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A MAXIMUM PRINCIPLE FOR A FOURTH-ORDER DIRICHLET PROBLEM ON SMOOTH DOMAINS

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Our main result is that for any bounded smooth domain $\Omega \subset \mathbb{R}^n$ there exists a positive-weight function w and an interval I such that for $\lambda \in I$ and $\Delta^2 u = \lambda w u + f$ in Ω with $u = \frac{\partial}{\partial \nu} u = 0$ on $\partial\Omega$ the following holds: if f is positive, then u is positive. The proofs are based on the construction of an appropriate weight function w with a corresponding strongly positive eigenfunction and on a converse of the Krein–Rutman theorem. For the Dirichlet bilaplace problem above with $\lambda = 0$ the Boggio–Hadamard conjecture from around 1908 claimed that positivity is preserved on convex 2-dimensional domains and was disproved by counterexamples from Duffin and Garabedian some 40 years later. With $w = 1$ not even the first eigenfunction is in general positive. So by adding a certain weight function our result shows a striking difference: not only is a corresponding eigenfunction positive but also a fourth-order "maximum principle" holds for some range of λ.

1. Introduction

Consider for $\Omega \subset \mathbb{R}^n$ a bounded domain with a smooth boundary $\partial\Omega$ and $\lambda \in \mathbb{R}$ the fourth-order Dirichlet problem

$$\begin{cases} (\Delta^2 - \lambda w)u = f & \text{in } \Omega, \\ u = \dfrac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with weight function $w > 0$. Here ν is the exterior normal on $\partial\Omega$ and Ω is a domain, whenever it is open and connected. For (1) with $\lambda = 0$ and $\Omega = B$, a ball in $\mathbb{R}^n$, Boggio [1905] constructed explicit Green's functions G_B. Since his Green's functions are positive, one finds for any f for which the corresponding solution is well-defined through $u(x) = \int_B G_B(x, y) f(y)\, dy$ that

$$f \geq 0 \quad \text{implies} \quad u \geq 0,$$

and not only for $\lambda = 0$ but even for λ in some interval. By introducing an appropriate weight w that depends on the domain, we derive such kind of positivity-preserving property (PPP) on general domains for some range of λ. For $\lambda = 0$ and $\Omega \subset \mathbb{R}^2$ (1) is called *the clamped plate problem* [Hadamard 1968a].

Concerning that just-mentioned interval for λ, if (1) is positivity-preserving for $\lambda = 0$ on a domain Ω as above, then by a Krein–Rutman theorem, see [Gazzola et al. 2010, page 63], there is a first and simple eigenvalue $\lambda_1 \in \mathbb{R}^+$ for the biharmonic eigenvalue problem. Moreover, $\rho = \lambda_1^{-1}$ is the spectral radius of

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the corresponding solution operator for $\lambda = 0$ and by a Neumann series expansion [Grunau and Sweers 1998, Proposition 4.1] one finds that PPP holds for all $\lambda \in [0, \lambda_1)$. The eigenfunction φ_1 for λ_1 is of fixed sign, and hence can be chosen positive. For $\partial\Omega$ smooth, the function φ_1 is then even strongly positive in the sense that for some $c > 0$

$$\varphi_1(x) \geq c\, d(x, \partial\Omega)^2 \quad \text{for all } x \in \Omega. \tag{2}$$

Here $d(\,\cdot\,, \partial\Omega)$ is the distance to the boundary

$$d(x, \partial\Omega) := \inf_{y \in \partial\Omega} |x - y|. \tag{3}$$

In [Schnieders and Sweers 2020] a converse of the Krein–Rutman theorem is shown for (1) with $w = 1$ on arbitrary smooth and bounded domains Ω:

If there exists a simple eigenvalue λ_j to the biharmonic eigenvalue problem with the corresponding eigenfunction strongly positive in the sense of (2)*, then* (1) *is positivity-preserving for λ in a left neighbourhood of λ_j.*

Although there are domains besides balls for which there exists an eigenfunction that satisfies (2), see [Sweers 2001], for most domains there is no positive eigenfunction. In this article we overcome that restriction by introducing an appropriate weight function w that is positive. With this w we prove the existence of a simple eigenvalue $\lambda_{j,w}$ and a corresponding eigenfunction $\varphi_{j,w}$ for the weighted eigenvalue problem

$$\begin{cases} \Delta^2 \varphi = \lambda w \varphi & \text{in } \Omega, \\ \varphi = \dfrac{\partial}{\partial \nu}\varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\varphi_{j,w}$ is strongly positive as in (2). We will do this for arbitrary bounded smooth domains Ω in any dimension. As a consequence and by arguing as in [Schnieders and Sweers 2020], we find for (1) a positivity-preserving property if λ is in a left neighbourhood of $\lambda_{j,w}$.

Remark 1. Although the positive eigenfunction for most Ω will correspond to the first eigenvalue, [Duffin and Shaffer 1952; Coffman et al. 1979] give an example where such an eigenvalue is the third one. See also [Schnieders and Sweers 2020]. So, we suppose that the eigenvalue $\lambda_{j,w}$ is the j-th eigenvalue, where eigenvalues are counted with their multiplicity. Hence $0 < \lambda_{1,w} \leq \lambda_{2,w} \leq \cdots \leq \lambda_{j,w} \leq \cdots \to \infty$.

The precise statements and main result of the present article are presented in the following theorem and corollary:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{4,\gamma}$ for some $\gamma \in (0, 1)$. Then, there exists a strictly positive-weight function $w \in C^{0,\gamma}(\bar{\Omega})$, meaning* $\min_{x\in\bar{\Omega}} w(x) > 0$, *such that the eigenvalue problem* (4) *has the simple eigenvalue $\lambda_{j,w} = 1$ with an eigenfunction $\varphi_{j,w} \in C^{4,\gamma}(\bar{\Omega}) \cap C_0^1(\bar{\Omega})$ satisfying $\varphi_{j,w}(x) \geq d(x, \partial\Omega)^2$ for all $x \in \Omega$.*

The existence of a strictly positive weight w with a strongly positive eigenfunction $\varphi_{j,w}$ is established in Section 2, more precisely in Proposition 9 below. Since the corresponding eigenvalue is not necessarily simple, we have to consider an eventual small perturbation of the weight function. In Section 4 we describe a perturbation procedure so that the slightly changed weight is still positive, the eigenfunction remains strongly positive and the corresponding eigenvalue becomes simple.

Remark 3. The generic simplicity of the spectrum for the clamped plate equation with respect to domain was proved in [Ortega and Zuazua 2000; Pereira 2004]. Other results for generic simplicity under perturbations with respect to the coefficients can be found in [Albert 1975; Teytel 1999].

With the w-variant of the main theorem from [Schnieders and Sweers 2020] we find the following positivity-preserving property for the biharmonic Dirichlet problem in (1).

Corollary 4 (PPP). *Let Ω, w and $\lambda_{j,w} = 1$ be as in Theorem 2. Then there is $\lambda_c < \lambda_{j,w}$ such that for $0 \le f \in L^2(\Omega)$ with f nontrivial and u the solution of* (1):

(1) *If $\lambda \in [\lambda_c, \lambda_{j,w})$, then $u > 0$ in Ω.*

(2) *If $\lambda \in (\lambda_c, \lambda_{j,w})$, then a Hopf type result holds: there exists $c_{f,\lambda} > 0$ such that*

$$u(x) \ge c_{f,\lambda}\, d(x, \partial\Omega)^2 \quad \textit{for all } x \in \Omega.$$

Proof. With the existence of a simple eigenvalue $\lambda_{j,w} = 1$ with a strongly positive eigenfunction $\varphi_{j,w}$ from Theorem 2 one may continue with the estimates in Theorem 16 and find statement (1) for $\lambda \in [\lambda_{j,w} - C_2/C_3, \lambda_{j,w})$ and (2) for $\lambda \in (\lambda_{j,w} - C_2/C_3, \lambda_{j,w})$. □

Remark 5. As already mentioned, the positivity-preserving property does not hold true for the biharmonic Dirichlet problem on general domains Ω,

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \dfrac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Hadamard [1968b] reported on discussions with Boggio and conjectured that at least on convex domains (in $\mathbb{R}^2$) there should be a positivity-preserving property for (5). The first, by now well-known counterexample was established by Duffin [1949], who considered the biharmonic Dirichlet problem on an infinitely long strip. Garabedian [1951] showed that the Green's function changes sign in the case that the underlying domain is a sufficiently eccentric ellipse. For a survey see [Sweers 2001]. An interesting family of domains concerning PPP are the limaçons of Pascal. Hadamard calculated an explicit Green's function for those limaçons in [Hadamard 1968a, Supplement] and, as was shown in [Dall'Acqua and Sweers 2005], those functions are positive only when the limaçon is not too far from the disk. Other known examples with PPP for (5) are based on perturbations of Boggio's results [1905] for balls. See [Gazzola et al. 2010, Chapter 6].

One notices that if (5) is not positivity-preserving for a domain Ω and $\lambda = \lambda_c$, with λ_c as described in Corollary 4, then a Hopf principle fails for the solution to (1). Moreover, for $\lambda < \lambda_c$ one expects some negativity close to the boundary since this is the same phenomenon that appears for the limaçons which are close to the cardioid.

Remark 6. When asked about a physical meaning of the weighted problem, we recall that (5) for $n = 2$ is used to model the deviation u of a thin plate due to a force density f that is clamped at its boundary.

The eigenvalues here correspond to resonances due to exterior induced vibrations and the weight w would be a measure for the stiffness of the plate. This stiffness could be x-dependent, although then the corresponding differential equation should be $\Delta(w^{-1}\Delta u) = \lambda u + f$. A second-order term $b\Delta u$ in the equation also appears when modelling a prestressed plate and fixing the horizontal movements at the boundary. The value of b can have either sign, although for reinforced concrete no engineer would like $b > 0$. If we forget about the third-order terms and compensate the second-order term by prestressing appropriately, the present weight produces a plate that is very stiff near the boundary and rather flexible in the interior.

The structure of the paper is as follows. In Section 2 we introduce a specific weight with which we get a positive eigenfunction with corresponding eigenvalue $\lambda = 1$ for the eigenvalue problem in (4). Next in Section 3, we will describe the adapted setting and adjust and expand the results in [Schnieders and Sweers 2020] to the weighted biharmonic problem (1). Finally in Section 4, we prove that by perturbing the initial weight function slightly, we obtain a simple eigenvalue with a positive eigenfunction.

2. Construction of weight and eigenfunction

In this section we will construct an explicit weight function that guarantees the existence of a positive eigenfunction. To this end we suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^{4,\gamma}$ for some $\gamma \in (0, 1)$. We start with one special positive combination u, f for (5). Let $\boldsymbol{e} : \overline{\Omega} \to \mathbb{R}$ be the solution of

$$\begin{cases} -\Delta \boldsymbol{e} = 1 & \text{in } \Omega, \\ \quad\ \boldsymbol{e} = 0 & \text{on } \partial\Omega. \end{cases}$$

It holds that $\boldsymbol{e} \in C^{4,\gamma}(\overline{\Omega})$; see [Gilbarg and Trudinger 1983, Theorem 6.19]. Using the maximum principle for the laplacian, it follows that $\boldsymbol{e} > 0$ in Ω, and with Hopf's boundary point lemma [Gilbarg and Trudinger 1983, Section 3.2] and the mean value theorem, we obtain constants $c_1, c_2 > 0$ such that

$$c_1 d(x) \le \boldsymbol{e}(x) \le c_2 d(x) \quad \text{for all } x \in \Omega, \tag{6}$$

where we let $d(x) := d(x, \partial\Omega)$ from (3). In [Gilbarg and Trudinger 1983, Lemma 14.16] one finds that $d \in C^{4,\gamma}$ near $\partial\Omega$ follows from $\partial\Omega \in C^{4,\gamma}$.

A direct computation shows

$$\boldsymbol{e}^2 = \frac{\partial}{\partial\nu}\boldsymbol{e}^2 = 0$$

on $\partial\Omega$ and

$$\begin{aligned} \Delta^2 \boldsymbol{e}^2 &= 2(-\Delta)\biggl((-\Delta\boldsymbol{e})\boldsymbol{e} - \sum_{i=1}^{n}\Bigl(\frac{\partial \boldsymbol{e}}{\partial x_i}\Bigr)^2\biggr) = 2(-\Delta)\biggl(\boldsymbol{e} - \sum_{i=1}^{n}\Bigl(\frac{\partial \boldsymbol{e}}{\partial x_i}\Bigr)^2\biggr) \\ &= 2 + 4\sum_{i=1}^{n}\Bigl(\frac{\partial \boldsymbol{e}}{\partial x_i}\frac{\partial \Delta\boldsymbol{e}}{\partial x_i}\Bigr) + 4\sum_{i,j=1}^{n}\Bigl(\frac{\partial^2 \boldsymbol{e}}{\partial x_i \partial x_j}\Bigr)^2 = 2 + 4\sum_{i,j=1}^{n}\Bigl(\frac{\partial^2 \boldsymbol{e}}{\partial x_i \partial x_j}\Bigr)^2 =: f. \end{aligned} \tag{7}$$

Note that the function $f \in C^{2,\gamma}(\overline{\Omega})$ is strictly positive on $\overline{\Omega}$.

Example 7. For $\Omega = B_R(0)$ we find

$$\boldsymbol{e}(x) = \frac{R^2 - \|x\|^2}{2n} \quad \text{and} \quad f(x) = 2 + \frac{4}{n}.$$

The main idea of the construction of the weighted problem with positive eigenfunction is the following: If we define $\tilde{w} = f/\boldsymbol{e}^2$, then the function $\boldsymbol{e}^2$ would be a solution to

$$\begin{cases} (-\Delta)^2 \boldsymbol{e}^2 = \tilde{w}\boldsymbol{e}^2 & \text{in } \Omega, \\ \boldsymbol{e}^2 = \dfrac{\partial}{\partial\nu}\boldsymbol{e}^2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $\boldsymbol{e}^2$ is a weighted eigenfunction with corresponding eigenvalue $\lambda = 1$ and there is a constant $c > 0$ such that $\boldsymbol{e}^2(x) \geq c\, d(x)^2$ for all $x \in \Omega$. We notice that f is strictly positive and $\boldsymbol{e}^2$ behaves like $d(x)^2$ near the boundary. So the weight function $\tilde{w}$ is unbounded and especially not Hölder-continuous on Ω. In order to deduce estimates for the Green's function and positivity results we will apply a converse of the Krein–Rutman theorem. In Section 3 we need regularity results from Agmon–Douglis–Nirenberg results, and Hölder-continuity of the weight function is necessary.

So, the combination of the positive functions $\boldsymbol{e}^2$ and f is not directly suitable and we need a combination where both functions grow like $d(x)^2$ near the boundary. In order to achieve this we modify f and consider $f_\varepsilon : \overline{\Omega} \to \mathbb{R}$ defined by

$$f_\varepsilon(x) = \chi_\varepsilon(d(x))^2 f(x), \tag{8}$$

where $\varepsilon > 0$ is small enough and $\chi_\varepsilon \in C^\infty(\mathbb{R}; \mathbb{R})$ is an ε-sized mollification of the sign-function. A sketch of χ_ε can be found in Figure 1. For ε small one finds $f_\varepsilon \in C^{2,\gamma}(\overline{\Omega})$ and on $\partial\Omega$ that

$$f_\varepsilon = \frac{\partial}{\partial\nu} f_\varepsilon = 0 \quad \text{and} \quad \frac{\partial^2}{\partial\nu^2} f_\varepsilon > 0.$$

Remark 8. The function χ_ε is constructed with the usual mollifiers $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ with support in $[-\varepsilon, \varepsilon]$ and defined by

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)$$

and

$$\varphi(t) = \begin{cases} c_m^{-1} \exp\left(-\dfrac{1}{1-t^2}\right) & \text{for } |t| < 1, \\ 0 & \text{for } |t| \geq 1, \end{cases} \qquad \text{with } c_m = \int_{-1}^{1} \exp\left(-\frac{1}{1-s^2}\right) ds.$$

With $\operatorname{sign}(t) = t/|t|$ for $t \neq 0$ we define the function

$$\chi_\varepsilon(t) = (\varphi_\varepsilon * \operatorname{sign})(t) \quad \text{for } t \in \mathbb{R}.$$

Note that $\chi_\varepsilon \in C^\infty(\mathbb{R})$ satisfies

$$\chi_\varepsilon(0) = 0, \quad \chi_\varepsilon'(0) = \frac{2}{c_m e}\varepsilon^{-1}, \quad \text{and} \quad \chi_\varepsilon(t) = 1 \quad \text{for } t > \varepsilon.$$

Moreover

$$\min\left(\frac{t}{\varepsilon}, 1\right) \leq \chi_\varepsilon(t) \leq \min\left(\frac{2t/(c_m e)}{\varepsilon}, 1\right) \quad \text{for } t \geq 0. \tag{9}$$

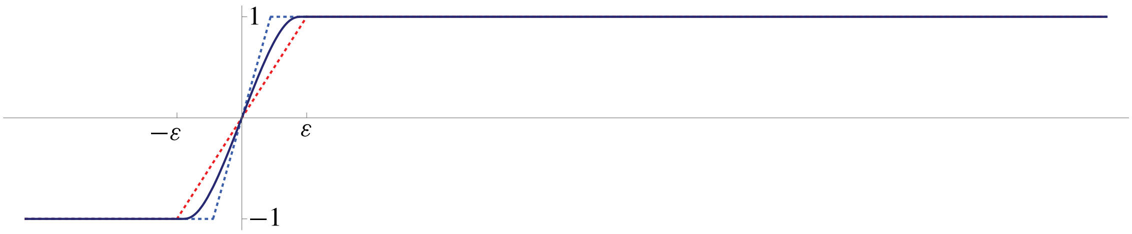

Figure 1. Sketch of χ_ε as mollified sign-function with the estimates from (9).

Letting u_ε be the solution of

$$\begin{cases} \Delta^2 u_\varepsilon = f_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = \dfrac{\partial}{\partial \nu} u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

we shall prove with the next proposition that $w_\varepsilon = f_\varepsilon / u_\varepsilon$ is well-defined and that $\varphi = u_\varepsilon$ with $\lambda = 1$ is an appropriate eigenfunction of the eigenvalue problem

$$\begin{cases} \Delta^2 \varphi = \lambda w_\varepsilon \varphi & \text{in } \Omega, \\ \varphi = \dfrac{\partial}{\partial \nu} \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Theorem 2 will follow from this result except for the simplicity of the eigenvalue.

Proposition 9. *Let f, f_ε, u_ε be defined in* (7), (8) *and* (10). *Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the following holds*:

(1) $w_\varepsilon := f_\varepsilon / u_\varepsilon \in C^{0,\gamma}(\overline{\Omega})$ *and* $\min\{w_\varepsilon(x) : x \in \overline{\Omega}\} > 0$.

(2) $\varphi := u_\varepsilon$ *is a strongly positive eigenfunction in the sense of* (2), *with eigenvalue* $\lambda = 1$, *for the weighted eigenvalue problem* (11).

Remark 10. One may guess that generically the eigenvalue $\lambda = 1$ is simple for $\varepsilon \in (0, \varepsilon_0)$. We need, however, that the eigenvalue is simple and not just generically. To obtain this we may fix $\varepsilon = \frac{1}{2}\varepsilon_0$ and proceed by an appropriate perturbation of f_ε for this fixed ε. This is done in Section 4 and yields a simple eigenvalue 1.

Proof. Let $\Omega(\varepsilon) = \{x \in \Omega : d(x) < \varepsilon\}$. One directly checks that for any $p \in [1, \infty)$ it holds that

$$\|f_\varepsilon - f\|_{L^p(\Omega)} \le \|f\|_{L^\infty(\Omega)} |\Omega(\varepsilon)|^{1/p} \to 0 \quad \text{for } \varepsilon \downarrow 0. \tag{12}$$

Since (12) holds, we find by Agmon–Douglis–Nirenberg [Gazzola et al. 2010, Theorem 2.20] that

$$\|u_\varepsilon - \boldsymbol{e}^2\|_{W^{4,p}(\Omega)} \le C_{ADN} \|f_\varepsilon - f\|_{L^p(\Omega)} \to 0 \quad \text{for } \varepsilon \downarrow 0. \tag{13}$$

By Sobolev imbedding [Adams and Fournier 2003, Theorem 4.12] and taking $p > n$ it follows that

$$\|u_\varepsilon - \boldsymbol{e}^2\|_{C^3(\overline{\Omega})} \to 0 \quad \text{for } \varepsilon \downarrow 0. \tag{14}$$

Using the mean value theorem, we find

$$\boldsymbol{e}(x)^2 - u_\varepsilon(x) \le \|u_\varepsilon - \boldsymbol{e}^2\|_{C^2(\overline{\Omega})}\, d(x)^2,$$

and applying (6) we obtain

$$u_\varepsilon(x) \geq \boldsymbol{e}(x)^2 - \|u_\varepsilon - \boldsymbol{e}^2\|_{C^2(\bar{\Omega})}\, d(x)^2 \geq (c_1^2 - \|u_\varepsilon - \boldsymbol{e}^2\|_{C^2(\bar{\Omega})})\, d(x)^2.$$

So there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the function u_ε is strongly positive and uniformly in the sense that $\tilde{c}_1 > 0$ exists, not depending on ε, it satisfies

$$u_\varepsilon(x) \geq \tilde{c}_1\, d(x)^2 \quad \text{in } \Omega. \tag{15}$$

Using the mean value theorem, we also find a constant $\tilde{c}_2 > 0$, also independent of ε, such that

$$u_\varepsilon(x) \leq \tilde{c}_2\, d(x)^2 \quad \text{in } \Omega. \tag{16}$$

Hence, we find by (15) an upper bound for f_ε:

$$f_\varepsilon(x) \leq \tilde{c}\varepsilon^{-2} u_\varepsilon(x) \tag{17}$$

for some constant $\tilde{c} > 0$. For $\varepsilon \in (0, \varepsilon_0)$ the function $\varphi_1 = u_\varepsilon$ is a strictly positive eigenfunction of (11) for $\lambda = 1$ and $0 \leq w_\varepsilon = f_\varepsilon / u_\varepsilon$. We also find by applying (7), (9), (16) and (17) that

$$0 < \frac{2}{\tilde{c}_2\, d^2} \min(d^2 \varepsilon^{-2}, 1) \leq \frac{f_\varepsilon}{u_\varepsilon} = w_\varepsilon \leq \tilde{c}\varepsilon^{-2} < \infty,$$

so $\min w_\varepsilon > 0$, and since $u_\varepsilon \in C^3(\bar{\Omega})$ and $f_\varepsilon \in C^{2,\gamma}(\bar{\Omega})$, we obtain $w_\varepsilon \in C^{0,\gamma}(\bar{\Omega})$. □

Remark 11. Note that even if one considers a small perturbation of f_ε, respectively w_ε, one obtains a positive eigenfunction with the eigenvalue $\lambda = 1$. For example, by setting $\tilde{f}_\varepsilon(x) = f_\varepsilon(x) + tq(x)$ for $t \in \mathbb{R}$ with $|t|$ small and $q \in C_c^\infty(\Omega)$, one finds

$$\tilde{u}_\varepsilon(x) = u_\varepsilon + t u_q(x),$$

where u_q is the solution to

$$\begin{cases} \Delta^2 u_q = q & \text{in } \Omega, \\ u_q = \dfrac{\partial}{\partial \nu} u_q = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, also

$$\tilde{w}_\varepsilon = \frac{f_\varepsilon + tq}{u_\varepsilon + t u_q} = w_\varepsilon + \sum_{k=1}^{\infty} t^k (-1)^k \left(\frac{u_q}{u_\varepsilon}\right)^{k-1} \frac{1}{u_\varepsilon}(u_q w_\varepsilon - q)$$

is a real analytic function of t. Analogously to (12)-(15) it follows that $\tilde{u}_\varepsilon \geq 0$ in Ω for sufficiently small $|t|$. We use this fact in Section 4.

3. Adapting to the weight

In this section we will present the new weighted setting and the results from [Schnieders and Sweers 2020] adjusted to the weighted case. The results in that paper depend strongly on the estimates in [Grunau et al. 2011] for the Green's function for $\lambda = 0$, which use the positive function $H_n : \bar{\Omega} \times \bar{\Omega} \to [0, \infty]$

defined by

$$H_n(x,y)=\begin{cases}(d(x)^2d(y)^2)^{1-n/4}\min\left(1,\dfrac{d(x)^2d(y)^2}{|x-y|^4}\right)^{n/4} & \text{for } 1\le n<4,\\ \log\left(1+\dfrac{d(x)^2d(y)^2}{|x-y|^4}\right) & \text{for } n=4,\\ |x-y|^{4-n}\min\left(1,\dfrac{d(x)^2d(y)^2}{|x-y|^4}\right) & \text{for } n>4.\end{cases} \tag{18}$$

The functions H_n give the asymptotic behaviour of the biharmonic Green's function on bounded smooth domains $\Omega\subset\mathbb{R}^n$ besides a rank-1 perturbation.

Notation 12. Throughout the paper:

(1) Calligraphic $\mathcal{H}_n : L^2(\Omega)\to L^2(\Omega)$ denotes the operator defined by

$$(\mathcal{H}_n f)(x)=\int_\Omega H_n(x,y)\,f(y)\,dy.$$

(2) For $w\in C^{0,\gamma}(\overline{\Omega})$ a positive weight as in Section 2 we write $\tilde f := f/w$ for $f\in L^2(\Omega)$ and we let $G_{\lambda,w}(\,\cdot\,,\cdot\,)$ denote the Green's function for

$$\begin{cases}(\Delta^2-\lambda w)u=w\,\tilde f & \text{in } \Omega,\\ u=\dfrac{\partial}{\partial\nu}u=0 & \text{on } \partial\Omega;\end{cases} \tag{19}$$

that is, $u(x)=\int_\Omega G_{\lambda,w}(x,y)\tilde f(y)\,dy$ solves (19) if defined. By $G_{0,1}$ we mean the Green's function for the biharmonic Dirichlet problem (5) and if we write $G_{\lambda,1}$, we consider the Green's function for (19) without a weight function, i.e., $w\equiv 1$.

(3) For $\mathcal{A},\mathcal{B}: L^2(\Omega)\to L^2(\Omega)$ we write $\mathcal{A}\ge\mathcal{B}$ whenever, for all $f\in L^2(\Omega)$ with $f(x)\ge 0$ a.e., it holds that

$$(\mathcal{A}f)(x)\ge(\mathcal{B}f)(x)\text{ a.e.} \tag{20}$$

If $\mathcal{A},\mathcal{B}$ are defined through kernels, i.e., $(\mathcal{A}f)(x)=\int_\Omega A(x,y)f(x)\,dx$, and these kernels are continuous except maybe for the diagonal $x=y$, then $A(x,y)\ge B(x,y)$ for all $x\ne y\in\Omega$ implies (20).

One finds for $G_{0,1}$ the Green's function for (1) with $\lambda=0$ that for c large enough

$$G_{0,1}(x,y)+cd(x)^2d(y)^2\sim H_n(x,y)\quad\text{for all } x,y\in\Omega.$$

In [Schnieders and Sweers 2020] such estimate was first extended to $G_{\lambda,1}$ for any bounded interval in $\mathbb{R}$ below λ_1. In a second step the asymptotic behaviour of the constant c was studied for $\lambda\uparrow\lambda_1$. We have to adapt these results for the weighted problem and can do so by similar lemmata to those in [Schnieders and Sweers 2020, Sections 4–7].

Remark 13. The weight w does not influence the arguments in the proofs of the results in [Schnieders and Sweers 2020]. The results are consequences of estimates for the Green's function and since there exist two constants $c_{w,1}, c_{w,2} > 0$ such that

$$c_{w,1} \le w(x) \le c_{w,2} \quad \text{for all } x \in \Omega,$$

we can follow the steps with only adjusted constants.

We will obtain the two following results, which are variations of [Schnieders and Sweers 2020, Theorems 1, 2]:

Theorem 14. *Suppose that $\Omega \subset \mathbb{R}^n$ with $n \ge 2$ is a bounded domain with $\partial\Omega \in C^{4,\gamma}$ for some $\gamma \in (0, 1)$. Suppose $0 < w \in C^{0,\gamma}(\bar{\Omega})$ and let $\{\lambda_{i,w}\}_{i\in\mathbb{N}^+}$ denote the eigenvalues for* (4) *and take $M, \delta \in \mathbb{R}^+$. Set*

$$I_{M,\delta} = [-M, M] \setminus \bigcup_{i=1}^{\infty} (\lambda_{i,w} - \delta, \lambda_{i,w} + \delta).$$

Let $G_{\lambda,w}$ be the Green's function for (19)*. Then there are $c_1, c_2, c_3 > 0$, depending on the domain, M, δ and w, such that for all $\lambda \in I_{M,\delta}$ the following estimate holds*:

$$c_1 \, H_n(x, y) \le G_{\lambda,w}(x, y) + c_2 \, d(x)^2 \, d(y)^2 \le c_3 \, H_n(x, y) \quad \text{for all } x, y \in \Omega. \tag{21}$$

Remark 15. For $w = 1$ and $\lambda = 0$ this result can be found in [Grunau et al. 2011, Theorem 1]. For $w = 1$ and $\lambda \in I = [-M, \lambda_1 - \delta]$ see [Schnieders and Sweers 2020, Theorem 1].

Theorem 16. *Suppose that $\Omega \subset \mathbb{R}^n$ with $n \ge 2$ is a bounded domain with $\partial\Omega \in C^{4,\gamma}$ for some $\gamma \in (0, 1)$. Let $\delta > 0$. Suppose $0 < w \in C^{0,\gamma}(\bar{\Omega})$ and that $\lambda_{j,w}$ is a simple eigenvalue of* (4) *with the corresponding eigenfunction $\varphi_{j,w}$ strongly positive as in* (2)*. Suppose $I_\delta = [\lambda_{j,w} - \delta, \lambda_{j,w})$ contains no eigenvalue. Let $G_{\lambda,w}$ be the Green's function for* (19)*. Then there exist $C_1, C_2, C_3 > 0$, depending on Ω, δ and w, such that for all $\lambda \in I_\delta$*

$$G_{\lambda,w}(x, y) \ge C_1 \, H_n(x, y) + \left(\frac{C_2}{\lambda_{j,w} - \lambda} - C_3 \right) \varphi_{j,w}(x) \varphi_{j,w}(y) \quad \text{for all } x, y \in \Omega. \tag{22}$$

Proof of Theorems 14 and 16. The proofs use the estimate from [Grunau et al. 2011] just as [Schnieders and Sweers 2020] does. Instead of using a Weyl-type asymptotics for the growth rate of eigenvalues, we exploit here regularity results and Sobolev imbeddings.

• We first recall the standard arguments for existence and the relation with corresponding eigenvalues. Let $L^2_w(\Omega)$ denote the Hilbert space $(L^2(\Omega), \langle \cdot, \cdot \rangle_{L^2_w})$,

$$\langle u, v \rangle_{L^2_w(\Omega)} := \int_\Omega u(x) v(x) w(x) \, dx,$$

equivalent with the standard inner product since $w \in C^{0,\gamma}(\bar{\Omega})$ satisfies $w > 0$ on $\bar{\Omega}$.

A weak solution to (1) for $f \in L^2(\Omega)$ is defined by $u \in W^{2,2}_0(\Omega)$ such that

$$\int_\Omega (\Delta u \Delta v - \lambda w u v - \tilde{f} w v) \, dx = 0 \quad \text{for all } v \in W^{2,2}_0(\Omega). \tag{23}$$

We obtain that the standard norm on $W_0^{2,2}(\Omega)$ is equivalent to the norm

$$\|u\| := \|\Delta u\|_{L^2(\Omega)} - \lambda\sqrt{\langle u, u\rangle_{L^2_w(\Omega)}} \quad \text{for any } \lambda \le 0.$$

Hence, by the Riesz representation theorem there exists a solution $u_{\lambda,w}$ to (23) for every $\tilde f \in L^2_w(\Omega)$ and $\lambda \le 0$. The solution operator $\mathcal{G}_{\lambda,w}$, i.e., $u_{\lambda,w} = \mathcal{G}_{\lambda,w}\tilde f$ solves (19), is well-defined on $L^2_w(\Omega)$. Using the results by Agmon–Douglis–Nirenberg [Gazzola et al. 2010, Theorems 2.19, 2.10], we find that

$$\mathcal{G}_{\lambda,w} : L^2_w(\Omega) \to W^{4,2}(\Omega) \cap W_0^{2,2}(\Omega)$$

is an isomorphism for $\lambda \le 0$.

With $\mathcal{I}$ the compact imbedding from $W^{4,2}(\Omega)$ to $L^2_w(\Omega)$, one finds $\mathcal{I} \circ \mathcal{G}_{0,w}$ is compact and it is the inverse operator of $A_w : D(A_w) \subset L^2_w(\Omega) \to L^2_w(\Omega)$ defined by

$$D(A_w) = W^{4,2}(\Omega) \cap W_0^{2,2}(\Omega) \quad \text{with } A_w = \frac{1}{w}\Delta^2.$$

Since $\mathcal{I} \circ \mathcal{G}_{0,w}$ is compact, the spectrum of A_w is discrete and since A_w is self-adjoint and positive, i.e., $\langle A_w u, u\rangle_{L^2_w(\Omega)} = \langle A_1 u, u\rangle_{L^2(\Omega)} > 0$ for $u \ne 0$, the spectrum consists of countably many real eigenvalues $\{\lambda_{i,w}\}_{i\in\mathbb{N}^+}$, with $0 < \lambda_{1,w} \le \lambda_{2,w} \le \cdots \to \infty$ and corresponding eigenfunctions $\{\varphi_{i,w}\}_{i\in\mathbb{N}^+}$. The eigenfunctions can be chosen such that they are orthonormal in the norm induced by $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2_w(\Omega)}$. By the Hilbert–Schmidt theorem we then find a complete orthonormal system of eigenfunctions, still denoted by $\{\varphi_{i,w}\}_{i\in\mathbb{N}}$, and such that for $\lambda \notin \{\lambda_{i,w}\}_{i\in\mathbb{N}^+}$ and $\tilde f \in L^2_w(\Omega)$

$$\mathcal{G}_{\lambda,w}\tilde f = \sum_{i=1}^{\infty} \frac{1}{\lambda_{i,w} - \lambda}\langle \varphi_{i,w}, \tilde f\rangle_{L^2_w(\Omega)}\,\varphi_{i,w}.$$

- Next we recall an asymptotic formula for $\mathcal{G}_{\lambda,w}$ that uses $\mathcal{G}_{0,1}$. If $|\lambda| < \lambda_{1,w}$ and $u_{\lambda,w} = \mathcal{G}_{\lambda,w}\tilde f$, then also

$$u_{\lambda,w} = \mathcal{G}_{0,w}(\lambda u_{\lambda,w} + \tilde f) = \mathcal{G}_{0,1}(\lambda w u_{\lambda,w} + w\,\tilde f),$$

which is equivalent to

$$(\mathcal{I} - \lambda\mathcal{G}_{0,1}(w\,\cdot\,))u_{\lambda,w} = \mathcal{G}_{0,1}(w\tilde f),$$

where $\mathcal{G}_{0,1}$ is the solution operator for (5). For $\lambda \in (-\lambda_{1,w}, \lambda_{1,w})$ we may invert $\mathcal{I} - \lambda\mathcal{G}_{0,1}(w\,\cdot\,)$ and by using a Neumann series we obtain

$$u_{\lambda,w} = \sum_{k=0}^{\infty} \lambda^k (\mathcal{G}_{0,1}(w\,\cdot\,))^{k+1}\tilde f. \tag{24}$$

We can still find a similar expression when $|\lambda| > \lambda_{1,w}$ when we single out the lower eigenfunctions. Let $\lambda_{m,w}$ be the smallest eigenvalue larger than M and we may use for $\lambda \in (-\lambda_{m,w}, \lambda_{m,w}) \setminus \{\lambda_{i,w}\}_{i<m}$ the expression

$$u_{\lambda,w} = \underbrace{\sum_{i=1}^{m} \frac{1}{\lambda_{i,w} - \lambda}\mathcal{P}_i\tilde f}_{I} + \sum_{k=0}^{\infty} \lambda^k (\mathcal{G}_{0,1}(w\,\cdot\,))^{k+1}\mathcal{P}_{m,+}\tilde f, \tag{25}$$

with the following orthogonal projections in $L^2_w(\Omega)$:

$$(\mathcal{P}_i v)(x) := \varphi_{i,w}(x) \int_\Omega \varphi_{i,w}(y)\, v(y)\, w(y)\, dy,$$
$$\mathcal{P}_{i,+} := \mathcal{I} - \mathcal{P}_1 - \cdots - \mathcal{P}_i.$$

We may suppose that $\lambda_{m,w} > M \geq \lambda_{j,w}$.

• In order to estimate I in (25) we will use $\mathcal{D} : L^2(\Omega) \to L^2(\Omega)$, defined by

$$(\mathcal{D} f)(x) := d(x)^2 \int_\Omega f(y) d(y)^2\, dy. \tag{26}$$

With the mean value theorem, we get for every $\varphi \in C^2(\overline{\Omega}) \cap C^1_0(\overline{\Omega})$ and all $x \in \Omega$

$$|\varphi(x)| \leq \|\varphi\|_{C^2(\overline{\Omega})} d(x)^2. \tag{27}$$

Since (27) holds for each eigenfunction $\varphi_{i,w}$ there exists $c_i > 0$ such that

$$-c_i \mathcal{D} \leq \mathcal{P}_i \leq c_i \mathcal{D}. \tag{28}$$

• We split the series on the right of (25) into a finite part with singular behaviour *III* and an infinite remainder *II* that can be estimated by $\mathcal{D}$. The splitting for those λ above is as follows:

$$\sum_{k=0}^{\infty} \lambda^k (\mathcal{G}_{0,1}(w\,\cdot))^{k+1} \mathcal{P}_{m,+} \tilde{f} = \underbrace{\sum_{k=2k_n}^{\infty} \lambda^k \mathcal{G}_{0,w}^{k+1} \mathcal{P}_{m,+} \tilde{f}}_{II} + \underbrace{\sum_{k=0}^{2k_n-1} \lambda^k \mathcal{G}_{0,w}^{k+1} \mathcal{P}_{m,+} \tilde{f}}_{III}, \tag{29}$$

where $k_n = \left[\frac{1}{8}(n+4)\right] + 1$.

• This number k_n is determined as follows. With $\partial\Omega \in C^{4,\gamma}$ the regularity results of Agmon–Douglis–Nirenberg state that for all $p \in (1, \infty)$

$$\mathcal{G}_{0,1} : L^p(\Omega) \to W^{4,p}(\Omega) \cap W^{2,p}_0(\Omega),$$
$$\mathcal{G}_{0,1} : C^{0,\gamma}(\overline{\Omega}) \to C^{4,\gamma}(\overline{\Omega}) \cap C^1_0(\overline{\Omega})$$

are bounded operators. Alternating such a regularity result with a Sobolev imbedding [Adams and Fournier 2003, Theorem 4.12],

$$W^{4,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for } 4 - \frac{n}{p} > -\frac{n}{q},$$
$$W^{4,p}(\Omega) \hookrightarrow C^{k,\gamma}(\overline{\Omega}) \quad \text{for } 4 - \frac{n}{p} > k + \gamma,$$

one finds after $k_n = \left[\frac{1}{8}(n+4)\right] + 1$ iterations that

$$(\mathcal{G}_{0,1}(w\,\cdot))^{k_n} : L^2_w(\Omega) \to W^{4,p}(\Omega) \cap W^{2,p}_0(\Omega)$$

is bounded for some $p > \frac{1}{2}n$ (and $p \geq 2$): there is $c > 0$ such that

$$\|(\mathcal{G}_{0,1}(w\,\cdot))^{k_n} f\|_{W^{4,p}(\Omega)\cap W^{2,p}_0(\Omega)} \leq c \|f\|_{L^2_w(\Omega)} \quad \text{for all } f \in L^2(\Omega). \tag{30}$$

Since $W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ imbeds in $C^2(\bar{\Omega}) \cap C_0^1(\bar{\Omega})$ for $p > \frac{1}{2}n$ there exists $\tilde{c} > 0$ such that for all $\varphi \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$

$$\sup_{x\in\Omega}\left|\frac{\varphi(x)}{d(x)^2}\right| \le \|\varphi\|_{C^2(\bar{\Omega})} \le \tilde{c}\|\varphi\|_{W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega)}. \tag{31}$$

By combining (30) and (31) we find for $C = c\tilde{c}$ that

$$|(\mathcal{G}_{0,1}(w\,\cdot))^{k_n} f(x)| \le C\|f\|_{L^2_w(\Omega)} d(x)^2 \quad \text{for all } f \in L^2(\Omega). \tag{32}$$

• This number k_n not only allows the estimate in (32) but also allows us to have a dual estimate by working in Sobolev spaces with a negative coefficient. By duality one finds that also

$$(\mathcal{G}_{0,1}(w\,\cdot)^*)^{k_n} : (W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega))^* \to L^2_w(\Omega)$$

is bounded with k_n as above for some $p > \frac{1}{2}n$ (and $p \ge 2$). Therefore we find a constant c such that for all $g \in (W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega))^*$

$$\|(\mathcal{G}_{0,1}(w\,\cdot)^*)^{k_n} g\|_{L^2_w(\Omega)} \le c\|g\|_{(W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega))^*}. \tag{33}$$

Since $p \ge 2$, one has $L^2(\Omega) \subset (W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega))^*$ in the sense that $f \in L^2(\Omega)$ determines a continuous linear mapping on $W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$. Indeed, we have $\langle f, \cdot\rangle_{L^2_w(\Omega)} \in (W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega))^*$ for all $f \in L^2(\Omega)$, see [Adams and Fournier 2003, Paragraph 3.13]. For $f \in L^2(\Omega)$ the symmetry of the kernel implies that $\mathcal{G}_{0,1}(w\,\cdot)^* f = \mathcal{G}_{0,1}(w\,f)$. Moreover, using the imbedding in (31) one obtains for $p > \frac{1}{2}n$

$$\begin{aligned}\|f\|_{(W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega))^*} &:= \sup\left\{\int_\Omega f(x)w(x)\varphi(x)\,dx : \varphi \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \text{ with } \|\varphi\|_{W^{4,p}(\Omega)} \le 1\right\}\\ &\le \tilde{c}\sup\left\{\int_\Omega f(x)w(x)\varphi(x)\,dx : \varphi \in C^2(\bar{\Omega}) \cap C_0^1(\bar{\Omega}) \text{ with } \|\varphi\|_{C^2(\bar{\Omega})} \le 1\right\}.\end{aligned} \tag{34}$$

With (27) and $c_{w,2}$ as in Remark 13 we find that for all $f \in L^2(\Omega)$ and $\varphi \in C^2(\bar{\Omega}) \cap C_0^1(\bar{\Omega})$ with $\|\varphi\|_{C^2(\bar{\Omega})} \le 1$

$$\int_\Omega f(x)w(x)\varphi(x)\,dx \le \int_\Omega |f(x)|\,w(x)\,|\varphi(x)|\,dx \le c_{w,2}\int_\Omega |f(x)|\,d(x)^2\,dx. \tag{35}$$

Inequality (34) and (35) imply

$$\|f\|_{(W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega))^*} \le \tilde{c}\,c_{w,2}\int |f(x)|\,d(x)^2\,dx. \tag{36}$$

By combining (33) and (36) we find a constant $C > 0$ such that

$$\|(\mathcal{G}_{0,1}(w\,\cdot))^{k_n} f\|_{L^2_w(\Omega)} \le C\int |f(x)|d(x)^2\,dx \quad \text{for all } f \in L^2(\Omega). \tag{37}$$

• For part *II* in (29) we write

$$\sum_{k=2k_n}^{\infty} \lambda^k \mathcal{G}_{0,w}^{k+1} \mathcal{P}_{m,+} \tilde{f} = \lambda^{2k_n} \mathcal{G}_{0,w}^{k_n} \left(\sum_{k=0}^{\infty} \lambda^k \mathcal{G}_{0,w}^{k+1} \mathcal{P}_{m,+} \right) \mathcal{G}_{0,w}^{k_n} \tilde{f},$$

with the middle series denoting a bounded operator in $L_w^2(\Omega)$. With (32) and (37) we find that $\tilde{c}_m > 0$ exists with

$$\left| \sum_{k=2k_n}^{\infty} \lambda^k \mathcal{G}_{0,w}^{k+1} \mathcal{P}_{m,+} f(x) \right| \leq \tilde{c}_m (\mathcal{D}|f|)(x) \quad \text{for all } f \in L^2(\Omega).$$

This means that there is $C_m > 0$ such that

$$-C_m \, \mathcal{D} \leq \sum_{k=2k_n}^{\infty} \lambda^k \mathcal{G}_{0,w}^{k+1} \mathcal{P}_{m,+} \leq C_m \, \mathcal{D}. \tag{38}$$

• We are left with an estimate for *III* in (29). We refer to [Schnieders and Sweers 2020, Corollary 9, Lemma 10, 11], from which it follows that for each $k \geq 1$ there are $c_{1,k}, c_{2,k}, c_{3,k}, c_{4,k}, c_{5,k} > 0$ such that

$$c_{1,k}\mathcal{D} \leq c_{2,k}\mathcal{H}_n^k \leq \mathcal{G}_{0,1}^k + c_{3,k}\mathcal{D} \leq c_{4,k}\mathcal{H}_n^k \leq c_{5,k}\mathcal{H}_n.$$

Since we consider such estimates for only finitely many terms, the additional factor w only results in adapted constants and we again find

$$c_1\mathcal{H}_n - c_2\mathcal{D} \leq \sum_{k=0}^{2k_n-1} \lambda^k \mathcal{G}_{0,w}^{k+1} \mathcal{P}_{m,+} \leq c_3\mathcal{H}_n - c_4\mathcal{D}. \tag{39}$$

• We may wrap up our estimates to finish the proof for both theorems. If $|\lambda| < \lambda_{m,w}$ and $|\lambda - \lambda_{i,w}| > \delta$ for all $i \leq m$, then we may combine (38), (28) and (39) to find through the splitting in (25) and (29) that

$$\mathcal{G}_{\lambda,w} \geq C_{1,m}\mathcal{H}_n - C_{2,m}\mathcal{D}.$$

This shows Theorem 14.

For $\lambda \in [\lambda_{j,w} - \delta, \lambda_{j,w})$ we also single out $\mathcal{P}_j$ in *I* and find for those λ uniform constants $C_0, C_1, C_2 \in \mathbb{R}^+$ such that

$$\mathcal{G}_{\lambda,w} \geq C_0 \frac{1}{\lambda_{j,w} - \lambda} \mathcal{P}_j + C_1\mathcal{H}_n - C_2\mathcal{D}. \tag{40}$$

Only here we will use that besides $\varphi_{j,w} \in C^{4,\gamma}(\bar{\Omega}) \cap C_0^1(\bar{\Omega})$ this function $\varphi_{j,w}$ is strongly positive and hence there are $c_1, c_2 > 0$ such that

$$c_1\mathcal{D} \leq \mathcal{P}_j \leq c_2\mathcal{D}. \tag{41}$$

The estimate in (22) follows from (40) and (41) and this completes the proof of Theorem 16. □

4. Simplicity of the eigenvalue

If $\lambda_{w_\varepsilon} = 1$ is a simple eigenvalue of (11), we consider problem (1) with $w = w_\varepsilon$ and find a positivity-preserving property for λ in a small left neighbourhood of λ_{w_ε}. If the multiplicity of $\lambda_{w_\varepsilon} = 1$ is greater than or equal to 2, we will show the simplicity of the eigenvalue after a small perturbation of the weight function w_ε.

The perturbations we consider start from the function f_ε defined in (8).

Definition 17. Let f_ε, u_ε be as in Proposition 9 and Remark 10. For $q \in C_c^\infty(\Omega)$ and $t \in \mathbb{R}$ with $|t|$ small, set

$$w_{tq} = \frac{f_\varepsilon + tq}{u_\varepsilon + t\mathcal{G}_{0,1}(q)}, \tag{42}$$

where $\mathcal{G}_{0,1}$ is the solution operator for (5), and define $A(tq) : W_0^{2,2}(\Omega) \cap W^{4,2}(\Omega) \to L^2(\Omega)$ by

$$A(tq) = \Delta^2 - w_{tq}. \tag{43}$$

We will consider the t-dependent eigenvalue problems

$$\mathcal{A}(tq) : \begin{cases} (\Delta^2 - w_{tq})\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = \dfrac{\partial}{\partial\nu}\varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{44}$$

Remark 18. Note that the multiplicity of the eigenvalue $\lambda_{w_\varepsilon} = 1$ for (11) coincides with the multiplicity of $\lambda = 0$ for (44) with $t = 0$, since $w_0 = w_\varepsilon$.

Assuming that $\lambda = 0$ is an eigenvalue of multiplicity $m \geq 2$ for (44) with $t = 0$, one finds by [Kato 1980, Theorem 3.9, Chapter 7] or [Rellich 1969, pages 76–77] the existence of an interval $(-t_0, t_0) \subset \mathbb{R}$ and m real analytic functions

$$t \mapsto (\lambda_{i,t,q}, \varphi_{i,t,q}) : (-t_0, t_0) \to \mathbb{R} \times C_0^1(\bar{\Omega}) \cap C^{4,\gamma}(\bar{\Omega}) \quad \text{for } i \in \{1, \dots, m\},$$

with:

(1) $(\lambda_{i,t,q}, \varphi_{i,t,q})$ are pairs of eigenvalues and eigenfunctions for $\mathcal{A}(tq)$ for all $i \in \{1, \dots, m\}$.

(2) $\{\varphi_{i,0,q}\}_{i=1}^m$ is an orthogonal system and so $\{\varphi_{i,t,q}\}_{i=1}^m$ is independent for $|t|$ small.

(3) $\lambda_{i,0,q} = 0$ for all $i \in \{1, \dots, m\}$.

With our construction we may fix the first one by

$$\varphi_{1,t,q} = u_\varepsilon + t\mathcal{G}_{0,1}(q) \tag{45}$$

and find

$$\lambda_{1,t,q} = 0 \quad \text{for all } t \in (-t_0, t_0).$$

We will show that there exists q_1 such that

$$\lambda'_{k,0,q_1} := \left(\frac{\partial}{\partial t}\lambda_{k,t,q_1}\right)_{t=0} \neq 0$$

for at least one $k \in \{2, \dots, m\}$. In that case one finds for some small positive t_1 that $\lambda_{k,t_1,q_1} \neq 0$ and hence that 0 is an eigenvalue of multiplicity at most $m-1$ for $\mathcal{A}(t_1 q_1)$. If the multiplicity of the eigenvalue 0 for $\mathcal{A}(t_1 q_1)$ is 1, we are done. Otherwise we repeat our arguments for $\mathcal{A}(t_1 q_1 + tq)$. After $k \leq m-1$ steps we have found an eigenvalue problem $\mathcal{A}(t_1 q_1 + \cdots + t_k q_k)$ having 0 as a simple eigenvalue. The idea of the proof was inspired by [Albert 1975; Teytel 1999].

Lemma 19. *Suppose that* 0 *is an eigenvalue of multiplicity* $m \geq 2$ *for problem* (44) *with* $t = 0$. *Then there exist* $k \in \{2, \dots, m\}$ *and* $q_1 \in C_c^\infty(\Omega)$ *such that*

$$\left(\frac{\partial}{\partial t}\lambda_{k,t,q_1}\right)_{|t=0} \neq 0.$$

Proof. Suppose that $\lambda'_{k,0,q} = 0$ for all $k \in \{2, \dots, m\}$ and $q \in C_c^\infty(\Omega)$. Note that $\lambda'_{1,t,q} = 0$ by construction. Differentiation with respect to t of

$$A(tq)\varphi_{k,t,q} = \lambda_{k,tq}\varphi_{k,t,q} \quad \text{for all } k \in \{1, \dots, m\}$$

yields

$$(A(tq) - \lambda_{k,t,q})\frac{\partial}{\partial t}\varphi_{k,t,q} = \left(\frac{\partial}{\partial t}w_{tq} + \lambda'_{k,t,q}\right)\varphi_{k,t,q}$$

and setting $t = 0$, we find using (42), (43) and $\lambda'_{k,0,q} = 0$ that

$$A(0)\left(\frac{\partial}{\partial t}\varphi_{k,t,q}\right)_{|t=0} = \frac{1}{u_\varepsilon}(q - w_0\mathcal{G}_{0,1}(q))\varphi_{k,0,q}.$$

Hence, we obtain that $(1/u_\varepsilon)(q - w_0\mathcal{G}_{0,1}(q))\varphi_{k,0,q}$ is in the range of $A(0)$ for all $q \in C_c^\infty(\Omega)$. Since every eigenfunction in $\ker(A(0))$ can be written in the form $\sum_{k=1}^m c_k\varphi_{k,0,q}$ and $A(0)$ is self-adjoint, it follows that

$$\frac{1}{u_\varepsilon}(q - w_0\mathcal{G}_{0,1}(q))\psi_1 \perp \ker(A(0)) \quad \text{for all } \psi_1 \in \ker(A(0)),$$

or in other words

$$\int_\Omega \frac{1}{u_\varepsilon}(q - w_0\mathcal{G}_{0,1}(q))\psi_1\,\psi_2\,dx = 0 \quad \text{for all } \psi_1, \psi_2 \in \ker(A(0)).$$

Since $G_{0,1}(x, y) = G_{0,1}(y, x)$, we obtain

$$\begin{aligned} 0 &= \int_\Omega \frac{1}{u_\varepsilon}(q - w_0\mathcal{G}_{0,1}(q))\psi_1\,\psi_2\,dx \\ &= \int_\Omega \left(q(x) - w_0(x)\int_\Omega G_{0,1}(x, y)q(y)\,dy\right)\frac{\psi_1(x)\,\psi_2(x)}{u_\varepsilon(x)}\,dx \\ &= \int_\Omega q(x)\left(\frac{\psi_1(x)\,\psi_2(x)}{u_\varepsilon(x)} - \mathcal{G}_{0,1}\left(w_0\frac{\psi_1\,\psi_2}{u_\varepsilon}\right)(x)\right)dx \end{aligned} \tag{46}$$

and we can use the fundamental lemma of calculus of variations to find for all $\psi_1, \psi_2 \in \ker(A(0))$ that

$$\frac{\psi_1(x)\,\psi_2(x)}{u_\varepsilon(x)} - \mathcal{G}_{0,1}\left(w_0\frac{\psi_1\psi_2}{u_\varepsilon}\right)(x) = 0.$$

So if ψ_1 and ψ_2 are eigenfunctions of $\mathcal{A}(0)$ with $\lambda = 0$ in (44), then also

$$\tilde{\psi}_{1,2} := \frac{\psi_1\,\psi_2}{u_\varepsilon} \tag{47}$$

is an eigenfunction for $\mathcal{A}(0)$ with $\lambda = 0$. This is obvious for $\psi_1 = u_\varepsilon$, since then $\tilde{\psi}_{1,2} = \psi_2$, but it is not to be expected for all $\psi_1, \psi_2 \in \ker(A(0))$. Indeed, we will show that this cannot be true. Therefore fix some eigenfunction $\psi \in \ker(A(0))\backslash\{0\}$ orthogonal to u_ε.

Let $x_0 \in \Omega$ be a point on a nodal line of ψ. Indeed the existence of the nodal line follows since u_ε is positive with ψ orthogonal. Suppose that $\beta_0 \in [1, \infty]$ is the largest constant such that

$$\lim_{x \to x_0} \frac{\psi(x)}{|x - x_0|^\beta} = 0 \quad \text{for all } \beta < \beta_0.$$

Here $\beta_0 \geq 1$ follows from the fact that ψ is differentiable and $\psi(x_0) = 0$. By repeating (47) we find nonzero eigenfunctions $\{\psi_n\}_{n \in \mathbb{N}}$ defined by

$$\psi_n(x) = \left(\frac{\psi(x)}{u_\varepsilon(x)}\right)^n \psi(x)$$

and $\beta_n = (n+1)\beta_0$ is the largest constant in $[1, \infty]$ such that

$$\lim_{x \to x_0} \frac{\psi_n(x)}{|x - x_0|^\beta} = 0 \quad \text{for all } \beta < \beta_n. \tag{48}$$

Since the multiplicity is m, there is $m_0 \leq m$ such that ψ_{m_0} is a linear combination of the previous ones. Since any such linear combination inherits the behaviour as in (48) of the lowest-order term ψ_n, one finds a contradiction for $\beta_0 < \infty$. Hence ψ_n and also $D^\alpha \psi_n$, with $|\alpha| \leq 4$ and $n \geq |\alpha|$, contain a factor ψ and satisfy

$$\lim_{x \to x_0} \frac{D^\alpha \psi_n(x)}{|x - x_0|^\beta} = 0 \quad \text{for all } \beta \in \mathbb{R}. \tag{49}$$

One finds by the unique continuation theorem of [Shirota 1960] that $\psi_n \equiv 0$ for $n \geq 4$ and hence that $\psi \equiv 0$, a contradiction. So there exists $q_1 \in C_c^\infty(\Omega)$ and $k \in \{1, \ldots, m\}$ such that $\lambda'_{k,0,q_1} \neq 0$. □

The previous lemma implies:

Corollary 20. *Let ε be fixed as in Remark 10. Then there is $q^* \in C_c^\infty(\Omega)$ such that*

(1) $w^* = (f_\varepsilon + q^*)/(u_\varepsilon + \mathcal{G}_{0,1}(q^*)) \in C^{0,\gamma}(\overline{\Omega})$ *is strictly positive on* $\overline{\Omega}$, *and*

(2) $\varphi = u_\varepsilon + \mathcal{G}_{0,1}(q^*)$ *is a strongly positive eigenfunction in the sense of* (2) *for*

$$\begin{cases} (\Delta^2 - w^*)\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = \dfrac{\partial}{\partial\nu}\varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

with simple eigenvalue $\lambda = 0$.

Proof. If the multiplicity of the eigenfunction $\varphi = u_\varepsilon$ for the weight function $w = f_\varepsilon/u_\varepsilon$ is $m \geq 2$ we may proceed as in Lemma 19 and find q_1 such that for $t_1 > 0$ small enough, problem $\mathcal{A}(t_1 q_1)$ contains a positive weight and has a positive eigenfunction φ_{1,t_1,q_1} with eigenvalue 0 of multiplicity at most $m - 1$. Repeating the argument now starting with $\mathcal{A}(t_1 q_1)$ as in (43) and considering $\mathcal{A}_1(tq) = \mathcal{A}(t_1 q_1 + tq)$, we may again reduce the multiplicity. After at most $k \leq m - 1$ steps the multiplicity for $\mathcal{A}(q^*)$ with

$$q^* = t_1 q_1 + t_2 q_2 + \cdots + t_k q_k,$$

with $t_1 \gg t_2 \gg \cdots \gg t_k > 0$, has reduced to 1. □

Using this result, the proof of Theorem 2 is complete.

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