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**THE GAUSS–BONNET–CHERN MASS
OF HIGHER-CODIMENSION GRAPHS**

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We give an explicit formula for the Gauss–Bonnet–Chern mass of an asymptotically flat graph of arbitrary codimension and use it to prove the positive mass theorem and the Penrose inequality for graphs with flat normal bundle.

1. Introduction

A complete Riemannian manifold (M^n, g) , $n \geq 3$, is said to be asymptotically flat of order τ (with one end) if there exists a compact subset K of M and a diffeomorphism $\Psi : M \setminus K \rightarrow \mathbb{R}^n \setminus \bar{B}_1(0)$, introducing coordinates in $M \setminus K$, say $x = (x_1, \dots, x_n)$, such that, in these coordinates,

$$(1) \quad g_{ij} = \delta_{ij} + \sigma_{ij}$$

and

$$(2) \quad |\sigma_{ij}| + |x| |\sigma_{ij,k}| + |x|^2 |\sigma_{ij,kl}| = O(|x|^{-\tau}),$$

where the σ_{ij} 's are the coefficients of σ with respect to x , $\sigma_{ij,k} = \partial \sigma_{ij} / \partial x_k$, $\sigma_{ij,kl} = \partial^2 \sigma_{ij} / \partial x_k \partial x_l$, and $|\cdot|$ is the standard Euclidean norm. The ADM mass of (M, g) , introduced by Arnowitt, Deser and Misner in [Arnowitt et al. 1961] (see also [Jaramillo and Gourgoulhon 2011]) is defined by

$$(3) \quad m_{\text{ADM}} = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{S_r} (g_{ij,i} - g_{ii,j}) v^j dS_r,$$

where ω_{n-1} is the volume of the $(n-1)$ -dimensional unit sphere, S_r is the Euclidean coordinate sphere of radius r , dS_r is the volume form of S_r induced by the Euclidean metric, and $v = r^{-1}x$ is the outward unit normal to S_r (with respect to the Euclidean metric).

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It is known that if $\tau > (n-2)/2$ and the scalar curvature of (M, g) is integrable, then the limit (3) exists, is finite, and is a geometric invariant, that is, two coordinate systems satisfying (1) and (2) yield the same value for it [Bartnik 1986; Chruściel 1986].

One of the most important conjectures in mathematical general relativity is the famous positive mass conjecture (PMC):

Conjecture 1. *If (M^n, g) , $n \geq 3$, is an asymptotically flat Riemannian manifold of order $\tau > (n-2)/2$ whose scalar curvature is nonnegative and integrable, then the ADM mass of (M, g) is nonnegative. Moreover, if the mass is zero, then (M, g) is isometric to the Euclidean space (\mathbb{R}^n, δ) .*

The PMC was settled for $n \leq 7$ in [Schoen and Yau 1979], for (M, g) conformally flat in [Schoen and Yau 1988], and for M spin in [Witten 1981] (see also [Parker and Taubes 1982] and [Choquet-Bruhat 1984]). Very elegant proofs for the case when (M, g) is a Euclidean graph were given in [Lam 2011] (see also [de Lima and Girão 2015]) for graphs of codimension one and in [Mirandola and Vitória 2015] for graphs of arbitrary codimension with flat normal bundle (notice that the case of graphs also follows from Witten's argument, since a Euclidean graph is spin). The case of Euclidean hypersurfaces (not necessarily graphs), including the rigidity statement, was treated in [Huang and Wu 2013], under appropriate decay conditions.

The Penrose inequality (PI) is a conjectured sharpening of the PMC when (M, g) has a compact boundary Γ which is an outermost minimal hypersurface.

Conjecture 2. *If (M^n, g) , $n \geq 3$, is an asymptotically flat Riemannian manifold of order $\tau > (n-2)/2$ whose scalar curvature is nonnegative (and integrable), and Γ is a (possibly disconnected) outermost minimal hypersurface of area A , then*

$$m_{\text{ADM}} \geq \frac{1}{2} \left(\frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Furthermore, if the equality holds, then (M, g) is isometric to the Riemannian Schwarzschild manifold.

The PI was proved by Huisken and Ilmanen [2001] for $n = 3$ and Γ connected, and by Bray [2001] for $n = 3$ and general Γ . Bray and Lee [2009] established the conjecture for $n \leq 7$, with the extra requirement that M be spin for the rigidity statement. The case of Euclidean graphs of codimension one was treated by Lam [2011] (see also [de Lima and Girão 2015]) and generalized by Mirandola and Vitória for graphs of arbitrary codimension with flat normal bundle [2015]. The equality case for graphs of codimension one was treated in [Huang and Wu 2015a].

In [Ge et al. 2014b], a new mass for asymptotically flat Riemannian manifolds, named Gauss–Bonnet–Chern mass, was introduced. For a positive integer $q < n/2$,

consider the q -th Gauss–Bonnet curvature, denoted $L_{(q)}$, and defined by

$$(4) \quad L_{(q)} = \frac{1}{2^q} \delta_{b_1 b_2 \dots b_{2q}}^{a_1 a_2 \dots a_{2q}} \left(\prod_{s=1}^q R_{a_{2s-1} a_{2s}}^{b_{2s-1} b_{2s}} \right) = P_{(q)}^{ijkl} R_{ijkl},$$

where R is the Riemann curvature tensor of (M, g) and $P_{(q)}$, which has the same symmetries of the Riemann tensor (see [Ge et al. 2014b, Section 3]), is given by

$$(5) \quad P_{(q)}^{ijkl} = \frac{1}{2^q} \delta_{b_1 b_2 \dots b_{2q-3} b_{2q-2} b_{2q-1} b_{2q}}^{a_1 a_2 \dots a_{2q-3} a_{2q-2} i j} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}^{b_{2s-1} b_{2s}} \right) g^{b_{2q-1} k} g^{b_{2q} l}.$$

Remark 3. One can considerably simplify this complicated tensorial expression by rewriting it in the language of double forms, which are a special type of vector valued forms (see [Labbi 2007], for example).

The q -th Gauss–Bonnet–Chern mass (GBC mass) of (M, g) is defined by

$$(6) \quad m_q = c_q(n) \lim_{r \rightarrow \infty} \int_{S_r} P_{(q)}^{ijkl} g_{jk,l} v_i dS_r,$$

where

$$(7) \quad c_q(n) = \frac{(n - 2q)!}{2^{q-1} (n - 1)! \omega_{n-1}}$$

and S_r, dS_r, v and ω_{n-1} are as in the definition of the ADM mass.

As observed in [Ge et al. 2014b], m_1 coincides with the ADM mass. In the same article, the authors show that, if $\tau > (n - 2q)/(q + 1)$ and $L_{(q)}$ is integrable, then the limit (6) exists, is finite, and is a geometric invariant. Next, we state versions of the PMC and PI for the GBC mass. We start with the version of the PMC.

Conjecture 4. *Let n and q be integers such that $n \geq 3$ and $1 \leq q < n/2$. If (M^n, g) is an asymptotically flat Riemannian manifold of order $\tau > (n - 2q)/(q + 1)$ whose q -th Gauss–Bonnet curvature $L_{(q)}$ is nonnegative and integrable, then the q -th GBC mass of (M, g) is nonnegative. Moreover, if the mass is zero, then (M, g) is isometric to the Euclidean space (\mathbb{R}^n, δ) .*

Before we state the analogue of the PI, we recall the Riemannian manifold known as the q -th Riemannian Schwarzschild [Ge et al. 2014b, Section 6], which is $(\mathbb{R} \times \mathbb{S}^{n-1}, g_{\text{Sch}}^q)$ with

$$g_{\text{Sch}}^q = \left(1 + \frac{m}{2r^{\frac{n}{q}-2}} \right)^{\frac{4q}{n-2q}} (dr^2 + r^2 d\theta^2),$$

where $d\theta^2$ is the round metric on \mathbb{S}^{n-1} and $m \in \mathbb{R}$ is the mass parameter. Let $r_0 = (2m)^{q/(n-2q)}$. The hypersurface $r = r_0$ is an outermost minimal hypersurface of area $A = \omega_{n-1} r_0^{n-1}$, and the q -th GBC mass of $(\mathbb{R} \times \mathbb{S}^{n-1}, g_{\text{Sch}}^q)$ is $m_q = m^q$.

Thus, for the q -th Riemannian Schwarzschild manifold, one has

$$m_q = \frac{1}{2^q} \left(\frac{A}{\omega_{n-1}} \right)^{\frac{n-2q}{n-1}}.$$

We can now state the version of the PI for the GBC mass.

Conjecture 5. *Let n and q be integers such that $n \geq 3$ and $1 \leq q < n/2$. If (M^n, g) is an asymptotically flat Riemannian manifold of order $\tau > (n - 2q)/(q + 1)$ whose q -th Gauss–Bonnet curvature $L_{(q)}$ is nonnegative and integrable, and Γ is a (possibly disconnected) outermost minimal hypersurface of area A , then*

$$m_q \geq \frac{1}{2^q} \left(\frac{A}{\omega_{n-1}} \right)^{\frac{n-2q}{n-1}},$$

where m_q is the q -th GBC mass. Moreover, if the equality holds, then (M, g) is isometric to the q -th Riemannian Schwarzschild manifold.

We now turn to the special case of graphs. Let Ω be a (possibly empty) bounded open subset of \mathbb{R}^n such that $\Sigma = \partial\Omega$ is the union of finitely many smooth connected hypersurfaces. Let $f : \mathbb{R}^n \setminus \Omega \rightarrow \mathbb{R}^m$ be a continuous map such that its restriction to $\mathbb{R}^n \setminus \bar{\Omega}$ is smooth. Let f^α , $1 \leq \alpha \leq m$, be the components of f and let f_i^α , f_{ij}^α and f_{ijk}^α denote the first, second and third partial derivatives of f^α on $\mathbb{R}^n \setminus \bar{\Omega}$, where $1 \leq i, j, k \leq n$. The map f is said to be asymptotically flat of order τ if

$$(8) \quad |f_i^\alpha(x)| + |f_{ij}^\alpha(x)| |x| + |f_{ijk}^\alpha(x)| |x|^2 = O(|x|^{-\tau/2}),$$

for each $\alpha \in \{1, \dots, m\}$ and each triple (i, j, k) with $1 \leq i, j, k \leq n$.

We assume throughout the paper that

$$M = \{(x, f(x)) : x \in \mathbb{R}^n \setminus \Omega, f(x) \in \mathbb{R}^m\},$$

the graph of f , is a smooth submanifold with (possibly empty) boundary and that $g_{ij} = \delta_{ij} + f_i^\alpha f_j^\alpha$, the metric induced by the Euclidean metric on \mathbb{R}^{n+m} , extends to a smooth metric on M . Notice that if f is asymptotically flat of order τ , then from (8) we get that (M, g) is asymptotically flat of order τ .

Conjectures 4 and 5 have been proved for graphs of codimension one [Ge et al. 2014b]. When $q = 2$, Li, Wei and Xiong proved these conjectures for graphs of higher codimension with flat normal bundle [Li et al. 2014]. Conjecture 4 is also known to be true for conformally flat manifolds [Ge et al. 2014a].

The purpose of the present article is to prove Conjectures 4 and 5 for a family of higher-codimension Euclidean graphs (without the rigidity statements). This family includes the graphs with flat normal bundle. The exposition follows closely the ones given in [Lam 2011; Mirandola and Vitório 2015; Ge et al. 2014b; Li et al. 2014]. Before stating our main results, we need to introduce some notation.

Denote by $\{e_i\}_{i=1}^n$ the standard basis of \mathbb{R}^n and by $\{e_\alpha\}_{\alpha=1}^m$ the standard basis of \mathbb{R}^m . The coordinate vector fields on M are given by $\partial_i = (e_i, f_i^\alpha e_\alpha)$, and the vector fields $\eta_\alpha = (-Df^\alpha, e_\alpha)$, where Df^α denotes the Euclidean gradient of f^α , give us a (global) frame field for the normal bundle of M . We denote by B the second fundamental form of M , by B_α its α -th component with respect to the frame $\{\eta_\alpha\}_{\alpha=1}^m$, and by A_α the shape operator with respect to η_α . Also, let $U = (U_{\alpha\beta})$ be the metric on the normal bundle induced by the Euclidean metric $\langle \cdot, \cdot \rangle$ on \mathbb{R}^{n+m} . The components of U are given by

$$U_{\alpha\beta} = \delta_{\alpha\beta} + \langle Df^\alpha, Df^\beta \rangle.$$

The inverse of U is denoted by $(U^{\alpha\beta})$.

Recall the Gauss and the Ricci equations, which are, respectively, given by

$$(9) \quad R_{ijkl} = \langle B_{ik}, B_{jl} \rangle - \langle B_{il}, B_{jk} \rangle$$

and

$$(10) \quad \langle R_{\alpha\beta}^\perp(X), Y \rangle = \langle [A_\beta, A_\alpha](X), Y \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean metric on \mathbb{R}^{n+m} , R^\perp is the normal curvature operator and

$$[A_\beta, A_\alpha] = A_\alpha \circ A_\beta - A_\beta \circ A_\alpha.$$

We denote by $T_{(2q-1)}$ the Newton tensor of order $(2q-1)$ and denote by $T_{(2q-1)\alpha}$ its α -th component with respect to the frame $\{\eta_\alpha\}_{\alpha=1}^m$ (see [Grosjean 2002] and [Cao and Li 2007]). The expression for $T_{(2q-1)}$ in coordinates is

$$(11) \quad T_{(2q-1)i}{}^j = \frac{1}{(2q-1)!} \delta_{b_1 \dots b_{2q-1} i}^{a_1 \dots a_{2q-1} j} \left(\prod_{s=1}^{q-1} \langle B_{a_{2s-1}}^{b_{2s-1}}, B_{a_{2s}}^{b_{2s}} \rangle \right) B_{a_{2q-1}}^{b_{2q-1}},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean metric on \mathbb{R}^{n+m} . As we will see in Section 2, if M has flat normal bundle, then $T_{(2q-1)\alpha}$ commutes with A_β , for $1 \leq \alpha, \beta \leq m$.

We can now state the main results of the article. The first of them is this:

Theorem 6. *Let n and q be integers such that $n \geq 3$ and $1 \leq q < n/2$, and let (M, g) be the graph of an asymptotically flat map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ of order $\tau > (n-2q)/(q+1)$. If the q -th Gauss–Bonnet curvature $L_{(q)}$ of (M, g) is integrable, then the q -th Gauss–Bonnet–Chern mass m_q satisfies*

$$(12) \quad m_q = \frac{1}{2} c_q(n) \int_M (L_{(q)} + (2q-1)! \langle [T_{(2q-1)\alpha}, A_\beta] \cdot e_\alpha^\top, e_\beta^\top \rangle) \frac{1}{\sqrt{G}} dM,$$

where $c_q(n)$ is the constant (7), G is the determinant of (g_{ij}) ,

$$[T_{(2q-1)\alpha}, A_\beta] = T_{(2q-1)\alpha} \circ A_\beta - A_\beta \circ T_{(2q-1)\alpha}$$

is the commutator of the operators $T_{(2q-1)\alpha}$ and A_β , and e_α^\top is the tangent part

(along the graph M) of the canonical lift to $\mathbb{R}^{n+m} \equiv \mathbb{R}^n \times \mathbb{R}^m$ of the standard frame field on \mathbb{R}^m . Moreover, if M has flat normal bundle and $L_{(q)}$ is nonnegative, then m_q is nonnegative.

Remark 7. Notice that, since the graph structure is used in the definition of the vector fields η_α , the tensor $[T_{(2q-1)\alpha}, A_\beta]$ is defined only for graphs. It is desirable to find an expression similar to (12) that holds for any asymptotically flat submanifold (not necessarily a graph), but we were unable to do it. One strategy in order to do this is to rewrite $[T_{(2q-1)\alpha}, A_\beta]$ in such a way that it also makes sense for submanifolds which are not necessarily graphs and try proving that (12) also holds in this case. Another strategy is to find a similar expression for the mass by considering, instead of the vector field (25) used to get (12), one which is defined for any submanifold (compare, for example, the vector fields considered in [Lam 2011] and [de Lima and Girão 2015]).

Let $\Sigma \subset \mathbb{R}^n$ be an orientable hypersurface and let ξ be a unit normal vector field along Σ (chosen to point outwards, whenever this makes sense). The r -th mean curvature of Σ is defined as the r -th elementary symmetric function on the principal curvatures of Σ . Alternatively, if K is the second fundamental form of $\Sigma \subset \mathbb{R}^n$, then

$$(13) \quad H_r = \frac{1}{r!} \delta_{b_1 \dots b_r}^{a_1 \dots a_r} \prod_{s=1}^r K_{a_s}^{b_s}.$$

The hypersurface $\Sigma \subset \mathbb{R}^n$ is called *strictly p -mean convex*, $1 \leq p \leq n - 1$, if $H_r > 0$ for all $1 \leq r \leq p$. Our second main result is the following:

Theorem 8. *Let n and q be integers such that $n \geq 3$ and $1 \leq q < n/2$. Let Ω be a bounded and open subset of \mathbb{R}^n such that $\Sigma = \partial\Omega$ is the union of finitely many smooth connected hypersurfaces. Let $f : \mathbb{R}^n \setminus \Omega \rightarrow \mathbb{R}^m$ be an asymptotically flat map of order $\tau > (n - 2q)/(q + 1)$, and let (M, g) be the graph of f . Assume that f extends smoothly to an open set containing $\mathbb{R}^n \setminus \Omega$ and that f is constant along each connected component of Σ . If the q -th Gauss–Bonnet curvature $L_{(q)}$ is integrable, then the q -th Gauss–Bonnet–Chern mass m_q satisfies*

$$m_q = \frac{1}{2} c_q(n) \int_M (L_{(q)} + (2q - 1)! ([T_{(2q-1)\alpha}, A_\beta] \cdot e_\alpha^\top, e_\beta^\top)) \frac{1}{\sqrt{G}} dM + \frac{1}{2} (2q - 1)! c_q(n) \int_\Sigma \left(\frac{|Df|^2}{1 + |Df|^2} \right)^q H_{(2q-1)} d\Sigma,$$

where

$$|Df|^2 = \sum_{\alpha=1}^m |Df^\alpha|^2$$

and $H_{(2q-1)}$ is the $(2q - 1)$ -th mean curvature of Σ .

Our third main result is the following:

Theorem 9. *Let n and q be integers such that $n \geq 3$ and $1 \leq q < n/2$. Let Ω be a bounded and open subset of \mathbb{R}^n such that $\Sigma = \partial\Omega$ is the union of finitely many smooth hypersurfaces. Let*

$$f : \mathbb{R}^n \setminus \Omega \rightarrow \mathbb{R}^m$$

be an asymptotically flat map of order

$$\tau > (n - 2q)/(q + 1),$$

and let (M, g) be the graph of f . Assume that f is constant along each connected component of Σ and that

$$|Df| \rightarrow \infty \text{ as } x \rightarrow \Sigma.$$

If the q -th Gauss–Bonnet curvature $L_{(q)}$ is integrable, then the q -th Gauss–Bonnet–Chern mass m_q satisfies

$$(14) \quad m_q = \frac{1}{2} c_q(n) \int_M (L_{(q)} + (2q - 1)! ([T_{(2q-1)\alpha}, A_\beta] \cdot e_\alpha^\top, e_\beta^\top)) \frac{1}{\sqrt{G}} dM \\ + \frac{1}{2} (2q - 1)! c_q(n) \int_\Sigma H_{(2q-1)} d\Sigma,$$

where $H_{(2q-1)}$ is the $(2q-1)$ -th mean curvature of Σ . Furthermore, if M has flat normal bundle, $L_{(q)}$ is nonnegative and each component of Σ is star-shaped and strictly $(2q-1)$ -mean convex, then

$$(15) \quad m_q \geq \frac{1}{2^q} \left(\frac{A}{\omega_{n-1}} \right)^{\frac{n-2q}{n-1}}.$$

Remark 10. As explained in [Ge et al. 2014b, Remark 5.1], when $\Sigma \subset \mathbb{R}^n$ is strictly mean convex, the condition

$$(16) \quad |Df| \rightarrow \infty \text{ as } x \rightarrow \partial\Omega$$

holds if and only if $\Gamma = \partial M$ is an outermost minimal hypersurface. Therefore, this is a natural assumption.

Remark 11. Geometrically, condition (16) is equivalent to saying that along each connected component of ∂M , the graph M meets orthogonally the hyperplane that contains that component (see [de Sousa 2016]).

2. Auxiliary results

Let n and q be positive integers such that $n \geq 3$ and $1 \leq q < n/2$. Throughout this section, the tensors $P_{(q)}$ and $T_{(2q-1)}$ are defined by (5) and (11), respectively. Also, unless stated otherwise, we will follow the notation introduced in Section 1.

If Ω is not empty, we assume, throughout this section, that f extends smoothly to an open set containing $\mathbb{R}^n \setminus \Omega$.

Lemma 12. *Under the notation introduced above, the following identities hold:*

$$(17) \quad g_{jk,l} = f_{jl}^\alpha f_k^\alpha + f_j^\alpha f_{kl}^\alpha$$

$$(18) \quad e_\alpha^\top = \nabla f^\alpha = g^{ij} f_j^\alpha \partial_i = U^{\alpha\beta} f_i^\beta \partial_i$$

$$(19) \quad A_\alpha \partial_i = f_{ik}^\alpha g^{kj} \partial_j$$

$$(20) \quad B(\partial_i, \partial_j) = f_{ij}^\alpha U^{\alpha\beta} \eta^\beta$$

$$(21) \quad (B_\alpha)_{ij} = f_{ij}^\alpha$$

$$(22) \quad \Gamma_{ij}^k = g^{kl} f_l^\alpha f_{ij}^\alpha$$

$$(23) \quad \nabla e_\alpha^\top = \langle B, e_\alpha \rangle$$

Proof. Identities (17) to (22) are proven in [Mirandola and Vitório 2015] and [Li et al. 2014] and identity (23) is proved in [Palais and Terng 1988, Proposition 4.1.1]. \square

On an open set that contains $\mathbb{R}^n \setminus \Omega$, consider the vector field $X_{(q)}$ given by

$$(24) \quad X_{(q)} = X_{(q)}^i \partial_i = P_{(q)}^{ijkl} g_{j,k,l} \partial_i.$$

Proposition 13. *It holds that*

$$(25) \quad X_{(q)} = \frac{1}{2}(2q-1)! T_{(2q-1)\alpha} \cdot e_\alpha^\top.$$

Proof. By (24) and (17) we have

$$X_{(q)}^i = P_{(q)}^{ijkl} (f_{jl}^\alpha f_k^\alpha + f_j^\alpha f_{kl}^\alpha).$$

Using the antisymmetry of $P_{(q)}^{ijkl}$ with respect to the indices k and l , we have

$$X_{(q)}^i = P_{(q)}^{ijkl} f_{jl}^\alpha f_k^\alpha.$$

Combining this identity with (5), (19) and (18), we find

$$\begin{aligned} X_{(q)}^i &= \frac{1}{2^q} \delta_{b_1 b_2 \dots b_{2q-3} b_{2q-2} c d}^{a_1 a_2 \dots a_{2q-3} a_{2q-2} i j} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}^{b_{2s-1} b_{2s}} \right) g^{ck} g^{dl} f_{jl}^\alpha f_k^\alpha \\ &= \frac{1}{2^q} \delta_{b_1 b_2 \dots b_{2q-3} b_{2q-2} c d}^{a_1 a_2 \dots a_{2q-3} a_{2q-2} i j} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}^{b_{2s-1} b_{2s}} \right) g^{dl} f_{jl}^\alpha g^{ck} f_k^\alpha \\ &= \frac{1}{2^q} \delta_{b_1 b_2 \dots b_{2q-3} b_{2q-2} c d}^{a_1 a_2 \dots a_{2q-3} a_{2q-2} i j} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}^{b_{2s-1} b_{2s}} \right) (A_\alpha)_j^d g^{ck} f_k^\alpha \\ &= \frac{1}{2^q} \delta_{b_1 b_2 \dots b_{2q-3} b_{2q-2} c d}^{a_1 a_2 \dots a_{2q-3} a_{2q-2} i j} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}^{b_{2s-1} b_{2s}} \right) (A_\alpha)_j^d (\nabla f^\alpha)^c. \end{aligned}$$

Hence, using (9), (11), (18) and switching i with j and c with d , we find

$$\begin{aligned}
X_{(q)}^i &= \frac{2^{q-1}}{2^q} \delta_{b_1 b_2 \dots b_{2q-3} b_{2q-2} c d}^{a_1 a_2 \dots a_{2q-3} a_{2q-2} j i} \left(\prod_{s=1}^{q-1} \langle B_{a_{2s-1}}^{b_{2s-1}}, B_{a_{2s}}^{b_{2s}} \rangle \right) (A_\alpha)_j^d (\nabla f^\alpha)^c \\
&= \frac{1}{2} \delta_{b_1 b_2 \dots b_{2q-3} b_{2q-2} d c}^{a_1 a_2 \dots a_{2q-3} a_{2q-2} j i} \left(\prod_{s=1}^{q-1} \langle B_{a_{2s-1}}^{b_{2s-1}}, B_{a_{2s}}^{b_{2s}} \rangle \right) (A_\alpha)_j^d (\nabla f^\alpha)^c \\
&= \frac{1}{2} (2q-1)! (T_{(2q-1)\alpha})_c^i (\nabla f^\alpha)^c \\
&= \frac{1}{2} (2q-1)! (T_{(2q-1)\alpha}) \cdot \nabla f^\alpha)^i \\
&= \frac{1}{2} (2q-1)! (T_{(2q-1)\alpha}) \cdot e_\alpha^\top)^i. \quad \square
\end{aligned}$$

The next identity is a higher-codimensional version of Proposition 3.5(b) in [Reilly 1973] (see also [Alfás et al. 2006, Section 8]).

Proposition 14. *It holds that*

$$\operatorname{div}_e X = \frac{1}{2} L_{(q)} + \frac{1}{2} (2q-1)! \left[(T_{(2q-1)\alpha})_\beta^\top, A_\beta \right] \cdot e_\alpha^\top, e_\beta^\top),$$

where div_e denotes the Euclidean divergence.

Proof. Using the identity

$$\operatorname{div}_e X_{(q)} = \partial_i X_{(q)}^i$$

and the identities (18), (21) and (22), we have

$$\begin{aligned}
\operatorname{div}_g X_{(q)} &= \nabla_i X_{(q)}^i = \partial_i X_{(q)}^i + \Gamma_{ij}^i X_{(q)}^j \\
&= \operatorname{div}_e X_{(q)} + (e_\beta^\top)^i (B_\beta)_{ij} X_{(q)}^j \\
&= \operatorname{div}_e X_{(q)} + \langle A_\beta \cdot X_{(q)}, e_\beta^\top \rangle \\
&= \operatorname{div}_e X_{(q)} + \frac{1}{2} (2q-1)! \langle (A_\beta \circ T_{(2q-1)\alpha}) \cdot e_\alpha^\top, e_\beta^\top \rangle.
\end{aligned}$$

By the expression for the vector field $X_{(q)}$ established in the previous proposition, it follows that

$$\begin{aligned}
\operatorname{div}_g X_{(q)} &= \frac{1}{2} (2q-1)! \operatorname{div}_g (T_{(2q-1)\beta}) \cdot e_\beta^\top \\
&= \frac{1}{2} (2q-1)! \left[\operatorname{div}_g (T_{(2q-1)\beta}) \cdot e_\beta^\top + T_{(2q-1)\beta} \cdot \nabla e_\beta^\top \right].
\end{aligned}$$

By (18) and (23), the identities

$$\nabla e_\beta^\top = \langle B, e_\beta \rangle = U^{\gamma\alpha} \langle \eta_\gamma, e_\beta \rangle B_\alpha = U^{\beta\alpha} B_\alpha$$

hold. Therefore, the Gauss equation together with identities (11) and (4) give

$$T_{(2q-1)\beta} \cdot \nabla e_\beta^\top = U^{\beta\alpha} T_{(2q-1)\beta} \cdot B_\alpha = \frac{1}{(2q-1)!} L_{(q)}.$$

Thus,

$$\operatorname{div}_e X_{(q)} = \frac{1}{2}L_{(q)} + \frac{1}{2}(2q-1)! \left[\operatorname{div}_g (T_{(2q-1)\beta}) \cdot e_\beta^\top - \langle (A_\beta \circ T_{(2q-1)\alpha}) \cdot e_\alpha^\top, e_\beta^\top \rangle \right].$$

Recall that the Newton tensors of a submanifold of Euclidean space are divergence free (see, for example, Lemmata 3.1 and 3.2 of [Cao and Li 2007]) and that each of the fields in the normal frame is given by the expression $\eta_\beta = (-Df^\beta, e_\beta)$. Hence, using identity (18), we have

$$\begin{aligned} (\operatorname{div}_g T_{(2q-1)\beta})_j &= \nabla_i (T_{(2q-1)\beta})^i_j = \nabla_i \langle (T_{(2q-1)})^i_j, \eta_\beta \rangle \\ &= \langle \nabla_i^\perp (T_{(2q-1)})^i_j, \eta_\beta \rangle + \langle (T_{(2q-1)})^i_j, \nabla_i^\perp \eta_\beta \rangle \\ &= \langle (\operatorname{div} T_{(2q-1)})_j, \eta_\beta \rangle + U^{\gamma\alpha} (T_{(2q-1)\alpha})^i_j \langle \eta_\gamma, \bar{D}_i \eta_\beta \rangle \\ &= (T_{(2q-1)\alpha})^i_j U^{\gamma\alpha} f_k^\gamma f_{ik}^\beta = (T_{(2q-1)\alpha})^i_j (e_\alpha^\top)^k (B_\beta)_{ik} \\ &= \langle (T_{(2q-1)\alpha} \circ A_\beta) \cdot e_\alpha^\top \rangle_j, \end{aligned}$$

where \bar{D} is the Levi-Civita connection of the ambient space $\mathbb{R}^{n+m} \equiv \mathbb{R}^n \times \mathbb{R}^m$. Therefore,

$$\begin{aligned} \operatorname{div}_e X_{(q)} &= \frac{1}{2}L_{(q)} + \frac{1}{2}(2q-1)! \langle (T_{(2q-1)\alpha} \circ A_\beta - A_\beta \circ T_{(2q-1)\alpha}) \cdot e_\alpha^\top, e_\beta^\top \rangle \\ &= \frac{1}{2}L_{(q)} + \frac{1}{2}(2q-1)! \langle [T_{(2q-1)\alpha}, A_\beta] \cdot e_\alpha^\top, e_\beta^\top \rangle. \quad \square \end{aligned}$$

Proposition 15. *For a level set $\Sigma \subset \mathbb{R}^n$ in the domain of a Euclidean graph, the identity*

$$\langle X_{(q)}, \xi \rangle = -\frac{1}{2}(2q-1)! \left(\frac{|Df|^2}{1+|Df|^2} \right)^q H_{(2q-1)}$$

holds, where ξ denotes a unit normal vector field along Σ (chosen to point outwards, whenever this makes sense).

Proof. We have

$$\langle X_{(q)}, \xi \rangle = \frac{1}{2}(2q-1)! (T_{(2q-1)\alpha} \cdot e_\alpha^\top)^i \xi_i.$$

Let $x \in \Sigma$. Rotate the coordinates such that, at x , $e_1 = \xi$ and $\{e_A\}_{A=2}^n$ is an orthonormal frame for the tangent space of Σ at x . With respect to this new frame $\{e_i\}_{i=1}^n$ on \mathbb{R}^n ,

$$\xi_i = \delta_i^1,$$

for $i = 1, \dots, n$. Thus,

$$\langle X_{(q)}, \xi \rangle = \frac{1}{2}(2q-1)! (T_{(2q-1)\alpha} \cdot e_\alpha^\top)^1.$$

As in [Li et al. 2014, Section 4], we find that the inverse of g is given by

$$\begin{aligned} g^{11} &= \frac{1}{1+|Df|^2}, \\ g^{A1} &= 0, \\ g^{AB} &= \delta^{AB}. \end{aligned}$$

It follows that

$$e_\alpha^\top = \frac{f_1^\alpha}{1 + |Df|^2} \partial_1 = \frac{\langle Df^\alpha, \xi \rangle}{1 + |Df|^2} \partial_1.$$

Therefore,

$$\langle X_{(q)}, \xi \rangle = \frac{1}{2} (2q - 1)! \frac{\langle Df^\alpha, \xi \rangle}{1 + |Df|^2} (T_{(2q-1)\alpha})_1^1.$$

Since, by (9) and (11),

$$(T_{(2q-1)\alpha})_1^1 = \frac{1}{2^{q-1}} \frac{1}{(2q - 1)!} \delta_{b_1 \dots b_{2q-1}}^{a_1 \dots a_{2q-1}} \left(\prod_{s=1}^{q-1} R_{A_{2s-1} A_{2s}}^{b_{2s-1} b_{2s}} \right) (A_\alpha)_{A_{2q-1}}^{b_{2q-1}},$$

using the antisymmetry of $\delta_{b_1 \dots b_{2q-1}}^{a_1 \dots a_{2q-1}}$ we find that

$$(T_{(2q-1)\alpha})_1^1 = \frac{1}{2^{q-1}} \frac{1}{(2q - 1)!} \delta_{B_1 \dots B_{2q-1}}^{A_1 \dots A_{2q-1}} \left(\prod_{s=1}^{q-1} R_{A_{2s-1} A_{2s}}^{B_{2s-1} B_{2s}} \right) (A_\alpha)_{A_{2q-1}}^{B_{2q-1}}.$$

Recall that the generalized Kronecker delta is a determinant. Using the $2q$ -th column to expand it, we find

$$\delta_{B_1 \dots B_{2q-1}}^{A_1 \dots A_{2q-1}} = \delta_{B_1 \dots B_{2q-1}}^{A_1 \dots A_{2q-1}}.$$

Hence,

$$(T_{(2q-1)\alpha})_1^1 = \frac{1}{2^{q-1}} \frac{1}{(2q - 1)!} \delta_{B_1 \dots B_{2q-1}}^{A_1 \dots A_{2q-1}} \left(\prod_{s=1}^{q-1} R_{A_{2s-1} A_{2s}}^{B_{2s-1} B_{2s}} \right) (A_\alpha)_{A_{2q-1}}^{B_{2q-1}}.$$

Let \hat{R} denote the Riemann curvature tensor of Σ , and denote by K and \tilde{K} , respectively, the second fundamental form of Σ as a hypersurface of \mathbb{R}^n and the second fundamental form of $f(\Sigma)$ as a hypersurface of (M, g) . By equations (4.3) and (4.4) of [Li et al. 2014], we have

$$\tilde{K} = \frac{K}{\sqrt{1 + |Df|^2}}$$

and

$$R_{AB}{}^{CD} = \frac{|Df|^2}{1 + |Df|^2} \hat{R}_{AB}{}^{CD}.$$

Plugging this into the expression for $(T_{(2q-1)\alpha})_1^1$, we find

$$(T_{(2q-1)\alpha})_1^1 = \frac{1}{2^{q-1}} \frac{1}{(2q - 1)!} \left(\frac{|Df|^2}{1 + |Df|^2} \right)^{q-1} \times \delta_{B_1 \dots B_{2q-1}}^{A_1 \dots A_{2q-1}} \left(\prod_{s=1}^{q-1} \hat{R}_{A_{2s-1} A_{2s}}^{B_{2s-1} B_{2s}} \right) (A_\alpha)_{A_{2q-1}}^{B_{2q-1}}.$$

From

$$\eta^\alpha = e_\alpha - Df^\alpha = e_\alpha - \langle Df^\alpha, \xi \rangle \xi,$$

it follows that

$$(A_\alpha)_A^B = -\langle Df^\alpha, \xi \rangle K_A^B.$$

Also, the Gauss equation applied to $\Sigma \subset \mathbb{R}^n$ yields

$$\hat{R}_{AB}^{CD} = K_A^C K_B^D - K_A^D K_B^C.$$

We then conclude that

$$\begin{aligned} (T_{(2q-1)\alpha})_1^1 &= -\frac{1}{2^{q-1}} \frac{1}{(2q-1)!} \langle Df^\alpha, \xi \rangle \left(\frac{|Df|^2}{1+|Df|^2} \right)^{q-1} \\ &\quad \times \delta_{B_1 \dots B_{2q-1}}^{A_1 \dots A_{2q-1}} \left(\prod_{s=1}^{q-1} \hat{R}_{A_{2s-1} A_{2s}}^{B_{2s-1} B_{2s}} \right) (K)_{A_{2q-1}}^{B_{2q-1}} \\ &= -\frac{1}{(2q-1)!} \langle Df^\alpha, \xi \rangle \left(\frac{|Df|^2}{1+|Df|^2} \right)^{q-1} \\ &\quad \times \delta_{B_1 \dots B_{2q-1}}^{A_1 \dots A_{2q-1}} \left(\prod_{s=1}^{q-1} K_{A_{2s-1}}^{B_{2s-1}} K_{A_{2s}}^{B_{2s}} \right) K_{A_{2q-1}}^{B_{2q-1}} \\ &= -\langle Df^\alpha, \xi \rangle \left(\frac{|Df|^2}{1+|Df|^2} \right)^{q-1} H_{(2q-1)}, \end{aligned}$$

where we have used the expression (13) to obtain the last equality. It follows that

$$\begin{aligned} \langle X_{(q)}, \xi \rangle &= \frac{1}{2} (2q-1)! \frac{\langle Df^\alpha, \xi \rangle}{1+|Df|^2} (T_{(2q-1)\alpha})_1^1 \\ &= -\frac{1}{2} (2q-1)! \frac{\langle Df^\alpha, \xi \rangle^2}{1+|Df|^2} \left(\frac{|Df|^2}{1+|Df|^2} \right)^{q-1} H_{(2q-1)} \\ &= -\frac{1}{2} (2q-1)! \left(\frac{|Df|^2}{1+|Df|^2} \right)^q H_{(2q-1)}, \end{aligned}$$

where, to obtain the last equality, we have used that

$$Df^\alpha = \langle Df^\alpha, \xi \rangle \xi$$

implies

$$|Df|^2 = \sum_\alpha \langle Df^\alpha, \xi \rangle^2. \quad \square$$

Remark 16. In Proposition 15, the expression $|Df|^2/(1+|Df|^2)$ is the cosine of the angle between the graph and the hyperplane containing its boundary (see [de Sousa 2016]).

3. Proof of the theorems

Suppose first that M has no boundary. Let S_r be an Euclidean coordinate sphere of radius r . By (6), (24) and the divergence theorem, we have

$$\begin{aligned} m_q &= c_q(n) \lim_{r \rightarrow \infty} \int_{S_r} X_{(q)}^i \xi_i dS_r \\ &= c_q(n) \int_{\mathbb{R}^n} \operatorname{div}_e X_{(q)} dV, \end{aligned}$$

where dV denotes the Euclidean volume form. Thus, invoking Proposition 14 and using that

$$(26) \quad dV = \frac{1}{\sqrt{G}} dM,$$

we find

$$m_q = \frac{1}{2} c_q(n) \int_M (L_{(q)} + (2q - 1)! \langle [T_{(2q-1)\alpha}, A_\beta] \cdot e_\alpha^\top, e_\beta^\top \rangle) \frac{1}{\sqrt{G}} dM,$$

which is exactly the first part of Theorem 6.

To prove the second part of Theorem 6, notice that, from equations (3) and (6) of [Andrzejewski et al. 2016], the tensor $T_{(2q-1)\alpha}$ can be written as a polynomial on the A_α 's. Also, if M has flat normal bundle, then the Ricci equation (10) yields

$$(27) \quad [A_\alpha, A_\beta] = 0,$$

for all $\alpha, \beta \in \{1, \dots, m\}$. Thus, using (27) several times, we find

$$[T_{(2q-1)\alpha}, A_\beta] = 0,$$

for all $\alpha, \beta \in \{1, \dots, m\}$. Hence, (12) becomes

$$m_q = \frac{1}{2} c_q(n) \int_M L_{(q)} \frac{1}{\sqrt{G}} dM.$$

Therefore, if $L_{(q)}$ is nonnegative, then m_q is nonnegative. This finishes the proof of Theorem 6.

Suppose now that ∂M is not empty and that f can be extended to a smooth map on some open set containing $\mathbb{R}^n \setminus \Omega$. This assumption allows us to use the results of Section 2. Equations (6), (24) and the divergence theorem yield

$$\begin{aligned} m_q &= c_q(n) \lim_{r \rightarrow \infty} \int_{S_r} X_{(q)}^i v_i dS_r \\ &= c_q(n) \int_{\mathbb{R}^n \setminus \Omega} \operatorname{div}_e X_{(q)} dV - c_q(n) \int_\Sigma \langle X_{(q)}, \xi \rangle d\Sigma. \end{aligned}$$

Invoking Propositions 14 and 15 and (26), we get

$$(28) \quad m_q = \frac{1}{2}c_q(n) \int_M (L_{(q)} + (2q - 1)!([T_{(2q-1)\alpha}, A_\beta] \cdot e_\alpha^\top, e_\beta^\top)) \frac{1}{\sqrt{G}} dM \\ + \frac{1}{2}(2q - 1)!c_q(n) \int_\Sigma \left(\frac{|Df|^2}{1 + |Df|^2} \right)^q H_{(2q-1)} d\Sigma.$$

This finishes the proof of Theorem 8.

Let us now prove Theorem 9. We cannot use (28) directly, since, by hypothesis,

$$|Df| \rightarrow \infty \text{ as } x \rightarrow \Sigma,$$

and hence, it is not possible to extend f to a smooth function on some open set containing $\mathbb{R}^n \setminus \Omega$. To circumvent this problem, we proceed as in the last section of [Mirandola and Vítório 2015]. Namely, we consider an approximating sequence

$$F^k = (f^{1,k}, \dots, f^{m,k}) : \mathbb{R}^n \setminus \Omega \rightarrow \mathbb{R}^m,$$

$k \in \mathbb{N}$, of smooth maps such that each F^k extends to a smooth map on some open set containing $\mathbb{R}^n \setminus \Omega$. We then apply (28) to each F^k and take the limit as $k \rightarrow \infty$, reaching (14).

It remains to prove inequality (15). If Σ has only one component then, by a result of Guan and Li [2009], it holds that

$$(29) \quad \frac{1}{2}(2q - 1)!c_q(n) \int_\Sigma H_{(2q-1)} d\Sigma \geq \frac{1}{2^q} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2q}{n-1}},$$

with equality holding if and only if Σ is a round sphere.

Suppose now that Σ has more than one component. Recall that if x_1, \dots, x_j are nonnegative real numbers and $0 \leq s < 1$ then

$$(30) \quad \sum_{i=1}^j x_i^s \geq \left(\sum_{i=1}^j x_i \right)^s,$$

with equality holding if and only if at most one of the x_i 's is positive (see [Huang and Wu 2015b, Proposition 5.2]). Inequality (15) then follows by combining inequalities (29) and (30).

Remark 17. Unfortunately our methods are not suitable to deal with the equality cases, that is, to prove the rigidity statements contained in Conjectures 4 and 5. If equality holds in Theorem 6, then we can only conclude that the Gauss–Bonnet curvature $L_{(q)}$ is identically zero. If equality holds in Theorem 9, then we can only conclude that $L_{(q)}$ is zero on M and that Σ has only one component which is a round sphere.

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