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In this paper, we construct a double chain complex generated by certain graphs and a chain map from that to the Chevalley–Eilenberg double complex of the differential graded Lie algebra (dgl) of symplectic derivations on a free dgl. It is known that the target of the map is related to characteristic classes of fibrations. We can describe some characteristic classes of fibrations whose fiber is a 1-punctured even-dimensional manifold by linear combinations of graphs though the cohomology of the dgl of derivations.

1. Introduction

The Chevalley–Eilenberg complex of the limit of the Lie algebra of symplectic derivations on (graded) free Lie algebras is isomorphic to the graph complex defined by the cyclic Lie operad (details in [Kontsevich 1994; 1993; Conant and Vogtmann 2003; Hamilton 2006]). In this paper, we introduce an extension of (the dual of) the construction to a Lie algebra of symplectic derivations on free differential graded Lie algebras (dgl's). Let (W, ω) be a finite-dimensional graded vector space with symmetric inner product of even degree N and δ a differential of degree -1 on the completed free Lie algebra $\hat{L}W$ satisfying the symplectic condition $\delta\omega = 0$. An important example is the case that $(\hat{L}W, \delta)$ is a Chen dgl model of an even-dimensional manifold and ω is its intersection form. We construct a W -labeled graph complex $C_{\text{com}}^{\bullet,\bullet}(W)$ and a chain map

$$C_{\text{com}}^{\bullet,\bullet}(W) \rightarrow C_{\text{CE}}^{\bullet,\bullet}(\text{Der}_{\omega}(\hat{L}W))$$

to the Chevalley–Eilenberg (double) complex $C_{\text{CE}}^{\bullet,\bullet}(\text{Der}_{\omega}(\hat{L}W))$ of the differential graded Lie algebra $(\text{Der}_{\omega}(\hat{L}W), \text{ad}(\delta))$ of symplectic derivations on $\hat{L}W$. Furthermore, from the nonlabeled part $C_{\text{com}}^{\bullet,\bullet}(N, Z)$ of the graph complex, which depends on only the integer N and the set Z of degrees of W , we can obtain a chain map

$$C_{\text{com}}^{\bullet,\bullet}(N, Z) \subset C_{\text{com}}^{\bullet,\bullet}(W)^{\text{Sp}(W,\delta)} \rightarrow C_{\text{CE}}^{\bullet,\bullet}(\text{Der}_{\omega}(\hat{L}W))^{\text{Sp}(W,\delta)},$$

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where $\text{Sp}(W, \delta)$ is the group of graded linear isomorphisms of W preserving ω and δ . In the case of $N = 0$ and $Z = \{0\}$, the map corresponds to the Kontsevich one [1994; 1993].

The construction above gives characteristic classes of fibrations. It is known that characteristic classes of simply connected fibrations are related to Lie algebras of derivations [Schlessinger and Stasheff 2012; Tanré 1983]. In nonsimply connected cases, we got relations between characteristic classes and Lie algebras of derivations as in [Matsuyuki and Terashima 2016; Kajiura et al. 2016]. In this paper, we consider the case that the boundary of a fiber is a sphere. For a simply connected compact manifold X with $\partial X = S^{n-1}$, let $\text{aut}_\partial(X)$ be the monoid of self-homotopy equivalences of X fixing the boundary pointwisely and $\text{aut}_{\partial,0}(X)$ its connected component containing id_X . According to [Berglund and Madsen 2014], the isomorphism

$$H^\bullet(B \text{aut}_{\partial,0}(X); \mathbb{Q}) \simeq H_{\text{CE}}^\bullet(\text{Der}_\omega^+(L_X))$$

is obtained. Here L_X is a cofibrant dgl model of X . The underlying Lie algebra of L_X is generated by the linear dual W of the suspension of the reduced cohomology of X . So the graph complex above gives the invariant part of the cohomology $H_{\text{CE}}^\bullet(\text{Der}_\omega^+(L_X))$ with respect to the action of the group $\text{Sp}(W, \delta)$ of automorphisms of W with intersection form preserving the differential δ of L_X . Using the Serre spectral sequence for the fibration

$$B \text{aut}_{\partial,0}(X) \rightarrow B \text{aut}_\partial(X) \rightarrow B\pi_0(\text{aut}_\partial(X)),$$

the image of the natural map $H^\bullet(B \text{aut}_\partial(X); \mathbb{Q}) \rightarrow H^\bullet(B \text{aut}_{\partial,0}(X); \mathbb{Q})$ is included in the invariant part. We give a chain map

$$C_{\text{com}}^{\bullet,\bullet}(N, Z)_+ \rightarrow C_{\text{CE}}^{\bullet,\bullet}(\text{Der}_\omega^+(L_X))^{\text{Sp}(W,\delta)}$$

using a positive truncated version $C_{\text{com}}^{\bullet,\bullet}(W)_+$ of $C_{\text{com}}^{\bullet,\bullet}(W)$. Considering W -labeled graphs, we can also obtain a W -labeled version $C_{\text{com}}^{\bullet,\bullet}(W)_+$ and a chain map

$$C_{\text{com}}^{\bullet,\bullet}(W)_+ \rightarrow C_{\text{CE}}^{\bullet,\bullet}(\text{Der}_\omega^+(L_X)).$$

2. Preliminary

In this paper, all vector spaces are over a field K whose characteristic is zero. A field K is regarded as a \mathbb{Z} -graded vector space all of whose elements have degree 0.

For a finite set U , the number of elements in U is denoted by $\#U$.

All tensor products of linear maps between \mathbb{Z} -graded vector spaces contain their signs: for homogeneous linear maps $f : A \rightarrow V$ and $g : B \rightarrow W$ between \mathbb{Z} -graded

vector spaces, we set

$$(f \otimes g)(a, b) := (-1)^{g^a} f(a) \otimes g(b)$$

for $a \in A$ and $b \in B$. (We often denote by $|a|$ the degree of an element a . But we omit the symbol $|\cdot|$ of the degree when it appears in a power of -1 . For example, $(-1)^{g^a}$ means $(-1)^{|g||a|}$.)

Let V be a \mathbb{Z} -graded vector space. We denote by V^i the subspace of elements of V of *cohomological degree* i and $V_i = V^{-i}$ the subspace of elements of *homological degree* i . Note that the *linear dual* $V^* = \text{Hom}(V, \mathbb{R})$ of V is graded by $(V^*)^i = \text{Hom}(V_i, \mathbb{R})$.

The p -fold suspension $V[p]$ of V for an integer p is defined by

$$V[p]^i := V^{i+p},$$

and elements of $V[p]^i$ are presented by $x\sigma$ for $x \in V^{i+p}$ using the symbol σ of cohomological degree $-p$. The p -suspension map $V \rightarrow V[p]$ is also denoted by σ . In this paper, the N -suspension σ for an even number N often appears. It is used for adjusting degrees of elements though we can ignore it when calculating signs.

Let V be a \mathbb{Z} -graded vector space and $\alpha : V \otimes V \rightarrow K$ be a nondegenerate bilinear map of (cohomological) degree n . Out of the two conditions

- (i) $\alpha(x, y) = (-1)^{xy} \alpha(y, x)$ for homogeneous elements $x, y \in V$, and
- (ii) $\alpha(x, y) = -(-1)^{xy} \alpha(y, x)$ for homogeneous elements $x, y \in V$,

the pair (V, α) is called a *symmetric vector space* with degree n if satisfying (i), and a *symplectic vector space* with degree n if satisfying (ii).

2A. Algebras and signs. Let V be a finite-dimensional \mathbb{Z} -graded vector space.

Definition 2.1. We define the following quotient algebras of the tensor algebra TV generated by V .

- The *symmetric algebra* SV generated by V is the \mathbb{Z} -graded commutative algebra which is the quotient algebra obtained from the \mathbb{Z} -graded tensor algebra TV by introducing the relation

$$xy = (-1)^{xy} yx$$

for $x, y \in V$. The image of $V^{\otimes k}$ for an integer k in SV is denoted by $S^k V$.

- The *exterior algebra* ΛV generated by V is the \mathbb{Z} -graded anticommutative algebra which is the quotient algebra obtained from the \mathbb{Z} -graded tensor algebra TV by introducing the relation

$$xy = -(-1)^{xy} yx$$

for $x, y \in V$. The image of $V^{\otimes k}$ for an integer k in ΛV is denoted by $\Lambda^k V$.

Definition 2.2. For distinct elements $v_1, \dots, v_k \in V$ and a permutation $\pi \in \mathfrak{S}_k$, the sign ϵ defined by the equation on $S^k V$

$$v_1 \cdots v_k = \epsilon \cdot v_{\pi(1)} \cdots v_{\pi(k)}$$

is called the *Koszul sign* of $(v_1, \dots, v_k) \mapsto (v_{\pi(1)}, \dots, v_{\pi(k)})$. Similarly the sign $\bar{\epsilon}$ defined by the same equation in $\Lambda^k V$ is called the *anti-Koszul sign*. Note the equation $\bar{\epsilon} = \text{sgn } \pi \cdot \epsilon$.

2B. Derivations. Let W be a finite-dimensional \mathbb{Z} -graded vector space.

Completed tensor algebras. We denote the *completed tensor algebra* by

$$\widehat{T}W := \prod_{r=0}^{\infty} W^{\otimes r}.$$

Its product μ and coproduct Δ are defined by

$$\mu(x_1 \otimes \cdots \otimes x_s, x_{s+1} \otimes \cdots \otimes x_r) = x_1 \otimes \cdots \otimes x_r,$$

$$\Delta(x_1 \otimes \cdots \otimes x_r) = \sum_{s=0}^r \sum_{\tau \in \text{Ush}(s, r-s)} \epsilon \cdot (x_{\tau(1)} \otimes \cdots \otimes x_{\tau(s)}) \otimes (x_{\tau(s+1)} \otimes \cdots \otimes x_{\tau(r)})$$

for homogeneous elements $x_1, \dots, x_r \in W$, where $\text{Ush}(s, r - s)$ is the set of $(r, s - r)$ -unshuffles and ϵ is the Koszul sign of the permutation $(x_1, \dots, x_r) \mapsto (x_{\tau(1)}, \dots, x_{\tau(r)})$ (Definition 2.2). The primitive part of $\widehat{T}W$ is the completed free Lie algebra $\widehat{L}W$. These algebras have the gradings defined by the grading of W .

Derivations on a completed tensor algebra. Let $\text{Der}(\widehat{L}W)$ be the Lie algebra of (continuous) derivations on the completed algebra $\widehat{T}W$. Given a symplectic inner product ω of degree N on W , we define the Lie algebra of *symplectic derivations* on $\widehat{T}W$

$$\text{Der}_{\omega}(\widehat{T}W) := \{D \in \text{Der}(\widehat{T}W); D(\omega) = 0\}.$$

Here ω is identified with the element of $\widehat{L}W$ described by

$$\sum_{i < j} \omega_{ij} [x^i, x^j],$$

where $\{x^i\}$ is a basis of W and matrix $(\omega_{ij})_{i,j}$ is the inverse matrix of $(\omega(x^i, x^j))_{i,j}$.

Since derivations on $\widehat{T}W$ are determined by the values on the generating space W , we get the isomorphism as graded vector space

$$\Phi_{\omega} : \text{Der}(\widehat{T}W) \simeq \text{Hom}(W, \widehat{T}W) \simeq \widehat{T}W \otimes W[-N] = \prod_{r=1} W^{\otimes r}[-N],$$

where the second isomorphism is induced by the isomorphism $\text{Hom}(W, \mathbb{R}) \simeq W[-N]$ derived from nondegeneracy of ω . Furthermore, we also have the identification by Φ_ω

$$\begin{aligned} \text{Der}^r(\widehat{T}W) &:= \{D \in \text{Der}(\widehat{T}W); D(W) \subset W^{\otimes(r+1)}\} \\ &\simeq \text{Hom}(W, W^{\otimes(r+1)}) \simeq W^{\otimes(r+2)}[-N]. \end{aligned}$$

Fixing a homogeneous basis x^1, \dots, x^m of W , the derivations $x^{i_1} \dots x^{i_k} \partial/\partial x^i$ ($1 \leq i_1, \dots, i_k, i \leq m$), which are these elements corresponding to the linear map $x^i \mapsto x^{i_1} \dots x^{i_k}$, comprise a basis of $\text{Der}^{k+1}(\widehat{T}W)$. On the basis, Φ_ω is described by

$$\Phi_\omega \left(x^{i_1} \dots x^{i_k} \frac{\partial}{\partial x^i} \right) = \sum_j \omega_{ij} x^{i_1} \dots x^{i_k} x^j \sigma^{-1},$$

where σ^{-1} is a symbol of the $(-N)$ -suspension which has homological degree $-N$.

By the identification Φ_ω , the space of symplectic derivations is described by

$$\text{Der}_\omega(\widehat{T}W) \stackrel{\Phi_\omega}{\simeq} \prod_{r=1}^{\infty} (W^{\otimes r})^{\mathbb{Z}/r\mathbb{Z}}[-N] = \prod_{r=1}^{\infty} W_{\text{cyc}}^{(r)}[-N].$$

Here $W_{\text{cyc}}^{(r)} := (W^{\otimes r})^{\mathbb{Z}/r\mathbb{Z}}$ is the space of invariant tensors by cyclic permutations of tensor factors, which is also defined in [Definition 3.2](#).

Therefore, the Lie algebra $\text{Der}_\omega(\widehat{L}W)$ of symplectic derivations on $\widehat{L}W$ is described by

$$\begin{aligned} \text{Der}_\omega(\widehat{L}W) &:= \text{Der}(\widehat{L}W) \cap \text{Der}_\omega(\widehat{T}W) \stackrel{\Phi_\omega}{\simeq} \prod_{r=2}^{\infty} W(r)[-N], \\ \text{Der}_\omega^{r+2}(\widehat{L}W) &:= \text{Der}^{r+2}(\widehat{L}W) \cap \text{Der}_\omega(\widehat{T}W) \stackrel{\Phi_\omega}{\simeq} W(r)[-N], \end{aligned}$$

where $W(r) := (LW \otimes W) \cap W_{\text{cyc}}^{(r)}$.

Through the isomorphism Φ_ω , the Lie algebra structure of $\text{Der}(\widehat{T}W)$ is described as follows:

Lemma 2.3. *Let $[\cdot, \cdot]$ be the Lie bracket of $\text{Der}(\widehat{T}W)$. Then the linear map $[\cdot, \cdot]_\omega := \sigma \Phi_\omega \circ [\cdot, \cdot] \circ (\Phi_\omega^{-1} \sigma^{-1})^{\otimes 2}$ is equal to*

$$\sum_{d_1+d_2=N} (\text{id} \otimes \omega_{(d_1, d_2)}) \left(\sum_{1 \leq t < r_2} \pi_{1;t}^{r_1, r_2} + \sum_{1 \leq s < r_1} \pi_{2;s}^{r_1, r_2} \right) : W^{\otimes r_1} \otimes W^{\otimes r_2} \rightarrow W^{\otimes r_1+r_2-2}$$

where $\omega_{(d_1, d_2)} : W \otimes W \rightarrow \mathbb{R}$ for integers d_1, d_2 is the composition of the projection $W \otimes W \rightarrow W_{d_1} \otimes W_{d_2}$ and the restriction of ω to $W_{d_1} \otimes W_{d_2}$, and $\pi_{1;j}^{r_1, r_2}, \pi_{2;i}^{r_1, r_2} :$

$W^{\otimes r_1} \otimes W^{\otimes r_2} \rightarrow W^{\otimes r_1+r_2}$ for $1 \leq i \leq r_1$ and $1 \leq j \leq r_2$ is defined by

$$\begin{aligned}\pi_{1;j}^{r_1,r_2}(a_1^{(1)} \cdots a_{r_1}^{(1)} \otimes a_1^{(2)} \cdots a_{r_2}^{(2)}) &= \epsilon \cdot a_1^{(2)} \cdots a_{j-1}^{(2)} a_1^{(1)} \cdots a_{r_1-1}^{(1)} a_{j+1}^{(2)} \cdots a_{r_2}^{(2)} a_{r_1}^{(1)} a_j^{(2)}, \\ \pi_{2;i}^{r_1,r_2}(a_1^{(1)} \cdots a_{r_1}^{(1)} \otimes a_1^{(2)} \cdots a_{r_2}^{(2)}) &= \epsilon' \cdot a_1^{(1)} \cdots a_{i-1}^{(1)} a_1^{(2)} \cdots a_{r_2-1}^{(2)} a_{j+1}^{(2)} \cdots a_{r_2-1}^{(2)} a_i^{(1)} a_{r_2}^{(2)}\end{aligned}$$

for homogeneous elements $a_1^{(1)}, \dots, a_{r_1}^{(1)}, a_1^{(2)}, \dots, a_{r_2}^{(2)}$. Here ϵ is the Koszul sign of the corresponding permutations.

Proof. Let x^1, \dots, x^m be a homogeneous basis of W . The Lie bracket for the basis is described by

$$\begin{aligned}\left[x^{i_1} \cdots x^{i_k} \frac{\partial}{\partial x^i}, x^{j_1} \cdots x^{j_l} \frac{\partial}{\partial x^j} \right] &= \sum_t \epsilon \delta_t^{j_1} x^{j_1} \cdots x^{j_{t-1}} x^{i_1} \cdots x^{i_k} x^{j_{t+1}} \cdots x^{j_l} \frac{\partial}{\partial x^j} \\ &\quad - \sum_s \epsilon' \delta_j^{i_s} x^{i_1} \cdots x^{i_{s-1}} x^{j_1} \cdots x^{j_l} x^{i_{s+1}} \cdots x^{i_k} \frac{\partial}{\partial x^i}\end{aligned}$$

where $\epsilon = (-1)^{(x^{i_1} + \cdots + x^{i_k} - x^i)(x^{j_1} + \cdots + x^{j_l - 1})}$, $\epsilon' = (-1)^{(x^{j_1} + \cdots + x^{j_l} - x^j)(x^{i_1} + \cdots + x^{i_k} - x^i)}$, and δ_j^i is the Kronecker's delta.

Then, for $A = x^{i_1} \cdots x^{i_{r_1}}$ and $B = x^{j_1} \cdots x^{j_{r_2}}$, we obtain

$$\begin{aligned}[A, B]_\omega &= \sum_t \epsilon x^{j_1} \cdots x^{i_1} \cdots x^{i_{r_1-1}} \cdots x^{j_{r_2}} \omega^{i_{r_1} j_t} + \sum_s \epsilon' x^{i_1} \cdots x^{j_1} \cdots x^{j_{r_2-1}} \cdots x^{i_{r_1}} \omega^{i_s j_{r_2}} \\ &= \sum_{d_1+d_2=N} (\text{id} \otimes \omega_{(d_1, d_2)}) \left(\sum_{1 \leq t < r_2} \pi_{1;t}^{r_1, r_2} + \sum_{1 \leq s < r_1} \pi_{2;s}^{r_1, r_2} \right) (A \otimes B),\end{aligned}$$

where ϵ and ϵ' are the Koszul signs of

$$\begin{aligned}(x^{i_1}, \dots, x^{i_{r_1}}, x^{j_1}, \dots, x^{j_{r_2}}) &\mapsto (x^{j_1}, \dots, x^{i_1}, \dots, x^{i_{r_1-1}}, \dots, x^{j_{r_2}}, x^{i_{r_1}}, x^{j_t}), \\ (x^{i_1}, \dots, x^{i_{r_1}}, x^{j_1}, \dots, x^{j_{r_2}}) &\mapsto (x^{i_1}, \dots, x^{j_1}, \dots, x^{j_{r_2-1}}, \dots, x^{i_{r_1}}, x^{i_s}, x^{j_{r_2}}),\end{aligned}$$

respectively. In the calculus above, note that we use the assumption that N is even. \square

The lemma above is needed to prove [Theorem 3.9](#).

Derivations on a dgl. Let δ be an element in $\text{Der}_\omega(\hat{L}W)$ of homological degree -1 such that $\delta^2 = 0$. Then $\text{ad}(\delta)$ is a differential operator on $\text{Der}_\omega(\hat{L}W)$.

In the case that W_0 is positively graded, i.e., $W_i = 0$ for $i \leq 0$, we can regard that $\delta \in \text{Der}_\omega(LW)$ since δ is described by only finite sums. Then we often consider the positive truncation $(\text{Der}_\omega^+(LW), \text{ad}(\delta))$ of the chain complex $(\text{Der}_\omega(LW), \text{ad}(\delta))$ defined by

$$\text{Der}_\omega^+(LW)_i := \begin{cases} \text{Der}_\omega(LW)_i & (i > 2), \\ \text{Ker}(\text{ad}(\delta))_1 & (i = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Definition 2.4 (Chevalley–Eilenberg complex). Let (L, δ) be a dgl. We define the Chevalley–Eilenberg complex as

$$C_{\text{CE}}^{p,q}(L) := (\Lambda^p L^*)^q,$$

where $\Lambda^\bullet L^*$ is the exterior algebra generated by the graded vector space L^* . The first differential d_{CE} is defined by the formula for $c \in C_{\text{CE}}^{p,q}(L)$ and $D_1, \dots, D_{p+1} \in L$,

$$(d_{\text{CE}}c)(D_1, \dots, D_{p+1}) = \sum_{i < j} \bar{\epsilon} \cdot c([D_i, D_j], D_1, \dots, \widehat{D}_i, \dots, \widehat{D}_j, \dots, D_{p+1}),$$

where $\bar{\epsilon} = (-1)^{D_i(D_1 + \dots + D_{i-1}) + D_j(D_1 + \dots + D_{j-1}) + D_i D_j + i + j - 1}$, and the second differential L_δ derived from δ is defined by

$$L_\delta = i_\delta d_{\text{CE}} - d_{\text{CE}} i_\delta,$$

using the *interior product* defined by

$$(i_\delta c)(D_1, \dots, D_p) = c(\delta, D_1, \dots, D_p),$$

for $c \in C_{\text{CE}}^{p+1,q}(L)$ and $D_1, \dots, D_p \in L$. Then the triple $(C_{\text{CE}}^{\bullet,\bullet}(L), d_{\text{CE}}, L_\delta)$ is a double complex.

We will consider the Chevalley–Eilenberg complexes of dgls $(\text{Der}_\omega(\widehat{L}W), \text{ad}(\delta))$ and $(\text{Der}_\omega^+(LW), \text{ad}(\delta))$, and the invariant space $C_{\text{CE}}^{\bullet,\bullet}(\text{Der}_\omega^+(LW))^{\text{Sp}(W,\delta)}$, where $\text{Sp}(W, \delta)$ is the group of symplectic linear isomorphisms $W \rightarrow W$ preserving δ .

2C. A dgl model with symplectic form of manifolds. In this subsection, we review a Chen dgl model of a manifold. Let X be a smooth manifold. Put $A = A^\bullet(X)$ and $H = H_{DR}^\bullet(X)$. Fix a homotopy transfer diagram

$$\begin{array}{c} \curvearrowright \\ A \rightleftarrows H \end{array}$$

e.g., in the case that X is a closed manifold, it is obtained by using the Hodge decomposition of the de Rham complex A . Since A is a commutative differential graded algebra (dga) with symmetric form (intersection form), H has the structure of a minimal cyclic C_∞ -algebra by the diagram (details in [Kontsevich and Soibelman 2001; Merkulov 1999; Kadeishvili 2009; Markl et al. 2002; Hamilton and Lazarev 2004] for instance).

Let I be the intersection form on H , m the cyclic C_∞ -algebra structure on H obtained by the homotopy transfer diagram, and $s : H \rightarrow H[1]$ the suspension map. We denote $V = H[1]^*$. Defining the suspension of m_i by $\bar{m}_i := s \circ m_i \circ (s^{-1})^{\otimes i}$ for all $i \geq 1$ and of I by $\omega := I \circ (s^{-1})^{\otimes 2}$, then the duals of these define the symplectic inner product ω on $H[1]^*$ of degree $N = n - 2$ and the linear map $\delta_i : V \rightarrow V^{\otimes n}$

of homological degree -1 . Thus, extending the unique derivation $\bar{\delta}_i : \hat{L}V \rightarrow \hat{L}V$ by the Leibniz rule, then we have the derivation of homological degree -1

$$\bar{\delta} := \sum_{i=1}^{\infty} \bar{\delta}_i \in \text{Der}_{\omega}(\hat{L}V).$$

Furthermore we can prove that $\bar{\delta}$ is a differential since m satisfies the A_{∞} -relations and *quadratic*, i.e., $\bar{\delta}(V) \subset \prod_{i \geq 2} V^{\otimes i}$, since (H, I, m) is minimal.

The Chen dgl model is a reduced version of the construction. Suppose X is connected, and put

$$W := H[1]_{\geq 0}^* = H_+(X; \mathbb{R})[-1].$$

Then we have the restriction $\delta : \hat{L}W \rightarrow \hat{L}W$ of $\bar{\delta}$ and $\omega : W^{\otimes 2} \rightarrow \mathbb{R}$. If X is simply connected, we can restrict the differential δ on the free Lie algebra $LW \subset \hat{L}W$ since $\delta(w)$ for $w \in W$ has only finitely many nontrivial terms.

Theorem 2.5 [Chen 1977]. *For a simply connected closed manifold X with base point $*$, the dgl (LW, δ) is a Quillen model of X , i.e., there is a Lie algebra isomorphism*

$$H_*(LW, \delta) \simeq \pi_*(\Omega X) \otimes \mathbb{Q}.$$

3. Graph complex

3A. Orientation and ordering of graded sets. The set of *orderings* on a set U is defined by

$$\text{Ord}(U) := \{(u_1, \dots, u_k) \in U^{\times k}; U = \{u_1, \dots, u_k\}\},$$

where $k := \#U$.

Definition 3.1. Let U be a \mathbb{Z} -graded set, i.e., a finite set U given a map $|\cdot| : U \rightarrow \mathbb{Z}$.

- The graded vector space generated by U is denoted by $\mathbb{R}U$.
- The symmetric algebra generated by U is denoted by $SU := S(\mathbb{R}U)$.
- The exterior algebra generated by U is denoted by $\Lambda U := \Lambda(\mathbb{R}U)$.

For an element $(u_1, \dots, u_k) \in \text{Ord}(U)$, we denote the image of $u_1 \otimes \dots \otimes u_k$ in ΛU by $[u_1, \dots, u_k]$. The 1-dimensional vector space generated by this element is written by

$$O(U) := \langle [u_1, \dots, u_k] \rangle \subset \Lambda U.$$

Definition 3.2. Let V be a \mathbb{Z} -graded vector space. We define the subspace $V_{\text{cyc}}^{(k)}$ of *cyclic tensors* in $V^{\otimes k}$ by the image of the map $[\cdot, \dots, \cdot]_{\text{cyc}} : V^{\otimes k} \rightarrow V^{\otimes k}$ obtained by

$$x_1 \otimes \dots \otimes x_k \mapsto \sum_{\tau \in \mathbb{Z}/k\mathbb{Z}} \epsilon \cdot x_{\tau(1)} \otimes \dots \otimes x_{\tau(k)},$$

where $\mathbb{Z}/k\mathbb{Z}$ is identified with the group of cyclic permutations and ϵ is the Koszul sign of $(x_1, \dots, x_k) \mapsto (x_{\tau(1)}, \dots, x_{\tau(k)})$. For a \mathbb{Z} -graded set U , we denote

$$\text{Cyc}(U) := \langle [u_1, \dots, u_k]_{\text{cyc}}; (u_1, \dots, u_k) \in \text{Ord}(U) \rangle \subset (\mathbb{R}U)_{\text{cyc}}^{(k)}.$$

3B. Definition of graph complex. Let W be a finite-dimensional symplectic vector space with form ω of degree N and suppose N is even and $Z := \{a \in \mathbb{Z}; W_a \neq 0\} \subset \{0, \dots, N\}$. Our labeled graph complex depends on (W, ω) .

Definition of graphs.

Definition 3.3. An N -graded graph Γ consists of the following information:

- The set $H(\Gamma)$ of half-edges.
- The set $V(\Gamma)$ of vertices. It is a partition of the set $H(\Gamma)$; i.e.,

$$H(\Gamma) = \coprod_{v \in V(\Gamma)} v, \quad v \neq \emptyset \quad (v \in V(\Gamma)).$$

The number $\#v$ of elements of any $v \in V(\Gamma)$ is called the *valency* of v . A vertex with valency > 1 is called an *internal vertex*, and one with valency 1 is called an *external vertex*. The sets of internal and external vertices are denoted by $V_i(\Gamma)$ and $V_e(\Gamma)$, respectively.

- The set $E(\Gamma)$ of edges. It is a partition of the set $H(\Gamma)$ such that the number of elements of any $e \in E(\Gamma)$ is two, i.e.,

$$H(\Gamma) = \coprod_{e \in E(\Gamma)} e, \quad \#e = 2 \quad (e \in E(\Gamma)).$$

- The cohomological *degree of half-edges*. It is a map $|\cdot| : H(\Gamma) \rightarrow Z$ such that $|h_1| + |h_2| = N$ for an edge $e = \{h_1, h_2\} \in E(\Gamma)$. Then the cohomological degrees of vertices and edges are defined by

$$|v| := |h_1| + \dots + |h_r| - N, \quad |e| := N,$$

for $v = \{h_1, \dots, h_r\} \in V(\Gamma)$ and $e \in E(\Gamma)$.

- The division of the set $V_i(\Gamma)$ of internal vertices to two disjoint sets

$$V_i(\Gamma) = V_n(\Gamma) \sqcup V_s(\Gamma)$$

such that all elements in $V_s(\Gamma)$ have cohomological degree -1 and valency ≥ 3 . An element of $V_n(\Gamma)$ is called a *normal vertex*, and one of $V_s(\Gamma)$ is called a *special vertex*.

The set of isomorphism classes of such graphs is denoted by $\mathcal{G}(N)$. Here an isomorphism between N -graded graphs is a bijection between the sets of half-edges preserving all information of N -graded graphs.

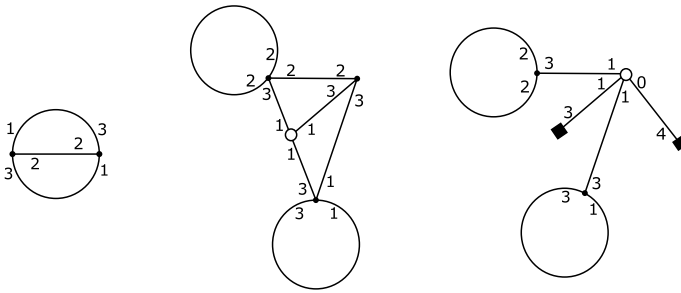


Figure 1. Examples of 4-graded graphs.

Example 3.4. In the case of $N = 4$ and $Z = \{0, 1, 2, 3, 4\}$, we can give examples of 4-graded graphs in Figure 1. In these figures,

- a black vertex • means a normal vertex, a white vertex \circ a special vertex, and a square vertex \blacksquare a univalent vertex, and
- a number drawn beside a half-edge is its degrees.

Decoration on vertices. We shall give the relation equivalent to the dual of vertices defined by the cyclic Lie operad as in [Conant and Vogtmann 2003; Hamilton 2006; Markl 1999].

Definition 3.5. Let Γ be an N -graded graph.

- We introduce to $\text{Cyc}(v)[N]$ for $v \in V_i(\Gamma)$ the *commutativity relation*

$$S_{v,hr;s}(o) := \sum_{\tau \in \text{Sh}(s,r-s-1)} o^{\tau(v,hr)} = 0,$$

$$o^{\tau(v,hr)} := \epsilon[h_{\tau(1)}, \dots, h_{\tau(r-1)}, h_r]_{\text{cyc}} \sigma,$$

for $r - 1 > s > 0$ and $o = [h_1, \dots, h_r]_{\text{cyc}} \sigma \in \text{Cyc}(v)[N]$, where $\text{Sh}(p, q)$ is the set of (p, q) -shuffles, σ is the symbol of the N -fold suspension, and ϵ is the Koszul sign. Then we denote by $C(v) = \text{Cyc}(v)[N]/(\text{com. rel.})$ the obtained space. This relation for $r = 3, 4$ is described in Figure 2. (In the case of $r = 3$, it is the AS-relation for Jacobi diagrams.)

Decoration on N -graded graphs. Set

$$\tilde{O}_{\text{com}}(W, \Gamma) := \bigodot_{e \in E(\Gamma)} O(e) \otimes \bigodot_{u \in V_e(\Gamma)} W[-N]_{|u|} \otimes \bigwedge_{v^s \in V_s(\Gamma)} C(v^s) \otimes \bigwedge_{v \in V_n(\Gamma)} C(v),$$

where

$$\bigodot_{u \in U} V(u) := \left\{ v_{u_1} \cdots v_{u_k} \in S^k \left(\bigoplus_{u \in U} V(u) \right); v_{u_i} \in V(u_i), (u_1, \dots, u_k) \in \text{Ord}(U) \right\},$$

$$\bigwedge_{u \in U} V(u) := \left\{ v_{u_1} \cdots v_{u_k} \in \Lambda^k \left(\bigoplus_{u \in U} V(u) \right); v_{u_i} \in V(u_i), (u_1, \dots, u_k) \in \text{Ord}(U) \right\}$$

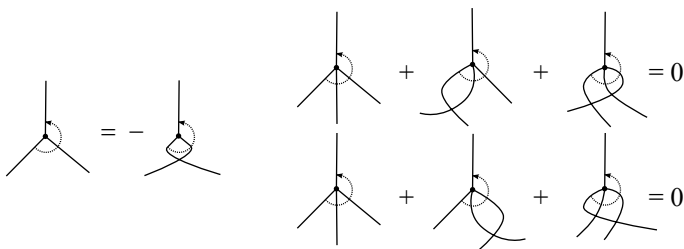


Figure 2. Commutativity ($r = 3, 4$). (Koszul signs are omitted in figures.)

for a family $(V(u))_{u \in U}$ of \mathbb{Z} -graded vector spaces indexed by a finite set U . This tensor product consists of four factors: the first factor means directions of edges of Γ , the second factor W -labels of external vertices of Γ , the third factor (equivalence classes of) cyclic orderings on special vertices of Γ , and the fourth factor the same on normal vertices of Γ . Note that $W[-N]_{|u|} = W_{|h|}[-N]$ for an external vertex $u = \{h\}$.

We need to identify elements of $\tilde{O}_{\text{com}}(W, \Gamma)$ by the symmetry of Γ . An automorphism α of an N -graded graph $\Gamma \in \mathcal{G}(N)$ induces the linear isomorphism $C(v) \rightarrow C(\alpha(v))$ for $v \in V_i(\Gamma)$ described by

$$[h_1, \dots, h_k]_{\text{cyc}} \mapsto [\alpha(h_1), \dots, \alpha(h_k)]_{\text{cyc}},$$

and the identity map $W[-N]_{|u|} \rightarrow W[-N]_{|\alpha(u)|} = W[-N]_{|u|}$ for $u \in V_e(\Gamma)$. Therefore, the automorphism group of Γ acts on the vector space $\tilde{O}_{\text{com}}(W, \Gamma)$ by the induced permutation of half-edges. Then the coinvariant vector space of $\tilde{O}_{\text{com}}(W, \Gamma)$ by this action is denoted by $O_{\text{com}}(W, \Gamma)$. We often consider an element o of $O_{\text{com}}(W, \Gamma)$ described by the form

$$\begin{aligned} o &= [o_1, \dots, o_l; w_1, \dots, w_{k_e}; c_1^s, \dots, c_{k_s}^s; c_1, \dots, c_{k_n}] \\ &:= (o_1 \cdots o_l) \otimes (w_1 \cdots w_{k_e}) \otimes (c_1^s \cdots c_{k_s}^s) \otimes (c_1 \cdots c_{k_n}) \end{aligned}$$

where $w_i \in W[-N]_{|u_i|}$ and

$$o_i = [\hat{o}_i], \quad c_i^s = [\hat{c}_i^s]_{\text{cyc}} \sigma, \quad c_i = [\hat{c}_i]_{\text{cyc}} \sigma,$$

for $\hat{o}_i \in \text{Ord}(e_i)$, $\hat{c}_i \in \text{Ord}(v_i)$, and $\hat{c}_i^s \in \text{Ord}(v_i^s)$. Such element o is called an *orientation* of Γ , a pair (Γ, o) is an *oriented graph*, and the information

$$\hat{o} = (\hat{o}_1, \dots, \hat{o}_l; w_1, \dots, w_{k_e}; \hat{c}_1^s, \dots, \hat{c}_{k_s}^s; \hat{c}_1, \dots, \hat{c}_{k_n})$$

is called a *lift* of an orientation $o = [\hat{o}]$ on Γ . The vector space $O_{\text{com}}(W, \Gamma)$ is generated by orientations.

Example 3.6. In the case of $N = 4$ and $Z = \{0, 1, 2, 3, 4\}$, we can give examples of decorated 4-graded graphs in [Figure 3](#). In these figures,

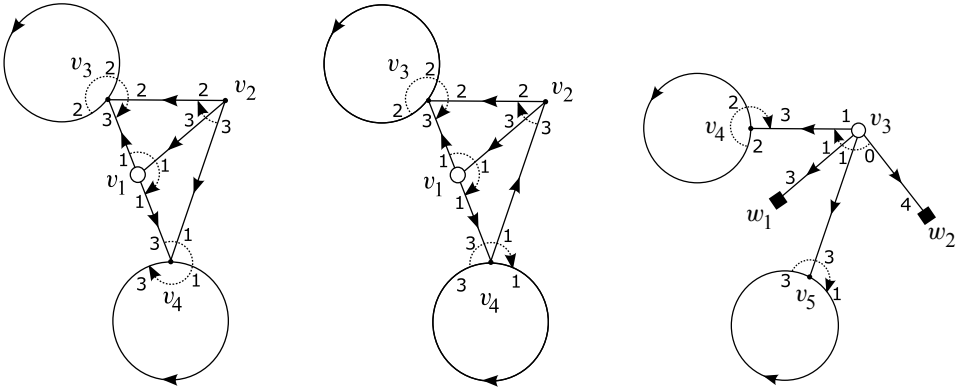


Figure 3. Left and center: nonlabeled examples (Γ, o_1) and (Γ, o_2) . The ordering of vertices is $v_1 v_2 v_3 v_4$ and $v_1 v_2 v_4 v_3$. Right: a labeled example. The ordering of vertices and labels is $(w_1 \omega^{-1})(w_2 \omega^{-1}) v_3 v_4 v_5$.

- an arrow on an edge means a direction, and
- an arc drawn around a vertex is an ordering of half-edges incident to this vertex.

In Figure 3, left and center, the degrees of vertices are $v_1 = -1$, $v_2 = 4$, $v_3 = 5$, and $v_4 = 4$. In the space $O(\Gamma)$, we have

$$o_1 = (-1)^{5 \cdot 4 + 1} (-1)^{3 \cdot 1 + 1} (-1)^{3 \cdot (3 + 1 + 1)} o_2 = o_2,$$

where the signs $(-1)^{5 \cdot 4 + 1}$, $(-1)^{3 \cdot 1 + 1}$, $(-1)^{3 \cdot (3 + 1 + 1)}$ are coming from changes of the ordering of vertices, the direction of the edge between v_2 and v_4 , and the ordering of half-edges incident to v_4 , respectively.

In Figure 3, right, elements $w_1 \in W_3$ and $w_2 \in W_4$ are labels of univalent vertices v_1, v_2 (their names v_1, v_2 of vertices are omitted in the figure). Note their degrees $|v_1| = |w_1 \sigma^{-1}| = -1$ and $|v_2| = |w_2 \sigma^{-1}| = 0$.

Definition of the bigraded vector space $\widehat{C}_{\text{com}}^{p,q}(W)$. The cohomological bidegree $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ of $\Gamma \in \mathcal{G}(N)$ is defined by

$$p = \#V_n(\Gamma), \quad q = \sum_{v \in V_n(\Gamma)} |v| = \#V_s(\Gamma) + N(\#E(\Gamma) - \#V(\Gamma)) - \sum_{u \in V_e(\Gamma)} |u|,$$

and the bidegree of elements in $O_{\text{com}}(W, \Gamma)$ is defined by that of Γ . We define the space of N -graded ribbon graphs by

$$\widehat{C}_{\text{com}}^{p,q}(W) := \bigoplus_{\Gamma \in \mathcal{G}(N)} O_{\text{com}}(W, \Gamma), \quad \widehat{C}_{\text{com}}^{p,q}(W) := \bigoplus_{\Gamma \in \mathcal{G}^{p,q}(W)} O_{\text{com}}(W, \Gamma),$$

where $\mathcal{G}^{p,q}(W)$ is the subset of $\mathcal{G}(N)$ consisting of N -graded graphs of degree (p, q) . Then $\widehat{C}_{\text{com}}^{\bullet,\bullet}(W)$ can be regarded as a bigraded vector space. We often denote an element in $\widehat{C}_{\text{com}}^{\bullet,\bullet}(W)$ corresponding to $o \in O_{\text{com}}(W, \Gamma)$ for $\Gamma \in \mathcal{G}(N)$ by (Γ, o) .

Definition of the first differential d . We define a linear map $d_{v;h^1,h^2}^{a,b} : O_{\text{com}}(W, \Gamma) \rightarrow \widehat{C}_{\text{com}}^{\bullet,\bullet}(W)$ for an N -graded graph $\Gamma \in \mathcal{G}(N)$, a normal vertex $v \in V_n(\Gamma)$, two distinct half-edges h^1, h^2 incident to v , and $a, b \in \mathbb{Z}$ satisfying $a + b = N$. For an order h_1, \dots, h_r of half-edges incident to v such that $h^1 = h_r$ and $h^2 = h_i$, put

$$\begin{aligned} d_{v;h^1,h^2}^{a,b}(\Gamma, [\cdot; \cdot; \cdot; [h_1, \dots, h_r]\sigma, \cdot]) \\ = (\Gamma_{v;h^1,h^2}^{a,b}, [\cdot, [h', h'']; \cdot; \cdot; [h_1, \dots, h_i, h']\sigma, [h'', h_{i+1}, \dots, h_r]\sigma, \cdot]). \end{aligned}$$

Here σ is the N -fold suspension, and the N -graded graph $\Gamma_{v;h^1,h^2}^{a,b}$ is defined by

$$\begin{aligned} H(\Gamma_{v;h^1,h^2}^{a,b}) &= H(\Gamma) \sqcup \{h', h''\}, & V(\Gamma_{v;h^1,h^2}^{a,b}) &= (V(\Gamma) \setminus \{v\}) \sqcup \{v', v''\}, \\ V_s(\Gamma_{v;h^1,h^2}^{a,b}) &= V_s(\Gamma), & E(\Gamma_{v;h^1,h^2}^{a,b}) &= E(\Gamma) \sqcup \{e_0\}, \end{aligned}$$

where $v' = \{h_1, \dots, h_i, h'\}$, $v'' = \{h'', h_{i+1}, \dots, h_r\}$, $e_0 = \{h', h''\}$, $|h'| = a$, and $|h''| = b$. Note that the equation above is enough to define the operator $d_{v;h^1,h^2}^{a,b}$ and the operator is well-defined. A picture of the map $d_{v;h^1,h^2}^{a,b}$ is described in Figure 4.

Then we obtain the linear map $d : \widehat{C}_{\text{com}}^{\bullet,\bullet}(W) \rightarrow \widehat{C}_{\text{com}}^{\bullet,\bullet}(W)$ by

$$d_v(\Gamma, o) := \frac{1}{2} \sum_{a+b=N} \sum_{h^1 \neq h^2 \in v} d_{v;h^1,h^2}^{a,b}(\Gamma, o), \quad d(\Gamma, o) := \sum_{v \in V_n(\Gamma)} d_v(\Gamma, o).$$

The map d can be also described by

$$d_v(\Gamma, o) = \sum_{a+b=N} \sum_{0 \leq s < t < r} d_{v;h_s,h_t}^{a,b}(\Gamma, o),$$

where $o = [\cdot; \cdot; \cdot; [h_1, \dots, h_r]\sigma, \cdot]$ and $v = \{h_1, \dots, h_r\}$. Note the relation

$$d_{v;h^1,h^2}^{a,b}(\Gamma, o) = d_{v;h^2,h^1}^{b,a}(\Gamma, o)$$

for half-edges $h^1 \neq h^2 \in v$. Here well-definedness of d is proved by the relation with the commutativity relation:

Proposition 3.7. *Using the notations above, $d_v S_{v,h_r,i}(\Gamma, o)$ is equal to zero under the commutativity relation.*

Proof. For integers p, q , we define the linear ordered set $[p, q]$ by $\{p < p+1 < \dots < q-1 < q\}$. If $p > q$, put $[p, q] = \emptyset$. For partially ordered sets P_1, P_2 , we denote their direct sum by $P_1 + P_2$ (in the category of posets), and their ordinal sum by $P_1 \oplus P_2$. Then a (p, q) -shuffle is equivalent to the inverse of an order-preserving bijection $[1, p] + [p+1, p+q] \rightarrow [1, p+q]$.

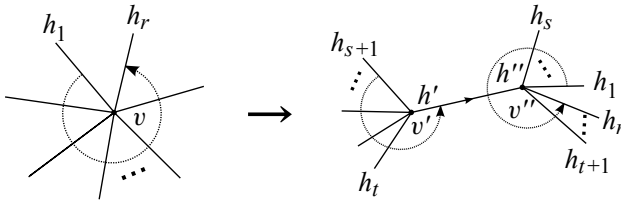


Figure 4. The operator $d_{v;h_s,h_t}^{a,b}$.

Let $\tau^{-1} : [1, i] + [i + 1, r - 1] \rightarrow [1, r - 1]$ be an $(i, r - i - 1)$ -shuffle and $0 \leq s < t < r$ integers. Put $L = \tau([s + 1, t])$ and $l = t - s$.

If $\tau(s + 1), \dots, \tau(t)$ are $\leq i$, then we have $\tau(s + m) = \tau(s + 1) + (m - 1)$ for $1 \leq m \leq t - s$ since $[1, i] \rightarrow \tau^{-1}([1, i])$ is an isomorphism between posets. Put $a = \tau(s + 1) - 1$. Then we obtain the shuffle τ_2 by τ :

$$\begin{array}{ccc}
 [1, i - l + 1] + [i - l + 2, r - l] & \xrightarrow{\tau_2^{-1}} & [1, r - l] \\
 \uparrow \text{canonical isom.} & & \uparrow \text{canonical isom.} \\
 [1, a] \oplus \{*\} \oplus [a + l, i] + [i + 1, r - 1] & \xrightarrow{\text{bij.}} & [1, s] \oplus \{*\} \oplus [t + 1, r - 1] \\
 \uparrow & & \uparrow \\
 [1, i] + [i + 1, r - 1] & \xrightarrow{\tau^{-1}} & [1, r - 1]
 \end{array}$$

The shuffle τ can be recovered from a pair (a, l, τ_2) , where $\{a + 1, \dots, a + l\} \subset [1, i]$ and τ_2 is an $(i - l + 1, r - i - 1)$ -shuffle.

Similarly, if $\tau(s + 1), \dots, \tau(t)$ are $\geq i + 1$, we can obtain a triple (a, l, τ_2) , where $\{a + 1, \dots, a + l\} \subset [i + 1, r - 1]$ and τ_2 is an $(i - l + 1, r - i - 1)$ -shuffle.

Otherwise, put $p = \#(L \cap [1, i])$. Then we obtain the shuffle τ_1 by restricting τ :

$$\begin{array}{ccc}
 [1, p] + [p + 1, l] & \xrightarrow{\tau_1^{-1}} & [1, l] \\
 \downarrow \text{canonical isom.} & & \downarrow \text{canonical isom.} \\
 L & \xrightarrow{\tau^{-1}} & \tau^{-1}(L) \\
 \downarrow & & \downarrow \\
 [1, i] + [i + 1, r - 1] & \xrightarrow{\tau^{-1}} & [1, r - 1]
 \end{array}$$

We consider $\bar{L} = ([1, i] + [i + 1, r - 1]) \setminus L$ and the order-preserving bijection $\rho^{-1} : \bar{L} \rightarrow [1, s] \oplus [t + 1, r - 1]$ defined by the restriction of τ^{-1} . The shuffle τ is recovered from a pair (ρ, τ_1) , where $\rho^{-1} : \bar{L} \rightarrow [1, s] \oplus [t + 1, r - 1]$ is an order-preserving bijection and τ_1 is a $(p, l - p)$ -shuffle.

Thus, we have

$$\begin{aligned}
 d_v S_{v,h_r;i}([h_1, \dots, h_r]\sigma) &= \sum_{l=1}^{r-1} \left(\sum_{p=1}^{l-1} \sum_{\rho} \sum_{\tau_1} o_{\rho}^{\tau_1^{(v',h')}} + \sum_a \sum_{\tau_2} o_{a,l}^{\tau_2^{(v'',h_r)}} \right) \\
 &= \sum_{l=1}^{r-1} \left(\sum_{p=1}^{l-1} \sum_{\rho} S_{v',h';p}(o_{\rho}) + \sum_a S_{v'',h_r;i-l+1}(o_{a,l}) \right),
 \end{aligned}$$

where $L = \{1, \dots, r - 1\} \setminus \bar{L} = \{u_1 < \dots < u_p \text{ as integers}\}$,

$$\begin{aligned}
 o_{\rho} &= \epsilon \left[[h_{u_1}, \dots, h_{u_p}, h']\sigma, [h_{\rho(1)}, \dots, h_{\rho(s)}, h'', h_{\rho(t+1)}, \dots, h_{\rho(r-1)}, h_r]\sigma \right], \\
 o_{a,l} &= \epsilon' \left[[h_{a+1}, \dots, h_{a+l}, h']\sigma, [h_1, \dots, h_a, h'', h_{a+l+1}, \dots, h_r]\sigma \right],
 \end{aligned}$$

and ϵ, ϵ' are appropriate Koszul signs. (In these equations, the subscripts cyc are omitted.) □

Definition of the second differential L. For $\Gamma \in \mathcal{G}(N)$, let $i_v(\Gamma)$ be the N -graded graph obtained by converting a normal vertex v of degree -1 to a special vertex. We define the linear map $i_v : O_{\text{com}}(W, \Gamma) \rightarrow O_{\text{com}}(W, i_v(\Gamma))$ for $o \in O_{\text{com}}(W, \Gamma)$ such that

$$i_v(\Gamma, [\cdot; \cdot; \cdot; c, \cdot]) = (i_v(\Gamma), [\cdot; \cdot; \cdot, c; \cdot])$$

for $c \in C(v)$ if v has degree -1 and valency ≥ 3 , and $i_v(\Gamma, o) = 0$ if v does not. Since the relation

$$i_{v_1} S_{v_2,h_r;k}(\Gamma, o) = S_{v_2,h_r;k} i_{v_1}(\Gamma, o)$$

for $v_1, v_2 \in V_i(\Gamma)$ holds clearly, the map i_v is well-defined. Then the linear map $L : \widehat{C}_{\text{com}}^{\bullet,\bullet}(W) \rightarrow \widehat{C}_{\text{com}}^{\bullet,\bullet}(W)$ is defined by

$$L := id - di,$$

where the linear map $i : \widehat{C}_{\text{com}}^{\bullet,\bullet}(W) \rightarrow \widehat{C}_{\text{com}}^{\bullet,\bullet}(W)$ is obtained by

$$i(\Gamma, o) := \sum_{v \in V_n(\Gamma)} i_v(\Gamma, o).$$

The map L is also described by

$$L(\Gamma, o) = \sum_{v \in V_n(\Gamma)} (i_{v'} + i_{v''})d_v(\Gamma, o)$$

since $i_u d_v = d_v i_u$ for normal vertices $u \neq v$.

Then $d, i,$ and L have (cohomological) bidegree $(1, 0), (-1, 1),$ and $(0, 1),$ respectively.

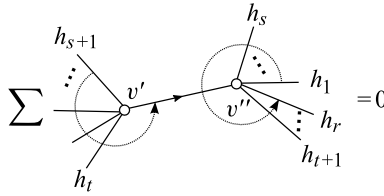


Figure 5. A_∞ -relation.

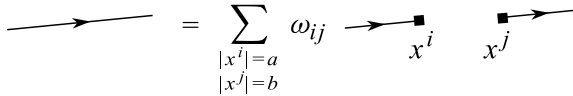


Figure 6. Cut-off relation.

Definition of the underlying bigraded vector space $C_{\text{com}}^{\bullet,\bullet}(W)$. The space $C_{\text{com}}^{\bullet,\bullet}(W)$ is the quotient space of $\widehat{C}_{\text{com}}^{\bullet,\bullet}(W)$ by

- (A_∞ -relation) $R_v(\Gamma, o) := i_{v'}i_{v''}d_v(\Gamma, o) = 0$
for $\Gamma \in \mathcal{G}(N)$ and a normal vertex v (of degree -2). This relation is described in [Figure 5](#).
- (*Cut-off relation*) For $\Gamma \in \mathcal{G}(N)$ and $e = \{h_1, h_2\} \in E(\Gamma)$, we define the N -graded graph Γ_e as

$$\begin{aligned} H(\Gamma_e) &= H(\Gamma) \sqcup \{\bar{h}_1, \bar{h}_2\}, \\ E(\Gamma_e) &= (E(\Gamma) \setminus \{e\}) \sqcup \{\{h_1, \bar{h}_1\}, \{h_2, \bar{h}_2\}\}, \\ V(\Gamma_e) &= V(\Gamma) \sqcup \{\{\bar{h}_1\}, \{\bar{h}_2\}\}, \\ |\bar{h}_1| &= N - |h_1| =: a, \quad |\bar{h}_2| = N - |h_2| =: b. \end{aligned}$$

Then

$$(\Gamma, [[h_1, h_2], \cdot, \cdot; \cdot; \cdot]) = \sum_{\substack{|x^i|=a \\ |x^j|=b}} \omega_{ij}(\Gamma_e, [[h_1, \bar{h}_1], [\bar{h}_2, h_2], \cdot; x^i\sigma^{-1}, x^j\sigma^{-1}, \cdot; \cdot; \cdot]),$$

where $\{x^i\}$ is a homogeneous basis of W and (ω_{ij}) is the inverse matrix of $(\omega(x^i, x^j))$. This relation is described in [Figure 6](#).

Note that $C_{\text{com}}^{\bullet,\bullet}(W)$ is generated by W -labeled graphs with only one internal vertex by cut-off relation.

On well-definedness of three operators d, i, L on $C_{\text{com}}^{\bullet,\bullet}(W)$. The endomorphisms d, i , and L of $\widehat{C}_{\text{com}}^{\bullet,\bullet}(W)$ induce endomorphisms of $C_{\text{com}}^{\bullet,\bullet}(W)$ by the equations

$$dR_v(\Gamma, o) = \sum_{u \neq v} R_v d_u(\Gamma, o), \quad iR_v(\Gamma, o) = \sum_{u \neq v} R_v i_u(\Gamma, o)$$

for a normal vertex v of an N -graded graph Γ .

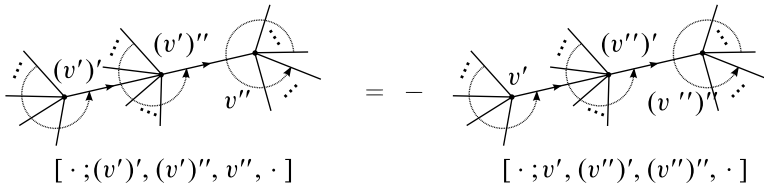


Figure 7. $d_{v'}d_v(\Gamma, o) = -d_{v''}d_v(\Gamma, o)$.

On two differentials d, L on $C_{\text{com}}^{\bullet, \bullet}(W)$.

Proposition 3.8. *The bigraded vector space $C_{\text{com}}^{\bullet, \bullet}(W)$ is a double complex with respect to differentials d and L . We call $C_{\text{com}}^{\bullet, \bullet}(W)$ a double graph complex.*

Proof. First, we show the equation $d^2 = 0$. It is proved in the same way as Kontsevich’s original graph complex. For a normal vertex v of an N -graded graph (Γ, o) , let v', v'' be new vertices obtained by splitting at v . Then

$$d_{v'}d_v(\Gamma, o) = -d_{v''}d_v(\Gamma, o), \quad d_u d_v(\Gamma, o) = -d_v d_u(\Gamma, o)$$

for $u \neq v$. The first equation is shown by **Figure 7**. In the figure, v' and v'' are defined such that the direction of the new edge is from v' to v'' , and $(v')', (v')'', (v'')', (v'')''$ are also defined in the same way. So we obtain $d^2(\Gamma, o) = 0$ by cancellation.

Next, we show $L^2 = 0$. From the equation in $\widehat{C}_{\text{com}}^{\bullet, \bullet}(W)$

$$\begin{aligned} (iL - Li)(\Gamma, o) &= \left(\sum_u i_u(i_{v'} + i_{v''})d_v - \sum_{u \neq v} (i_{v'} + i_{v''})d_v i_u \right)(\Gamma, o) \\ &= \sum_v (i_{v''}i_{v'} + i_{v'}i_{v''})d_v(\Gamma, o) \\ &= 2 \sum_v R_v(\Gamma, o), \end{aligned}$$

we obtain the relation $iL - Li = 0$ in $C_{\text{com}}^{\bullet, \bullet}(W)$. So the equations

$$\begin{aligned} L^2 &= (id - di)L = idL - diL = idL - dLi = idid - did, \\ L^2 &= L(id - di) = Lid - Ldi = iLd - Ldi = -idid + did \end{aligned}$$

hold. Then we obtain $L^2 = 0$. Since $Ld + dL = -did + did = 0$ holds by definition of L , we get the proposition. □

3C. Construction of the map to Chevalley–Eilenberg complexes. Let (W, ω) and Z be as in **Section 3B** and δ be a symplectic and quadratic differential of homological degree -1 on $\hat{L}W$. In this section, the Lie algebra $\text{Der}_\omega(\hat{L}W)$ of symplectic derivations is denoted by \mathfrak{D} . We construct a double chain map

$$C_{\text{com}}^{\bullet, \bullet}(W) \rightarrow C_{\text{CE}}^{\bullet, \bullet}(\mathfrak{D})$$

from the graph complex $C_{\text{com}}^{\bullet, \bullet}(W)$ to the Chevalley–Eilenberg complex of the dgl $(\mathfrak{D}, \text{ad}(\delta))$.

Let (Γ, o) be an oriented graph and \hat{o} be a lift of o . Put

$$\begin{aligned} k &= \#V(\Gamma), & k_e &= \#V_e(\Gamma), & k_s &= \#V_s(\Gamma), & k_n &= \#V_n(\Gamma), \\ (r_1, \dots, r_k) &:= \underbrace{(1, \dots, 1)}_{k_e}, a_1, \dots, a_{k_s+k_n} \\ &:= \underbrace{(1, \dots, 1)}_{k_e}, \#v_1^s, \dots, \#v_{k_s}^s, \#v_1, \dots, \#v_{k_n}. \end{aligned}$$

We denote by $\tau(\hat{o})$ the linear isomorphism (the permutation of factors of the tensor product)

$$W^{\otimes r_1} \otimes \dots \otimes W^{\otimes r_k} \rightarrow W^{\otimes 2} \otimes \dots \otimes W^{\otimes 2} = (W^{\otimes 2})^{\otimes l}$$

corresponding to the permutation of half-edges

$$(h_1, \dots, h_{k_e}, \hat{c}_1^s, \dots, \hat{c}_{k_s}^s, \hat{c}_1, \dots, \hat{c}_{k_n}) \mapsto (\hat{o}_1, \dots, \hat{o}_l).$$

Then we define the linear map $\alpha(\Gamma, \hat{o})$ of cohomological degree $(l - k)N$ by composing these maps

$$\begin{aligned} \alpha(\Gamma, \hat{o}) : W[-N]^{\otimes k_e} \otimes \text{Der}_\omega(\hat{L}W)^{\otimes (k_s+k_n)} &\xrightarrow{\text{proj.}} W[-N]^{\otimes k_e} \otimes \bigotimes_{i=1}^{k_s+k_n} \text{Der}_\omega^{a_i+2}(\hat{L}W) \\ &\xrightarrow{\Phi} W[-N]^{\otimes k_e} \otimes \bigotimes_{i=1}^{k_s+k_n} W(a_i)[-N] \subset \bigotimes_{i=1}^k (W^{\otimes r_i}[-N]) \\ &\xrightarrow{\sigma^{\otimes k}} \bigotimes_{i=1}^k W^{\otimes r_i} \xrightarrow{\tau(\hat{o})} (W^{\otimes 2})^{\otimes l} \xrightarrow{\omega_E} \mathbb{R}, \end{aligned}$$

where $\Phi := \text{id}_{W[-N]}^{\otimes k_e} \otimes \Phi_\omega^{\otimes (k_s+k_n)}$, $\omega_E := \omega_{e_1} \otimes \dots \otimes \omega_{e_l}$, and $\omega_{e_j} := \omega_{(|h_1^{e_j}|, |h_2^{e_j}|)}$ if $e_j = \{h_1^{e_j}, h_2^{e_j}\}$. Here we denote by $\omega_{(d_1, d_2)}$ for integers d_1, d_2 the composition of the projection $W \otimes W \rightarrow W_{d_1} \otimes W_{d_2}$ and the restriction of ω to $W_{d_1} \otimes W_{d_2}$. The map $\alpha(\Gamma, \hat{o})$ is independent of a choice of linear orders of half-edges representing cyclic orders, and compatible with the commutativity relation.

We define the map $\hat{\psi}(\Gamma, \hat{o}) : \mathfrak{D}^{\otimes k_n} \rightarrow \mathbb{R}$ by

$$\hat{\psi}(\Gamma, \hat{o})(D_1, \dots, D_{k_n}) := \alpha(\Gamma, \hat{o})(w_1, \dots, w_{k_e}, \underbrace{\delta, \dots, \delta}_{k_s}, D_1, \dots, D_{k_n})$$

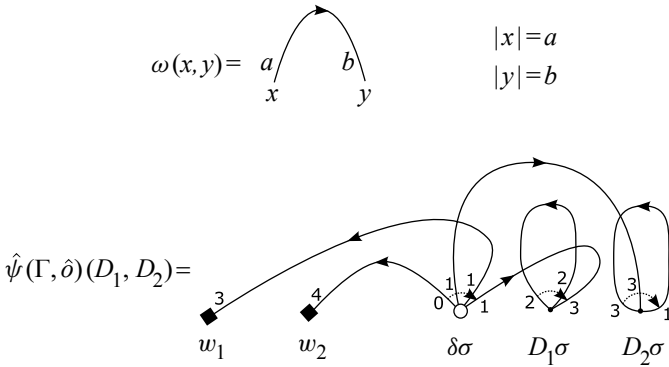


Figure 8. An example of $\hat{\psi}(\Gamma, \hat{\sigma})(D_1, D_2)$. (Γ is the decorated graph in Figure 3, right.)

for $D_i \in \mathcal{D}$. (An example of $\hat{\psi}(\Gamma, \hat{\sigma})$ is described in Figure 8.) Restricting the map¹ on the exterior algebra, we can get the map

$$\psi(\Gamma, o) = \hat{\psi}(\Gamma, \hat{\sigma}) \circ \text{Alt}_{k_n} : \Lambda^{k_n} \mathcal{D} \rightarrow \mathbb{R}.$$

The map is independent of a representation $\hat{\sigma}$ of o by the definition of an orientation. So we obtain the map $\psi : C_{\text{com}}^{\bullet, \bullet}(W) \rightarrow C_{\text{CE}}^{\bullet, \bullet}(\mathcal{D})$.

Well-definedness of ψ is proved by the correspondence through ψ between relations in the graph complex $C_{\text{com}}^{\bullet, \bullet}(W)$ corresponding to properties of derivations:

graph complex	derivations
cyclicity	symplectic derivation
commutativity	Lie derivation
A_∞ -relation	$\delta^2 = 0$
cut-off	symplectic form

By definition, it is clear except for the A_∞ -relation. The correspondence for the A_∞ -relation is proved in the end of the proof of the following theorem.

Theorem 3.9. *The map $\psi : C_{\text{com}}^{\bullet, \bullet}(W) \rightarrow C_{\text{CE}}^{\bullet, \bullet}(\mathcal{D})$ is a double chain map.*

Proof. First, we shall show that $d_{\text{CE}}\psi = \psi d$ on $\widehat{C}_{\text{com}}^{\bullet, \bullet}(W)$. To prove this, we need Lemma 2.3.

¹For a graded vector space V , the injective map $\text{Alt}_n : \Lambda^n V \rightarrow V^{\otimes n}$ is defined by

$$\text{Alt}_n(v_1 \cdots v_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \bar{\epsilon}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

for $v_1, \dots, v_n \in V$, where $\bar{\epsilon}(\sigma)$ is the corresponding anti-Koszul sign.

For an oriented graph (Γ, o) , we define the two lifts $\hat{\delta}^1, \hat{\delta}^2$ on $\Gamma_{v_i; h_\nu, h_\mu}^{s,t}$ as

$$\begin{aligned} \hat{\delta}^1 &= ((h', h''), \cdot; \cdot; \cdot; v_1, \dots, v'_i, v''_i, \dots, v_p), \\ \hat{\delta}^2 &= ((h'', h'), \cdot; \cdot; \cdot; v_1, \dots, v''_i, v'_i, \dots, v_p), \\ v'_i &= (h^i_{\nu+1}, \dots, h^i_\mu, h'), \quad v''_i = (h^i_1, \dots, h^i_\nu, h'', h^i_{\mu+1}, \dots, h^i_{r_i}), \end{aligned}$$

where $r_i = \#v_i$. The signs ϵ_i are defined by the equations

$$o^1 := \epsilon_1[\hat{\delta}^1], \quad o^2 := \epsilon_2[\hat{\delta}^2], \quad d_{v_i; h_\nu, h_\mu}^{s,t} o = (-1)^{i-1} o^1 = (-1)^{i-1} o^2.$$

So we obtain

$$\begin{aligned} d(\Gamma, o) &= \sum_{i=1}^k \sum_{\nu < \mu} \sum_{a+b=N} (-1)^{i-1} (\Gamma_{v_i; h_\nu, h_\mu}^{a,b}, o^1) \\ &= \sum_{i=1}^k \sum_{\nu < \mu} \sum_{a+b=N} (-1)^{i-1} (\Gamma_{v_i; h_\nu, h_\mu}^{a,b}, o^2). \end{aligned}$$

Note that

$$d_{CE}(\chi \circ \text{Alt}_p) = \frac{1}{2} \sum_{s=1}^p (-1)^{s-1} \chi \circ (1^{\otimes s-1} \otimes [\cdot, \cdot] \otimes 1^{\otimes p-s}) \circ \text{Alt}_{p+1}$$

for a linear map $\chi : W^{\otimes r_1}[-N] \otimes \dots \otimes W^{\otimes r_p}[-N] \rightarrow \mathbb{R}$ and the antisymmetrization Alt_p for p -components. So we should prove

$$\begin{aligned} \hat{\psi}(\Gamma, \hat{\delta}) \circ (1^{\otimes i-1} \otimes [\cdot, \cdot] \otimes 1^{\otimes p-i-1}) \\ = \sum_{\nu < \mu} \sum_{a+b=N} (\epsilon_1 \hat{\psi}(\Gamma_{v_i; h_\nu, h_\mu}^{a,b}, \hat{\delta}^1) + \epsilon_2 \hat{\psi}(\Gamma_{v_i; h_\nu, h_\mu}^{a,b}, \hat{\delta}^2) \circ \tau), \end{aligned}$$

where the map τ means the permutation

$$\begin{aligned} X_1 \otimes \dots \otimes (x_{\nu+1} \dots x_\mu x') \otimes (x_1 \dots x_\nu x'' x_{\mu+1} \dots x_{r_i}) \otimes \dots \otimes X_p \\ \mapsto \epsilon \cdot X_1 \otimes \dots \otimes (x_1 \dots x_\nu x' x_{\mu+1} \dots x_{r_i}) \otimes (x_{\nu+1} \dots x_\mu x'') \otimes \dots \otimes X_p \end{aligned}$$

and ϵ is the Koszul sign. It follows from the equations

$$\begin{aligned} \hat{\psi}(\Gamma, \hat{\delta}) \circ (1^{\otimes i-1} \otimes \sigma^{-1}(1 \otimes \omega_{(a,b)}) \pi_{1;t}^{r',r''} \sigma^{\otimes 2} \otimes 1^{\otimes p-i-1}) &= \epsilon_1 \hat{\psi}(\Gamma_{v_i; h_\nu, h_\mu}^{a,b}, \hat{\delta}^1), \\ \hat{\psi}(\Gamma, \hat{\delta}) \circ (1^{\otimes i-1} \otimes \sigma^{-1}(1 \otimes \omega_{(a,b)}) \pi_{2;t}^{r',r''} \sigma^{\otimes 2} \otimes 1^{\otimes p-i-1}) &= \epsilon_2 \hat{\psi}(\Gamma_{v_i; h_\nu, h_\mu}^{a,b}, \hat{\delta}^2) \circ \tau, \end{aligned}$$

for $r' = \mu - \nu + 1, r'' = r - \mu + \nu + 1$, and $t = \nu + 1$. The first equation is verified as follows: we have by the definition of $\hat{\psi}$

$$\omega(x', x'') \hat{\psi}(\Gamma, \hat{\delta})(X_1, \dots, X_p) = \epsilon_1 \hat{\psi}(\Gamma_{v_i; h_\nu, h_\mu}^{a,b}, \hat{\delta}^1)(X_1, \dots, X'_i, X''_i, \dots, X_p)$$

for $X_s \in W^{\otimes r_s}$, $x' \in W_a$, and $x'' \in W_b$. Here we put $X'_i = x_{v+1} \cdots x_{\mu} x' \sigma^{-1}$ and $X''_i = x_1 \cdots x_{\nu} x'' x_{\mu+1} \cdots x_r \sigma^{-1}$ for $X_i = x_1 \cdots x_r \sigma^{-1}$. So we obtain the first equation from

$$\epsilon_1 X_i \omega(x', x'') = \sigma^{-1} (1 \otimes \omega) \pi_{1;t}^{r',r''} \otimes \sigma^{\otimes 2} (X'_i \otimes X''_i).$$

The second is also verified in the same way.

Next, we shall prove $i_{\delta} \psi = \psi i$ on $\widehat{C}_{\text{com}}^{\bullet, \bullet}(W)$. The ordering

$$\hat{\delta}_i := (\cdot; \cdot; \cdot; v_i, v_1, \dots, \hat{v}_i, \dots, v_p)$$

is a lift of $\bar{\epsilon}_i \cdot o$, where $\bar{\epsilon}_i$ is the anti-Koszul sign of the permutation

$$(v_1, \dots, v_p) \mapsto (v_i, v_1, \dots, \hat{v}_i, \dots, v_p).$$

So we have

$$\begin{aligned} \psi i(\Gamma, o)(X_1, \dots, X_{p-1}) &= \sum_{s=1}^j \bar{\epsilon}_i \cdot \alpha(i_{v_i}(\Gamma), \hat{\delta}_i)(w_1, \dots, w_{k_e}, \underbrace{\delta, \dots, \delta}_{k_s+1}, \text{Alt}_{p-1}(X_1, \dots, X_{p-1})) \\ &= \sum_{s=1}^j \sum_{\pi \in \mathfrak{S}_{p-1}} \bar{\epsilon} \cdot \alpha(\Gamma, \hat{\delta})(w_1, \dots, w_{k_e}, \underbrace{\delta, \dots, \delta}_{k_s}, X_{\pi(1)}, \dots, \delta, \dots, X_{\pi(p-1)}) \\ &= \alpha(\Gamma, \hat{\delta})(w_1, \dots, w_{k_e}, \underbrace{\delta, \dots, \delta}_{k_s}, \text{Alt}_p(\delta, X_1, \dots, X_{p-1})) \\ &= i_{\delta} \psi(\Gamma, o)(X_1, \dots, X_{p-1}) \end{aligned}$$

where $\bar{\epsilon}$ is the anti-Koszul sign of

$$(\delta, X_1, \dots, X_{p-1}) \mapsto (X_{\pi(1)}, \dots, \delta, \dots, X_{\pi(p-1)}).$$

From the discussion above, the relation $\psi(R_v(\Gamma, o)) = 0$ follows from

$$\psi(R_v(\Gamma, o)) = \psi(i_{v'} i_{v''} d_v(\Gamma, o)) = \psi(\Gamma, o)([\delta, \delta], \cdot) = 0.$$

Thus, ψ induces the map $\psi : C_{\text{com}}^{\bullet, \bullet}(W) \rightarrow C_{\text{CE}}^{\bullet, \bullet}(\mathcal{D})$. Furthermore, since ψ is commutative with d and i , so is L . So we complete the proof. \square

The group $\text{Sp}(W, \delta)$ acts on $C_{\text{com}}^{\bullet, \bullet}(W)$ by the action on their labels. Then, the chain map $\psi : C_{\text{com}}^{\bullet, \bullet}(W) \rightarrow C_{\text{CE}}^{\bullet, \bullet}(\mathcal{D})$ is $\text{Sp}(W, \delta)$ -equivariant clearly. In particular, we can consider the $\text{Sp}(W, \delta)$ -invariant part $C_{\text{com}}^{\bullet, \bullet}(W)^{\text{Sp}(W, \delta)}$ of the complex $C_{\text{com}}^{\bullet, \bullet}(W)$. It has the double subcomplex $C_{\text{com}}^{\bullet, \bullet}(N, Z)$ consisting of N -graded graphs which have no external vertex. This complex $C_{\text{com}}^{\bullet, \bullet}(N, Z)$ does not depend on the symplectic vector space W . It depends only a range Z of degrees and a degree N of a symplectic inner product.

Remark 3.10. We can define the associative version of $C_{\text{com}}^{\bullet,\bullet}(W)$ as follows. Set

$$\begin{aligned} \tilde{O}_{\text{ass}}(W, \Gamma) &:= \bigoplus_{e \in E(\Gamma)} \mathcal{O}(e) \otimes \bigoplus_{u \in V_e(\Gamma)} W[-N]_{|u|} \\ &\quad \otimes \bigwedge_{v^s \in V_s(\Gamma)} \text{Cyc}(v^s)[N] \otimes \bigwedge_{v \in V_n(\Gamma)} \text{Cyc}(v)[N], \\ C_{\text{ass}}^{\bullet,\bullet}(W) &:= \bigoplus_{\Gamma \in \mathcal{G}(N)} O_{\text{ass}}(W, \Gamma), \quad O_{\text{ass}}(W, \Gamma) := \tilde{O}_{\text{ass}}(W, \Gamma)_{\text{Aut}(\Gamma)}. \end{aligned}$$

Then $(C_{\text{ass}}^{\bullet,\bullet}(W), d, L)$ is also a double $\text{Sp}(W, \delta)$ -chain complex, and the chain map

$$C_{\text{ass}}^{\bullet,\bullet}(W) \rightarrow C_{\text{CE}}^{\bullet,\bullet}(\text{Der}_\omega(\widehat{T}W))$$

can be defined in the same way. In this case, we can also consider the double sub-complex $C_{\text{ass}}^{\bullet,\bullet}(N, Z)$ which consists of N -graded graphs without external vertices.

4. Applications and examples

Examples of relations between our chain map and a known notion are given in the following two examples.

Example 4.1. For a cyclic minimal A_∞ -algebra (H, I, m) with even degree, putting $W := H^*[-1]$, we have the map $C_{\text{ass}}^{\bullet,\bullet}(W) \rightarrow C_{\text{CE}}^{\bullet,\bullet}(\text{Der}_\omega(\widehat{T}W))$. Here $\widehat{T}W$ is the dual of the bar construction of (H, I, m) . The map induced by the chain map

$$C_{\text{ass}}^{0,\bullet}(N, Z) \rightarrow C_{\text{CE}}^{0,\bullet}(\text{Der}_\omega(\widehat{T}W)) = \mathbb{R}$$

is known as the Kontsevich cocycle [Kontsevich 1994; Penkava and Schwarz 1995; Hamilton and Lazarev 2008] of a cyclic A_∞ -algebra (H, I, m) .

Example 4.2. In the case of $Z = \{0\}$ and $\delta = 0$, the chain map

$$C_{\text{ass}}^{\bullet,0}(0, \{0\}) \rightarrow C_{\text{CE}}^{\bullet,0}(\text{Der}_\omega(\widehat{T}W))^{\text{Sp}(W)}$$

is equal to Kontsevich’s chain map [1994; 1993].

In the case that W is positively graded, we define a chain complex $C_{\text{com}}^{\bullet,\bullet}(W)_+$ by

$$C_{\text{com}}^{\bullet,\bullet}(W)_+ = C_{\text{com}}^{\bullet,\bullet}(W)/(\text{positivity}),$$

where the positivity relation is as follows:

- (Positivity) (i) a graph which has a normal vertex v satisfying $|v| \leq 0$ is zero, and (ii) $(i_{v'} + i_{v''})d_v(\Gamma, o) = 0$ for an oriented graph (Γ, o) and a normal vertex v of degree 0.

The differentials d, L are also defined on $C_{\text{com}}^{\bullet,\bullet}(W)_+$, while i is not.

Proposition 4.3. *The operators d, L induce the differentials on $C_{\text{com}}^{\bullet,\bullet}(W)_+$.*

Proof. It is clear that these operators are compatible with the former condition (i) of the positivity relation. Note that, to prove compatibility with L for a graph including a vertex with degree 0, we need to use (ii).

We shall prove they are compatible with (ii). First, we shall calculate the image of (ii) by the operator d . For $\Gamma \in \mathcal{G}(N)$ and a normal vertex v of degree 0, we have

$$\begin{aligned} d(i_{v'} + i_{v''})d_v &= d_{v''}i_{v'}d_v + d_{v'}i_{v''}d_v + \sum_{u \neq v', v''} d_u(i_{v'} + i_{v''})d_v \\ &= d_{v''}i_{v'}d_v + d_{v'}i_{v''}d_v - \sum_{u \neq v} (i_{v'} + i_{v''})d_v d_u. \end{aligned}$$

Here we used the equations in the proof of [Proposition 3.8](#). For a splitting of v such that $|v'| = -1$, $d_{v''}i_{v'}d_v$ must have a nonpositive vertex since $|v''| = 1$. In the same way, $d_{v'}i_{v''}d_v$ must also have a nonpositive vertex. So $d(i_{v'} + i_{v''})d_v$ is equal to zero under the positivity relation.

Next, we shall calculate the image of (ii) by the operator L :

$$\begin{aligned} L(i_{v'} + i_{v''})d_v &= \sum_u (i_{u'} + i_{u''})d_u(i_{v'} + i_{v''})d_v \\ &= (i_{(v')'} + i_{(v'')'})d_{v''}i_{v'}d_v + (i_{(v')'} + i_{(v'')'})d_{v'}i_{v''}d_v \\ &\quad - \sum_{u \neq v} (i_{v'} + i_{v''})d_v(i_{u'} + i_{u''})d_u \\ &= (i_{(v')'} + i_{(v'')'})i_{v'}d_{v''}d_v + (i_{(v')'} + i_{(v'')'})i_{v''}d_{v'}d_v \\ &\quad - \sum_{u \neq v} (i_{v'} + i_{v''})d_v(i_{u'} + i_{u''})d_u. \end{aligned}$$

By changing names of vertices like in the proof of [Proposition 3.8](#), we get

$$(i_{(v'')'} + i_{(v')'})i_{v'}d_{v''}d_v = -(i_{(v')'} + i_{v''})i_{(v')'}d_{v'}d_v = -R_{v'}d_{v'}d_v - i_{(v')'}i_{v''}d_{v'}d_v,$$

and

$$\begin{aligned} (i_{(v')'} + i_{(v'')'})i_{v''}d_{v'}d_v + (i_{(v')'} + i_{(v'')'})i_{v'}d_{v''}d_v &= -R_{v''}d_{v''}d_v + i_{(v')'}i_{v''}d_{v'}d_v \\ &= -R_{v''}d_{v''}d_v - i_{(v'')'}i_{(v')'}d_{v''}d_v \\ &= -R_{v'}d_{v'}d_v - R_{v''}d_{v''}d_v. \end{aligned}$$

Using the A_∞ -relation, $L(i_{v'} + i_{v''})d_v$ is equal to zero under the positivity relation. \square

Then we can also get the chain map

$$\psi_+ : C_{\text{com}}^{\bullet, \bullet}(W)_+ \rightarrow C^*(\text{Der}_\omega^+(LW))$$

induced by ψ .

Example 4.4. Suppose $X = \#_g(S^n \times S^n) \setminus \text{Int } D^{2n}$. Its Quillen model is described by

$$L_X = L(u_1, v_1, \dots, u_g, v_g) \quad (\deg u_i = \deg v_i = n - 1), \quad \delta = 0,$$

$$\omega(u_i, v_j) = \delta_{ij}, \quad \omega(u_i, u_j) = \omega(v_i, v_j) = 0.$$

It means $N = 2n - 2$, $W = \langle u_1, v_1, \dots, u_g, v_g \rangle$, and $Z = \{n - 1\}$. Then the dgl $(\text{Der}_\omega^+(L_X), 0)$ is a Quillen model of $B \text{aut}_{\partial, 0}(X)$ (which is proved in [Berglund and Madsen 2014]). In this case, we can forget all special vertices in the graph complex since $\delta = 0$. So we have the chain map

$$C_{\text{com}}^{\bullet\bullet}(2n - 2, \{n - 1\})_+ / (\text{special vertices}) \rightarrow C_{\text{CE}}^{\bullet\bullet}(\text{Der}_\omega^+(L_X))^{\text{Sp}(W)}.$$

This map is constructed by [Berglund and Madsen 2014], and it is proved that the map is an isomorphism under the limit $g \rightarrow \infty$.

Example 4.5. Suppose $X = \mathbb{C}P^3 \setminus \text{Int } D^6$. Its Quillen model is described by

$$L_X = L(u_1, u_2) \quad (\deg u_i = 2i - 1), \quad \delta = \frac{1}{2}[u_1, u_1] \frac{\partial}{\partial u_2},$$

$$\omega(u_1, u_2) = \omega(u_2, u_1) = 1.$$

It means $N = 4$, $W = \langle u_1, u_2 \rangle$, and $Z = \{1, 3\}$. Then the dgl $(\text{Der}_\omega^+(L_X), \delta)$ is a Quillen model of $B \text{aut}_{\partial, 0}(X)$. Since $\text{Sp}(W, \delta) = 1$, we have the chain map

$$C_{\text{com}}^{\bullet\bullet}(W)_+ \rightarrow C_{\text{CE}}^{\bullet\bullet}(\text{Der}_\omega^+(L_X)) = C_{\text{CE}}^{\bullet\bullet}(\text{Der}_\omega^+(L_X))^{\text{Sp}(W, \delta)}.$$

We shall define a certain sub dgl \mathfrak{d} of $\text{Der}_\omega(L_X)$. Put

$$A_1 = \frac{1}{2}[u_1, u_1] \frac{\partial}{\partial u_2}, \quad A_2 = \frac{1}{2}[u_2, u_2] \frac{\partial}{\partial u_1}$$

$$B_1 = \frac{1}{2}[u_1, u_1] \frac{\partial}{\partial u_1} + [u_1, u_2] \frac{\partial}{\partial u_2}, \quad B_2 = [u_1, u_2] \frac{\partial}{\partial u_1} + \frac{1}{2}[u_2, u_2] \frac{\partial}{\partial u_2}.$$

Then we have

$$\delta(A_1) = \delta(B_1) = \delta(B_2) = 0,$$

$$\delta(A_2) = \frac{1}{2}[[u_1, u_1], u_2] \frac{\partial}{\partial u_1} + \frac{1}{2}[[u_2, u_2], u_1] \frac{\partial}{\partial u_2} = [A_1, A_2] = -[B_1, B_2] =: C,$$

$$[A_i, B_j] = [A_i, A_i] = [B_j, B_j] = 0 \quad (i, j = 1, 2),$$

$$\deg A_1 = -1, \quad \deg A_2 = 5, \quad \deg B_1 = 1, \quad \deg B_2 = 3, \quad \deg C = 4.$$

Here we put $\delta(Z) := [\delta, Z]$ for simplicity. By the relation above,

$$\mathfrak{d} := \langle A_1, A_2, B_1, B_2, C \rangle = \text{Der}_\omega^1(L_X) \oplus \text{Der}_\omega^2(L_X)$$

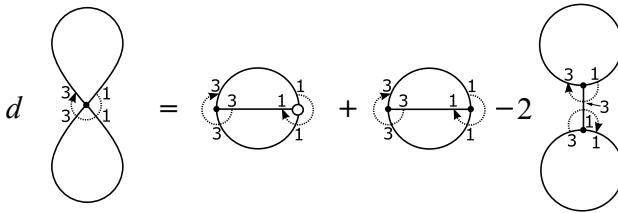


Figure 9. The relation of graphs (the orientations are omitted).

is a sub dgl of $\text{Der}_\omega(L_X)$. Its positive truncation \mathfrak{d}^+ is described by

$$\mathfrak{d}^+ = \langle A_2, B_1, B_2, C \rangle,$$

$$\delta(A_2) = -[B_1, B_2] = C, \quad \delta(B_1) = \delta(B_2) = \delta(C) = 0,$$

$$[A_2, B_i] = [A_2, A_2] = [B_i, B_i] = [A_2, C] = [B_i, C] = 0 \quad (i = 1, 2).$$

Let x, y_1, y_2, z be the suspension of the dual basis of A_2, B_1, B_2, C . Then the Chevalley–Eilenberg complex of the dgl \mathfrak{d}^+ is written by

$$C_{\text{CE}}^{\bullet, \bullet}(\mathfrak{d}^+) = \Lambda(x, y_1, y_2, z) \quad (\deg x = 6, \deg y_1 = 2, \deg y_2 = 4, \deg z = 5),$$

$$dx = dy_1 = dy_2 = 0, \quad dz = x - y_1y_2$$

and its total cohomology

$$H_{\text{CE}}^{\bullet}(\mathfrak{d}^+) = \Lambda(x, y_1, y_2)/(x - y_1y_2).$$

Since \mathfrak{d}^+ is the rank ≤ 2 part of $\text{Der}_\omega^+(L_X)$, the map $H_{\text{CE}}^{\bullet}(\text{Der}_\omega^+(L_X)) \rightarrow H_{\text{CE}}^{\bullet}(\mathfrak{d}^+)$ induced by the inclusion has a section. So nontrivial classes in $H_{\text{CE}}^{\bullet}(\mathfrak{d}^+)$ give nontrivial classes in $H_{\text{CE}}^{\bullet}(\text{Der}_\omega^+(L_X))$.

The relation $dz = x - y_1y_2$ in the Chevalley–Eilenberg complex is corresponding to the relation in the graph complex $C_{\text{com}}^{\bullet, \bullet}(W)_+$ described in Figure 9. Here the classes x and y_1y_2 correspond to the first term and the sum of the second and third terms in the figure. Note that y_1 and y_2 do not correspond to graphs without external vertices. According to the positivity relation, all the trivalent graphs appearing in the right-hand side are cycles since the degrees of two half-edges incident to a permitted bivalent vertex in $C_{\text{com}}^{\bullet, \bullet}(W)_+$ must be 3.

Example 4.6. Suppose $X = \mathbb{C}P^4 \setminus \text{Int } D^8$. Its Quillen model is described by

$$L_X = L(u_1, u_2, u_3) \quad (\deg u_i = 2i - 1), \quad \delta = \frac{1}{2}[u_1, u_1] \frac{\partial}{\partial u_2} + [u_1, u_2] \frac{\partial}{\partial u_3},$$

$$\omega(u_2, u_2) = \omega(u_1, u_3) = 1.$$

It means $N = 6$, $W = \langle u_1, u_2, u_3 \rangle$, and $Z = \{1, 3, 5\}$. Then the dgl $(\text{Der}_\omega^+(L_X), \delta)$ is a Quillen model of $B \text{aut}_{\partial,0}(X)$. Defining the linear transformation τ by $\tau(u_1) = -u_1$, $\tau(u_2) = u_2$, and $\tau(u_3) = -u_3$, we have $\text{Sp}(W, \delta) = \{1, \tau\}$. So $C_{\text{com}}^{\bullet,\bullet}(W)^{\text{Sp}(W,\delta)}$ is generated by graphs labeled by u_1, u_2, u_3 satisfying $\#\{u_1, u_3\text{-labeled vertex}\}$ even. For simplicity, we put

$$[u_{i_1} \cdots u_{i_k}] := [u_{i_1}, \dots, u_{i_k}]_{\text{cyc}} = \sum_{s=1}^k (-1)^{s(k-s)} u_{i_{s+1}} \cdots u_{i_k} u_{i_1} \cdots u_{i_s} \in W_{\text{cyc}}^k.$$

Using notations in [Section 3C](#), we can take a basis of $W(2)$

$$[u_i u_j] \quad (\{i < j\} \subset \{1, 2, 3\}),$$

a basis of $W(3)$

$$\frac{1}{3}[u_i u_i u_i], [u_i u_j u_j], [u_i u_i u_j] \quad (\{i < j\} \subset \{1, 2, 3\}), \quad [u_1 u_2 u_3] + [u_1 u_3 u_2],$$

and a basis of $W(4)$

$$[u_i u_i u_j u_j] \quad (i < j),$$

$$[u_1 u_1 u_2 u_3] + [u_1 u_1 u_3 u_2], [u_1 u_2 u_2 u_3] - [u_1 u_3 u_2 u_2], [u_1 u_2 u_3 u_3] - [u_1 u_3 u_3 u_2].$$

We put the corresponding rank-0, rank-1, and rank-2 bases of $\text{Der}_\omega(L_X)$

$$P_{ij}, A_{iii}, A_{ijj}, A_{iij}, A_{123}, B_{ijj}, B_{1123}, B_{1223}, B_{1233},$$

and these dual bases p_{ij} , x_{ijk} , and y_{ijkl} of P_{ij} , A_{ijk} , and B_{ijkl} . Then by direct calculation we have the equations in $C_{\text{CE}}^{\bullet,\bullet}(\text{Der}_\omega^+(L_X))$

$$dy_{1122} = x_{222} - 2x_{123} + x_{122}x_{113} - x_{122}x_{122},$$

$$dy_{2233} = x_{333}x_{122} + x_{233}x_{222} - x_{223}x_{223} - 2x_{123}x_{233} + x_{133}x_{223} + 2p_{23}y_{1233},$$

$$dy_{1133} = x_{233} - x_{133}x_{113} - x_{123}x_{123} - 2p_{23}y_{1123},$$

$$dy_{1123} = x_{223} - x_{133} - p_{23}y_{1122},$$

$$dy_{1223} = x_{233} + x_{223}x_{122} + x_{123}x_{123} - x_{223}x_{113} - x_{123}x_{222} - x_{133}x_{122} + p_{23}y_{1123},$$

$$dy_{1233} = x_{333} + x_{233}x_{122} - x_{123}x_{223} - x_{233}x_{113} - x_{123}x_{122} + p_{23}y_{1133}.$$

Here all terms appearing in the right-hand side of the equations are cocycles. For example, the fifth relation is corresponding to the relation in the graph complex $C_{\text{com}}^{\bullet,\bullet}(W)_+$ described in [Figure 10](#). In [Figure 10](#), the image by ψ_+ of each graph appearing the last term of the right-hand side is zero.

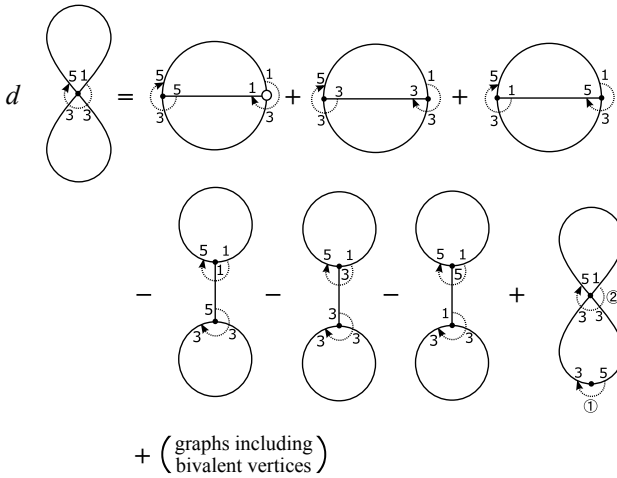


Figure 10. The relation of graphs (① and ② mean the orientation of vertices and the other orientations are omitted).

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