-\eta\},\\
&\Omega_{\text{in}}^r := \{x \in \Omega :\ \eta < x_2 < b-\eta,\ x_1 - \underline{x}_1(x_2) < \eta\},\\
&\Omega_{\text{in}}^s := \{x \in \Omega :\ x_2 < 2\eta \ \vee\ b-2\eta < x < b\},\\
&\Omega_{\text{in}} = \Omega_{\text{in}}^r \cup \Omega_{\text{in}}^s.
\end{aligned} \tag{66}$$

Therefore, Γ_{in}^r and Ω_{in}^r are, respectively, a part of Γ_{in} and its neighborhood, which are at some fixed distance from singularity points. In particular, we have

$$|\underline{x}_1'(x_2)| \le C(\eta) \quad \text{for } (\underline{x}_1(x_2), x_2) \in \Gamma_{\text{in}}^r. \tag{67}$$

Notice also that $\Gamma_{\text{in}}^s = \Omega_{\text{in}}^s \cap \Gamma_{\text{in}}$. Next, for $y = (y_1, y_2)$ let us denote by γ_y a characteristic curve of the operator $(\partial_{x_1} + U \cdot \nabla)$ connecting y with Γ_{in}, i.e., a solution to (43) with $\gamma_y(0) = y$. Furthermore, we denote the intersection of γ_y with Γ_{in} by $y^{\text{in}}(y)$ (see Figure 3). We will need the following auxiliary result:

Lemma 14. *Assume that*

$$y \in \Omega_{\text{in}}^r, \quad \underline{x} \in \Gamma_{\text{in}}^r. \tag{68}$$

Then

$$|\underline{x} - y^{\text{in}}(y)| + |y^{\text{in}}(y) - y| \le C|\underline{x} - y|, \tag{69}$$

$$|\gamma_y| \le (1+E)|y_1 - \underline{x}_1(y_2)| \tag{70}$$

for some constant $C > 0$ and a small constant E.

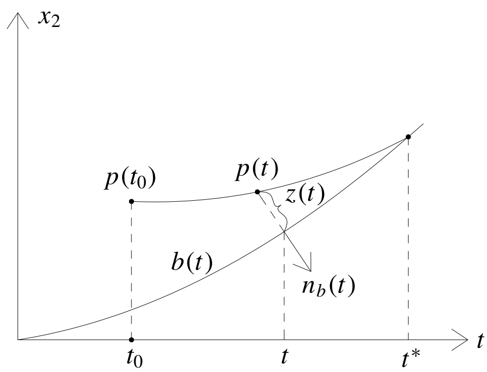

Figure 3. Explanation of notation used in Lemma 15.

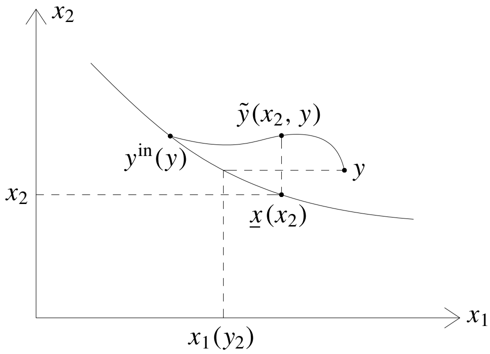

Figure 4. Illustration of notation. Case (b): intersection point $\tilde{y}$ well-defined.

Proof. Without loss of generality we can restrict to assuming that x_2 is a decreasing function of x_1 on Γ_{in} as the opposite case is analogous. Then it is convenient to distinguish three cases:

$$\begin{aligned} &\text{(a)}\ \ \underline{x}_1 \le y_1^{\mathrm{in}}(y) && \text{(see Figure 4).} \\ &\text{(b)}\ \ y_1^{\mathrm{in}}(y) < \underline{x}_1 < y_1 && \text{(see Figure 3).} \\ &\text{(c)}\ \ \underline{x}_1 \ge y_1. \end{aligned} \tag{71}$$

In the case (a) (69) is obvious with $C = 2$ as $|\underline{x} - y^{\mathrm{in}}(y)| \le |\underline{x} - y|$ and $|y^{\mathrm{in}}(y) - y| \le |\underline{x} - y|$. The case (c) is treated similarly to (b); therefore we don't show it on a separate figure and focus on the proof for (b). In that case it is convenient to define by $\tilde{y}(x_2, y)$ the intersection of a line $\{(\underline{x}_1(x_2), t) : t \in \mathbb{R}\}$ with a characteristic γ_y connecting Γ_{in} with y. Such an intersection is well-defined in the case (b); see Figure 3. We have for some $\vartheta \in [\underline{x}_2, y_2]$

$$\begin{aligned} |\underline{x} - y^{\mathrm{in}}(y)| &\le C(|\underline{x}_1 - y_1^{\mathrm{in}}(y)| + |\underline{x}_2 - y_2^{\mathrm{in}}(y)|) \\ &= (C\underline{x}_1'(\vartheta) + 1)|\underline{x}_2 - y_2^{\mathrm{in}}(y)| \\ &\le C|\underline{x}_2 - \tilde{y}_2(x_2, y)| \le C|\underline{x} - \tilde{y}(x_2, y)| \le C|\underline{x} - y|, \end{aligned} \tag{72}$$

where we have used (67). Therefore also

$$|y^{\mathrm{in}}(y) - y| \le |y^{\mathrm{in}}(y) - \underline{x}| + |\underline{x} - y| \le C|\underline{x} - y|,$$

which completes the proof of the first assertion. To show the second one we can assume without loss of generality that $y_2^{\mathrm{in}}(y) \ge y_2$. We have

$$|y_2^{\mathrm{in}}(y) - y_2| = \int_{y_1^{\mathrm{in}}(y)}^{y_1} U_2(t, \gamma(t))\, dt \le \|U\|_\infty |y_1^{\mathrm{in}}(y) - y_1|;$$

therefore using again (67) we get

$$\begin{aligned} |y_1 - y_1^{\mathrm{in}}(y)| &\le |y_1 - \underline{x}_1(y_2)| + |\underline{x}_1(y_2) - y_1^{\mathrm{in}}(y)| + |y_2^{\mathrm{in}}(y) - y_2| \\ &\le |y_1 - \underline{x}_1(y_2)| + (|\underline{x}_1'(\vartheta)| + 1)|y_2^{\mathrm{in}}(y) - y_2| \\ &\le |y_1 - \underline{x}_1(y_2)| + (|\underline{x}_1'(\vartheta)| + 1)\|U_2\|_\infty |y_1 - y_1^{\mathrm{in}}(y)|, \end{aligned}$$

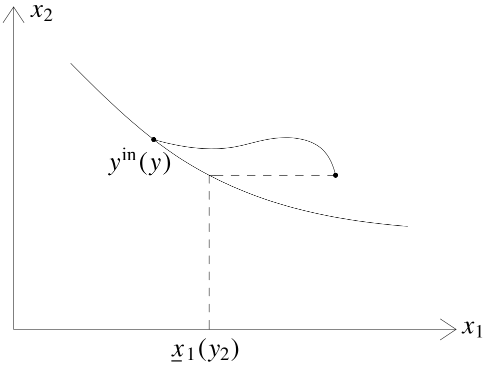

Figure 5. Case (a): no intersection point $\tilde{y}$.

so for sufficiently small $\|U\|_\infty$ we obtain

$$|y_1 - y_1^{\text{in}}(y)| \le (1+E)|y_1 - \underline{x}_1(y_2)|,$$

and to complete the proof it is enough to note that

$$|\gamma_y| = \int_{y_1^{\text{in}}(y)}^{y_1} \sqrt{[1+U_1(\gamma_y(s))]^2 + (U_2(\gamma_y(s)))^2}\, ds \le (1+E)|y_1 - y_1^{\text{in}}(y)|. \qquad \square$$

We shall emphasize that (69) holds except in a neighborhood of singularity points. The point is that $|\underline{x}_1'(x_2)|$ is unbounded as we approach the singularity. Therefore, as

$$|\underline{x}(x_2) - y^{\text{in}}(y_2)| \ge |\underline{x}_1(x_2) - y_1^{\text{in}}(y_2)| = |\underline{x}_1'(\vartheta)||x_2 - y_2| \tag{73}$$

for some $\vartheta \in [x_2, y_2]$, we no longer have (69). In order to control $|\underline{x} - y^{\text{in}}(y)|$ we need an appropriate estimate on a length of a characteristic connecting y with the boundary and here the assumptions (16) and (A1) come into play. The required result is:

Lemma 15. *Let $b(t) = (t, \underline{x}_2(-t))$ and let $p(t)$ satisfy*

$$\dot{p}(t) = (1 + U^1(p(t)), U^2(p(t))), \quad p(t_0) = x_2^0, \tag{74}$$

where x_2^0 is a given, small positive number, t_0 is sufficiently small so that

$$p(s) > b(s) \quad \textit{for } s \le t_0,$$

and U satisfies the assumptions of Proposition 9. Let t_* denote the first coordinate of the first intersection of $p(t)$ and $b(t)$ (see Figure 5). Then

$$t_* - t_0 \le C(x_2^0 - b(t_0))^\delta,$$

where δ is from (17) and $C = C(U, \Omega)$.

Remark 16. We will apply the above lemma with $t = -x_1$ in some neighborhood of a singularity point $(0, 0)$ in the proof of Lemma 17 which will follow; therefore we consider a “backward” characteristic of (46).

Proof of Lemma 15. Assume first that $t_0 = 0$. Let us define

$$A(t) = \sqrt{1 + |\underline{x}_2'(t)|^2}.$$

Then the unit normal to $b(t)$ is $n_b(t) = A(t)^{-1}(-\underline{x}_2'(t), 1)$. Let $z(t)$ denote the distance between $b(t)$ and $p(t)$ measured along $n_b(t)$ (see Figure 5); that is,

$$p(t) = b(t) + n_b(t)z(t).$$

Differentiating this identity in t and multiplying the resulting equation by $n_b(t)$ we get

$$\dot{p}(t) \cdot n_b(t) = \dot{z}(t), \tag{75}$$

which by (74) can be rewritten as

$$\dot{z}(t) = U \cdot n_b(t, z) - \frac{\underline{x}_2'(t)}{A(t)}. \tag{76}$$

We have

$$|U \cdot n_b(t, z)| \le |U \cdot n_b(t, 0)| + \|\nabla U\|_{L_\infty} z;$$

therefore by (76)

$$\dot{z}(t) \le |U \cdot n_b(t, 0)| - \frac{\underline{x}_2'(t)}{A(t)} + \|\nabla U\|_{L_\infty} z(t). \tag{77}$$

Now, due to (64) and (65) we have

$$|U \cdot n_b(t, 0)| \le \frac{\tilde{\kappa}|\underline{x}_2'(t)|}{\sqrt{1+\lambda^2}} < \tilde{\kappa}\frac{|\underline{x}_2'(t)|}{A(t)}; \tag{78}$$

therefore

$$\dot{z}(t) \le -(1-\tilde{\kappa})\frac{\underline{x}_2'(t)}{A(t)} + E_U z(t) \le -\vartheta + E_U z(t), \tag{79}$$

with $\vartheta = (1-\tilde{\kappa})/\sqrt{1+\lambda^2}$ and $E_U = \|\nabla U\|_{L_\infty}$. Now t_* must satisfy

$$\int_0^{t^*} \dot{z}(t)\, dt = -x_2^0; \quad \text{therefore} \quad x_2^0 = \vartheta \underline{x}_2(t_*) - E_U \underbrace{\int_0^{t_*} z(t)\, dt}_{Z(t_*)}. \tag{80}$$

In order to simplify the above condition observe that

$$z(t) \le x_2^0 e^{E_U t} \le C x_2^0, \quad p(t) - b(t) \le C z(t),$$

and therefore

$$Z(t_*) \le t \sup_{s \le t}[p(t) - b(t)] \le C_2 t_* x_2^0. \tag{81}$$

Moreover, by (16) we have

$$\underline{x}_2(t_*) \ge C_1 t_*^N. \tag{82}$$

In view of (81) and (82), if we define $\tilde{t}_*$ by

$$P_{x_2^0}(\tilde{t}_*) := C_1 \vartheta \tilde{t}_*^N - C_2 E_U x_2^0 \tilde{t}_* - x_2^0 = 0,$$

then, because of (80), (81) and (82),

$$t_* \leq \tilde{t}_*. \tag{83}$$

We claim that there exists $1 \leq C_3 = C_3(C_1, C_2, \Omega)$ such that if $t_1 = C_3(x_2^0)^{1/N}$ then $P(t_1) \geq 0$. Indeed, we have

$$P(t_1) = x_2^0[C_1\vartheta C_3 - 1 - C_2 E_U (C_3 x_2^0)^{1/N}] \geq C_3(C_1 - C_2 E_U (x_2^0)^{1/N}) - 1 > 0$$

for sufficiently large $C_3 = C_3(C_1, C_2, \Omega)$. Therefore,

$$\tilde{t}_* \leq t_1 \leq C_3(x_2^0)^{1/N},$$

which together with (83) completes the proof for $t_0 = 0$ as $\delta = 1/N$. For $t_0 > 0$ the reasoning is the same modulo minor adjustments, replacing x_2^0 with $-z(t_0)$ in (80). We are now in a position to prove the estimate on the boundary term on the left-hand side of (53).

Lemma 17. *Let the assumptions of Proposition 9 hold. Then*

$$\int_\Omega dy \int_{\Gamma_{\text{in}}} \frac{|w(x) - w(y)|^p (n_1 + U \cdot n)(x)}{\phi_\epsilon(x, y)}\, dS(x) \\ \leq C[\|H\|_{L_\infty(\Omega)} + \|w\|_{L_p(\Omega)} + \|w\|_{L_\infty(\Omega)} + \|w_{\text{in}}\|_{W_p^s(\Gamma_{\text{in}})} + \|w_{\text{in}}\|_{W_p^r(\Gamma_{\text{in}}^s)}]^p. \tag{84}$$

Proof. First of all, due to (52) and (A1) we have

$$\left|\frac{U \cdot n}{n_1}\right| \leq \kappa,$$

which gives

$$|n_1 + U \cdot n| \leq (1 + \kappa) n_1. \tag{85}$$

Therefore it is enough to show (84) for $U \cdot n = 0$. As

$$\begin{aligned} dS(x_2)|_{\Gamma_{\text{out}}} &= \sqrt{1 + [\bar{x}_1'(x_2)]^2}\, dx_2, \\ dS(x_2)|_{\Gamma_{\text{in}}} &= \sqrt{1 + [\underline{x}_1'(x_2)]^2}\, dx_2, \end{aligned} \tag{86}$$

(52) yields

$$dS(x_2)|_{\Gamma_{\text{in}}} = -\frac{dx_2}{n_1}, \quad n_2 dS(x_2)|_{\Gamma_{\text{in}}} = \underline{x}_1'(x_2) dx_2. \tag{87}$$

By (87), assuming $U \cdot n = 0$ the left-hand side of (84) can be rewritten as

$$\int_\Omega \int_{\Gamma_{\text{in}}} \frac{|w(x) - w(y)|^p n_1(x)}{\phi_\epsilon(x, y)}\, dS(x)\, dy = \int_\Omega dy \underbrace{\int_0^b -\frac{|w(\underline{x}_1(x_2), x_2) - w(y)|^p}{\phi_\epsilon((\underline{x}_1(x_2), x_2), y)}\, dx_2}_{I_\epsilon(y)}, \tag{88}$$

and it remains to show the estimate (84) for (88). When we take $\epsilon \to 0$, we can expect some problems only when y is close to Γ_{in}. Therefore we write (88) as

$$\int_\Omega I_\epsilon(y)\, dy = \int_{\Omega_{\text{in}}^r} I_\epsilon(y)\, dy + \int_{\Omega_{\text{in}}^s} I_\epsilon(y)\, dy + \int_{\Omega \setminus \Omega_{\text{in}}} I_\epsilon(y)\, dy =: I_1^r + I_1^s + I_2,$$

where the sets Ω^r_{in}, Ω^s_{in} and Ω_{in} are defined in (66). The set $\Omega \setminus \Omega_{\text{in}}$ consists of points at the distance from Γ_{in} bounded from below by η. Therefore the I_2 integral is straightforward as we have $|(\underline{x}_1(x_2), x_2) - y| > \eta$ for $y \in \Omega \setminus \Omega_{\text{in}}$ and so

$$I_2 \le C(\eta)[\|w_{\text{in}}\|_{L_p(\Gamma_{\text{in}})} + \|w\|_{L_p}]^p. \tag{89}$$

The estimates for I^r_2 and I^s_1 are more involved; we show them in detail.

Estimate of I^r_1: Let $\underline{x}(x_2) = (\underline{x}_1(x_2), x_2) \in \Gamma_{\text{in}}$. We have

$$\frac{|w(\underline{x}(x_2)) - w(y)|^p}{|\underline{x}(x_2) - y|^{2+sp}} \le C(p)\left[\frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y))|^p}{|\underline{x}(x_2) - y|^{2+sp}} + \frac{|w(y^{\text{in}}(y)) - w(y)|^p}{|\underline{x}(x_2) - y|^{2+sp}}\right]. \tag{90}$$

For the first term on the right-hand side of (90) we have

$$\begin{aligned}\int_{\Omega^r_{\text{in}}} dy \int_0^b &\frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y))|^p}{|\underline{x}(x_2) - y|^{2+sp}}\, dx_2 \\ &= \int_{2\eta}^{b-2\eta} dx_2 \left[\int_{V_\eta(x_2)} \frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y))|^p}{|\underline{x}(x_2) - y|^{2+sp}}\, dy + \int_{\Omega^r_{\text{in}} \setminus V_\eta(x_2)} \frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y))|^p}{|\underline{x}(x_2) - y|^{2+sp}}\, dy\right],\end{aligned} \tag{91}$$

where

$$V_\eta(x_2) = \{y \in \Omega^r_{\text{in}} : \ x_2 - \eta < y_2 < x_2 + \eta\}, \tag{92}$$

which implies that in the second integral $|\underline{x}(x_2) - y|$ is bounded from below by η; therefore

$$\begin{aligned}\int_0^b dx_2 \int_{\Omega^r_{\text{in}} \setminus V_\eta(x_2)} \frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y))|^p}{|\underline{x}(x_2) - y|^{2+sp}}\, dy &\le C\left[\int_0^b |w(\underline{x}(x_2))|^p\, dx_2 + \int_0^b |w(y^{\text{in}}(y))|^p\, dy_2\right] \\ &\le C\|w_{\text{in}}\|_{L_p(\Gamma_{\text{in}})},\end{aligned}$$

where in the last inequality we have used an obvious inequality

$$dy_2 \le dS(y_2). \tag{93}$$

In order to treat the first integral in (91) we apply (69)–(70). Due to (70) we can assume without loss of generality for this estimate that γ_y is a straight line $y_2 \equiv \text{const}$; therefore $y^{\text{in}}(y) = \underline{y}(y)$. Then by the definitions of $V_\eta(x_2)$ and Ω^r_{in} ((92) and (66), respectively)

$$\begin{aligned}\int_{V_\eta(x_2)} &\frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y))|^p}{|\underline{x}(x_2) - y|^{2+sp}}\, dy \\ &= \int_{x_2-\eta}^{x_2+\eta} \int_{\underline{y}_1(y_2)}^{\underline{y}_1(y_2)+\eta} \frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y_2))|^p}{|\underline{x}(x_2) - y|^{2+sp}}\, dy_1\, dy_2 \\ &\le \int_{x_2-\eta}^{x_2+\eta} |w(\underline{x}(x_2)) - w(y^{\text{in}}(y_2))|^p\, dy_2 \int_{\underline{y}_1(y_2)}^{\underline{y}_1(y_2)+\eta} (y_1 - \underline{y}_1(y) + |\underline{x}(x_2) - y^{\text{in}}(y_2)|)^{-2-sp}\, dy_1 \\ &\le \int_{x_2-\eta}^{x_2+\eta} \frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y_2))|^p}{|\underline{x}(x_2) - y^{\text{in}}(y_2)|^{1+sp}}\, dy_2,\end{aligned}$$

where in the first inequality we have used (69). Therefore, as we have additionally (93),

$$\int_{2\eta}^{b-2\eta} dx_2 \int_{V_\eta(x_2)} \frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y))|^p}{|\underline{x}(x_2) - y|^{2+sp}} \, dy \le C\|w\|_{W^s_p(\Gamma_{\text{in}})}. \tag{94}$$

Notice that in this part of the estimate the norm of boundary data appears naturally.

In the second term on the right-hand side of (90) we express $w(y) - w(y^{\text{in}}(y))$ by an integral along γ_y using (46). Applying Jensen's inequality we get

$$\begin{aligned}
\int_{\Omega^r_{\text{in}}} dy \int_0^b \frac{|w(y^{\text{in}}(y)) - w(y)|^p}{|\underline{x}(x_2) - y|^{2+sp}} \, dx_2 &\le \int_{\Omega^r_{\text{in}}} dy \int_0^b \frac{|\int_{\gamma_y} (H - w)\, dl_{\gamma_y}|^p}{|\underline{x}(x_2) - y|^{2+sp}} \, dx_2 \\
&\le (\|H\|_{L_\infty} + \|w\|_{L_\infty})^p \int_{\Omega^r_{\text{in}}} dy \int_0^b \frac{|\gamma_y|^p}{|\underline{x}(x_2) - y|^{2+sp}} \, dx_2 \\
&\le C(\|H\|_{L_\infty} + \|w\|_{L_\infty})^p \int_{\Omega^r_{\text{in}}} dy \int_0^b \frac{|y^{\text{in}}(y) - y|^p}{|\underline{x}(x_2) - y|^{2+sp}} \, dx_2 \\
&\le C(\|H\|_{L_\infty} + \|w\|_{L_\infty})^p \int_0^b dx_2 \int_{\Omega^r_{\text{in}}} \frac{|\underline{x}(x_2) - y|^p}{|\underline{x}(x_2) - y|^{2+sp}} \, dy,
\end{aligned} \tag{95}$$

where we have used (69) and (70). The last integral is finite for $s < 1$; we remember Ω^r_{in} is two-dimensional. Combining (90), (94) and (95) we get

$$I^r_1 \le C(\|H\|_{L_\infty(\Omega)} + \|w\|_{L_\infty(\Omega)} + \|w_{\text{in}}\|_{W^s_p(\Gamma_{\text{in}})})^p. \tag{96}$$

Estimate of I^s_1: Around the singularity points we have to proceed more carefully. If we consider again separately the cases (71), then in the case (a) (69) still holds with $C = 2$ so we can repeat directly the previous approach and it remains to consider the cases (b) and (c). The key difference is that, as we already explained, $|\underline{x}'_1(x_2)|$ is unbounded. Therefore we cannot repeat directly (94) and (95). We have to control $|\underline{x} - y^{\text{in}}(y)|$ and here we use the assumption (17). Recall the notions of γ_y and $\tilde{y}$ introduced in (43) and after (71), respectively; see Figure 3. Let us denote by $\gamma_{\tilde{y}}$ the part of the γ_y connecting Γ_{in} with $\tilde{y}(x_2, y)$. We modify (90) adding the point $\tilde{y}(x_2, y)$:

$$\begin{aligned}
&\frac{|w(\underline{x}(x_2)) - w(y)|^p}{|\underline{x}(x_2) - y|^{2+sp}} \\
&\quad \le C(p)\left[\frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y))|^p}{|\underline{x}(x_2) - y|^{2+sp}} + \frac{|w(y^{\text{in}}(y)) - w(\tilde{y}(x_2, y))|^p}{|\underline{x}(x_2) - y|^{2+sp}} + \frac{|w(\tilde{y}(x_2, y)) - w(y)|^p}{|\underline{x}(x_2) - y|^{2+sp}}\right].
\end{aligned} \tag{97}$$

We start with the second and third terms on the right-hand side of (97). Much as before, in both terms we replace the value of w with an integral along γ_y. The last term is analogous to I^r_1 and we get

$$\int_{\Omega^s_{\text{in}}} dy \int_0^b \frac{|w(\tilde{y}(x_2, y)) - w(y)|^p}{|\underline{x}(x_2) - y|^{2+sp}} \, dx_2 \le C(\|H\|_{L_\infty(\Omega)} + \|w\|_{L_\infty(\Omega)})^p \quad \text{for } s < 1. \tag{98}$$

In the second term we have

$$
\begin{aligned}
|w(\tilde{y}(x_2, y)) - w(y^{\text{in}}(y))| &= \left| \int_{\gamma_{\tilde{y}}} (H - w)\, dl_{\gamma_{\tilde{y}}} \right| \\
&\le (\|H\|_{L_\infty} + \|w\|_{L_\infty}) \int_{y_1^{\text{in}}(y)}^{\underline{x}_1(x_2)} \sqrt{[1 + U_1(\gamma_y(s))]^2 + (U_2(\gamma_y(s)))^2}\, ds \\
&\le C(\|H\|_{L_\infty} + \|w\|_{L_\infty}) |y_1^{\text{in}}(y) - \underline{x}_1(x_2)|.
\end{aligned}
$$

Now we estimate the latter term. This is the most subtle part of the whole proof, and which has been carried out in Lemma 15. Applying this result with $t_0 = -\underline{x}_1(x_2)$ we have

$$
|y_1^{\text{in}}(y) - \underline{x}_1(x_2)| \le C|x_2 - \tilde{y}_2(x_2, y)|^{\delta}, \tag{99}
$$

where δ is from (17). Next,

$$
|x_2 - \tilde{y}_2(x_2, y)| \le |x_2 - y_2| + \|U_2\|_{L_\infty(\Omega)} |x_1 - y_1| \le C|\underline{x} - y|;
$$

therefore

$$
\begin{aligned}
\int_{\Omega_{\text{in}}^s} dy \int_0^b \frac{|w(y^{\text{in}}(y)) - w(\tilde{y}(x_2, y))|^p}{|\underline{x}(x_2) - y|^{2+sp}} &\le \int_{\Omega_{\text{in}}^s} dy \int_0^b \frac{|\int_{\gamma_{\tilde{y}}} (H - w)\, dl_{\gamma_{\tilde{y}}}|^p}{|\underline{x}(x_2) - y|^{2+sp}}\, dx_2 \\
&\le C(\|H\|_{L_\infty} + \|w\|_{L_\infty})^p \int_{\Omega_{\text{in}}^s} dy \int_0^b \frac{|x_2 - \tilde{y}_2(x_2, y)|^{\delta p}}{|\underline{x}(x_2) - y|^{2+sp}}\, dx_2 \\
&\le C(\|H\|_{L_\infty} + \|w\|_{L_\infty})^p \int_{\Omega_{\text{in}}^s} dy \int_0^b |\underline{x}(x_2) - y|^{\delta p - 2 - sp}\, dx_2,
\end{aligned}
$$

where in the second inequality we used (99). The last integral is finite for $s < \delta$ and we conclude

$$
\int_{\Omega_{\text{in}}^s} dy \int_0^b \frac{|w(y^{\text{in}}(y)) - w(\tilde{y}(x_2, y))|^p}{|\underline{x}(x_2) - y|^{2+sp}}\, dx_2 \le C(\|H\|_{L_\infty} + \|w\|_{L_\infty})^p \quad \text{for } s < \delta. \tag{100}
$$

It remains to estimate the first term on the right-hand side of (97). For this purpose we assume $\underline{x} \ne y^{\text{in}}(y)$ since otherwise this term vanishes. By Fubini's theorem we have $dy = dS(y_2)\, dy_1$; therefore we can write

$$
\begin{aligned}
\int_{\Omega_{\text{in}}^s} dy \int_0^b \frac{|w(\underline{x}(x_2)) - w(y^{\text{in}}(y))|^p}{|\underline{x}(x_2) - y|^{2+sp}}\, dx_2 &\le C \int_{\Gamma_{\text{in}}^s} dS(y_2) \int_0^b dx_2 \int_{\underline{y}_1(y_2)}^{\bar{y}_1(y_2)} \frac{|w(\underline{x}(x_2)) - w(\underline{y}(y_2))|^p}{[|\underline{x}_1(x_2) - y_1| + |x_2 - y_2|]^{2+sp}}\, dy_1 \\
&\le C \int_{\Gamma_{\text{in}}^s} dS(y_2) \int_0^b dx_2 \frac{|w(\underline{x}(x_2)) - w(\underline{y}(y_2))|^p}{|x_2 - y_2|^{1+sp}}.
\end{aligned} \tag{101}
$$

Introducing $h = x_2 - y_2$ and using the imbedding (8) we get

$$
\begin{aligned}
\int_{\Gamma_{\text{in}}^s} dS(y_2) \int_0^b dx_2 \frac{|w(\underline{x}(x_2)) - w(\underline{y}(y_2))|^p}{|x_2 - y_2|^{1+sp}} &\le C \int_0^{h_0} \frac{dh}{|h|^{1+sp}} \int_{\Gamma_{\text{in}}^{s,h}} |w(\underline{x}(y_2 + h)) - w(\underline{y}(y_2))|^p\, dS(y_2) \\
&\le C \|w\|_{B_{p,\infty}^r(\Gamma_{\text{in}}^s)} \int_0^{h_0} h^{-1-sp} \sup_{y_2 \in (0,\eta)} |\underline{x}(y_2 + h) - \underline{y}(y_2)|^{rp}\, dh \\
&\le C \|w\|_{W_p^r(\Gamma_{\text{in}}^s)} \int_0^{h_0} h^{-1-sp} \sup_{y_2 \in (0,\eta)} |\underline{x}(y_2 + h) - \underline{y}(y_2)|^{rp}\, dh,
\end{aligned} \tag{102}
$$

where

$$h_0 = \sup\{x_2 : (\underline{x}_1(x_2), x_2) \in \Gamma^s_{\text{in}}\} \quad \text{and} \quad \Gamma^{s,h}_{\text{in}} = \{\underline{x}(x_2) : \underline{x}(x_2+h) \in \Gamma^s_{\text{in}}\}.$$

In the second inequality in (102) we have used (7) and in the last one (8). Now we apply Lemma 15 with

$$t_0 = -x_1, \quad p(t_0) = (y_1^{\text{in}}(y_2), y_2+h), \quad p(t) = \gamma_{p(t_0)}(-t),$$

where $\gamma_y(\,\cdot\,)$ is a characteristic passing through y defined in (43). Then Lemma 15 implies

$$|\underline{x}(y_2+h) - \underline{y}(y_2)| \le C|\underline{x}(y_2+h) - p(t_0)| \le C|h|^\delta, \tag{103}$$

which together with (102) gives

$$\int_{\Gamma^s_{\text{in}}} dS(y_2) \int_0^b dx_2 \frac{|w(\underline{x}(x_2)) - w(\underline{y}(y_2))|^p}{|x_2-y_2|^{1+sp}} \le C\|w\|_{W^r_p(\Gamma^s_{\text{in}})} \int_0^{h_0} h^{\delta rp-1-sp}. \tag{104}$$

The last integral is finite provided

$$\delta r > s,$$

which implies the second relation in (24). Combining (98), (100) and (104) we conclude

$$I^s_1 \le C(\|H\|_{L_\infty} + \|w\|_{L_\infty} + \|w_{\text{in}}\|_{W^r_p(\Gamma_s)})^p. \tag{105}$$

Putting together (89), (96) and (105) we get (84) with $U\cdot n = 0$, which completes the proof due to (85). □

Proof of Proposition 9. Now we are ready to close the estimate (47). Combining (53) with (84) we obtain

$$\|w\|^p_{W^s_p} \le C(s,p,\Omega)\big[\|H\|_{W^s_p}\|w\|^{p-1}_{W^s_p} + \|w\|^p_{L_\infty} + \|w\|^p_{L_p} \\ + \|w_{\text{in}}\|_{W^s_p(\Gamma_{\text{in}})} + \|w_{\text{in}}\|_{W^r_p(\Gamma_s)} + \|\nabla U\|_{L_\infty}\|w\|^p_{W^s_p}\big] \tag{106}$$

and applying the interpolation inequality (6) we conclude (47). □

Remark 18. Notice that from (106) it follows that the constant in (47) is of the form

$$C = \frac{C(s,p,\Omega)}{1-\|\nabla U\|_{L_\infty}};$$

therefore under the assumption of smallness of $\|\nabla U\|_{L_\infty}$ we can assume that it does not depend on $\|\nabla U\|_{L_\infty}$.

Remark 19. The proof of Proposition 9 is quite technical; therefore it may be unclear to the reader how it can be extended to a more general domain. To clarify it, we shall emphasize that the most subtle part is the estimate of $|w(\underline{x}) - w(y)|^p/|\underline{x}-y|^{2+sp}$, where $\underline{x}$ is on the boundary and y is inside Ω, both close to singularity. The most delicate point is to estimate the distance between the point where a characteristic crossing y intersects the boundary and $\underline{x}$. This was carried out in Lemma 15. On the other hand, a larger distance between $\underline{x}$ and y does not create additional problems. Therefore the proof presented here can be extended to the general case of multiple singularity points. We have to consider again separately the "regular" part of the boundary where $\underline{x}_2'(x_1)$ is bounded by a given constant and therefore we get the estimates (89), (96), and neighborhoods of each singularity point where we repeat the estimate (105).

This concludes the proof of Lemma 15. □

2.3. ***Linear estimate in $W_p^{1+s}\times W_p^s$ and solution of the linear system.*** In this section we solve the linear problem (36) with right-hand side of regularity determined in (37). First we show the a priori estimate:

Proposition 20. *Let (u,w) solve the system (36) with the right-hand side of regularity determined in (37), with s, p, and the boundary data satisfying the assumptions of Theorem 5. Then we have*

$$\|u\|_{W_p^{1+s}(\Omega)}+\|w\|_{W_p^s(\Omega)}\le C\,[\|F\|_{L_p(\Omega)}+\|G\|_{W_p^s(\Omega)}+\|w_{\mathrm{in}}\|_{W_p^s(\Gamma_{\mathrm{in}})\cap W_p^r(\Gamma_s)}+\|B\|_{W_p^{1-1/p}(\Gamma)}]. \tag{107}$$

Remark 21. The main difficulty in the proof of (107) lies in the steady transport equation which has been dealt with in Proposition 9. The rest of the proof uses classical results from the theory of elliptic problems and can be divided into several steps.

Proof of Proposition 20. Step I: Let us take the rotation of $(36)_1$. Then we get the system

$$\begin{aligned} -\mu\Delta\operatorname{rot}u&=\operatorname{rot}(F-\partial_{x_1}u) \quad \text{in } \Omega,\\ \operatorname{rot}u&=\Big(2\chi-\frac{f}{\nu}\Big)u\cdot\tau+\frac{B}{\mu} \quad \text{at } \Gamma. \end{aligned} \tag{108}$$

In order to show the boundary relation for the vorticity we differentiate (36c) in the tangential direction and apply (36d). The details are shown in [Mucha and Rautmann 2006] in the case $u\cdot n=d,\ B=0$. A minor modification for our case $u\cdot n=0,\ B\neq 0$, yields (108). We find the following estimate for the solution to (108):

$$\|\operatorname{rot}u\|_{W_p^1(\Omega)}\le C(\|F\|_{L_p(\Omega)}+\|u,B\|_{W_p^{1-1/p}(\Gamma)}+\|\partial_{x_1}u\|_{L_p(\Omega)}). \tag{109}$$

The solvability of (108) belongs to the classical theory of elliptic linear problems; hence we omit the details of this construction.

Step II: Consider the Helmholtz decomposition of u:

$$u=\nabla\phi+\nabla^\perp A. \tag{110}$$

We have $0=n\cdot\nabla^\perp A=\tau\cdot\nabla A$. Therefore A satisfies

$$\Delta A=\operatorname{rot}u,\qquad A|_\Gamma=\text{const}. \tag{111}$$

As $\partial\Omega\in C^3$, from (111) we obtain

$$\|A\|_{W_p^3(\Omega)}\le C\|\operatorname{rot}u\|_{W_p^1(\Omega)}. \tag{112}$$

Substituting (110) into (36a) we get

$$\nabla(-(2\mu+\nu)\operatorname{div}u+Kw)=F-\partial_{x_1}u+\mu\Delta\nabla^\perp A+(\mu+\nu)\nabla\operatorname{div}\nabla^\perp A=:\bar F.$$

By (109) and (112) we have

$$\|\bar F\|_{L_p(\Omega)}\le C[\|F\|_{L_p(\Omega)}+\|\partial_{x_1}u\|_{L_p(\Omega)}+\|u,B\|_{W_p^{1-1/p}(\Gamma)}];$$

therefore setting

$$-(2\mu+\nu)\operatorname{div}u+Kw=H \tag{113}$$

we have for any $\delta > 0$

$$\begin{aligned}\|H\|_{W_p^s(\Omega)} &\le \delta\|\nabla H\|_{L_p(\Omega)} + C(\delta)\|H\|_{L_2(\Omega)}\\ &\le \delta[\|F\|_{L_p(\Omega)} + \|\partial_{x_1} u\|_{L_p(\Omega)} + \|u, B\|_{W_p^{1-1/p}(\Gamma)}] + C(\delta)\|\nabla u, w\|_{L_2(\Omega)}.\end{aligned} \tag{114}$$

Step III: Adding (36b) to (113) with suitable scaling we obtain

$$\partial_{x_1} w + U \cdot \nabla w + \bar{K} w = \bar{G}, \tag{115}$$

with $\bar{K} = K/(2\mu + \nu)$ and

$$\|\bar{G}\|_{W_p^s(\Omega)} \le \|G\|_{W_p^s(\Omega)} + [\text{RHS of (114)}].$$

Here we meet the main mathematical challenge of our result; we have to solve this transport-like problem in a domain which touches the characteristics at the ends of Γ_{in}. Its solvability is given by Proposition 9. In particular, the estimate (47) yields

$$\|w\|_{W_p^s(\Omega)} \le C(\|G\|_{W_p^s(\Omega)} + \|w_{\text{in}}\|_{W_p^s(\Gamma_{\text{in}}) \cap W_p^r(\Gamma_s)}). \tag{116}$$

Remark 22. Notice that Remark 18 implies that we can assume that the constant in (116) does not depend on $\|U\|_{W_\infty^1}$.

Step IV: We collect the elements of our estimation. In order to get the information about the velocity we use estimates for $\operatorname{rot} u$ given by (109) and about $\operatorname{div} u$ coming from (113), (114) and (116). We obtain

$$\begin{aligned}\|u\|_{W_p^{1+s}(\Omega)} &\le C(\|\operatorname{rot} u\|_{W_p^s(\Omega)} + |\operatorname{div} u\|_{W_p^s(\Omega)})\\ &\le C(\|F\|_{L_p(\Omega)} + \|G\|_{W_p^s(\Omega)} + \|B\|_{W^{1-1/p}(\Gamma)} + \|\nabla u, w\|_{L_2(\Omega)} + \|\partial_{x_1} u\|_{L_p} + \|u\|_{W^{1-1/p}(\Gamma)})\\ &\qquad + \delta(\|u\|_{W_p^{1+s}(\Omega)} + \|w\|_{W_p^s(\Omega)}).\end{aligned} \tag{117}$$

The L_2 term on the right-hand side is treated with the energy estimate (38). To the boundary term $\|u\|_{W^{1-1/p}(\Gamma)}$ we apply first the trace theorem and then the interpolation inequality (6) to get

$$\|u\|_{W_p^{1-1/p}(\Gamma)} \le C\|u\|_{W_p^s(\Omega)} \le \delta\|u\|_{W_p^{1+s}(\Omega)} + C(\delta)\|\nabla u\|_{L_2(\Omega)}. \tag{118}$$

The term $\partial_{x_1} u$ is treated similarly with the interpolation inequality. We conclude (107) and complete the proof. □

Now it is a matter of standard theory to show the existence for the linear system (36).

Proposition 23. *Let the right-hand side of (36) be of regularity specified in (37) with s, p, and the boundary data satisfying the assumptions of Theorem 5. Then there exists a unique solution to (36) satisfying the estimate (107).*

Proof. As we have the a priori estimate, the proof of existence is standard. We start with showing the weak solutions using the Galerkin method and the energy estimate (38). Then, using our a priori estimate we show that our solution has the required regularity. □

2.4. ***Estimate for the nonlinear system.*** We are now ready to close the estimate in $W_p^{1+s} \times W_p^s$ for the nonlinear system. To this end we combine the linear estimate (107) with the bounds (32) on the right-hand side of the nonlinear problem obtaining

$$\|u\|_{W_p^{1+s}(\Omega)} + \|w\|_{W_p^s(\Omega)} \leq E(\|u\|_{W_p^{1+s}(\Omega)} + \|w\|_{W_p^s(\Omega)} + 1) + C(\|u\|_{W_p^{1+s}(\Omega)} + \|w\|_{W_p^s(\Omega)})^2,$$

where E is a small constant dependent on the data and we can assume (see Remark 22) that C does not depend on $\|u\|_{W^1_\infty(\Omega)}$. For sufficiently small data we conclude

$$\|u\|_{W_p^{1+s}(\Omega)} + \|w\|_{W_p^s(\Omega)} \leq C(\|u\|_{W_p^{1+s}(\Omega)} + \|w\|^2_{W_p^s(\Omega)} + E), \tag{119}$$

where C does not depend on $(\|w\|_{W_p^s(\Omega)}, \|u\|_{W_p^{1+s}(\Omega)})$. This estimate will be crucial in showing the existence and uniqueness of solutions in the next section.

Remark 24. Notice that the estimate (119) holds without any assumptions on relation between μ and K. However, this relation will come into play in the following section.

3. Proof of Theorem 5

In order to prove our main result we apply an iterative scheme. The idea is to combine the estimate (119) with the Cauchy condition in some weaker space. In [Piasecki 2010; Piasecki and Pokorný 2014] this space was $H^1 \times L_2$. However, we need some information on the gradient of the density which is not available in our framework. Here we overcome this obstacle showing the convergence in $L_2 \times H^{-1}$. However, then we have to estimate $\|\pi(\rho^n) - \pi(\rho^{n-1})\|_{H^{-1}}$ in terms of $\|\rho^n - \rho^{n-1}\|_{H^{-1}}$ and for this purpose we have to assume (25). A similar approach was applied in the context of uniqueness of weak solutions in the nonstationary case in [Hoff 2006], where the constraint (25) also appeared. Under this assumption, setting

$$v^{n+1} = u^{n+1} + u_0 + \bar{v}$$

we can define the sequence of approximations in the following way:

$$\begin{aligned}
\operatorname{div}(w^{n+1}v^{n+1}\otimes v^{n+1}) - \mu\Delta u^{n+1} - (\mu+\nu)\nabla \operatorname{div} u^{n+1} + K\nabla w^{n+1} &= 0 && \text{in } \Omega, && (120a)\\
w^{n+1}_{x_1} + \operatorname{div}((u^n+u_0)w^{n+1}) &= -\operatorname{div}(u^n+u_0) && \text{in } \Omega, && (120b)\\
n\cdot 2\mu \boldsymbol{D}(u^{n+1})\cdot\tau + f u^{n+1}\cdot\tau &= B && \text{on } \Gamma, && (120c)\\
n\cdot u^{n+1} &= 0 && \text{on } \Gamma, && (120d)\\
w^{n+1} &= w_{\text{in}} && \text{on } \Gamma_{\text{in}}. && (120e)
\end{aligned}$$

We start with the following auxiliary result:

Lemma 25. *Assume that ϕ solves*

$$(\partial_{x_1} + U\cdot\nabla)\phi = f, \quad \phi|_{\Gamma_{\text{out}}} = 0, \tag{121}$$

with $f \in H^1_0(\Omega)$ *and sufficiently small* $U \in W^1_\infty(\Omega)$ *such that*

$$U \cdot n|_{\Gamma_{\text{in}}} < 0, \quad U \cdot n|_{\Gamma_{\text{out}}} > 0. \tag{122}$$

Then

$$\|\nabla \phi\|_{L_2(\Omega)} \le C_\Omega \|f\|_{H^1_0(\Omega)}, \tag{123}$$

where $C_\Omega \sim \max\{|\Omega|, |\Omega|^2\}$.

Proof. The boundary condition (120b) implies

$$0 = \partial_\tau \phi|_{\Gamma_{\text{out}}} = \tau_1 \phi_{x_1} + \tau_2 \phi_{x_2}|_{\Gamma_{\text{out}}}. \tag{124}$$

On the other hand we have

$$(1+u^1)\phi_{x_1} + u^2 \phi_{x_2}|_{\Gamma_{\text{out}}} = 0. \tag{125}$$

Subtracting (124) multiplied by u^2 from (125) multiplied by τ^2 we get

$$0 = [(1+u^1)\tau^2 - u^2\tau^1]\phi_{x_1} = -[(1+u^1)n^1 + u^2 n^2]\phi_{x_1} = 0;$$

therefore $\phi_{x_1}|_{\Gamma_{\text{out}}} = 0$ due to (122). Combining this identity with (124) we conclude

$$\nabla \phi|_{\Gamma_{\text{out}}} = 0. \tag{126}$$

Next we differentiate (120a) with respect to x_2 and multiply by ϕ_{x_2}. Setting $V = [1+U^1, U^2]$ we obtain

$$\tfrac{1}{2} V \cdot \nabla \phi^2_{x_2} = f_{x_2}\phi_{x_2} - V_{x_2}\phi_{x_2} \cdot \nabla \phi. \tag{127}$$

Now let us define

$$\Omega_a = \Omega \cap \{x_1 > a\}, \quad I_a = \Omega \cap \{x_1 = a\}.$$

Integrating (127) over Ω_a we get

$$\frac{1}{2}\int_{\partial\Omega_{x_1}} (V \cdot n)\phi^2_{x_2}\, d\sigma = \int_{\Omega_{x_1}} \left[\frac{1}{2}\phi^2_{x_2} \operatorname{div} V + f_{x_2}\phi_{x_2} + V_{x_2}\phi_{x_2} \cdot \nabla\phi\right] dx. \tag{128}$$

Notice that by (126) and the condition $V \cdot n|_{\Gamma_{\text{in}}} < 0$ we have

$$\int_{\partial\Omega \cap \partial\Omega_{x_1}} (V \cdot n)\phi^2_{x_2}\, d\sigma \le 0.$$

Therefore the smallness of U^1 implies

$$\int_{\partial\Omega_{x_1}} (V \cdot n)\phi^2_{x_2}\, d\sigma = -\int_{I_{x_1}} (1+U^1)\phi^2_{x_2}\, dx_2 + \int_{\partial\Omega \cap \partial\Omega_{x_1}} (V \cdot n)\phi^2_{x_2}\, d\sigma \le -C_U \int_{I_{x_1}} \phi^2_{x_2}\, d\sigma$$

for some $C_U > 0$. The latter inequality combined with (128) gives for any $\epsilon > 0$

$$\sup_{x_1} \int_{I_{x_1}} \phi^2_{x_2}\, dx_2 \le \|\nabla U\|_{L_\infty(\Omega)} \|\nabla\phi\|_{L_\infty(\Omega)} + \epsilon \|\phi_{x_2}\|^2_{L_2(\Omega)} + \frac{C}{\epsilon}\|\nabla f\|^2_{L_2(\Omega)}.$$

Using again the smallness of U we obtain

$$\|\phi_{x_2}\|^2_{L_2(\Omega)} \le |\Omega|[(\epsilon + \|\nabla U\|_{L_\infty(\Omega)})\|\nabla\phi\|^2_{L_2(\Omega)} + C(\epsilon)\|\nabla f\|^2_{L_2(\Omega)}]. \tag{129}$$

Finally from (120a) we have

$$\|\phi_{x_1}\|^2_{L_2(\Omega)} \le \left\|\frac{f}{1+U^1}\right\|^2_{L_2(\Omega)} + \left\|\frac{U^2}{1+U^1}\right\|_{L_\infty(\Omega)} \|\phi_{x_2}\|^2_{L_2(\Omega)}.$$

Combining this inequality with (129) we get

$$\|\nabla\phi\|^2_{L_2(\Omega)} \le [|\Omega|(\epsilon + \nabla U\|_{L_\infty(\Omega)}) + C\|U\|_{L_\infty(\Omega)}]\|\nabla\phi\|^2_{L_2(\Omega)} + \left[\frac{C|\Omega|}{\epsilon} + C_p\right]\|\nabla f\|^2_{L_2(\Omega)},$$

where C_p is the constant from the Poincaré inequality. Now, provided $\|U\|_{W^1_\infty}$ is sufficiently small, to close the estimate we need $\epsilon \sim 1/|\Omega|$. Taking into account that $C_p \sim |\Omega|$ we conclude (123). □

The following proposition implies convergence of the sequence (u^n, w^n) in $L_2 \times H^{-1}$.

Proposition 26. *Assume that μ is sufficiently large compared to $|\Omega|$, $\|\bar{v}\|_{L_\infty}$ and K. Then the sequence (u^n, w^n) defined by (120) satisfies*

$$\|u^{n+1} - u^n\|_{L_2(\Omega)} + \|w^{n+1} - w^n\|_{H^{-1}(\Omega)} \le M[\|u^n - u^{n-1}\|_{L_2(\Omega)} + \|w^n - w^{n-1}\|_{H^{-1}(\Omega)}], \tag{130}$$

with $M < 1$.

Remark 27. In order to track the dependence of μ on $\|\bar{v}\|_{L_\infty}$ we consider again (11) with a general constant v^*.

Proof. Subtracting (120b) for two consecutive steps we get

$$(w^{n+1} - w^n)_{x_1} + \operatorname{div}[(u^n + u_0)(w^{n+1} - w^n)] = -\operatorname{div}((w^n + 1)(u^n - u^{n-1})).$$

We test this equation with ϕ given by (121) with $U = u^n + u_0$ and f such that

$$\Delta f = w^{n+1} - w^n.$$

We obtain

$$\begin{aligned}\int_\Omega (u^n - u^{n-1})(w_n + 1)\cdot\nabla\phi\,dx &= -\int_\Omega (w^{n+1} - w^n)(\partial_{x_1} + (u^n + u_0)\cdot\nabla)\phi\,dx \\ &= -\int_\Omega f\Delta f\,dx = \|w^{n+1} - w^n\|^2_{H^{-1}}.\end{aligned}$$

Therefore by (123) we obtain

$$\|w^{n+1} - w^n\|^2_{H^{-1}} \le \|w^n + 1\|_{L_\infty}\|u^n - u^{n-1}\|_{L_2}\|\nabla\phi\|_{L_2} \le C_1 C_\Omega \|u^n - u^{n-1}\|_{L_2}\|\nabla f\|_{L_2},$$

where C_Ω is the constant from (123). Now, since $\|\nabla f\|_{L_2} \le C\|w^{n+1} - w^n\|_{H^{-1}}$, we conclude

$$\|w^{n+1} - w^n\|_{H^{-1}(\Omega)} \le C_2 C_\Omega \|u^n - u^{n-1}\|_{L_2(\Omega)}. \tag{131}$$

Now we have to estimate $\|u^n - u^{n-1}\|_{L_2(\Omega)}$ in terms of $\|w^n - w^{n-1}\|_{H^{-1}(\Omega)}$. Subtracting (120a) for two consecutive steps we obtain

$$-\mu\Delta(u^{n+1}-u^n)-(\mu+\nu)\nabla\operatorname{div}(u^{n+1}-u^n) = -K\nabla(w^{n+1}-w^n)-\operatorname{div}[(w^{n+1}-w^n)(v^{n+1}\otimes v^{n+1})] \\ -\operatorname{div}[w^n(v^{n+1}\otimes(v^{n+1}-v^n)+(v^{n+1}-v^n)\otimes v^n)]. \quad (132)$$

Notice that as $u^{n+1}-u^n$ satisfy homogeneous boundary conditions we have in the weak sense

$$\int_\Omega \psi\cdot[\mu\Delta u+(\mu+\nu)\nabla\operatorname{div} u]\,dx = \int_\Omega u\cdot[\mu\Delta\psi+(\mu+\nu)\nabla\operatorname{div}\psi]\,dx$$

for any $\psi\in H^2$ such that $\psi|_\Gamma = 0$. Therefore testing (132) with ψ being a solution to the problem

$$-\mu\Delta\psi-(\mu+\nu)\nabla\operatorname{div}\psi = \mu(u^{n+1}-u^n), \qquad \psi|_\Gamma = 0,$$

we get

$$\begin{aligned}\mu\|u^{n+1}-u^n\|^2_{L_2(\Omega)} &= \int_\Omega (w^{n+1}-w^n)(K\operatorname{div}\psi + v^{n+1}\otimes v^{n+1}:\nabla\psi)\,dx \\ &\qquad + \int_\Omega w^n[v^{n+1}\otimes(v^{n+1}-v^n)+(v^{n+1}-v^n)\otimes v^n]:\nabla\psi\,dx \\ &\le C_v[\|w^{n+1}-w^n\|_{H^{-1}(\Omega)}\|\nabla\psi\|_{L_2(\Omega)} + \|u^{n+1}-u^n\|_{L_2(\Omega)}\|\nabla\psi\|_{L_2(\Omega)}] \\ &\le C_{v,K}[\|w^{n+1}-w^n\|_{H^{-1}(\Omega)}\|u^{n+1}-u^n\|_{L_2(\Omega)} + \|u^{n+1}-u^n\|^2_{L_2(\Omega)}],\end{aligned}$$

where $C_{v,K} = C_{v,K}(\|v^{n+1}\|_{L_\infty(\Omega)}, K)$. In the above display we have used the facts that $v^{n+1}-v^n = u^{n+1}-u^n$ and $\|\nabla\psi\|_{L_2(\Omega)} \le C\|u^{n+1}-u^n\|_{L_2(\Omega)}$. Therefore, assuming $\mu > C_v$ we obtain

$$\|u^{n+1}-u^n\|_{L_2(\Omega)} \le \frac{C_{v,K}}{\mu - C_{v,K}}\|w^{n+1}-w^n\|_{H^{-1}(\Omega)}.$$

Combining this estimate with (131) we conclude (130) with $M = C_2C_\Omega C_{v,K}/(\mu - C_{v,K})$. Therefore $M<1$ provided the viscosity is sufficiently large compared to $|\Omega|$, $\|\bar v\|_{L_\infty(\Omega)}$, and K. □

The solvability of the linear system established in Section 3 implies that the sequence (120) is well-defined. Moreover, setting

$$A_n = \|u^n\|_{W_p^{1+s}(\Omega)} + \|w^n\|_{W_p^s(\Omega)},$$

by (119) we have

$$A_{n+1} \le CA_n^2 + E$$

and we can assume $E \le 1/(2C)$. Then the sequence starting with $(u^0, w^0) = ([0,0],0)$ satisfies

$$A_n \le 2E, \tag{133}$$

where E is the constant from (119). Proposition 26 implies

$$(u^n, w^n) \to (u, w) \quad \text{in } L_2(\Omega)\times H^{-1}(\Omega).$$

On the other hand, (133) yields up to a subsequence $(u^n, w^n) \rightharpoonup (\bar u, \bar w)$ in $W_p^{1+s}(\Omega)\times W_p^s(\Omega)$. By the definition of (u^n, w^n), the limit (u, w) is a solution to (30)–(31) in the sense of Definition 7. The uniqueness is shown in the same way as Proposition 26. Namely, taking two solutions (u^1, w^1) and (u^2, w^2) for the same data we show

$$\|u^1 - u^2\|_{L^2(\Omega)} + \|w^1 - w^2\|_{H^{-1}(\Omega)} = 0. \qquad \square$$

4. Concluding remarks

The solutions considered here are located somehow between weak and "traditional" regular solutions satisfying the equations almost everywhere. The result for the steady transport equation given by Proposition 9 is obtained for a general class of boundary singularities showing that the choice of the fractional Sobolev–Slobodetskii spaces is in a sense natural for the problem under consideration. The price we pay is that we have to assume linearity of the pressure. Getting rid of this constraint seems an interesting open problem. It is likely that the existence itself could be shown for more general pressure laws using for example approximation with more regular solutions which give some information about the gradient of the density. However, such a result would be highly technical and not really meaningful without uniqueness, which is more challenging and seems to require some novel approach to treat more general pressure laws. Finally we shall mention that the assumed C^3 regularity of the domain required to solve an elliptic problem appearing in the estimate for the velocity is not optimal and could be relaxed at the price of additional technicalities. However, as we are rather interested in a careful investigation of the boundary singularity in the stationary transport equation which is independent of global regularity of the boundary (for example a C^∞ boundary can fail to satisfy (16)), we keep this regularity assumption.

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Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O'Neil of the Courant Institute.

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