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A FREE BOUNDARY PROBLEM DRIVEN BY THE BIHARMONIC OPERATOR

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We consider the minimization of the functional

$$J[u] := \int_\Omega \left(|\Delta u|^2 + \chi_{\{u>0\}}\right)$$

in the admissible class of functions

$$\mathcal{A} := \left\{u \in W^{2,2}(\Omega) : u - u_0 \in W_0^{1,2}(\Omega)\right\}.$$

Here, Ω is a smooth and bounded domain of $\mathbb{R}^n$ and $u_0 \in W^{2,2}(\Omega)$ is a given function defining the Navier type boundary condition.

When $n = 2$, the functional J can be interpreted as a sum of the linearized Willmore energy of the graph of u and the area of $\{u > 0\}$ on the xy-plane.

The regularity of a minimizer u and that of the free boundary $\partial\{u > 0\}$ are very complicated problems. The most intriguing part of this is to study the structure of $\partial\{u > 0\}$ near singular points, where $\nabla u = 0$ (of course, at the nonsingular free boundary points where $\nabla u \neq 0$, the free boundary is locally C^1 smooth).

The scale invariance of the problem suggests that, at the singular points of the free boundary, quadratic growth of u is expected. We prove that u is quadratically nondegenerate at the singular free boundary points using a refinement of Whitney's cube decomposition, which applies, if, for instance, the set $\{u > 0\}$ is a John domain.

The optimal growth is linked with the approximate symmetries of the free boundary. More precisely, if at small scales the free boundary can be approximated by zero level sets of a quadratic degree two homogeneous polynomial, then we say that $\partial\{u > 0\}$ is rank-2 flat.

Using a dichotomy method for nonlinear free boundary problems, we also show that, at the free boundary points $x \in \Omega$, where $\nabla u(x) = 0$, the free boundary is either well approximated by zero sets of quadratic polynomials, i.e., $\partial\{u > 0\}$ is rank-2 flat, or u has quadratic growth.

More can be said if $n = 2$, in which case we obtain a monotonicity formula and show that, at the singular points of the free boundary where the free boundary is not well approximated by level sets of quadratic polynomials, the blow-up of the minimizer is a homogeneous function of degree two.

In particular, if $n = 2$ and $\{u > 0\}$ is a John domain, then we get that the blow-up of the free boundary is a cone; and in the one-phase case, it follows that $\partial\{u > 0\}$ possesses a tangent line in the measure theoretic sense.

Differently from the classical free boundary problems driven by the Laplacian operator, the one-phase minimizers present structural differences with respect to the minimizers, and one notion is not included into the other. In addition, one-phase minimizers arise from the combination of a volume type free boundary problem and an obstacle type problem, hence their growth condition is influenced in a nonstandard way by these two ingredients.

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1. Introduction

1A. ***Mathematical framework and motivations.*** In this paper we consider the problem of minimizing the functional

$$J[u] = J[u,\Omega] := \int_\Omega \big(|\Delta u|^2 + \chi_{\{u>0\}}\big) \tag{1-1}$$

over the admissible class of functions

$$\mathcal{A} := \big\{u \in W^{2,2}(\Omega) : u - u_0 \in W_0^{1,2}(\Omega)\big\}. \tag{1-2}$$

Here, Ω is a smooth and bounded domain of $\mathbb{R}^n$ and $u_0 \in W^{2,2}(\Omega)$ is a given function defining the Navier type boundary condition (see, e.g., the "hinged problem" on the right-hand side of Figure 1(a) and on page 84 of [Sweers 2009], or Figure 1.5 on page 6 of [Ganguli 2017], or the monograph [Gazzola et al. 2010] for additional information on this condition, which can be interpreted as a weak form of two boundary conditions: $u = u_0$ along $\partial\Omega$ and $\Delta u = 0$ along $\partial\Omega \cap \{u \neq 0\}$).

More precisely, we study here two different types of minimization problems related to the functional in (1-1), namely the *minimizers* in the class $\mathcal{A}$ introduced in (1-2), as well as the minimizers among all the nonnegative functions in $\mathcal{A}$ (that will be called *one-phase minimizers* and thoroughly discussed from Definition 1.2 on). An important feature of the problem that we study is that these two types of minimizers are different and exhibit different[1] features.

[1] As a matter of fact, most of the results presented here will concern minimizers (see in particular Theorems 1.1, 1.7, 1.8, 1.10, and 1.11); some results will include, basically at the same time, both minimizers and one-phase minimizers (see Theorems 1.3, 1.12, and 1.13), and one result (namely Theorem 1.14) will focus specifically on the case of one-phase minimizers. Yet, we believe it was worth stressing the distinction between minimizers and one-phase minimizers, since it is a special characteristic of the fourth order equations and provides a conceptual difference with respect to the more extensively studied case of second order

The functional in (1-1) is clearly related to the biharmonic operator, which provides classical models for rigidity problems with concrete applications, for instance, in the construction of suspension bridges, see, e.g., [McKenna and Walter 1987]. Other classical applications of the biharmonic operator arise in the study of steady state incompressible fluid flows at small Reynolds numbers under the Stokes flow approximation assumption, see, e.g., formula (1) in [Mardanov and Zaripov 2016].

Moreover, the functional in (1-1) provides a linearized model for the Willmore problem which asks to find an immersion/embedding M in $\mathbb{R}^3$ that minimizes the Willmore energy

$$W(M)=\int_M H^2\,dA,$$

where H denotes the mean curvature. The linearization of this energy density gives

$$H^2\,dA=\tfrac{1}{4}(\Delta u)^2\,dx\,dy+\text{lower order terms}.$$

In this context, our problem can be regarded as a free boundary problem for the linearized Willmore energy, where the surface M has a flat part on the xy-plane.

We also refer to the very recent work in [Da Lio et al. 2020] for a problem related to the minimization of the Willmore energy functional with prescribed boundary, boundary Gauss map, and area. See also the recent contributions in [Miura 2016; 2017] for the one-dimensional analysis of the global properties of the solutions of free boundary problems involving the curvature of a curve.

In the setting of (1-1), an additional motivation for us comes from the study of the degenerate/unstable obstacle problem, see [Caffarelli 1980; Monneau and Weiss 2007]. Indeed, we will see in Corollary 4.2 that u is globally almost subharmonic in Ω, i.e., there exists $\hat{C}>0$ (possibly depending also on the energy of the minimizer) such that $\Delta u\geqslant -\hat{C}$. Therefore, the function $\Delta u:=f$ is bounded from below. Accordingly, we can relate our problem to an obstacle problem with unknown right-hand side, namely determine u and $f\geqslant -\hat{C}$ such that

$$\begin{cases}\Delta u=f & \text{in } \Omega,\\ u=|\nabla u|=0 & \text{on } \partial\{u>0\},\\ f=1 & \text{on } \partial\{u>0\}.\end{cases} \tag{1-3}$$

The principal difference from the classical obstacle problem is that f may change sign in Ω and degenerate on the free boundary points, since the last condition in (1-3) is satisfied in a generalized sense: for this reason, it does not follow from the classical obstacle problem theory that u is quadratically nondegenerate.

Motivation for (1-1) also comes from the limit as $\varepsilon\to 0$ of the singularly perturbed bi-Laplacian equation

$$\Delta^2 u^\varepsilon=-\frac{1}{\varepsilon}\beta\Big(\frac{u^\varepsilon}{\varepsilon}\Big), \tag{1-4}$$

where β is a compactly supported nonnegative function with finite total mass, see [Dipierro et al. 2019]. Equation (1-4) can be seen as the biharmonic counterpart to classical combustion models, see, e.g., [Petrosyan 2002].

equations. In particular, while one-phase minimizers exhibit nontrivial zero sets, the same does not happen for the minimizers (see Proposition B.1). Let us also mention that one-phase minimizers are perhaps less justified by physical motivations, since one is adding an extra "obstacle condition" precisely at the discontinuity level of the potential, nevertheless we think they also deserve further mathematical investigation besides the one carried out in the present paper.

1B. ***Comparison with the existing literature.*** Free boundary problems are, of course, a classical topic of investigation, nevertheless only few results are available concerning the case of equations of order higher than two, and there seems to be no investigation at all for the free boundary problem in (1-1).

Other types of free boundary problems for higher order operators have been considered in [Mawi 2014]. Moreover, obstacle problems involving biharmonic operators have been studied in [Frehse 1973; Caffarelli and Friedman 1979; Caffarelli et al. 1981; 1982; Adams and Vandenhouten 2000; Pozzolini and Léger 2008; Novaga and Okabe 2015; 2016; Aleksanyan 2019]; but, till now, we are not aware of any previous investigation of free boundary problems dealing with higher order operators combined with "bulk" volume terms as in (1-1) here.

Of course, one of the striking differences in our framework, as opposed to the case of the Alt–Caffarelli functional (see [Alt and Caffarelli 1981])

$$J_{\mathrm{AC}}[u] := \int_\Omega \big(|\nabla u|^2 + \chi_{\{u>0\}}\big),$$

is the lack of a maximum principle and the Harnack inequality for higher order operators. This, in our setting, reflects to the fact that the set $\{u<0\}$ may be nonempty, even under the boundary condition $u_0 \geqslant 0$. This is one of the peculiarities of the situation involving the bi-Laplacian, and it makes the mathematical treatment of the problem extremely difficult (and this is likely to be the reason for which there are not many results in the direction of free boundary regularity in the framework that we consider here).

Thus, the main difficulties in our setting, in comparison with the existing literature, follow from the fact that major elliptic methods based on a maximum principle, the Harnack inequality, and propagation of ellipticity cannot be applied. Moreover, many classical tools, such as domain variations, have not been fully analyzed yet; and, in any case, cannot provide consequences which are as strong as in the classical framework. For instance, the main result that we obtain by domain variation (given in details in Lemma 4.4) is that, for any $\phi = (\phi^1, \dots, \phi^n) \in C_0^\infty(\Omega)$,

$$2\int_\Omega \Delta u(x) \sum_{m=1}^{n} (2\nabla u_m(x)\cdot\nabla\phi^m(x) + u_m(x)\Delta\phi^m(x))\,dx = \int_\Omega \big(|\Delta u(x)|^2 + \chi_{\{u>0\}}(x)\big)\operatorname{div}\phi(x)\,dx. \tag{1-5}$$

As customary, we denote by $u_m = \partial_m u = \partial_{x_m} u$ the partial derivative of u with respect to the mth variable. Then, in the classical literature, the standard argument leading to the monotonicity formula for the Alt–Caffarelli problem would be to choose ϕ of a particular form, see [Weiss 1998]. More precisely, for $\varepsilon > 0$, the classical idea would be to consider

$$\eta(x) := \begin{cases} 1 & \text{if } x \in B_r(x_0),\\ \dfrac{r+\varepsilon-|x-x_0|}{\varepsilon} & \text{if } x \in B_{r+\varepsilon}(x_0)\setminus B_r(x_0),\\ 0 & \text{otherwise,} \end{cases}$$

where $x_0 \in \partial\{u>0\}$, and take $\phi(x) := x\eta(x)$ in identity (1-5). Note that

$$\nabla\phi(x) = \begin{cases} \mathbb{I} & \text{if } x \in B_r(x_0),\\ \mathbb{I}\eta - \dfrac{1}{\varepsilon}\dfrac{(x-x_0)\otimes(x-x_0)}{|x-x_0|} & \text{if } x \in B_{r+\varepsilon}(x_0)\setminus B_r(x_0),\\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{I} \in \mathrm{Mat}_{n\times n}$ is the identity matrix. However, in our case, identity (1-5) contains the term $\Delta\phi$ which is not defined on the boundary of the ring $B_{r+\varepsilon}(x_0) \setminus B_r(x_0)$, and this creates an important conceptual difficulty. Thus, to overcome this issue, one needs to perform a series of ad hoc integration by parts. This strategy, however, has to deal with the possible generation of third order derivatives of the minimizers, which also cannot be controlled. Therefore, these terms need to be suitably smoothened and simplified via appropriate cancellations.

In this setting, the lack of monotonicity formulas can also be seen as a counterpart to the lack of Pohozhaev type inequalities, and our approach bypasses this kind of difficulty.

As a matter of fact, we will establish a new monotonicity formula in dimension 2, which will lead to Theorem 1.12. In addition, differently from the harmonic case, there are no estimates available in the literature for the biharmonic measure, and this makes the free boundary analysis significantly more complicated. We will overcome these difficulties by Theorem 1.10.

Moreover, in terms of barrier and test functions, an additional difficulty of the biharmonic setting is given by the fact that the function $\max\{u, v\}$ is not an admissible competitor, having possibly infinite energy, so we cannot consider the maximal and minimal solutions.

The analysis of nondegeneracy and optimal regularity of minimizers and of their free boundary is also a novel ingredient with respect to the classical literature, and nothing seemed to be known before about these important questions.

1C. *Main results.* In what follows, we will denote by $\{u > 0\}$ the positivity set of u and by $\partial\{u > 0\}$ its free boundary. The main results of this paper are the following:

- If $z \in \partial\{u > 0\}$ and $\nabla u(z) = 0$, then either $\partial\{u > 0\}$ can be approximated by the zero level sets of a quadratic homogeneous polynomial of degree two, or u has quadratic growth at z.
- If $n = 2$, there exists a monotonicity formula, and we can classify the homogeneous one-phase solutions of degree two.
- We provide various sufficient conditions for strong nondegeneracy in terms of a suitable refinement of Whitney's cube decomposition (c-covering). For instance, we show that if $\{u > 0\}$ is a John domain (see the definition in Section 7B), then $\partial\{u > 0\}$ possesses a measure theoretic tangent line.

A road map of this article is displayed in Figure 1.

1C.1. *BMO estimates for the Laplacian of the minimizers and free boundary conditions.* In further details, the first regularity result that we establish is a BMO estimate on the Laplacian of the minimizers.

Theorem 1.1. *Let u be a minimizer of the functional J defined in (1-1). Then, we have $\Delta u \in BMO_{\mathrm{loc}}(\Omega)$.*

We also introduce a notion of one-phase minimizer, in the following setting:

Definition 1.2. We say that u is a one-phase minimizer of J if it minimizes the functional J in (1-1) among the nonnegative admissible functions $\mathcal{A}_+ := \{u \in \mathcal{A} : u \geqslant 0 \text{ in } \Omega\}$, $\mathcal{A}$ being as in (1-2).

Interestingly, one-phase minimizers, as given in Definition 1.2, arise from a combination of a biharmonic free boundary problem and an obstacle problem. We also observe that, in general, minimizers of J which

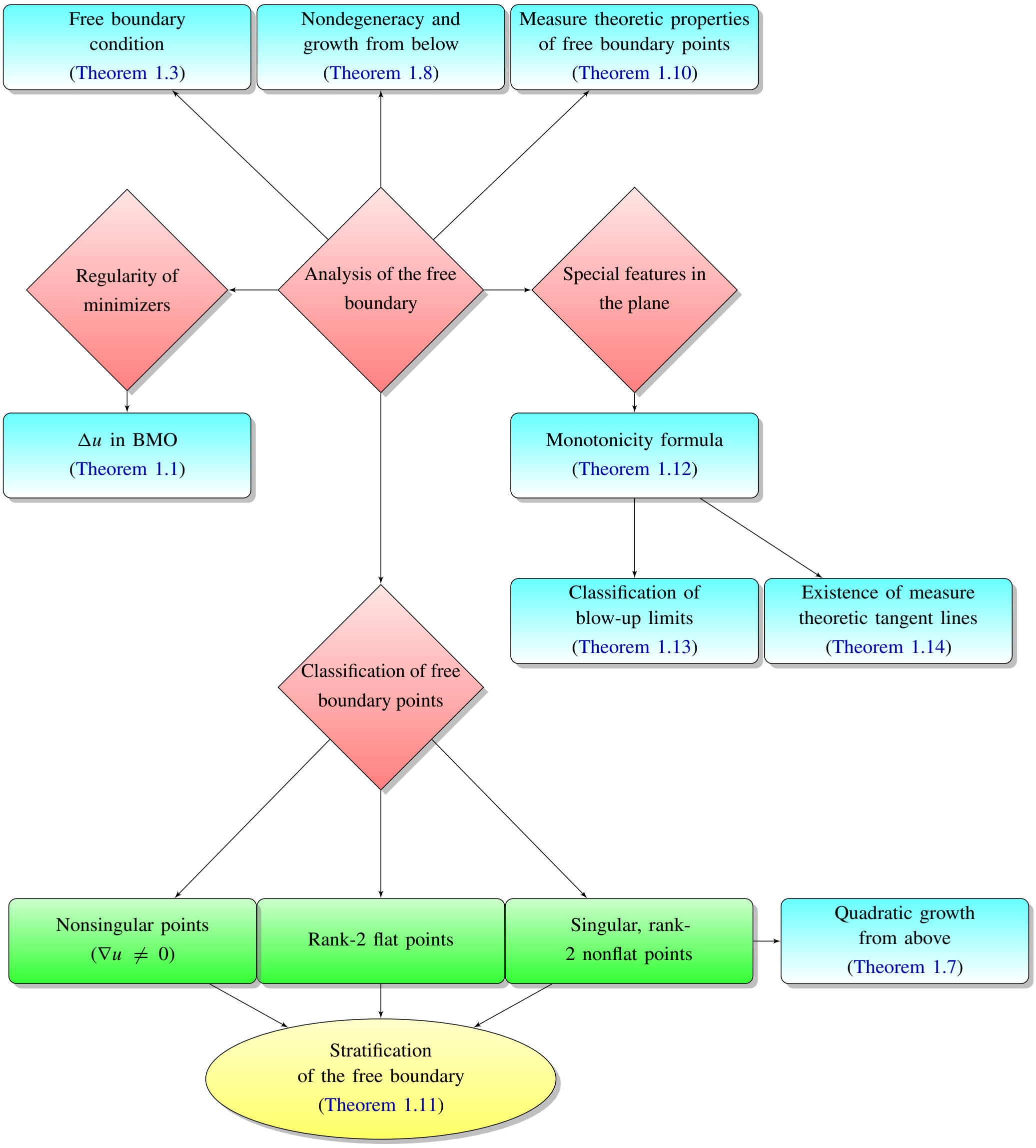

Figure 1. A road map of this article.

happen to be nonnegative do not naturally develop open regions in which the minimizer vanishes (see Proposition B.1 for a concrete result), while one-phase minimizers do (hence, the notion of minimizers that are nonnegative and the notion of one-phase minimizers are structurally very different in this framework, due to the lack of a maximum principle).

We stress that one-phase minimizers, as given in Definition 1.2, are not necessarily minimizers over $\mathcal{A}$. This fact produces significant differences, with respect to the classical case of free boundary problems driven by the Laplacian, and requires some nonstandard techniques to overcome the lack of structure provided, in the classical case, by super-harmonic functions.

We also observe that, in the classical Alt–Caffarelli problem [Alt and Caffarelli 1981] a nonnegative boundary datum produces, in general, considerable portions of the domain in which the minimizer vanishes, but in our case minimizers with nonnegative (and even strictly positive) boundary data may produce regions with considerable negative phases. This difference between zero and strictly negative phases is indeed one of the typical features of our problem, and it is also due to the characteristic function in (1-1). Specifically, the Alt–Caffarelli problem [Alt and Caffarelli 1981] with nonnegative datum typically produces large zero phases, while in most of the situations that one can imagine, our minimizers with nonnegative data have negligible zero sets (but nonnegligible negative sets): the role of one-phase minimizers in our setting is precisely to create natural conditions to produce nonnegligible zero sets (the reader may also consider looking immediately at the examples in Section 5 to see these phenomena of zero and negative phases in simple, but concrete, cases).

Given the higher order structure of the biharmonic functional, the minimizers satisfy a free boundary condition which is richer, and more complicated, than in the harmonic case. To express it in a general form, suppose that the free boundary (locally) separates two regions, say $\Omega^{(1)}$ and $\Omega^{(2)}$, of the domain Ω, with $\partial\Omega^{(1)} = \partial\Omega^{(2)} = \partial\{u > 0\}$: in this case, the minimizer u can be seen as the result of the junction of two functions, say $u^{(1)}$ and $u^{(2)}$, from each side of the free boundary, with $u^{(1)}$ and $u^{(2)}$ not changing sign. In this notation, for $i \in \{1, 2\}$, we set

$$\lambda^{(i)} := \begin{cases} 1 & \text{if } u^{(i)} > 0 \text{ in } \Omega^{(i)}, \\ 0 & \text{if } u^{(i)} \leqslant 0 \text{ in } \Omega^{(i)}. \end{cases} \tag{1-6}$$

Then, we have the following result describing the free boundary condition in this framework:

Theorem 1.3. *Let u be either a minimizer or a continuous one-phase minimizer of the functional J defined in (1-1). Assume that*

$$\partial\{|u| > \varepsilon\} \text{ is of class } C^1, \tag{1-7}$$

for all $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 > 0$. Then, for any $\phi = (\phi^1, \dots, \phi^n) \in C_0^\infty(\Omega)$,

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\partial(\Omega\cap\{|u|>\varepsilon\})}\Big(\big(|\Delta u^{(1)}|^2+\lambda^{(1)}\big)\phi\cdot\nu-2\sum_{m=1}^n\big(\phi^m(\Delta u^{(1)}\nabla u_m^{(1)}-u_m^{(1)}\nabla\Delta u^{(1)})\cdot\nu+\Delta u^{(1)}u_m^{(1)}\nabla\phi^m\cdot\nu\big)\Big)\\ &=\lim_{\varepsilon\to 0}\int_{\partial(\Omega\cap\{|u|>\varepsilon\})}\Big(\big(|\Delta u^{(2)}|^2+\lambda^{(2)}\big)\phi\cdot\nu-2\sum_{m=1}^n\big(\phi^m(\Delta u^{(2)}\nabla u_m^{(2)}-u_m^{(2)}\nabla\Delta u^{(2)})\cdot\nu+\Delta u^{(2)}u_m^{(2)}\nabla\phi^m\cdot\nu\big)\Big), \end{aligned} \tag{1-8}$$

where ν is the exterior normal to $\Omega^{(1)}$.

Furthermore, if $u \in C^1(\Omega)\cap C^3(\overline{\Omega^{(1)}})\cap C^3(\overline{\Omega^{(2)}})$ and $\partial\{|u| > \varepsilon\}$ approaches $\partial\{|u| > 0\} = \partial\{u > 0\} = \partial\{u < 0\} = \{u = 0\}$ in the C^1-sense, we have that

$$\begin{cases} \Delta u^{(1)}u_m^{(1)} = \Delta u^{(2)}u_m^{(2)} \\ \big(|\Delta u^{(1)}|^2+\lambda^{(1)}\big)\nu_m - 2(\Delta u^{(1)}\nabla u_m^{(1)} - u_m^{(1)}\nabla\Delta u^{(1)})\cdot\nu \\ \qquad\qquad = \big(|\Delta u^{(2)}|^2+\lambda^{(2)}\big)\nu_m - 2(\Delta u^{(2)}\nabla u_m^{(2)} - u_m^{(2)}\nabla\Delta u^{(2)})\cdot\nu, \end{cases} \tag{1-9}$$

for any $m \in \{1, \dots, n\}$, on $\partial\{u > 0\}$.

Concrete examples of this free boundary condition will also be presented in Section 5 (of course, the reader is welcome to jump to these examples, before diving into all the rather technical details of this paper, if she or he wants to immediately have a close-to-intuition approach to the model and the problems discussed in this paper, as well as to develop some feeling on how minimizers may be expected to look).

As already discussed in Section 1B, one of the principal features of the problem that we consider in the present work is that it does not share the standard properties of its "sibling" Alt–Caffarelli problem [Alt and Caffarelli 1981], such as nondegeneracy, linear growth, etc. Moreover, the existing techniques fail because of the involvement of higher order derivatives.

However, the scale invariance of the functional suggests that the optimal regularity of u must be $C^{1,1}$. This is also supported by the computations that we have for the one-dimensional case (see Remark 4.5 and the explicit examples in Section 5).

1C.2. *Notion of rank-2 flatness, the role played by quadratic polynomials, and dichotomy arguments.* To study the free boundary points of the minimizers, it is useful to distinguish between regular and singular points. Related to this, suppose that $x \in \partial\{u > 0\}$, then there are two possible cases:

- $\nabla u(x) \neq 0$, then $\partial\{u > 0\}$ is C^1 near x.
- $\nabla u(x) = 0$, then we expect u to grow quadratically, and the free boundary may have self-intersections.

To analyze these situations, we introduce the following setting:

Definition 1.4. If $x \in \partial\{u > 0\}$ and $\nabla u(x) = 0$, then we say that x is a singular free boundary point. The set of singular points is denoted by $\partial_{\text{sing}}\{u > 0\}$.

Clearly the singular points are the most interesting points of the free boundary to study. In order to overcome all the difficulties mentioned in Section 1B and to study the regularity of u and that of the free boundary $\partial\{u > 0\}$, we employ a dichotomy argument which was introduced in [Dipierro and Karakhanyan 2018]. The idea is to exploit a suitable notion[2] of "flatness" and distinguish between points where the free boundary is flat and points where it is nonflat, according to this *new notion*.

To this aim, we let

$$\mathrm{HD}(A, B) := \max\Big\{ \sup_{a\in A} \mathrm{dist}(a, B),\ \sup_{b\in B} \mathrm{dist}(b, A)\Big\} \tag{1-10}$$

be the Hausdorff distance of two sets $A, B \subset \mathbb{R}^n$.

We also let P_2 be the set of all homogeneous polynomials of degree two, i.e.,

$$P_2 := \left\{ p(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j,\ \text{ for any } x \in \mathbb{R}^n \text{ with } \|p\|_{L^\infty(B_1)} = 1 \right\}, \tag{1-11}$$

[2]We stress that the "flatness" condition that we consider here is not related to a geometric idea of flatness as being close to a hyperplane. In general, the "flat" objects that we consider look like boundaries of cones and their special feature is related to the "rank-2" notion of flatness, that is being close to zero sets of homogeneous polynomials of degree 2. With this respect, the usual notion of flatness intended as proximity to hyperplanes can be interpreted as a "rank-1 flatness". We maintained the name of "flatness" also for the rank-2 case in order to make the comparison with the classical elliptic free boundary theory easier and more transparent.

where a_{ij} is a symmetric $n \times n$ matrix. Moreover, given $p \in P_2$ and $x_0 \in \mathbb{R}^n$, we set $p_{x_0}(x) := p(x - x_0)$ and

$$S(p, x_0) := \{x \in \mathbb{R}^n : p_{x_0}(x) = 0\}. \tag{1-12}$$

We observe that the set $S(p, x_0)$ defined in (1-12) is a cone with vertex at x_0, i.e., if $x \in S(p, x_0)$ then, for every $t > 0$, it holds that $x_0 + t(x - x_0) \in S(p, x_0)$.

With this notation, we set:

Definition 1.5. Let $\delta > 0$, $R > 0$, and $x_0 \in \partial\{u > 0\}$. We say that $\partial\{u > 0\}$ is (δ, R)-rank-2 flat at x_0 if, for every $r \in (0, R]$, there exists $p \in P_2$ such that

$$\mathrm{HD}\big(\partial\{u > 0\} \cap B_r(x_0), S(p, x_0) \cap B_r(x_0)\big) < \delta r.$$

Now, given $r > 0$, $x_0 \in \partial\{u > 0\}$, and $p \in P_2$, we let

$$h_{\min}(r, x_0, p) := \mathrm{HD}\big(\partial\{u > 0\} \cap B_r(x_0), S(p, x_0) \cap B_r(x_0)\big). \tag{1-13}$$

Then, we define the rank-2 flatness at level $r > 0$ of $\partial\{u > 0\}$ at x_0 as follows: We set

$$h(r, x_0) := \inf_{p \in P_2} h_{\min}(r, x_0, p), \tag{1-14}$$

and we introduce the following notation:

Definition 1.6. Let $\delta > 0$, $r > 0$, and $x_0 \in \partial\{u > 0\}$. We say that $\partial\{u > 0\}$ is δ-rank-2 flat at level r at x_0 if $h(r, x_0) < \delta r$.

In view of Definitions 1.5 and 1.6, we can say that $\partial\{u > 0\}$ is (δ, R)-rank-2 flat at $x_0 \in \partial\{u > 0\}$ if and only if, for every $r \in (0, R]$, it is δ-rank-2 flat at level r at x_0.

We stress that the notion of "flatness" introduced in Definitions 1.5 and 1.6 does not refer to a geometric property of being "close to linear," but rather to a proximity to level sets of quadratic polynomials (that is, from the linguistic perspective, one should not separate the adjective "flat" from its own specification "rank-2"). Roughly speaking, our objective is to exploit quadratic objects to describe the minimizers, and our typical strategy would be to distinguish between points of the free boundary where the free boundary itself "looks like the level set of a quadratic polynomial" (i.e., it is in some sense rank-2 flat), and the "other points" of the free boundary, proving in the latter case that then it is the minimizer itself to possess some similarities, in terms of growth, with "quadratic objects". The reason for which we used the terminology of "flatness" to describe these "quadratic" (rather than "linear") scenarios is to maintain some jargon coming from the classical case in [Alt and Caffarelli 1981] and to interpret the notion of flatness as the one describing the "deviation" from a well-understood case (that is, the linear case in [Alt and Caffarelli 1981] and the quadratic case here).

Of course, making precise these results in our setting requires the development of a rather technical terminology, and detailed formulations of these ideas will be provided in Theorems 1.7, 1.8, 1.10, and 1.11.

In this framework, we now state the following result concerning the quadratic growth of u at δ-rank-2 nonflat points of the free boundary:

Theorem 1.7. *Let $n \geqslant 2$ and u be a minimizer of the functional J defined in (1-1). Let $D \subset\subset \Omega$, $\delta > 0$, and let $x_0 \in \partial\{u > 0\} \cap D$ such that $|\nabla u(x_0)| = 0$ and $\partial\{u > 0\}$ is not δ-rank-2 flat at x_0 at any level $r > 0$.*

Then, u has at most quadratic growth at x_0, bounded from above in dependence on δ.

1C.3. *Further results on the quadratic growth of the minimizers.* Now we turn our attention to the nondegeneracy properties of the minimizers. First of all, setting as usual $u^+(x) := \max\{u(x), 0\}$, we provide a weak form of nondegeneracy, investigating the validity of statements of this form:

$$\text{If } B \subset \{u > 0\} \text{ is a ball touching } \partial\{u > 0\},\ \text{ then } \sup_B u^+ \geqslant C[\operatorname{diam}(B)]^2 \tag{1-15}$$

for some $C > 0$ (possibly depending on dimension, on the domain, and on the datum u_0).

We consider this as a weak form of nondegeneracy as opposed to the one in which B is centered at free boundary points, which we call strong nondegeneracy.

We establish that (1-15) is satisfied, and, more generally, that the positive density of the positivity set is sufficient to ensure at least quadratic growth from the free boundary. The precise result we obtain is:

Theorem 1.8. *Let u be a minimizer of the functional J defined in (1-1). Then*:

1° *If $x_0 \in \partial\{u > 0\}$ and*

$$\liminf_{\rho \to 0} \frac{|B_\rho(x_0) \cap \{u > 0\}|}{|B_\rho|} \geqslant \theta_* \tag{1-16}$$

for some $\theta_* > 0$, then

$$\sup_{B_r(x_0)} |u| \geqslant \bar{c} r^2,$$

as long as $B_r(x_0) \subset\subset \Omega$, for some $\bar{c} > 0$ depending on θ_*, n, $\operatorname{dist}(B_r(x_0), \Omega)$, and $\|u_0\|_{W^{2,2}(\Omega)}$.

2° *If $x_0 \in \{u > 0\}$ and $r := \operatorname{dist}(x_0, \partial\{u > 0\})$, then*

$$\sup_{B_r(x_0)} u^+ \geqslant \bar{c} r^2,$$

as long as $B_r(x_0) \subset\subset \Omega$, for some $\bar{c} > 0$ depending on n, $\operatorname{dist}(B_r(x_0), \Omega)$, and $\|u_0\|_{W^{2,2}(\Omega)}$.

We observe that the claim in **2°** is exactly the statement in (1-15).

Sufficient conditions for the density estimate in (1-16) to hold will be discussed in Section 7B, where we also recall and compare the notions of the weak c-covering condition and Whitney's covering. In addition, in Section 7C we will relate the nondegeneracy properties with a fine analysis of the biharmonic measure, which in turn produces some regularity results on the free boundary.

It is also convenient to consider "vanishing" free boundary points, in the following sense:

Definition 1.9. Let u be a minimizer of the functional J defined in (1-1), and let $x_0 \in \partial\{u > 0\} \cap B_1$. We say that $\partial\{u > 0\}$ is vanishing rank-2 flat at x_0 if there exist sequences $\delta_k \to 0$ and $r_k \to 0$ such that

$$h(r_k, x_0) \leqslant \delta_k r_k, \tag{1-17}$$

where h is defined in (1-14).

Notice, in particular, that condition (1-17) is equivalent to $\lim_{k \to +\infty} \dfrac{h(r_k, x_0)}{r_k} = 0$, and this justifies the name of "vanishing" in Definition 1.9.

Theorem 1.10. *Let u be a minimizer of the functional J defined in (1-1). Then*:

1° *The set of vanishing rank-2 flat points of the free boundary has zero measure in Ω.*

2° *If $D \subset\subset \Omega$ and there exists $\bar{c} > 0$ such that*

$$\liminf_{r\to 0} \frac{\sup_{B_r(x)} |u|}{r^2} \geqslant \bar{c} \tag{1-18}$$

for every $x \in \partial\{u > 0\} \cap \overline{D}$, then $\partial\{u > 0\}$ has zero measure, and for any $\delta > 0$, the set of free boundary points that are not δ-rank-2 flat has finite $(n-2)$-dimensional Hausdorff measure.

In general, we can restate the previous results in a dichotomy form: roughly speaking, the free boundary in the vicinity of singular points is either "flat" with respect to the level sets of homogeneous polynomial of degree two, being "close" to the level sets of quadratic polynomials, or "nonflat" and in this case the growth from the free boundary is quadratic. To formalize these notions, we decompose the class P_2 introduced in (1-11) as

$$P_2 = \bigcup_{i=1}^{n} P_2^i,$$

where $P_2^i := \{p \in P_2 : \mathrm{Rank}(D^2 p) = i\}$. As we will see, in our setting, the above notion will play a useful role since if $x_0 \in \partial\{u > 0\}$, with $|\nabla u(x_0)| = 0$, and $\partial\{u > 0\}$ is rank-2 flat at x_0, then there exists $p \in P_2$ such that the blow-up of $\partial\{u > 0\}$ at x_0 is the zero set of p. We separate out some interesting cases:

- If $\mathrm{Rank}(D^2 p) = n$ and $D^2 p \geqslant 0$, then the free boundary is a singleton.
- If $\mathrm{Rank}(D^2 p) = 1$, then the free boundary is a hyperplane in $\mathbb{R}^n$, i.e., a codimension 1 plane in $\mathbb{R}^n$ and after some rotation of coordinates we can write $p(x) = \alpha(x_1^+)^2$, where $\alpha \in \mathbb{R}$ is a normalizing constant.
- If $\mathrm{Rank}(D^2 p) = n$ and $D^2 p$ has eigenvalues of opposite signs, then the free boundary has self intersection. For instance, if $n = 2$, then $p(x) = \alpha(x_1^2 - x_2^2)$, where $\alpha \in \mathbb{R}$ is a normalizing constant.

Roughly speaking, in this setting the classes P_2^i detect the approximate symmetries of the free boundary at small scales.

Now, let $\mathcal{F} \subseteq \partial_{\mathrm{sing}}\{u > 0\}$ be the set of singular free boundary points that are vanishing rank-2 flat and

$$\mathcal{N} := (\partial\{u > 0\} \setminus \mathcal{F}) \cap \Big\{|\nabla u| = 0\Big\} = \partial_{\mathrm{sing}}\{u > 0\} \setminus \mathcal{F}.$$

In this framework, the main result in the stratification setting reads as follows:

Theorem 1.11. *Let u be a minimizer of J. We have*:

- *For any $z \in \mathcal{F}$, there exist $r_k \to 0$ and $p \in P_2^i$, for some $i \in \{1, \dots, n\}$, such that*

$$\lim_{k\to+\infty} \mathrm{HD}\big((\partial E_k) \cap B_R, \{p = 0\} \cap B_R\big) = 0 \tag{1-19}$$

  for every fixed $R > 0$, where

$$E_k := \{x \in \mathbb{R}^n : z + r_k x \in \{u > 0\}\}.$$

Furthermore, u^+ is strongly nondegenerate at z, namely

$$\sup_{B_r(z)} u^+ \geqslant cr^2$$

for some $c > 0$, as long as $B_r(z) \subset\subset \Omega$, with c possibly depending on n, $\operatorname{dist}(z, \partial\Omega)$ *and u.*

- *For any $z \in \mathcal{N}$, there exists $C_z > 0$, possibly depending on n,* $\operatorname{dist}(z, \partial\Omega)$*, and $\|u\|_{W^{2,2}(\Omega)}$, such that*

$$|u(x)| \leqslant C_z |x - z|^2 \tag{1-20}$$

near z.

1C.4. *Monotonicity formula and classification of blow-up limits.* To analyze and classify the free boundary properties of the minimizers of J and their blow-up limits, it would be extremely desirable to have suitable monotonicity formulas. Different from the classical case, in our setting no general result of this type is known. To overcome this difficulty, we focus on the two-dimensional case, for which we prove:

Theorem 1.12. *Let $n = 2$ and $\tau > 0$ such that $B_\tau \subset\subset \Omega$. Let $u : \Omega \to \mathbb{R}$, with $0 \in \partial\{u > 0\}$ and $\nabla u(0) = 0$, be*

- *either: a minimizer of the functional J, with 0 not (δ, τ)-rank-2 flat in the sense of Definition 1.5,*
- *or: a one-phase minimizer of the functional J with $u \in C^{1,1}(\Omega)$, and such that $\partial\{u > 0\}$ has null Lebesgue measure.*

Then, there exists a function $E : (0, \tau) \to \mathbb{R}$, which is bounded, nondecreasing, and such that, for any $\tau_2 > \tau_1 > 0$,

$$E(\tau_2) - E(\tau_1) = \int_{\tau_1}^{\tau_2} \left\{ \frac{1}{r^2} \int_{\partial B_r} \left[\left(\frac{u_{\theta r}}{r} - \frac{2u_\theta}{r^2} \right)^2 + \left(u_{rr} - \frac{3u_r}{r} + \frac{4u}{r^2} \right)^2 \right] \right\} dr. \tag{1-21}$$

The explicit value of the function E is given by

$$E(r) = \int_{\partial B_r} \left(\frac{\Delta u\, u_r}{2r^2} - \frac{5u_r^2}{2r^3} - \frac{\Delta u u}{r^3} + \frac{6uu_r}{r^4} + \frac{u_\theta u_{\theta r}}{r^4} - \frac{4u^2}{r^5} - \frac{3u_\theta^2}{2r^5} \right) + \frac{1}{4r^2} \int_{B_r} \big(|\Delta u|^2 + \chi_{\{u>0\}} \big). \tag{1-22}$$

Furthermore, if E is constant in $(0, \tau)$, then u is a homogeneous function of degree two in B_τ.

We stress that the C^1 assumption on u in Theorem 1.12 is taken only in the case of one-phase minimizers, while for minimizers no additional regularity assumption is required in Theorem 1.12.

Given $x_0 \in \partial\{u > 0\}$, we consider the blow-up sequence of u at x_0, defined as

$$u_k(x) := \frac{u(x_0 + \rho_k x)}{\rho_k^2}, \tag{1-23}$$

where $\rho_k \to 0$ as $k \to +\infty$.

In this setting, we can classify blow-up limits of minimizers in the plane.

Theorem 1.13. *Let $n = 2$. Let $B_r \subset\subset \Omega$. Let $x_0 \in \Omega$ and $u : \Omega \to \mathbb{R}$, with $x_0 \in \partial_{\text{sing}}\{u > 0\}$. Assume that either u is a minimizer of the functional J, with*

$$\partial\{u > 0\} \textit{ not } \delta\textit{-rank-2 flat at } x_0 \textit{ at any level} \tag{1-24}$$

for some $\delta > 0$, *or that* u *is a one-phase minimizer of the functional* J *with* $u \in C^{1,1}(\Omega)$, *and such that* $\partial\{u > 0\}$ *has null Lebesgue measure. Then every blow-up limit of* u *at* x_0 *is either a homogeneous function of degree two*, *or it is identically zero.*

One of the main issues in the free boundary analysis is that, even in the one-phase problem, the topological and measure theoretic boundaries of $\{u > 0\}$ may not coincide. On the other hand, the following is a regularity result for the one-phase free boundary in the plane:

Theorem 1.14. *Let* $n = 2$. *Suppose that* $B_1 \subset\subset \Omega$. *Assume that* u *is a one-phase minimizer for* J, *that*

$$u \in C^{1,1}(B_1), \tag{1-25}$$

and that $\partial\{u > 0\}$ *has null Lebesgue measure. Suppose that* $0 \in \partial_{\text{sing}}\{u > 0\}$. *Assume also that*, *for every* $\bar{x} \in \partial\{u > 0\} \cap B_1$,

$$\liminf_{\rho \to 0^+} \frac{\sup_{B_\rho(\bar{x})} u}{\rho^2} \geqslant c \tag{1-26}$$

for some $c > 0$, *for all* $\rho \in (0, 1)$, *and that*

$$\limsup_{\rho \to 0} \frac{|B_\rho \cap \{u > 0\}|}{|B_\rho|} < 1. \tag{1-27}$$

Then there exists $r_0 > 0$ *such that at every point* $\bar{x}$ *of* $\partial\{u > 0\} \cap B_{r_0}$ *the free boundary possesses a unique approximate tangent line in measure theoretic sense*, *namely if* D *is the symmetric difference of the sets* $\{u > 0\}$ *and a suitable rotation of* $\{(x - \bar{x}) \cdot e_1 > 0\}$, *we have that*

$$\lim_{\rho \to 0^+} \frac{|B_\rho(\bar{x}) \cap D|}{|B_\rho(\bar{x})|} = 0.$$

We think that it is an interesting open problem to detect suitable conditions guaranteeing that the $C^{1,1}$-assumptions taken in Theorems 1.12, 1.13, and 1.14 are fulfilled.

Moreover, in our setting, Theorems 1.1, 1.7, 1.8, 1.10, and 1.11 are obtained specifically for the minimizers, and Theorem 1.14 specifically for the one -phase minimizers, while Theorems 1.3, 1.12, and 1.13 are valid for both minimizers and one-phase minimizers. Though the minimization setting is, in our case, structurally different from that of one-phase minimization due to the lack of Maximum Principle, we think that it is an interesting open problem to unify as much as possible the theory of minimizers with that of one-phase minimizers.

It is also an interesting problem to detect the optimal regularity of the solutions and their free boundaries.

1D. ***Organization of the paper.*** The rest of the paper is organized as follows: Section 2 contains the main existence result. In Section 3 we provide the proof of the local BMO estimate for the Laplacian of the minimizers, as given by Theorem 1.1. In Section 4 we present some structural properties of the minimizers which are based on the first variation of the functional J. As a consequence, we also obtain the free boundary condition, and we prove Theorem 1.3. In Section 5, we discuss some one-dimensional examples. Section 6 contains a dichotomy argument which leads to the proof of Theorem 1.7. Section 7 is devoted to nondegeneracy considerations and to the proof of Theorems 1.8 and 1.10. In Section 8 we consider the stratification of the free boundary, reformulating some results obtained in Section 6, and, in particular, we prove Theorem 1.11. Section 9 focuses on the monotonicity formula and contains the proof of Theorem 1.12. In Section 10 we present an application of such a monotonicity formula, proving the homogeneity of the blow-up limits, and establishing Theorem 1.13. Then, Section 11 focuses on explicit two-dimensional regularity and classification results and contains the proof of Theorem 1.14. The paper ends with two appendices which collect some ancillary observations.

2. Existence of minimizers

The following result exploits the direct method of the calculus of variations to obtain the existence of the minimizers for our problem. Due to the presence of several technical aspects in the proof, we provide the argument in full details.

Lemma 2.1. *The functional in (1-1) attains a minimum over $\mathcal{A}$.*

Proof. Let $u_k \in \mathcal{A}$ be a minimizing sequence, namely

$$\lim_{k\to+\infty} J[u_k] = \inf_{v\in\mathcal{A}} J[v]. \tag{2-1}$$

For large k, we can suppose that

$$J[u_k] \leqslant J[u_0]+1 \leqslant \int_\Omega \big(|\Delta u_0|^2+1\big) \leqslant C \tag{2-2}$$

for some $C>0$. Also, since $u_k\in\mathcal{A}$, we know from (1-2) that $u_k^* := u_k - u_0 \in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$. Let also $v_k^* := \Delta u_k^* \in L^2(\Omega)$. In this way, we have that

$$\begin{cases} \Delta u_k^* = v_k^* & \text{in } \Omega,\\ u_k^* = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, by elliptic regularity (see Theorem 4 on page 317 of [Evans 1998]), we know that

$$\|u_k^*\|_{W^{2,2}(\Omega)} \leqslant C'\big(\|v_k^*\|_{L^2(\Omega)} + \|u_k^*\|_{L^2(\Omega)}\big) \tag{2-3}$$

for some $C'>0$. Also (see Theorem 6 on page 306 of [Evans 1998]), one has that

$$\|u_k^*\|_{L^2(\Omega)} \leqslant C''\|v_k^*\|_{L^2(\Omega)} \tag{2-4}$$

for some $C''>0$. Therefore, in light of (2-3) and (2-4) we conclude

$$\|u_k^*\|_{W^{2,2}(\Omega)} \leqslant C'''\|v_k^*\|_{L^2(\Omega)} = C'''\|\Delta u_k^*\|_{L^2(\Omega)}$$

for some $C'''>0$. This and (2-2) imply

$$\|u_k^*\|_{W^{2,2}(\Omega)} \leqslant C''''$$

for some $C''''>0$. Therefore, we can suppose, up to a subsequence, that

$$u_k^* \text{ converges to some } u^* \text{ weakly in } W^{2,2}(\Omega), \tag{2-5}$$

and then, by compact embedding,

$$u_k^* \text{ converges strongly to } u^* \text{ in } W^{1,2}(\Omega). \tag{2-6}$$

Since $u_k^* \in W_0^{1,2}(\Omega)$, this implies that also $u^* \in W_0^{1,2}(\Omega)$. As a consequence, recalling (1-2), we know

$$u := u^* + u_0 \text{ belongs to } \mathcal{A}. \tag{2-7}$$

Furthermore, by (2-5), it holds that u_k converges to u weakly in $W^{2,2}(\Omega)$. In particular, u_k is bounded in $W^{2,2}(\Omega)$, and therefore, for any $i \in \{1, \dots, n\}$, it holds that $\partial_i^2 u_k$ is bounded in $L^2(\Omega)$. This yields that $\partial_i^2 u_k$ converges to some w_i weakly in $L^2(\Omega)$. This and

$$\text{the strong convergence of } u_k \text{ to } u \text{ in } W_0^{1,2}(\Omega) \subset L^2(\Omega) \tag{2-8}$$

(recall (2-6)) imply that, for any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega w_i\, \varphi = \lim_{k\to+\infty} \int_\Omega \partial_i^2 u_k\, \varphi = \lim_{k\to+\infty} \int_\Omega u_k\, \partial_i^2 \varphi = \int_\Omega u\, \partial_i^2 \varphi,$$

which shows that $w_i = \partial_i^2 u$.

Accordingly, we have that $\partial_i^2 u_k$ converges to $\partial_i^2 u$ weakly in $L^2(\Omega)$. Therefore, we have

$$\begin{aligned} 0 &\leqslant \lim_{k\to+\infty} \int_\Omega |\Delta(u_k - u)|^2 \\ &= \lim_{k\to+\infty} \int_\Omega |\Delta u_k|^2 + \int_\Omega |\Delta u|^2 - 2\int_\Omega \Delta u_k \Delta u = \lim_{k\to+\infty} \int_\Omega |\Delta u_k|^2 - \int_\Omega |\Delta u|^2. \end{aligned} \tag{2-9}$$

Now, up to a subsequence, recalling (2-8), we can suppose that u_k converges to u a.e. in Ω, and therefore, $\liminf_{k\to+\infty} \chi_{\{u_k>0\}} \geqslant \chi_{\{u>0\}}$ a.e. in Ω. Consequently, by Fatou's Lemma,

$$\liminf_{k\to+\infty} \int_\Omega \chi_{\{u_k>0\}} \geqslant \int_\Omega \chi_{\{u>0\}}.$$

Combining this with (2-9), we see that (2-1) provides

$$J[u] \leqslant \liminf_{k\to+\infty} J[u_k] = \inf_{v\in\mathcal{A}} J[v].$$

This and (2-7) imply that u is the desired minimizer. □

By taking into account a nonnegative constraint in the minimizing sequence in the proof of Lemma 2.1, one also obtains an existence result for the one-phase problem[3] with datum $u_0 \geqslant 0$.

[3]To prove the existence of one-phase minimizers, one can proceed as in the proof of Lemma 2.1, but considering in this case a minimizing sequence $u_k \in \mathcal{A}_+$: notice that $\mathcal{A}_+$ was introduced in Definition 1.2, and this space is nonempty due to the sign of u_0. Also, the sequence $u_k \in \mathcal{A}_+$ is not obtained by a minimizing sequence in $\mathcal{A}$ by taking its positive part, but simply by the usual procedure of minimizing the energy functional in the given domain $\mathcal{A}_+$. The argument given in the proof of Lemma 2.1 leads to a function $u \in \mathcal{A}$ such that $u_k \to u$ weakly in $W^{2,2}(\Omega)$, strongly to u^* in $W^{1,2}(\Omega)$, and a.e. in Ω, up to a subsequence, with $J[u] \leqslant \liminf_{k\to+\infty} J[u_k] = \inf_{v\in\mathcal{A}_+} J[v]$. Since $u(x) = \lim_{k\to+\infty} u_k(x) \geqslant 0$ for a.e. $x \in \Omega$, it follows that $u \in \mathcal{A}_+$, hence u is the desired one-phase minimizer.

3. BMO estimates and proof of Theorem 1.1

The goal of this section is to show that the minimizers of (1-1) have a Laplacian which is a function of locally bounded mean oscillation, and thus prove Theorem 1.1.

Proof of Theorem 1.1. We fix $R_0 > R > r > 0$ and $x_0 \in \Omega$ such that the ball $B_{2R_0}(x_0)$ is contained in Ω, and we consider the function h that solves

$$\begin{cases} \Delta^2 h = 0 & \text{in } B_{2R}(x_0), \\ h = u & \text{on } \partial B_{2R}(x_0), \\ \nabla h = \nabla u & \text{on } \partial B_{2R}(x_0). \end{cases}$$

The existence of h follows from Green's formula for biharmonic functions, see page 48 in [Gazzola et al. 2010], or by minimizing energy with

$$h - u \in W_0^{2,2}(B_{2R}(x_0)). \tag{3-1}$$

We also extend h outside $B_{2R}(x_0)$ to be equal to u in $\Omega \setminus B_{2R}(x_0)$. We observe that the function h is an admissible competitor for u, since

$$h \in W^{2,2}(\Omega). \tag{3-2}$$

Indeed, if $v := h - u$, we see from (3-1) and the extension results in classical Sobolev spaces (see, e.g., Proposition IX.18 in [Brezis 1983]) that $v \in W^{2,2}(\Omega)$. Since $u \in W^{2,2}(\Omega)$, the claim in (3-2) follows.

Then, by the minimality of u, we have that $J[u] \leqslant J[h]$; that is,

$$\int_{B_{2R}(x_0)} |\Delta u|^2 + \chi_{\{u>0\}} \leqslant \int_{B_{2R}(x_0)} |\Delta h|^2 + \chi_{\{h>0\}},$$

which in turn yields

$$\int_{B_{2R}(x_0)} |\Delta u|^2 - |\Delta h|^2 \leqslant C R^n \tag{3-3}$$

for some $C > 0$. Also, by (3-1), and since $\Delta^2 h = 0$ in $B_{2R}(x_0)$, we get

$$\begin{aligned} \int_{B_{2R}(x_0)} |\Delta u|^2 - |\Delta h|^2 &= \int_{B_{2R}(x_0)} (\Delta u - \Delta h)(\Delta u + \Delta h) \\ &= \int_{B_{2R}(x_0)} (\Delta u - \Delta h)\Delta u = \int_{B_{2R}(x_0)} |\Delta u - \Delta h|^2. \end{aligned}$$

From this and (3-3), we obtain

$$\int_{B_{2R}(x_0)} |\Delta u - \Delta h|^2 \leqslant C R^n. \tag{3-4}$$

Now we introduce the notation

$$(\Delta u)_{x_0,r} := \fint_{B_r(x_0)} \Delta u(x)\, dx,$$

and we observe that, by Hölder's inequality,

$$|(\Delta u)_{x_0,r} - (\Delta h)_{x_0,r}|^2 \leqslant \left(\fint_{B_r(x_0)} |\Delta u - \Delta h| \right)^2 \leqslant \fint_{B_r(x_0)} |\Delta u - \Delta h|^2,$$

which implies that

$$\int_{B_r(x_0)} |(\Delta h)_{x_0,r} - (\Delta u)_{x_0,r}|^2 \leqslant \int_{B_r(x_0)} |\Delta u - \Delta h|^2. \tag{3-5}$$

Moreover, since the function $H := \Delta h$ is harmonic in $B_{2R}(x_0)$, we have the following Campanato type estimate: there exists $\alpha > 0$ and a universal constant $C > 0$ such that

$$⨍_{B_r(x_0)} |\Delta h - (\Delta h)_{x_0,r}|^2 \leqslant C\Big(\frac{r}{R}\Big)^{\alpha} ⨍_{B_R} |\Delta h - (\Delta h)_{x_0,R}|^2,$$

see, e.g., formula (1.13) on page 96 in [Giaquinta 1983] (see also the notation on page 92 there).

Hence, using also the triangle inequality and recalling (3-4) and (3-5),

$$\begin{aligned}
&\int_{B_r(x_0)} |\Delta u - (\Delta u)_{x_0,r}|^2 \\
&= \int_{B_r(x_0)} |\Delta u - \Delta h + \Delta h - (\Delta h)_{x_0,r} + (\Delta h)_{x_0,r} - (\Delta u)_{x_0,r}|^2 \\
&\leqslant C\bigg(\int_{B_r(x_0)} |\Delta u - \Delta h|^2 + \int_{B_r(x_0)} |\Delta h - (\Delta h)_{x_0,r}|^2 + \int_{B_r(x_0)} |(\Delta h)_{x_0,r} - (\Delta u)_{x_0,r}|^2\bigg) \\
&\leqslant C\bigg(R^n + \Big(\frac{r}{R}\Big)^{\alpha+n} \int_{B_R(x_0)} |\Delta h - (\Delta h)_{x_0,R}|^2\bigg) \\
&= C\bigg(R^n + \Big(\frac{r}{R}\Big)^{\alpha+n} \int_{B_R(x_0)} |\Delta h - \Delta u + \Delta u - (\Delta u)_{x_0,R} + (\Delta u)_{x_0,R} - (\Delta h)_{x_0,R}|^2\bigg) \\
&\leqslant C\bigg[R^n + \Big(\frac{r}{R}\Big)^{\alpha+n}\bigg(\int_{B_R(x_0)} |\Delta h - \Delta u|^2 + \int_{B_R(x_0)} |\Delta u - (\Delta u)_{x_0,R}|^2 + \int_{B_R(x_0)} |(\Delta u)_{x_0,R} - (\Delta h)_{x_0,R}|^2\bigg)\bigg] \\
&\leqslant C\bigg[R^n + \Big(\frac{r}{R}\Big)^{\alpha+n}\bigg(\int_{B_R(x_0)} |\Delta h - \Delta u|^2 + \int_{B_R(x_0)} |\Delta u - (\Delta u)_{x_0,R}|^2\bigg)\bigg] \\
&\leqslant C\bigg[R^n + \Big(\frac{r}{R}\Big)^{\alpha+n}\bigg(R^n + \int_{B_R(x_0)} |\Delta u - (\Delta u)_{x_0,R}|^2\bigg)\bigg] \leqslant C\bigg[R^n + \Big(\frac{r}{R}\Big)^{\alpha+n} \int_{B_R(x_0)} |\Delta u - (\Delta u)_{x_0,R}|^2\bigg].
\end{aligned} \tag{3-6}$$

We can therefore exploit Lemma 2.1 in Chapter 3 on page 86 of [Giaquinta 1983] (see also Lemma 3.1 in [Dipierro and Karakhanyan 2018] and Theorem 1.1 in [Dipierro et al. 2017]), used here with

$$\phi(\rho) := \int_{B_\rho(x_0)} |\Delta u - (\Delta u)_{x_0,\rho}|^2, \quad \beta := n, \quad a := \alpha + n, \quad \beta := n, \quad A := C, \quad \text{and} \quad B := C.$$

Thus writing (3-6) in the form

$$\phi(r) \leqslant C\Big[R^\beta + \Big(\frac{r}{R}\Big)^a \phi(R)\Big] = A\Big[\Big(\frac{r}{R}\Big)^a + \varepsilon\Big]\phi(R) + BR^\beta,$$

and hence deducing that $\phi(r) \leqslant C\big[\big(\frac{r}{R}\big)^\beta \phi(R) + r^\beta\big]$, up to renaming constants, that provides

$$\int_{B_r(x_0)} |\Delta u - (\Delta u)_{x_0,r}|^2 \leqslant Cr^n, \tag{3-7}$$

for a suitable $C > 0$, possibly depending on u, x_0, R_0, which gives the desired result. ∎

4. First variation of J, free boundary condition, and proof of Theorem 1.3

We consider the first variation of the functional in (1-1). Of course, the main problem is to take into account variations performed by a test function whose support intersects the free boundary of u, since in this case the lack of regularity of the characteristic function plays an important role. Therefore, it is useful to know that the set $\{u > 0\}$ is an open subset of Ω, which, in the case of minimizers, follows from

$$u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega) \text{ for any } \alpha \in (0, 1), \tag{4-1}$$

which, in turn, follows from the fact that

$$u \in W^{2,p}_{\mathrm{loc}}(\Omega) \text{ for any } p \in (1, +\infty), \tag{4-2}$$

in virtue of Theorem 1.1 and the Calderón–Zygmund regularity theory (we think that it is an interesting open problem to establish whether (4-1) and (4-2) are also fulfilled by one-phase minimizers).

The main structural properties of the minimizers which are based on the first variation of the functional are given by the following result:

Lemma 4.1. *Let u be a minimizer of J. Then u is weakly super-biharmonic in Ω (i.e., $\Delta^2 u \leqslant 0$ in the sense of distributions) and biharmonic in $\{u > 0\} \cup \{u < 0\}^\circ$, where E° denotes the interior of E.*

Similarly, if u is a one-phase minimizer of J and B is an open ball contained in $\{u \geqslant a\}$, with $a > 0$, then u is biharmonic in B.

Proof. We prove the claims assuming that u is a minimizer (the one-phase problem can be treated similarly). Define $u_\varepsilon := u - \varepsilon\phi$, where $0 \leqslant \phi \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ and ε is a small parameter to be fixed below. Using the comparison of the energies of u and u_ε, and recalling (1-1), we get

$$\int_\Omega \big(|\Delta u|^2 - |\Delta u - \varepsilon\Delta\phi|^2\big) \leqslant \int_\Omega (\chi_{\{u-\varepsilon\phi>0\}} - \chi_{\{u>0\}}).$$

Note that $\{u - \varepsilon\phi > 0\} \subset \{u > 0\}$, provided that $\varepsilon > 0$. Consequently, we have

$$0 \geqslant \int_\Omega \big(|\Delta u|^2 - |\Delta u - \varepsilon\phi|^2\big) = 2\varepsilon \int_\Omega \Delta u\, \Delta\phi - \varepsilon^2 \int_\Omega (\Delta u)^2. \tag{4-3}$$

Dividing both sides of the last inequality by $\varepsilon > 0$ and then letting $\varepsilon \to 0$, we get that $\int_\Omega \Delta u\, \Delta\phi \leqslant 0$. If we take $\phi \in C^\infty_0(\Omega)$, this gives that u is super-biharmonic. In addition, if we suppose that $\operatorname{supp}\phi \subset \{u > 0\}$, then from (4-3) we deduce, without any sign assumption on ε, that $\int_\Omega \Delta u\, \Delta\phi = 0$. ☐

Concerning the statement of Lemma 4.1, it is interesting to remark that one-phase minimizers are not necessarily super-biharmonic (an explicit counterexample to this fact is discussed on page 898).

The basic analytic structure of the minimizers is then completed by the following result:

Corollary 4.2. *Let u be a minimizer of J. For every bounded subdomain $\Omega' \subset\subset \Omega$, there exists $C > 0$, depending only on n, such that*

$$\Delta u \geqslant -\frac{C\|\Delta u\|_{L^1(\Omega)}}{(\operatorname{dist}(\Omega', \partial\Omega))^n} \quad \text{in } \Omega'.$$

Proof. Let $r := \frac{1}{2}\operatorname{dist}(\Omega', \partial\Omega)$ and, for all $y \in \Omega'$, define the function

$$\phi(y) := \fint_{B_r(y)} \Delta u(x)\, dx.$$

Thanks to (4-2), we see that ϕ is continuous on the compact set $\overline{\Omega}'$. Therefore, there exists $y_0 \in \overline{\Omega}'$ such that $\min_{\overline{\Omega}'} \phi(y) = \phi(y_0)$. Then, for any $y \in \Omega'$,

$$\phi(y) \geqslant \phi(y_0) \geqslant -\fint_{B_r(y_0)} |\Delta u(x)|\, dx \geqslant -\frac{\|\Delta u\|_{L^1(\Omega)}}{|B_r|}. \tag{4-4}$$

As a consequence, since u is super-biharmonic, thanks to Lemma 4.1, we obtain the desired estimate by the mean value inequality for weak subsolutions of the Laplace equation (see, e.g., [Serrin 2011] and [Littman 1963]). More precisely, if v is weakly super-harmonic in Ω, we know from Theorem A in [Littman 1963] that there exists a sequence of smooth super-harmonic functions v_h in Ω' that converge to v a.e. in Ω' and in $L^1(\Omega')$. Consequently, a.e. $y \in \Omega'$,

$$v(y) = \lim_{h\to 0} v_h(y) \geqslant \lim_{h\to 0} \fint_{B_r(y_0)} v_h(x)\, dx = \fint_{B_r(y_0)} v(x)\, dx. \tag{4-5}$$

Then, choosing $v := \Delta u$ and applying (4-4), we find that

$$\Delta u(y) \geqslant \fint_{B_r(y)} \Delta u(x)\, dx = \phi(y) \geqslant -\frac{\|\Delta u\|_{L^1(\Omega)}}{|B_r|}. \qquad \square$$

For the sake of completeness, we observe that the statement of Corollary 4.2 can be strengthen by showing, under additional regularity assumptions, that minimizers are super-harmonic, according to:

Proposition 4.3. *Let u be a minimizer of J. Assume that*

$$u \in C(\overline{\Omega}). \tag{4-6}$$

Assume also that

$$\Delta u \text{ is } C^1 \text{ in a neighborhood of } \partial\Omega, \tag{4-7}$$

and that

$$\partial\Omega \cap \{|u| > 0\} \text{ is dense in } \partial\Omega. \tag{4-8}$$

Then,

$$\Delta u \geqslant 0 \quad \text{a.e. in } \Omega. \tag{4-9}$$

We think that the result of Proposition 4.3 is helpful to understand the geometric structure of the minimizers: nevertheless, since it is not used in the rest of this paper, we deferred its proof to Appendix C.

In Example 4 of Section 5 (see page 903), we will further discuss the result of Proposition 4.3, also in view of the free boundary conditions provided by Theorem 1.3 and of the bi-harmonicity properties outside the free boundary discussed in Lemma 4.1.

Next we compute the first domain variation (for this, we use the notation in which subscripts denote differentiation and superscripts denote coordinates).

Lemma 4.4. *Let u be a minimizer or a one-phase minimizer of J. For any $\phi = (\phi^1, \dots, \phi^n) \in C_0^\infty(\Omega)$,*

$$2\int_\Omega \Delta u(x) \sum_{m=1}^n \big(2\nabla u_m(x)\cdot\nabla\phi^m(x) + u_m(x)\Delta\phi^m(x)\big)\,dx = \int_\Omega \big(|\Delta u(x)|^2 + \chi_{\{u>0\}}(x)\big)\operatorname{div}\phi(x)\,dx. \tag{4-10}$$

Proof. Fix $\varepsilon \in \mathbb{R}$ (to be taken with $|\varepsilon|$ small in the sequel). Let

$$u_\varepsilon(x) := u(x + \varepsilon\phi(x)). \tag{4-11}$$

Notice that u_ε is an admissible competitor for u (in case we are dealing with the one-phase problem, observe that $u_\varepsilon \geqslant 0$ if $u \geqslant 0$).

For any $i \in \{1, \dots, n\}$, we have

$$\begin{aligned}
\partial_i u_\varepsilon &= \sum_{m=1}^n u_m(\delta_{mi} + \varepsilon\phi_i^m) \\
\partial_{ii} u_\varepsilon &= \sum_{m,l=1}^n u_{ml}(\delta_{li} + \varepsilon\phi_i^l)(\delta_{mi} + \varepsilon\phi_i^m) + \sum_{m=1}^n u_m \varepsilon \phi_{ii}^m \\
&= u_{ii} + \varepsilon\Big[\sum_{m,l=1}^n (u_{ml}\phi_i^l\delta_{mi} + u_{ml}\phi_i^m\delta_{li}) + \sum_{m=1}^n u_m\phi_{ii}^m\Big] + \varepsilon^2 \sum_{m,l=1}^n u_{ml}\phi_i^l\phi_i^m \\
&= u_{ii} + \varepsilon\sum_{m=1}^n (2u_{mi}\phi_i^m + u_m\phi_{ii}^m) + \varepsilon^2 \sum_{m,l=1}^n u_{ml}\phi_i^l\phi_i^m.
\end{aligned}$$

We use the change of variable $y := x + \varepsilon\phi(x)$. Noticing $\phi(x) = \phi(y - \varepsilon\phi(x)) = \phi(y) + O(\varepsilon)$, we get

$$\begin{aligned}
J[u_\varepsilon] &= \int_\Omega \bigg\{ \bigg| \sum_{i=1}^n \Big[u_{ii}(x+\varepsilon\phi(x)) + \varepsilon \sum_{m=1}^n \big(2u_{mi}(x+\varepsilon\phi(x))\phi_i^m(x) + u_m(x+\varepsilon\phi(x))\phi_{ii}^m(x)\big)\Big] \\
&\qquad\qquad + o(\varepsilon)\bigg|^2 + \chi_{\{u>0\}}(x+\varepsilon\phi(x))\bigg\}\,dx \\
&= \int_\Omega \bigg\{ \bigg| \sum_{i=1}^n \Big[u_{ii}(y) + \varepsilon\sum_{m=1}^n \big(2u_{mi}(y)\phi_i^m(y) + u_m(y)\phi_{ii}^m(y)\big)\Big] + o(\varepsilon)\bigg|^2 + \chi_{\{u>0\}}(y)\bigg\} \\
&\qquad\qquad \times (1 - \varepsilon\operatorname{div}\phi(y) + o(\varepsilon))\,dy \\
&= \int_\Omega \bigg\{ \bigg| \sum_{i=1}^n u_{ii}(y) + \varepsilon \sum_{i,m=1}^n \big(2u_{mi}(y)\phi_i^m(y) + u_m(y)\phi_{ii}^m(y)\big)\bigg|^2 + \chi_{\{u>0\}}(y)\bigg\}(1 - \varepsilon\operatorname{div}\phi(y))\,dy + o(\varepsilon) \\
&= \int_\Omega \bigg\{ \sum_{i,j=1}^n u_{ii}(y)u_{jj}(y) + 2\varepsilon\sum_{i,j,m=1}^n \big(2u_{jj}(y)u_{mi}(y)\phi_i^m(y) + u_{jj}(y)u_m(y)\phi_{ii}^m(y)\big) + \chi_{\{u>0\}}(y)\bigg\} \\
&\qquad\qquad \times (1 - \varepsilon\operatorname{div}\phi(y))\,dy + o(\varepsilon) \\
&= J[u] - \varepsilon\int_\Omega \bigg\{ \big(|\Delta u(y)|^2 + \chi_{\{u>0\}}(y)\big)\operatorname{div}\phi(y) \\
&\qquad\qquad - 2\Delta u(y)\sum_{m=1}^n \big(2\nabla u_m(y)\cdot\nabla\phi^m(y) + u_m(y)\Delta\phi^m(y)\big)\bigg\}\,dy + o(\varepsilon).
\end{aligned}$$

Thus, taking the derivative in ε and evaluating it at $\varepsilon = 0$, we obtain (4-10), as desired. □

As a consequence of Lemma 4.4, we obtain the free boundary condition of Theorem 1.3:

Proof of Theorem 1.3. We use the notation

$$g(x) := |\Delta u(x)|^2 + \chi_{\{u>0\}}(x), \quad G^m(x) := \Delta u(x)\nabla u_m(x), \quad \text{and} \quad H^m(x) := \Delta u(x) u_m(x)$$

for each $m \in \{1, \dots, n\}$.

We let $\phi \in C_0^\infty(\Omega)$, and we claim that

$$g \operatorname{div} \phi - 4\sum_{m=1}^{n} G^m \cdot \nabla\phi^m - 2\sum_{m=1}^{n} H^m \Delta\phi^m = 0 \quad \text{a.e. in } \Omega. \tag{4-12}$$

To check this, we recall:

$$\text{If } f \in W^{1,1}_{\text{loc}}(\Omega), \text{ then } \nabla f = 0 \text{ a.e. in } \{x \in \Omega : f = 0\}, \tag{4-13}$$

see, e.g., Theorem 6.19 in [Lieb and Loss 2001] (used here with $A := \{0\}$). Then, first of all, since $u \in W^{2,2}(\Omega)$, we deduce from (4-13) that

$$\nabla u(x) = 0 \quad \text{for all } x \in \{u = 0\} \setminus Z, \tag{4-14}$$

for a suitable Z of null measure. Furthermore, for every $j \in \{1, \dots, n\}$, we have that $\partial_j u \in W^{1,2}(\Omega)$. Accordingly, using (4-13) once again, we find

$$\nabla \partial_j u(x) = 0 \quad \text{for all } x \in \{\partial_j u = 0\} \setminus Z_j, \tag{4-15}$$

with Z_j of null measure.

We also remark that

$$\{\partial_j u = 0\} \supseteq \{u = 0\} \setminus Z,$$

thanks to (4-14), and therefore (4-15) yields

$$\nabla \partial_j u(x) = 0 \quad \text{for all } x \in \{u = 0\} \setminus (Z \cup Z_j). \tag{4-16}$$

Hence, defining $Z^\star := Z \cup Z_1 \cup \dots Z_n$, we have that $Z^\star$ has null measure and, by (4-14) and (4-16),

$$D^2 u(x) = 0 \quad \text{for every } x \in \{u = 0\} \setminus Z^\star. \tag{4-17}$$

Moreover, if $x \in \{u = 0\}$, then $\chi_{\{u>0\}}(x) = 0$. This and (4-17) give that $g = G^m = H^m = 0$ in $\{u = 0\} \setminus Z^\star$, which in turn yields (4-12), as desired.

As a consequence of (4-12) and of the Monotone Convergence Theorem, we deduce that

$$\int_\Omega \left(g \operatorname{div} \phi - 4\sum_{m=1}^{n} G^m \cdot \nabla\phi^m - 2\sum_{m=1}^{n} H^m \Delta\phi^m \right) = \lim_{\varepsilon \to 0} \int_{\Omega \cap \{|u|>\varepsilon\}} \left(g \operatorname{div} \phi - 4\sum_{m=1}^{n} G^m \cdot \nabla\phi^m - 2\sum_{m=1}^{n} H^m \Delta\phi^m \right).$$

Therefore, recalling (4-10) and (1-7), we find that

$$\begin{aligned}0 &= \int_\Omega \Big(g\ \operatorname{div}\phi - 4\sum_{m=1}^n G^m \cdot \nabla\phi^m - 2\sum_{m=1}^n H^m \Delta\phi^m \Big)\\ &= \lim_{\varepsilon\to 0}\int_{\Omega\cap\{|u|>\varepsilon\}} \Big(g\ \operatorname{div}\phi - 4\sum_{m=1}^n G^m \cdot \nabla\phi^m - 2\sum_{m=1}^n H^m \Delta\phi^m \Big)\\ &= \lim_{\varepsilon\to 0}\int_{\Omega\cap\{|u|>\varepsilon\}} \Big(\operatorname{div}(g\,\phi) - 4\sum_{m=1}^n \operatorname{div}(\phi^m G^m) - 2\sum_{m=1}^n \operatorname{div}(H^m\nabla\phi^m)\\ &\qquad\qquad + 4\sum_{m=1}^n \phi^m \operatorname{div} G^m + 2\sum_{m=1}^n \nabla H^m\cdot\nabla\phi^m - \nabla g\cdot\phi\Big).\end{aligned} \tag{4-18}$$

We remark that, in $\{|u|>\varepsilon\}$,

$$\begin{aligned}&4\sum_{m=1}^n \phi^m \operatorname{div} G^m + 2\sum_{m=1}^n \nabla H^m\cdot\nabla\phi^m - \nabla g\cdot\phi\\ &= \sum_{m=1}^n \big(4\phi^m(\nabla\Delta u\cdot\nabla u_m + \Delta u\Delta u_m) + 2(u_m\nabla\Delta u + \Delta u\nabla u_m)\cdot\nabla\phi^m - 2\Delta u\Delta u_m\phi^m\big)\\ &= \sum_{m=1}^n \big(4\nabla\Delta u\cdot\nabla u_m\phi^m + 2\Delta u\Delta u_m\phi^m + 2(u_m\nabla\Delta u + \Delta u\nabla u_m)\cdot\nabla\phi^m\big)\\ &= \sum_{m=1}^n \big(4\nabla\Delta u\cdot\nabla u_m\phi^m + 2\Delta u\Delta u_m\phi^m + 2\operatorname{div}(\phi^m(u_m\nabla\Delta u + \Delta u\nabla u_m)) - 2\operatorname{div}(u_m\nabla\Delta u + \Delta u\nabla u_m)\phi^m\big)\\ &= 2\sum_{m=1}^n \big(\operatorname{div}(\phi^m(u_m\nabla\Delta u + \Delta u\nabla u_m)) - u_m\Delta^2 u\phi^m\big) = 2\sum_{m=1}^n \operatorname{div}(\phi^m(u_m\nabla\Delta u + \Delta u\nabla u_m)),\end{aligned}$$

by virtue of Lemma 4.1. As a consequence, we see that

$$\begin{aligned}\int_{\Omega\cap\{|u|>\varepsilon\}}\Big(4\sum_{m=1}^n \phi^m \operatorname{div} G^m + 2\sum_{m=1}^n \nabla H^m\cdot\nabla\phi^m - \nabla g\cdot\phi\Big) &= 2\sum_{m=1}^n \int_{\Omega\cap\{|u|>\varepsilon\}} \operatorname{div}(\phi^m(u_m\nabla\Delta u + \Delta u\nabla u_m))\\ &= 2\sum_{m=1}^n \int_{\partial(\Omega\cap\{|u|>\varepsilon\})} \phi^m(u_m\nabla\Delta u + \Delta u\nabla u_m)\cdot\nu,\end{aligned}$$

where ν is the exterior normal to $\Omega\cap\{|u|>\varepsilon\}$. Hence, using this information in (4-18), we obtain

$$\begin{aligned}0 &= \lim_{\varepsilon\to 0}\int_{\partial(\Omega\cap\{|u|>\varepsilon\})}\Big(g\,\phi\cdot\nu - \sum_{m=1}^n\big(4\phi^m G^m\cdot\nu + 2H^m\nabla\phi^m\cdot\nu - 2\phi^m(u_m\nabla\Delta u + \Delta u\nabla u_m)\cdot\nu\big)\Big)\\ &= \lim_{\varepsilon\to 0}\int_{\partial(\Omega\cap\{|u|>\varepsilon\})}\Big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)\phi\cdot\nu - 2\sum_{m=1}^n\big(\phi^m(\Delta u\nabla u_m - u_m\nabla\Delta u)\cdot\nu + \Delta u u_m\nabla\phi^m\cdot\nu\big)\Big).\end{aligned}$$

This gives (1-8). To obtain (1-9), use the two different scales of the test function ϕ^m and its derivative. □

Remark 4.5. We point out if $n=1$, when the free boundary divides regions of positivity and nonpositivity of u (say, $u^{(1)}>0$ and the interior of $u^{(2)}\leqslant 0$), formula (1-9) gives the free boundary conditions

$$\ddot{u}^{(1)}\dot{u}^{(1)}=\ddot{u}^{(2)}\dot{u}^{(2)} \tag{4-19}$$

$$2\dot{u}^{(1)}\dddot{u}^{(1)}-|\ddot{u}^{(1)}|^2+1=2\dot{u}^{(2)}\dddot{u}^{(2)}-|\ddot{u}^{(2)}|^2. \tag{4-20}$$

Also, since $u\in W^{2,2}(\Omega)$ and $n=1$, by standard embedding results we already know that $u\in C^1(\Omega)$. This, in view of (4-19), implies that either $\dot{u}=0$ at a free boundary point, or $\ddot{u}^{(1)}=\ddot{u}^{(2)}$. That is, either u has horizontal tangent at a free boundary point, or it is C^2 across the free boundary point. Hence, from (4-20), we have the following one-dimensional dichotomy for the free boundary points:

$$\text{either: } \dot{u}=0 \text{ and } |\ddot{u}^{(1)}|^2-|\ddot{u}^{(2)}|^2=1, \tag{4-21}$$

$$\text{or: } \dot{u}\neq 0,\, u \text{ is } C^2 \text{ across and } \dddot{u}^{(1)}=\dddot{u}^{(2)}-\frac{1}{2\dot{u}}\,. \tag{4-22}$$

5. Some examples in dimension 1

Example 5.1. To better understand Remark 4.5, we can sketch some one-dimensional computations. Namely, we let $n=1$, consider an interval $\Omega:=(0,A)$, with $A>0$, and prescribe the Navier conditions $u(0)=\ddot{u}(0)=0$, $u(A)=1$ and $\ddot{u}(A)=0$. We look for one-phase minimizers of J with such boundary conditions.

In this case, by the finiteness of the energy and Sobolev embedding, we know that the one-phase minimizer is $C^1(0,A)$; also the free boundary points are minimal point for u, and therefore

$$\dot{u}=0 \text{ at any free boundary point.} \tag{5-1}$$

Accordingly, condition (4-21) prescribes that

$$\ddot{u}^+=1. \tag{5-2}$$

Let us see how such condition emerges from energy considerations. We suppose that the problem develops a free boundary and we denote by $a\in(0,A)$ the largest free boundary point, i.e., $u(a)=0$ and $u>0$ in (a,A). From Lemma 4.1, we know that $\ddddot{u}=0$ in (a,A), and so u is a polynomial of degree 3 in (a,A). Consequently, we can write, for any $x\in(a,A)$,

$$u(x)=\alpha(x-a)+\beta(x-a)^2+\gamma(x-a)^3.$$

Recalling (5-1), we conclude that $\alpha=0$. Imposing the boundary conditions at the point $x=A$, we find

$$\beta=\frac{3}{2(A-a)^2} \quad\text{and}\quad \gamma=-\frac{1}{2(A-a)^3},$$

and therefore,

$$u(x)=\frac{3(x-a)^2}{2(A-a)^2}-\frac{(x-a)^3}{2(A-a)^3}. \tag{5-3}$$

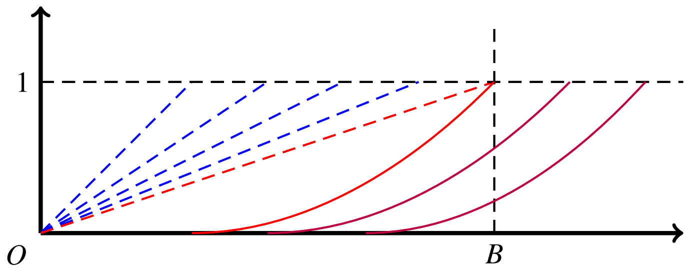

Figure 2. The one-phase minimizers of a one-dimensional problem, in dependence of the right endpoint. We stress that the ones with a nontrivial zero level sets are one-phase minimizers, but not minimizers (see Proposition B.1).

The goal is then to choose $a \in (0, A)$ in order to minimize the energy contribution of u in (a, A), namely we want to minimize the function

$$\begin{aligned}\Phi(a) &:= \int_a^A |\ddot{u}(x)|^2\,dx + (A-a)\\ &= \int_a^A \left|\frac{3}{(A-a)^2} - \frac{3(x-a)}{(A-a)^3}\right|^2 dx + (A-a)\\ &= 9\int_a^A \left|\frac{(A-a)-(x-a)}{(A-a)^3}\right|^2 dx + (A-a)\\ &= \frac{9}{(A-a)^6}\int_a^A |A-x|^2\,dx + (A-a) = \frac{3}{(A-a)^3} + (A-a),\end{aligned}$$

which attains its minimum for

$$a = A - \sqrt{3}. \tag{5-4}$$

That is, comparing with the linear function $\ell(x) := \dfrac{x}{A}$, we have that

$$A = J[\ell] \geqslant J[u] \geqslant \Phi(a) \geqslant \Phi(A-\sqrt{3}) = \frac{1}{\sqrt{3}} + \sqrt{3}.$$

This means that when $A < \frac{1}{\sqrt{3}} + \sqrt{3} =: B$, the problem does not develop any free boundary; when $A = B$ the problem has two minimizers, and when $A > B$ the minimizer in (5-3) becomes

$$u(x) = \frac{(x-a)^2}{2} - \frac{(x-a)^3}{2\cdot 3^{3/2}}, \tag{5-5}$$

for which $\ddot{u}(a^+) = 1$. This checks (5-2) in this case.

The description of the different one-phase minimizers in dependence of the endpoint A is sketched in Figure 2. It is also worth pointing out:

$$\text{The one-phase minimizers described here are } \textit{not} \text{ super-biharmonic,} \tag{5-6}$$

and this creates a major difference with respect to the case of minimizers, compare with Lemma 4.1: indeed, if $\varphi \in C_0^\infty((0, A), [0, +\infty))$ and $A > \frac{1}{\sqrt{3}} + \sqrt{3}$, from (5-4) and (5-5) we see that

$$\int_0^A \ddot{u}\ddot{\varphi} = \int_a^A \Big(1 - \frac{x-a}{\sqrt{3}}\Big)\ddot{\varphi} = \Big(1 - \frac{A-a}{\sqrt{3}}\Big)\dot{\varphi}(A) - \dot{\varphi}(a) - \int_a^A \frac{d}{dx}\Big(1 - \frac{x-a}{\sqrt{3}}\Big)\dot{\varphi}$$
$$= 0 - \dot{\varphi}(a) + \frac{1}{\sqrt{3}}\int_a^A \dot{\varphi} = -\dot{\varphi}(a) - \frac{\varphi(a)}{\sqrt{3}},$$

which has no sign, thus proving (5-6).

Example 5.2. Having clarified condition (4-21) in a concrete example, we aim now at clarifying the role of condition (4-22). Such condition is, in a sense, more unusual, since it prescribes the matching of the second derivatives at the free boundary points with nontrivial slopes, with the bulk term of the energy producing a discontinuity on the third derivatives.

To understand this phenomenon in a concrete example, we fix a small parameter $\varepsilon > 0$ and minimize the energy functional

$$J[u] = \int_{-1}^{1} \big(|\ddot{u}(x)|^2 + \varepsilon\chi_{\{u>0\}}(x)\big)\,dx,$$

subject to the Navier conditions

$$u(-1) = -1, \quad \ddot{u}(-1) = 0, \quad u(1) = 1, \quad \ddot{u}(1) = 0. \tag{5-7}$$

If we call u_ε such minimizer, we can bound the energy of u_ε with that of the identity function. This produces a uniform bound for u_ε in $W^{2,2}((-1, 1))$, which implies that u_ε converges in $C^1((-1, 1))$ to the identity function as $\varepsilon \to 0$. Consequently, for a fixed and small $\varepsilon > 0$, we can find some $a \in (-1, 1)$, which depends on ε, such that

$$u_\varepsilon(x) = \begin{cases} \underline{\alpha}(a-x) + \underline{\beta}(a-x)^2 + \underline{\gamma}(a-x)^3 & \text{if } x \in (-1, a), \\ \bar{\alpha}(x-a) + \bar{\beta}(x-a)^2 + \bar{\gamma}(x-a)^3 & \text{if } x \in [a, 1). \end{cases}$$

The condition that $u_\varepsilon \in C^1((-1, 1))$ (with derivative close to 1 when ε is small) implies that $-\underline{\alpha} = \bar{\alpha} = \alpha$, for some $\alpha > 0$ (which depends on ε and it is close to 1 when ε is small). Imposing the boundary conditions in (5-7), we find

$$\underline{\beta} = -\frac{3(1-\alpha(1+a))}{2(1+a)^2}, \quad \underline{\gamma} = \frac{1-\alpha(1+a)}{2(1+a)^3}, \quad \bar{\beta} = \frac{3(1-\alpha(1-a))}{2(1-a)^2}, \quad \bar{\gamma} = \frac{\alpha(1-a)-1}{2(1-a)^3}. \tag{5-8}$$

Therefore, the energy of u_ε corresponds to the function

$$\Psi(a,\alpha) := J[u_\varepsilon] = \int_{-1}^{a} |2\underline{\beta} + 6\underline{\gamma}(a-x)|^2\,dx + \int_a^1 |2\bar{\beta} + 6\bar{\gamma}(x-a)|^2\,dx + \varepsilon(1-a)$$
$$= \Big(\frac{3(1-\alpha(1+a))}{(1+a)^3}\Big)^2 \int_{-1}^{a} |1+x|^2\,dx + \Big(\frac{3(1-\alpha(1-a))}{(1-a)^3}\Big)^2 \int_a^1 |1-x|^2\,dx + \varepsilon(1-a)$$
$$= \frac{3(1-\alpha(1+a))^2}{(1+a)^3} + \frac{3(1-\alpha(1-a))^2}{(1-a)^3} + \varepsilon(1-a).$$

Thus, we have to minimize such function for $(a,\alpha)\in(-1,1)\times(0,+\infty)$, and in fact we know that such minimum is localized at $(0,1)$ when $\varepsilon=0$. Therefore, to find the minima of Ψ, we solve the system

$$\begin{cases} 0=\partial_a\Psi=\dfrac{12a(\alpha a^4(\alpha+2)+2a^2(2\alpha-\alpha^2+3)+\alpha^2-6\alpha+6)}{(1-a^2)^4}-\varepsilon, \\ 0=\partial_\alpha\Psi=12\,\dfrac{\alpha-1-a^2(1+\alpha)}{(1-a^2)^2}. \end{cases} \tag{5-9}$$

The latter equation produces

$$a^2=\frac{\alpha-1}{1+\alpha}. \tag{5-10}$$

We notice that, by (5-8),

$$\frac{2}{3}(\bar\beta-\underline{\beta})=\frac{1-\alpha(1-a)}{(1-a)^2}+\frac{1-\alpha(1+a)}{(1+a)^2}=\frac{2((\alpha+1)a^2-\alpha+1)}{(1-a^2)^2}.$$

Hence, in view of (5-10),

$$\frac{2}{3}(\bar\beta-\underline{\beta})=\frac{2((\alpha+1)\frac{\alpha-1}{1+\alpha}-\alpha+1)}{(1-a^2)^2}=\frac{2(\alpha-1-\alpha+1)}{(1-a^2)^2}=0,$$

and so $\bar\beta=\underline{\beta}$. This says that the second derivatives match at the free boundary point, in agreement with the condition in (4-22).

In addition, by (5-8),

$$\begin{aligned} 4\alpha(\bar\gamma+\underline{\gamma}) &= 2\alpha\left(\frac{\alpha(1-a)-1}{(1-a)^3}+\frac{1-\alpha(1+a)}{(1+a)^3}\right) \\ &= -\frac{4\alpha a\,(a^2(2\alpha+1)-2\alpha+3)}{(1-a^2)^3}=-\frac{4\alpha a\,(-2\alpha a^4-a^4+4\alpha a^2-2a^2-2\alpha+3)}{(1-a^2)^4}. \end{aligned} \tag{5-11}$$

On the other hand, the first equation in (5-9) says that

$$\frac{12a}{(1-a^2)^4}=\frac{\varepsilon}{\alpha a^4(\alpha+2)+2a^2(2\alpha-\alpha^2+3)+\alpha^2-6\alpha+6}.$$

Using this information in (5-11), we deduce that

$$12\alpha(\bar\gamma+\underline{\gamma})=-\frac{\varepsilon\,\alpha\,(-2\alpha a^4-a^4+4\alpha a^2-2a^2-2\alpha+3)}{\alpha a^4(\alpha+2)+2a^2(2\alpha-\alpha^2+3)+\alpha^2-6\alpha+6}. \tag{5-12}$$

Moreover, in view of (5-10), we have

$$-2\alpha a^4-a^4+4\alpha a^2-2a^2-2\alpha+3=\frac{4}{(1+\alpha)^2},\quad \alpha a^4(\alpha+2)+2a^2(2\alpha-\alpha^2+3)+\alpha^2-6\alpha+6=\frac{4\alpha}{(1+\alpha)^2}.$$

Hence, we insert these identities into (5-12) and we find that

$$2\dot u(a)\,(\,\dddot u(a^+)-\dddot u(a^-))=12\alpha(\bar\gamma+\underline{\gamma})=-\frac{\varepsilon\alpha\dfrac{4}{(1+\alpha)^2}}{\dfrac{4\alpha}{(1+\alpha)^2}}=-\varepsilon,$$

in agreement with the third derivative prescription in (4-22).

Example 5.3. As a variation of Example 5.2, we point out that positive data can yield minimizers which change sign, thus providing an important difference with respect to the classical cases in which the energy is driven by the standard Dirichlet form. This example is interesting also because it shows that, in our framework, this "loss of Maximum Principle" can occur even when the domain is a ball (in fact, even in one dimension, when the domain is an interval) and even when the data is strictly positive.

In this sense, this example is instructive since it shows that, even in domains in which the Maximum Principle holds for biharmonic equations (such as the ball, as established in [Boggio 1905]), the Maximum Principle can be violated in our framework due to the important role played by the "bulk" term in the energy functional.

To construct our example, we take $A > 0$ and we look for minimizers in $(-A, A)$ with boundary conditions $u(A) = u(-A) = 1$ and $\ddot{u}(A) = \ddot{u}(-A) = 0$.

First of all, we observe that

$$J[u] \leqslant C, \tag{5-13}$$

for some $C > 0$ independent of A. To this end, we take $\phi \in C^\infty(\mathbb{R}, [0, +\infty))$ such that $\phi(x) = 0$ for all $x \leqslant 1$ and $\phi(x) = x - 2$ for all $x \geqslant \frac{5}{2}$. Then, assuming $A \geqslant 5$, we define

$$v(x) := \begin{cases} \phi(x+3-A) & \text{if } x \in (A-4, A], \\ 0 & \text{if } x \in [-A+4, A-4], \\ \phi(-x+3-A) & \text{if } x \in [-A, -A+4). \end{cases}$$

We observe $v(A) = \phi(3) = 3-2 = 1$ and $v(-A) = \phi(-x+3-A) = \phi(3) = 1$. Moreover $\ddot{v}(A) = \ddot{\phi}(3) = 0$ and $\ddot{v}(-A) = \ddot{\phi}(3) = 0$. Therefore,

$$\begin{aligned} J[u] \leqslant J[v] &\leqslant \int_{-A}^{A} \big(|\ddot{v}|^2 + \chi_{\{v>0\}}\big) = \int_{[-A,-A+4)\cup(A-4,A]} \big(|\ddot{v}|^2 + \chi_{\{v>0\}}\big) \\ &\leqslant \int_{[-A,-A+4)} |\ddot{\phi}(-x+3-A)|^2\,dx + \int_{(A-4,A]} |\ddot{\phi}(x+3-A)|^2\,dx \\ &= \int_{[-1,3)} |\ddot{\phi}(y)|^2\,dx + \int_{(-1,3]} |\ddot{\phi}(y)|^2\,dx + 8 \\ &\leqslant 8(\|\phi\|_{C^2([-1,3])} + 1), \end{aligned}$$

which proves (5-13).

Now we show that, if A is sufficiently large, then:

$$\text{The minimizer } u \text{ cannot be strictly positive in } (-A, A). \tag{5-14}$$

To check this, we argue by contradiction, supposing that $u > 0$ in $(-A, A)$. Therefore, $\ddot{\ddot{u}} = 0$, and hence u must be a polynomial of degree 3, namely

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

As a consequence,

$$0 = \ddot{u}(\pm A) = 2a_2 \pm 6a_3 A,$$

and hence

$$2a_2 + 6a_3 A = 0 = 2a_2 - 6a_3 A,$$

which yields $a_3 = 0$ and as a result $a_2 = 0$. Accordingly,

$$1 = u(\pm A) = a_0 \pm a_1 A,$$

giving that

$$a_0 + a_1 A = 1 = a_0 - a_1 A,$$

and therefore $a_1 = 0$, which also implies $a_0 = 1$. In this way, we found $u(x) = 1$ for all $x \in (-A, A)$, and consequently $J[u] = 2A$. This is in contradiction with (5-13) as long as A is sufficiently large, and so we have established (5-14).

We now strengthen (5-14) by proving:

$$\text{The set } \{u < 0\} \text{ is nonempty.} \tag{5-15}$$

For this, we first use (5-14) to find a point $\bar{x} \in (-A, A)$ such that $u(\bar{x}) \leqslant 0$. If $u(\bar{x}) < 0$ we are done, hence we can suppose $0 = u(\bar{x}) \leqslant u(x)$ for all $x \in (-A, A)$. By the finiteness of the energy and Sobolev embedding, we know the one-phase minimizer is $C^{1,\alpha}(0, A)$, for some $\alpha \in (0, 1)$. In particular, we can take $\bar{x}$ as large as possible in the zero set of u, finding that $u > 0$ in $(\bar{x}, A]$, and therefore we can write

$$0 < u(x) \leqslant C_0 \, |x - \bar{x}|^{1+\alpha} \quad \text{for all } x \in (\bar{x}, A],$$

for some $C_0 > 0$. Notice also that

$$1 = u(A) - u(\bar{x}) \leqslant \|u\|_{C^1((-A,A))}(A - \bar{x}),$$

and therefore, $A - \bar{x} \geqslant c_0$, for some $c_0 > 0$.

Now, given $\varepsilon > 0$, to be taken conveniently small in what follows, we define

$$\delta := \left(\frac{\varepsilon}{C_0}\right)^{1/(1+\alpha)}, \tag{5-16}$$

and in this way $\delta < c_0$ if ε is sufficiently small. Furthermore, we observe if $x \in (\bar{x}, \bar{x} + \delta] \subset (\bar{x}, A]$, then

$$0 < u(x) \leqslant C_0 \delta^{1+\alpha} = \varepsilon,$$

that is, $(\bar{x}, \bar{x} + \delta] \subseteq \{0 < u \leqslant \varepsilon\}$. For this reason,

$$\delta \leqslant |\{0 < u \leqslant \varepsilon\}| = |\{u > 0\}| - |\{u > \varepsilon\}|. \tag{5-17}$$

We now define

$$u_\varepsilon(x) := \frac{u(x) - \varepsilon}{1 - \varepsilon},$$

and we point out that

$$u_\varepsilon(\pm A) = \frac{u(\pm A) - \varepsilon}{1 - \varepsilon} = \frac{1 - \varepsilon}{1 - \varepsilon} = 1 \quad \text{and} \quad \ddot{u}_\varepsilon(\pm A) = \frac{\ddot{u}(\pm A)}{1 - \varepsilon} = 0.$$

This says that u_ε is a competitor for u, hence, recalling (5-13) and (5-17),

$$0 \leqslant J[u_\varepsilon] - J[u] = \int_{-A}^{A} \big(|\ddot{u}_\varepsilon|^2 - |\ddot{u}|^2 + \chi_{\{u_\varepsilon>0\}} - \chi_{\{u>0\}}\big) = \int_{-A}^{A} \left(\left| \frac{\ddot{u}}{1-\varepsilon} \right|^2 - |\ddot{u}|^2 + \chi_{\{u>\varepsilon\}} - \chi_{\{u>0\}} \right)$$
$$\leqslant \frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2} \int_{-A}^{A} |\ddot{u}|^2 + |\{u > \varepsilon\}| - |\{u > 0\}| \leqslant C_1 \varepsilon - \delta,$$

for some $C_1 > 0$.

From this and (5-16), it follows that

$$C_1 \geqslant \frac{\delta}{\varepsilon} = \frac{1}{\varepsilon} \left(\frac{\varepsilon}{C_0} \right)^{1/(1+\alpha)} = \frac{1}{C_0^{1/(1+\alpha)} \, \varepsilon^{\alpha/(1+\alpha)}},$$

which produces a contradiction when ε is sufficiently small and thus completes the proof of (5-15).

Example 5.4. A natural question arising from Proposition 4.3 (in view of of Lemma 4.1 and (4-21)) is whether a function $u \in C^{1,1}([-1, 1])$ satisfying

$$\begin{cases} \ddddot{u} = 0 \quad \text{in } \{u > 0\} \cup \{u < 0\}, \\ \ddot{u}(-1) = \ddot{u}(1) = 0, \\ u > 0 \quad \text{in } (0, 1) \quad \text{and} \quad u < 0 \quad \text{in } (-1, 0), \\ \dot{u}(0) = 0 \quad \text{and} \quad |\ddot{u}(0^+)|^2 - |\ddot{u}(0^-)|^2 = 1. \end{cases} \tag{5-18}$$

needs necessarily to satisfy

$$\ddot{u} \geqslant 0 \quad \text{a.e. in } (-1, 1). \tag{5-19}$$

Were a statement like this true, the result of Proposition 4.3 could be strengthened (at least in dimension 1) by taking into account not only minimizers but solutions of Navier equations with prescribed free boundary conditions. The following example shows this is not the case, namely (5-18) does not imply (5-19): Let

$$u(x) := \begin{cases} \dfrac{\sqrt{2}\, x^2(3-x)}{6} & \text{if } x \in [0, 1], \\ -\dfrac{x^2(3+x)}{6} & \text{if } x \in [-1, 0). \end{cases}$$

We remark that

$$\ddot{u}(x) = \begin{cases} \sqrt{2}(1-x) & \text{if } x \in (0, 1], \\ -(1+x) & \text{if } x \in [-1, 0), \end{cases}$$

from which the system in (5-18) plainly follows.

Nevertheless, the claim in (5-19) does not hold, since $\ddot{u} < 0$ in $(-1, 0)$.

6. A dichotomy argument and proof of Theorem 1.7

We remark that if u is a minimizer of J in Ω in the admissible class in (1-2) and Ω' is a subdomain of Ω, then it is not necessarily true that u is a minimizer of J in Ω' in the admissible class in (1-2) with Ω replaced by Ω'. This is due to the fact that the admissible class in (1-2) with Ω replaced by Ω' does not prevent the Laplacian of $u - u_0$ to become singular at $\partial\Omega'$, and this provides an important difference with respect to the classical cases dealing with the standard Dirichlet energy. To circumvent this problem, we will consider local minimizers in subdomains:

Definition 6.1. Let Ω' be a subdomain of Ω with smooth boundary. We say that u is a local minimizer in Ω' if, in the notation of (1-1),

$$J[u,\Omega'] \leqslant J[v,\Omega']$$

for every $v \in W^{2,2}(\Omega')$ such that $v-u \in W^{2,2}_0(\Omega')$.

In this way, we have:

Lemma 6.2. *If u is a minimizer in Ω, then it is a local minimizer in every subdomain $\Omega' \subset\subset \Omega$ with smooth boundary.*

Proof. Let $v \in W^{2,2}(\Omega')$ such that $v-u \in W^{2,2}_0(\Omega')$. By the extension results in classical Sobolev spaces (see, e.g., Proposition IX.18 in [Brezis 1983]), we can extend v outside Ω' by setting $v(x) := u(x)$ for all $x \in \Omega \setminus \Omega'$, and we have that $v-u \in W^{2,2}_0(\Omega) \subseteq W^{1,2}_0(\Omega)$. In particular, recalling (1-2), we have that $v \in \mathcal{A}$, and thus,

$$0 \leqslant J[v,\Omega]-J[u,\Omega] = \int_{\Omega\setminus\Omega'} \big(|\Delta u|^2+\chi_{\{u>0\}}-|\Delta v|^2+\chi_{\{v>0\}}\big)+J[v,\Omega']-J[u,\Omega'].$$

Since $u=v$ in $\Omega\setminus\Omega'$, this gives that $0 \leqslant J[v,\Omega']-J[u,\Omega']$, as desired. □

Before proving Theorem 1.7, we show a result concerning the convergence of the blow-up sequence of a minimizer.

Lemma 6.3. *Let $D \subset\subset \Omega$. Let $u_k \in W^{2,2}(D)$, with $k \in \mathbb{N}$, be a sequence of local minimizers of*

$$\int_D \big(|\Delta u_k|^2 + M_k \chi_{\{u_k>0\}}\big), \tag{6-1}$$

with $M_k \in (0,1)$, such that $0 \in \partial\{u_k>0\}$ and $|\nabla u_k(0)|=0$.

Fix $R>0$ such that $B_{5R} \subset\subset D$, and suppose that

$$\sup_{B_{4R}} u_k \leqslant C_0(R), \tag{6-2}$$

$$\|\Delta u_k\|_{L^1(B_{4R})} \leqslant \hat{C}_0(R), \tag{6-3}$$

for some $C_0(R)$, $\hat{C}_0(R)>0$.

Then, there exists a positive constant $C(R)$, independent of k, such that

$$\|u_k\|_{W^{2,2}(B_R)} \leqslant C(R), \tag{6-4}$$

$$\|\Delta u_k\|_{BMO(B_R)} \leqslant C(R), \tag{6-5}$$

for any $k \in \mathbb{N}$.

Furthermore, if $u_k : \mathbb{R}^n \to \mathbb{R}$ and the minimization property in (6-1) holds true in any domain $D \subset \mathbb{R}^n$, and the corresponding assumptions in (6-2) and (6-3) are satisfied, then there exists $u_0 : \mathbb{R}^n \to \mathbb{R}$ such that, up to subsequences, as $k \to +\infty$, $u_k \to u_0$ in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, for any $\alpha \in (0,1)$.

Proof. To check (6-4), we observe that, in virtue of Lemma 4.1,

$$\int_{B_{2R}} \Delta u_k \, \Delta\phi \leqslant 0, \tag{6-6}$$

for any $\phi \in W_0^{2,2}(B_{2R})$. Now, we take $\xi \in C_0^\infty(B_{2R}, [0, 1])$ such that

$$\xi = 1 \text{ in } B_R, \quad |\nabla\xi| \leqslant \frac{C}{R}, \quad \text{and} \quad |D^2\xi| \leqslant \frac{C}{R^2}, \tag{6-7}$$

for some $C > 0$. We set $m_k := \min_{B_{4R}} u_k$ and choose $\phi := (u_k - m_k)\xi^2 \geqslant 0$ in (6-6). In this way, setting

$$I_1 := 2\int_{B_{2R}} \Delta u_k \, \nabla u_k \cdot \nabla\xi^2 \quad \text{and} \quad I_2 := \int_{B_{2R}} (u_k - m_k)\, \Delta u_k \, \Delta\xi^2,$$

we have that

$$0 \geqslant \int_{B_{2R}} \Delta u_k \, \Delta\big((u_k - m_k)\xi^2\big) = \int_{B_{2R}} (\Delta u_k)^2\xi^2 + I_1 + I_2. \tag{6-8}$$

Thanks to Corollary 4.2, we can use the standard method to prove Caccioppoli's inequality, namely we take $\eta \in C_0^\infty(B_{4R}, [0, 1])$ such that $\eta = 1$ in B_{2R} and $|\nabla\eta| \leqslant \frac{C}{R}$ and we infer from Corollary 4.2 and (6-3)

$$\begin{aligned}\hat{C}\int_{B_{4R}} (u_k - m_k)\,\eta^2 &\geqslant -\int_{B_{4R}} \Delta u_k\,(u_k - m_k)\,\eta^2\\ &= \int_{B_{4R}} |\nabla u_k|^2\eta^2 + \int_{B_{4R}} 2\eta\,(u_k - m_k)\,\nabla\eta\cdot\nabla u_k\\ &\geqslant \frac{1}{2}\int_{B_{4R}} |\nabla u_k|^2\eta^2 - C\int_{B_{4R}} (u_k - m_k)^2|\nabla\eta|^2. \end{aligned}\tag{6-9}$$

We remark that, in view of Corollary 4.2 and (6-3), we can choose here $\hat{C}$ proportional to $\tilde{C}(R)/R^n$. Hence, the result in (6-9) yields that

$$\int_{B_{2R}} |\nabla u_k|^2 \leqslant \frac{C}{R^2}\int_{B_{4R}} (u_k - m_k)^2 + C\int_{B_{4R}} (u_k - m_k) \tag{6-10}$$

for some $C > 0$, possibly varying from line to line.

Hence, by Young's inequality, (6-7) and (6-10), we get

$$\begin{aligned}|I_1| &\leqslant 2\left(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{1}{\varepsilon}\int_{B_{2R}} |\nabla u_k|^2\,|\nabla\xi|^2\right)\\ &\leqslant 2\left(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{C}{\varepsilon\,R^2}\int_{B_{2R}} |\nabla u_k|^2\right)\\ &\leqslant 2\left(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{C}{\varepsilon\,R^4}\int_{B_{4R}} (u_k - m_k)^2 + \frac{C}{R^2}\int_{B_{4R}} (u_k - m_k)\right). \end{aligned}\tag{6-11}$$

Furthermore, noticing that $(u_k-m_k)\Delta u_k|\nabla\xi|^2 \geqslant -\hat{C}(u_k-m_k)|\nabla\xi|^2$, thanks to Corollary 4.2, and making again use of Young's inequality, we obtain that

$$\begin{aligned} I_2 &= \int_{B_{2R}} (u_k-m_k)\,\Delta u_k\,(2\xi\,\Delta\xi+|\nabla\xi|^2) \\ &\geqslant 2\int_{B_{2R}} (u_k-m_k)\,\Delta u_k\,\xi\,\Delta\xi - \hat{C}\int_{B_{2R}} (u_k-m_k)|\nabla\xi|^2 \\ &\geqslant -2\biggl(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{1}{\varepsilon}\int_{B_{2R}} (u_k-m_k)^2(\Delta\xi)^2\biggr) - \hat{C}\int_{B_{2R}} (u_k-m_k)|\nabla\xi|^2 \\ &\geqslant -2\biggl(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{C}{\varepsilon R^4}\int_{B_{2R}} (u_k-m_k)^2\biggr) - \frac{C}{R^2}\int_{B_{2R}} (u_k-m_k). \end{aligned}$$

From this, (6-8), and (6-11), we conclude

$$\begin{aligned} \int_{B_{2R}} (\Delta u_k)^2\xi^2 \leqslant 2\biggl(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{C}{\varepsilon R^4}\int_{B_{2R}} (u_k-m_k)^2 + \frac{C}{R^2}\int_{B_{4R}} (u_k-m_k)\biggr) \\ +2\biggl(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{C}{\varepsilon R^4}\int_{B_{2R}} (u_k-m_k)^2\biggr) + \frac{C}{R^2}\int_{B_{2R}} (u_k-m_k), \end{aligned}$$

which, in turn, implies that

$$(1-4\varepsilon)\int_{B_{2R}} (\Delta u_k)^2\xi^2 \leqslant \frac{C}{\varepsilon R^4}\int_{B_{2R}} (u_k-m_k)^2 + \frac{C}{R^2}\int_{B_{2R}} (u_k-m_k) \leqslant \frac{C}{\varepsilon} + C,$$

where the last step follows from (6-2). Choosing $\varepsilon = \frac{1}{8}$ and recalling (6-7), we obtain

$$\int_{B_R} (\Delta u_k)^2 \leqslant \int_{B_{2R}} (\Delta u_k)^2\xi^2 \leqslant C,$$

up to renaming $C > 0$, that does not depend on k. This implies the desired estimate in (6-4).

Moreover, the estimate in (6-5) follows from the BMO estimates in Section 3.

Finally, from the uniform estimate in (6-4), we can apply a customary compactness argument to conclude that there exists a function u_0 such that, up to a subsequence, $u_k \to u_0$ in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, for any $\alpha \in (0,1)$, as $k \to +\infty$. This completes the proof of Lemma 6.3. □

With this, we are now in the position of completing the proof of Theorem 1.7.

Proof of Theorem 1.7. We suppose that $B_1(x_0) \subset\subset D$, with x_0 as in the statement of Theorem 1.7. We claim that there exist an integer $k_0 > 0$ and a structural constant $C > 0$, depending only on δ, n, and $\mathrm{dist}(D,\Omega)$, such that the following inequality holds:

$$\sup_{B_{2^{-k-1}}(x_0)} |u| \leqslant \max\biggl\{\frac{C}{2^{2k}}, \frac{\sup_{B_{2^{-k}}(x_0)}|u|}{2^2}, \dots, \frac{\sup_{B_{2^{-k+m}}(x_0)}|u|}{2^{2(m+1)}}, \dots, \frac{\sup_{B_1(x_0)}|u|}{2^{2(k+1)}}\biggr\}, \tag{6-12}$$

for any $k \geqslant k_0$.

Indeed, if (6-12) fails, then, for any $j \in \mathbb{N}$, there exist singular free boundary points $x_j \in D$, integers k_j, and minimizers u_j (with $\|u_j\|_{W^{2,2}(\Omega)} = \|u\|_{W^{2,2}(\Omega)}$ be given) such that

$$\sup_{B_{2^{-k_j-1}}(x_j)} |u_j| > \max\left\{ \frac{j}{2^{2k_j}}, \frac{\sup_{B_{2^{-k_j}}(x_j)} |u_j|}{2^2}, \dots, \frac{\sup_{B_{2^{-k_j+m}}(x_j)} |u_j|}{2^{2(m+1)}}, \dots, \frac{\sup_{B_1(x_j)} |u_j|}{2^{2(k_j+1)}} \right\}. \tag{6-13}$$

We denote by $S_j := \sup_{B_{2^{-k_j-1}}}(x_j)|u_j|$, and we consider the scaled functions

$$v_j(x) := \frac{u_j(x_j + 2^{-k_j}x)}{S_j}.$$

In this way, (6-13) gives that

$$1 > \max\left\{ \frac{j}{2^{2k_j} S_j}, \frac{\sup_{B_1} |v_j|}{2^2}, \dots, \frac{\sup_{B_{2^m}} |v_j|}{2^{2(m+1)}}, \dots, \frac{\sup_{B_{2^{k_j}}} |v_j|}{2^{2(k_j+1)}} \right\}. \tag{6-14}$$

From this, we have that the functions v_j satisfy the following properties:

$$\sup_{B_{1/2}} v_j = 1, \quad v_j(0) = |\nabla v_j(0)| = 0, \quad \sup_{B_{2^m}} |v_j| \leqslant 4 \cdot 2^{2m} \text{ for any } m < k_j, \quad \sigma_j := \frac{1}{2^{2k_j} S_j} < \frac{1}{j}. \tag{6-15}$$

We also remark that, from the scaling properties of the functional J, we have

$$\int_{B_R} \left(|\Delta v_j|^2 + \sigma_j^2 \chi_{\{v_j>0\}}\right) = 2^{k_j n} \sigma_j^2 \int_{B_{R2^{-k_j}}(x_j)} \left(|\Delta u_j|^2 + \chi_{\{u_j>0\}}\right), \tag{6-16}$$

for every fixed $R < 2^{k_j}$.

We claim that

$$v_j \text{ is a local minimizer in } B_R. \tag{6-17}$$

Indeed, by Lemma 6.2, we know that u is a local minimizer in $B_{R2^{-k_j}}(x_j)$. Hence, if w_j is such that $w_j - v_j \in W_0^{2,2}(B_R)$, we define, for all $y \in B_{R2^{-k_j}}(x_j)$,

$$W_j(y) := S_j\, w_j(2^{k_j}(y - x_j)).$$

In this way, we have that $W_j \in W_0^{2,2}(B_{R2^{-k_j}}(x_j))$, thus yielding, in light of (6-16), that

$$\begin{aligned} 0 &\geqslant 2^{k_j n} \sigma_j^2 \left(\int_{B_{R2^{-k_j}}(x_j)} \left(|\Delta u_j|^2 + \chi_{\{u_j>0\}}\right) - \int_{B_{R2^{-k_j}}(x_j)} \left(|\Delta W_j|^2 + \chi_{\{W_j>0\}}\right) \right) \\ &= \int_{B_R} \left(|\Delta v_j|^2 + \sigma_j^2 \chi_{\{v_j>0\}}\right) - \int_{B_R} \left(|\Delta w_j|^2 + \sigma_j^2 \chi_{\{w_j>0\}}\right). \end{aligned}$$

This completes the proof of (6-17).

Now, by assumption, u_j is not δ-rank-2 flat at each level $r = 2^{-k}$, for any $k \geqslant 1$, at x_j. As a consequence, v_j is not δ-rank-2 flat in B_1. So, recalling (1-14) and Definition 1.6, this means that

$$h(1, 0) = \inf_{p \in P_2} h_{\min}(1, x_0, p) \geqslant \delta. \tag{6-18}$$

Also, we have that condition (6-2) is guaranteed in this case, in view of (6-15). In addition, we have that (6-3) holds true here, since, in view of (6-15), if $20R \in [2^{m-1}, 2^m]$,

$$v_j - \min_{B_{20R}} v_j \leqslant 2 \sup_{B_{20R}} |v_j| \leqslant 2 \sup_{B_{2^m}} |v_j| \leqslant 8 \cdot 2^{2m} \leqslant 8 \cdot (40R)^2 \leqslant CR^2,$$

and consequently, by Lemma A.1 and (6-14),

$$\begin{aligned}
&\int_{B_{5R}} |\Delta v_j(x)|\,dx \\
&= \int_{B_{5R}} \frac{2^{-2k_j}|\Delta u_j(x_j+2^{-k_j}x)|}{S_j}\,dx = \frac{2^{(n-2)k_j}}{S_j}\int_{B_{5R2^{-k_j}}(x_j)} |\Delta u_j(y)|\,dy \\
&\leqslant \frac{2^{nk_j}}{2^{2k_j}S_j}\int_{B_{5R2^{-k_j}}(x_j)} |D^2u_j(y)|\,dy = \frac{CR^n}{2^{2k_j}S_j} ⨍_{B_{5R2^{-k_j}}(x_j)} |D^2u_j(y)|\,dy \\
&\leqslant \frac{CR^n}{2^{2k_j}S_j}\sqrt{⨍_{B_{5R2^{-k_j}}(x_j)} |D^2u_j(y)|^2\,dy} \\
&\leqslant \frac{CR^n}{2^{2k_j}S_j}\left(\frac{2^{4k_j}}{R^4}⨍_{B_{20R2^{-k_j}}(x_j)} \big(u_j - \min_{B_{20R2^{-k_j}}(x_j)} u_j\big)^2 + \frac{2^{2k_j}}{R^2}⨍_{B_{20R2^{-k_j}}(x_j)} \big(u_j - \min_{B_{20R2^{-k_j}}(x_j)} u_j\big)\right)^{1/2} \\
&= \frac{CR^n}{2^{2k_j}S_j}\left(\frac{2^{4k_j}S_j^2}{R^4}⨍_{B_{20R}} \big(v_j - \min_{B_{20R}} v_j\big)^2 + \frac{2^{2k_j}S_j}{R^2}⨍_{B_{20R}} \big(v_j - \min_{B_{20R}} v_j\big)\right)^{1/2} \\
&\leqslant \frac{CR^n}{2^{2k_j}S_j}\big(2^{4k_j}S_j^2 + 2^{2k_j}S_j\big)^{1/2} \leqslant CR^n + \frac{CR^n}{\sqrt{2^{2k_j}S_j}} \leqslant CR^n + \frac{CR^n}{\sqrt{j}} \leqslant CR^n.
\end{aligned}$$

Therefore, recalling (6-16), from Lemma 6.3, applied here with $M_j := \sigma_j^2$, we know that, up to a subsequence, still denoted by v_j, there exists a function v_∞ such that

$$v_j \to v_\infty \text{ in } W^{2,2}(B_R) \cap C^{1,\alpha}(B_R), \quad \text{for any } \alpha \in (0,1), \text{ as } j \to +\infty. \tag{6-19}$$

Moreover, we have that $\Delta v_j \in BMO(B_R)$ uniformly. Consequently $v_\infty \in W^{2,2}(B_R) \cap C^{1,\alpha}(B_R)$, for all $\alpha \in (0,1)$, and $\Delta v_\infty \in BMO(B_R)$. Furthermore,

$$\Delta^2 v_\infty = 0 \text{ in } \mathbb{R}^n, \quad \sup_{B_{1/2}} v_\infty = 1, \quad |v_\infty(x)| \leqslant 8|x|^2 \text{ for any } x \in \mathbb{R}^n, \quad v_\infty(0) = |\nabla v_\infty(0)| = 0. \tag{6-20}$$

Let now $f := \Delta v_\infty$, then we have that f is harmonic in $\mathbb{R}^n$. Moreover, by Lemma A.1 and the second line in (6-20), we see that, for any $r > 0$,

$$\frac{1}{r^n}\int_{B_r} |D^2 v_\infty|^2 \leqslant \frac{C}{r^{n+4}}\int_{B_r} (v_\infty - \min_{B_{4r}} v_\infty)^2 + \frac{C}{r^{n+2}}\int_{B_r} (v_\infty - \min_{B_{4r}} v_\infty) \leqslant C,$$

up to renaming $C > 0$. Thus, from the Liouville Theorem we infer that f must be constant, i.e., $\Delta v_\infty = C_0$, for some $C_0 \in \mathbb{R}$.

Consequently, $v_\infty - \frac{C_0}{2n}|x|^2$ is harmonic in $\mathbb{R}^n$ with quadratic growth. Hence, by using the Liouville Theorem once again, we have that $v_\infty(x) = g(x) + \frac{C_0}{2n}|x|^2$, where g is a second order polynomial. Moreover, since $\nabla v_\infty(0) = 0$, we deduce that $g = cp$, for some $c \in \mathbb{R}$ and $p \in P_2$ (recall (1-11)).

Therefore, we can write

$$v_\infty(x) = x \cdot Ax,$$

for some constant and symmetric matrix A. Consequently, recalling the notation in (1-12),

$$\partial\{v_\infty > 0\} = S(p, 0) \tag{6-21}$$

for some $p \in P_2$. On the other hand, from our construction in (6-18), we have

$$\mathrm{HD}\big(\partial\{v_j > 0\} \cap B_1, S(p, 0) \cap B_1\big) \geqslant \delta$$

(recall the definitions of HD and $h_{\min}$ in (1-10) and (1-13), respectively). As a consequence, there exist points $z_j \in \partial\{v_j > 0\} \cap B_1$ such that

$$\mathrm{dist}(z_j, S(p, 0)) \geqslant \delta. \tag{6-22}$$

Now we extract a converging sequence, still denoted z_j, such that $z_j \to z_0$ as $j \to +\infty$, and we see from the uniform convergence of v_j given in (6-19) that $v_\infty(z_0) = 0$, which implies that $z_0 \in S(p, 0)$, thanks to (6-21). On the other hand, we also have that $\mathrm{dist}(z_0, S(p, 0)) \geqslant \delta$, in virtue of (6-22). Therefore, we reach a contradiction, and so the proof of Theorem 1.7 is finished. □

7. Nondegeneracy and proof of Theorems 1.8 and 1.10

In this section we deal with weak and strong nondegeneracy properties of the minimizers. Due to the lack of Harnack inequalities for biharmonic functions, the strong nondegeneracy result does not follow immediately from the weak one, unless we impose some additional conditions on the set $\{u > 0\}$.

7A. ***Weak nondegeneracy and proof of Theorem 1.8.*** Here we prove the weak nondegeneracy for u^+, according to the statement in Theorem 1.8.

Proof of Theorem 1.8. We prove the claims in **1**° and **2**° together, distinguishing the different structures of the two cases when needed.

After rescaling u by defining $r^{-2}u(x_0 + rx)$, we may assume without loss of generality that $r = 1$ and $x_0 = 0$. Also, denote by

$$\gamma := \sup_{B_1} |u|. \tag{7-1}$$

We remark that in the setting of **2**°, we have

$$B_1 \subseteq \{u > 0\}, \tag{7-2}$$

and therefore,

$$\gamma := \sup_{B_1} u. \tag{7-3}$$

We also remark that, in the setting of **1**°, in light of (1-16), we have

$$|\{u > 0\} \cap B_{1/16}| \geqslant \theta_* \, |B_{1/16}|. \tag{7-4}$$

As a matter of fact, in case **2**°, the statement in (7-4) is also true, with $\theta_* := 1$, as a consequence of (7-2). Hence, we will exploit (7-4) in both the cases **1**° and **2**°, with the convention that $\theta_* = 1$ in the latter case.

We also point out that, in $B_{1/8}$

$$u - \min_{B_{1/8}} u \leqslant 2\gamma. \tag{7-5}$$

Indeed, in case **1**°, the claim in (7-5) follows from (7-1). Instead, in case **2**°, we exploit (7-2) to write that

$$u - \min_{B_{1/8}} u \leqslant u \leqslant \gamma,$$

thus completing the proof of (7-5).

Now, let $\psi \in C^\infty(\mathbb{R}^n, [0, 1])$ such that $\psi = 0$ in $B_{1/16}$, $\psi > 0$ in $\mathbb{R}^n \setminus \overline{B}_{1/16}$, and $\psi = 1$ in $\mathbb{R}^n \setminus B_{1/8}$. Set $v := \psi u$. Then $u - v \in W_0^{2,2}(B_{1/8})$, and so v is a competitor for u in $B_{1/8}$. Therefore, from the local minimality of u (as warranted by Lemma 6.2) we have that

$$\int_{B_{1/8}} \big(|\Delta u|^2 + \chi_{\{u>0\}}\big) \leqslant \int_D \big(|\Delta v|^2 + \chi_{\{v>0\}}\big),$$

where $D := B_{1/8} \setminus \overline{B}_{1/16}$. From this, and recalling the definitions of v and ψ, we obtain that

$$\begin{aligned}
|\{u > 0\} \cap B_{1/16}| &\leqslant \int_{B_{1/16}} \big(|\Delta u|^2 + \chi_{\{u>0\}}\big) \\
&\leqslant \int_D \big(|\Delta v|^2 + \chi_{\{v>0\}}\big) - \int_D \big(|\Delta u|^2 + \chi_{\{u>0\}}\big) = \int_D \big(|\Delta v|^2 - |\Delta u|^2\big) \\
&\leqslant \int_D |\Delta v|^2.
\end{aligned}$$

Hence, using Lemma A.1 and (7-5), it follows that

$$\begin{aligned}
|\{u > 0\} \cap B_{1/16}| &\leqslant \int_D (u \Delta\psi + 2\nabla u \nabla \psi + \psi \Delta u)^2 \\
&\leqslant 2\|\psi\|_{C^2(B_{1/8})} \int_{B_{1/8}} u^2 + 4|\nabla u|^2 + |D^2 u|^2 \\
&\leqslant C\|\psi\|_{C^2(B_{1/8})} \int_{B_{1/8}} \Big(\big(u - \min_{B_{1/8}} u\big)^2 + \big(u - \min_{B_{1/8}} u\big)\Big) \\
&\leqslant C\|\psi\|_{C^2(B_{1/8})} \, \gamma(1+\gamma),
\end{aligned} \tag{7-6}$$

for some $C > 0$, possibly varying from line to line.

Combining this with (7-4) and (7-6), we conclude that

$$\gamma(1+\gamma) \geqslant \frac{|\{u > 0\} \cap B_{1/16}|}{C\|\psi\|_{C^2(B_{1/8})}} \geqslant \frac{\theta_* \, |B_{1/16}|}{\|\psi\|_{C^2(B_{1/8})}},$$

which gives the desired result (using (7-1) in case **1**° and (7-3) in case **2**°). □

7B. ***Whitney's covering.*** Here we recall the Whitney's decomposition method, to obtain suitable conditions which allow us to use Theorem 1.8 (in our setting, the structural assumptions of Theorem 1.8 will be provided by formula (7-7)). Suppose that $E \subset \mathbb{R}^n$ is a nonempty compact set, then $\mathbb{R}^n \setminus E$ can be represented as a union of closed dyadic cubes Q_j^k with mutually disjoint interiors

$$\mathbb{R}^n \setminus E = \bigcup_{k\in\mathbb{Z}} \bigcup_{j=1}^{N_k} Q_j^k$$

such that

$$c_1 \leqslant \frac{\operatorname{dist}(Q_j^k, E)}{\operatorname{diam} Q_j^k} \leqslant c_2$$

for two universal constants $c_1, c_2 > 0$. Here Q_j^k is a cube with side length equal to 2^{-k}.

Let now $E := \{u \leqslant 0\} \cap \overline{Q_1(x_0)}$, where $Q_1(x_0)$ is the unit cube centered at $x_0 \in \partial\{u > 0\}$, and consider the Whitney's decomposition for $\mathbb{R}^n \setminus E$. Let $k_0 \in \mathbb{N}$ be fixed, and suppose that for every $k \geqslant k_0$ there exists $c > 0$ such that, for some Q_j^k, we have

$$\operatorname{dist}(x_0, Q_j^k) \leqslant c2^{-k}. \tag{7-7}$$

Then u^+ is strongly nondegenerate at x_0. To see this, for every large k let us take a cube Q_j^k such that (7-7) holds. Then, if x_1 is the center of Q_j^k, we have that $u(x_1) > 0$ and $\operatorname{dist}(x_1, \partial\{u > 0\}) \geqslant 2^{-k-1}$, Hence, in view of claim **2**° of Theorem 1.8, we find that

$$\sup_{B_{2^{-k-1}}(x_1)} u^+ \geqslant \bar{c}(2^{-k-1})^2 = \frac{\bar{c}}{4} 2^{-2k}. \tag{7-8}$$

On the other hand, by (7-7), we see that

$$|x_0 - x_1| \leqslant c2^{-k} + \sqrt{n}2^{-k} = (c + \sqrt{n})2^{-k},$$

and accordingly $B_{c^* 2^{-k}}(x_0) \supseteq B_{2^{-k-1}}(x_1)$, with $c^* := c + \sqrt{n} + \frac{1}{2}$. Therefore, by (7-8),

$$\sup_{B_{c^* 2^{-k}}(x_0)} u^+ \geqslant \frac{\bar{c}}{4} 2^{-2k}.$$

Definition 7.1. If (7-7) holds, then we say $\partial\{u > 0\}$ satisfies a weak c-covering condition at $x_0 \in \partial\{u > 0\}$.

We remark that the standard c-covering condition, introduced in [Martio and Vuorinen 1987], is stronger than (7-7) and indeed it requires that

$$\operatorname{dist}\Big(x_0, \bigcup_{j=1}^{N_k} Q_j^k\Big) \leqslant c2^{-k}.$$

Moreover, it is known that the weak c-covering condition of Definition 7.1 is satisfied by the John domains, see [Martio and Vuorinen 1987].

In order to recall the definition of a John domain, we let $0 < \alpha \leqslant \beta < \infty$. A domain $D \subset \mathbb{R}^n$ is called an (α, β)-John domain, denoted by $D \in \mathcal{J}(\alpha, \beta)$, if there exists $x_0 \in D$ such that every $x \in D$ has a rectifiable path $\gamma : [0, d] \to D$ with arc length as parameter such that $\gamma(0) = x$, $\gamma(d) = x_0$, $d \leqslant \beta$ and

$$\operatorname{dist}(\gamma(t), \partial D) \geqslant \frac{\alpha}{d} t, \quad \text{for all } t \in [0, d].$$

The point x_0 is called a center of D. A domain D is called a John domain if $D \in \mathcal{J}(\alpha, \beta)$ for some α and β. The class of all John domains in $\mathbb{R}^n$ is denoted by $\mathcal{J}$. For more on such coverings and applications of Whitney's decompositions we refer to [Martio and Vuorinen 1987].

Alternative sufficient geometric conditions on $\{u > 0\}$ guaranteeing the strong nondegeneracy of u can be given. Note that in order to pass from weak to strong nondegeneracy at some $z \in \partial\{u > 0\}$, it is enough to have a small ball $B' \subset B_r(z) \cap \{u > 0\}$ and $c > 0$ such that diam $B' \geqslant cr$ for every small r, since this guarantees (1-16).

Definition 7.2. We say that $\partial\{u > 0\}$ satisfies a nonuniform interior cone condition if for every $x \in \partial\{u > 0\}$ there exist a positive number $r_x > 0$ and a cone K_x with vertex at x, such that $B_{r_x}(x) \cap K_x \subset \{u > 0\}$.

We also say that $\partial\{u > 0\}$ satisfies a uniform interior cone condition if there exist a positive number $r > 0$ and a cone K with vertex at 0, such that for every $x \in \partial\{u > 0\}$ we have $B_r(x) \cap (x + K) \subset \{u > 0\}$.

From our observation above and Theorem 1.8, we immediately obtain the following result:

Corollary 7.3. *Let u be a minimizer for J in Ω, and $x_0 \in \Omega$. Suppose that $\{u > 0\}$ satisfies the interior cone condition at $x_0 \in \partial\{u > 0\}$, then $|u|$ is nondegenerate at x_0. Moreover, if $\{u > 0\}$ satisfies the uniform interior cone condition and $B_1 \subset \Omega$, then*

$$\sup_{B_r(z)} u^+ \geqslant C_0 r^2,$$

for any $z \in \partial\{u > 0\} \cap B_1$, for some $C_0 > 0$.

7C. ***The biharmonic measure and proof of Theorem 1.10.*** In this subsection, we describe the main features of the measure induced by the bi-Laplacian of a minimizer. For this, we observe that, since, by Lemma 4.1, Δu is super-harmonic:

$$\text{There exists a nonnegative measure } \mathcal{M}_u \text{ such that } -\Delta^2 u = \mathcal{M}_u. \tag{7-9}$$

Hence, for any $\psi \in C_0^\infty(\Omega)$, we have that

$$\int_\Omega \mathcal{M}_u \psi = \int_\Omega (-\Delta u) \Delta\psi. \tag{7-10}$$

Recalling the notion of flatness introduced in Definition 1.6, we have the following:

Lemma 7.4. *Let u be a minimizer of the functional J defined in (1-1), let $\delta > 0$, and let $x_0 \in \partial\{u > 0\}$ such that $\nabla u(x_0) = 0$ and $\partial\{u > 0\}$ is not δ-rank-2 flat at x_0 at any level $r > 0$ with $B_r(x_0) \subset\subset \Omega$. Then,*

$$\mathcal{M}_u(B_r(x_0)) \leqslant Cr^{n-2} \tag{7-11}$$

for any $r > 0$ as above, for some $C > 0$.

Proof. Without loss of generality, we take $x_0 = 0$. We consider a function $\psi_0 \in C_0^\infty(B_2, [0, 1])$, with $\psi_0 = 1$ in B_1, and we let $\psi(x) := \psi_0(x/r)$. In this way, $\psi = 1$ in B_r and $|D^2\psi| \leqslant C/r^2$ for some $C > 0$.

We now exploit (7-10) with such ψ. Then, by Corollary A.2, we have

$$\mathcal{M}_u(B_r) \leqslant \int_\Omega \mathcal{M}_u \psi = \int_\Omega (-\Delta u)\Delta\psi \leqslant \sqrt{\int_{B_{2r}} |\Delta u|^2}\sqrt{\int_{B_{2r}} |\Delta\psi|^2} \leqslant C r^{n/2} r^{(n-4)/2},$$

which implies the desired result, up to renaming $C > 0$. □

We remark that a full counterpart of Lemma 7.4 does not hold for the one-phase problem (in particular, $\mathcal{M}_u$ as defined in (7-9) and (7-10) does not need to have a sign, see (5-6)). Nevertheless, the following result holds:

Lemma 7.5. *Let u be a one-phase minimizer of J. Assume that $u \in C^{1,1}(\Omega)$ and $\partial\{u > 0\}$ has null Lebesgue measure. Let $\varphi \in C_0^\infty(B_1, [0, 1])$ with*

$$\int_{B_1} \varphi = 1.$$

For any $\delta > 0$, let

$$\varphi_\delta(x) := \frac{1}{\delta^n}\varphi\left(\frac{x}{\delta}\right),$$

and $u_\delta := u * \varphi_\delta$. Then, for any $\Omega' \subset\subset \Omega$, we have that

$$\lim_{\delta\to 0}\int_{\Omega'} \Delta^2 u_\delta \, u_\delta = 0.$$

Proof. Let

$$\Gamma_\delta := \bigcup_{p\in\partial\{u>0\}} B_\delta(p).$$

We claim:

$$\text{If } x \in \Omega \setminus \Gamma_\delta, \text{ then } \Delta^2 u(x) = 0. \tag{7-12}$$

To prove this, we argue by contradiction, and we suppose that there exists $x \in \Omega \setminus \Gamma_\delta$ such that

$$\Delta^2 u(x) \text{ is either not defined or not null.} \tag{7-13}$$

We observe that:

$$\text{There exists } \rho, a > 0 \text{ such that } u \geqslant a \text{ in } B_\rho(x). \tag{7-14}$$

Because, if not, for any $k \in \mathbb{N}$, there exists x_k such that $|x - x_k| + u(x_k) \leqslant 1/k$, and thus $u(x) = 0$. Since x lies outside Γ_δ, it cannot be a free boundary point, hence u must vanish in a neighborhood of x. Consequently, $\Delta^2 u$ vanishes in a neighborhood of x, and this is in contradiction with (7-13), thus proving (7-14).

Then, from (7-14) and Lemma 4.1, it follows that u is biharmonic in $B_\rho(x)$. Once again, this is in contradiction with (7-13), and thus the proof of (7-12) is complete.

Now, by taking δ sufficiently small, we suppose that the distance from Ω' to $\partial\Omega$ is larger than δ. Thus, from (7-12) we obtain that, if $x \in \Omega' \setminus \Gamma_{2\delta}$ and $y \in B_\delta$, then $x - y \in \Omega' \setminus \Gamma_\delta$, hence $\Delta^2 u(x-y) = 0$.

Consequently, for every $x \in \Omega' \setminus \Gamma_{2\delta}$,

$$\Delta^2 u_\delta(x) = \int_{B_\delta} \Delta^2 u(x-y)\, \varphi_\delta(y)\, dy = 0.$$

This implies that

$$\int_{\Omega'} \Delta^2 u_\delta\, u_\delta = \int_{\Omega' \cap \Gamma_{2\delta}} \Delta^2 u_\delta\, u_\delta. \tag{7-15}$$

We also remark that

$$\begin{aligned} |\Delta^2 u_\delta(x)| &\leqslant \int_{B_\delta} |u(x-y)|\, |\Delta^2 \varphi_\delta(y)|\, dy = \frac{1}{\delta^{n+4}} \int_{B_\delta} |u(x-y)| \left|\Delta^2 \varphi\Big(\frac{y}{\delta}\Big)\right| dy \\ &= \frac{1}{\delta^4} \int_{B_1} |u(x-\delta y)|\, |\Delta^2 \varphi(y)|\, dy \leqslant \frac{C}{\delta^4} \int_{B_1} u(x-\delta y)\, dy, \end{aligned} \tag{7-16}$$

for some $C > 0$. Now, if $x \in \Gamma_{2\delta}$ and $y \in B_1$, we have that there exists $p \in \partial\{u > 0\} \subseteq \{u = 0\}$ such that $|p - x| \leqslant 2\delta$ and accordingly $|(x - \delta y) - p| \leqslant |x - p| + \delta \leqslant 3\delta$. Then, in this setting, the regularity of u implies that

$$u(x - \delta y) \leqslant 9 \|u\|_{C^{1,1}(\Omega)} \delta^2. \tag{7-17}$$

In particular, recalling (7-16), we find that, if $x \in \Gamma_{2\delta}$,

$$|\Delta^2 u_\delta(x)| \leqslant \frac{C}{\delta^2}, \tag{7-18}$$

up to renaming $C > 0$, also depending on $\|u\|_{C^{1,1}(\Omega)}$.

From (7-17) we also deduce that, if $x \in \Gamma_{2\delta}$,

$$|u_\delta(x)| \leqslant \int_{B_1} u(x-\delta y)\, \varphi(y)\, dy \leqslant 9 \|u\|_{C^{1,1}(\Omega)} \delta^2.$$

Using this information and (7-18) we conclude that, if $x \in \Gamma_{2\delta}$,

$$|\Delta^2 u_\delta(x)\, u_\delta(x)| \leqslant C,$$

and therefore,

$$\left| \int_{\Omega' \cap \Gamma_{2\delta}} \Delta^2 u_\delta\, u_\delta \right| \leqslant C |\Omega' \cap \Gamma_{2\delta}|,$$

up to renaming $C > 0$ once again.

This and (7-15) provide

$$\left| \int_{\Omega'} \Delta^2 u_\delta\, u_\delta \right| \leqslant C |\Omega' \cap \Gamma_{2\delta}|.$$

Hence, taking the limit as $\delta \to 0$,

$$\lim_{\delta \to 0} \left| \int_{\Omega'} \Delta^2 u_\delta\, u_\delta \right| \leqslant |\Omega' \cap \partial\{u > 0\}|. \qquad \square$$

Now we prove a counterpart of (7-11) at nondegenerate points of the free boundary of the minimizers. For this, recalling the setting in formula (1-14), we let $\mathcal{N}_\delta$ be the set of free boundary points x with the property that there exists $r_x > 0$ small enough such that $h(r, x) \geqslant \delta r$ for every $r < r_x$. Moreover, in the spirit of Definition 1.4, we also denote by

$$\mathcal{N}_\delta^{\mathrm{sing}} := \{x \in \mathcal{N}_\delta : \nabla u(x) = 0\}.$$

Lemma 7.6. *Let u be a minimizer of J. Let $D \subset \Omega$ and suppose that there exists $\bar{c} > 0$ such that*

$$\liminf_{r\to 0} \frac{\sup_{B_r(x)} |u|}{r^2} \geqslant \bar{c} \tag{7-19}$$

for every $x \in \partial\{u > 0\} \cap \overline{D}$. Then there exists $c_0(\delta) > 0$, depending on n, δ, $\bar{c}$, $\|u\|_{W^{2,2}(\Omega)}$, and $\operatorname{dist}(\overline{D}, \partial\Omega)$, such that

$$\liminf_{r\to 0} \frac{\mathcal{M}_u(B_r(x))}{r^{n-2}} \geqslant c_0(\delta), \quad \text{for any } x \in \mathcal{N}_\delta^{\mathit{sing}}. \tag{7-20}$$

Proof. We argue by contradiction. If (7-20) fails, then there exists a sequence $x_j \in \mathcal{N}_\delta^{\mathrm{sing}}$ such that

$$\liminf_{r\to 0} \frac{\mathcal{M}_u(B_r(x_j))}{r^{n-2}} < \varepsilon_j, \tag{7-21}$$

with $\varepsilon_j \to 0$. Since $x_j \in \mathcal{N}_\delta^{\mathrm{sing}}$, there exists a sequence $r_j \to 0$ such that

$$h(r_j, x_j) \geqslant \delta r_j. \tag{7-22}$$

Now we define

$$U_j(x) := \frac{u(x_j + r_j x)}{r_j^2}.$$

By construction, recalling (7-19), we have that $\{U_j\}$ is nondegenerate with quadratic growth, i.e., there exists $C > 0$ independent of j such that

$$\frac{1}{C} R^2 \leqslant \sup_{B_R} |U_j| \leqslant C R^2 \quad \text{for any } R < \frac{1}{r_j}. \tag{7-23}$$

Moreover, by (7-21) and (7-22), we see that

$$h(1, 0) \geqslant \delta \quad \text{and} \quad \mathcal{M}_{U_j}(B_R) \leqslant \varepsilon_j R^{n-2} \to 0 \tag{7-24}$$

for every fixed $R > 0$.

As a consequence, using a customary compactness argument, we can extract a converging subsequence, still denoted by U_j, such that $U_j \to U_0$ locally uniformly as $j \to +\infty$. Then (7-24) translates into

$$h(1, 0) \geqslant \delta \quad \text{and} \quad \mathcal{M}_{U_0}(B_R) = 0 \tag{7-25}$$

for every fixed $R > 0$. In other words, in view of (7-23), we have that U_0 is an entire nontrivial biharmonic function with quadratic growth.

On the other hand, applying Corollary A.2 we also have that

$$\int_{B_R} |D^2u|^2 \leqslant CR^n.$$

This, together with the Liouville Theorem, implies that

$$U_0 \text{ is a quadratic polynomial.} \tag{7-26}$$

Accordingly, there exists $\alpha \in \mathbb{R}$ such that $p := \alpha U_0 \in P_2$ (recall the notation in (1-11)). From (7-25), we conclude that

$$\mathrm{HD}(S(p,0) \cap B_1, \partial\{U_0 > 0\} \cap B_1) \geqslant \delta,$$

which is a contradiction with (7-26). □

We are now in position to complete our analysis of the free boundary regularity results which follow from the study of the biharmonic measure by proving Theorem 1.10.

Proof of Theorem 1.10. We start by proving **1°**. For this, let $D \subset\subset \Omega$ and $x \in \mathcal{F}_\delta := (\partial\{u > 0\} \cap D) \setminus \mathcal{N}_\delta$, where $\mathcal{N}_\delta$ has been introduced before Lemma 7.6. Then there exists $r_x > 0$ such that

$$|\partial\{u > 0\} \cap B_{r_x}(x)| \leqslant C(n)\delta r_x^n,$$

where $C(n)$ is a dimensional constant. In this way, we can cover $\mathcal{F}_\delta$ with balls $B_{r_x}(x)$, and we can then extract a Besicovitch covering such that

$$|\mathcal{F}_\delta \cap D| \leqslant C(n)\delta\, |D|. \tag{7-27}$$

Then, sending $\delta \to 0$ the result in **1°** follows.

We now focus on **2°**. In this case, thanks to (1-18) we can use Lemma 7.6 and find a Besicovitch covering by balls $B_{r_x}(x)$ of $\mathcal{N}_\delta^{\mathrm{sing}}$ such that

$$c_0(\delta) \sum r_x^{n-2} \leqslant \mathcal{M}_u(D') < \infty, \tag{7-28}$$

where $D' \Supset D$ is a subdomain of Ω such that

$$\mathrm{dist}(D, \partial D') < \sup_{x \in \partial\{u>0\} \cap D} r_x := r_0.$$

Therefore, letting $r_0 \to 0$ in (7-28), we get that

$$\mathcal{H}^{n-2}(\mathcal{N}_\delta^{\mathrm{sing}} \cap D) < +\infty. \tag{7-29}$$

Furthermore, since the free boundary is C^1 near points in $\mathcal{N}_\delta \setminus \mathcal{N}_\delta^{\mathrm{sing}}$, we have that

$$\mathcal{H}^{n-2}((\mathcal{N}_\delta \setminus \mathcal{N}_\delta^{\mathrm{sing}}) \cap D) < +\infty,$$

which, together with (7-29), implies that

$$\mathcal{H}^{n-2}(\mathcal{N}_\delta \cap D) < +\infty. \tag{7-30}$$

This gives the second claim in **2°**. We now prove the first claim in **2°**. For this, we use (7-27) and (7-30) to obtain

$$|\partial\{u>0\}\cap D| \leqslant |\mathcal{F}_\delta \cap D| + |\mathcal{N}_\delta \cap D| = |\mathcal{F}_\delta \cap D| \leqslant C(n)\delta\,|D|.$$

Then, sending $\delta \to 0$, we complete the proof of **2°**. □

Remark 7.7. If $\{u>0\}$ is a John domain, then $|u|$ is nondegenerate by the discussion in Section 7B. Alternatively, as in Theorem 1.8, if $\{u>0\}$ has uniformly positive Lebesgue density then $|u|$ is nondegenerate.

8. Stratification of free boundary and proof of Theorem 1.11

In this section we reformulate some results obtained in Section 6 related to the dichotomy between the notion of rank-2 flatness and the quadratic growth of the minimizer.

For this, to describe an appropriate flatness rate of the minimizers, we recall Definition 1.4 and we also define a suitable class, in the following way:

Definition 8.1. Fix $r>0$. We say that $u \in \mathcal{P}_r$ if:

- $u \in W^{2,2}(B_r)$ is a minimizer of J in (1-1) in B_r, among functions $v \in W^{2,2}(B_r)$, and $v-u \in W_0^{1,2}(B_r)$,
- $0 \in \partial_{\text{sing}}\{u>0\}$.

If, in addition, given $\delta>0$, the free boundary is not (δ, r)-rank-2 flat at 0, then we say that $u \in \mathcal{P}_r(\delta)$.

In the setting of Definition 8.1, Theorem 1.7 can be reformulated as follows:

Proposition 8.2. *Let $u \in \mathcal{P}_r(\delta)$. Then there exist $r_0>0$ and $C>0$, possibly depending on n, δ, r, and $\|u\|_{W^{2,2}(\Omega)}$, such that*

$$|u(x)| \leqslant C|x|^2, \quad \textit{for any } x \in B_{r_0}.$$

Moreover, recalling the definition of $h(r, x_0)$ in (1-14), a refinement of Theorem 1.7 can be formulated as:

Theorem 8.3. *Let $u \in \mathcal{P}_1$. Let $\delta \in (0,1)$, $k>10$, and $r_k := 2^{-k}$. Then, either $h(0, r_k) < \delta r_k$, or there exists $C>0$, possibly depending on n, δ, and $\|u\|_{W^{2,2}(\Omega)}$, such that*

$$\sup_{B_{r_k/2}} |u| \leqslant C r_k^2.$$

We are now ready to complete the proof of Theorem 1.11.

Proof of Theorem 1.11. Notice that (1-19) and (1-20) follow as a consequence of Theorem 8.3. Therefore, to complete the proof of Theorem 1.11, it only remains to prove that u^+ is strongly nondegenerate at $z \in \mathcal{F}$. After rescaling $U_r(x) := r^{-2}u(z+rx)$, we see that it is enough to show that

$$\sup_{B_1} U_r^+ \geqslant \hat{C}, \tag{8-1}$$

for some $\hat{C}>0$ (which here can depend on n, $\operatorname{dist}(z, \partial\Omega)$ and the minimizer u itself).

To check this, we first prove:

$$\begin{gathered}\text{If } p \text{ is a homogeneous polynomial of degree two,}\\ \text{then } \{p=0\} \text{ is contained in the union of finitely many hypersurfaces.}\end{gathered} \tag{8-2}$$

Indeed, up to a linear transformation, and possibly exchanging the order of the variables, we can suppose

$$p(x) = \sum_{i=1}^{n} a_i x_i^2,$$

with $(a_1, \dots, a_m) \in \mathbb{R} \setminus \{0\}$ and $a_{m+1} = \dots = a_n = 0$, for some $m \in \{1, \dots, n\}$. Therefore the zero set of p is obtained by the zero set of the polynomial

$$\mathbb{R}^m \ni x \mapsto \tilde{p}(x) = \sum_{i=1}^{m} a_i x_i^2,$$

up to a Cartesian product with an $(n-m)$-dimensional linear space. Also:

$$\text{If } x \in \{\tilde{p} = 0\}, \text{ then } tx \in \{\tilde{p} = 0\} \text{ for all } t \in \mathbb{R}. \tag{8-3}$$

Therefore,

$$\{\tilde{p} = 0\} = \big\{tx,\ x \in \{\tilde{p} = 0\} \cap \mathbb{S}^{m-1}\big\}. \tag{8-4}$$

Furthermore,

$$\{\nabla \tilde{p} = 0\} = \{(2a_1x_1, \dots, 2a_mx_m) = 0\} = \{0\}. \tag{8-5}$$

Therefore, by (8-5), in the vicinity of any $x \in \{\tilde{p} = 0\} \cap \mathbb{S}^{m-1}$, the set $\{\tilde{p} = 0\}$ is an $(m-1)$-dimensional surface, which, in view of (8-3), is transverse to $\mathbb{S}^{m-1}$. Consequently, we have $\{\tilde{p} = 0\} \cap \mathbb{S}^{m-1}$ is the union of $(m-2)$-dimensional surfaces. In addition, from (8-5) we know that these surfaces cannot accumulate to each other, and so $\{\tilde{p} = 0\} \cap \mathbb{S}^{m-1}$ is the union of finitely many $(m-2)$-dimensional surfaces.

This and (8-4) imply that $\{\tilde{p} = 0\}$ is the union of finitely many $(m-1)$-dimensional surfaces. Accordingly, we have that $\{p = 0\}$ is the union of finitely many surfaces of dimension $(m-1) + (n-m) = n-1$. This completes the proof of (8-2).

We also stress that, in light of (8-3), the intersection of the hypersurfaces described in (8-2) and $\mathbb{S}^{n-1}$ have codimension 1 inside $\mathbb{S}^{n-1}$. In particular, for every $p \in \mathbb{S}^{n-1}$ outside these hypersurfaces there exists $\rho(p) \in (0, \frac{1}{2})$ such that $B_{\rho(p)}(p)$ does not intersect these hypersurfaces.

Given $x \in B_1 \setminus \{0\}$, we now use the notation $\hat{x} := x/|x|$, and we claim:

$$\text{There exists } x_* \in B_{1/2} \setminus \{0\} \text{ such that } U_r(x_*) > 0 \text{ and } \hat{x}_* \text{ lies outside the hypersurfaces (8-2).} \tag{8-6}$$

Indeed, we can assume that $|B_{1/2} \cap \{U_r > 0\}| > 0$ (otherwise $u \leqslant 0$, contradicting the assumption that $z \in \partial\{u > 0\}$), and from this we obtain (8-6).

From (8-6), we deduce that $B_{\rho(\hat{x}_*)}(\hat{x}_*)$ does not intersect the hypersurfaces in (8-2). Hence, by (8-3), setting $r(x_*) := |x_*|\rho(\hat{x}_*)$, we see that $B_{r(x_*)}(x_*)$ does not intersect the hypersurfaces in (8-2). Then, from (1-19), it follows that if $r = r_k$ is sufficiently small, then $B_{r(x_*)/2}(x_*)$ does not intersect $\partial\{U_r > 0\}$. For this reason, since $U_r(x_*) > 0$, we conclude that $B_{r(x_*)/2}(x_*) \subseteq \{U_r > 0\}$.

Consequently, we are in the position of using claim **2°** in Theorem 1.8, thus obtaining that

$$\sup_{B_{r(x_*)/2}(x_*)} U_r^+ \geqslant \bar{c}\left(\frac{r(x_*)}{2}\right)^2 = \frac{\bar{c}\,(r(x_*))^2}{4} = \frac{\bar{c}\,(\rho(\hat{x}_*))^2}{4}\,|x_*|^2 \quad \text{for some } \bar{c} > 0. \tag{8-7}$$

Now we claim that

$$B_1 \supseteq B_{r(x_*)/2}(x_*). \tag{8-8}$$

Indeed, if $y \in B_{r(x_*)/2}(x_*)$, we have

$$|y| \leqslant |y - x_*| + |x_*| \leqslant \frac{r(x_*)}{2} + |x_*| = \frac{\rho(\hat{x}_*)\,|x_*|}{2} + |x_*| \leqslant \frac{|x_*|}{4} + |x_*| = \frac{5|x_*|}{4} \leqslant \frac{5}{8} < 1,$$

thus proving (8-8).

Then, from (8-7) and (8-8) we obtain

$$\sup_{B_1} U_r^+ \geqslant \frac{\bar{c}\,(\rho(\hat{x}_*))^2}{4}\,|x_*|^2 =: \hat{C},$$

and (8-1) follows, as desired. □

9. Monotonicity formula: proof of Theorem 1.12

This section is devoted to the proof of Theorem 1.12, which is based on a series of careful integration by parts aimed at spotting suitable integral cancellations. In addition, some "high order of differentiability" terms naturally appear in the computations, which need to be suitably removed in order to rigorously make sense of the formal manipulations. We start with some general computations valid in $\mathbb{R}^n$, then, from (9-24) on, we specialize to the case $n = 2$. In this part of the paper, for the sake of shortness, we suppose that the assumptions of Theorem 1.12 are always satisfied without further mentioning them. Without loss of generality, we also suppose that $B_2 \subset\subset \Omega$. Then, we have the following identity:

Lemma 9.1. *For every $r_1, r_2 \in (0, 3/2)$,*

$$4\int_{r_1}^{r_2} R(r)\,dr + 2T(r_2) - 2T(r_1) + D(r_2) - D(r_1) = 0, \tag{9-1}$$

where

$$\begin{aligned}
R(r) &:= \frac{1}{r^{n+1}} \sum_{m=1}^{n} \int_{B_r} \Delta u\, \nabla u_m \cdot e_m - \sum_{m=1}^{n} \int_{\partial B_r} \Delta u \nabla u_m \cdot \frac{x^m\, x}{r^{n+2}} = \frac{1}{r^{n+1}} \int_{B_r} |\Delta u|^2 - \frac{1}{r^n} \int_{\partial B_r} \Delta u\, \partial_r^2 u,\\
T(r) &:= \sum_{m=1}^{n} \int_{\partial B_r} \Delta u\, u_m \frac{x^m}{r^{n+1}} = \frac{1}{r^n} \int_{\partial B_r} \Delta u\, \partial_r u,\\
D(r) &:= \frac{1}{r^n} \int_{B_r} \big(|\Delta u|^2 + \chi_{\{u>0\}}\big),
\end{aligned} \tag{9-2}$$

and the notation $\partial_r := (x/|x|) \cdot \nabla$ has been used.

Proof. Fix $r \in (0, 3/2)$. We let $\delta > 0$ (to be taken as small as we wish in what follows), and consider a smooth function $\eta = \eta_\delta$ supported in $B_{r+\delta}$. Fixed $\varepsilon > 0$, we also consider the mollifier $\rho_\varepsilon(x) := (1/\varepsilon^n)\rho\big(x/\varepsilon\big)$, for a given even function $\rho \in C_0^\infty(B_1)$. We also define $\phi = (\phi^1, \dots, \phi^n) : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mathbb{R}^n \ni x = (x^1, \dots, x^n) \longmapsto \phi^m(x) := (\psi^m * \rho_\varepsilon)(x), \quad \text{where } \psi^m(x) := x^m \eta(x).$$

Let also

$$F^m(x) := \Delta u(x)\, u_m(x). \tag{9-3}$$

In view of (4-1) and (4-2) (if u is a minimizer), or recalling that u is assumed to be in $C^{1,1}(\Omega)$ (if u is a one-phase minimizer), we know that

$$F^m \in L^p(B_1) \quad \text{for every } p \in (1, +\infty).$$

We observe that ψ^m is supported in $B_{r+\delta}$, and so ϕ^m is supported in $B_{r+\delta+\varepsilon} \subset B_1$, as long as δ and ε are sufficiently small. Consequently,

$$\begin{aligned}\int_\Omega \Delta u\, u_m \Delta\phi^m &= \int_{\mathbb{R}^n} \Delta u\, u_m \Delta\phi^m = \int_{\mathbb{R}^n} F^m(\Delta\psi^m * \rho_\varepsilon) = \iint_{\mathbb{R}^n\times B_\varepsilon(x)} F^m(x)\Delta\psi^m(y)\rho_\varepsilon(x-y)\,dx\,dy\\ &= \iint_{B_\varepsilon(x)\times\mathbb{R}^n} F^m(x)\Delta\psi^m(y)\rho_\varepsilon(y-x)\,dx\,dy = \iint_{\mathbb{R}^n} (F^m * \rho_\varepsilon)(y)\Delta\psi^m(y)\,dy\\ &= \int_\Omega F^m_\varepsilon \Delta\psi^m = -\int_\Omega \nabla F^m_\varepsilon \cdot \nabla\psi^m,\end{aligned} \tag{9-4}$$

with

$$F^m_\varepsilon := F^m * \rho_\varepsilon. \tag{9-5}$$

Similarly, we have

$$\int_\Omega \Delta u \nabla u_m \cdot \nabla\phi^m = \int_\Omega \Delta u \nabla u_m \cdot (\nabla\psi^m * \rho_\varepsilon) = \int_\Omega ((\Delta u \nabla u_m) * \rho_\varepsilon) \cdot \nabla\psi^m. \tag{9-6}$$

Also,

$$\begin{aligned}\int_\Omega \big(|\Delta u|^2 + \chi_{\{u>0\}}\big)\operatorname{div}\phi &= \sum_{m=1}^n \int_\Omega \big(|\Delta u|^2 + \chi_{\{u>0\}}\big)(\psi^m_m * \rho_\varepsilon)\\ &= \int_\Omega \Big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big) * \rho_\varepsilon\Big)\operatorname{div}\psi.\end{aligned}$$

Then, we plug this information, (9-4), and (9-6) into (4-10), and we see that

$$\begin{aligned}0 &= 2\int_\Omega \Delta u \sum_{m=1}^n \big(2\nabla u_m \cdot \nabla\phi^m + u_m \Delta\phi^m\big) - \int_\Omega \big(|\Delta u|^2 + \chi_{\{u>0\}}\big)\operatorname{div}\phi\\ &= 4\sum_{m=1}^n \int_\Omega ((\Delta u \nabla u_m) * \rho_\varepsilon)\cdot\nabla\psi^m - 2\sum_{m=1}^n \int_\Omega \nabla F^m_\varepsilon \cdot \nabla\psi^m - \int_\Omega \Big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big) * \rho_\varepsilon\Big)\operatorname{div}\psi.\end{aligned} \tag{9-7}$$

Since the latter identity only involves the first derivatives of ψ^m, up to an approximation argument we can choose η to be the radial Lipschitz function defined by

$$\eta(x) := \begin{cases} 1 & \text{if } x \in B_r,\\ \dfrac{r+\delta-|x|}{\delta} & \text{if } x \in B_{r+\delta}\setminus B_r,\\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_{r+\delta}.\end{cases}$$

In this way, we have

$$\nabla\eta(x) = -\frac{x}{\delta\,|x|}\,\chi_{B_{r+\delta}\setminus B_r}(x) \quad \text{and} \quad \nabla\psi^m(x) = e_m\eta(x) - \frac{x^m\,x}{\delta\,|x|}\,\chi_{B_{r+\delta}\setminus B_r}(x),$$

which also gives that

$$\operatorname{div}\psi(x) = n\eta(x) - \frac{|x|}{\delta}\,\chi_{B_{r+\delta}\setminus B_r}(x).$$

Therefore, we infer from (9-7) that

$$\begin{aligned}0 = 2\sum_{m=1}^{n}\int_{B_r}\big(2((\Delta u\nabla u_m)*\rho_\varepsilon) - \nabla F^m_\varepsilon\big)\cdot e_m - n\int_{B_r}\big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)*\rho_\varepsilon\big)\\ +2\sum_{m=1}^{n}\int_{B_{r+\delta}\setminus B_r}\big(2((\Delta u\nabla u_m)*\rho_\varepsilon) - \nabla F^m_\varepsilon\big)\cdot\Big(e_m\eta(x) - \frac{x^m\,x}{\delta\,|x|}\Big)\\ -\int_{B_{r+\delta}\setminus B_r}\big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)*\rho_\varepsilon\big)\Big(n\eta(x) - \frac{|x|}{\delta}\Big).\end{aligned}$$

Then, sending $\delta\to 0^+$, we deduce that

$$\begin{aligned}0 &= 2\sum_{m=1}^{n}\int_{B_r}\big(2((\Delta u\nabla u_m)*\rho_\varepsilon) - \nabla F^m_\varepsilon\big)\cdot e_m - n\int_{B_r}\big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)*\rho_\varepsilon\big)\\ &\qquad -2\sum_{m=1}^{n}\int_{\partial B_r}\big(2((\Delta u\nabla u_m)*\rho_\varepsilon) - \nabla F^m_\varepsilon\big)\cdot\frac{x^m\,x}{r} + r\int_{\partial B_r}\big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)*\rho_\varepsilon\big)\\ &= 2\sum_{m=1}^{n}\int_{B_r} G^m_\varepsilon\cdot e_m - n\int_{B_r}\big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)*\rho_\varepsilon\big)\\ &\qquad -2\sum_{m=1}^{n}\int_{\partial B_r} G^m_\varepsilon\cdot\frac{x^m\,x}{r} + r\int_{\partial B_r}\big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)*\rho_\varepsilon\big),\end{aligned} \tag{9-8}$$

where

$$G^m_\varepsilon := 2((\Delta u\nabla u_m)*\rho_\varepsilon) - \nabla F^m_\varepsilon. \tag{9-9}$$

Furthermore, letting

$$D_\varepsilon(r) := \frac{1}{r^n}\int_{B_r}\big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)*\rho_\varepsilon\big), \tag{9-10}$$

we have

$$D'_\varepsilon(r) = \frac{1}{r^n}\int_{\partial B_r}\big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)*\rho_\varepsilon\big) - \frac{n}{r^{n+1}}\int_{B_r}\big(\big(|\Delta u|^2 + \chi_{\{u>0\}}\big)*\rho_\varepsilon\big). \tag{9-11}$$

Thus, we multiply (9-8) by $1/r^{n+1}$, and we exploit (9-11) to conclude that

$$0 = \frac{2}{r^{n+1}}\sum_{m=1}^{n}\int_{B_r} G^m_\varepsilon\cdot e_m - 2\sum_{m=1}^{n}\int_{\partial B_r} G^m_\varepsilon\cdot\frac{x^m\,x}{r^{n+2}} + D'_\varepsilon(r) = 2Z_\varepsilon(r) + D'_\varepsilon(r), \tag{9-12}$$

where

$$Z_\varepsilon(r) := \frac{1}{r^{n+1}} \sum_{m=1}^{n} \int_{B_r} G_\varepsilon^m \cdot e_m - \sum_{m=1}^{n} \int_{\partial B_r} G_\varepsilon^m \cdot \frac{x^m x}{r^{n+2}}. \tag{9-13}$$

Now, in light of (9-3), we observe that ∇F_m (and thus ∇F_m^ε) involves third derivatives, and therefore we aim at "lowering the order of derivative" of this term from (9-13) in view of (9-9) (and this goal will be accomplished via a suitable averaging procedure). To this end, we observe that

$$\int_{B_r} \nabla F_\varepsilon^m \cdot e_m = \int_{B_r} \operatorname{div}(F_\varepsilon^m e_m) = \int_{\partial B_r} F_\varepsilon^m e_m \cdot \frac{x}{r} = \int_{\partial B_r} F_\varepsilon^m \frac{x^m}{r}. \tag{9-14}$$

We notice that the last term in (9-14) does not contain any third order derivatives. As for the boundary term in (9-8) that involves the third derivative, we have that

$$\begin{aligned}
\int_{\partial B_r} \nabla F_\varepsilon^m \cdot \frac{x^m x}{r^{n+2}} &= \int_{\partial B_1} \nabla F_\varepsilon^m(rx) \cdot \frac{x^m x}{r} \\
&= \int_{\partial B_1} \partial_r (F_\varepsilon^m(rx)) \cdot \frac{x^m}{r} \\
&= \frac{d}{dr}\left\{ \int_{\partial B_1} F_\varepsilon^m(rx) \frac{x^m}{r} \right\} + \int_{\partial B_1} F_\varepsilon^m(rx) \frac{x^m}{r^2} \\
&= \frac{d}{dr}\left\{ \int_{\partial B_r} F_\varepsilon^m \frac{x^m}{r^{n+1}} \right\} + \int_{\partial B_r} F_\varepsilon^m \frac{x^m}{r^{n+2}}.
\end{aligned}$$

As a consequence, using the latter identity, (9-9) and (9-14), we find that

$$\int_{B_r} G_\varepsilon^m \cdot e_m = 2\int_{B_r} ((\Delta u \nabla u_m) * \rho_\varepsilon) \cdot e_m - \int_{B_r} \nabla F_\varepsilon^m \cdot e_m = 2\int_{B_r} ((\Delta u \nabla u_m) * \rho_\varepsilon) \cdot e_m - \int_{\partial B_r} F_\varepsilon^m \frac{x^m}{r}$$

$$\begin{aligned}
\int_{\partial B_r} G_\varepsilon^m \cdot \frac{x^m x}{r^{n+2}} &= 2\int_{\partial B_r} ((\Delta u \nabla u_m) * \rho_\varepsilon) \cdot \frac{x^m x}{r^{n+2}} - \int_{\partial B_r} \nabla F_\varepsilon^m \cdot \frac{x^m x}{r^{n+2}} \\
&= 2\int_{\partial B_r} ((\Delta u \nabla u_m) * \rho_\varepsilon) \cdot \frac{x^m x}{r^{n+2}} - \int_{\partial B_r} F_\varepsilon^m \frac{x^m}{r^{n+2}} - \frac{d}{dr}\left\{ \int_{\partial B_r} F_\varepsilon^m \frac{x^m}{r^{n+1}} \right\}.
\end{aligned}$$

From this and (9-13), we obtain

$$\begin{aligned}
Z_\varepsilon(r) &= \frac{2}{r^{n+1}} \sum_{m=1}^{n} \int_{B_r} ((\Delta u \nabla u_m) * \rho_\varepsilon) \cdot e_m - 2\sum_{m=1}^{n} \int_{\partial B_r} ((\Delta u \nabla u_m) * \rho_\varepsilon) \cdot \frac{x^m x}{r^{n+2}} + \sum_{m=1}^{n} \frac{d}{dr}\left\{ \int_{\partial B_r} F_\varepsilon^m \frac{x^m}{r^{n+1}} \right\} \\
&= 2R_\varepsilon(r) + T_\varepsilon'(r),
\end{aligned}$$

with

$$\begin{aligned}
R_\varepsilon(r) &:= \frac{1}{r^{n+1}} \sum_{m=1}^{n} \int_{B_r} ((\Delta u \nabla u_m) * \rho_\varepsilon) \cdot e_m - \sum_{m=1}^{n} \int_{\partial B_r} ((\Delta u \nabla u_m) * \rho_\varepsilon) \cdot \frac{x^m x}{r^{n+2}}, \\
T_\varepsilon(r) &:= \sum_{m=1}^{n} \left\{ \int_{\partial B_r} F_\varepsilon^m \frac{x^m}{r^{n+1}} \right\} = \sum_{m=1}^{n} \left\{ \int_{\partial B_r} (\Delta u\, u_m) * \rho_\varepsilon \frac{x^m}{r^{n+1}} \right\},
\end{aligned} \tag{9-15}$$

where we have also used (9-3) and (9-5).

Consequently, integrating (9-12),

$$0=2\int_{r_1}^{r_2} Z_\varepsilon(r)\,dr+D_\varepsilon(r_2)-D_\varepsilon(r_1)=4\int_{r_1}^{r_2} R_\varepsilon(r)\,dr+2T_\varepsilon(r_2)-2T_\varepsilon(r_1)+D_\varepsilon(r_2)-D_\varepsilon(r_1). \tag{9-16}$$

Comparing (9-2) with (9-15), we see that $R_\varepsilon\to R$ and $T_\varepsilon\to T$ as $\varepsilon\to 0$, thanks to (4-1) and (4-2).

We thereby obtain the desired claim in (9-1) by passing to the limit the identity in (9-16). □

We also point out the following useful calculation:

Lemma 9.2. *In the notation stated by (9-2), we have that*

$$4\int_{r_1}^{r_2}\Big(\frac{1}{r^n}\int_{\partial B_r}\Delta u\,\Big(2\frac{u_r}{r}-\partial_r^2 u-2\frac{u}{r^2}\Big)\Big)\,dr-4V(r_2)+4V(r_1)+2T(r_2)-2T(r_1)+D(r_2)-D(r_1)=0, \tag{9-17}$$

where

$$V(r):=\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta u u. \tag{9-18}$$

Proof. For any smooth function v,

$$\begin{aligned}\int_{B_r}|\Delta v|^2&=\int_{B_r}(\operatorname{div}(\Delta v\nabla v)-\nabla\Delta v\cdot\nabla v)=\int_{\partial B_r}\Delta v\,v_r-\int_{B_r}\nabla\Delta v\cdot\nabla v\\&=\int_{\partial B_r}\Delta v\,v_r-\int_{B_r}\operatorname{div}(v\nabla\Delta v)+\int_{B_r}\Delta^2 v\,v=\int_{\partial B_r}\Delta v\,v_r-\int_{\partial B_r}v\,\Delta v_r+\int_{B_r}\Delta^2 v\,v.\end{aligned} \tag{9-19}$$

We also observe that

$$\begin{aligned}\frac{d}{dr}\Big(\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v v\Big)&=\frac{d}{dr}\Big(\frac{1}{r^2}\int_{\partial B_1}\Delta v(r\theta)v(r\theta)\Big)\\&=-\frac{2}{r^3}\int_{\partial B_1}\Delta v(r\theta)v(r\theta)+\frac{1}{r^2}\int_{\partial B_1}\Delta v_r(r\theta)v(r\theta)+\frac{1}{r^2}\int_{\partial B_1}\Delta v(r\theta)v_r(r\theta)\\&=-\frac{2}{r^{n+2}}\int_{\partial B_r}\Delta v\,v+\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v_r\,v+\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v\,v_r.\end{aligned}$$

From this and (9-19), we obtain that, for any smooth function v,

$$\begin{aligned}\frac{1}{r^{n+1}}\int_{B_r}|\Delta v|^2-\frac{1}{r^n}\int_{\partial B_r}\Delta v\partial_r^2 v&=\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v\,v_r-\frac{1}{r^{n+1}}\int_{\partial B_r}v\,\Delta v_r+\frac{1}{r^{n+1}}\int_{B_r}\Delta^2 v\,v-\frac{1}{r^n}\int_{\partial B_r}\Delta v\,\partial_r^2 v\\&=\frac{1}{r^n}\int_{\partial B_r}\Delta v\Big(2\frac{v_r}{r}-\partial_r^2 v-2\frac{v}{r^2}\Big)+\frac{1}{r^{n+1}}\int_{B_r}\Delta^2 v v-\frac{d}{dr}\Big(\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v v\Big).\end{aligned}$$

Integrating this identity and setting

$$V_v(r):=\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v v, \tag{9-20}$$

we thereby obtain that

$$\int_{r_1}^{r_2}\left(\frac{1}{r^{n+1}}\int_{B_r}|\Delta v|^2-\frac{1}{r^n}\int_{\partial B_r}\Delta v\,\partial_r^2 v\right)dr \\ =\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r}\Delta v\left(2\frac{v_r}{r}-\partial_r^2 v-2\frac{v}{r^2}\right)+\frac{1}{r^{n+1}}\int_{B_r}\Delta^2 v\,v\right)dr-V_v(r_2)+V_v(r_1). \tag{9-21}$$

The idea is now to take v as a mollification of u and use either (7-9) (if u is a minimizer) or Lemma 7.5 (if u is a one-phase minimizer). In this way, the term $\int_{B_r}\Delta^2 v\,v$ approaches either $\int_{B_r}u\,\mathcal{M}_u$, in the notation of (7-9) (if u is a minimizer), or 0 (if u is a one-phase minimizer, due to Lemma 7.5).

To make the notation uniform, we therefore define $\mathcal{M}_u^*:=\mathcal{M}_u$ if u is a minimizer and $\mathcal{M}_u^*:=0$ if u is a one-phase minimizer: then, approximating u, passing to the limit (9-21), and comparing (9-20) with (9-18), we can write

$$\int_{r_1}^{r_2}\left(\frac{1}{r^{n+1}}\int_{B_r}|\Delta u|^2-\frac{1}{r^n}\int_{\partial B_r}\Delta u\,\partial_r^2 u\right)dr \\ =\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r}\Delta u\left(2\frac{u_r}{r}-\partial_r^2 u-2\frac{u}{r^2}\right)-\frac{1}{r^{n+1}}\int_{B_r}u\,\mathcal{M}_u^*\right)dr-V(r_2)+V(r_1).$$

That is, recalling (9-2),

$$\int_{r_1}^{r_2}R(r)\,dr=\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r}\Delta u\left(2\frac{u_r}{r}-\partial_r^2 u-2\frac{u}{r^2}\right)-\frac{1}{r^{n+1}}\int_{B_r}u\,\mathcal{M}_u^*\right)dr-V(r_2)+V(r_1).$$

From this and (9-1) we obtain that

$$2T(r_1)-2T(r_2)+D(r_1)-D(r_2) \\ =4\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r}\Delta u\left(2\frac{u_r}{r}-\partial_r^2 u-2\frac{u}{r^2}\right)-\frac{1}{r^{n+1}}\int_{B_r}u\,\mathcal{M}_u^*\right)dr-4V(r_2)+4V(r_1). \tag{9-22}$$

Now we claim that

$$\int_{B_r}u\,\mathcal{M}_u^*=0. \tag{9-23}$$

For this, since $\mathcal{M}_u^*=0$ in the one-phase problem, we can suppose that u is a minimizer, in which case $\mathcal{M}_u^*=\mathcal{M}_u$. Then, let us fix $\delta\in(0,1)$. From Lemma 4.1, we know that

$$-\int_{B_r\cap\{|u|\geqslant\delta\}}u\,\mathcal{M}_u=\int_{B_r\cap\{u\geqslant\delta\}}u\,\Delta^2 u+\int_{B_r\cap\{u\leqslant-\delta\}}u\,\Delta^2 u=0.$$

Therefore, exploiting Lemma 7.4,

$$\left|\int_{B_r}u\,\mathcal{M}_u\right|=\left|\int_{B_r\cap\{|u|<\delta\}}u\,\mathcal{M}_u\right|\leqslant\delta\mathcal{M}_u(B_r)\leqslant C\delta r^{n-2},$$

for some $C>0$. Then, sending $\delta\to 0^+$, we obtain (9-23) as desired.

Then, the identities in (9-22) and (9-23) lead to (9-17). □

Now we restrict the previous calculations to the case $n=2$, and we complete the proof of (1-21).

Proof of (1-21). Using polar coordinates (r,θ), we compute

$$
\begin{aligned}
-\frac{1}{r^n}\int_{\partial B_r}\Delta u\Big(2\frac{u_r}{r}-\partial_r^2 u-2\frac{u}{r^2}\Big) &= \int_{\partial B_1}\frac{1}{r}\Delta u\Big(u_{rr}-2\frac{u_r}{r}+2\frac{u}{r^2}\Big)\\
&= \int_{\partial B_1}\frac{1}{r}\Big(u_{rr}+\frac{u_r}{r}+\frac{u_{\theta\theta}}{r^2}\Big)\Big(u_{rr}-2\frac{u_r}{r}+2\frac{u}{r^2}\Big)=A(r)+B(r),
\end{aligned}
\tag{9-24}
$$

where

$$
A(r):=\int_{\partial B_1}\frac{1}{r^3}u_{\theta\theta}\Big(u_{rr}-2\frac{u_r}{r}+2\frac{u}{r^2}\Big)\quad\text{and}\quad B(r):=\int_{\partial B_1}\frac{1}{r}\Big(u_{rr}+\frac{u_r}{r}\Big)\Big(u_{rr}-2\frac{u_r}{r}+2\frac{u}{r^2}\Big).
\tag{9-25}
$$

Now, we perform several integrations by parts involving the terms related to $A(r)$. First of all, we see that

$$
\begin{aligned}
\frac{1}{r^3}\int_{\partial B_1}u_{\theta\theta}u_{rr} &= -\frac{1}{r^3}\int_{\partial B_1}u_\theta u_{\theta rr}\\
&= -\frac{d}{dr}\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}+\int_{\partial B_1}\frac{u_{r\theta}^2}{r^3}-3\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}.
\end{aligned}
\tag{9-26}
$$

Similarly, we have that

$$
-2\int_{\partial B_1}\frac{1}{r^4}u_{\theta\theta}u_r=2\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}=2\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}
\tag{9-27}
$$

and

$$
2\int_{\partial B_1}\frac{1}{r^5}u_{\theta\theta}u=-2\int_{\partial B_1}\frac{u_\theta^2}{r^5}.
\tag{9-28}
$$

Combining (9-26), (9-27), and (9-28), and recalling (9-25), we get

$$
\begin{aligned}
A(r) &= -\frac{d}{dr}\Big(\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}\Big)+\int_{\partial B_1}\frac{u_{r\theta}^2}{r^3}-3\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}+2\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}-2\int_{\partial B_1}\frac{u_\theta^2}{r^5}\\
&= -\frac{d}{dr}\Big(\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}\Big)+\int_{\partial B_1}\frac{u_{r\theta}^2}{r^3}-\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}-2\int_{\partial B_1}\frac{u_\theta^2}{r^5}\\
&= -\frac{d}{dr}\Big(\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}\Big)+\int_{\partial B_1}\frac{1}{r^3}\Big(u_{\theta r}-\frac{2u_\theta}{r}\Big)^2+3\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}-6\int_{\partial B_1}\frac{u_\theta^2}{r^5}\\
&= -\frac{d}{dr}\Big(\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}+\frac{3}{2}\int_{\partial B_1}\frac{u_\theta^2}{r^4}\Big)+\int_{\partial B_1}\frac{1}{r^3}\Big(u_{\theta r}-\frac{2u_r}{r}\Big)^2\\
&= -\frac{d}{dr}\Big(\int_{\partial B_r}\frac{u_\theta u_{r\theta}}{r^4}+\frac{3}{2}\int_{\partial B_r}\frac{u_\theta^2}{r^5}\Big)+\int_{\partial B_r}\frac{1}{r^4}\Big(u_{\theta r}-\frac{2u_r}{r}\Big)^2.
\end{aligned}
\tag{9-29}
$$

From (9-25), we also compute that

$$
\begin{aligned}
B(r) &= \int_{\partial B_1} \frac{1}{r}\left(u_{rr}^2 - \frac{2u_{rr}u_r}{r} + \frac{2uu_{rr}}{r^2} + \frac{u_r u_{rr}}{r} - \frac{2u_r^2}{r^2} + \frac{2uu_r}{r^3}\right) \\
&= \int_{\partial B_1} \frac{1}{r}\left(u_{rr}^2 - \frac{u_{rr}u_r}{r} + \frac{2uu_{rr}}{r^2} - \frac{2u_r^2}{r^2} + \frac{2uu_r}{r^3}\right) \\
&= \int_{\partial B_1} \frac{1}{r}\left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2 + \frac{1}{r}\left(\frac{5u_r u_{rr}}{r} - \frac{6uu_{rr}}{r^2} - \frac{11u_r^2}{r^2} + \frac{26uu_r}{r^3} - \frac{16u^2}{r^4}\right) \\
&= \int_{\partial B_1} \frac{1}{r}\left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2 + \frac{d}{dr}\left(\int_{\partial B_1} \frac{5u_r^2}{2r^2} - \int_{\partial B_1} \frac{6uu_r}{r^3} + \int_{\partial B_1} \frac{4u^2}{r^4}\right) \\
&= \int_{\partial B_r} \frac{1}{r^2}\left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2 + \frac{d}{dr}\left(\int_{\partial B_r} \frac{5u_r^2}{2r^3} - \int_{\partial B_r} \frac{6uu_r}{r^4} + \int_{\partial B_r} \frac{4u^2}{r^5}\right).
\end{aligned} \tag{9-30}
$$

Using (9-29) and (9-30), we conclude that

$$
A(r) + B(r) = \frac{1}{r^2}\int_{\partial B_r}\left[\left(\frac{u_{\theta r}}{r} - \frac{2u_r}{r^2}\right)^2 + \left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2\right] + W'(r), \tag{9-31}
$$

where

$$
W(r) := \int_{\partial B_r}\left(\frac{5u_r^2}{2r^3} - \frac{6uu_r}{r^4} + \frac{4u^2}{r^5} - \frac{u_\theta u_{r\theta}}{r^4} - \frac{3u_\theta^2}{2r^5}\right). \tag{9-32}
$$

Now, from (9-17) and (9-24), we see that

$$
\begin{aligned}
-4V(r_2) + 4V(r_1) + 2T(r_2) - 2T(r_1) + D(r_2) - D(r_1) &= -4\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r} \Delta u\left(2\frac{u_r}{r} - \partial_r^2 u - 2\frac{u}{r^2}\right)\right) dr \\
&= 4\int_{r_1}^{r_2} (A(r) + B(r))\, dr.
\end{aligned}
$$

This and (9-31) give that

$$
\begin{aligned}
-V(r_2) + V(r_1) + \frac{T(r_2) - T(r_1)}{2} &+ \frac{D(r_2) - D(r_1)}{4} - W(r_2) + W(r_1) \\
&= \int_{r_1}^{r_2}\left\{\frac{1}{r^2}\int_{\partial B_r}\left[\left(\frac{u_{\theta r}}{r} - \frac{2u_r}{r^2}\right)^2 + \left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2\right]\right\}.
\end{aligned} \tag{9-33}
$$

Recalling (1-22), (9-2), (9-18), and (9-32), we see that

$$
\begin{aligned}
&-V(r) + \frac{T(r)}{2} + \frac{D(r)}{4} - \int_{\partial B_r}\left(\frac{5u_r^2}{2r^3} - \frac{6uu_r}{r^4} + \frac{4u^2}{r^5} - \frac{u_\theta u_{r\theta}}{r^4} - \frac{3u_\theta^2}{2r^5}\right) \\
&= -\frac{1}{r^3}\int_{\partial B_r} \Delta u u + \frac{1}{2r^2}\int_{\partial B_r} \Delta u\, \partial_r u + \frac{1}{4r^2}\int_{B_r}\left(|\Delta u|^2 + \chi_{\{u>0\}}\right) - \int_{\partial B_r}\left(\frac{5u_r^2}{2r^3} - \frac{6uu_r}{r^4} + \frac{4u^2}{r^5} - \frac{u_\theta u_{r\theta}}{r^4} - \frac{3u_\theta^2}{2r^5}\right) \\
&= E(r).
\end{aligned}
$$

This and (9-33) establish (1-21), as desired. □

Now, since the proof of (1-21) has been completed, to finish the proof of Theorem 1.12, we only need to show that the function E defined in (1-22) is bounded and to check that if E is constant then u is a homogeneous function of degree two.

These goals will be accomplished by the following arguments:

Proof of the boundedness of E. To show that E is bounded, we claim that there exist $C > 0$ and a sequence $r_k \to 0^+$ such that

$$\int_{\partial B_{r_k}} \left(\frac{|\nabla u|^2}{r_k^3} + \frac{|D^2 u|^2}{r_k} \right) \leqslant C. \tag{9-34}$$

The proof of (9-34) needs to distinguish the case in which u is a minimizer from the case in which u is a one-phase minimizer. Suppose first that u is a one-phase minimizer. Then, since $u(0) = 0 \leqslant u(x)$ for any $x \in \Omega$ and u is assumed to be $C^{1,1}(\Omega)$, we can write $|\nabla u(x)| \leqslant C|x|$ and $|D^2 u(x)| \leqslant C$, for some $C > 0$, from which (9-34) plainly follows in this case.

Now, we prove (9-34) assuming that u is a minimizer. We argue by contradiction, supposing that (9-34) does not hold. Then, for any $\bar{C} > 0$ there exists $\bar{r} \in (0, 1)$ such that for any $r \in (0, \bar{r})$ we have

$$\int_{\partial B_r} \left(\frac{|\nabla u|^2}{r^3} + \frac{|D^2 u|^2}{r} \right) \geqslant \bar{C}.$$

This, Corollary A.2 (if u is a minimizer) or the fact that u is assumed to be in $C^{1,1}(\Omega)$ (if u is a one-phase minimizer) imply that, for a suitable $C > 0$,

$$\begin{aligned}
C &\geqslant \frac{1}{\bar{r}^4} \int_{B_{\bar{r}}} |\nabla u|^2 + \frac{1}{\bar{r}^2} \int_{B_{\bar{r}}} |D^2 u|^2 \\
&= \frac{1}{\bar{r}^4} \int_0^{\bar{r}} \left(\int_{\partial B_r} |\nabla u|^2 \right) dr + \frac{1}{\bar{r}^2} \int_0^{\bar{r}} \left(\int_{\partial B_r} |D^2 u|^2 \right) dr = \frac{1}{\bar{r}} \int_0^{\bar{r}} \left(\int_{\partial B_r} \frac{|\nabla u|^2}{\bar{r}^3} + \int_{\partial B_r} \frac{|D^2 u|^2}{\bar{r}} \right) dr \\
&\geqslant \frac{1}{\bar{r}} \int_{\bar{r}/2}^{\bar{r}} \left(\int_{\partial B_r} \frac{|\nabla u|^2}{\bar{r}^3} + \int_{\partial B_r} \frac{|D^2 u|^2}{\bar{r}} \right) dr \\
&\geqslant \frac{1}{8\bar{r}} \int_{\bar{r}/2}^{\bar{r}} \left(\int_{\partial B_r} \frac{|\nabla u|^2}{r^3} + \int_{\partial B_r} \frac{|D^2 u|^2}{r} \right) dr \\
&\geqslant \frac{\bar{C}}{16},
\end{aligned}$$

which is a contradiction if $\bar{C}$ is suitably large, and this establishes (9-34).

As a consequence, using the Cauchy–Schwarz inequality, Theorem 1.7, and (9-34),

$$\begin{aligned}
&\int_{\partial B_{r_k}} \left| \frac{\Delta u\, u_r}{2r_k^2} - \frac{5u_r^2}{2r_k^3} - \frac{\Delta u u}{r_k^3} + \frac{6uu_r}{r_k^4} + \frac{u_\theta u_{\theta r}}{r_k^4} - \frac{4u^2}{r_k^5} - \frac{3u_\theta^2}{2r_k^5} \right| \\
&\leqslant C \int_{\partial B_{r_k}} \left(\frac{|D^2 u|\,|\nabla u|}{r_k^{1/2} r_k^{3/2}} + \frac{|\nabla u|^2}{r_k^3} + \frac{|\Delta u|}{r_k^{1/2} r_k^{1/2}} + \frac{|\nabla u|}{r_k^{3/2} r_k^{1/2}} + \frac{1}{r_k} \right) \leqslant C \int_{\partial B_{r_k}} \left(\frac{|\nabla u|^2}{r_k^3} + \frac{|D^2 u|^2}{r_k} + \frac{1}{r_k} \right) \leqslant C,
\end{aligned}$$

for some $C > 0$, possibly varying from line to line.

Using this, (1-22), and Corollary A.2 (if u is a minimizer) or the assumption that $u \in C^{1,1}(\Omega)$ (if u is a one-phase minimizer), we deduce that

$$\begin{aligned}|E(r_k)| &\leqslant \int_{\partial B_{r_k}} \left| \frac{\Delta u\, u_r}{2r_k^2} - \frac{5u_r^2}{2r_k^3} - \frac{\Delta u u}{r_k^3} + \frac{6uu_r}{r_k^4} + \frac{u_\theta u_{\theta r}}{r_k^4} - \frac{4u^2}{r_k^5} - \frac{3u_\theta^2}{2r_k^5} \right| + \frac{1}{4r_k^2} \int_{B_{r_k}} \big(|\Delta u|^2 + \chi_{\{u>0\}}\big) \\ &\leqslant C + \frac{1}{4r_k^2} \int_{B_{r_k}} \chi_{\{u>0\}} \leqslant C, \end{aligned} \tag{9-35}$$

up to renaming $C > 0$.

Now, fix $r \in (0, 1)$. Let $\bar{k}$ sufficiently large, such that $r_{\bar{k}} \in (0, r)$. From (1-21), we know that $E(r_{\bar{k}}) \leqslant E(r) \leqslant E(1)$. Hence, by (9-35),

$$-C \leqslant E(r) \leqslant E(1). \qquad \square$$

Having already checked the validity of the monotonicity formula in (1-21) and the fact that E is bounded, in order to complete the proof of Theorem 1.12, we only need to show that if E is constant in $(0, \tau)$, then u is a homogeneous function of degree two. This is now a simple consequence of (1-21). The detailed argument goes as follows:

Proof of the case of constant E. Suppose now that E is constant in $(0, \tau)$. Then, by (1-21),

$$-\frac{\partial}{\partial\theta}\left(-\frac{u_r}{r} + \frac{2u}{r^2}\right) = \frac{u_{r\theta}}{r^2} - \frac{2u_\theta}{r} = 0 \quad \text{and} \quad -r\,\frac{\partial}{\partial r}\left(-\frac{u_r}{r} + \frac{2u}{r^2}\right) = u_{rr} - \frac{3u_r}{r} + \frac{4u}{r^2} = 0,$$

which, in turn, gives that

$$\nabla\left(-\frac{u_r}{r} + 2\frac{u}{r^2}\right) = 0.$$

Consequently, the function $-\dfrac{u_r}{r} + \dfrac{2u}{r^2}$ is constant for $|x| \in (0, \tau)$, hence we write

$$-\frac{u_r}{r} + \frac{2u}{r^2} = c, \tag{9-36}$$

for some $c \in \mathbb{R}$.

Now we define

$$v(r, \theta) := u(r, \theta) + cr^2 \log r. \tag{9-37}$$

Using (9-36), we obtain

$$v_r = u_r + 2cr \log r + cr = \frac{2u}{r} + 2cr \log r = \frac{2v}{r}.$$

Integrating this equation, fixed $\bar{r} \in (0, \tau)$, we find that

$$v(r, \theta) = \frac{r^2\, v(\bar{r}, \theta)}{\bar{r}^2}.$$

This and (9-37) provide

$$u(r, \theta) = \frac{r^2\, v(\bar{r}, \theta)}{\bar{r}^2} - cr^2 \log r.$$

Therefore, exploiting Theorem 1.7 (if u is a minimizer) or the assumption that $u \in C^{1,1}(\Omega)$ (if u is a one-phase minimizer),

$$C \geqslant \frac{|u(r,\theta)|}{r^2} \geqslant |c|\,|\log r| - \frac{|v(\bar r,\theta)|}{\bar r^2},$$

for some $C > 0$, and therefore,

$$|c| \leqslant \lim_{r\to 0} \frac{|v(\bar r,\theta)|}{\bar r^2\,|\log r|} + \frac{C}{|\log r|} = 0.$$

Hence, we get that $c = 0$ and, as a consequence, we can write (9-36) as

$$-\frac{u_r}{r} + \frac{2u}{r^2} = 0$$

for any $x \in B_\tau$, and therefore $\nabla u(x)\cdot x = 2u(x)$ for any $x \in B_\tau$. Observing that this is the Euler equation for homogeneous functions of degree two, we thus obtain the homogeneity of u. The proof of Theorem 1.12 is thereby complete. □

We finish this section by an explicit computation of the energy, E, for the homogeneous functions of degree two on the plane. It will be used later in the proof of Theorem 1.14.

Lemma 9.3. *Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a cone in $\mathbb{R}^2$, written in polar coordinates as*

$$\mathcal{C} = \{(r,\theta) \in (0,+\infty) \times (\theta_1,\theta_2)\},$$

for some $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$.

Let $u : \mathcal{C} \to \mathbb{R}$ be a homogeneous function of the form $u(x) = r^2 g(\theta)$, with $g \in C^2([\theta_1,\theta_2])$, $g > 0$ in (θ_1,θ_2), and

$$g(\theta_1) = g(\theta_2) = 0 \quad \textit{and} \quad g'(\theta_1) = g'(\theta_2) = 0.$$

Assume also that Δu is constant in $\mathcal{C}$. Then, for any $r > 0$,

$$\int_{\mathcal{C}\cap\partial B_r}\Big(\frac{\Delta u\, u_r}{2r^2} - \frac{5u_r^2}{2r^3} - \frac{\Delta u u}{r^3} + \frac{6uu_r}{r^4} + \frac{u_\theta u_{\theta r}}{r^4} - \frac{4u^2}{r^5} - \frac{3u_\theta^2}{2r^5}\Big) + \frac{1}{4r^2}\int_{\mathcal{C}\cap B_r}\big(|\Delta u|^2 + \chi_{\{u>0\}}\big) = \frac{\pi}{4}\,\frac{|\{u>0\}\cap B_r|}{|B_r|} = \frac{\theta_2-\theta_1}{8}. \tag{9-38}$$

Proof. By assumption, in $\mathcal{C}$ we have that

$$C_0 = \Delta u = 4g + g'', \tag{9-39}$$

for some $C_0 \in \mathbb{R}$, and

$$\frac{\Delta u\, u_r}{2r^2} - \frac{5u_r^2}{2r^3} - \frac{\Delta u u}{r^3} + \frac{6uu_r}{r^4} + \frac{u_\theta u_{\theta r}}{r^4} - \frac{4u^2}{r^5} - \frac{3u_\theta^2}{2r^5} = \frac{(4g+g'')g}{r} - \frac{10g^2}{r} - \frac{(4g+g'')g}{r} + \frac{12g^2}{r} + \frac{2(g')^2}{r} - \frac{4g^2}{r} - \frac{3(g')^2}{2r} = -\frac{2g^2}{r} + \frac{(g')^2}{2r}.$$

Therefore, after an integration by parts, and recalling (9-39), we have that

$$\begin{aligned}\int_{\mathcal{C}\cap\partial B_r}\Big(\frac{\Delta u\, u_r}{2r^2}-\frac{5u_r^2}{2r^3}-\frac{\Delta u u}{r^3}+\frac{6uu_r}{r^4}+\frac{u_\theta u_{\theta r}}{r^4}-\frac{4u^2}{r^5}-\frac{3u_\theta^2}{2r^5}\Big)&\\ =\int_{\theta_1}^{\theta_2}\Big(-2g^2+\frac{(g')^2}{2}\Big)=\int_{\theta_1}^{\theta_2}\Big(-2g^2-\frac{g''g}{2}\Big)&=-\frac{1}{2}\int_{\theta_1}^{\theta_2} g(4g+g'')\\ =-\frac{C_0}{2}\int_{\theta_1}^{\theta_2} g=\frac{C_0}{8}\int_{\theta_1}^{\theta_2}(g''-C_0)&=-\frac{C_0^2\,(\theta_2-\theta_1)}{8}.\end{aligned}\tag{9-40}$$

On the other hand,

$$\frac{1}{4r^2}\int_{\mathcal{C}\cap B_r}|\Delta u|^2=\frac{1}{8}\int_{\theta_1}^{\theta_2}(4g+g'')^2=\frac{C_0^2\,(\theta_2-\theta_1)}{8}.$$

This and (9-40) provide

$$\int_{\mathcal{C}\cap\partial B_r}\Big(\frac{\Delta u\, u_r}{2r^2}-\frac{5u_r^2}{2r^3}-\frac{\Delta u u}{r^3}+\frac{6uu_r}{r^4}+\frac{u_\theta u_{\theta r}}{r^4}-\frac{4u^2}{r^5}-\frac{3u_\theta^2}{2r^5}\Big)+\frac{1}{4r^2}\int_{\mathcal{C}\cap B_r}\big(|\Delta u|^2+\chi_{\{u>0\}}\big)=\frac{1}{4r^2}\int_{B_r}\chi_{\{u>0\}},$$

which proves (9-38). □

10. Monotonicity formula: homogeneity of the blow-up limits and proof of Theorem 1.13

In this section, we apply the results in Theorem 1.12 to study the homogeneity properties of the blow-up limits of the minimizers of J at free boundary points with vanishing gradient, thus proving Theorem 1.13.

Proof of Theorem 1.13. Suppose that u does not vanish identically. We let

$$Q(u,x):=Q(u,r,\theta)=\Big(-\frac{u_{r\theta}}{r}+2\frac{u_\theta}{r^2}\Big)^2+\Big(u_{rr}-3\frac{u_r}{r}+4\frac{u}{r^2}\Big)^2.\tag{10-1}$$

Note that Q is invariant with respect to quadratic scaling. Indeed, if we define, for any $s>0$,

$$u_s(x):=\frac{u(sx)}{s^2},$$

we have that

$$\begin{aligned}Q(u_s,x)&=\Big(-\frac{(u_s)_{r\theta}}{r}+2\frac{(u_s)_\theta}{r^2}\Big)^2+\Big((u_s)_{rr}-3\frac{(u_s)_r}{r}+4\frac{u_s}{r^2}\Big)^2\\ &=\Big(-\frac{u_{r\theta}(sx)}{sr}+2\frac{u_\theta(sx)}{(sr)^2}\Big)^2+\Big(u_{rr}(sx)-3\frac{u_r(sx)}{sr}+4\frac{u(sx)}{(sr)^2}\Big)^2=Q(u,sx).\end{aligned}\tag{10-2}$$

Now, in view of (1-21) and (10-1), we observe that

$$\begin{aligned}E(\tau_2)-E(\tau_1)&=\int_{\tau_1}^{\tau_2}\Big\{\frac{1}{r^2}\int_{\partial B_r}\Big[\Big(\frac{u_{\theta r}}{r}-\frac{2u_\theta}{r^2}\Big)^2+\Big(u_{rr}-\frac{3u_r}{r}+\frac{4u}{r^2}\Big)^2\Big]\Big\}\,dr\\ &=\int_{\tau_1}^{\tau_2}\Big(\frac{1}{r^2}\int_{\partial B_r}Q(u,x)\,dx\Big)\,dr.\end{aligned}\tag{10-3}$$

As a consequence, for any $s > 0$, using the changes of variables $\rho = r/s$ and $y = x/s$, and making use of (10-2), we see that

$$\begin{aligned} E(s\tau_2) - E(s\tau_1) &= \int_{s\tau_1}^{s\tau_2} \left(\frac{1}{r^2} \int_{\partial B_r} Q(u, x)\, dx \right) dr \\ &= \int_{\tau_1}^{\tau_2} \left(\frac{1}{\rho^2} \int_{\partial B_\rho} Q(u, sy)\, dy \right) d\rho \\ &= \int_{\tau_1}^{\tau_2} \left(\frac{1}{\rho^2} \int_{\partial B_\rho} Q(u_s, y)\, dy \right) d\rho. \end{aligned} \tag{10-4}$$

On the other hand, by Theorem 1.12, we know that E is monotone and bounded, and therefore the limit as $\vartheta \to 0^+$ of $E(\vartheta)$ exists and it is finite. Consequently, we have that

$$E(s\tau_2) - E(s\tau_1) \to 0, \quad \text{as } s \to 0.$$

Hence, recalling (10-4), we conclude that

$$\int_{\tau_1}^{\tau_2} \left(\frac{1}{\rho^2} \int_{\partial B_\rho} Q(u_s, y)\, dy \right) d\rho \to 0, \quad \text{as } s \to 0. \tag{10-5}$$

Also, by compactness (ensured here, if u is a minimizer, by (1-24), which in turns allows us to exploit Corollary A.2, and, if u is a one-phase minimizer by the assumption that $u \in C^{1,1}(\Omega)$), we have that u_s converges to some u_0, up to a subsequence. Therefore, by (10-5),

$$\int_{\tau_1}^{\tau_2} \left(\frac{1}{\rho^2} \int_{\partial B_\rho} Q(u_0, y)\, dy \right) d\rho = 0$$

for all $\tau_2 > \tau_1 > 0$. Thus, since $Q \geqslant 0$, due to (10-1), it follows that $Q(u_0, y) = 0$. Consequently, by (10-3), we have that the function E relative to the minimizer u_0 is identically constant. Therefore, in view of the last claim in Theorem 1.12, it follows that u_0 is a homogeneous function of degree two. □

11. Regularity of the free boundary in two dimensions: explicit computations, classification results in $2D$, and proof of Theorem 1.14

In this section we study the regularity of free boundary of minimizers in dimension 2. Some of the results presented rely on direct calculations, while others are obtained by the monotone quantity E that has been analyzed in Theorems 1.12 and 1.13. In this setting, we have the following classification result for one-phase minimizers:

Theorem 11.1. *Let $u \in C^1(\mathbb{R}^n)$ be a one-phase local minimizer in any ball of $\mathbb{R}^n$, with $0 \in \partial_{\text{sing}}\{u > 0\}$. Let $u = r^2 g(\theta)$, where (r, θ) denotes the polar coordinates. Then, the following dichotomy holds:*

- *either u is a homogeneous polynomial of degree two,*
- *or, up to a rotation, $u(x) = a(x_1^+)^2$ for some $a > 0$.*

Proof. A direct computation shows that

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{1}{r^2}\,\Delta_{\mathbb{S}^1} u = 2g + 2g + g'' = g'' + 4g. \tag{11-1}$$

Accordingly, by Lemma 4.1, we have that, in the positivity set of u,

$$r^2 \Delta^2 u = \frac{d^2}{d\theta^2}(g'' + 4g) = 0.$$

From this, we deduce that

$$g''(\theta) + 4g(\theta) = c_1\theta + c_2, \quad \text{for all } \theta \in \{g \neq 0\}, \tag{11-2}$$

for some constants c_1 and c_2. We notice that (11-2) has explicit solution

$$g(\theta) = \frac{c_1\theta}{4} + \frac{c_2}{4} + c_3\cos(2\theta) + c_4\sin(2\theta) = \frac{c_1\theta}{4} + \frac{c_2}{4} + c_3(\cos^2\theta - \sin^2\theta) + 2c_4\sin\theta\cos\theta \tag{11-3}$$

for some constants c_3 and c_4.

Since 0 is a free boundary point for u, we have that g cannot vanish identically. Hence, we distinguish some cases, depending on the number of zeros of g. First of all, we consider the cases in which either $g > 0$ for all $\theta \in [0, 2\pi)$ or g vanishes only at one point. Then, in this case the free boundary is contained in a ray and, up to a rotation, we can assume that $g(\theta) > 0$ for all $\theta \in (0, 2\pi)$ and so (11-3) is valid for all $\theta \in (0, 2\pi)$. The periodicity of g then implies that

$$0 = \lim_{\theta\to 0^+} g(\theta) - \lim_{\theta\to 2\pi^-} g(\theta) = -\frac{c_1\pi}{2},$$

and so $c_1 = 0$. As a consequence, by (11-3),

$$u(r,\theta) = \frac{c_2 r^2}{4} + c_3 r^2(\cos^2\theta - \sin^2\theta) + 2r^2 c_4\sin\theta\cos\theta = \frac{c_2(x_1^2 + x_2^2)}{4} + c_3(x_1^2 - x_2^2) + 2c_4 x_1 x_2,$$

which is a homogeneous polynomial of degree two, thus proving the desired claim in this case.

Now we suppose that g vanishes at least at two points, say, up to a rotation, θ_0 and $-\theta_0$, for some $\theta_0 \in (0, \pi)$, that is

$$g(\theta) > 0 \text{ for all } \theta \in (-\theta_0, \theta_0) \quad \text{and} \quad g(\theta_0) = g(-\theta_0) = 0. \tag{11-4}$$

Then, by (11-3),

$$0 = g(\pm\theta_0) = \pm\frac{c_1\theta_0}{4} + \frac{c_2}{4} + c_3\cos(2\theta_0) \pm c_4\sin(2\theta_0). \tag{11-5}$$

By the assumptions that $u \in C^1(\mathbb{R}^n)$ and $g \geqslant 0$, we also know that

$$0 = g'(\pm\theta_0) = \frac{c_1}{4} \mp 2c_3\sin(2\theta_0) + 2c_4\cos(2\theta_0). \tag{11-6}$$

Then, we obtain from (11-5) and (11-6) the system

$$\begin{cases} \dfrac{c_1\theta_0}{4} + c_4 \sin(2\theta_0) = 0, \\ \dfrac{c_2}{4} + c_3 \cos(2\theta_0) = 0, \\ c_3 \sin(2\theta_0) = 0, \\ \dfrac{c_1}{4} + 2c_4 \cos(2\theta_0) = 0. \end{cases} \tag{11-7}$$

Now, if

$$\theta_0 \neq \pi/2, \tag{11-8}$$

from (11-7), we have that necessarily $c_3 = 0$, and accordingly

$$\begin{cases} \dfrac{c_1\theta_0}{4} + c_4 \sin(2\theta_0) = 0, \\ \dfrac{c_2}{4} = 0, \\ \dfrac{c_1}{4} + 2c_4 \cos(2\theta_0) = 0. \end{cases}$$

This implies that $c_2 = 0$, and so (11-3) becomes

$$g(\theta) = \frac{c_1\theta}{4} + c_4 \sin(2\theta).$$

In particular $g(0) = 0$, which is in contradiction with (11-4).

This says that the case in (11-8) must be ruled out, and thus $\theta_0 = \pi/2$ (and the positivity sets of u are either one or two half-planes). In this way, the system in (11-7) reduces to

$$\begin{cases} \dfrac{c_1\pi}{8} = 0, \\ \dfrac{c_2}{4} - c_3 = 0, \\ \dfrac{c_1}{4} - 2c_4 = 0, \end{cases}$$

which leads to $c_1 = c_4 = 0$ and $c_2/4 = c_3$. Substituting these conditions into (11-3), we obtain that, for all $\theta \in (-\pi/2, \pi/2)$,

$$g(\theta) = c_3(1 + \cos(2\theta)) = c_3(1 + \cos^2\theta - \sin^2\theta),$$

and therefore, for all $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 > 0$,

$$u(x) = 2c_3 x_1^2.$$

This gives that either u is a homogeneous polynomial of degree two, or $u(x) = a\,(x_1^+)^2$ for some $a > 0$, or

$$u(x) = \begin{cases} ax_1^2 & \text{if } x_1 \geqslant 0, \\ bx_1^2 & \text{if } x_1 < 0, \end{cases}$$

with $a, b \in (0, +\infty)$ and

$$a \neq b. \tag{11-9}$$

To complete the proof of the desired result, we need to exclude this case. To this end, we observe that

$$\big(|\Delta u(0^+,1)|^2+1\big)-2\big(\Delta u(0^+,1)u_{11}(0^+,1)-u_1(0^+,1)\Delta u(0^+,1)\big)=((2a)^2+1)-2((2a)^2+0)=1-4a^2,$$

and similarly,

$$\big(|\Delta u(0^-,1)|^2+1\big)-2\big(\Delta u(0^-,1)u_{11}(0^-,1)-u_1(0^-,1)\Delta u(0^-,1)\big)=1-4b^2.$$

These identities and the free boundary condition (1-9) computed at the point $(0,1)$, where according to the definition in (1-6) we have $\lambda^{(1)}=\lambda^{(2)}=1$, lead to

$$1-4a^2=1-4b^2,$$

which provides $a^2=b^2$, and thus $a=b$. This contradicts with (11-9), and the desired result follows. □

With this, we are now in the position of completing the proof of Theorem 1.14.

Proof of Theorem 1.14. Let E be as in Theorem 1.12, and let[4]

$$E(0):=\lim_{\rho\to 0^+}E(\rho). \tag{11-10}$$

Let $\bar{x}\in\partial\{u>0\}$. Suppose that $u_{0,\bar{x}}$ is a blow-up of u at $\bar{x}$. Notice that $u_{0,\bar{x}}$ cannot be identically equal to zero, due to (1-26). Then by Theorem 11.1 we know that, after some rotation of coordinates,

$$u_{0,\bar{x}}\text{ must be either }\quad \frac{a_1(x_1-\bar{x}_1)^2+a_2(x_2-\bar{x}_2)^2}{2},\quad \frac{a(x_1-\bar{x}_1)^2}{2},\quad\text{or}\quad \frac{a((x_1-\bar{x}_1)^+)^2}{2}, \tag{11-11}$$

with $a_1,a_2,a>0$ (say, possibly depending on $\bar{x}$, though the free boundary conditions in Theorem 1.3 have to be fulfilled).

In particular, from (11-11), we know that

$$\Delta u\text{ is constant in the positivity cone of }u. \tag{11-12}$$

Now, from (1-25), we know that, if

$$u_{k,\bar{x}}(x):=\frac{u(\bar{x}+\rho_k x)}{\rho_k^2}, \tag{11-13}$$

with $\rho_k\to 0^+$, then, up to a subsequence,

$$u_{k,\bar{x}}\to u_{0,\bar{x}}\text{ in }C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n), \tag{11-14}$$

as $k\to+\infty$, for any $\alpha\in(0,1)$.

We claim that

$$u_{0,0}\text{ must necessarily be }\frac{a(x_1^+)^2}{2}, \tag{11-15}$$

[4] We observe that the limit in (11-10) exist, due to the monotonicity of E, recall Theorem 1.12.

namely the first and the second possibilities in (11-11) are excluded at the origin. To prove (11-15), we argue by contradiction. If not, by (11-14) and (11-11), necessarily

$$\frac{u(\rho_k x)}{\rho_k^2} = u_{k,0}(x) \to \left\{\text{either } \frac{a_1 x_1^2 + a_2 x_2^2}{2} \text{ or } \frac{a x_1^2}{2}\right\} =: u_{0,0}(x)$$

in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore, using the change of variable $y := \rho_k x$,

$$\lim_{k\to+\infty} \frac{|B_{\rho_k} \cap \{u > 0\}|}{|B_{\rho_k}|} = \lim_{k\to+\infty} \frac{1}{|B_{\rho_k}|} \int_{B_{\rho_k}\cap\{u>0\}} dx = \lim_{k\to+\infty} \frac{1}{|B_1|} \int_{B_1\cap\{u_{k,0}>0\}} dy = \frac{1}{|B_1|} \int_{B_1\cap\{u_{0,0}>0\}} dy = 1.$$

This is a contradiction with (1-27), and so (11-15) is proved.

We let $E_{k,\bar{x}}$ be the monotone function in (1-22) for $u_{k,\bar{x}}$ (while $E_{\bar{x}}$ denotes the same type of function for u centered at the point $\bar{x}$). Let also $E_{0,\bar{x}}$ be the monotone function in (1-22) for $u_{0,\bar{x}}$. In view of (11-14), we have that

$$E_{0,\bar{x}}(r) = \lim_{k\to+\infty} E_{k,\bar{x}}(r). \tag{11-16}$$

We remark that (1-22) is compatible with the blow-up scaling, namely

$$E_{k,\bar{x}}(r) = E_{\bar{x}}(\rho_k r).$$

As a consequence, by (11-10) and (11-16),

$$E_{0,\bar{x}}(r) = \lim_{k\to+\infty} E_{\bar{x}}(\rho_k r) = E_{\bar{x}}(0). \tag{11-17}$$

We now classify the free boundary points according to the monotone function induced by their blow-up limits. For this, we introduce the following notation: recalling (11-11), we say that $\bar{x}$ is Type-1 if, up to a rotation,

$$u_{0,\bar{x}}(x) = \frac{a_1(x_1 - \bar{x}_1)^2 + a_2(x_2 - \bar{x}_2)^2}{2}.$$

Similarly, we say that $\bar{x}$ is Type-2 if

$$u_{0,\bar{x}}(x) = \frac{a(x_1 - \bar{x}_1)^2}{2},$$

and Type-3 if

$$u_{0,\bar{x}}(x) = \frac{a((x_1 - \bar{x}_1)^+)^2}{2}.$$

In this notation, (11-15) says that the origin is Type-3.

Now, in light of (1-22) and Lemma 9.3 (which can be utilized here thanks to (11-12)), we have that

$$E_{0,\bar{x}}(r) = \begin{cases} \dfrac{\pi}{4}, & \text{if } \bar{x} \text{ is either Type-1 or Type-2,} \\ \dfrac{\pi}{8}, & \text{if } \bar{x} \text{ is Type-3.} \end{cases} \tag{11-18}$$

In particular, the monotone function E is minimized for Type-3 free boundary points.

Moreover, we have the semicontinuity property: if $x_j \in \partial\{u > 0\}$ and $x_j \to x_0$ as $j \to +\infty$, then

$$\limsup_{j\to+\infty} E_{x_j}(0) \leqslant E_{x_0}(0). \tag{11-19}$$

Indeed, by the monotonicity of E in Theorem 1.12 and (1-22), for any $r \in (0, 1)$ we have that

$$\limsup_{j\to+\infty} E_{x_j}(0) \leqslant \limsup_{j\to+\infty} E_{x_j}(r) = E_{x_0}(r).$$

Then, we take the limit as $r \to 0^+$, and we obtain (11-19), as desired.

Now we claim that there exists $r_0 > 0$ such that:

$$\text{For any } \bar{x} \in \partial\{u > 0\} \cap B_{r_0}, \text{ we have that } E_{\bar{x}}(0) = E_0(0). \tag{11-20}$$

In other words, in B_{r_0} all free boundary points must be of Type-3. To prove this we argue by contradiction: If not, there exists a sequence of points $\bar{x}_j \in \partial\{u > 0\}$ such that $\bar{x}_j \to 0$ as $j \to +\infty$ and

$$E_{\bar{x}_j}(0) \neq E_0(0). \tag{11-21}$$

From (11-11), (11-15), (11-17), (11-18), and (11-21), we deduce that

$$\left\{\frac{\pi}{8}, \frac{\pi}{4}\right\} \ni E_{0,\bar{x}_j}(r) = E_{\bar{x}_j}(0) \neq E_0(0) = E_{0,0}(r) = \frac{\pi}{8},$$

and accordingly

$$E_{\bar{x}_j}(0) = E_{0,\bar{x}_j}(r) = \frac{\pi}{4} > \frac{\pi}{8} = E_{0,0}(r) = E_0(0).$$

This gives that

$$\lim_{j\to+\infty} E_{\bar{x}_j}(0) = \frac{\pi}{4} > \frac{\pi}{8} = E_0(0),$$

which is in contradiction with (11-19), and so the proof of (11-20) is complete.

Then, by (11-18) and (11-20), it follows that if $\bar{x} \in \partial\{u > 0\} \cap B_{r_0}$, then $\bar{x}$ must necessarily be Type-3, i.e., up to rotations, $u_{0,\bar{x}}(x) = a((x_1 - \bar{x}_1)^+)^2/2$, which is the desired result. □

Appendix A. Decay estimate for D^2u

Here we provide some decay estimates for the gradient and the Hessian of a local minimizer of the functional J in (1-1).

Lemma A.1. *Let $n \geqslant 2$, u be a minimizer for the functional J defined in (1-1), and $x_0 \in \partial\{u > 0\}$. Assume that $B_{\bar{R}} \subset\subset \Omega$. Then, there exist $R_0 \in (0, \bar{R})$ and $C > 0$, depending only on n, $\|u\|_{W^{2,2}(\Omega)}$ and* $\operatorname{dist}(B_{\bar{R}}, \partial\Omega)$, *such that*

$$\frac{1}{R^{n+2}} \int_{B_R(x_0)} |\nabla u|^2 + \frac{1}{R^n} \int_{B_R(x_0)} |D^2u|^2 \leqslant \frac{C}{R^{n+4}} \int_{B_{4R}(x_0)} (u - m)^2 + \frac{C}{R^{n+2}} \int_{B_{4R}} (u - m),$$

for any $R \in (0, R_0)$, where

$$m := \min\nolimits_{B_{4R}(x_0)} u. \tag{A-1}$$

Proof. Without loss of generality we suppose that $x_0 = 0$. Recalling Lemma 4.1, we have that, for any $\phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, with $\phi \geqslant 0$, it holds that

$$0 \geqslant \int_\Omega \Delta u \, \Delta\phi. \tag{A-2}$$

Now, let $\phi \in C_0^\infty(\Omega)$, with $\phi \geqslant 0$, and define $\phi_\varepsilon := \phi * \rho_\varepsilon$, where $\rho_\varepsilon(x) := (1/\varepsilon^n)\rho(x/\varepsilon)$, for any $x \in \mathbb{R}^n$, is a mollifying kernel, for any $\varepsilon \in (0, 1)$. We also set $u_\varepsilon := u * \rho_\varepsilon$. Then, if $\operatorname{dist}(\operatorname{supp}\phi, \partial\Omega) \gg \varepsilon$, we can use (A-2) and make an integration by parts twice to obtain that

$$\begin{aligned} 0 \geqslant \int_\Omega \Delta u\, \Delta\phi_\varepsilon &= \int_\Omega \Delta u (\Delta\phi) * \rho_\varepsilon = \int_\Omega \Delta u(x)\Big(\int_\Omega \rho_\varepsilon(x-y)\Delta\phi(y)dy\Big)dx \\ &= \int_\Omega \Delta\phi(y)\Big(\int_\Omega \rho_\varepsilon(x-y)\Delta u(x)dx\Big)dy = \int_\Omega \Delta\phi\, \Delta u_\varepsilon \\ &= \sum_{i,j=1}^n \int_\Omega \phi_{ii}(u_\varepsilon)_{jj} = \sum_{i,j=1}^n \int_{\partial\Omega} \phi_i (u_\varepsilon)_{jj}\nu^i - \sum_{i,j=1}^n \int_\Omega \phi_i (u_\varepsilon)_{ijj} \\ &= \sum_{i,j=1}^n \int_{\partial\Omega} \phi_i (u_\varepsilon)_{jj}\nu^i - \sum_{i,j=1}^n \int_{\partial\Omega} \phi_i (u_\varepsilon)_{ij}\nu^j + \sum_{i,j=1}^n \int_\Omega \phi_{ij}(u_\varepsilon)_{ij} \\ &= \sum_{i,j=1}^n \int_\Omega \phi_{ij}(u_\varepsilon)_{ij}. \end{aligned} \tag{A-3}$$

Moreover, we observe that

$$\lim_{\varepsilon\to 0} \int_\Omega \phi_{ij}(u_\varepsilon)_{ij} = \int_\Omega \phi_{ij}u_{ij}.$$

From this and (A-3), we have that

$$\sum_{i,j=1}^n \int_\Omega \phi_{ij}u_{ij} \leqslant 0. \tag{A-4}$$

Now, we choose $\phi := (u-m)\eta^2$, where m is as in (A-1), and η is a standard cut-off function supported in $B_{2R} \subset\subset \Omega$, such that $\eta = 1$ in B_R and $\eta = 0$ outside B_{2R}. Therefore, we see that $\phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $\phi \geqslant 0$. With this choice,

$$\phi_{ij} = u_{ij}\eta^2 + 2u_i\eta_j\eta + 2u_j\eta_i\eta + (u-m)(\eta^2)_{ij}.$$

If we plug this into (A-4), we have that

$$\sum_{i,j=1}^n \int_\Omega \big(u_{ij}\eta^2 + 4u_i\eta_j\eta + (u-m)(\eta^2)_{ij}\big)\, u_{ij} \leqslant 0.$$

That is, rearranging the terms and integrating by parts,

$$\begin{aligned} \sum_{i,j=1}^n \int_\Omega u_{ij}^2\eta^2 &\leqslant -\sum_{i,j=1}^n \int_\Omega (4u_{ij}u_i\eta_j\eta + (u-m)u_{ij}(\eta^2)_{ij}) \\ &= -\sum_{i,j=1}^n \int_\Omega 4(u_{ij}\eta)u_i\eta_j + \sum_{i,j=1}^n \int_\Omega ((u-m)u_i(\eta^2)_{ijj} + u_iu_j(\eta^2)_{ij}) \\ &\leqslant 2\delta \sum_{i,j=1}^n \int_\Omega u_{ij}^2\eta^2 + \frac{8}{\delta}\sum_{i,j=1}^n \int_\Omega u_i^2\eta_j^2 + \sum_{i,j=1}^n \int_\Omega ((u-m)u_i(\eta^2)_{ijj} + u_iu_j(\eta^2)_{ij}), \end{aligned} \tag{A-5}$$

where the last line follows from a suitable application of the Hölder inequality, for some $\delta > 0$.

Now, by direct computations we have

$$(\eta^2)_{ij} = 2\eta_i\eta_j + 2\eta\eta_{ij} \quad \text{and} \quad (\eta^2)_{ijj} = 2\eta_{ij}\eta_j + 2\eta_i\eta_{jj} + 2\eta_j\eta_{ij} + 2\eta\eta_{ijj},$$

and therefore,

$$|(\eta^2)_{ij}| \leqslant \frac{C}{R^2} \quad \text{and} \quad |(\eta^2)_{ijj}| \leqslant \frac{C}{R^3},$$

for some $C > 0$.

As a consequence, plugging this information into (A-5) and using the Hölder inequality, we obtain that

$$\begin{aligned}(1-2\delta)\sum_{i,j=1}^{n}\int_{\Omega}u_{ij}^2\eta^2 &\leqslant \frac{8}{\delta}\sum_{i,j=1}^{n}\int_{\Omega}u_i^2\eta_j^2 + \sum_{i,j=1}^{n}\int_{\Omega}\big((u-m)u_i(\eta^2)_{ijj} + u_iu_j(\eta^2)_{ij}\big)\\ &\leqslant \frac{C}{\delta R^2}\int_{B_{2R}}|\nabla u|^2 + \frac{C}{R^3}\int_{B_{2R}}(u-m)|\nabla u| + \frac{C}{R^2}\int_{B_{2R}}|\nabla u|^2\\ &\leqslant \left(1+\frac{1}{\delta}\right)\frac{C}{R^2}\int_{B_{2R}}|\nabla u|^2 + \frac{C}{R^4}\int_{B_{2R}}(u-m)^2, \end{aligned} \tag{A-6}$$

up to renaming C. Since $\Delta u \geqslant -C$ (up to renaming constants, recall Corollary 4.2), then from the Caccioppoli inequality (see, e.g., (6-10)) we get that

$$\int_{B_{2R}}|\nabla u|^2 \leqslant \frac{C}{R^2}\int_{B_{4R}}(u-m)^2 + C\int_{B_{4R}}(u-m),$$

which implies that

$$\frac{1}{R^{n+2}}\int_{B_{2R}}|\nabla u|^2 \leqslant \frac{C}{R^{n+4}}\int_{B_{4R}}(u-m)^2 + \frac{C}{R^{n+2}}\int_{B_{4R}}(u-m). \tag{A-7}$$

Moreover, from (A-6) and (A-7), we conclude that

$$\frac{1-2\delta}{R^n}\sum_{i,j=1}^{n}\int_{B_R}u_{ij}^2 \leqslant \frac{1-2\delta}{R^n}\sum_{i,j=1}^{n}\int_{\Omega}u_{ij}^2\eta^2 \leqslant \frac{C}{R^{n+4}}\int_{B_{4R}}(u-m)^2 + \frac{C}{R^{n+2}}\int_{B_{4R}}(u-m)$$

up to renaming $C > 0$. Putting together this and (A-7), we obtain the desired estimate. □

Corollary A.2. *Let $n \geqslant 2$, $\delta > 0$, and u be a minimizer of the functional J defined in (1-1) in Ω. Assume that $B_{\bar{R}} \subset\subset \Omega$. Let $x_0 \in \partial\{u > 0\}$ such that $\nabla u(x_0) = 0$ and $\partial\{u > 0\}$ is not δ-rank-2 flat at x_0 at any level $r > 0$ in the sense of Definition 1.6.*

Then, there exist $R_0 \in (0, \bar{R})$ and $C > 0$, depending only on n, $\|u\|_{W^{2,2}(\Omega)}$, and $\operatorname{dist}(B_{\bar{R}}, \partial\Omega)$, such that

$$\frac{1}{R^{n+2}}\int_{B_R(x_0)}|\nabla u|^2 + \frac{1}{R^n}\int_{B_R(x_0)}|D^2u|^2 \leqslant C,$$

for any $R < R_0$.

Proof. The estimate follows from Lemma A.1 and the quadratic growth of u, as given by Theorem 1.7. □

Appendix B. a remark on the one-phase problem

here, we show that the one-phase problem, as presented in Definition 1.2, and the analysis of the minimizers which happen to be nonnegative are structurally very different questions. indeed, while a "typical" one-phase minimizer exhibits nontrivial open regions in which it vanishes, the free minimizers that are nonnegative do not show the same phenomena. as a prototype result for this, we point out the following observation:

Proposition B.1. *Suppose that* $0 \in \omega$, $u \in c^{1,1}(\omega)$ *is such that* $u > 0$ *in* $\omega \cap \{x_n > 0\}$ *and* $u = 0$ *in* $\omega \cap \{x_n \leqslant 0\}$. *then,* u *cannot be a local minimizer for the functional* j *in* ω *in the class of admissible functions* $\dashv$ *given in* (1-2).

Proof. without loss of generality, we can assume that $b_2 \subset\subset \omega$. let $\varphi \in c_0^\infty(b_2, [0, 1])$ be such that $\varphi = 1$ in b_1. let also $\varepsilon \in (0, 1)$ and $u_\varepsilon := u - \varepsilon\varphi$.

we observe that the regularity of u and the fact that $u(x', 0) = 0 \leqslant u(y)$ for any x' such that $(x', 0) \in b_2$ and any $y \in b_2$ give that, for every $x = (x', x_n) \in b_1$,

$$u(x) \leqslant k x_n^2,$$

for some $k > 0$. consequently, for every $x \in b_1$ with $|x_n| < \sqrt{\varepsilon/k}$ we have that

$$u_\varepsilon(x) \leqslant k x_n^2 - \varepsilon < 0.$$

this gives that, for any $x \in (-1/n, 1/n)^{n-1} \times (0, \sqrt{\varepsilon/K}) =: Q_\varepsilon$,

$$u_\varepsilon(x) < 0 < u(x),$$

as long as $\varepsilon > 0$ is sufficiently small.

Also note that $u_\varepsilon \leqslant u$ and so $\{u_\varepsilon > 0\} \subseteq \{u > 0\}$. Accordingly, computing the energy functional in B_2,

$$\begin{aligned} J[u_\varepsilon] - J[u] &= \int_{B_2} \big(|\Delta u_\varepsilon|^2 - |\Delta u|^2\big) + |B_2 \cap \{u_\varepsilon > 0\}| - |B_2 \cap \{u > 0\}| \\ &= \int_{B_2} \big(|\Delta u - \varepsilon \Delta\varphi|^2 - |\Delta u|^2\big) - |B_2 \cap \{u_\varepsilon \leqslant 0 < u\}| \\ &\leqslant \int_{B_2} \big(\varepsilon^2 |\Delta\varphi|^2 - 2\varepsilon \Delta u \Delta\varphi\big) - |Q_\varepsilon| \leqslant C\varepsilon - \Big(\frac{2}{n}\Big)^{n-1} \sqrt{\frac{\varepsilon}{K}} < 0, \end{aligned}$$

provided that ε is small enough. □

Appendix C. Proof of an auxiliary result

For completeness, in this appendix we provide the proof of Proposition 4.3.

Proof of Proposition 4.3. Given $\delta > 0$, let $p \in \partial\Omega$ with $|u(p)| > \delta$. By (4-6), we can find $\rho > 0$ such that

$$\overline{\Omega} \cap B_\rho(p) \subset \Big\{ |u(p)| > \frac{\delta}{2} \Big\}. \tag{C-1}$$

Let $\phi \in C_0^\infty(B_\rho(p))$, with $\phi = 0$ along $\partial\Omega$. For each $\varepsilon \in \mathbb{R}$ with $|\varepsilon| < \dfrac{\delta}{4(1+\|\phi\|_{L^\infty(\mathbb{R}^n)})}$, we let $u_\varepsilon := u + \varepsilon\phi$. We observe that

$$\chi_{\{u_\varepsilon>0\}} = \chi_{\{u>0\}}, \quad \text{in } \Omega. \tag{C-2}$$

Indeed, if $x \in \Omega \setminus B_\rho(p)$, we have that $\phi(x) = 0$, and thus $u_\varepsilon(x) = u(x)$, proving (C-2) in this case. If instead $x \in \Omega \cap B_\rho(p)$, by (C-1) we can assume $u(x) > \delta/2$ (the case $u(x) < -\delta/2$ being similar). Then,

$$u_\varepsilon(x) \geqslant u(x) - \varepsilon\|\phi\|_{L^\infty} > \frac{\delta}{2} - \frac{\delta\,\|\phi\|_{L^\infty}}{4(1+\|\phi\|_{L^\infty(\mathbb{R}^n)})} > \frac{\delta}{4},$$

and hence

$$\chi_{\{u_\varepsilon>0\}}(x) = 1 = \chi_{\{u>0\}}(x),$$

completing the proof of (C-2).

As a byproduct of (C-2), we have that

$$0 \leqslant J[u_\varepsilon] - J[u] = \int_{\Omega\cap B_\rho(p)} \big(|\Delta u + \varepsilon\Delta\phi|^2 - |\Delta u|^2\big) = \int_{\Omega\cap B_\rho(p)} \big(2\varepsilon\Delta u\Delta\phi + \varepsilon^2|\Delta\phi|^2\big)$$

yielding that

$$\int_{\Omega\cap B_\rho(p)} \Delta u\Delta\phi = 0. \tag{C-3}$$

That is, defining $v := \Delta u$, we have that v is weakly harmonic in $\Omega\cap B_\rho(p)$, hence harmonic in $\Omega\cap B_\rho(p)$, and therefore, v is smooth in $\Omega\cap B_\rho(p)$, up to the boundary. Hence, we deduce from (4-7) and (C-3) that

$$0 = \int_{\Omega\cap B_\rho(p)} v\Delta\phi = \int_{\Omega\cap B_\rho(p)} (\operatorname{div}(v\nabla\phi) - \operatorname{div}(\phi\nabla v)) = \int_{(\partial\Omega)\cap B_\rho(p)} (v\partial_\nu\phi - \phi\partial_\nu v) = \int_{(\partial\Omega)\cap B_\rho(p)} v\partial_\nu\phi.$$

Therefore, since v is continuous on $(\partial\Omega)\cap B_\rho(p)$, thanks to (4-7), we find that $v(p) = 0$.

By taking δ arbitrary, we thus conclude that $v = 0$ on $(\partial\Omega)\cap\{|u| > 0\}$. This and (4-8) give that

$$v = 0 \quad \text{along } \partial\Omega. \tag{C-4}$$

Now we prove (4-9) by arguing by contradiction: We define $V := -v = -\Delta u$, and we suppose that

$$M := \sup_\Omega V > 0.$$

Now, we use (4-7), and we find some $\rho > 0$ such that V is continuous in a ρ-neighborhood of $\partial\Omega$ that we denote by $\mathcal{O}_\rho$. Thus, V is uniformly continuous in $\mathcal{O}_{\rho/2}$. In particular, there exists $\delta \in (0, \rho/2)$ such that if $x, y \in \mathcal{O}_\delta$ with $|x-y| \leqslant \delta$, then $|V(x) - V(y)| \leqslant \frac{M}{2}$.

Consequently, taking $y \in \partial\Omega$ and recalling (C-4), we find that

$$|V(x)| \leqslant \frac{M}{2} \quad \text{for every } x \in \mathcal{O}_\delta, \tag{C-5}$$

and, as a result,

$$0 < M = \sup_\Omega V = \sup_{\Omega\setminus\mathcal{O}_\delta} V. \tag{C-6}$$

Furthermore, in view of Lemma 4.1, for every $\phi \in C_0^\infty(\Omega, [0, +\infty))$,

$$\int_\Omega V\,\Delta\phi = -\int_\Omega \Delta u\,\Delta\phi \geqslant 0,$$

hence V is weakly subharmonic. From this, (C-6), and Theorem B in [Littman 1963], we deduce that $V = M$ a.e. in Ω. This is in contradiction with (C-5), hence the claim in (4-9) is established. □

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References

[Adams and Vandenhouten 2000] D. R. Adams and R. F. Vandenhouten, "Stability for polyharmonic obstacle problems with varying obstacles", *Comm. Partial Differential Equations* **25**:7-8 (2000), 1171–1200. MR Zbl

[Aleksanyan 2019] G. Aleksanyan, "Regularity of the free boundary in the biharmonic obstacle problem", *Calc. Var. Partial Differential Equations* **58**:6 (2019), art. id. 206. MR Zbl

[Alt and Caffarelli 1981] H. W. Alt and L. A. Caffarelli, "Existence and regularity for a minimum problem with free boundary", *J. Reine Angew. Math.* **325** (1981), 105–144. MR Zbl

[Boggio 1905] T. Boggio, "Sulle funzioni di Green d'ordine *m*", *Rend. Circ. Mat. Palermo* **20** (1905), 97–135. Zbl

[Brezis 1983] H. Brezis, *Analyse fonctionnelle: théorie et applications*, Masson, Paris, 1983. MR Zbl

[Caffarelli 1980] L. A. Caffarelli, "Compactness methods in free boundary problems", *Comm. Partial Differential Equations* **5**:4 (1980), 427–448. MR Zbl

[Caffarelli and Friedman 1979] L. A. Caffarelli and A. Friedman, "The obstacle problem for the biharmonic operator", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **6**:1 (1979), 151–184. MR Zbl

[Caffarelli et al. 1981] L. A. Caffarelli, A. Friedman, and A. Torelli, "The free boundary for a fourth order variational inequality", *Illinois J. Math.* **25**:3 (1981), 402–422. MR Zbl

[Caffarelli et al. 1982] L. A. Caffarelli, A. Friedman, and A. Torelli, "The two-obstacle problem for the biharmonic operator", *Pacific J. Math.* **103**:2 (1982), 325–335. MR Zbl

[Da Lio et al. 2020] F. Da Lio, F. Palmurella, and T. Rivière, "A resolution of the Poisson problem for elastic plates", *Arch. Ration. Mech. Anal.* **236**:3 (2020), 1593–1676. MR Zbl

[Dipierro and Karakhanyan 2018] S. Dipierro and A. L. Karakhanyan, "Stratification of free boundary points for a two-phase variational problem", *Adv. Math.* **328** (2018), 40–81. MR Zbl

[Dipierro et al. 2017] S. Dipierro, A. Karakhanyan, and E. Valdinoci, "A nonlinear free boundary problem with a self-driven Bernoulli condition", *J. Funct. Anal.* **273**:11 (2017), 3549–3615. MR Zbl

[Dipierro et al. 2019] S. Dipierro, A. L. Karakhanyan, and E. Valdinoci, "Limit behaviour of a singular perturbation problem for the biharmonic operator", *Appl. Math. Optim.* **80**:3 (2019), 679–713. MR Zbl

[Evans 1998] L. C. Evans, *Partial differential equations*, Grad. Stud. Math. **19**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

[Frehse 1973] J. Frehse, "On the regularity of the solution of the biharmonic variational inequality", *Manuscripta Math.* **9** (1973), 91–103. MR Zbl

[Ganguli 2017] R. Ganguli, *Finite element analysis of rotating beams: physics based interpolation*, Springer, 2017. MR Zbl

[Gazzola et al. 2010] F. Gazzola, H.-C. Grunau, and G. Sweers, *Polyharmonic boundary value problems*, Lect. Notes Math. **1991**, Springer, 2010. MR Zbl

[Giaquinta 1983] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Ann. of Math. Stud. **105**, Princeton Univ. Press, 1983. MR Zbl

[Lieb and Loss 2001] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Grad. Stud. Math. **14**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[Littman 1963] W. Littman, "Generalized subharmonic functions: monotonic approximations and an improved maximum principle", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (3) **17**:3 (1963), 207–222. MR Zbl

[Mardanov and Zaripov 2016] R. F. Mardanov and S. K. Zaripov, "Solution of Stokes flow problem using biharmonic equation formulation and multiquadrics method", *Lobachevskii J. Math.* **37**:3 (2016), 268–273. MR Zbl

[Martio and Vuorinen 1987] O. Martio and M. Vuorinen, "Whitney cubes, p-capacity, and Minkowski content", *Expo. Math.* **5**:1 (1987), 17–40. MR Zbl

[Mawi 2014] H. Mawi, "A free boundary problem for higher order elliptic operators", *Complex Var. Elliptic Equ.* **59**:7 (2014), 937–946. MR Zbl

[McKenna and Walter 1987] P. J. McKenna and W. Walter, "Nonlinear oscillations in a suspension bridge", *Arch. Ration. Mech. Anal.* **98**:2 (1987), 167–177. MR Zbl

[Miura 2016] T. Miura, "Singular perturbation by bending for an adhesive obstacle problem", *Calc. Var. Partial Differential Equations* **55**:1 (2016), art. id. 19. MR Zbl

[Miura 2017] T. Miura, "Overhanging of membranes and filaments adhering to periodic graph substrates", *Phys. D* **355** (2017), 34–44. MR Zbl

[Monneau and Weiss 2007] R. Monneau and G. S. Weiss, "An unstable elliptic free boundary problem arising in solid combustion", *Duke Math. J.* **136**:2 (2007), 321–341. MR Zbl

[Novaga and Okabe 2015] M. Novaga and S. Okabe, "Regularity of the obstacle problem for the parabolic biharmonic equation", *Math. Ann.* **363**:3-4 (2015), 1147–1186. MR Zbl

[Novaga and Okabe 2016] M. Novaga and S. Okabe, "The two-obstacle problem for the parabolic biharmonic equation", *Nonlinear Anal.* **136** (2016), 215–233. MR Zbl

[Petrosyan 2002] A. Petrosyan, "On existence and uniqueness in a free boundary problem from combustion", *Comm. Partial Differential Equations* **27**:3-4 (2002), 763–789. MR Zbl

[Pozzolini and Léger 2008] C. Pozzolini and A. Léger, "A stability result concerning the obstacle problem for a plate", *J. Math. Pures Appl.* (9) **90**:6 (2008), 505–519. MR Zbl

[Serrin 2011] J. Serrin, "Weakly subharmonic function", *Boll. Unione Mat. Ital.* (9) **4**:3 (2011), 347–361. MR Zbl

[Sweers 2009] G. Sweers, "A survey on boundary conditions for the biharmonic", *Complex Var. Elliptic Equ.* **54**:2 (2009), 79–93. MR Zbl

[Weiss 1998] G. S. Weiss, "Partial regularity for weak solutions of an elliptic free boundary problem", *Comm. Partial Differential Equations* **23**:3-4 (1998), 439–455. MR Zbl

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