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We study the regularity of asymptotically hyperbolic metrics in general dimensions. By carefully constructing harmonic coordinates near the boundary at infinity, a method pioneered by Anderson, we show that, for $m \geq 3$, a $C^{m,\alpha}$ asymptotically hyperbolic metric that satisfies the asymptotic Einstein condition $\|E(g_+)\|_{g_+} = \|\mathrm{Ric}_{g_+} + n g_+\|_{g_+} = o(\rho^2)$ is in fact $C^{m+2,\alpha}$, provided that its Weyl curvature is $C^{m,\alpha}$ and the metric on the boundary that represents its conformal infinity is $C^{m+2,\alpha}$.

1. Introduction

It is well known that there are very close connections between the hyperbolic space $\mathbb{H}^{n+1}$ and its boundary. In recent years, however, many mathematicians tend to be interested in conformally compact Einstein manifolds with negative scalar curvature instead of hyperbolic space. The physics community has also become interested in the compact Einstein manifolds since the introduction of the AdS/CFT correspondence proposed by Maldacena in the theory of quantum gravity in theoretic physics. In this paper, we mainly discuss the boundary regularity problem when the Weyl curvature of the compactification has some regularity.

Let M be the interior of a compact $(n+1)$-dimensional manifold $\overline{M}$ with nonempty boundary ∂M. We say a complete metric g_+ on M is $C^{m,\alpha}$(or $W^{k,p}$) conformally compact if there exists a defining function ρ on $\overline{M}$ such that the conformally equivalent metric

$$g = \rho^2 g_+$$

can extend to a $C^{m,\alpha}$(or $W^{k,p}$) Riemannian metric on $\overline{M}$. The defining function is smooth on $\overline{M}$ and satisfies

$$\begin{cases} \rho > 0 & \text{in } M, \\ \rho = 0 & \text{on } \partial M, \\ d\rho \neq 0 & \text{on } \partial M. \end{cases} \tag{1-1}$$

Here $C^{m,\alpha}$ and $W^{k,p}$ are the usual Hölder space and the Sobolev space.

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The induced metric $h = g|_{\partial M}$ is called the boundary metric associated to the compactification g. The defining function is unique up to a multiplication by a positive function on $\overline{M}$. So the conformal class $[g]$ is uniquely determined by g_+, and the conformal class $[h]$ is uniquely determined by (M, g_+). We call $[h]$ the conformal infinity of g_+. We are interested in Einstein manifolds, which means the metric g_+ also satisfies

$$E(g_+) = \mathrm{Ric}_{g_+} + n g_+ = 0. \tag{1-2}$$

The boundary regularity problem was first raised by Fefferman and Graham in 1985. Namely, given a conformally compact Einstein manifold (M, g_+) and its compactification g, if the boundary metric h is in $C^{m,\alpha}$, is there a $C^{m,\alpha}$ compactification of g_+? In fact, Fefferman and Graham noticed that if $\dim M = n+1$ is odd, the boundary regularity in general breaks down at order n. When $\dim M = n+1$ is even, the $C^{m,\alpha}$ compactification should exist. M. T. Anderson [2003; 2008], solved the problem in dimension 4 by using the Bach equation in dimension 4. He only assumed the original compactification g is in $W^{2,p}$ for some $p > 4$. I'm not sure whether the $W^{2,p}$ condition is good enough for the manifold to improve the boundary regularity. Helliwell [2008] solved the issue in all even dimensions by following Anderson's method. He considered the Fefferman–Graham ambient obstruction tensor instead of the Bach tensor in higher dimensions. Helliwell assumed the compactification g is at least in $C^{n,\alpha}$ for a $(n+1)$ smooth manifold. It means the original compactification is $C^{3,\alpha}$ for a smooth manifold of dimension 4.

In this paper, we follow Anderson's approach to study the boundary regularity in general dimensions. As observed in [Fefferman and Graham 1985], if $\dim M = n+1$ is odd, there are log terms in the asymptotic expansion of g near ∂M at the order n. If we add a condition of g, the log term may be ruled out. By studying (3-8), we find that if the Weyl curvature and scalar curvature of g are smooth enough near boundary, the log term may not exist. In [Jin 2017] and [Shi and Tian 2005], (2-5) tells us that in harmonic coordinates, the regularity of a metric can be improved to two orders higher than the regularity of its Weyl curvature locally in the sense of conformal transformation. Our main aim is to extend the result to the manifolds with boundary.

At the end of this paper, we prove that the regularity of defining function is the same as the new structure on $\overline{M}$ for Einstein case. This extends the result in [Helliwell 2008], where it was proved that the regularity of defining functions is the same as the original compactification.

The equation (2-5) holds for all manifolds, not just for Einstein manifolds. We don't need to use the Einstein equation in the interior of $\overline{M}$. We focus on metrics that satisfy the condition of that Einstein equation vanishing to finite order near boundary, that is

$$\|E(g_+)\|_{g_+} = \|\mathrm{Ric}_{g_+} + n g_+\|_{g_+} = o(\rho^2). \tag{1-3}$$

The main result is as follows:

Theorem 1.1. *Let (M, g_+) be a conformally compact $(n+1)$-manifold with a $C^{m,\alpha}$ conformal compactification $g = \rho^2 g_+$ in a given C^∞ atlas $\{y^\beta\}_{\beta=0}^n$ of $\overline{M}$ near ∂M $(m \geq 3, 0 < \alpha < 1)$. $\|\mathrm{Ric}_{g_+} + n g_+\|_{g_+} = o(\rho^2)$. ρ is a C^∞ defining function of y^β. If the boundary metric $h = g|_{\partial M} \in C^{m+2,\alpha}(\partial M)$ and the Weyl curvature W of g is in $C^{m,\alpha}(\overline{M})$ in the atlas y^β, then there exists atlas $\{x^\beta\}$ of $\overline{M}$ near ∂M and in the atlas $\{x^\beta\}$, g_+ has a $C^{m+2,\alpha}$ compactification $\tilde{g} = \tilde{\rho}^2 g_+$ with boundary metric h. The atlas $\{x^\beta\}$ forms a $C^{m+3,\alpha}$ structure of $\overline{M}$. Furthermore, if g_+ is Einstein, $\tilde{\rho}$ is a $C^{m+2,\alpha}$ function in x-coordinates.*

Remark 1.2. If $\dim M = n+1$ is even, (M, g_+) is Einstein and $m+2 \geq n$, then the defining function in Theorem 1.1, $\tilde{\rho}$, is a $C^{m+3,\alpha}$ function in x-coordinates.

In [Chruściel et al. 2005], Chruściel, Delay, Lee and Skinner showed a good result of the boundary regularity of conformal compact Einstein manifolds. They proved that if the boundary metrics are smooth, the C^2 conformally compact Einstein metrics have conformal compactifications that are smooth up to the boundary in the sense of $C^{1,\lambda}$ diffeomorphism in dimension 3 and all even dimensions, and polyhomogeneous in odd dimensions greater than 3. The C^2 condition is of course weaker than the $C^{n,\alpha}$ condition in Helliwell's paper. I think the C^2 condition is probably sharp. However, their result only holds for the smooth case. It is unknown whether their method can be used to prove the finite boundary regularity. In this paper, by assuming a condition of Weyl tensors, we solve the finite regularity problem for a conformal compact manifold, which is not Einstein, only satisfies (1-3). Furthermore, by observing a calculation in Section 3B we find that the log term of the formal power series of a Weyl tensor vanishes if and only if the obstruction tensor of the metric vanishes. So if we assume the Weyl tensors are in C^{n-2} in Theorem A in [Chruściel et al. 2005] when $n+1$ is odd and greater than 3, we can obtain an extended smooth result. That is:

Remark 1.3. Let (M, g_+) be a conformally compact Einstein $(n+1)$-manifold with a C^2 conformal compactification $g = \rho^2 g_+$. If the boundary metric $h = g|_{\partial M}$ is smooth, then for any $\lambda > 0$, there exists $R > 0$ and a $C^{1,\lambda}$ diffeomorphism $\Phi : \overline{M}_R \to \overline{M}$ such that

$$\Phi^* g_+ = \rho^{-2}(d\rho^2 + G(\rho)),$$

where $\overline{M}_R = \partial M \times [0, R]$, $\{G(\rho) : 0 < \rho \leq R\}$ is a one-parameter family of smooth Riemannian metrics on ∂M.

If $\dim M$ is even or equal to 3, then $\Phi^* g_+$ is conformally compact of class C^∞.

If $\dim M$ is odd and greater than 3, the Weyl tensors of $\rho^2 \Phi^* g_+$ are of class C^{n-2}, then $\Phi^* g_+$ is conformally compact of class C^∞.

The outline of this paper is as follows. In Section 2, we introduce the constant scalar curvature compactification and construct a kind of harmonic coordinates near ∂M. The regularity of the metric does not change in the above two steps. We also study the relationship between Ricci curvature and Weyl curvature for constant scalar curvature.

In Section 3, we provide some literature reviews concerning the study of conformally compact Einstein manifolds and asymptotically hyperbolic metrics satisfying (1-3), including the change of curvature under conformal transformation, the existence and regularity of geodesic defining function. We also show the reason why the boundary regularity in general breaks down at order n when $\dim M = n+1$ is odd. Then after some simple computations, we get that the Weyl curvature has an influence on the regularity in geodesic coordinates. Besides, for conformally compact Einstein manifolds, by calculating the formal power series of the Weyl tensor in geodesic coordinates, we obtain that the obstruction tensor of the conformal metric vanishes if and only if the formal power series of the Weyl tensor doesn't contain an $x^{n-2}\log x$ term. This improves Theorem A in [Chruściel et al. 2005].

Section 4 is dedicated to the derivation of some boundary conditions, including the Dirichlet conditions for g_{ij} and Ricci curvature, and the Neumann conditions for $g^{0\beta}$. We use the geodesic defining function as a transition tool to calculate the Dirichlet conditions for Ricci curvature. The regularity will drop one order when we change the defining function into geodesic defining function. The geodesic compactification ought to be at least C^2 so that we can calculate some curvature tensor. That's why we need $m \geq 3$ in Theorem 1.1. Then with the property of harmonic coordinates, we obtain the Neumann condition for $g^{0\beta}$.

Finally, in Section 5, we use the theory of elliptic system to prove Theorem 1.1. We improve the regularity of the conformal metric and defining function in the new coordinates.

2. Basic geometry equations in harmonic coordinates

In this section, we discuss some basic geometry equations for the manifold $(\overline{M}, g)$. Before doing that, we choose an appropriate conformal compactification to make the scalar curvature constant near boundary and to construct harmonic coordinates for the metric.

2A. *Constant scalar curvature compactification.*

Lemma 2.1. *Let (M, g_+) be a conformally compact n-manifold. M has a $C^{2,\alpha}$ conformal compactification $g = \rho^2 g_+$ and $h = g|_{\partial M}$ is the boundary metric. Then there exists a $C^{2,\alpha}$ constant scalar curvature compactification $\hat{g} = \hat{\rho}^2 g_+$ with boundary metric h.*

Proof. We only need to solve a Yamabe problem with Dirichlet data. Let $\hat{g} = u^{4/(n-2)}g$; then we consider the equation

$$\begin{cases} \Delta_g u - \dfrac{n-2}{4(n-1)} Su + \dfrac{n-2}{4(n-1)} \lambda u^{(n+2)/(n-2)} = 0, \\ u > 0 \qquad \text{in } \overline{M}, \\ u \equiv 1 \qquad \text{on } \partial M. \end{cases} \tag{2-1}$$

Let $\lambda = -1$; from [Ma 1995] we know that the equation always has a $C^{2,\alpha}$ solution. Then $\hat{g} = u^{4/(n-2)}g$ is also in $C^{2,\alpha}$. The boundary metric h is not changed since $u \equiv 1$ on ∂M. □

By the standard theory for elliptic equations, if g is in $C^{m,\alpha}(\overline{M})$ for some $\alpha \in (0, 1)$, any such solution u is in $C^{m,\alpha}(\overline{M})$. Then $\hat{\rho} = u^{2/(n-2)}\rho$ and $\hat{g}$ are also in $C^{m,\alpha}$. The Weyl tensor $\hat{W} = u^{4/(n-2)}W$ of $\hat{g}$ are in $C^{m,\alpha}$. In the following, we don't distinguish g and $\hat{g}$. When we refer to the compactification g, we mean that the scalar curvature S of g is constant.

2B. ***The harmonic coordinates near boundary.*** We call the coordinates $\{x^\beta\}_{\beta=0}^n$ harmonic coordinates with respect to g when $\Delta_g x^\beta = 0$ for $0 \le \beta \le n$. We are now going to construct harmonic coordinates near ∂M which are also harmonic when restricted on $(\partial M, h)$. In the following, if there are no special instructions, any use of indices will follow the convention that Roman indices will range from 1 to n, while Greek indices range from 0 to n.

Firstly, for any point $p \in \partial M$, there is smooth structure $\{y^\beta\}$. It is easy to construct harmonic coordinates $\{x^i\}$ with respect to $(\partial M, h)$. When h is in $C^{m+2,\alpha}$, we have that x^i are $C^{m+3,\alpha}(\partial M)$ functions of y^β. Then

$$h_{ij} = h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \in C^{m+2,\alpha}(\partial M).$$

Then by solving a local Dirichlet problem: $\Delta_g x^i = 0$, with the boundary condition x^i as above, we can extend x^i to $\overline{M}$. Similarly, we can choose a harmonic defining function x^0 which satisfies $\Delta_g x^0 = 0$, $x^0|_{\partial M} = 0$. Then $\{x^\beta\}_{\beta=0}^n$ form harmonic coordinates with respect to g in a neighborhood of ∂M. When g is in $C^{m,\alpha}$, x^β are $C^{m+1,\alpha}$ functions of y^β. Then

$$g_{\alpha\beta} = g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) \in C^{m,\alpha}(\overline{M}).$$

Therefore when we change the coordinates $\{y^\beta\}$ to $\{x^\beta\}$, the regularities of g, the Weyl tensor on $\overline{M}$ and h on ∂M are unchanged.

2C. ***Geometry equations.*** Now we study the relationship between Weyl tensors and Ricci tensors. For any $(n+1)$-manifold, $0 \le i, j, k, l, h, m \le n$, we have

$$R_{ijkl} = \frac{1}{n-1}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{jk}g_{il} - R_{il}g_{jk}) - \frac{S}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}) + W_{ijkl}. \tag{2-2}$$

On the other hand, we have the second Bianchi identity for g.

$$0 = \mathrm{Bian}(g, \mathrm{Ric}) = g^{jh}\left(R_{jk,h} - \tfrac{1}{2}R_{jh,k}\right)$$

When the scalar curvature S is constant, we get

$$g^{jh}R_{jk,h} = 0. \tag{2-3}$$

From (2-2) and (2-3), we have

$$R_{ik,l} - R_{il,k} = \frac{n-1}{n-2}g^{jh}W_{ijkl,h}.$$

It follows

$$g^{lt}R_{ik,lt} - g^{lt}R_{il,kt} = \frac{n-1}{n-2}g^{lt}g^{jh}W_{ijkl,ht}.$$

Note that

$$R_{il,kt} = R_{il,tk} + \mathrm{Ric} * Rm, \quad g^{lt}R_{il,tk} = (g^{lt}R_{il,t})_k = 0.$$

Where $\mathrm{Ric} * Rm$ refers to a bilinear form of R_{ij} and R_{ijkl}. We finally get

$$g^{lt}R_{ik,lt} = g * \mathrm{Ric} * Rm + g * g * (\partial^2 W + \Gamma * \partial W + \Gamma * \Gamma * W + \partial\Gamma * W). \tag{2-4}$$

If $g \in C^{m,\alpha}$, in harmonic coordinates $\{x^\beta\}$, (2-4) can be written in the form:

$$g^{\gamma\tau}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial x^\tau}R_{\alpha\beta} = Q(g, \partial g, \partial^2 g, W, \partial W, \partial^2 W), \tag{2-5}$$

where Q is a polynomial. Even when $m = 3$, we can define the first and second covariant derivatives of curvature in the sense of distribution (see [Jin 2017; Shi and Tian 2005]). In this case (2-5) still holds. For more details about geometric equations, see [Petersen 1998].

3. The conformal infinity of asymptotically hyperbolic metrics

We will discuss some background material for conformally compact metrics in this section. Just as in the definition in introduction, we let ρ be a defining function for M, and set

$$g = \rho^2 g_+.$$

We assume ρ is $C^{m,\alpha}$ on $\overline{M}$ in the initial atlas $\{y^\beta\}$, and that m, α are defined as in Theorem 1.1. We have that g_+ satisfies

$$E(g_+) = \mathrm{Ric}_{g_+} + ng_+, \quad \|E(g_+)\|_{g_+} = o(\rho^2).$$

The curvature of g can be expressed as the following formulas:

$$K_{ab} = \frac{K_{+ab}+|\nabla\rho|^2}{\rho^2} - \frac{1}{\rho}[D^2\rho(e_a,e_a)+D^2\rho(e_b,e_b)]. \tag{3-1}$$

$$\mathrm{Ric} = -(n-1)\frac{D^2\rho}{\rho} + \left[\frac{n(|\nabla\rho|^2-1)}{\rho^2} - \frac{\Delta\rho}{\rho}\right]g + E(g_+), \tag{3-2}$$

$$S = -2n\frac{\Delta\rho}{\rho} + n(n+1)\frac{|\nabla\rho|^2-1}{\rho^2} + \mathrm{tr}_g\, E(g_+). \tag{3-3}$$

Here D^2 is the Hessian. $E(g_+)$ satisfies $\|\mathrm{tr}_g\, E(g_+)\| = \|E(g_+)\|_g = o(1)$, which implies the tensor $E = E(g_+)$ vanishes on ∂M.

When g is at least C^2 in $\overline{M}$, $|\nabla\rho| \to 1$ as $\rho \to 0$, and $|K_{+ab}+1| = O(\rho^2)$. Hence a C^2 conformally compact Einstein manifold is asymptotically hyperbolic. On ∂M, we have $|\nabla\rho| = 1$. Let $D^2\rho|_{\partial M} = A$ denote the second fundamental form of ∂M in $(\overline{M}, g)$. The equation (3-2) further implies that ∂M is umbilic, i.e., $A = \lambda h$, for some function λ on ∂M, see Lemma 4.2.

3A. ***Geodesic conformal compactification.*** As noted in the introduction, defining functions are unique only up to multiplication by positive functions on $\overline{M}$. From the formulas (3-2) and (3-3), we can see that it is special if the defining function r and its compactification $\bar{g} = r^2 g_+$ satisfies

$$|\overline{\nabla} r|_{\bar{g}} \equiv 1$$

in a neighborhood of ∂M. Such defining functions are called geodesic defining functions. In fact, the defining functions always exist and were first introduced by Graham and Lee [1991].

Lemma 3.1. *Let g be a C^2 conformal compactification of (M, g_+), $g = \rho^2 g_+$. $h = g|_{\partial M}$ is the boundary metric. Then g_+ has a unique geodesic conformal compactification with the same boundary metric h.*

Proof. Let $r = u\rho$, $\bar{g} = r^2 g_+$. The lemma is equivalent to the equation:

$$\begin{cases} 2(\nabla\rho)(\log u) + \rho|\nabla \log u|_g^2 = \frac{1-|\nabla\rho|_g^2}{\rho}, \\ u \equiv 1 \qquad \text{on } \partial M. \end{cases} \tag{3-4}$$

By general theory of first order partial differential equations, we know that it has a unique positive solution in a collar neighborhood U of ∂M. □

However, that $\bar{g}$ may not be as smooth as g. When g is in $C^{m,\alpha}$, we only have $u \in C^{m-1,\alpha}$, hence $\bar{g}$ is only a $C^{m-1,\alpha}$ conformal compactification. Here we refer the reader to [Lee 1995, Lemma 5.1] for more details concerning the Hölder regularity.

It is obvious that when $\|\mathrm{Ric}_{g_+} + n g_+\|_{g_+} = o(\rho^2)$ for the defining function ρ, we also have $\|\mathrm{Ric}_{g_+} + n g_+\|_{g_+} = o(r^2)$ for the geodesic defining function.

In the following of this section, we assume g_+ is Einstein, i.e.,

$$E(g_+) = \mathrm{Ric}_{g_+} + ng_+ = 0.$$

Under this condition, the terms $E(g_+)$ and $\mathrm{tr}_g E(g_+)$ in (3-2) and (3-3) do not exist.

When M has a C^2 geodesic conformal compactification $\bar{g}$, it is very convenient for us to do some calculations. From (3-2) and (3-3), we know that the second fundamental form $\bar{A}$ given by $\bar{A} = \bar{D}^2 r = 0$ on ∂M, i.e., ∂M is totally geodesic in $\overline{M}$. The Gauss lemma implies that $\bar{g}$ can split in U,

$$\bar{g} = dr^2 + g_r$$

for a 1-parameter family g_r of metrics on ∂M.

Now we choose the local coordinates $(r, x^1, x^2, \ldots, x^n)$ near ∂M to study the regularity of $\bar{g}$ near the boundary ∂M. Using (3-2) and the Gauss–Codazzi equation, we finally get

$$\begin{aligned}(3\text{-}5)\quad & r\partial_r^2 \bar{g}_{ij} - (n-1)\partial_r \bar{g}_{ij} - \bar{g}^{kl}\partial_r \bar{g}_{kl}\bar{g}_{ij} - r\bar{g}^{kl}\partial_r \bar{g}_{kl}\partial_r \bar{g}_{ij} \\ & \qquad + \frac{r}{2}\bar{g}^{kl}\partial_r \bar{g}_{ik}\partial_r \bar{g}_{jl} - 2r\overline{\mathrm{Ric}}(g_r)_{ij} = 0.\end{aligned}$$

Here $\overline{\mathrm{Ric}}(g_r)_{ij}$ denotes the Ricci tensor for the induced metric on level sets of r. We assume that $\bar{g}$ is smooth enough so that we could make its expansion from (3-5). Let $r = 0$, we can derive that $\partial_r \bar{g}(0) = 0$. By using mathematical induction, we differentiate (3-5) $p-1$ times with respect to r

$$\begin{aligned}(3\text{-}6)\quad & r\partial_r^{p+1}\bar{g}_{ij} + (p-n)\partial_r^p \bar{g}_{ij} - \bar{g}^{kl}(\partial_r^p \bar{g}_{kl})\bar{g}_{ij} \\ & \qquad = Q_1(\bar{g}^{-1}, \partial_r^q \bar{g}, \partial_r^q \bar{g}\partial_s^2 \bar{g}) + rQ_2(\bar{g}^{-1}, \partial_r^q \bar{g}, \partial_r^p \bar{g}, \partial_r^{p-2}\partial_s^2 \bar{g}).\end{aligned}$$

Here $q < p$ and $\partial_s \bar{g}$ is the differential of $\bar{g}$ with respect to $x^i (1 \le i \le n)$. Q_1, Q_2 are the third polynomials.

Setting $r = 0$, (3-6) implies that $\partial_r^p \bar{g}(0) = 0$ when p is odd and that $\partial_r^p \bar{g}(0)$ is uniquely determined by each step when p is even. However, this will break down when $p = n$ and $n+1$ is odd. In this case we only have $\bar{g}^{kl}(\partial_r^n \bar{g}_{kl}) = 0$ at $r = 0$. It gives no further information at this order. To be specific, $\partial_r \bar{g}(0) = \partial_r^3 \bar{g}(0) = \cdots = \partial_r^{n-1}\bar{g}(0) = 0$, which means

$$\partial_r^{n-1}\bar{g}(r) = O(1).$$

Let's consider (3-6) when $p = n-1$,

$$(3\text{-}7)\qquad \partial_r^n \bar{g}_{ij} = \frac{\partial_r^{n-1}\bar{g}_{ij} - \bar{g}^{kl}(\partial_r^{n-1}\bar{g}_{kl})\bar{g}_{ij} + Q_1}{r} + Q_2.$$

If $\partial_r^{n-1}\bar{g}(r) \neq O(r)$, there will be the log term in the expansion for g_r. Note that $\partial_r^{n-1}\bar{g}(r) = O(r)$ is the necessary condition to ensure $\bar{g}$ is n-th differentiable.

3B. ***Weyl tensor in geodesic coordinates.*** In this subsection we shall show that if the Weyl tensor $\overline{W}$ of $\bar{g}$ is $(n-2)$-th differentiable, then $\partial_r^{n-1}\bar{g}(r) = O(r)$ holds.

We begin with (2-4) by taking $x^l = r$, $1 \le i, k \le n$.

In the coordinates $(r, x^1, x^2, \ldots, x^n)$, we have

$$\bar{R}_{ik,r} = \partial_r^3 \bar{g} + \partial_r \bar{g} * \partial_r^2 \bar{g} * \bar{g} + \partial_r \bar{g} * \partial_r \bar{g} * \partial_r \bar{g} * \bar{g} * \bar{g} + \partial_r P + \partial_r \bar{g} * P,$$
$$\bar{R}_{ir,k} = \partial_r P * P + \partial_r \bar{g} * \partial_r^2 \bar{g} * \bar{g} + \partial_r \bar{g} * \partial_r \bar{g} * \partial_r \bar{g} * \bar{g} * \bar{g} + \partial_r \bar{g} * \bar{g} * \overline{\mathrm{Ric}}.$$

Here $P = P(\bar{g}, \partial_{x^i}\bar{g}, \partial^2_{x^i}\bar{g})$ is a polynomial for $1 \le i \le n$. From (2-4) we have:

$$\partial_r^3 \bar{g} = \partial_r \bar{g} * \partial_r^2 \bar{g} * \bar{g} + \partial_r \bar{g} * \partial_r \bar{g} * \partial_r \bar{g} * \bar{g} * \bar{g} + \partial_r P * P + \partial_r \bar{g} * \bar{g} * \overline{\mathrm{Ric}} + \bar{g} * \nabla \overline{W}. \tag{3-8}$$

We already know that $\partial_r g = \partial_r^3 g = \cdots \partial_r^{n-3} = O(r)$. Then differentiating (3-8) $n-4$ times with respect to r, we get that each term on the right-hand side is $O(r)$ apart from $\bar{g} * \nabla \overline{W}$. The left side is $\partial_r^{n-1}\bar{g} = O(1)$. Hence $\partial_r^{n-3}\overline{W} = O(1)$. When $\overline{W} \in C^{n-2}$, we get that $\partial_r^{n-3}\overline{W} = O(r)$. So the left-hand side is $\partial_r^{n-1}\bar{g} = O(r)$. This gives the necessary condition that $\bar{g}$ has C^n regularity or higher.

We calculate the formal power series of Weyl tensor in geodesic coordinates. We have $g = dr^2 + g_r$. From (3-6), (3-7) and [Graham 2000], when n is odd,

$$g_r = h + g^{(2)}r^2 + (\text{even powers of } r) + g^{(n-1)}r^{n-1} + g^{(n)}r^n + \cdots. \tag{3-9}$$

When n is even,

$$g_r = h + g^{(2)}r^2 + (\text{even powers of } r) + g^{(n)}r^n + f r^n \log r + \cdots, \tag{3-10}$$

where $g^{(2i)}$ and f are two tensors on ∂M and f is trace free and determined by h locally. We now consider the case when n is even. If $f = 0$, g is at least in C^n. Let $1 \le i, j \le n$.

$$W_{irjr} = R_{irjr} - \frac{1}{n-1}(R_{ik} + R_{rr}g_{ij}) + \frac{S}{n(n-1)}g_{ij} \tag{3-11}$$

The formal power series of the Weyl tensor contain r and $\log r$ and we only need to check the coefficients of $r^{n-2}\log r$. A short calculation reveals that the coefficients of $r^{n-2}\log r$ of R_{irjr} are

$$\mathrm{coeff}(R_{irjr}) = -\frac{n(n-1)}{2} f_{ij}.$$

And also

$$\mathrm{coeff}(R_{ij}) = -\frac{n(n-1)}{2} f_{ij},$$
$$\mathrm{coeff}(R_{rr}) = -\frac{n(n-1)}{2} h^{st} f_{st},$$
$$\mathrm{coeff}(S) = -\frac{n(n-1)}{2} 2h^{st} f_{st}.$$

When coeff$(W_{irjr}) = 0$, by (3-11) we finally derive

$$f_{ij} - \frac{1}{n}(h^{st} f_{st}) h_{ij} = 0. \tag{3-12}$$

Since f is trace free, $f \equiv 0$. This gives the n-th regularity of g. From Theorem A in [Chruściel et al. 2005], we know the log term is the only obstruction of the smoothness, and hence Remark 1.3 holds.

4. The boundary condition

In this section, we derive a boundary problem for g and Ricci curvature of a conformal compact Einstein manifold in the harmonic coordinates as defined in Section 2. The compactification we use here will not be a geodesic compactification, but rather one which has constant scalar curvature. We do it locally, that is, for any $p \in \partial M$, there is a neighborhood V which contains p and local atlas $\{x^\beta\}$. Let $D = V \cap \partial M$ be the boundary portion. $W \in C^{m,\alpha}(V)$, $g \in C^{m,\alpha}(V)$, $h \in C^{m+2,\alpha}(D)$. We will give the Dirichlet and Neumann boundary conditions of g and Ric(g) on D.

4A. ***Dirichlet boundary conditions on g_{ij}.***

$$g_{ij} = h_{ij}. \tag{4-1}$$

4B. ***Dirichlet boundary conditions on R_{ij}.*** We claim that

$$R_{ij} = \frac{n-1}{n-2}(\mathrm{Ric}_h)_{ij} + \left(\frac{1}{2n}S - \frac{1}{2(n-2)}S_h\right)h_{ij} + \frac{n-1}{2n^2}H^2 h_{ij}. \tag{4-2}$$

Here Ric_h and S_h are the Ricci curvature and scalar curvature of $(\partial M, h)$, and H is the mean curvature, $H = g^{ij}A_{ij}$. We use the following three lemmas to prove (4-2).

Lemma 4.1. *Let $\bar{g} = r^2 g_+$ be a C^2 geodesic compactification of (M, g_+) with boundary metric h on ∂M. Then on ∂M,*

$$\bar{S} = \frac{n}{n-1}S_h. \tag{4-3}$$

$$\bar{R}_{ij} = \frac{n-1}{n-2}(\mathrm{Ric}_h)_{ij} - \frac{1}{2(n-1)(n-2)}S_h h_{ij}. \tag{4-4}$$

Proof. Since we only need to study the Ricci curvature on $T\partial M$, we can choose the coordinates $(r, x^1, \ldots, x^n)$ in V where $(x^1, \ldots, x^n)$ are harmonic with respect to h when restricted on D. We have $\bar{g} = dr^2 + g_r$, i.e.,

$$g_{ri} = g^{ri} = 0, \quad g_{rr} = g^{rr} = 1.$$

Since the second fundamental form $\bar{A} = \bar{D}^2 r$ vanishes on ∂M,

$$\begin{aligned}\bar{R}_{ij} &= \bar{g}^{\alpha\beta}\bar{R}_{i\alpha\beta j} \\ &= \bar{g}^{kl}((R_h)_{iklj} + \bar{A}_{il}\bar{A}_{kj} - \bar{A}_{ij}\bar{A}_{kl}) + \bar{R}_{irrj} \\ &= (R_h)_{ij} + \bar{R}_{irrj}.\end{aligned} \tag{4-5}$$

Taking the trace for i, j, we get

$$\bar{R}_{rr} = \tfrac{1}{2}(\bar{S} - S_h). \tag{4-6}$$

For $\bar{R}_{irrj}$, we have:

$$\begin{aligned}\bar{R}_{irrj} &= \bar{g}(\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_r}\partial_r, \partial_j) - \bar{g}(\overline{\nabla}_{\partial_r}\overline{\nabla}_{\partial_i}\partial_r, \partial_j) - \bar{g}(\overline{\nabla}_{[\partial_r,\partial_i]}\partial_r, \partial_j) \\ &= -\partial_r\bar{g}(\overline{\nabla}_{\partial_i}\partial_r, \partial_j) + \bar{g}(\overline{\nabla}_{\partial_i}\partial_r, \overline{\nabla}_{\partial_r}\partial_j) \\ &= -\partial_r\bar{A}_{ij}.\end{aligned} \tag{4-7}$$

From (3-2) and (3-3),

$$\bar{R}_{ij} = -(n-1)\frac{\bar{A}_{ij}}{r} - \frac{\bar{\Delta} r}{r}\bar{g}_{ij} + E_{ij}, \tag{4-8}$$

$$\bar{S} = -2n\frac{\bar{\Delta} r}{r} + \mathrm{tr}_g\, E. \tag{4-9}$$

The tensor E and $\mathrm{tr}_g\, E$ vanish on the boundary, hence on ∂M,

$$\bar{S} = -2n\partial_r\bar{\Delta} r. \tag{4-10}$$

Taking the trace on (4-8),

$$\bar{S} - \bar{R}_{rr} = -(n-1)\frac{\overline{H}}{r} - n\frac{\bar{\Delta} r}{r} + \bar{g}^{ij}E_{ij}.$$

By l'Hôspital's rule, with $r \to 0$,

$$\partial_r\bar{A}_{ij}|_{r=0} = -\frac{1}{n-1}(\bar{R}_{ij} + \partial_r\bar{\Delta} r\bar{g}_{ij})|_{r=0}, \tag{4-11}$$

$$\partial_r\bar{\Delta} r|_{r=0} = -\frac{1}{n}(\bar{S} - \bar{R}_{rr} + (n-1)\partial_r\overline{H})|_{r=0}. \tag{4-12}$$

At last, we only need to calculate $\partial_r\overline{H}|_{r=0}$.

$$\begin{aligned}\partial_r\overline{H}|_{r=0} &= \partial_r(\bar{g}^{ij}\bar{A}_{ij})|_{r=0} = (\partial_r\bar{g}^{ij})\bar{A}_{ij}|_{r=0} + \bar{g}^{ij}(\partial_r\bar{A}_{ij})|_{r=0} \\ &= 0 + \bar{g}^{ij}(-\bar{R}_{irrj}) = -\bar{R}_{rr}.\end{aligned} \tag{4-13}$$

Combining the formulas above, we finally conclude (4-3) and (4-4). □

Lemma 4.2. *Let $g = \rho^2 g_+$ be a $C^{3,\lambda}$ conformal compactification and $\bar{g} = r^2 g_+$ be the $C^{2,\lambda}$ geodesic conformal compactification of (M, g_+) with the same boundary metric $g|_{\partial M} = \bar{g}|_{\partial M} = h$. If $r = u\rho$, $A = D^2\rho$, then $A|_{\partial M} = -u_r h$.*

Proof. We begin with the coordinates $(r, x^1, x^2, \ldots, x^n)$. On ∂M, $0 = \bar{A}_{ij} = -\overline{\Gamma}^r_{ij}$. The connection ∇ and $\overline{\nabla}$ of g and $\bar{g}$ have the following relationship:

$$\Gamma^r_{ij} = \overline{\Gamma}^r_{ij} - \frac{1}{u}(\delta^r_j u_i + \delta^r_i u_j - g_{ij}u_r) = \frac{1}{u}u_r h_{ij}.$$

We have $g = u^{-2}\bar{g}$, $\operatorname{grad}_g = u^2 \operatorname{grad}_{\bar{g}}$.

$$\begin{aligned} A_{ij} &= D^2\rho(\partial_i, \partial_j) = g(\nabla_{\partial_i}\nabla\rho, \partial_j) = -g(\nabla\rho, \nabla_{\partial_i}\partial_j) \\ &= -\Gamma^r_{ij} g(\nabla\rho, \partial_r) = -\Gamma^r_{ij}\bar{g}(\overline{\nabla}\rho, \partial_r) \\ &= -\Gamma^r_{ij}\bar{g}\left(\overline{\nabla}\left(\frac{r}{u}\right), \partial_r\right) = -\Gamma^r_{ij}\bar{g}\left(\frac{u\overline{\nabla}r - r\overline{\nabla}u}{u^2}, \partial_r\right) \\ &= -\Gamma^r_{ij}\bar{g}(\overline{\nabla}r, \overline{\nabla}r) = -u_r h_{ij}. \end{aligned} \tag{4-14}$$

□

Lemma 4.2 tells us $u_r = -H/n$. Since $u|_{\partial M} \equiv 1$, we get

$$\overline{\nabla}u = -\frac{H}{n}\overline{\nabla}r.$$

Lemma 4.3. *We define g, $\bar{g}$ as in Lemma 4.2. Then on ∂M,*

$$\begin{aligned} R_{ir} &= -\frac{n-1}{n}\frac{\partial H}{\partial x_i}, \\ R_{rr} &= \tfrac{1}{2}(S - S_h) - \frac{n-1}{2n}H^2, \\ R_{ij} &= \bar{R}_{ij} + \left(\frac{1}{2n}(S - \bar{S})\right)h_{ij} + \frac{n-1}{2n^2}H^2 h_{ij}. \end{aligned} \tag{4-15}$$

Proof. By the standard formulas for conformal changes of the metric $g = u^{-2}\bar{g}$, the Ricci curvature of g and $\bar{g}$ are related by

$$\operatorname{Ric} = \overline{\operatorname{Ric}} + (n-1)\frac{\overline{D}^2 u}{u} + \left(\frac{\bar{\Delta}u}{u} + \frac{n|\overline{\nabla}u|^2_{\bar{g}}}{u^2}\right)\bar{g},$$

Since

$$\begin{aligned} \bar{\Delta}u &= \operatorname{div}\overline{\nabla}u = \operatorname{div}\left(-\frac{H}{n}\overline{\nabla}r\right) = -\frac{\partial_r H}{n}, \\ \overline{D}^2 u(\partial_i, \partial_j) &= 0, \\ \overline{D}^2 u(\partial_i, \partial_r) &= u_{ir} = -\frac{1}{n}\frac{\partial H}{\partial x_i}, \\ \overline{D}^2 u(\partial_r, \partial_r) &= -\frac{\partial_r H}{n} = \bar{\Delta}u; \end{aligned}$$

then on ∂M,

$$R_{ir} = \bar{R}_{ir} - \frac{n-1}{n}\frac{\partial H}{\partial x_i} = -\frac{n-1}{n}\frac{\partial H}{\partial x_i},$$

$$R_{rr} = \bar{R}_{rr} + n\bar{\Delta}u + \frac{H^2}{n},$$

$$R_{ij} = \bar{R}_{ij} + \left(\bar{\Delta}u + \frac{H^2}{n}\right)h_{ij}. \tag{4-16}$$

Therefore

$$S = \bar{S} + 2n\bar{\Delta}u + \frac{n+1}{n}H^2.$$

This implies

$$\bar{\Delta}u = \frac{1}{2n}\left(S - \bar{S} - \frac{n+1}{n}H^2\right). \tag{4-17}$$

Lemma 4.3 follows from (4-16) and (4-17). □

Finally, Lemmas 4.1 and 4.3 imply the formula (4-2).

4C. ***Neumann boundary conditions on*** $g^{0\alpha}$**.** We use the harmonic coordinates $\{x^\beta\}_{\beta=0}^n$ as defined in Section 2. Let $N = \nabla x_0/|\nabla x_0|$ be the unit norm vector on ∂M. In coordinates $\{x^\beta\}_{\beta=0}^n$,

$$N = (g^{00})^{-1/2} g^{0\beta}\partial_\beta.$$

These are the Neumann conditions:

$$N(g^{00}) = -2Hg^{00}. \tag{4-18}$$

$$N(g^{0i}) = -Hg^{0i} + \tfrac{1}{2}(g^{00})^{-1/2} g^{i\beta}\partial_\beta g^{00}. \tag{4-19}$$

Proof. Let $\{e_i\}_{i=1}^n$ be the orthonormal basis at a given point $p \in \partial M$.

$$0 = \Delta x^\alpha = \operatorname{div} \nabla x^\alpha = \sum_{i=1}^n g(\nabla_{e^i}\nabla x^\alpha, e_i) + g(\nabla_N \nabla x^\alpha, N) \tag{4-20}$$

Write $\nabla x^\alpha = (\nabla x^\alpha)^T + (\nabla x^\alpha)^N$, where

$$(\nabla x^\alpha)^T = \nabla_h x^\alpha, \quad (\nabla x^\alpha)^N = g(\nabla x^\alpha, N)N = (g^{00})^{-1/2} g^{0\alpha} N.$$

We have

$$0 = \Delta_h x^j = \operatorname{div} \nabla_h x^j = \sum_{i=1}^n h(\nabla^h_{e^i}(\nabla x^j)^T, e_i) = \sum_{i=1}^n g(\nabla_{e^i}(\nabla x^j)^T, e_i),$$

which also holds for $j = 0$ as $x^0 \equiv 0$ on ∂M. Then (4-20) turns into

$$\sum_{i=1}^n g(\nabla_{e^i}((g^{00})^{-1/2} g^{0\alpha} N), e_i) + g(\nabla_N \nabla x^\alpha, N) = 0 \tag{4-21}$$

The first term is just $(g^{00})^{-1/2}g^{0\alpha}\sum_{i=1}^{n}A(e_i,e_i)=H(g^{00})^{-1/2}g^{0\alpha}$ and the second term is

$$
\begin{aligned}
g(\nabla_N\nabla x^\alpha,N)&=\frac{1}{|\nabla x^0|^2}g(\nabla_{\nabla x^0}\nabla x^\alpha,\nabla x^0)\\
&=\frac{1}{|\nabla x^0|^2}[\nabla x^0(g(\nabla x^\alpha,\nabla x^0))-g(\nabla x^\alpha,\nabla_{\nabla x^0}\nabla x^0)]\\
&=\frac{1}{|\nabla x^0|^2}[\nabla x^0(g^{0\alpha})-g(\nabla x^0,\nabla_{\nabla x^\alpha}\nabla x^0)]\\
&=\frac{1}{|\nabla x^0|}N(g^{0\alpha})-\frac{1}{2|\nabla x^0|^2}\nabla x^\alpha(g^{00})
\end{aligned}
\tag{4-22}
$$

We know $|\nabla x^0|=(g^{00})^{1/2}$, then (4-21) is just

$$
Hg^{0\alpha}+N(g^{0\alpha})-\tfrac{1}{2}(g^{00})^{-1/2}\nabla x^\alpha(g^{00})=0. \tag{4-23}
$$

Letting $\alpha=0$, we have $N(g^{00})=-2Hg^{00}$. Letting $\alpha=i$, we have

$$
N(g^{0i})=-Hg^{0i}+\tfrac{1}{2}(g^{00})^{-1/2}g^{i\beta}\partial_\beta g^{00}. \qquad \square
$$

4D. ***Dirichlet boundary conditions on $R_{\alpha\beta}$.*** We already know the formulas of R_{ij} on ∂M and the mixed components R_{ri} and R_{rr} of Ricci Curvature in the coordinates $(r,x^1,x^2,\dots,x^n)$ in Section 4B. That is,

$$
\begin{aligned}
R_{ir}&=-\frac{n-1}{n}\frac{\partial H}{\partial x_i},\\
R_{rr}&=\tfrac{1}{2}(S-S_h)-\frac{n-1}{2n}H^2.
\end{aligned}
$$

Now we study the the mixed components R_{0i} and R_{00} of Ricci Curvature in the harmonic coordinates $(x^0,x^1,\dots,x^n)$. In fact, as the vector $N=\nabla x_0/|\nabla x_0|$ is also the unit norm vector on ∂M with respect to g, we have $N=\overline{\nabla}r=\nabla r$. That is, $\nabla r=(g^{00})^{-1/2}g^{0\beta}\partial_\beta$. Then

$$
R_{0i}=-(g^{00})^{-1/2}\,\frac{n-1}{n}\frac{\partial H}{\partial x_i}-\frac{g^{0j}}{g^{00}}R_{ij} \tag{4-24}
$$

$$
R_{00}=\frac{1}{(g^{00})^2}\left(g^{0i}g^{0j}R_{ij}+g^{00}\left(\tfrac{1}{2}(S-S_h)-\frac{n-1}{2n}H^2\right)\right) \tag{4-25}
$$

4E. ***Neumann boundary conditions on R_{0i}.*** The Dirichlet conditions for R_{0i} in (4-24) are not good because there are still second order differential terms of metric on the right side. Now we consider the differential terms of Ricci curvature.

Since the scalar curvature is constant, with the second Bianchi identity, we have

$$
0=\tfrac{1}{2}S_{,\alpha}=g^{\eta\beta}R_{\eta\alpha,\beta}=g^{\eta\beta}\partial_\beta R_{\alpha\eta}-g^{\eta\beta}\Gamma^\tau_{\alpha\beta}R_{\eta\tau}.
$$

Then

$$g^{0\beta}\partial_\beta R_{0\alpha} = -g^{j\beta}\partial_\beta R_{j\alpha} + g^{\eta\beta}\Gamma^\tau_{\alpha\beta}R_{\eta\tau}.$$

Letting $\alpha = i$, we get

$$N(R_{0i}) = (g^{00})^{-1/2}(-g^{j\beta}\partial_\beta R_{ji} + g^{\eta\beta}\Gamma^\tau_{i\beta}R_{\eta\tau}). \tag{4-26}$$

5. Proof of the main theorem

In this section, we prove the main theorem. Suppose $m \geq 3$ and $\alpha \in (0, 1)$. For any point $p \in \partial M$, we select the harmonic coordinates $\{x^\beta\}_{\beta=0}^n$ in its neighborhood V. Let $D = V \cap \partial M$ be the boundary portion. Now we have $g \in C^{m,\alpha}(V)$, $h \in C^{m+2,\alpha}(D)$, the Weyl tensor $W \in C^{m,\alpha}(V)$.

5A. *Regularity of the metric.*

Step 1 (regularity of the Ricci curvature). We begin with the elliptic equation (2-5), with the right side of (2-5) in $C^{m-2,\alpha}$.

On ∂M, we already derived the formulas of $R_{\alpha\beta}$ in Section 3. We have $H = g^{ij}A_{ij}$, where $A_{ij} = \frac{1}{2}(g^{00})^{1/2}g^{0\beta}(\partial_\beta g_{ij} - \partial_i g_{\beta j} - \partial_j g_{\beta i}) \in C^{m-1,\alpha}(D)$. Then the Dirichlet condition of R_{ij} and R_{00} given by (4-4) and (4-25) shows that

$$R_{ij} \in C^{m-1,\alpha}(D), \quad R_{00} \in C^{m-1,\alpha}(D).$$

By standard elliptic regularity theory,

$$R_{ij} \in C^{m-1,\alpha}(V), \quad R_{00} \in C^{m-1,\alpha}(V).$$

Then by the Neumann boundary conditions on R_{0i} given by (4-26), we also have $R_{0i} \in C^{m-1,\alpha}(V)$ since $N(R_{0i}) \in C^{m-2,\alpha}(D)$.

In the following, we prove that $g_{\alpha\beta} \in C^{m+1,\alpha}(V)$ when $R_{\alpha\beta} \in C^{m-1,\alpha}(V)$. If it holds, we can get that $g_{\alpha\beta} \in C^{m+2,\alpha}(V)$ by repeating the steps.

Step 2 (regularity of g_{ij}). In harmonic coordinates, from [DeTurck and Kazdan 1981], we have

$$\Delta g_{ij} = -2R_{ij} + Q(g, \partial g).$$

Here Q contains a quadratic term of g and ∂g. Hence $\Delta g_{ij} \in C^{m-1,\alpha}(V)$, together with the boundary condition $g_{ij} = h_{ij} \in C^{m+2,\alpha}(D)$. From the standard elliptic theory in [Gilbarg and Trudinger 1977], we conclude that $g_{ij} \in C^{m+1,\alpha}(V)$.

Step 3 (regularity of $g_{0\beta}$). In Section 4, we obtain the Neumann boundary conditions of $g^{0\beta}$ which are expressed by mean curvature H. Since $H \in C^{m-1,\alpha}(D)$, we can't improve the regularity of $g^{0\beta}$ with this condition. Now we are going to calculate

the oblique derivative of $g_{0\beta}$ on ∂M. As $g^{0\alpha}g_{\alpha\beta}=\delta^0_\beta$,

$$\begin{aligned}
0 &= N(g^{0\alpha}g_{\alpha\beta}) = N(g^{00}g_{0\beta}) + N(g^{0j}g_{j\beta})\\
&= g^{00}N(g_{0\beta}) + g_{0\beta}N(g^{00}) + g_{j\beta}N(g^{0j}) + g^{0j}N(g_{j\beta})\\
&= g^{00}N(g_{0\beta}) + g_{0\beta}(-2Hg^{00})\\
&\qquad + g_{j\beta}\big(\tfrac{1}{2}(g^{00})^{-1/2}g^{j\tau}\partial_\tau g^{00} - Hg^{0j}\big) + g^{0j}N(g_{j\beta})\\
&= g^{00}N(g_{0\beta}) - 2Hg_{0\beta}g^{00}\\
&\qquad + \tfrac{1}{2}(g^{00})^{-1/2}(\delta^\tau_\beta - g_{0\beta}g^{0\tau})\partial_\tau g^{00} - H(-g_{0\beta}g^{00}) + g^{0j}N(g_{j\beta})\\
&= g^{00}N(g_{0\beta}) + \tfrac{1}{2}(g^{00})^{-1/2}\partial_\beta g^{00} + g^{0j}N(g_{j\beta}) - H\delta^0_\beta.
\end{aligned} \tag{5-1}$$

Letting $\beta=0$, we get

$$g^{00}N(g_{00}) + \tfrac{1}{2}(g^{00})^{-1/2}\partial_0 g^{00} + g^{0j}N(g_{0j}) - H = 0. \tag{5-2}$$

Letting $\beta=i$, we get

$$g^{00}N(g_{0i}) + \tfrac{1}{2}(g^{00})^{-1/2}\partial_i g^{00} + g^{0j}N(g_{ij}) = 0. \tag{5-3}$$

Now we consider the elliptic system of g^{00}, g_{01}, g_{02}, …, g_{0n}:

$$\begin{cases}
\Delta g^{00} = Q(g, \partial g, \mathrm{Ric}),\\
\Delta g_{01} = -\frac{1}{2}R_{01} + Q(g, \partial g),\\
\quad\vdots\\
\Delta g_{0n} = -\frac{1}{2}R_{0n} + Q(g, \partial g).
\end{cases} \tag{5-4}$$

Through the formulas (4-18), (5-3), the expression of H and the regularities of g_{ij}, we obtain the boundary condition:

$$\begin{cases}
N(g^{00}) - 2Pg^{ij}\partial_i g_{0j} \in C^{m,\alpha}(D),\\
N(g_{01}) + \frac{1}{2P}\partial_1 g^{00} \in C^{m,\alpha}(D),\\
\quad\vdots\\
N(g_{0n}) + \frac{1}{2P}\partial_n g^{00} \in C^{m,\alpha}(D),
\end{cases} \tag{5-5}$$

where $P=(g^{00}(x))^{3/2}$. We are going to prove:

Lemma 5.1. *Let $u^0=g^{00}$, $u^i=g_{0i}$, $i=1,2,\dots,n$. The equations* (5-4) *can be written as the formula*

$$L_{\alpha\beta}u^\beta(x) = f_\alpha(x), \quad x\in V.$$

The boundary conditions (5-5) *can be written as*

$$B_{\alpha\beta}u^\beta(x) = g_\alpha(x), \quad x\in D.$$

Then the operator L is proper elliptic and the boundary operator B satisfies the complementing condition with respect to the system (L, B).

Proof. For any $n+1$ vector $\xi = (\xi_0, \xi_1, \ldots, \xi_n)$, we consider the principal part of L

$$L'_{\alpha\beta}(x,\xi) = \begin{bmatrix} g^{\alpha\beta}\xi_\alpha\xi_\beta & 0 & \cdots & 0 \\ 0 & g^{\alpha\beta}\xi_\alpha\xi_\beta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g^{\alpha\beta}\xi_\alpha\xi_\beta \end{bmatrix} \tag{5-6}$$

Then

$$L_{\alpha\beta}(x,\xi) = \det(L'_{\alpha\beta}(x,\xi)) = |\xi|_g^{2(n+1)}.$$

For any $\xi \neq 0$, $L_{\alpha\beta}(x,\xi) \neq 0$, so L is elliptic.

For each pair of ξ and ξ' of linearly independent vectors, the equation

$$L_{\alpha\beta}(x, \xi + z\xi') = 0$$

is equivalent to

$$(z^2 \cdot |\xi'|_g^2 + 2 < \xi, \xi' >_g z + |\xi|_g^2)^{n+1} = 0.$$

It has $n+1$ roots with positive imaginary part and $n+1$ with negative imaginary part. So L is proper elliptic.

For any $x_0 \in D$, let $n = (1, 0, \ldots, 0)$ denote the unit normal at x_0 and $\xi = (0, \xi_1, \ldots, \xi_n)$ denote any nonzero real vector tangent to D at x_0. Let $z_s^+(x_0, \xi)$, $s = 0, 1, \ldots, n$ be the roots of $L_{\alpha\beta}(x_0, \xi + zn) = 0$ with positive imaginary part.

Define

$$L_0^+(x_0, \xi; z) = \prod_{s=0}^{n} (z - z_s^+(x_0, \xi).$$

Let $L^{\alpha\beta}(x_0, \xi + zn)$ be matrix adjoint to $L'_{\alpha\beta}(x_0, \xi + zn)$. Now we define

$$Q_{r\beta} = B'_{r\alpha}(x_0, \xi + zn) \cdot L^{\alpha\beta}(x_0, \xi + zn)$$

as polynomials in z, where $B'_{r\alpha}$ is the principal part of B.

Then B satisfies the complementing condition with respect to the system (L, B) if and only if the rows of the Q matrix are linearly independent modulo $L_0^+(x_0, \xi; z)$, that is, the polynomial

$$\sum_{r=0}^{n} C_r Q_{r\beta}(x_0, \xi; z) \equiv 0 \ (\mathrm{mod}\, L_0^+)$$

only if C_r are all 0.

By a direct computation, we can derive $z_s = \sqrt{-1}(|\xi|_h/\sqrt{g^{00}})$ for $s = 0, 1, \ldots, n$. Then

$$L_0^+(x_0, \xi; z) = \left(z - \sqrt{-1}\frac{|\xi|_h}{\sqrt{g^{00}}}\right)^{n+1}.$$

From the above, we can get

$$L'_{\alpha\beta}(x_0, \xi + zn) = |\xi + zn|_g^2 \cdot \delta_{\alpha\beta} = (z^2 g^{00} + |\xi|_h^2) \cdot \delta_{\alpha\beta}.$$

Its adjoint matrix is

$$L^{\alpha\beta}(x_0, \xi + zn) = (z^2 g^{00} + |\xi|_h^2)^n \cdot \delta^{\alpha\beta}.$$

The principal part of B is

$$B'_{\alpha\beta}(x, \xi) = \begin{bmatrix} z & -2Pg^{i1}\xi_i & -2Pg^{i2}\xi_i & \cdots & -2Pg^{in}\xi_i \\ \frac{1}{2P}\xi_1 & z & 0 & \cdots & 0 \\ \frac{1}{2P}\xi_2 & 0 & z & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2P}\xi_n & 0 & 0 & \cdots & z \end{bmatrix} \tag{5-7}$$

Then

$$\begin{aligned} \sum_{r=0}^{n} C_r Q_{r\beta} &= \sum_{r=0}^{n} C_r B'_{r\alpha} \cdot L^{\alpha\beta} \\ &= \sum_{r=0}^{n} C_r B'_{r\alpha} \cdot (z^2 g^{00} + |\xi|_h^2)^n \cdot \delta^{\alpha\beta} \\ &= \sum_{r=0}^{n} C_r B'_{r\beta} \cdot (z^2 g^{00} + |\xi|_h^2)^n \\ &\equiv 0 \quad \left(\mathrm{mod}\left(z - \sqrt{-1}\frac{|\xi|_h}{\sqrt{g^{00}}}\right)^{n+1}\right). \end{aligned} \tag{5-8}$$

This implies

$$z - \sqrt{-1}\frac{|\xi|_h}{\sqrt{g^{00}}} \Big| \sum_{r=0}^{n} C_r B'_{r\beta} \tag{5-9}$$

for any $0 \le \beta \le n$.

When $\beta \ge 1$, (5-9) implies

$$z - \sqrt{-1}\frac{|\xi|_h}{\sqrt{g^{00}}} \Big| \frac{C_0}{2P}\xi_\beta + C_\beta z.$$

Then

$$C_\beta = -\frac{C_0\sqrt{g^{00}}\xi_\beta}{2P|\xi|_h\cdot\sqrt{-1}}. \tag{5-10}$$

When $\beta = 0$, (5-9) implies

$$z - \sqrt{-1}\frac{|\xi|_h}{\sqrt{g^{00}}}\Big|C_0 z - 2PC_1 g^{i1}\xi_i - 2PC_2 g^{i2}\xi_i - \cdots - 2PC_n g^{in}\xi_i.$$

With (5-10), we have

$$z - \sqrt{-1}\frac{|\xi|_h}{\sqrt{g^{00}}}\Big|C_0 z + \frac{C_0\sqrt{g^{00}}|\xi|_h}{\sqrt{-1}}. \tag{5-11}$$

By a linear transformation, we can make $g^{00}(x_0) \neq 1$, for all $x_0 \in D$. Then (5-11) shows that $C_0 = 0$, and (5-10) implies $C_r = 0$, $r = 1, 2, \ldots, n$. □

Lemma 5.1 and Theorem 6.3.7 in [Morrey 1966] tell us that $g^{00}, g_{01}, \ldots, g_{0n}$ are all in $C^{m+1,\alpha}$. Then the boundary condition (5-2) can be written as

$$g^{00}N(g_{00}) = Q(g, \partial g^{00}, \partial g_{0i}, \partial g_{ij}) \in C^{m,\alpha}(D).$$

With the elliptic equation

$$\Delta g_{00} = -2R_{00} + Q(g, \partial g) \in C^{m-1,\alpha}(V),$$

we finally derive $g_{00} \in C^{m+1,\alpha}$.

Now we know that $g \in C^{m+1,\alpha}$. Back to step 1, we have $A_{ij} \in C^{m,\alpha}(D)$, thus $R_{\alpha\beta} \in C^{m,\alpha}(V)$. Repeating the steps above, we can get $g_{\alpha\beta} \in C^{m+2,\alpha}(V)$. Then we complete the proof.

5B. ***Regularity of the structure and the defining function.*** We already proved that g is in $C^{m+2,\alpha}$ in structure $\{x^\beta\}$. It is trivial to show that $\{x^\beta\}$ is a $C^{m+3,\alpha}$ structure of $\overline{M}$.

In Section 2, we obtain that u is in $C^{m,\alpha}(y)$ if the scalar curvature is constant. When we change the y-coordinates to harmonic coordinates x, we notice that x^β are $C^{m+1,\alpha}$ functions of y^β. So the defining function $\rho \in C^{m,\alpha}(x)$. Since the initial compactification g is smooth in M and the initial defining function is smooth in y-coordinates, then $\rho \in C^\infty(x)$ in M.

For any $p \in \partial M$, consider the neighborhood V of p and $D = \partial M \cap V$. By a linear transformation, we can assume that at p, $g_{\alpha\alpha} = 1$, $g_{ij} = g_{02} = g_{03} = \cdots = g_{0n} = 0 (i \neq j)$, $g_{01} = -\delta$ for some $\delta > 0$ and $1 - \delta$ is a very small positive number. When g_+ is Einstein, from (3-2) and (3-3),

$$\mathrm{Ric} - \frac{Sg}{n+1} = -(n-1)\frac{D^2\rho}{\rho} + \frac{n-1}{n+1}\frac{\Delta\rho}{\rho}g.$$

In local coordinates, acting on $(\partial/\partial x^0, \partial/\partial x^1)$, we have

$$\Delta\rho - (n+1)\cdot g_{01}^{-1}\cdot D^2\rho\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}\right) = \frac{n+1}{n-1}\cdot g_{01}^{-1}\cdot\rho\left(\mathrm{Ric}_{01} - \frac{Sg_{01}}{n+1}\right) \tag{5-12}$$

If $1-\delta$ is small enough, we can find that the left-hand side of (5-12) is a uniformly elliptic operator on ρ locally. Since $\rho R_{01} \in C^{m,\alpha}(\overline{M})$, $\rho|_D \equiv 0$, so in V, $\rho \in C^{m+2,\alpha}(x)$.

To prove Remark 1.2, we only need to show that $\rho R_{01} \in C^{m+1,\alpha}(\overline{M})$. When $\dim M = n+1$ is even, we can define the obstruction tensor $\mathcal{O}_{ij}$, in local coordinates:

$$\mathcal{O}_{ij} = \Delta^{(n+1)/2-2}(P_{ij,k}{}^{k} - P_{ik,j}{}^{k}) + Q_n, \tag{5-13}$$

where P_{ij} is defined by $P_{ij} = \frac{1}{2}R_{ij} - \frac{S}{12}g_{ij}$ and Q_n denotes quadratic and higher terms in metric involving at most n-th derivatives. $\mathcal{O}_{ij}$ is conformally invariant of weight $2-n$ and if g_{ij} is conformal to an Einstein metric, then $\mathcal{O}_{ij} = 0$ (see more in [Graham and Hirachi 2005]).

Since the scalar curvature of $(\overline{M}, g)$ is constant, (5-13) can be written in the form:

$$\Delta^{(n+1)/2-1}R_{ij} = Q_n. \tag{5-14}$$

Now we consider the function ρR_{01}. Through a direct computation, $\Delta(\rho R_{ij}) = \rho\Delta R_{ij} + Q_3^2$. Here Q_3^2 denotes quadratic and higher terms in the metric involving at most 3-rd derivatives and in ρ involving at most 2-nd derivatives.

Then we use iterative method to obtain that $\Delta^k(\rho R_{ij}) = \rho\Delta^k R_{ij} + Q_{2k+1}^{2k}$ for $1 \le k \le (n+1)/2 - 1$. Letting $k = (n+1)/2 - 1$, we have an elliptic equation of second order with Dirichlet boundary condition:

$$\begin{cases} \Delta(\Delta^{(n+1)/2-2}(\rho R_{01})) = Q_n^{n-1} & \text{in } \overline{M}, \\ \Delta^{(n+1)/2-2}(\rho R_{01})|_{\partial M} = Q_{n-2}^{n-3}. \end{cases} \tag{5-15}$$

Since g and ρ are all in $C^{m+2,\alpha}$, $m+2 \ge n$, we have

$$\Delta^{(n+1)/2-2}(\rho R_{01}) \in C^{m+2-(n-2),\alpha}(\overline{M}).$$

Then we consider the equation

$$\begin{cases} \Delta(\Delta^{(n+1)/2-3}(\rho R_{01})) \in C^{m+2-(n-2),\alpha}(\overline{M}) \\ \Delta^{(n+1)/2-3}(\rho R_{01})|_{\partial M} = Q_{n-4}^{n-5} \in C^{m+2-(n-4),\alpha}(\partial M) \end{cases} \tag{5-16}$$

So we have $\Delta^{(n+1)/2-3}(\rho R_{01}) \in C^{m+2-(n-4),\alpha}(\overline{M})$

If we keep using the equation, we finally get $\rho R_{01} \in C^{m+1,\alpha}(\overline{M})$, which implies $\rho \in C^{m+3,\alpha}(\overline{M})$ by (5-12).

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