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SEMICLASSICAL RESOLVENT ESTIMATES FOR HÖLDER POTENTIALS

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We first prove semiclassical resolvent estimates for the Schrödinger operator in $\mathbb{R}^d$, $d \ge 3$, with real-valued potentials which are Hölder with respect to the radial variable. Then we extend these resolvent estimates to exterior domains in $\mathbb{R}^d$, $d \ge 2$, and real-valued potentials which are Hölder with respect to the space variable. As an application, we obtain the rate of the decay of the local energy of the solutions to the wave equation with a refraction index which may be Hölder, Lipschitz or just L^∞.

1. Introduction and statement of results

In this paper we are going to study the resolvent of the Schrödinger operator

$$P(h) = -h^2\Delta + V(x),$$

where $0 < h \le 1$ is a semiclassical parameter, Δ is the negative Laplacian in $\mathbb{R}^d$, $d \ge 2$, and $V \in L^\infty(\mathbb{R}^d)$ is a real-valued potential satisfying the condition

$$V(x) \le p(|x|), \tag{1.1}$$

where $p(r) > 0$, $r \ge 0$, is a decreasing function such that $p(r) \to 0$ as $r \to \infty$. More precisely, we are interested in bounding the quantity

$$g_s^\pm(h,\varepsilon) := \log\big\|(|x|+1)^{-s}(P(h) - E \pm i\varepsilon)^{-1}(|x|+1)^{-s}\big\|_{L^2\to L^2}$$

from above by an explicit function of h, independent of ε, without imposing extra assumptions on the function p. Here $L^2 := L^2(\mathbb{R}^d)$, $0 < \varepsilon < 1$, $s > \frac{1}{2}$ is independent of h and $E > 0$ is a fixed energy level independent of h. Instead, we impose some regularity on the potential with respect to the radial variable $r = |x|$. Note that throughout this paper, the space C^1 will denote the Lipschitz functions, that is, the ones with first derivatives belonging to L^∞ (and not necessarily continuous).

We will first extend the result of [Datchev 2014] to a larger class of potentials. Recall that in [Datchev 2014] the bound

$$g_s^\pm(h,\varepsilon) \le Ch^{-1} \tag{1.2}$$

is proved when $d \ge 3$, with some constant $C > 0$ independent of h and ε, for potentials $V \in C^1(\overline{\mathbb{R}^+})$ with respect to the radial variable r and satisfying (1.1) with $p(|x|) = C_1(|x|+1)^{-\delta}$, as well as the condition

$$\partial_r V(x) \le C_2(|x|+1)^{-\beta}, \tag{1.3}$$

where $C_1, C_2, \delta > 0$ and $\beta > 1$ are some constants. We will prove the following:

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Theorem 1.1. *Let $d \geq 3$ and suppose that the potential V satisfies the conditions* (1.1) *and* (1.3). *Then there exists a constant $C > 0$ independent of h and ε but depending on s, E and the function p, such that the bound* (1.2) *holds for all $0 < h \leq 1$.*

Note that the bound (1.2) was first proved for smooth potentials in [Burq 2002]. A high-frequency analog of (1.2) on Riemannian manifolds was also proved in [Burq 1998] and [Cardoso and Vodev 2002]. When $d = 2$, the bound (1.2) is proved in [Shapiro 2019] for potentials $V \in C^1(\mathbb{R}^2)$ satisfying (1.1) with $p(|x|) = C_1(|x|+1)^{-\delta}$ as well as the condition

$$|\nabla V(x)| \leq C_2(|x|+1)^{-\beta}, \tag{1.4}$$

where $C_1, C_2, \delta > 0$ and $\beta > 1$ are some constants.

On the other hand, for compactly supported L^∞ potentials without any regularity, the weaker bound

$$g_s^\pm(h, \varepsilon) \leq Ch^{-4/3}\log(h^{-1}) \tag{1.5}$$

was proved for $0 < h \ll 1$ in [Klopp and Vogel 2019] and [Shapiro 2020] when $d \geq 2$. When $d \geq 3$, the bound (1.5) has been extended in [Vodev 2019] to potentials satisfying the condition

$$|V(x)| \leq C_3(|x|+1)^{-\delta}, \tag{1.6}$$

where $C_3 > 0$ and $\delta > 3$ are some constants. Note that (1.5) has been recently proved in [Galkowski and Shapiro 2020] for potentials satisfying (1.6) with $\delta > 2$. For potentials satisfying (1.6) with $1 < \delta \leq 3$, the much weaker bound

$$g_s^\pm(h, \varepsilon) \leq Ch^{-(2\delta+5)/(3(\delta-1))}\big(\log(h^{-1})\big)^{1/(\delta-1)} \tag{1.7}$$

was proved in [Vodev 2020c].

In the present paper we show that the bound (1.5) can be improved if some small regularity of the potential is assumed. To be more precise, given $0 < \alpha < 1$ and $\beta > 0$, we introduce the space $C_\beta^\alpha(\overline{\mathbb{R}^+})$ of all Hölder functions a such that

$$\sup_{r' \geq 0:\, 0 < |r-r'| \leq 1} \frac{|a(r) - a(r')|}{|r-r'|^\alpha} \leq C(r+1)^{-\beta}, \quad \forall r \in \overline{\mathbb{R}^+},$$

for some constant $C > 0$. We now suppose that the function $V(r, w) := V(rw)$ satisfies the condition

$$V(\cdot\,, w) \in C_4^\alpha(\overline{\mathbb{R}^+}), \quad 0 < \alpha < 1, \tag{1.8}$$

uniformly in $w \in \mathbb{S}^{d-1}$.

Theorem 1.2. *Let $d \geq 3$, and suppose that the potential V satisfies the conditions* (1.1) *and* (1.8). *Then there exists a constant $C > 0$ independent of h and ε but depending on s, E and the function p, such that the bound*

$$g_s^\pm(h, \varepsilon) \leq Ch^{-4/(\alpha+3)}\log(h^{-1}) + C \tag{1.9}$$

holds for all $0 < h \leq 1$.

The proofs of the above theorems are based on the global Carleman estimates proved in [Vodev 2020c], but with different phase and weight functions (see Theorem 4.1). In fact, in the case of Hölder or Lipschitz potentials, we need to construct better phase functions, and hence get better Carleman estimates. Such functions are constructed in Section 2, modifying the construction in [Vodev 2020c] in a suitable way. In order for the Carleman estimates (see (4.1) and (4.6) below) to hold, the phase and weight functions must satisfy some inequalities (see (2.5), (2.9) and (2.21) below), so most of the proofs of the above theorems consist of verifying these inequalities. Note also that the above theorems have been recently proved in [Galkowski and Shapiro 2020] by using similar Carleman estimates, but with a better choice of the phase function. Consequently, the bound (1.9) is proved in [Galkowski and Shapiro 2020] for a larger class of α-Hölder potentials. On the other hand, it is shown in [Vodev 2020a] that the logarithmic term in the right-hand side of (1.9) can be removed for radial potentials.

We next extend the above results to arbitrary obstacles and all dimensions $d \geq 2$. To do so, we need to replace the conditions (1.3) and (1.8) by stronger ones. To be more precise, we let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a connected domain with smooth boundary $\partial\Omega$ such that $\mathbb{R}^d \setminus \Omega$ is compact. Let $r_0 > 0$ be such that $\mathbb{R}^d \setminus \Omega \subset \{x \in \mathbb{R}^d : |x| \leq r_0\}$. Given a real-valued potential $V \in L^\infty(\Omega)$ satisfying (1.1) for $|x| \geq r_0$, we denote by $P(h)$ the Dirichlet self-adjoint realization of the operator $-h^2\Delta + V(x)$ on the Hilbert space $L^2(\Omega)$. We define the quantity $g_s^\pm$ in the same way as above with $L^2 = L^2(\Omega)$. Given $0 < \alpha \leq 1$ and $\beta > 0$, we introduce the space $C^\alpha_\beta(\overline{\Omega})$ of all Hölder functions a such that

$$\sup_{x' \in \overline{\Omega}:\, 0 < |x-x'| \leq 1} \frac{|a(x) - a(x')|}{|x - x'|^\alpha} \leq C(|x| + 1)^{-\beta}, \quad \forall x \in \overline{\Omega},$$

for some constant $C > 0$. Note that the case $\alpha = 1$ corresponds to the Lipschitz functions. We suppose that

$$V \in C^\alpha_\beta(\overline{\Omega}), \quad 0 < \alpha \leq 1, \beta > 1. \tag{1.10}$$

Theorem 1.3. *Let $d \geq 2$, and suppose that the potential $V \in L^\infty(\Omega)$ satisfies (1.1) for $|x| \geq r_0$. If V satisfies (1.10) with $\alpha = 1$ and $\beta > 1$, then the bound (1.2) holds for all $0 < h \leq 1$. If V satisfies (1.10) with $0 < \alpha < 1$ and $\beta = 4$, then the bound (1.9) holds for all $0 < h \leq 1$.*

To prove this theorem we follow the same strategy as in [Vodev 2020b], where the bound (1.5) is proved in all dimensions $d \geq 2$ for potentials $V \in L^\infty(\Omega)$ satisfying (1.6). It consists of gluing up two different types of estimates — one in a compact set coming from the local Carleman estimates proved in [Lebeau and Robbiano 1995] (see Theorem 3.1) with a global Carleman estimate outside a sufficiently big compact (see Theorem 4.2). This is carried out in Section 4.

Theorem 1.3 together with Theorem 1.1 of [Vodev 2020b] allow us to get uniform bounds for the resolvent of the Dirichlet self-adjoint realization, G, of the operator $-n(x)^{-1}\Delta$ in the Hilbert space $H = L^2(\Omega, n(x)dx)$, where $n \in L^\infty(\Omega)$ is a real-valued function called the refraction index, satisfying the conditions

$$n_1 \leq n(x) \leq n_2 \quad \text{in } \Omega, \tag{1.11}$$

with some constants $n_1, n_2 > 0$, and

$$|n(x) - 1| \le C(|x| + 1)^{-\delta} \quad \text{in } \Omega, \tag{1.12}$$

with some constants $C, \delta > 0$. More precisely, we have the following:

Corollary 1.4. *Suppose that the function n satisfies the conditions* (1.11) *and* (1.12). *Then, given any* $s > \frac{1}{2}$ *and* $\lambda_0 > 0$, *there is a constant* $C > 0$ *depending on s and* λ_0 *such that the estimate*

$$\big\|(|x|+1)^{-s}(G - \lambda^2 \pm i\varepsilon)^{-1}(|x|+1)^{-s}\big\|_{H \to H} \le e^{C\psi(\lambda)} \tag{1.13}$$

holds for all $\lambda \ge \lambda_0$ *uniformly in* ε, *where* $\psi(\lambda) = \lambda^{4/3}\log(\lambda+1)$ *if* $n \in L^\infty(\Omega)$ *satisfies* (1.12) *with* $\delta > 3$, $\psi(\lambda) = \lambda^{4/(\alpha+3)}\log(\lambda+1)$ *if* $n \in C_4^\alpha(\overline{\Omega})$ *with* $0 < \alpha < 1$ *and* $\psi(\lambda) = \lambda$ *if* $n \in C_\beta^1(\overline{\Omega})$ *with* $\beta > 1$.

To get (1.13) we apply the theorems mentioned above with $h = \lambda_0/\lambda$, $V = \lambda_0^2(1-n)$, $E = \lambda_0^2$ and ε replaced by $\varepsilon h^2 n$.

Using Corollary 1.4 one can extend the result of [Shapiro 2018] on the local energy decay of the solutions of the wave equation

$$\begin{cases} (n(x)\partial_t^2 - \Delta)u(t,x) = 0 & \text{in } \mathbb{R} \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ u(0,x) = f_1(x),\ \partial_t u(0,x) = f_2(x) & \text{in } \Omega. \end{cases} \tag{1.14}$$

Given any $r_0 \gg 1$, denote $\Omega_{r_0} = \{x \in \Omega : |x| \le r_0\}$.

Corollary 1.5. *Suppose that the function n satisfies* (1.11) *and that* $n = 1$ *outside some compact subset of* Ω. *Then, the solution* $u(t,x)$ *to* (1.14) *with compactly supported initial data* $(f_1, f_2) \in H_0^2(\Omega) \times H_0^1(\Omega)$ *satisfies the estimate*

$$\|\nabla u(t,\cdot)\|_{L^2(\Omega_{r_0})} + \|\partial_t u(t,\cdot)\|_{L^2(\Omega_{r_0})} \le C\omega(t)\big(\|f_1\|_{H^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\big) \tag{1.15}$$

for $t \gg 1$, *where*

$$\omega(t) = \left(\frac{\log\log t}{\log t}\right)^{3/4}.$$

Suppose, in addition, that $n \in C^\alpha(\overline{\Omega})$ *with* $0 < \alpha \le 1$. *Then, the estimate* (1.15) *holds with*

$$\omega(t) = \left(\frac{\log\log t}{\log t}\right)^{(\alpha+3)/4}$$

if $0 < \alpha < 1$ *and with* $\omega(t) = (\log t)^{-1}$ *if* $\alpha = 1$. *The estimate* (1.15) *remains valid when* $\Omega = \mathbb{R}^d$.

Remark 1.6. In view of the recent results in [Vodev 2020a], when $\Omega = \mathbb{R}^d$, $d \ge 3$ and the function n depends only on the radial variable r, the estimate (1.15) holds with $\omega(t) = (\log t)^{-3/4}$ if $n \in L^\infty$ and with $\omega(t) = (\log t)^{-(\alpha+3)/4}$ if n is α-Hölder in r.

Note that estimates similar to (1.15) were first proved by [Burq 1998] in the case $n \equiv 1$. Note also that an analog of the above theorem is proved by [Shapiro 2018] in the case $\Omega = \mathbb{R}^d$. Then, an estimate similar to (1.15) is proved with $\omega(t)$ replaced by $(\log t)^{-3/4+\varepsilon}$, $\varepsilon > 0$ being arbitrary. Moreover, if in addition the function n is supposed Lipschitz, then the decay rate is improved to $\omega(t)=(\log t)^{-1}$. The proof in [Shapiro 2018] is based on the resolvent estimates obtained in [Datchev 2014], [Shapiro 2019] and [Shapiro 2020].

The assumption that $n=1$ outside some compact set is only necessary to study the low-frequency behavior of the cut-off resolvent of G. Indeed, under this assumption one can easily see that this behavior is exactly the same as in the case when $n\equiv 1$, which in turn is well known (see Appendix B.2 of [Burq 1998]). Therefore, in this case the low-frequency analysis can be carried out in precisely the same way as in [Shapiro 2018]. Most probably, the condition (1.12) with $\delta>2$ would be enough. The high-frequency analysis, in our case, is also very similar to the one in [Shapiro 2018], with some slight modifications allowing to deduce from (1.13) the sharp decay rate $\omega(t)$ (instead of $(\log t)^{-3/4+\varepsilon}$).

2. Construction of the phase and weight functions

Let $\rho\in C_0^\infty([0,1])$, $\rho\ge 0$, be a real-valued function independent of h such that $\int_0^\infty \rho(\sigma)\,d\sigma=1$. If V satisfies (1.8), we approximate it by the function

$$V_\theta(r,w)=\theta^{-1}\int_0^\infty \rho\Big(\frac{r'-r}{\theta}\Big)V(r',w)\,dr'=\int_0^\infty \rho(\sigma)V(r+\theta\sigma,w)\,d\sigma,$$

where $\theta=h^{2/(\alpha+3)}$. Indeed, we have

$$\begin{aligned}|V(r,w)-V_\theta(r,w)|&\le\int_0^\infty \rho(\sigma)|V(r+\theta\sigma,w)-V(r,w)|\,d\sigma\\&\lesssim \theta^\alpha(r+1)^{-4}\int_0^\infty \sigma^\alpha\rho(\sigma)\,d\sigma\lesssim\theta^\alpha(r+1)^{-4}.\end{aligned}\tag{2.1}$$

This bound together with (1.1) implies

$$V_\theta(r,w)\le p(r)+\mathcal{O}\big((r+1)^{-4}\big).\tag{2.2}$$

Clearly, V_θ is C^1 with respect to the variable r, and its first derivative V'_θ is given by

$$\begin{aligned}V'_\theta(r,w)&=\theta^{-2}\int_0^\infty \rho'\Big(\frac{r'-r}{\theta}\Big)V(r',w)\,dr'\\&=\theta^{-1}\int_0^\infty \rho'(\sigma)V(r+\theta\sigma,w)\,d\sigma=\theta^{-1}\int_0^\infty \rho'(\sigma)(V(r+\theta\sigma,w)-V(r,w))\,d\sigma,\end{aligned}$$

where we have used that $\int_0^\infty \rho'(\sigma)\,d\sigma=0$. Hence,

$$|V'_\theta(r,w)|\lesssim\theta^{-1+\alpha}(r+1)^{-4}\int_0^\infty \sigma^\alpha|\rho'(\sigma)|\,d\sigma\lesssim\theta^{-1+\alpha}(r+1)^{-4}.\tag{2.3}$$

We now construct the weight function μ as follows:

$$\mu(r)=\begin{cases}(r+1)^{2k}-(r+1)^{2k_0} & \text{for } 0\le r\le a,\\ (a+1)^{2k}-(a+1)^{2k_0}+(a+1)^{-2s+1}-(r+1)^{-2s+1} & \text{for } r\ge a,\end{cases}$$

where $a=a_0h^{-m}$ with $a_0\gg 1$ independent of h, $m=0$ if V satisfies (1.3) and $m=2$ if V satisfies (1.8). We choose $k=\frac14\min\{1,\beta-1\}$, $k_0=0$ if V satisfies (1.3) and $k=1$, $k_0=\frac12$ if V satisfies (1.8). Furthermore,

s is independent of h such that

$$\frac{1}{2} < s < \begin{cases} \frac{1}{4}\min\{3, \beta+1\} & \text{if } V \text{ satisfies (1.3)}, \\ \frac{3}{4} & \text{if } V \text{ satisfies (1.8)}. \end{cases} \tag{2.4}$$

Clearly, the first derivative of μ is given by

$$\mu'(r) = \begin{cases} 2k(r+1)^{2k-1} - 2k_0(r+1)^{2k_0-1} & \text{for } 0 \le r < a, \\ (2s-1)(r+1)^{-2s} & \text{for } r > a. \end{cases}$$

Lemma 2.1. *For all $r > 0$, $r \neq a$, we have the inequalities*

$$2r^{-1}\mu(r) - \mu'(r) \ge 0, \tag{2.5}$$

$$\frac{\mu(r)^j}{\mu'(r)} \lesssim a^{2kj}(r+1)^{2s}, \tag{2.6}$$

for every $j \ge 0$.

Proof. It is shown in Section 2 of [Vodev 2020c] that when $k_0 = 0$ the inequality (2.5) holds for all $0 < k \le 1$. Here, we will prove it when $\nu := 2k - 2k_0 \ge 1$ and $0 < k \le 1$. For $r < a$ we have

$$\begin{aligned} 2\mu(r) - r\mu'(r) &= 2(1-k)(r+1)^{2k} - 2(1-k_0)(r+1)^{2k_0} + 2k(r+1)^{2k-1} - 2k_0(r+1)^{2k_0-1} \\ &= 2(r+1)^{2k_0-1}\big((1-k)(r+1)^{\nu+1} - (1-k_0)(r+1) + k(r+1)^{\nu} - k_0\big) \\ &= 2(r+1)^{2k_0-1}\big((1-k)r((r+1)^{\nu}-1) + (r+1)^{\nu} - \nu r/2 - 1\big) \\ &\ge 2(r+1)^{2k_0-1}\big((r+1)^{\nu} - \nu r/2 - 1\big) \ge \nu r(r+1)^{2k_0-1} > 0, \end{aligned}$$

where we have used the well-known inequality $(r+1)^{\nu} \ge \nu r + 1$, as long as $\nu \ge 1$. For $r > a$ the left-hand side of (2.5) is bounded from below by

$$2r^{-1}\big((a+1)^{2k} - (a+1)^{2k_0} - s\big) > 0,$$

provided a is taken large enough. To prove (2.6) observe that for $r < a$ we have

$$\mu'(r) \ge 2(k-k_0)(r+1)^{2k-1} \ge 2(k-k_0)(r+1)^{-1} \ge 2(k-k_0)(r+1)^{-2s},$$

which clearly implies the bound (2.6) with $j = 0$. This together with the fact that $\mu = \mathcal{O}(a^{2k})$ implies the bound (2.6) with any $j > 0$. □

We will now construct a phase function $\varphi \in C^1([0, +\infty))$ such that $\varphi(0) = 0$ and $\varphi(r) > 0$ for $r > 0$. We define the first derivative of φ by

$$\varphi'(r) = \begin{cases} \tau(r+1)^{-k} - \tau(a+1)^{-k} & \text{for } 0 \le r \le a, \\ 0 & \text{for } r \ge a, \end{cases}$$

where

$$\tau = \begin{cases} \tau_0 & \text{if } V \text{ satisfies (1.3)}, \\ \tau_0\theta^{2\alpha/3}h^{-1/3} & \text{if } V \text{ satisfies (1.8)}, \end{cases} \tag{2.7}$$

with some parameter $\tau_0 \gg 1$ independent of h to be fixed later on. We choose now the parameter a_0 of the form $a_0 = \tau_0^{\ell}$, where $\ell > 0$ is a constant such that $k\ell > 2$ and $(\beta - 2k - 2s)\ell > 2$. Note that the choice of the parameters k and s guarantees that $\beta - 2k - 2s > 0$.

Clearly, the first derivative of φ' satisfies

$$\varphi''(r) = \begin{cases} -k\tau(r+1)^{-k-1} & \text{for } 0 \le r < a, \\ 0 & \text{for } r > a. \end{cases}$$

Lemma 2.2. *For all $r \ge 0$ we have the bounds*

$$h^{-1}\varphi(r) \lesssim \begin{cases} h^{-1} & \text{if } V \text{ satisfies (1.3)}, \\ h^{-4/(\alpha+3)}\log(h^{-1}) + 1 & \text{if } V \text{ satisfies (1.8)}. \end{cases} \tag{2.8}$$

Proof. The lemma follows from the bounds

$$\max\varphi = \int_0^a \varphi'(r)\,dr \le \tau\int_0^a (r+1)^{-k}\,dr \lesssim \begin{cases} \tau a^{1-k} & \text{if } k < 1, \\ \tau\log a & \text{if } k = 1. \end{cases} \qquad \square$$

For $r > 0$, $r \ne a$, set

$$A(r) = \big(\mu\varphi'^2\big)'(r),$$
$$B(r) = B_1(r) + B_2(r),$$

where

$$B_1(r) = (r+1)^{-\beta}\mu(r) + p(r)\mu'(r),$$
$$B_2(r) = \frac{(\mu(r)\varphi''(r))^2}{h^{-1}\varphi'(r)\mu(r) + \mu'(r)},$$

with $\beta > 1$ if V satisfies (1.3) and

$$B_1(r) = \theta^{-1+\alpha}(r+1)^{-\beta}\mu(r) + (p(r) + (r+1)^{-\beta})\mu'(r),$$
$$B_2(r) = \frac{\big(\mu(r)(h^{-1}\theta^{\alpha}(r+1)^{-\beta} + |\varphi''(r)|)\big)^2}{h^{-1}\varphi'(r)\mu(r) + \mu'(r)},$$

with $\beta = 4$ if V satisfies (1.8). The following lemma will play a crucial role in the proof of the Carleman estimates (4.1) and (4.6) in the case $d \ge 3$:

Lemma 2.3. *Given any constant $C > 0$, there exists a positive constant $\tau_1 = \tau_1(C, E)$ such that for τ satisfying (2.7) with $\tau_0 \ge \tau_1$ and for all $0 < h \le 1$, we have the inequality*

$$A(r) - CB(r) \ge -\frac{E}{2}\mu'(r) \tag{2.9}$$

for all $r > 0$, $r \ne a$.

Proof. For $r < a$ we have

$$\begin{aligned} A(r) &= -\big((r+1)^{2k_0}\varphi'^2\big)' + \tau^2\partial_r\big(1-(r+1)^k(a+1)^{-k}\big)^2 \\ &= -2(r+1)^{2k_0}\varphi'(r)\varphi''(r) - 2k_0(r+1)^{2k_0-1}\varphi'(r)^2 - 2k\tau^2(r+1)^{k-1}(a+1)^{-k}\big(1-(r+1)^k(a+1)^{-k}\big) \\ &\ge 2\tau(k-k_0)(r+1)^{2k_0-k-1}\varphi'(r) - 2k\tau^2(r+1)^{k-1}(a+1)^{-k} \\ &\ge 2\tau(k-k_0)(r+1)^{2k_0-k-1}\varphi'(r) - \mathcal{O}\big(\tau^2a^{-k}\big)\mu'(r) \\ &\ge 2\tau(k-k_0)(r+1)^{2k_0-k-1}\varphi'(r) - \mathcal{O}\big(\tau_0^2a_0^{-k}\big)\mu'(r) \\ &\ge 2\tau(k-k_0)(r+1)^{2k_0-k-1}\varphi'(r) - \mathcal{O}\big(\tau_0^{-k\ell+2}\big)\mu'(r). \end{aligned}$$

Hence, taking τ_0 large enough, we can arrange the inequality

$$A(r) \geq 2\tau(k-k_0)(r+1)^{2k_0-k-1}\varphi'(r) - \frac{E}{4}\mu'(r) \tag{2.10}$$

for all $r < a$. Observe now that if $0 < r \leq a/2$, then

$$\varphi'(r) \geq \gamma\tau(r+1)^{-k} \tag{2.11}$$

with some constant $\gamma > 0$. By (2.10) and (2.11) we conclude

$$A(r) \geq \tilde{\gamma}\tau^2(r+1)^{-2(k-k_0)-1} - \frac{E}{4}\mu'(r) \tag{2.12}$$

for all $r \leq a/2$ with some constant $\tilde{\gamma} > 0$, and

$$A(r) \geq -\frac{E}{4}\mu'(r) \quad \text{for all } r \neq a. \tag{2.13}$$

We will now bound the function B_1 from above. Since the function p is decreasing, tending to zero, there is $b > 0$ such that

$$p(r) + (r+1)^{-\beta} \leq \frac{E}{9C} \quad \text{for } r \geq b.$$

Hence, for every $N > 0$, there is a constant $C_N > 0$ such that we have

$$\big(p(r) + (r+1)^{-\beta}\big)\mu'(r) \leq C_N(r+1)^{-N} + \frac{E}{9C}\mu'(r) \quad \text{for all } r \neq a. \tag{2.14}$$

Let $0 < r < a$. Then $\mu(r) < (r+1)^{2k}$, and in view of (2.14) with N big enough, we have

$$B_1(r) \leq \widetilde{C}(r+1)^{2k-\beta} + \frac{E}{9C}\mu'(r)$$

if V satisfies (1.3), and

$$B_1(r) \leq \widetilde{C}\theta^{-1+\alpha}(r+1)^{2k-\beta} + \frac{E}{9C}\mu'(r)$$

with $\beta = 4$ if V satisfies (1.8). Observe now that the choice of the parameters k, k_0 and θ guarantees that $\beta - 2k \geq 2(k-k_0)+1$ and $\theta^{-1+\alpha} = \theta^{4\alpha/3}h^{-2/3}$. Therefore, the above inequalities imply

$$B_1(r) \leq \mathcal{O}\big(\tau_0^{-2}\big)\tau^2(r+1)^{-2(k-k_0)-1} + \frac{E}{9C}\mu'(r) \quad \text{for } r \leq \frac{a}{2} \tag{2.15}$$

in both cases. Similarly, we get

$$B_1(r) \leq \mathcal{O}\big(\tau^2 a^{-\beta+1}\big)\mu'(r) + \frac{E}{9C}\mu'(r) \quad \text{for } \frac{a}{2} < r < a \tag{2.16}$$

and

$$B_1(r) \leq \mathcal{O}\big(\tau^2 a^{-\beta+2k+2s}\big)\mu'(r) + \frac{E}{9C}\mu'(r) \quad \text{for } r > a. \tag{2.17}$$

Since

$$\tau^2 a^{-\beta+1} < \tau^2 a^{-\beta+2k+2s} \leq \tau_0^2 a_0^{-\beta+2k+2s} = \tau_0^{-(\beta-2k-2s)\ell+2},$$

we obtain from (2.16) and (2.17),

$$B_1(r) \le \frac{E}{8C}\mu'(r) \quad \text{for } r > \frac{a}{2}, r \ne a, \tag{2.18}$$

provided τ_0 is taken large enough.

We will now bound the function B_2 from above. We will first consider the case when V satisfies (1.8). Let $0 < r \le a/2$. In view of (2.11), we have

$$\begin{aligned}
B_2(r) &\lesssim \frac{\mu(r)\big(h^{-2}\theta^{2\alpha}(r+1)^{-2\beta} + \varphi''(r)^2\big)}{h^{-1}\varphi'(r)} \\
&\lesssim h^{-1}\theta^{2\alpha}\frac{\mu(r)(r+1)^{-2\beta}}{\varphi'(r)} + h\frac{\mu(r)\varphi''(r)^2}{\varphi'(r)} \\
&\lesssim \tau^{-1}\theta^{2\alpha}h^{-1}(r+1)^{3k-2\beta} + h\tau(r+1)^{k-2} \\
&\lesssim \tau_0^{-3}\tau^2(r+1)^{-2(k-k_0)-1} + \tau(r+1)^{k-2},
\end{aligned}$$

where we have used that $5k - 2k_0 < 2\beta - 1$. Since $3k - 2k_0 - 1 > 0$, we have the inequality

$$(r+1)^{k-2} \le b^{3k-2k_0-1}(r+1)^{-2(k-k_0)-1} + b^{-k-1}(r+1)^{2k-1}$$

for every $b > 1$. We take b such that $b^{3k-2k_0-1} = b_0\tau$, where $b_0 > 0$ is a small parameter independent of τ and h to be fixed below. Then the above inequality takes the form

$$\tau(r+1)^{k-2} \lesssim b_0\tau^2(r+1)^{-2(k-k_0)-1} + \tau^{-2(1-k+k_0)/(3k-2k_0-1)}\mu'(r) \lesssim b_0\tau^2(r+1)^{-2(k-k_0)-1} + \tau_0^{-1}\mu'(r).$$

Thus, taking τ_0 big enough depending on b_0, E and C, we get the bound

$$B_2(r) \le \mathcal{O}\big(\tau_0^{-1} + b_0\big)\tau^2(r+1)^{-2(k-k_0)-1} + \frac{E}{8C}\mu'(r) \quad \text{for } 0 < r \le \frac{a}{2}. \tag{2.19}$$

When V satisfies (1.3), we have $3k - 2k_0 - 1 \le 0$, and hence

$$\tau(r+1)^{k-2} \le \tau(r+1)^{-2(k-k_0)-1} \le \tau_0^{-1}\tau^2(r+1)^{-2(k-k_0)-1}.$$

Therefore, the inequality (2.19) still holds in this case.

Let us now see that

$$B_2(r) \le \frac{E}{8C}\mu'(r) \quad \text{for } r > \frac{a}{2}, r \ne a. \tag{2.20}$$

Let $a/2 < r < a$. Since in this case $\mu(r)/\mu'(r) = \mathcal{O}(r)$, we get the bound

$$\begin{aligned}
B_2(r) &\lesssim \Big(\frac{\mu(r)}{\mu'(r)}\Big)^2\big(h^{-1}\theta^{\alpha}(r+1)^{-\beta} + |\varphi''(r)|\big)^2\mu'(r) \\
&\lesssim \big(h^{-2}\theta^{2\alpha}(r+1)^{2-2\beta} + \tau^2(r+1)^{-2k}\big)\mu'(r) \\
&\lesssim \big(h^{-2}a^{2-2\beta} + \tau^2 a^{-2k}\big)\mu'(r) \\
&\lesssim \big(h^{2m(\beta-1)-2}a_0^{2-2\beta} + h^{2m-2/3}\tau_0^2 a_0^{-2k}\big)\mu'(r) \\
&\lesssim \big(a_0^{2-2\beta} + \tau_0^2 a_0^{-2k}\big)\mu'(r) \lesssim \big(\tau_0^{-2\ell(\beta-1)} + \tau_0^{-2k\ell+2}\big)\mu'(r),
\end{aligned}$$

which clearly implies (2.20) in this case, provided τ_0 is taken big enough. Let $r > a$. Using (2.6) with $j = 1$, we get

$$\begin{aligned}B_2(r) &\lesssim \Big(\frac{\mu(r)}{\mu'(r)}\Big)^2 \big(h^{-1}\theta^\alpha (r+1)^{-\beta}\big)^2 \mu'(r)\\ &\lesssim h^{-2}a^{4k}(r+1)^{4s-2\beta}\mu'(r)\\ &\lesssim h^{-2}a^{4k+4s-2\beta}\mu'(r)\\ &\lesssim h^{2m(\beta-2k-2s)-2}a_0^{4k+4s-2\beta}\mu'(r)\\ &\lesssim a_0^{4k+4s-2\beta}\mu'(r) \lesssim \tau_0^{-2\ell(\beta-2k-2s)}\mu'(r),\end{aligned}$$

which again implies (2.20), provided τ_0 is taken big enough. Similarly, in the case when V satisfies (1.3), one concludes that the inequality (2.20) holds for all $r > 0$, $r \neq a$.

It is easy to see that for $r \le a/2$, the estimate (2.9) follows from (2.12), (2.15) and (2.19) by taking b_0 and τ_0^{-1} small enough, while for $r \ge a/2$, $r \neq a$, it follows from (2.13), (2.18) and (2.20). □

Remark 2.4. It is easy to see from the proof that when V satisfies (1.8), the inequality (2.9) holds as long as $\frac{1}{2} \le k \le 1$, $k_0 = k - \frac{1}{2}$. The choice $k = 1$, however, provides the best resolvent bound in the semiclassical regime, that is, for $0 < h \le h_0$ with some constant $0 < h_0 \ll 1$. When $h_0 < h \le 1$, the choice of k does not really matter, because in this case $g_s^\pm(h, \varepsilon)$ is bounded from above by a constant. For example, we may take $k = \frac{1}{2}$ and $k_0 = 0$.

The following lemmas will play a crucial role in the proof of the Carleman estimate (4.6) when $d = 2$:

Lemma 2.5. *Given any constants $C, r_0 > 0$, there exists a positive constant $\tau_1 = \tau_1(C, E, r_0)$ such that for τ satisfying (2.7) with $\tau_0 \ge \tau_1$ and for all $0 < h \le h_0$, $0 < h_0 < 1$ being a constant depending on E, r_0 and τ_0, we have the inequality*

$$A(r) - h^2 r^{-3}\mu(r) - C B(r) \ge -\frac{2E}{3}\mu'(r) \tag{2.21}$$

for all $r \ge r_0$, $r \neq a$.

Proof. For $r_0 \le r < a$, we have

$$h^2 r^{-3}\mu(r) \lesssim h^2(r+1)^{-3}\mu(r) \lesssim h^2(r+1)^{-2}\mu'(r) \le \frac{E}{6}\mu'(r),$$

provided h is taken small enough. For $r > a$, in view of (2.6) with $j = 1$, we have

$$h^2 r^{-3}\mu(r) \lesssim h^2 a^{2k}(r+1)^{2s-3}\mu'(r) \lesssim h^2 a^{2k+2s-3}\mu'(r) \lesssim h^{2-m(2k+2s-3)}a_0^{2k+2s-3}\mu'(r) \le \frac{E}{6}\mu'(r),$$

provided h is taken small enough, depending on a_0. Clearly, (2.21) follows from these inequalities and (2.9). □

It is easy to see from the proof that when V satisfies (1.3), the inequality (2.21) holds also for $h_0 < h \le 1$. This is no longer true when V satisfies (1.8), because in this case $2k + 2s - 3$ does not have the right sign. Therefore, to make (2.21) hold for h not necessarily small, we need to make a new choice of the parameters k and k_0 in order to change the sign of $2k + 2s - 3$ and for which Lemma 2.3 still holds.

Thus, in view of Remark 2.4, in the semiclassical regime ($0 < h \le h_0$) we take $k = 1$, $k_0 = \frac{1}{2}$, and in the classical regime ($h_0 < h \le 1$) we take $k = \frac{1}{2}$, $k_0 = 0$. To cover the second case we need the following:

Lemma 2.6. *If V satisfies* (1.8), *we take $k = \frac{1}{2}$ and $k_0 = 0$. Then, given any constants $C, r_0 > 0$, there exists a positive constant $\tau_1 = \tau_1(C, E, r_0)$ such that for τ satisfying* (2.7) *with $\tau_0 \ge \tau_1$ the inequality* (2.21) *holds for all $r \ge r_0$, $r \ne a$, and all $0 < h \le 1$.*

Proof. For $r_0 \le r \le a/2$, we have

$$h^2 r^{-3}\mu(r) \lesssim (r+1)^{-3}\mu(r) \lesssim (r+1)^{-3+2k} \lesssim (r+1)^{-2(k-k_0)-1}.$$

For $a/2 < r < a$, we have

$$h^2 r^{-3}\mu(r) \lesssim (r+1)^{-2}\mu'(r) \lesssim a^{-2}\mu'(r) \lesssim a_0^{-2}\mu'(r) \le \frac{E}{6}\mu'(r),$$

provided a_0 is taken big enough. For $r > a$, we have

$$h^2 r^{-3}\mu(r) \lesssim a^{2k}(r+1)^{2s-3}\mu'(r) \lesssim a^{2k+2s-3}\mu'(r) \lesssim a_0^{2k+2s-3}\mu'(r) \le \frac{E}{6}\mu'(r),$$

provided a_0 is taken big enough. Then, (2.21) easily follows from these inequalities and Remark 2.4. □

3. Carleman estimates for Hölder potentials on bounded domains

Throughout this section $X \subset \mathbb{R}^d$, $d \ge 2$, will be a bounded, connected domain with a smooth boundary ∂X. Introduce the operator

$$P(h) = -h^2\Delta + V(x),$$

where $0 < h \le 1$ is a semiclassical parameter and $V \in L^\infty(X)$ is a real-valued potential. Let $U \subset X$, $U \ne \varnothing$, be an arbitrary open domain, independent of h, such that $\partial U \cap \partial X = \varnothing$, and let $z \in \mathbb{C}$, $|z| \le C_0$, $C_0 > 0$ be a constant independent of h. We will also denote by H_h^1 the Sobolev space equipped with the semiclassical norm. Given any $0 < \alpha \le 1$, denote by $C^\alpha(\overline{X})$ the space of all functions a such that

$$\|a\|_{C^\alpha} := \sup_{x', x \in \overline{X} : 0 < |x - x'| \le 1} \frac{|a(x) - a(x')|}{|x - x'|^\alpha} < +\infty.$$

Theorem 3.1. *Let $V \in C^\alpha(\overline{X})$ with $0 < \alpha \le 1$. Then, there exists a positive constant γ depending on U, $\|V\|_{C^\alpha}$ and C_0, but independent of h, such that for all $0 < h \le 1$, we have the estimate*

$$\|u\|_{H_h^1(X)} \le e^{\gamma h^{-4/(\alpha+3)}}\|(P(h) - z)u\|_{L^2(X)} + e^{\gamma h^{-4/(\alpha+3)}}\|u\|_{H_h^1(U)} \tag{3.1}$$

for every $u \in H^2(X)$ such that $u|_{\partial X} = 0$.

It is proved in Section 2 of [Vodev 2020b] that for complex-valued potentials $V \in L^\infty(X)$, the estimate (3.1) holds with $\alpha = 0$. The proof is based on the local Carleman estimates proved in [Lebeau and Robbiano 1995]. We will follow the same strategy in the case of Hölder potentials as well. For such potentials we will get new local Carleman estimates by making use of the results of [Lebeau and Robbiano 1995]. To be more precise, we let $W \subset X$ be a small open domain and let x be local coordinates in W. If $\Gamma := \overline{W} \cap \partial X$ is not empty, we choose $x = (x_1, x')$, $x_1 > 0$ being the normal coordinate in W and x'

the tangential ones. Thus, in these coordinates Γ is given by $\{x_1 = 0\}$. Let $\phi, \phi_1 \in C^\infty(\overline{W})$ be real-valued functions such that $\operatorname{supp}\phi \subset \operatorname{supp}\phi_1 \subset \overline{W}$, $\phi_1 = 1$ on $\operatorname{supp}\phi$. When $V \in C^\alpha(\overline{X})$ with $0 < \alpha < 1$, we approximate the function $\phi_1 V$ by the smooth function

$$V_\theta(x) = \theta^{-1} \int_X \rho\Big(\frac{x' - x}{\theta}\Big)(\phi_1 V)(x')\, dx',$$

where $\rho \in C_0^\infty(|x| \le 1)$ is a real-valued function such that $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$ and $0 < \theta < 1$ is a small parameter to be fixed later on. The fact that $V \in C^\alpha(\overline{X})$ implies the bounds

$$|(\phi_1 V)(x) - V_\theta(x)| \lesssim \theta^\alpha, \tag{3.2}$$

$$|\partial_x^\beta V_\theta(x)| \lesssim \theta^{\alpha-1}, \tag{3.3}$$

for all multi-indices β such that $|\beta| = 1$. Set $\widetilde{V} = \theta^{1-\alpha}(V_\theta - z)$ if $V \in C^\alpha(\overline{X})$ with $0 < \alpha < 1$ and $\widetilde{V} = V - z$ if $V \in C^1(\overline{X})$. In view of (3.2) and (3.3), we have $\partial_x^\beta \widetilde{V}(x) = \mathcal{O}(1)$ uniformly in θ, for all multi-indices β such that $|\beta| \le 1$.

Now, let $\psi \in C^\infty(\overline{W})$ be a real-valued function independent of h and θ such that

$$\nabla\psi \ne 0 \quad \text{in } \overline{W}. \tag{3.4}$$

If $\Gamma \ne \emptyset$, we also suppose that

$$\frac{\partial\psi}{\partial x_1}(0, x') > 0 \quad \text{for all } x'. \tag{3.5}$$

We set $\varphi = e^{\lambda\psi}$, where $\lambda > 0$ is a big parameter to be fixed later on, independent of h and θ. Let $p(x, \xi) \in C^\infty(T^*W)$ be the principal symbol of the operator $-\Delta$, and let $0 < \tilde{h} \ll 1$ be a new semiclassical parameter. Then the principal symbol, $\tilde{p}_\varphi$, of the operator

$$e^{\varphi/\tilde{h}}\big(-\tilde{h}^2\Delta + \widetilde{V}\big)e^{-\varphi/\tilde{h}}$$

is given by the formula

$$\tilde{p}_\varphi(x, \xi) = p(x, \xi + i\nabla\varphi(x)) + \widetilde{V}(x).$$

An easy computation shows that given any constant $C > 0$, there is $\lambda = \lambda(C)$ such that the condition (3.4) for the function ψ implies the following condition for the function φ:

$$\{\operatorname{Re}\tilde{p}_\varphi, \operatorname{Im}\tilde{p}_\varphi\}(x, \xi) \ge c_1 \quad \text{for } |\xi| \le C, \tag{3.6}$$

with some constant $c_1 > 0$ independent of θ. On the other hand, if C is taken large enough, we can arrange the lower bound

$$|\tilde{p}_\varphi(x, \xi)| \ge c_2|\xi|^2 \quad \text{for } |\xi| \ge C, \tag{3.7}$$

with some constant $c_2 > 0$ independent of θ. If $\Gamma \ne \emptyset$, the condition (3.5) implies

$$\frac{\partial\varphi}{\partial x_1}(0, x') > 0 \quad \text{for all } x'. \tag{3.8}$$

Now, we are in position to use Propositions 1 and 2 of [Lebeau and Robbiano 1995], where the proof is based on the properties (3.6), (3.7) and (3.8).

Proposition 3.2. *Let the function u be as in Theorem 3.1. Then there exist constants* $C_1, \tilde{h}_0 > 0$ *such that for all* $0 < \tilde{h} \le \tilde{h}_0$, *we have the estimate*

$$\int_X \big(|\phi u|^2 + |\tilde{h}\nabla(\phi u)|^2\big) e^{2\varphi/\tilde{h}}\, dx \le C_1 \tilde{h}^{-1} \int_X \big|\big(-\tilde{h}^2\Delta + \widetilde{V}\,\big)(\phi u)\big|^2 e^{2\varphi/\tilde{h}}\, dx. \tag{3.9}$$

We take $\tilde{h} = h\theta^{(1-\alpha)/2}$ when $\alpha < 1$, and we rewrite the inequality (3.9) as follows:

$$\begin{aligned}
&\int_X \big(|\phi u|^2 + \theta^{1-\alpha}|h\nabla(\phi u)|^2\big) e^{2\varphi/h\theta^{(1-\alpha)/2}}\, dx \\
&\quad \le C_1 h^{-1}\theta^{3(1-\alpha)/2} \int_X |(-h^2\Delta + V_\theta - z)(\phi u)|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}}\, dx \\
&\quad \le C_1 h^{-1}\theta^{3(1-\alpha)/2} \int_X |(P(h) - z)(\phi u)|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}}\, dx \\
&\qquad\qquad + C_1 h^{-1}\theta^{3(1-\alpha)/2} \sup |\phi_1 V - V_\theta|^2 \int_X |\phi u|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}}\, dx \\
&\quad \le C_1 h^{-1}\theta^{3(1-\alpha)/2} \int_X |(P(h) - z)(\phi u)|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}}\, dx + C_2 h^{-1}\theta^{(3+\alpha)/2} \int_X |\phi u|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}}\, dx.
\end{aligned}$$

We now take $\theta = h^{2/(\alpha+3)}\kappa^{2/(1-\alpha)}$, where $\kappa > 0$ is a small parameter independent of h. Thus, taking κ small enough we can absorb the last term in the right-hand side of the above inequality. When $\alpha = 1$, we take $\tilde{h} = h\kappa$. Thus, we deduce the following from Proposition 3.2:

Proposition 3.3. *Let the function u be as in Theorem 3.1. Then there exist constants* $\widetilde{C}, \kappa_0 > 0$ *such that for all* $0 < \kappa \le \kappa_0$ *and all* $0 < h \le 1$, *we have the estimate*

$$\int_X \big(|\phi u|^2 + |h\nabla(\phi u)|^2\big) e^{2\varphi/\kappa h^{4/(\alpha+3)}}\, dx \le \widetilde{C}\kappa h^{-2(\alpha+1)/(\alpha+3)} \int_X |(P(h) - z)(\phi u)|^2 e^{2\varphi/\kappa h^{4/(\alpha+3)}}\, dx. \tag{3.10}$$

Now Theorem 3.1 follows from Proposition 3.3 in precisely the same way as in Section 2 of [Vodev 2020b], where the analysis is carried out in the particular case $\alpha = 0$. It is an easy observation that the general case requires no changes in the arguments, and therefore we omit the details.

4. Resolvent estimates

The following global Carlemann estimate is similar to that of [Vodev 2020c, Section 3] and can be proved in the same way. The proof will be carried out in Section 5. In what follows, we set $\mathcal{D}_r = -ih\partial_r$:

Theorem 4.1. *Let* $d \ge 3$, *and let the potential V satisfy* (1.1). *Let also V satisfy either* (1.3) *or* (1.8), *and let s satisfy* (2.4). *Then, for all* $0 < h \le 1$, $0 < \varepsilon \le 1$ *and for all functions* $f \in H^2(\mathbb{R}^d)$ *such that*

$$(|x| + 1)^s (P(h) - E \pm i\varepsilon) f \in L^2(\mathbb{R}^d),$$

we have the estimate

$$\begin{aligned}\|(|x|+1)^{-s}e^{\varphi/h}f\|_{L^2(\mathbb{R}^d)}+\|(|x|+1)^{-s}e^{\varphi/h}\mathcal{D}_r f\|_{L^2(\mathbb{R}^d)}\\ \le Ca^2h^{-1}\|(|x|+1)^{s}e^{\varphi/h}(P(h)-E\pm i\varepsilon)f\|_{L^2(\mathbb{R}^d)}+C\tau a(\varepsilon/h)^{1/2}\|e^{\varphi/h}f\|_{L^2(\mathbb{R}^d)},\end{aligned} \tag{4.1}$$

with a constant $C>0$ independent of h, ε and f.

Theorems 1.1 and 1.2 can be obtained from Theorem 4.1 in the same way as in Section 4 of [Vodev 2020c]. We will sketch the proof for the sake of completeness. It follows from the estimate (4.1) and Lemma 2.2 that for $0<h\le 1$ and s satisfying (2.4), we have the estimate

$$\|(|x|+1)^{-s}f\|_{L^2}\le M\|(|x|+1)^{s}(P(h)-E\pm i\varepsilon)f\|_{L^2}+M\varepsilon^{1/2}\|f\|_{L^2}, \tag{4.2}$$

where $M>0$ is given by

$$\log M=\begin{cases}Ch^{-1} & \text{if } V \text{ satisfies (1.3)},\\ Ch^{-4/(\alpha+3)}\log(h^{-1})+C & \text{if } V \text{ satisfies (1.8)},\end{cases}$$

with a constant $C>0$ independent of h and ε. On the other hand, since the operator $P(h)$ is symmetric, we have

$$\begin{aligned}\varepsilon\|f\|_{L^2}^2&=\pm\operatorname{Im}\langle (P(h)-E\pm i\varepsilon)f,f\rangle_{L^2}\\ &\le (2M)^{-2}\|(|x|+1)^{-s}f\|_{L^2}^2+(2M)^2\|(|x|+1)^{s}(P(h)-E\pm i\varepsilon)f\|_{L^2}^2,\end{aligned}$$

which yields

$$M\varepsilon^{1/2}\|f\|_{L^2}\le \tfrac{1}{2}\|(|x|+1)^{-s}f\|_{L^2}+2M^2\|(|x|+1)^{s}(P(h)-E\pm i\varepsilon)f\|_{L^2}. \tag{4.3}$$

By (4.2) and (4.3), we get

$$\|(|x|+1)^{-s}f\|_{L^2}\le 4M^2\|(|x|+1)^{s}(P(h)-E\pm i\varepsilon)f\|_{L^2}. \tag{4.4}$$

It follows from (4.4) that the resolvent estimate

$$\big\|(|x|+1)^{-s}(P(h)-E\pm i\varepsilon)^{-1}(|x|+1)^{-s}\big\|_{L^2\to L^2}\le 4M^2 \tag{4.5}$$

holds for all $0<h\le 1$ and s satisfying (2.4), and hence for all $s>\frac{1}{2}$ independent of h. Clearly, (4.5) implies the desired bounds for $g_s^{\pm}$.

Given any $r_0>0$, we denote $Y_{r_0}:=\{x\in\mathbb{R}^d:|x|\ge r_0\}$, and we let $\eta_{r_0}\in C^\infty(\mathbb{R})$ be such that $\eta_{r_0}(r)=0$ for $r\le r_0/3$ and $\eta_{r_0}(r)=1$ for $r\ge r_0/2$. We set $V_\eta(x):=\eta_{r_0}(|x|)V(x)$. To prove Theorem 1.3 we need the following:

Theorem 4.2. *Let $d\ge 3$, and let the potential V satisfy* (1.1) *for $|x|\ge r_0$. Let also V_η satisfy either* (1.3) *or* (1.8), *and let s satisfy* (2.4). *Then, for all $0<h\le 1$, $0<\varepsilon\le 1$ and for all functions $f\in H^2(Y_{r_0})$ such that $f=\partial_r f=0$ on ∂Y_{r_0} and*

$$(|x|+1)^{s}(P(h)-E\pm i\varepsilon)f\in L^2(Y_{r_0}),$$

we have the estimate

$$\begin{aligned}\|(|x|+1)^{-s}e^{\varphi/h}f\|_{L^2(Y_{r_0})}+\|(|x|+1)^{-s}e^{\varphi/h}\mathcal{D}_r f\|_{L^2(Y_{r_0})}\\ \le Ca^2h^{-1}\|(|x|+1)^{s}e^{\varphi/h}(P(h)-E\pm i\varepsilon)f\|_{L^2(Y_{r_0})}+C\tau a(\varepsilon/h)^{1/2}\|e^{\varphi/h}f\|_{L^2(Y_{r_0})},\end{aligned} \tag{4.6}$$

with a constant $C>0$ independent of h, ε and f.

Let $d=2$. If V_η satisfies (1.8) *and $k=1$, $k_0=\frac{1}{2}$, then* (4.6) *holds for $0<h\le h_0$ with some constant $0<h_0\ll 1$ depending on τ_0. If V_η satisfies* (1.8) *and $k=\frac{1}{2}$, $k_0=0$, or V_η satisfies* (1.3), *then* (4.6) *holds for all $0<h\le 1$.*

The proof of Theorem 4.2 is similar to that of Theorem 4.1 with some suitable modifications when $d=2$ and will be carried out in Section 5.

Theorem 1.3 can be derived from Theorems 3.1 and 4.2 in a way similar to the one developed in Section 5 of [Vodev 2020b]. Let $r_0>0$ be such that $Y_{r_0/3}\subset\Omega$. Fix r_j, $j=1,2,3,4$, such that $r_0<r_1<r_2<r_3<r_4$. Choose functions $\psi_1,\psi_2\in C^\infty(\mathbb{R}^d)$, depending only on the radial variable r, such that $\psi_1=1$ in $\mathbb{R}^d\setminus Y_{r_1}$, $\psi_1=0$ in Y_{r_2}, $\psi_2=1$ in $\mathbb{R}^d\setminus Y_{r_3}$, $\psi_2=0$ in Y_{r_4}. If s satisfies (2.4), we choose a function $\chi_s\in C^\infty(\bar{\Omega})$, $\chi_s>0$, such that $\chi_s(x)=|x|^{-s}$ on Y_{r_0}. Let $f\in H^2(\Omega)$ be such that $\chi_s^{-1}(P(h)-E\pm i\varepsilon)f\in L^2(\Omega)$ and $f|_{\partial\Omega}=0$. Set

$$\begin{aligned}\mathcal{Q}_0&=\|\chi_s^{-1}(P(h)-E\pm i\varepsilon)f\|_{L^2(\Omega)},\\ \mathcal{Q}_1&=\|f\|_{L^2(Y_{r_1}\setminus Y_{r_2})}+\|\mathcal{D}_r f\|_{L^2(Y_{r_1}\setminus Y_{r_2})},\\ \mathcal{Q}_2&=\|f\|_{L^2(Y_{r_3}\setminus Y_{r_4})}+\|\mathcal{D}_r f\|_{L^2(Y_{r_3}\setminus Y_{r_4})},\end{aligned}$$

and observe that

$$\|[P(h),\psi_j]f\|_{L^2}\lesssim\mathcal{Q}_j,\quad j=1,2.$$

We now apply Theorem 3.1 to the function $\psi_2 f$ with $X=\Omega\setminus Y_{r_4}$ and $U\subset X$ such that $U\cap\operatorname{supp}\psi_2=\varnothing$. Thus, we obtain

$$\begin{aligned}\|f\|_{H_h^1(\Omega\setminus Y_{r_3})}&\le\|\psi_2 f\|_{H_h^1(\Omega\setminus Y_{r_4})}\\ &\le e^{\gamma h^{-4/(\alpha+3)}}\|(P(h)-E\pm i\varepsilon)\psi_2 f\|_{L^2(\Omega\setminus Y_{r_4})}\\ &\le e^{\gamma h^{-4/(\alpha+3)}}\|(P(h)-E\pm i\varepsilon)f\|_{L^2(\Omega\setminus Y_{r_4})}+e^{\gamma h^{-4/(\alpha+3)}}\mathcal{Q}_2,\end{aligned} \tag{4.7}$$

with a constant $\gamma>0$ independent of h and τ_0. In particular, (4.7) implies

$$\mathcal{Q}_1\le e^{\gamma h^{-4/(\alpha+3)}}\mathcal{Q}_0+e^{\gamma h^{-4/(\alpha+3)}}\mathcal{Q}_2. \tag{4.8}$$

On the other hand, it is clear that if V satisfies (1.10) with $\alpha=1$ and $\beta>1$ (respectively, $0<\alpha<1$ and $\beta=4$), then V_η satisfies (1.3) (respectively, (1.8)). Therefore, we can apply Theorem 4.2 to the function $(1-\psi_1)f$ to obtain

$$\begin{aligned}
&\|(|x|+1)^{-s}e^{\varphi/h}f\|_{L^2(Y_{r_2})}+\|(|x|+1)^{-s}e^{\varphi/h}\mathcal{D}_r f\|_{L^2(Y_{r_2})}\\
&\quad\le \|(|x|+1)^{-s}e^{\varphi/h}(1-\psi_1)f\|_{L^2(Y_{r_1})}+\|(|x|+1)^{-s}e^{\varphi/h}\mathcal{D}_r(1-\psi_1)f\|_{L^2(Y_{r_1})}\\
&\quad\le Ca^2h^{-1}\|(|x|+1)^{s}e^{\varphi/h}(P(h)-E\pm i\varepsilon)(1-\psi_1)f\|_{L^2(Y_{r_1})}+C\tau a(\varepsilon/h)^{1/2}\|e^{\varphi/h}f\|_{L^2(Y_{r_1})}\\
&\quad\le Ca^2h^{-1}\|(|x|+1)^{s}e^{\varphi/h}(P(h)-E\pm i\varepsilon)f\|_{L^2(Y_{r_1})}+Ca^2h^{-1}e^{\varphi(r_2)/h}\mathcal{Q}_1\\
&\qquad\qquad+C\tau a(\varepsilon/h)^{1/2}\|e^{\varphi/h}f\|_{L^2(Y_{r_1})}
\end{aligned}\tag{4.9}$$

for all $0<h\le 1$. In particular, (4.9) implies

$$e^{\varphi(r_3)/h}\mathcal{Q}_2\le Ca^2h^{-1}e^{\max\varphi/h}\mathcal{Q}_0+C\tau a(\varepsilon/h)^{1/2}e^{\max\varphi/h}\|f\|_{L^2(\Omega)}+Ca^2h^{-1}e^{\varphi(r_2)/h}\mathcal{Q}_1. \tag{4.10}$$

We have

$$\varphi(r_3)-\varphi(r_2)=\tau\int_{r_2}^{r_3}\big((r+1)^{-k}-(a+1)^{-k}\big)\,dr\ge c\tau,$$

with some constant $c>0$. We deduce from (4.10)

$$\begin{aligned}
\mathcal{Q}_2\le \exp\Big(\tilde\beta h^{-4/(\alpha+3)}+\frac{\max\varphi}{h}\Big)\mathcal{Q}_0+\varepsilon^{1/2}\exp\Big(\tilde\beta h^{-4/(\alpha+3)}+\frac{\max\varphi}{h}\Big)\|f\|_{L^2(\Omega)}\\
+\tau_0^{2\ell}\exp\big((\beta-c\tau_0)h^{-4/(\alpha+3)}\big)\mathcal{Q}_1,
\end{aligned}\tag{4.11}$$

with a constant $\tilde\beta>0$ independent of h and a constant $\beta>0$ independent of h and τ_0. Combining (4.8) and (4.11) we get

$$\begin{aligned}
\mathcal{Q}_2\le \exp\Big((\tilde\beta+\gamma)h^{-4/(\alpha+3)}+\frac{\max\varphi}{h}\Big)\mathcal{Q}_0+\varepsilon^{1/2}\exp\Big(\tilde\beta h^{-4/(\alpha+3)}+\frac{\max\varphi}{h}\Big)\|f\|_{L^2(\Omega)}\\
+\tau_0^{2\ell}\exp\big((\beta+\gamma-c\tau_0)h^{-4/(\alpha+3)}\big)\mathcal{Q}_2.
\end{aligned}\tag{4.12}$$

Taking τ_0 big enough, independent of h, we can arrange that

$$\tau_0^{2\ell}\exp\big((\beta+\gamma-c\tau_0)h^{-4/(\alpha+3)}\big)\le\tau_0^{2\ell}\exp\big(-c\tau_0h^{-4/(\alpha+3)}/2\big)\le\tau_0^{2\ell}\exp\big(-c\tau_0/2\big)\le\tfrac12$$

for all $0<h\le 1$. Thus, we can absorb the last term in the right-hand side of (4.12) to conclude that

$$\mathcal{Q}_1+\mathcal{Q}_2\le \exp\Big(\beta_1h^{-4/(\alpha+3)}+\frac{\max\varphi}{h}\Big)\mathcal{Q}_0+\varepsilon^{1/2}\exp\Big(\beta_1h^{-4/(\alpha+3)}+\frac{\max\varphi}{h}\Big)\|f\|_{L^2(\Omega)}, \tag{4.13}$$

with a constant $\beta_1>0$ independent of h. By (4.7), (4.9) and (4.13) we obtain

$$\|\chi_s f\|_{L^2(\Omega)}\le N\mathcal{Q}_0+\varepsilon^{1/2}N\|f\|_{L^2(\Omega)}, \tag{4.14}$$

where

$$N=\exp\Big(\beta_2h^{-4/(\alpha+3)}+\frac{\max\varphi}{h}\Big),$$

with a constant $\beta_2 > 0$ independent of h. In the same way as above, using the fact that the operator $P(h)$ is symmetric, we get from (4.14) that the resolvent estimate

$$\left\| \chi_s (P(h) - E \pm i\varepsilon)^{-1} \chi_s \right\|_{L^2(\Omega) \to L^2(\Omega)} \le 4N^2 \tag{4.15}$$

holds for all $0 < h \le 1$, $0 < \varepsilon \le 1$ and s satisfying (2.4), which together with Lemma 2.2 clearly imply the desired bound.

5. Proofs of Theorems 4.1 and 4.2

The main point is to work with the polar coordinates $(r, w) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}$, $r = |x|$, $w = x/|x|$ and to use that $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^+ \times \mathbb{S}^{d-1}, r^{d-1}\, dr\, dw)$. In what follows in this section, we denote by $\|\cdot\|$ and $\langle \cdot , \cdot \rangle$ the norm and the scalar product in $L^2(\mathbb{S}^{d-1})$. We will make use of the identity

$$r^{(d-1)/2} \Delta r^{-(d-1)/2} = \partial_r^2 + \frac{\widetilde{\Delta}_w}{r^2}, \tag{5.1}$$

where $\widetilde{\Delta}_w = \frac{1}{4}(d-1)(d-3)$ and Δ_w denotes the negative Laplace–Beltrami operator on $\mathbb{S}^{d-1}$. Set $u = r^{(d-1)/2} e^{\varphi/h} f$ and

$$\mathcal{P}^{\pm}(h) = r^{(d-1)/2} (P(h) - E \pm i\varepsilon) r^{-(d-1)/2},$$
$$\mathcal{P}^{\pm}_{\varphi}(h) = e^{\varphi/h} \mathcal{P}^{\pm}(h) e^{-\varphi/h}.$$

Using (5.1) we can write the operator $\mathcal{P}^{\pm}(h)$ in the coordinates (r, w) as follows:

$$\mathcal{P}^{\pm}(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon + V,$$

where we have put $\mathcal{D}_r = -ih\partial_r$ and $\Lambda_w = -h^2 \widetilde{\Delta}_w$. Since the function φ depends only on the variable r, we get

$$\mathcal{P}^{\pm}_{\varphi}(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon - \varphi'^2 + h\varphi'' + 2i\varphi' \mathcal{D}_r + V.$$

We write $V = V_L + V_S$ with $V_L := V_\theta$ and $V_S := V - V_\theta$ if V satisfies (1.8), and $V_L := V$ and $V_S := 0$ if V satisfies (1.3). For $r > 0$, $r \ne a$, introduce the function

$$F(r) = -\langle (r^{-2}\Lambda_w - E - \varphi'(r)^2 + V_L(r, \cdot)) u(r, \cdot), u(r, \cdot) \rangle + \|\mathcal{D}_r u(r, \cdot)\|^2,$$

where $V_L(r, w) := V_L(rw)$. Then its first derivative is given by

$$\begin{aligned} F'(r) = {} & \frac{2}{r} \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle + ((\varphi')^2 - V_L)' \|u(r, \cdot)\|^2 - 2h^{-1} \operatorname{Im} \langle \mathcal{P}^{\pm}_{\varphi}(h) u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \\ & \pm 2\varepsilon h^{-1} \operatorname{Re} \langle u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle + 4h^{-1} \varphi' \|\mathcal{D}_r u(r, \cdot)\|^2 + 2h^{-1} \operatorname{Im} \langle (V_S + h\varphi'') u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle. \end{aligned}$$

Thus, we obtain the identity

$$
\begin{aligned}
(\mu F)' &= \mu' F + \mu F' \\
&= (2r^{-1}\mu - \mu')\langle r^{-2}\Lambda_w u(r,\cdot), u(r,\cdot)\rangle + (E\mu' + (\mu(\varphi')^2 - \mu V_L)')\|u(r,\cdot)\|^2 \\
&\qquad - 2h^{-1}\mu \operatorname{Im}\langle \mathcal{P}_\varphi^\pm(h)u(r,\cdot), \mathcal{D}_r u(r,\cdot)\rangle \pm 2\varepsilon h^{-1}\mu \operatorname{Re}\langle u(r,\cdot), \mathcal{D}_r u(r,\cdot)\rangle \\
&\qquad + 2h^{-1}\mu \operatorname{Im}\langle (V_S + h\varphi'')u(r,\cdot), \mathcal{D}_r u(r,\cdot)\rangle.
\end{aligned}
$$

Using that $\Lambda_w \geq 0$ as long as $d \geq 3$ together with (2.5), we get the inequality

$$
\begin{aligned}
&\mu' F + \mu F' \\
&\geq (E\mu' + (\mu(\varphi')^2 - \mu V_L)')\|u(r,\cdot)\|^2 + (\mu' + 4h^{-1}\varphi'\mu)\|\mathcal{D}_r u(r,\cdot)\|^2 \\
&\qquad - \frac{3h^{-2}\mu^2}{\mu'}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2 - \frac{\mu'}{3}\|\mathcal{D}_r u(r,\cdot)\|^2 - \varepsilon h^{-1}\mu\big(\|u(r,\cdot)\|^2 + \|\mathcal{D}_r u(r,\cdot)\|^2\big) \\
&\qquad - 3h^{-2}\mu^2(\mu' + 4h^{-1}\varphi'\mu)^{-1}\|(V_S + h\varphi'')u(r,\cdot)\|^2 - \frac{1}{3}(\mu' + 4h^{-1}\varphi'\mu)\|\mathcal{D}_r u(r,\cdot)\|^2 \\
&\geq (E\mu' + (\mu(\varphi')^2)' - T_L\mu - Z_L\mu')\|u(r,\cdot)\|^2 + \frac{\mu'}{3}\|\mathcal{D}_r u(r,\cdot)\|^2 - \frac{3h^{-2}\mu^2}{\mu'}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2 \\
&\qquad - \varepsilon h^{-1}\mu\big(\|u(r,\cdot)\|^2 + \|\mathcal{D}_r u(r,\cdot)\|^2\big) - 3h^{-2}\mu^2(\mu' + 4h^{-1}\varphi'\mu)^{-1}(Q_S + h|\varphi''|)^2\|u(r,\cdot)\|^2,
\end{aligned}
$$

where

$$
T_L = \mathcal{O}\big((r+1)^{-\beta}\big), \quad Z_L = p(r), \quad Q_S = 0
$$

if V satisfies (1.3),

$$
T_L = \mathcal{O}\big(\theta^{-1+\alpha}(r+1)^{-4}\big), \quad Z_L = p(r) + \mathcal{O}\big((r+1)^{-4}\big), \quad Q_S = \mathcal{O}\big(\theta^{\alpha}(r+1)^{-4}\big),
$$

if V satisfies (1.8), and we have used the bounds (2.1),(2.2) and (2.3) in the second case. Hence, we can rewrite the above inequality in the form

$$
\begin{aligned}
\mu' F + \mu F' &\geq \big(E\mu' + A(r) - CB(r)\big)\|u(r,\cdot)\|^2 + \frac{\mu'}{3}\|\mathcal{D}_r u(r,\cdot)\|^2 \\
&\qquad - \frac{3h^{-2}\mu^2}{\mu'}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2 - \varepsilon h^{-1}\mu\big(\|u(r,\cdot)\|^2 + \|\mathcal{D}_r u(r,\cdot)\|^2\big),
\end{aligned}
$$

with a suitable constant $C > 0$. Now we use Lemma 2.3 to conclude that

$$
\begin{aligned}
\mu' F + \mu F' &\geq \frac{E}{2}\mu'\|u(r,\cdot)\|^2 + \frac{\mu'}{3}\|\mathcal{D}_r u(r,\cdot)\|^2 - \frac{3h^{-2}\mu^2}{\mu'}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2 \\
&\qquad - \varepsilon h^{-1}\mu\big(\|u(r,\cdot)\|^2 + \|\mathcal{D}_r u(r,\cdot)\|^2\big). \qquad (5.2)
\end{aligned}
$$

We integrate this inequality with respect to r and use that $\mu(0) = 0$. We have

$$
\int_0^\infty (\mu' F + \mu F')\, dr = 0.
$$

Thus, we obtain the estimate

$$\frac{E}{2}\int_0^\infty \mu'\|u(r,\cdot)\|^2\,dr+\int_0^\infty \frac{\mu'}{3}\|\mathcal{D}_r u(r,\cdot)\|^2\,dr \le 3h^{-2}\int_0^\infty \frac{\mu^2}{\mu'}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2\,dr$$
$$+\varepsilon h^{-1}\int_0^\infty \mu\big(\|u(r,\cdot)\|^2+\|\mathcal{D}_r u(r,\cdot)\|^2\big)\,dr. \quad (5.3)$$

Using that $\mu=\mathcal{O}(a^2)$ together with (2.6), we get from (5.3)

$$\int_0^\infty (r+1)^{-2s}\big(\|u(r,\cdot)\|^2+\|\mathcal{D}_r u(r,\cdot)\|^2\big)\,dr \le Ca^4h^{-2}\int_0^\infty (r+1)^{2s}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2\,dr$$
$$+C\varepsilon h^{-1}a^2\int_0^\infty \big(\|u(r,\cdot)\|^2+\|\mathcal{D}_r u(r,\cdot)\|^2\big)\,dr, \quad (5.4)$$

with some constant $C>0$ independent of h and ε. On the other hand, we have the identity

$$\operatorname{Re}\int_0^\infty \langle 2i\varphi'\mathcal{D}_r u(r,\cdot),u(r,\cdot)\rangle\,dr=\int_0^\infty h\varphi''\|u(r,\cdot)\|^2\,dr,$$

and hence,

$$\operatorname{Re}\int_0^\infty \langle \mathcal{P}_\varphi^\pm(h)u(r,\cdot),u(r,\cdot)\rangle\,dr=\int_0^\infty \|\mathcal{D}_r u(r,\cdot)\|^2\,dr+\int_0^\infty \langle r^{-2}\Lambda_w u(r,\cdot),u(r,\cdot)\rangle\,dr$$
$$-\int_0^\infty (E+\varphi'^2)\|u(r,\cdot)\|^2\,dr+\int_0^\infty \langle Vu(r,\cdot),u(r,\cdot)\rangle\,dr$$
$$\ge \int_0^\infty \|\mathcal{D}_r u(r,\cdot)\|^2\,dr-\mathcal{O}(\tau^2)\int_0^\infty \|u(r,\cdot)\|^2\,dr.$$

This implies

$$\varepsilon h^{-1}a^2\int_0^\infty \|\mathcal{D}_r u(r,\cdot)\|^2\,dr \le \mathcal{O}(\tau^2)\varepsilon h^{-1}a^2\int_0^\infty \|u(r,\cdot)\|^2\,dr$$
$$+\gamma\int_0^\infty (r+1)^{-2s}\|u(r,\cdot)\|^2\,dr+\gamma^{-1}h^{-2}a^4\int_0^\infty (r+1)^{2s}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2\,dr \quad (5.5)$$

for every $\gamma>0$. Taking γ small enough, independent of h, τ and a and combining the estimates (5.4) and (5.5), we get

$$\int_0^\infty (r+1)^{-2s}\big(\|u(r,\cdot)\|^2+\|\mathcal{D}_r u(r,\cdot)\|^2\big)\,dr \le Ca^4h^{-2}\int_0^\infty (r+1)^{2s}\|\mathcal{P}_\varphi^\pm(h)u(r,\cdot)\|^2\,dr$$
$$+C\varepsilon h^{-1}a^2\tau^2\int_0^\infty \|u(r,\cdot)\|^2\,dr, \quad (5.6)$$

with a new constant $C>0$ independent of h and ε. Clearly, the estimate (5.6) implies (4.1).

The proof of Theorem 4.2 in the case when $d\ge 3$ goes very much like the proof of Theorem 4.1 above. The only difference in this case is that we have to integrate the function $F(r)$ from r_0 to ∞ and use that $F(r_0)=0$ by assumption. Thus, by Lemma 2.3 we conclude that the inequality (5.2) holds for all $r\ge r_0$.

When $d=2$, the operator Λ_w is no longer nonnegative. Instead, we will use that so is the operator $-\Delta_w$. Thus, it is easy to see that the above inequalities still hold with V_L replaced by $V_L - h^2(2r)^{-2}$. Since

$$h^2(\mu(r)(2r)^{-2})' = h^2\mu'(r)(2r)^{-2} - 2^{-1}h^2r^{-3}\mu(r) > -h^2r^{-3}\mu(r),$$

we can use Lemmas 2.5 and 2.6, instead of Lemma 2.3, to conclude that the inequality (5.2) remains valid for $r \geq r_0$ with $E/2$ replaced by $E/3$.

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