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EXPLICIT WHITTAKER DATA FOR ESSENTIALLY TAME SUPERCUSPIDAL REPRESENTATIONS

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Based on the ideas of Bushnell and Henniart, and of Paskunas and Stevens, we construct explicit Whittaker data for an essentially tame supercuspidal representation of $\mathrm{GL}_n(F)$, where F is a non-Archimedean local field.

1. Introduction

Let F be a non-Archimedean local field, V be an n-dimensional F-vector space, and G be the group $\mathrm{Aut}_F(V)$ of F-linear automorphisms of V, usually regarded as $\mathrm{GL}_n(F)$ by choosing a basis of V. Let π be a supercuspidal representation of G. As a classical result in [Gel'fand and Kajdan 1975], we know that π admits a unique Whittaker model. More precisely, take a tuple of Whittaker data (N, ψ) consisting of a maximal unipotent subgroup N of G and a nondegenerate character ψ of N, in the sense that its restriction to every simple root subgroup of N is nontrivial, then we have

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi) = 1.$$

As another classical result in [Bushnell and Kutzko 1993], we know that π is isomorphic to a compactly induced representation from a finite-dimensional representation Λ of an open compact-mod-center subgroup $\mathbf{J}$ of G; that is,

$$\pi \cong \mathrm{cInd}_{\mathbf{J}}^G \Lambda.$$

Using Frobenius reciprocity and Mackey's formula [Kutzko 1977], the existence and uniqueness of a Whittaker model is equivalent to the existence of a pair (N, ψ) as above such that

$$\mathrm{Hom}_{N\cap\mathbf{J}}(\psi, \Lambda) \neq 0, \tag{1-1}$$

and the pair is unique up to conjugation by $\mathbf{J}$ [Bushnell and Henniart 1998].

The above is the starting point of [Bushnell and Henniart 1998] in describing an explicit Whittaker function for a supercuspidal representation. This description, together with the result in [Paskunas and Stevens 2008], turn out to be useful in computing the epsilon factor for a certain pair of supercuspidal representations which

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differ only at the "tame level" (see [Paskunas and Stevens 2008, Section 7] and [Kim 2014, Theorem 4.4.3]). However, as pointed out in the introduction of [Paskunas and Stevens 2008], the proof of [Bushnell and Henniart 1998, Lemma 2.10] contains a gap, so they have to bypass the problem using a "black box" case (explained below) in [Paskunas and Stevens 2008].

The purpose of this paper is to construct explicit Whittaker data (N, ψ) for an essentially tame supercuspidal representation π. The essential tameness condition means that, by the definition in [Bushnell and Henniart 2005], if the group

$$\{\chi : \text{ unramified character of } F^\times \text{ such that } (\chi \circ \det) \otimes \pi \cong \pi\}$$

has order f, which is necessarily a divisor of n, then the residual characteristic p of F does not divide n/f. We will explain the advantage of restricting to the essentially tame case at the end of this introduction.

We summarize briefly the method of constructing our Whittaker data, mostly following Theorem 3.3 and Section 4.2 of [Paskunas and Stevens 2008]. Let θ be the simple character of a compact subgroup H^1 of G, in the sense of [Bushnell and Kutzko 1993, Section 3.2], underlying a chosen inducing type Λ of π. Associated to θ is an element $\beta \in A = \mathrm{End}_F(V)$ such that $E_0 = F[\beta]$ is a subfield of A and is tamely ramified over F in the essentially tame case. We will construct a maximal unipotent subgroup N satisfying

$$\theta|_{H^1 \cap N} = \psi_\beta|_{H^1 \cap N},$$

where $\psi_\beta : A \to \mathbb{C},\ x \mapsto \psi_F \circ \mathrm{tr}_{A/F}(\beta(x-1))$ with ψ_F being an additive character of F trivial on $\mathfrak{p}_F$ but not on $\mathfrak{o}_F$, together with other conditions in [Paskunas and Stevens 2008, Theorem 3.3].

The above unipotent subgroup N is defined by a particular ordered basis $\mathfrak{b}$ of V, given in (4-4). To describe it briefly, associated to the element β is a set $\{\beta_i\}_{i=0}^t$ of approximation elements such that $\sum_{i=0}^t \beta_i = \beta$ and, in the essentially tame case, that $E_j = F\bigl[\sum_{j \leq i} \beta_i\bigr]$ form a tower of intermediate (tamely ramified) extensions between E_0 and F. When $t = 0$, which is known as the minimal case in [Bushnell and Kutzko 1993, (1.4.15)], we define $\mathfrak{b}$ cyclicly using the element β, similar to the one defined in [Bushnell and Henniart 1998, 2.1 Proposition]. In the presence of multiple approximation elements (i.e., $t > 0$), we define $\mathfrak{b}$ cyclicly also, but in an inductive way along these elements. Our $\mathfrak{b}$ is different from the one defined in [Bushnell and Henniart 1998, 2.1 Proposition] at the level of the complexity of approximations.

It is unknown whether ψ_β can be extended to a character of N; however, using the matrix presentation of β with respect to $\mathfrak{b}$, we construct an analogous element $\alpha \in A$ such that $\psi = \psi_\alpha$ is a nondegenerate character of N. As the main result in Theorem 5.1, we will show that ψ satisfies

$$\psi|_{H^1 \cap N} = \psi_\beta|_{H^1 \cap N},$$

and that the conditions in [Paskunas and Stevens 2008, Theorem 3.3] are satisfied as well, which is enough to imply that (1-1) holds for our (N, ψ). As a note, the case when $[E_0 : F] = n$, i.e., E_0 is a maximal subfield in A, is the "black box" case deemed by [Paskunas and Stevens 2008]. Hence when E_0/F is moreover tamely ramified, our (N, ψ) serve as a "black box" character for the arguments in [Paskunas and Stevens 2008, Section 3].

The description of our basis $\mathfrak{b}$, combined with the results in [Paskunas and Stevens 2008; Kim 2014], provides a direct formula of the conductor of the epsilon factor for a certain pair of supercuspidal representations mentioned above and which are, as in our present paper, essentially tame. Such a formula can also be deduced from the general conductor formula in [Bushnell et al. 1998], obtained using the theory of intertwining operators of Shahidi, with an inductive calculation of a certain discriminant of β [Bushnell and Henniart 2003]. We will explain it briefly in the last section.

Note that, using the rational canonical form of β, we can of course extend ψ_β to a character of the maximal unipotent subgroup N_β defined using the cyclic basis generated by β (as in [Bushnell and Henniart 1998, 2.1 Proposition]). However, (N_β, ψ_β) may not be good Whittaker data; in particular, they do not give the correct conductor formula for the epsilon factors of pairs in Paskunas and Stevens' case.

Finally, we remark that the whole development of our main result requires E_0/F to be tamely ramified. As we will see, the tower of intermediate extensions $\{E_j\}_{j=0}^{t}$ between E_0 and F allows us to define inductively the basis $\mathfrak{b}$ in (4-4), and consequently decompose the simple character θ and the compact subgroup $H^1 \cap N$ inductively to derive our main result. The author believes that a more complex technique is required for the general case beyond the essentially tame case, and hopes to deal with it in his future work.

1A. ***Notations.*** Let F be a non-Archimedean local field with an algebraic closure denoted by $\bar{F}$. Denote by $\mathfrak{o}_F$ the ring of integers of F and by $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$. The residue field $\mathbf{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$ of F is a finite extension of $\mathbb{F}_p$. Denote by $v_F : F \to \mathbb{Z} \cup \{\infty\}$ the valuation of F.

If $r \in \mathbb{R}$, we denote by $r+$ the smallest integer strictly greater than r.

2. Tamely ramified extensions

The main purpose of this section is to gather some known facts concerning tamely ramified extensions. More importantly, we consider minimal elements in a tamely ramified extension, and study how they form bases with nice properties on lattice filtrations.

2A. ***Complementary subgroup.*** Let E/F be a tamely ramified extension of degree $n = n(E/F)$ and ramification degree $e = e(E/F)$. Put $f = f(E/F) = n/e$.

Throughout we fix a chosen uniformizer ϖ_F of F, and let $\boldsymbol{\mu}_F$ be the group of roots of unity with order coprime to p. We also fix ϖ_E and let $\boldsymbol{\mu}_E$ similarly for the field E, and assume that $\varpi_E^e \varpi_F^{-1} \in \boldsymbol{\mu}_E$. We define the complementary subgroup C_E of $E^\times$ to be the subgroup generated by ϖ_E and $\boldsymbol{\mu}_E$. It can be shown that C_E depends only on the choice of ϖ_F. Moreover, if K/F is an intermediate field extension in E, then $C_K \subseteq C_E$. We denote by C_F^{tame} the union of all C_E, with E ranges over all tamely ramified extensions of F.

If r is a positive integer, let U_F^r be the r-th unit group $1+\mathfrak{p}_F^r$. In general for $r \in \mathbb{R}$, we write $U_F^r = U_F^{\lceil r \rceil}$ where $\lceil r \rceil$ is the smallest integer $\geq r$, and write $U_F^{r+} = U_F^{\lceil r \rceil_+}$ where $\lceil r \rceil_+$ is the smallest integer $> r$. We define U_E^r similarly. Any element $b \in E^\times$ can be uniquely decomposed as cu where $c \in C_E$ and $u \in U_E^1$. We call c the *first term* of b.

2B. ***Minimality.*** At the beginning of this subsection, we only require E/F to be a finite separable extension of degree n. Later we will require E/F to be moreover tamely ramified.

Let $E = F[\alpha]$ for some $\alpha \in E$. Denote $e = e(E/F)$ and $v = v_E(\alpha)$. From [Kutzko and Manderscheid 1988, Proposition 1.5], we say that α is *minimal* over F if it satisfies

(I) $\gcd(v, e) = 1$, and

(II) any one of the following conditions:

  (a) $\mathfrak{o}_F[\beta] = \mathfrak{o}_K$, where K/F is the maximal unramified extension in E/F and $\beta = N_{E/K}(\alpha)/\varpi_F^v$.

  (b) The elements $\{x_j\}_{j=1}^n$, where $x_j = \alpha^j/\varpi_F^{\lfloor jv/e \rfloor}$, form an $\mathfrak{o}_F$-basis of $\mathfrak{o}_E$. In particular we have $\mathfrak{o}_E = \bigoplus_{j=0}^{n-1} \mathfrak{o}_F x_j$.

  (c) $\mathbf{k}_E = \mathbf{k}_F[\gamma + \mathfrak{p}_E]$, where $\gamma = x_e = \alpha^e/\varpi_F^v$ ([Bushnell and Kutzko 1993, (1.4.15)]).

By [Kutzko and Manderscheid 1988, Proposition 1.5], given (I), the three conditions in (II) are equivalent.

To incorporate the construction of simple characters, we recall another equivalent minimality condition from [Bushnell and Kutzko 1993]. Let V be a finite-dimensional E-vector space. We first regard V as an F-vector space and denote $A = \mathrm{End}_F(V)$. Let $\mathfrak{A}$ be an hereditary $\mathfrak{o}_F$-order in A, with Jacobson radical $\mathfrak{P}$ and normalized by $E^\times$. Let $v_{\mathfrak{A}}$ be the valuation on A associated with $\mathfrak{A}$. Let B be the centralizer of E in A, and denote $\mathfrak{B} = \mathfrak{A} \cap B$. Recall from [Bushnell and Kutzko 1993, 1.4] the $\mathfrak{o}_F$-lattice

$$\mathfrak{N}_k(\alpha, \mathfrak{A}) = \{x \in \mathfrak{A} : \alpha x - x\alpha \in \mathfrak{P}^k\}$$

and the critical exponent

$$k_0(\alpha, \mathfrak{A}) = \max\{k \in \mathbb{Z} : \mathfrak{N}_k(\alpha, \mathfrak{A}) \not\subset \mathfrak{B} + \mathfrak{P}\}.$$

By convention, if $\alpha \in F$, we put $k_0(\alpha, \mathfrak{A}) = \infty$. In fact, by [Bushnell and Kutzko 1993, (1.4.13)(ii)], the definition is independent of the vector space V. One important property is [Bushnell and Kutzko 1993, (1.4.15)]

$$v_{\mathfrak{A}}(\alpha) \leq k_0(\alpha, \mathfrak{A}), \tag{2-1}$$

with equality if and only if α is minimal over F.

Proposition 2.1. *An element $\alpha \in \bar{F}^\times$ with finite order modulo $F^\times$ coprime to p is minimal over F. In particular, any element in C_F^{tame} is minimal over F.*

Proof. The second statement follows directly from the first, so we focus on proving the first statement. The idea of the proof comes from [Reimann 1991, Lemma 2.8]. Let α have finite order modulo $F^\times$, and assume that $\alpha \notin F$ (otherwise the result is trivial). We will use condition (2-1) and, since the condition does not depend on the choice of the vector space V, we can assume V to be $E = F[\alpha]$ as an F-vector space, and denote A, $\mathfrak{A}$, and $\mathfrak{P}$ as above, so that $B = E$ and $\mathfrak{B} = \mathfrak{o}_E$. The statement can be proved if we show that

$$\mathfrak{N}_k(\alpha, \mathfrak{A}) = \mathfrak{o}_E + \mathfrak{P}^{k - v_{\mathfrak{A}}(\alpha)}$$

for all $k \in \mathbb{Z}$. Let $\tau(x) = \alpha x \alpha^{-1}$ for all $x \in A$, which is an F-algebra automorphism of A. We hence take $x \in \mathfrak{A}$ such that $\tau(x) - x \in \mathfrak{P}^{k - v_{\mathfrak{A}}(\alpha)}$. If m is the order of α in $\bar{F}^\times / F^\times$, we define

$$s : A \to A, \qquad X \mapsto \frac{1}{m} \sum_{i=0}^{m-1} \tau^i(X),$$

which is an F-linear projection onto E, and so $s(x) \in \mathfrak{o}_E$ as $m \in \mathfrak{o}_F^\times$. The relation

$$x = s(x) - \sum_{i=0}^{m-1} \sum_{j=1}^{i} \tau^{j-1}(\tau(x) - x)$$

implies that $x \in \mathfrak{o}_E + \mathfrak{P}^{k - v_{\mathfrak{A}}(\alpha)}$. The converse inclusion is straightforward. □

Corollary 2.2. *Suppose that the field $E = F[\alpha]$, for some $\alpha \in E$, is tamely ramified over F. Then α is minimal if and only if the first term of α also generates E over F.*

Proof. We write $\alpha = au$ for some $a \in C_E$ and $u \in U_E^1$. It is straightforward to see that α satisfies minimality conditions (I) and (c) if and only if a does the same for field E. □

We provide a useful calculation of the critical exponent of an element generating a tamely ramified field extension.

Proposition 2.3. *Suppose that $\beta \in A$ such that $E = F[\beta]$ is a tamely ramified extension of F, and $\mathfrak{A}$ is an $\mathfrak{o}_F$-hereditary order normalized by $E^\times$. Take $c \in C_E$ and denote $\gamma = \beta - c$. If $k_0(\gamma, \mathfrak{A}) < v_{\mathfrak{A}}(c)$, then*

$$k_0(\beta, \mathfrak{A}) = \begin{cases} k_0(\gamma, \mathfrak{A}) & \text{if } c \in F[\gamma], \\ v_{\mathfrak{A}}(c) & \text{otherwise.} \end{cases}$$

Proof. This can be derived from [Bushnell and Kutzko 1993, (2.2.8)]. □

2C. ***A special property.*** Suppose that V is an n-dimensional F-vector space containing an $\mathfrak{o}_F$-lattice chain $\mathcal{L}$. We call an F-basis $\{x_j\}_{j=1}^n$ of V an $\mathfrak{o}_F$-basis of $\mathcal{L}$, in the sense of [Bushnell and Kutzko 1993, (1.1.7)], if

(A) it is an $\mathfrak{o}_F$-basis of $\mathcal{L}(r)$ for some $r \in \mathbb{Z}$, and

(B) there exist $a(j, r) \in \mathbb{Z}$, for all $j = 1, \dots, n$ and $r \in \mathbb{Z}$, such that

$$\mathcal{L}(r) = \bigoplus_j \mathfrak{p}_F^{a(j,r)} x_j.$$

We may arrange the integers such that $a(j, r) \le a(j + 1, r)$.

For example, if V is a field extension $E = F[\alpha]$ as in the last section, then the set $\{x_j\}_{j=1}^n$ in the minimality condition (b) is an $\mathfrak{o}_F$-basis of $\{\mathfrak{p}_E^r\}_{r \in \mathbb{Z}}$. Indeed, suppose that $\{y_j\}$ is an ordered set equal to $\{x_j\}$ as a set but with the order rearranged such that $v_E(y_j) = i$ if $j = fi + k$ with $0 \le i < e$ and $1 \le k \le f$, then we have

$$\mathfrak{p}_E^r = \bigoplus_{i=t}^{e-1} \bigoplus_{k=1}^{f} \mathfrak{p}_F^s y_{fi+k} \oplus \bigoplus_{i=0}^{t-1} \bigoplus_{k=1}^{f} \mathfrak{p}_F^{s+1} y_{fi+k} \tag{2-2}$$

if $r = se + t$ for all $s \in \mathbb{Z}$ and $t = 0, \dots, e - 1$. Indeed we always have the inclusion $\supseteq$ for all $r \in \mathbb{Z}$, and we just have to show the equality for $r = 0, \dots, e - 1$ by periodicity. We of course have the equality for $r = 0$ and $r = e$. We then obtain the equality for other r by counting the $\mathbf{k}_F$-dimensions of successive quotients on both sides of (2-2).

For constructing Whittaker data, we require a special property. Denote by $v_{\mathcal{L}} : V \to \mathbb{Z} \cup \{\infty\}$ the associated valuation of $\mathcal{L}$. Let $\{u_j\}_{j=1}^n$ be an $\mathfrak{o}_F$-basis of $\mathcal{L}$ satisfying the following condition.

(*) For every $u = \sum_j a_j u_j \in V$ with $a_j \in F$, we have $v_{\mathcal{L}}(a_j u_j) \ge v_{\mathcal{L}}(u)$ for all j.

This condition leads to the following simple useful result:

Proposition 2.4. *Suppose further that $v_{\mathcal{L}}(u_i) \ge v_{\mathcal{L}}(u_j)$ if $i \le j$. For every $u = \sum_i a_i u_i \in V$, if $v_{\mathcal{L}}(u) > v_{\mathcal{L}}(u_i)$ for some i, then $a_j \in \mathfrak{p}_F$ for all $j \ge i$.*

Proof. This is because $v_{\mathcal{L}}(a_j u_j) \ge v_{\mathcal{L}}(u) > v_{\mathcal{L}}(u_i) \ge v_{\mathcal{L}}(u_j)$, where the first inequality comes from condition (*) above. □

For example, the basis of E in the minimality condition (b), and hence the cyclic basis $\{\alpha^i\}_{i=0}^{[E:F]-1}$, satisfy the condition (*), by observing from (2-2). If moreover $v_E(\alpha) \leq 0$, then the cyclic basis also satisfies the conclusion in Proposition 2.4.

3. Essentially tame supercuspidal representations

In this section, we recall the construction of essentially tame supercuspidal representations of G using admissible characters.

3A. ***Structure of admissible characters.*** Given a character ξ of $F^\times$, the level of ξ is the smallest integer $r = r_F(\xi) \geq 0$ such that $\xi|_{U_F^{r+1}}$ is trivial. We call ξ tamely ramified if $r = 0$.

Suppose that E/F is a tamely ramified extension and ξ is an admissible character of $E^\times$ over F in the sense of [Howe 1977], which means that for some intermediate subfield K between E and F,

- if ξ factors through $N_{E/K}$, then $E = K$, and
- if $\xi|_{U_E^1}$ factors through $N_{E/K}$, then E/K is unramified.

From [Howe 1977, Corollary of Lemma 11] we know that an admissible character ξ admits a factorization

$$\xi = \xi_{-1}(\xi_0 \circ N_{E/E_0}) \cdots (\xi_t \circ N_{E/E_t})(\xi_{t+1} \circ N_{E/F}), \tag{3-1}$$

with notations specified as follows.

- We have a tower of field extensions

$$E = E_{-1} \supseteq E_0 \supsetneq E_1 \cdots \supsetneq E_t \supsetneq E_{t+1} = F, \tag{3-2}$$

  and each ξ_i is a character of $E_i^\times$. This tower is uniquely determined by ξ.
- Let r_i be the level of $\xi_i \circ N_{E/E_i}$, then $r = r_{t+1}$ is the level of ξ. We assume that ξ_{t+1} is trivial if $r_{t+1} = r_t$. We call the increasing sequence of integers $r_0 < \cdots < r_t$ the *jumps* of ξ, which are uniquely determined by ξ. For later computation, we put $r_{-1} = 0$.
- If $E_0 = E$, then we replace $(\xi_0 \circ N_{E/E_0})\xi_{-1}$ by ξ_0. If $E_0 \subsetneq E$, then we assume that ξ_{-1} is tamely ramified and E/E_0 is unramified.

We put $\xi\xi_{-1}^{-1} = \Xi_0 \circ N_{E/E_0}$, where $\Xi_0 = \xi_0(\xi_1 \circ N_{E_0/E_1}) \cdots (\xi_t \circ N_{E_0/E_t})(\xi_{t+1} \circ N_{E_0/F})$. Note that the jumps $\{r_i\}_{i=0}^t$ depend only on $\Xi_0 \circ N_{E/E_0}|_{U_E^1}$, and are invariant under the Galois action on ξ.

We fix an additive character ψ of F, which is assumed to be trivial on $\mathfrak{p}_F$ but not on $\mathfrak{o}_F$. For any tamely ramified extension K/F, we write $\psi_K = \psi_F \circ \mathrm{tr}_{K/F}$.

We recall several results from [Moy 1986, Section 2.2]. For $i = 0, \dots, t+1$, suppose that s_i is the level of ξ_i, which means that $s_i e(E/E_i) = r_i$; then there is $\beta_i \in \mathfrak{p}_{E_i}^{-s_i} - \mathfrak{p}_{E_i}^{-s_i+}$ such that

$$\xi_i(x) = \psi_{E_i}(\beta_i(x-1)), \quad \text{for all } x \in U_{E_i}^{s_i/2+}. \tag{3-3}$$

This β_i, depending on the choice of ξ_i, can be regarded as in $\mathfrak{p}_E^{-r_i} - \mathfrak{p}_E^{-r_i+}$ and chosen mod $\mathfrak{p}_E^{(-r_i/2)+}$. Let $c_i \in C_{E_i}$ be the first term of ξ_i. For $i = 0, \dots, t$, each character ξ_i is generic over E_{i+1}, in the sense that

$$E_{i+1}[c_i] = E_i, \tag{3-4}$$

which implies that

$$\gcd(s_i, e(E_i/E_{i+1})) = 1. \tag{3-5}$$

We write

$$\beta = \beta(\xi) = \beta_0 + \cdots + \beta_{t+1}. \tag{3-6}$$

Note that $v_E(\beta) = -r$, the level of ξ. When $r = 0$, i.e., ξ is tamely ramified, then all ξ_i, with $i = 0, \dots, t+1$, are trivial, and we take $\beta = 0$.

Proposition 3.1. *For $i = 0, \dots, t$:*

(i) $E_i = F[\beta_{t+1} + \cdots + \beta_i]$.

(ii) *Each $\beta_i \in E_i$ is minimal over E_{i+1}.*

Proof. We know (i) is true because we have a decreasing sequence (3-2) of field extensions, while (ii) follows from (3-4) and Corollary 2.2. □

3B. ***Construction of simple characters.*** We identify E as an n-dimensional vector space V and hence obtain an embedding $E \hookrightarrow A$. We define an hereditary order

$$\mathfrak{A} = \{X \in A : X\mathfrak{p}_E^k \subseteq \mathfrak{p}_E^k \text{ for all } k \in \mathbb{Z}\}$$

and its j-th radical

$$\mathfrak{P}_{\mathfrak{A}}^j = \{X \in A : X\mathfrak{p}_E^k \subseteq \mathfrak{p}_E^{k+j} \text{ for all } k \in \mathbb{Z}\}, \text{ for } j \in \mathbb{Z}.$$

We also extend the definition such that $\mathfrak{P}_{\mathfrak{A}}^r = \mathfrak{P}_{\mathfrak{A}}^{\lceil r \rceil}$ and $\mathfrak{P}_{\mathfrak{A}}^{r+} = \mathfrak{P}_{\mathfrak{A}}^{\lceil r \rceil +}$ for $r \in \mathbb{R}$. We then define the following subgroups in G,

$$U_{\mathfrak{A}} = U_{\mathfrak{A}}^0 = \mathfrak{A}^\times \quad \text{and} \quad U_{\mathfrak{A}}^j = 1 + \mathfrak{P}_{\mathfrak{A}}^j, \quad \text{for all } j \in \mathbb{Z}_{>0},$$

and define $U_{\mathfrak{A}}^r$ and $U_{\mathfrak{A}}^{r+}$ similarly for $r \in \mathbb{R}_{\geq 0}$. Finally, we define $\mathfrak{B}_i$, $\mathfrak{P}_{\mathfrak{B}_i}^r$, and $\mathfrak{P}_{\mathfrak{B}_i}^{r+}$ as the centralizers of E_i in $\mathfrak{A}$, $\mathfrak{P}_{\mathfrak{A}}^r$, and $\mathfrak{P}_{\mathfrak{A}}^{r+}$ respectively, and define the subgroups $U_{\mathfrak{B}_i}$, $U_{\mathfrak{B}_i}^r$, and $U_{\mathfrak{B}_i}^{r+}$ in $U_{\mathfrak{A}}$ as the centralizers of $E_i^\times$ in $U_{\mathfrak{A}}$, $U_{\mathfrak{A}}^r$, and $U_{\mathfrak{A}}^{r+}$ respectively.

Given an element $\alpha \in A$, we denote a map

$$\psi_\alpha : A \to \mathbb{C}, \qquad x \mapsto \psi_F \circ \mathrm{tr}_{A/F}(\alpha(x-1)).$$

If $v = v_{\mathfrak{A}}(\alpha) < 0$, then the restriction of ψ_α on $U_{\mathfrak{A}}^{-(v/2)+}$ defines a character, which is trivial on $U_{\mathfrak{A}}^{-v+}$.

Given an admissible character ξ of $E^\times$, we recall the construction of a simple character $\theta = \theta_\xi$, in the sense of [Bushnell and Kutzko 1993, Section 3.2], on the compact subgroup

$$H^1 = H^1_\xi := U^1_{\mathfrak{B}_0} U^{r_0/2+}_{\mathfrak{B}_1} \cdots U^{r_{t-1}/2+}_{\mathfrak{B}_t} U^{r_t/2+}_{\mathfrak{B}_{t+1}}$$

(note that $\mathfrak{B}_{t+1} = \mathfrak{A}$) and whose restriction onto U^1_E coincides with $\xi|_{U^1_E}$. Like ξ, this simple character also admits a factorization

$$\theta = \theta_0 \theta_1 \cdots \theta_{t+1}$$

such that

$$\theta_i|_{U^1_{\mathfrak{B}_0} U^{r_0/2+}_{\mathfrak{B}_1} \cdots U^{r_{i-1}/2+}_{\mathfrak{B}_i}} = \xi_i \circ \det_{B_i/E_i} \qquad \text{and} \qquad \theta_i|_{U^{r_i/2+}_{\mathfrak{B}_{i+1}} \cdots U^{r_t/2+}_{\mathfrak{B}_{t+1}}} = \psi_{\beta_i}.$$

It is well-defined since on the intersection $U^{r_i/2+}_{\mathfrak{B}_i}$ the characters are equal, by (3-3). Note that when $r = 0$, we take $\mathfrak{A} = \mathrm{M}_n(\mathfrak{o}_F)$ with $H^1 = U^1_{\mathfrak{A}}$, and θ is the trivial character of H^1.

Proposition 3.2. *The assignment $\xi|_{U^1_E} \mapsto \theta$ is well-defined, i.e., it is independent of the factorization* (3-1).

Proof. The verifying arguments are quite routine, so we only provide a brief idea as follows. Before we begin, in order to reduce the load of notations, we denote the restriction of any character ϕ of some $E^\times$ to U^1_E just by ϕ instead of $\phi|_{U^1_E}$, and similarly if we replace E by other fields.

First of all, remember that the jumps $\{r_i\}$ and the intermediate subfields $\{E_i\}$ in (3-2) are uniquely determined by ξ. Suppose we have another factorization of ξ whose factors are $\{\xi'_i\}_{i=-1}^{t+1}$, then we can inductively deduce that, for $i = 0, \dots, t+1$,

$$\xi_i^{-1} \xi'_i \phi_{i-1} = \phi_i \circ N_{E_i/E_{i+1}} \tag{3-7}$$

for some characters ϕ_i of $U^1_{E_{i+1}}$, each of whose level t_i is less than $s_i = r_i/e(E/E_i)$ because of (3-5). We remark that here we take ϕ_{-1} and ϕ_{t+1} to be trivial. In the additive level, suppose that

$$\phi_i(x) = \psi_{E_{i+1}}(\gamma_i(x-1)) \qquad \text{for all } x \in U^{t_i/2+}_{E_{i+1}}, \tag{3-8}$$

then (3-7) becomes, for $i = 0, \dots, t+1$,

$$\beta'_i + \gamma_{i-1} - \beta_i = \gamma_i \tag{3-9}$$

for some element $\gamma_i \in E_{i+1}$, and we take $\gamma_{-1} = \gamma_{t+1} = 0$.

Now we consider the restriction of θ to $U^{r_i/2+}_{\mathfrak{B}_{i+1}}$, on which each factor θ_j is equal to

$$\psi_{\beta_j} \text{ if } j \le i, \quad \text{and} \quad \xi_j \circ \det_{B_j/E_j} = (\xi_j \circ N_{E_i/E_j}) \circ \det_{B_i/E_i} \text{ if } j > i.$$

Similar results apply to each factor θ'_j of θ'. We then apply (3-7) and (3-9) to obtain

$$\theta(\theta')^{-1}|_{U^{r_i/2+}_{\mathfrak{B}_{i+1}}} = \phi_i \circ \det_{B_{i+1}/E_{i+1}} \cdot \psi^{-1}_{\gamma_i},$$

which is just trivial because of (3-8). Therefore, we have $\theta = \theta'$. □

Given ξ with β as in (3-6), we associate a stratum $[\mathfrak{A}, r, 0, \beta]$, in the sense of [Bushnell and Kutzko 1993, (1.5)], to ξ, where $r = -v_E(\beta)$. Note that we have taken $\beta = 0$ when ξ is tamely ramified, in which case the associated stratum is null $[\mathrm{M}_n(\mathfrak{o}_F), 0, 0, 0]$.

Proposition 3.3. (*i*) *If the level r of ξ is positive*, *then the stratum* $[\mathfrak{A}, r, 0, \beta]$ *is simple*, *with a sequence of approximation strata* $[\mathfrak{A}, r, r_i, \gamma_i]$, *where*

$$\gamma_i = \sum_{j=i}^{t+1} \beta_j,$$

and each with a derived stratum $[\mathfrak{B}_i, r_i, r_i - 1, c_i]$, *all in the sense of* [*Bushnell and Kutzko 1993,* (2.4.2)].

(*ii*) $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, *the set of simple characters in the sense of* [*Bushnell and Kutzko 1993,* (3.2.3)].

Proof. We first prove (i), which is to show that the sequence $[\mathfrak{A}, n, r_i, \gamma_i]$ satisfies the conditions in [Bushnell and Kutzko 1993, (2.4.1)]. In fact, many of the arguments are routine, mostly following from constructions. One technical part is [Bushnell and Kutzko 1993, (2.4.1)(iv)], where we have to show that

$$k_0(\gamma_i, \mathfrak{A}) = -r_i \quad \text{for each } i = 0, \dots, t.$$

We first decompose β term by term as $\sum_{i=1}^{r} a_i$ with $a_i \in C_E$ and $v_E(a_i) = -i$. Hence $\beta_i = \sum_{j=r_{i-1}+1}^{r_i} a_j$ and $a_{r_i} = c_i$. We now apply induction, assuming that $k_0(\gamma_{i+1}, \mathfrak{A}) = -r_{i+1}$, which is less than $v_E(c_i)$. By the second case of Proposition 2.3, we have $k_0(c_i + \gamma_{i+1}, \mathfrak{A}) = -r_i$. Now notice that each a_k with $k = r_{i-1}+1, \dots, r_i$, lies in $E_i = F\big[\gamma_{i+1} + \sum_{l=k}^{r_i} a_l\big]$. In particular $\gamma_{i+1} + \sum_{l=r_{i-1}+1}^{r_i} a_l = \gamma_i$, and so by the first case of Proposition 2.3, $k_0(\gamma_i, \mathfrak{A}) = -r_i$.

Once (i) is established, (ii) can be checked just by the definition in [Bushnell and Kutzko 1993, (3.2.3)]. The case for θ being trivial (when ξ is tamely ramified) is just by convention, so we move on to the positive level case. By induction along the approximation sequence in (i), it suffices to show that for each $i = 0, \dots, t+1$, we have

$$\Theta_i := \theta_i \cdots \theta_{t+1} \in \mathcal{C}(\mathfrak{A}, r_{i-1}/2+, \gamma_i).$$

Now the subgroup $H^{r_{i-1}/2+}$ is $U_{\mathfrak{B}_i}^{r_{i-1}/2+} \cdots U_{\mathfrak{B}_{t+1}}^{r_t/2+}$. For each $j \geq i$, the factor $\theta_j|_{H^{r_{i-1}/2+}}$ is equal to

$$\xi_j \circ \det_{B_j/E_j}|_{U_{\mathfrak{B}_i}^{r_{i-1}/2+} \cdots U_{\mathfrak{B}_j}^{r_{j-1}/2+}} \cdot \psi_{\beta_j}|_{U_{\mathfrak{B}_{j+1}}^{r_j/2+} \cdots U_{\mathfrak{B}_{t+1}}^{r_t/2+}}.$$

We hence check the conditions in [Bushnell and Kutzko 1993, (3.2.3)] for the character Θ_i.

(a) We have $\Theta_i|_{U_{\mathfrak{B}_i}^{r_{i-1}/2+}} = (\xi_i(\xi_{i+1} \circ N_{E_i/E_{i+1}}) \cdots (\xi_{t+1} \circ N_{E_i/F})) \circ \det_{B_i/E_i}$.

(b) The compact subgroup $H^{r_{i-1}/2+}$ is clearly normalized by

$$\mathfrak{K}(\mathfrak{B}_i) = \{x \in B_i^\times : x^{-1}\mathfrak{B}_i x = \mathfrak{B}_i\},$$

and so are the characters $\xi_j \circ \det_{B_j/E_j}$ and ψ_{β_j} for $j \geq i$.

(c) We have $H^{r_i/2+} = U_{\mathfrak{B}_{i+1}}^{r_i/2+} \cdots U_{\mathfrak{B}_{t+1}}^{r_t/2+}$, on which the factor θ_i is equal to ψ_{β_j}, and $\Theta_{i+1} \in \mathcal{C}(\mathfrak{A}, r_i/2+, \gamma_{i+1})$ by induction assumption. □

We show very briefly that our θ agrees with the one in [Bushnell and Henniart 2005, Section 2.3]. We will not go into detail as it incurs heavy definitions and notations from transfers [Bushnell and Kutzko 1993], endo-classes [Bushnell and Henniart 1996, Section 7], and tame liftings [Bushnell and Henniart 1996, Section 9], but only refer to the references as given.

Suppose that ξ is an admissible character of $E^\times$, with an associated stratum $[\mathfrak{A}, r, 0, \beta]$ as constructed in the previous section, and $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ is a simple character of a compact subgroup H^1 of G. Recall from [Bushnell and Henniart 2005, Section 2.3] that, if we write $\xi|_{U_E^1} = \Xi_0 \circ N_{E/E_0}$ for some character Ξ_0 of $U_{E_0}^1$, and denote the endo-classes of θ and Ξ_0 by $\mathcal{E}_F(\theta)$ and $\mathcal{E}_{E_0}(\Xi_0)$ respectively, then a specific simple character θ_0 is characterized by the condition that

$$\mathcal{E}_{E_0}(\Xi_0) \text{ is a } E_0/F\text{-lift of } \mathcal{E}_F(\theta_0).$$

Our simple character θ constructed above satisfies this condition, because of the relation

$$\theta|_{U_E^1} = \xi|_{U_E^1}$$

which is exactly the relation in [Bushnell and Henniart 1996, (9.2)] that defines the tame lifting of a simple character. Hence $\theta_0 = \theta$.

We continue to follow [Bushnell and Henniart 2005, Section 2.3]. On the compact mod-center subgroup

$$\mathbf{J} = \mathbf{J}_\xi := E^\times U_{\mathfrak{B}_0}^1 U_{\mathfrak{B}_1}^{r_0/2} \cdots U_{\mathfrak{B}_t}^{r_{t-1}/2} U_{\mathfrak{A}}^{r_t/2},$$

we define an extended maximal simple type $\Lambda = \Lambda_\xi$, which is a finite-dimensional irreducible representation, depending on ξ and whose restriction onto H^1 is a direct sum of $\theta = \theta_\xi$. We then put $\pi = \pi_\xi := \mathrm{cInd}_{\mathbf{J}}^G \Lambda$.

Proposition 3.4. (i) *The representation π is irreducible, supercuspidal, and essentially tame. Moreover, any such representation arises from the above construction.*

(ii) *We have $f(\pi) = n/e(E_0/F)$.*

(iii) *The isomorphism class of π depends only on the orbit of $(E/F, \xi)$ under Galois conjugation. (This orbit is called an admissible pair in [Bushnell and Henniart 2005].)*

Proof. All statements can be deduced from Proposition 2.3 and Theorem 2.3 of [Bushnell and Henniart 2005]. □

4. A special choice of ordered basis

We continue from the last section. In Section 4A, we provide the desired properties of our F-basis of E for constructing our explicit Whittaker data. In Section 4B, we construct such an ordered basis, and express the element β and the compact subgroup H^1 with respect to this basis. Finally, in Section 4C we provide a factorization of $H^1 \cap N$, where N is the maximal unipotent subgroup defined by this ordered basis, according to the one defined by θ.

4A. ***An inductively subordinate condition.*** We first consider a general situation. Suppose that E_0/F is a finite extension with a tower of subextensions $\{E_i\}_{i=0}^{t+1}$ similar to (3-2), except that we do not require E_0/F to be tamely ramified. Let V be an E_0-vector space with an $\mathfrak{o}_{E_0}$-lattice chain $\mathcal{L}$ in V. Suppose that, for each $i = 0, \dots, t+1$, there is an ordered E_i-decomposition of V as

$$V = \bigoplus_{j \in I^i} W_j^i \tag{4-1}$$

for an ordered set I^i of indices, such that the following conditions hold:

(I) There is a decomposition of ordered sets $I^{i+1} = \bigsqcup_{j \in I^i} I_j^{i+1}$ such that

$$W_j^i = \bigoplus_{k \in I_j^{i+1}} W_k^{i+1}. \tag{4-2}$$

(II) For each $r \in \mathbb{Z}$, we have

$$\mathcal{L}(r) = \bigoplus_{j \in I^i} \mathcal{L}(r) \cap W_j^i,$$

which means that the decomposition (4-1) conforms with $\mathcal{L}$ over E_i, in the sense of [Bushnell and Kutzko 1993, (7.1.1(i))] or [Bushnell and Henniart 1996, (10.5)].

We call the refining decompositions in (4-1) *inductively subordinate* to $\mathcal{L}$. Let $\mathcal{F}_{E_i}$ be the associated E_i-flag. By regarding all flags $\mathcal{F}_{E_i}$ as F-flags by restriction of scalars, we have a successive refinement of F-flags

$$\mathcal{F}_{E_0} \subset \cdots \subset \mathcal{F}_{E_t} \subseteq \mathcal{F}, \tag{4-3}$$

which gives rise to a tower of unipotent subgroups

$$N_{\mathcal{F}_{E_0}} \subset \cdots \subset N_{\mathcal{F}_{E_t}} \subseteq N = N_{\mathcal{F}}.$$

4B. ***The ordered basis.*** Let ξ be an admissible character of $E^\times$ over F, with $\{\beta_i\}_{i=0}^t$ the set of approximation elements as in Section 3A. For future computation, we define an extra element β_{-1} to be a primitive root of unity in $\boldsymbol{\mu}_E$ when $E \neq E_0$, and put $\beta_{-1} = 0$ when $E = E_0$.

We choose the following ordered F-basis $\mathfrak{b} = \{x_j\}_{j=1}^n$ of $V = E$,

$$x_1 = 1 \quad \text{and} \quad x_{j+1} = \beta_i x_j \tag{4-4}$$

for $i = -1, \ldots, t$, if j is a multiple of $[E_{i+1} : F]$ but not a multiple of $[E_i : F]$. Note that:

- $v_E(x_j) \leq v_E(x_k)$ for all $j > k$, with equality if and only if $E \neq E_0$ and k is a multiple of $[E_0 : F]$ with $j = k+1$, in which case $x_{k+1} = \beta_{-1} x_k$.
- If $\beta \notin F$, then $v_E(x_j) < 0$ for all $j > 1$.

We can also define this basis inductively as follows. Let

$$\mathfrak{b}^{-1} = \{1, \beta_{-1}, \beta_{-1}^2, \ldots, \beta_{-1}^{[E:E_0]-1}\}.$$

This is an ordered cyclic E_0-basis of E. For $i = 0, \ldots, t+1$, we define

$$\mathfrak{b}_{E_{i+1}}(\beta_i) = \{1, \beta_i, \beta_i^2, \ldots, \beta_i^{[E_i:E_{i+1}]-1}\}$$

and, if $\mathfrak{b}^{i-1}$ is ordered as $\{z_1, \ldots, z_{[E:E_i]}\}$, define

$$\mathfrak{b}_j^i = z_j \beta_i^{(j-1)([E_i:E_{i+1}]-1)} \mathfrak{b}_{E_{i+1}}(\beta_i) \tag{4-5}$$

for $j = 1, \ldots, [E : E_i]$. Each $\mathfrak{b}_j^i$ is an E_{i+1}-basis of an E_i-vector space, and

$$\mathfrak{b}^i = \mathfrak{b}_1^i \sqcup \cdots \sqcup \mathfrak{b}_{[E:E_i]}^i,$$

is an E_{i+1}-basis of E. Finally, we have $\mathfrak{b} = \mathfrak{b}^t$.

We hence define, for $i = 0, \ldots, t+1$ and $j = 1, \ldots, [E : E_t]$,

$$W_j^i = \operatorname{span}_F \mathfrak{b}_j^{i-1}, \tag{4-6}$$

which is an E_i-vector space of dimension 1. Condition (I) in the previous section is clearly satisfied.

Proposition 4.1. (i) *Let* $\mathcal{L}(r) = \mathfrak{p}_E^r$ *in* $V = E$; *then the refining decompositions defined by* (4-6) *are inductively subordinate to* $\mathcal{L}$.

(ii) *In particular, the condition of* Proposition 2.4 *is satisfied by the basis* $\mathfrak{b}$.

Proof. For (i), we claim that condition (II) is satisfied by (4-6). If we show that, for each $r \in \mathbb{Z}$,

$$\mathfrak{p}_E^r = \bigoplus_{j=1}^{[E:E_i]} \mathfrak{p}_{E_i}^{a_i(j,r)} z_j \quad (\text{here } \mathfrak{b}^{i-1} = \{z_1, \ldots, z_{[E:E_i]}\}) \tag{4-7}$$

for suitable integers $a_i(j,r)$, for $i = 0, \ldots, t+1$, then the claim is implied by the last i. The minimality of β_{-1} implies that (4-7) holds for $i = 0$, by the remark after Proposition 2.4 (indeed in this case $a_0(j,r) = r$ for all j). If (4-7) holds for some i, then by the same remark again (substituting E, F and α by E_i, E_{i+1} and β_i, respectively) and the minimality in Proposition 3.1, we show that, for $s \in \mathbb{Z}$,

$$\mathfrak{p}_{E_i}^{s-v} = \beta_i^{-(j-1)([E_i:E_{i+1}]-1)} \mathfrak{p}_{E_i}^s = \bigoplus_{k=0}^{[E_i:E_{i+1}]-1} \mathfrak{p}_{E_{i+1}}^{b_{i+1}^j(k,s)} \beta_i^k$$

for suitable integers $b_{i+1}^j(s,k)$ and where $v = (j-1)([E_i : E_{i+1}] - 1) v_{E_i}(\beta_i)$. We obtain

$$\mathfrak{p}_E^r = \bigoplus_{j,k} \mathfrak{p}_{E_{i+1}}^{b_{i+1}^j(k, a_i(r,j))} w_{j,k}$$

with $w_{j,k} = z_j \beta_i^{(j-1)([E_i:E_{i+1}]-1)+k}$ forming the basis $\mathfrak{b}^i$ by (4-5). Hence (4-7) holds for $i+1$.

For (ii), it is enough to show that (*) is satisfied. We again apply induction on i. Condition (*) is satisfied by the cyclic basis $\mathfrak{b}^{-1}$, and suppose it is satisfied by $\mathfrak{b}^{i-1}$, so that if $u = \sum_{z_j \in \mathfrak{b}^{i-1}} a_j z_j$ for $a_j \in E_i$ then $v_E(a_j z_j) \geq v_E(u)$. Write

$$a_j = \sum_{k=0}^{[E_i:E_{i+1}]-1} b_{j,k} y_{j,k}$$

for some $b_{j,k} \in E_{i+1}$ and $y_{j,k} = \beta_i^{(j-1)([E_i:E_{i+1}]-1)+k}$, then by applying (*) on the E_{i+1}-basis $\{y_{j,k}\}_k$ for E_i we obtain $v_{E_i}(b_{j,k} y_{j,k}) \geq v_{E_i}(a_j)$. Now

$$u = \sum_{w_{j,k} \in \mathfrak{b}^i} b_{j,k} w_{j,k},$$

with $w_{j,k}$ as above forming the basis $\mathfrak{b}^i$, and $v_E(b_{j,k} w_{j,k}) \geq v_E(a_j z_j) \geq v_E(u)$. Hence (*) is satisfied by $\mathfrak{b}^i$. □

We now provide some properties of the matrix presentations of the elements β and β_{-1}, and also the compact subgroup H^1, with respect to the ordered basis $\mathfrak{b}$.

Proposition 4.2. (i) *The matrix presentation $\beta_{j,k}$, where $j,k=1,\dots,n$, of β with respect to $\mathfrak{b}$ takes the form*

$$\beta_{j,k}\in\begin{cases}1+\mathfrak{p}_F & \text{if } j-k=1 \text{ and } k \text{ is not a multiple of } [E_0:F],\\ \mathfrak{p}_F & \text{if } j-k>1 \text{ or if } j-k=1 \text{ and } k \text{ is a multiple of } [E_0:F].\end{cases}$$

(ii) *When $E\neq E_0$, the matrix presentation $(\beta_{-1})_{j,k}$ of β_{-1} with respect to $\mathfrak{b}$ takes the form*

$$(\beta_{-1})_{j,k}\in\begin{cases}1+\mathfrak{p}_F & \text{if } j-k=1 \text{ and } k \text{ is a multiple of } [E_0:F],\\ \mathfrak{p}_F & \text{if } j-k>1 \text{ or if } j-k=1 \text{ and } k \text{ is not a multiple of } [E_0:F].\end{cases}$$

(iii) *In the matrix presentation of H^1 with respect to $\mathfrak{b}$, the entries in the strictly upper triangle belong to $\mathfrak{o}_F$.*

(We remark that, in cases (i) and (ii), we do not need to study the (j,k)-entries with $j\leq k$.)

Proof. To prove (i), for each $k=1,\dots,n$, we will determine where the entries of the k-th column of β belong with respect to $\mathfrak{b}$. Let $i=i(k)$ be the index such that k is a multiple of $[E_{i+1}:F]$ but not a multiple of $[E_i:F]$; then $x_{k+1}=\beta_i x_k$ by construction. If $E=E_0$, we want to show that the product

$$\beta x_k=\sum_{i=0}^{t+1}\beta_i x_k$$

lies in

$$\bigoplus_{l=1}^{k}Fx_l+x_{k+1}+\bigoplus_{l=k+1}^{n}\mathfrak{p}_F x_l.$$

First of all, we have

$$\beta_{t+1}x_k+\cdots+\beta_{i+1}x_k\in\bigoplus_{l=1}^{k}Fx_l, \tag{4-8}$$

because if we write $x_k=\beta_t^{m_t}\cdots\beta_{j+1}^{m_{j+1}}\beta_j^{m_j}$ for some integers $m_{t+1},\dots,m_j>0$, then $i\geq j-1$, and we see that $E_{i+1}x_k\in\bigoplus_{l=1}^{k}Fx_l$; in particular (4-8) holds. We then show that

$$\beta_{i-1}x_k+\cdots+\beta_0 x_k\in\bigoplus_{l=k+1}^{n}\mathfrak{p}_F x_l. \tag{4-9}$$

For all $j<i$,

$$v_E(\beta_j x_k)>v_E(\beta_i x_k)=v_E(x_{k+1}).$$

By Proposition 2.4, the coefficients of x_l, for $l \geq k+1$, of all $\beta_{i-1}x_k, \dots, \beta_0 x_k$, lie in $\mathfrak{p}_F$, and (4-9) holds. When $E \neq E_0$, the proof is similar, except that when k is a multiple of $[E_0 : F]$, we have $x_{k+1} = \beta_{-1}x_k$, and so

$$\beta x_k \in \bigoplus_{l=1}^{k} F x_l \oplus \bigoplus_{l=k+1}^{n} \mathfrak{p}_F x_l.$$

For (ii), the arguments are similar to above. If k is a multiple of $[E_0 : F]$, then $\beta_{-1}x_k = x_{k+1}$. Otherwise, we have $v_E(\beta_{-1}x_k) = v(x_k) > v(x_{k+1})$, and so $\beta_{-1}x_k \in \bigoplus_{l=1}^{k} F x_l \oplus \bigoplus_{l=k+1}^{n} \mathfrak{p}_F x_l$.

For (iii), notice that $\{x_j/\varpi_F^{\lfloor v_E(x_j)/e \rfloor}\}_{j=1}^{n}$ is an $\mathfrak{o}_F$-basis for the lattice chain $\mathcal{L}(r) = \mathfrak{p}_E^r$ in $V = E$. With this basis, the entries of $U_{\mathfrak{B}_i}$ for all i, hence those of H^1, belong to $\mathfrak{o}_F$. If we use the basis $\mathfrak{b} = \{x_j\}_{j=1}^{n}$ instead, then the (j,k)-entry is multiplied by $\varpi_F^{\lfloor v_E(x_j)/e \rfloor - \lfloor v_E(x_k)/e \rfloor}$. In the upper triangle consisting of (j,k)-entries where $j < k$, we have $v_E(x_j) > v_E(x_k)$, and so the (j,k)-entry with respect to $\mathfrak{b}$ is still in $\mathfrak{o}_F$. □

Corollary 4.3. $\psi_{\beta+\beta_{-1}}|_{N \cap H^1}(x) = \psi_F\left(\sum_{j=1}^{n-1} x_{j,j+1}\right)$, *where* $x_{j,k}$ *is the* (j,k)-*entry of the matrix presentation of* $x \in A$ *with respect to* $\mathfrak{b}$.

Proof. With respect to the basis $\mathfrak{b}$, it is easy to see that the entries of $\beta + \beta_{-1}$ in the lower sub-diagonal belong to $1 + \mathfrak{p}_F$, and those underneath belong to $\mathfrak{p}_F$. Also, N is defined by this ordered basis, and the entries of $N \cap H^1$ in the strictly upper triangle belong to $\mathfrak{o}_F$. Since ψ_F is trivial on $\mathfrak{p}_F$ but not on $\mathfrak{o}_F$, we have the desired result. □

As a remark, for a fixed β, there are other bases which also serve our purpose. For instance, we can take the basis constructed in the same way as $\mathfrak{b}$ but with all β_i replaced by their first terms c_i. One can prove, almost verbatim, that β takes the same form as in the proposition. Also, another factorization of ξ yields another set of elements $\{\beta_i\}$, and so another β, but the matrix presentation of that β takes the same form.

We end this subsection with a few examples.

Example 4.4. Let $[\mathfrak{A}, r, 0, \beta]$ be a minimal stratum, and let m be the degree of $E_0 = F[\beta]$ over F. For a positive integer d, let E be the unramified extension of E_0 of degree d, and take a primitive root of unity in $\boldsymbol{\mu}_E$. We construct the basis

$$\mathfrak{b} = \{1, \beta, \dots, \beta^{m-1}, \zeta\beta^{m-1}, \zeta\beta^{m}, \dots, \zeta\beta^{2m-2}, \zeta^2\beta^{2m-2} \dots, \zeta^{d-1}\beta^{d(m-1)}\}.$$

We consider the matrix of β relative to $\mathfrak{b}$. On the j-th column where j is not a multiple of m, the entries are all 0 except the $(j+1)$-th entry, which is 1. For $k = 1, \dots, d$, if $\beta^m = \phi(\beta)$ for some F-polynomial ϕ of degree $m-1$, then we have $\beta \cdot x_{km} = \beta \cdot \zeta^{k-1}\beta^{k(m-1)} = \zeta^{k-1}\beta^{(k-1)(m-1)}\phi(\beta)$, which lies in the F-span of x_j for $(k-1)m < j \le km$. Therefore, on the (km)-th column, the j-th entries for $j > km$ are all 0.

We then consider the matrix of ζ relative to $\mathfrak{b}$. On the j-th column where j is a multiple of m, the entries are all 0 except the $(j+1)$-th entry, which is 1. For $s = 1, \dots, m-1$, if $\beta^{-s} = \phi_s(\beta)$ for some F-polynomial ϕ_s of degree $m-1$, then minimality implies that its coefficients must lie in $\mathfrak{p}_F$. Now if $j = (k-1)m + l$, where $k = 1, \dots, d$ and $l = 1, \dots, m-1$, then

$$\zeta \cdot x_j = \zeta \cdot \zeta^{k-1} \beta^{(k-1)(m-1)+(l-1)} = \zeta^k \beta^{k(m-1)} \phi_{m-l}(\beta),$$

which lies in the $\mathfrak{p}_F$-span of x_i where $km < i \le (k+1)m$, in particular $i > j+1$. Therefore, on the j-th column, the i-th entry for $i \le j+1$ is 0, and lies in $\mathfrak{p}_F$ for $i > j+1$.

We hence see that the element $\zeta + \beta$ has the desired form as in Corollary 4.3. Note that we did not assume that E_0/F is tamely ramified in the minimal case: all we need to know is that the valuation of β is negative.

Example 4.5. We provide one more example for small n which exhibits the situation when multiple jumps are present. Let's take $n = 4$. As the minimal case is covered in the previous example, we assume that our simple stratum $[\mathfrak{A}, r, 0, \beta]$ has two jumps. Consider a tower of the form $E \supset K \supset F$ where $[E : K] = [K : F] = 2$. For simplicity, we only consider two extreme cases.

(i) Suppose that E/F is totally ramified, and so $p \neq 2$. We fix a uniformizer ϖ_F and choose ϖ_K and ϖ_E such that $\varpi_K^2 = a\varpi_F$ and $\varpi_E^2 = b\varpi_K$ for some $a, b \in \boldsymbol{\mu}_F$. Consider the element

$$\beta = \varpi_F^{-r} + \varpi_K^{-s} + \varpi_E^{-t},$$

where $4r > 2s > t > 0$ and both s and t are odd. The basis constructed by $\beta_0 = \varpi_E^{-t}$ and $\beta_1 = \varpi_K^{-s}$ is

$$\{1,\ \varpi_K^{-s},\ \varpi_E^{-t}\varpi_K^{-s},\ \varpi_E^{-t}\varpi_K^{-2s}\}.$$

The matrix of β takes the form

$$\begin{bmatrix} \varpi_F^{-r} & (a\varpi_F)^{-s} & b^{-t}(a\varpi_F)^{-(t+s)/2} & * \\ 1 & \varpi_F^{-r} & 0 & * \\ 0 & 1 & \varpi_F^{-r} & * \\ (a\varpi_F)^s & 0 & 1 & * \end{bmatrix}$$

(the last column is unimportant for our purposes).

(ii) Suppose now that E/F is unramified. Let $K = F[\zeta]$ and $E = K[\eta]$, where $\zeta, \eta \in \boldsymbol{\mu}_E$ and satisfy the equations $\zeta^2 = a\zeta + b$ and $\eta^2 = (c\zeta + d)\eta + (e\zeta + f)$ with all $a, \dots, f \in \mathfrak{o}_F$. Write $\varpi = \varpi_F$ and consider for example the element

$$\beta = \varpi^{-r} + \zeta\varpi^{-s} + \eta\varpi^{-t},$$

where $r > s > t > 0$. With the basis

$$\{1,\ \zeta\varpi^{-s},\ \zeta\eta\varpi^{-(t+s)},\ \zeta^2\eta\varpi^{-(t+2s)}\},$$

the matrix of β takes the form

$$\begin{bmatrix} \varpi^{-r} & b\varpi^{-2s} & b(ae+f)\varpi^{-(s+2t)} & * \\ 1 & \varpi^{-r}+a\varpi^{-s} & (ae+af+be)\varpi^{-2t} & * \\ (-a/b)\varpi^{s} & 1 & \varpi^{-r}+bc\varpi^{-t} & * \\ (1/b)\varpi^{2s} & 0 & 1+(ac+d)\varpi^{s-t} & * \end{bmatrix}.$$

4C. ***A factorization for maximal unipotent subgroups.*** We first work on a general situation. Let E/F be a finite extension and V be a finite-dimensional E-vector space, also regarded as an F-vector space. Denote $A = \mathrm{End}_F(V)$ and let $\mathfrak{A}$ be an hereditary $\mathfrak{o}_F$-order in A defined by an $\mathfrak{o}_E$-lattice chain $\mathcal{L}$ in V. Let B be the centralizer of E in A, and denote $\mathfrak{B} = \mathfrak{A} \cap B$.

We suppose that V admits an ordered decomposition $\bigoplus_j W_j$ into a direct sum of E-subspaces, and $\mathcal{F}_E$ is the associated E-flag. We further suppose that each W_j, viewed as an F-vector space, admits an ordered decomposition into a direct sum $\bigoplus_i W_j^i$ of F-subspaces, altogether forming an F-flag $\mathcal{F}$ in V. Let $M_{\mathcal{F}}$ be the subgroup of G stabilizing all W_j^i, let $N_{\mathcal{F}}$ be the unipotent subgroup in G defined by the flag $\mathcal{F}$, and let $N_{\mathcal{F}}^-$ be its opposite. Also, define $M_{\mathcal{F}_E}$, $N_{\mathcal{F}_E}$, and $N_{\mathcal{F}_E}^-$ similarly, using the flag $\mathcal{F}_E$.

Lastly, we suppose that the refining decompositions above are both subordinate to $\mathcal{L}$, in the sense of the conditions in Section 4A.

Proposition 4.6. (i) *For each positive integer k, the subgroups $U_{\mathfrak{A}}^k$ and $U_{\mathfrak{B}}^k$ admit an Iwahori decomposition*

$$U_{\mathfrak{A}}^k = (U_{\mathfrak{A}}^k \cap N_{\mathcal{F}_E})(U_{\mathfrak{A}}^k \cap M_{\mathcal{F}_E})(U_{\mathfrak{A}}^k \cap N_{\mathcal{F}_E}^-)$$

and similarly for $U_{\mathfrak{B}}^k$.

(ii) *For positive integers $k_1 < k_2$, we have*

$$(U_{\mathfrak{B}}^{k_1} U_{\mathfrak{A}}^{k_2}) \cap N_{\mathcal{F}} = (U_{\mathfrak{B}}^{k_1} \cap N_{\mathcal{F}_E})(U_{\mathfrak{A}}^{k_2} \cap N_{\mathcal{F}}).$$

Proof. Part (i) is given by [Bushnell and Henniart 1996, (10.4)] and noting that, if the decomposition conforms with $\mathcal{L}$ over E, it also conforms with $\mathcal{L}$ over F. For part (ii), we first prove the "maximal" case, i.e., when V is 1-dimensional over E, in which case $N_{\mathcal{F}_E}$ is trivial, and the right-hand side is $U_{\mathfrak{A}}^{k_2} \cap N_{\mathcal{F}}$. This is equal to the left-hand side by [Blondel and Stevens 2009, Lemma A.5 Appendix]. If E is not maximal in A, we can follow the idea in [Blondel and Stevens 2009, Corollary A.6 Appendix]. By (i), we have

$$(U_{\mathfrak{B}}^{k_1} U_{\mathfrak{A}}^{k_2}) \cap P_{\mathcal{F}_E} = (U_{\mathfrak{B}}^{k_1} \cap N_{\mathcal{F}_E})(U_{\mathfrak{A}}^{k_2} \cap N_{\mathcal{F}_E})(U_{\mathfrak{B}}^{k_1} \cap M_{\mathcal{F}_E})(U_{\mathfrak{A}}^{k_2} \cap M_{\mathcal{F}_E}),$$

and note that $N_{\mathcal{F}} \subset P_{\mathcal{F}_E}$, so

$$U_{\mathfrak{B}}^{k_1} U_{\mathfrak{A}}^{k_2} \cap N_{\mathcal{F}} = (U_{\mathfrak{B}}^{k_1} \cap N_{\mathcal{F}_E})(U_{\mathfrak{A}}^{k_2} \cap N_{\mathcal{F}_E})((U_{\mathfrak{B}}^{k_1} \cap M_{\mathcal{F}_E})(U_{\mathfrak{A}}^{k_2} \cap M_{\mathcal{F}_E}) \cap N_{\mathcal{F}}).$$

The last bracket lies in the maximal case for the Levi subgroup $M_{\mathcal{F}_E}$, and so is equal to $U_{\mathfrak{A}}^{k_2} \cap M_{\mathcal{F}_E} \cap N_{\mathcal{F}}$. Since $(U_{\mathfrak{A}}^{k_2} \cap N_{\mathcal{F}_E})(U_{\mathfrak{A}}^{k_2} \cap M_{\mathcal{F}_E} \cap N_{\mathcal{F}}) = U_{\mathfrak{A}}^{k_2} \cap N_{\mathcal{F}}$, we have the desired result. □

We further assume that E/F is tamely ramified, with a tower of intermediate subfields (3-2) coming from an admissible character. Let H^1 be the subgroup defined in Section 3B, and $N_{\mathcal{F}_{E_i}}$, for $i = 0, \dots, t+1$, be the maximal E_i-flags defined by the ordered decompositions in (4-6), which are inductively subordinate by Proposition 4.1.

Corollary 4.7. *Given a sequence of flags as in* (4-3), *then*

$$H^1 \cap N_{\mathcal{F}} = (U_{\mathfrak{B}_0}^1 \cap N_{\mathcal{F}_{E_0}}) \cdots (U_{\mathfrak{B}_t}^{r_{t-1}/2+} \cap N_{\mathcal{F}_{E_t}})(U_{\mathfrak{A}}^{r_t/2+} \cap N_{\mathcal{F}}).$$

Proof. By the inductive subordination, we apply Proposition 4.6 (ii) inductively. First regard $U_{\mathfrak{B}_0}^1 \cdots U_{\mathfrak{B}_t}^{r_{t-1}/2+}$ as a subgroup of $U_{\mathfrak{B}_t}^1$ and so

$$H^1 \cap N_{\mathcal{F}} = (U_{\mathfrak{B}_0}^1 \cdots U_{\mathfrak{B}_t}^{r_{t-1}/2+} \cap N_{\mathcal{F}_{E_t}})(U_{\mathfrak{A}}^{r_t/2+} \cap N_{\mathcal{F}}).$$

We can therefore apply induction on $U_{\mathfrak{B}_0}^1 \cdots U_{\mathfrak{B}_t}^{r_{t-1}/2+} \cap N_{\mathcal{F}_{E_t}}$ and obtain the desired result. □

Proposition 4.8. $\theta|_{N \cap H^1} = \psi_\beta|_{N \cap H^1}$.

Proof. For each i, we already know that θ_i is equal to ψ_{β_i} on

$$(U_{\mathfrak{B}_{i+1}}^{r_i/2+} \cap N_{\mathcal{F}_{E_t}}) \cdots (U_{\mathfrak{A}}^{r_t/2+} \cap N_{\mathcal{F}})$$

from its construction. It suffices to show that θ_i on

$$(U_{\mathfrak{B}_0}^1 \cap N_{\mathcal{F}_{E_0}}) \cdots (U_{\mathfrak{B}_i}^{r_{i-1}/2+} \cap N_{\mathcal{F}_{E_i}}),$$

which is $\xi_i \circ \det_{B_i/E_i}$, is also equal to ψ_{β_i}. Indeed, on all $N_{\mathcal{F}_{E_j}}$ for $j \le i$, the character $\det_{B_i/E_i}$ is trivial, while ψ_{β_i} is also trivial since $\beta_i \in M_{\mathcal{F}_{E_i}} \subset M_{\mathcal{F}_{E_j}}$. □

5. The main result

Let π be an essentially tame supercuspidal representation compactly induced by an extended maximal type $(\mathbf{J}, \Lambda)$ which contains a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ associated to an admissible character ξ, and $N = N_{\mathcal{F}}$ be the maximal unipotent subgroup defined by the F-flag

$$\mathcal{F} = \{V_j\}_{j=1}^n, \qquad \text{where } V_j = \bigoplus_{k=1}^{j} F x_k,$$

and $\{x_j\}_{j=1}^n$ is the ordered basis constructed in (4-4).

Let:

- α_0 be an element in $\mathrm{Mat}_n(F)$ whose matrix representation $(\alpha_0)_{j,k}$ with respect to $\mathfrak{b}$ is 1 if $j-k=1$ but is 0 if k is a multiple of $[E_0:F]$, and is 0 if $j-k>1$ (and can be anything if $j\leq k$).
- α_{-1} be an element in $\mathrm{Mat}_n(F)$ whose matrix representation $(\alpha_{-1})_{j,k}$ with respect to $\mathfrak{b}$ is 0 if $j-k=1$ but is 1 if k is a multiple of $[E_0:F]$, and is 0 if $j-k>1$ (and can be anything if $j\leq k$).
- $\alpha=\alpha_{-1}+\alpha_0$.

Hence, with the notation defined in [Paskunas and Stevens 2008], we have

$$\alpha\in X^+_{\mathcal{F}}:=\{x\in A: xV_i\subset V_{i+1}\text{ and } xV_i\not\subset V_i\text{ for all } i\},$$

and so by [Paskunas and Stevens 2008, Lemma 1.2] ψ_α defines a non-degenerate character of N. (Note that, in contrast, ψ_β may not extend to a character of the whole N.)

Theorem 5.1. $\mathrm{Hom}_{N\cap\mathbf{J}}(\psi_\alpha,\Lambda)\neq 0$.

Proof. We show that the condition at the beginning of [Paskunas and Stevens 2008, Section 4.2] is satisfied, then our result is implied by Corollary 4.13 of the same work. Hence it suffices to show that $(\mathcal{F},\psi_{\alpha_0},\alpha_{-1})$ satisfies the conditions (i)-(iv) in Theorem 3.3 of the same work.

(i) This condition is just $\mathcal{F}_{E_0}\subset\mathcal{F}$ in our notation, which is true by construction.

(ii) In Proposition 4.8, we showed that $\theta|_{N\cap H^1}=\psi_\beta|_{N\cap H^1}$. Now with the matrix representation of β and elements in H^1 in Proposition 4.2(i), we know that $\psi_\beta|_{N\cap H^1}=\psi_{\alpha_0}|_{N\cap H^1}$.

(iii) If $E=E_0$, then $N_{\mathcal{F}_{E_0}}$ is trivial and the result is clearly satisfied. If $E\neq E_0$, then ψ_{α_0} on $N_{\mathcal{F}_{E_0}}$ is trivial since the matrix entry $(\alpha_0)_{k,k+1}$ with respect to $\mathfrak{b}$ is 0 when k is a multiple of $[E_0:F]$.

(iv) The maximal unipotent subgroup $N_{\mathcal{F}_{E_0}}\cap U_{\mathfrak{B}_0}/U^1_{\mathfrak{B}_0}$ of $U_{\mathfrak{B}_0}/U^1_{\mathfrak{B}_0}$ is defined by the cyclic basis

$$\{\bar{1},\bar{\beta}_{-1},\bar{\beta}^2_{-1},\dots,\bar{\beta}^{[E:E_0]-1}_{-1}\},$$

where each $\bar{x}$ is $x+\mathfrak{P}_{\mathfrak{B}_0}\in U_{\mathfrak{B}_0}/U^1_{\mathfrak{B}_0}$ for $x\in U_{\mathfrak{B}_0}$. The character $\psi_{\beta_{-1}}$ clearly defines a nondegenerate character, by arguments similar to [Bushnell and Henniart 1998, 2.1]. This character is equal to $\psi_{\alpha_{-1}}$ by Proposition 4.2 (ii). □

5A. ***A formula for the Artin conductor.*** Suppose that (π_1,π_2) is a pair of essentially tame supercuspidal representations of $\mathrm{GL}_{n_i}(F)$, for $i=1,2$, such that their extended maximal simple types contain the same simple character, hence the same associated simple or null stratum. Recall the conductor of the epsilon factor for the pair (π_1, π_2) computed in [Paskunas and Stevens 2008; Kim 2014], which is

$$\mathfrak{f}(\pi_1 \times \pi_2) = \mathfrak{f}(\tau_1 \times \tau_2) + \frac{n_1 n_2}{e(E_0/F)[E_0 : F]} v_E(x_{[E_0:F]}/x_1),$$

where τ_i is a supercuspidal representation of $\mathrm{GL}_{n_i/[E_0:F]}(E_0)$, compactly induced from the "level-zero" component of the extended maximal simple type of π_i (see [Paskunas and Stevens 2008, Section 7]).

Let's compare this result with the calculation in [Bushnell et al. 1998, Theorem 6.5]. The conductor formula implies that

$$\mathfrak{f}(\pi_1 \times \pi_2) = \mathfrak{f}(\tau_1 \times \tau_2) + n_1 n_2 \frac{\mathfrak{c}(\beta)}{[E_0 : F]^2}.$$

Here $\mathfrak{c}(\beta)$ is a certain kind of "discriminant", whose value can be inductively computed from [Bushnell and Henniart 2003, 3.1] as

$$\frac{\mathfrak{c}(\beta_i)}{[E_i : F]^2} = \frac{\mathfrak{c}(\beta_{i+1})}{[E_{i+1} : F]^2} + \frac{k_0(\beta_i, \mathfrak{A})}{e(E_0/F)} \left(\frac{1}{[E_{i+1} : F]} - \frac{1}{[E_i : F]} \right).$$

We can rewrite it into a direct formula as

$$\mathfrak{c}(\beta) = \frac{[E_0 : F]}{e(E_0/F)} \sum_{i=0}^{t} ([E : E_{i+1}] - [E : E_i]) k_0(\beta_i, \mathfrak{A}).$$

In the essentially tame case, our result implies that

$$v_E(x_{[E_0:F]}/x_1) = \sum_{i=0}^{t} ([E : E_{i+1}] - [E : E_i]) v_E(\beta_i). \tag{5-1}$$

We can use Proposition 2.3 to see that our result (5-1) agrees with the calculation in the above literatures.

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